

Tutorial 5

Exercise 1

The I combinator simply returns its parameter. It can be used to extract data from encapsulated types and used for composition with other combinators.

Lambda calculus expression: $\lambda x. x$

Exercise 2

- 1 b. $\lambda a. a$
- 2 b. $\lambda a. (\lambda b. ba)$
- 3 b. $\lambda mn. mz$

Exercise 3

- 1 $(\lambda x. x)y = (\lambda x [x := y]. x)$
 $= x [x := y]$
 $= y \quad (\text{Beta Normal Form})$
- 2 $\lambda x. xx = \lambda x. xx \quad (\text{Beta Normal Form})$
- 3 $(\lambda z. zz)(\lambda y. yy) = zz [z := y. yy]$
 $= (\lambda y. yy)(\lambda y. yy) \quad (\text{Divergent})$
- 4 $(\lambda x. xx)y = xx [x := y]$
 $= yy \quad (\text{Beta Normal Form})$

Exercise 4

- 1 $(\lambda y. zy)a = zy [y := a]$
 $= za$
- 2 $(\lambda x. x)(\lambda x. x) = x [x := x. x]$
 $= \lambda x. x$
- 3 $(\lambda x. xy)(\lambda x. xx) = xy [x := x. xx]$
 $= (\lambda x. xx)y$
 $= xx [x := y]$
 $= yy$

$$\begin{aligned}
 4 \quad (\lambda z.z)(\lambda a.aa)(\lambda z.zb) &= z [z := a.aa] (\lambda z.zb) \\
 &= (\lambda a.aa)(\lambda z.zb) \\
 &= aa [a := z.zb] \\
 &= (\lambda z.zb)(\lambda z.zb) \\
 &= zb [z := z.zb] \\
 &= (\lambda z.zb)b \\
 &= zb [z := b] \\
 &= bb
 \end{aligned}$$

Exercise 5

$$1 \quad \lambda x.zx = z$$

$$2 \quad \lambda x.xz = \lambda x.xz$$

$$\begin{aligned}
 3 \quad (\lambda x.bx)(\lambda y.ay) &= bx [x := y.ay] \\
 &= b(\lambda y.ay) \\
 &= ba
 \end{aligned}$$

Exercise 6

1. $\lambda x.xxx$, 3. $\lambda xyz.xy(zx)$, 4. $\lambda xyz.xy(zxy)$ are combinators.

2. $\lambda xy.zx$ is not a combinator. z is a free variable.

Exercise 7

$$\begin{aligned}
 Yg &= \lambda f. (\lambda x.f(xx))(\lambda x.f(xx))g \\
 &= (\lambda x.f(xx))(\lambda x.f(xx)) [f := g] \\
 &= (\lambda x.g(xx))(\lambda x.g(xx)) \\
 &= g(xx) [x := x.g(xx)] \\
 &= g((\lambda x.g(xx))(\lambda x.g(xx))) \\
 &= g(Yg)
 \end{aligned}$$

Exercise 8

$$\begin{aligned}
 1 \quad \text{NOT FALSE} &= (\lambda x. \text{IF } x \text{ FALSE TRUE}) \text{ FALSE} \\
 &= \text{IF } x \text{ FALSE TRUE } [x := \text{FALSE}] \\
 &= \text{IF FALSE FALSE TRUE} \\
 &= (\lambda bff. bff) \text{ FALSE FALSE TRUE} \\
 &= bff [b := \text{FALSE}, f := \text{FALSE}, f := \text{TRUE}] \\
 &= \text{FALSE FALSE TRUE} \\
 &= (\lambda xy. y) \text{ FALSE TRUE} \\
 &= y [y := \text{TRUE}] \\
 &= \text{TRUE}
 \end{aligned}$$

$$\begin{aligned}
 2 \quad \text{OR TRUE FALSE} &= (\lambda xy. \text{IF } x \text{ TRUE } y) \text{ TRUE FALSE} \\
 &= \text{IF } x \text{ TRUE } y [x := \text{TRUE}, y := \text{FALSE}] \\
 &= \text{IF TRUE TRUE FALSE} \\
 &= (\lambda bff. bff) \text{ TRUE TRUE FALSE} \\
 &= bff [b := \text{TRUE}, f := \text{TRUE}, f := \text{FALSE}] \\
 &= \text{TRUE TRUE FALSE} \\
 &= (\lambda xy. x) \text{ TRUE FALSE} \\
 &= x [x := \text{TRUE}] \\
 &= \text{TRUE}
 \end{aligned}$$

$$\begin{aligned}
 3 \quad \text{AND TRUE TRUE} &= (\lambda xy. \text{IF } x \text{ } y \text{ FALSE}) \text{ TRUE TRUE} \\
 &= \text{IF } x \text{ } y \text{ FALSE } [x := \text{TRUE}, y := \text{TRUE}] \\
 &= \text{IF TRUE TRUE FALSE} \\
 &= (\lambda bff. bff) \text{ TRUE TRUE FALSE} \\
 &= bff [b := \text{TRUE}, f := \text{TRUE}, f := \text{FALSE}] \\
 &= \text{TRUE TRUE FALSE} \\
 &= (\lambda xy. x) \text{ TRUE FALSE} \\
 &= x [x := \text{TRUE}] \\
 &= \text{TRUE}
 \end{aligned}$$