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Efficiency of Supply Function Equilibrium in Networked Markets

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***Abstract*— Motivated by the deregulation of electricity market, we study the efficiency loss of the supply function equilibrium (SFE). Specifically, we consider a market where the demand is inelastic, the suppliers submit their supply functions, and a uniform price is set to clear the market. Existing literature has answers to this question without regard to the network structure of the market. However, in many markets (such as electricity markets), there is an underlying physical network that limits the market operations (e.g., in electricity markets, the transmission network determines how electricity flows through the network and thus puts constraints on the supply profile). Motivated by electricity markets, we study how network topology affects the efficiency of SFE. For general mesh networks, we show that the efficiency loss is upper bounded as in the case without networks. Interestingly, we find a class of radial networks, for which the efficiency loss is independent of the local network topology.**

I. INTRODUCTION

The traditional electricity market system uses the supply-follow-demand approach, where a central coordinator (Independent System Operator, or ISO) forecasts an inelastic demand profile for the next 24-hour period and the generation is scheduled to satisfy the demand at the minimal cost. Instead of the usual two-sided supply-demand market structure, the coordinator ISO is necessary here in electricity markets to guarantee that supply is equal to demand at all locations and at all times, because mismatch in demand and supply will cause the power system to deviate from its normal operating frequency and may lead to serious consequences such as blackout [1].

In the ideal scenario, the demand should be allocated among the suppliers by the market maker, such that the total production/generation cost is minimized. In most markets, however, the market maker neither knows the cost functions of the suppliers, nor could it dictate the amounts of supply provided by the suppliers. Motivated by the common practice in electricity markets, we assume that the market maker receives bids from suppliers in the form of *supply functions* (i.e., curves specifying suppliers' generation capabilities and associated costs), and then sets the price to clear the market.[1]

[1]There are other models for market competition, such as the Cournot model (i.e., the suppliers determine their supply levels) and the Bertrand model (i.e., the suppliers determine their prices for the supply). However, such competition model is not used in electricity markets.

The resulting market, however, can be very inefficient, as witnessed by the California electricity crisis [2], due to the fact that the suppliers have different objectives (i.e., maximizing their own profit) than the market maker, and that they can be strategic in submitting the bids. It is thus important to study the efficiency loss due to the strategic behavior of suppliers.

In the literature [3][4][5], the efficiency loss is usually quantified by price of anarchy (PoA), defined as the ratio of the total generation cost at the equilibrium to that at the social optimum. PoA is a number no smaller than 1, and a larger PoA indicates greater efficiency loss.

There has been active research in studying the efficiency loss in general markets [3][6] and in electricity markets [4][5][7][8][9]. The usual conclusion is that the efficiency loss is upper bounded, and vanishes as the number of suppliers increases. These works [3]–[9], however, focus on markets with no underlying network structure.

In many markets, there is an underlying physical network that limits how the supply can match the demand. For instance, in electricity markets, the transmission network topology and the flow limits of transmission lines put constraints on the amounts of electricity generated by the suppliers. In this work, we aim to study how the network affects the efficiency loss in networked markets. Some related works study the efficiency loss in electricity markets while considering the effect of the transmission network [10][11]. However, they adopt a Cournot competition model, where the suppliers determine the amounts of supply, instead of the more practical form of bidding a supply function.

II. MODEL

Our model is motivated by the common practice in deregulated electricity markets, where generators (i.e., suppliers of electricity) compete for the supply of demand. Specifically, we represent a power system as a graph $(\mathcal{N}, \mathcal{E})$. Each node in $\mathcal{N}$ is a bus with a generator or a load or both, and each edge in $\mathcal{E}$ represents a capacitated transmission line connecting two buses. We denote the set of buses with a generator by $\mathcal{N}_g$ and the set of buses with a load by $\mathcal{N}_\ell$. Without loss of generality, we assume that the load is inelastic, and denote the inelastic load profile by $\boldsymbol{d} = (d_j)_{j \in \mathcal{N}_\ell} \in \mathbb{R}_+^{|\mathcal{N}_\ell|}$ at each load bus.[2] If a load is elastic, we can model it as a generator and interpret its supply as the reduction of its load. The total demand is then given by $D \triangleq \sum_{j\in\mathcal{N}_\ell} d_j$.

To maintain supply-demand balance and thus stability of the electricity system, the generators need to fulfill the total demand D. Each generator n has a cost $c_n(s_n)$ in providing s_n unit of electricity, and has upper and lower limits $\bar{s}_n$ and $\underline{s}_n$ on its supply level reflecting its physical constraints. If the ISO knew the cost functions and could orchestrate the exact amount of supply by each generator, it would solve for the optimal supply profile $\boldsymbol{s} = (s_n)_{n\in\mathcal{N}_g}$ that minimizes the total generation cost

$$\sum_{n\in\mathcal{N}_g} c_n(s_n), \tag{1}$$

subject to the constraints that the total supply is equal to the total demand

$$\sum_{n\in\mathcal{N}_g} s_n = D, \tag{2}$$

that the generators are producing within their production limits,

$$\underline{s}_n \le s_n \le \bar{s}_n, \quad \forall n \in \mathcal{N}_g, \tag{3}$$

and more importantly, that the flow on each transmission line is within the line capacity. We adopt a direct current (DC) power flow model, such that the line flow constraints can be written as follows:

$$-\boldsymbol{f} \le \mathbf{A}_g \cdot \boldsymbol{s} + \mathbf{A}_\ell \cdot \boldsymbol{d} \le \boldsymbol{f}, \tag{4}$$

where $\boldsymbol{f} \in \mathbb{R}_+^{|\mathcal{E}|}$ is the vector of line capacities, $\mathbf{A}_g \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{N}_g|}$ and $\mathbf{A}_\ell \in \mathbb{R}^{|\mathcal{E}|\times|\mathcal{N}_\ell|}$ are shift-factor matrices. The shift-factor matrices $\mathbf{A}_\ell$ and $\mathbf{A}_g$ reflect the underlying transmission network topology and the physical properties of the lines, such as their admittance and impedance. Interested readers are referred to [1] and [11] for more details. The flows on each power line can go in either direction, and thus can be negative, which motivates the two-sided bound in (4).

In practice, the generators submit bids to the ISO to compete for the supply of demand. The bid is a parameterized supply function with the following form adopted from [3]:

$$S_n(p, w_n) = D - \frac{w_n}{p},$$

where $w_n \in \mathbb{R}_+$ is generator n's strategic action, and $p \in \mathbb{R}_+$ is the price. To clear the market, i.e., to find the price p satisfies the condition $\sum_{n\in\mathcal{N}_g} S_n(p, w_n) = D$, the ISO sets the price p as follows:

$$p(\boldsymbol{w}) = \frac{\sum_{n\in\mathcal{N}_g} w_n}{(|\mathcal{N}_g| - 1)\, D}.$$

We observe that here the market clearing price is affected by each generator's strategy w_n. Each generator n chooses its action w_n to maximize its own payoff:

$$\begin{aligned} u_n\left(w_n, \boldsymbol{w}_{-n}\right) \;\; &\triangleq \;\; p\left(w_n, \boldsymbol{w}_{-n}\right) \cdot S_n\left[p\left(w_n, \boldsymbol{w}_{-n}\right), w_n\right] \\ &- \;\; c_n\left(S_n\left[p(w_n, \boldsymbol{w}_{-n}), w_n\right]\right), \end{aligned}$$

where $\boldsymbol{w}_{-n}$ is the action profile of all the generators other than generator n.

An action profile $\boldsymbol{w}$ is a supply function equilibrium (SFE), if each generator n's action w_n maximizes its payoff, subject to the constraints (2)–(4), i.e., each generator cannot unilaterally deviate and improve its own payoff.

III. Summary of Results

We study general mesh networks and a class of radial (tree) networks. First, for general mesh networks, we proved that the supply profile at any SFE is unique, and is the solution to a convex optimization problem. The convex optimization problem has the same constraints (2)–(4), with a modified objective function based on the total cost in (1). This result allows us to compute the equilibrium supply profile, compare it against the social welfare optimality and hence the efficiency loss efficiently. Moreover, we proved that the PoA is upper bounded, and that the upper bound decreases with the number of generators.

Second, we consider a class of radial networks. We refer to each subtree starting from a child of the root and the edge connecting the child with the root as a *branch.* This special class of radial networks has homogeneity among each branch, where the nodes and lines within the same branch are the same (i.e., the same cost function, the same demand, the same upper and lower bounds of supply, and the same line flow limits). Different branches can have various parameters and cost functions. We proved that at both the socially optimal and equilibrium supply profiles, the supply levels of nodes within a branch are the same, and are independent of the network topology of each individual branch.

We formally state our results for the special radial networks as follows.

Proposition 1: In a radial network with identical nodes and lines within each branch, the following statements hold for both socially optimal and equilibrium supply profiles:

- The nodes within a branch have the same supply.
- The only possibly congested line (i.e., a line with a binding line flow constraint) is the one connected to the root.

One implication of Proposition 1 is that the local network topology in a branch does not matter. For illustration, we show in Fig. 1 two networks with different local network topologies. Based on Proposition 1, these two networks have the same socially optimal and equilibrium supply profiles.

Another implication of Proposition 1 is that we can compute the socially optimal and equilibrium supply profiles efficiently. Suppose that there are one root (branch 0) and K branches (branches $1, \ldots, K$). Denote the number of nodes in branch k by N^k, the demand in branch k by d^k, the cost function by $c^k(\cdot)$, the lower and upper bounds on generator capacity by $\underline{s}^k$ and $\bar{s}^k$, and the line flow limit by f^k. We index the root node by 0. Then the socially optimal supply

[2] We use the notation $\mathbb{R}_+$ to denote the set of positive real numbers.

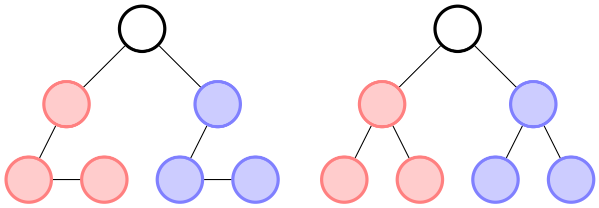

Fig. 1. Two radial networks that have different local topologies but the same socially optimal and equilibrium supply profiles. The nodes with the same color are homogeneous, while the nodes with different colors can be different.

profile is the solution to the following optimization problem:

$$\min_{\{s^k\}_{k=0}^K} \quad \sum_{k=0}^{K} N^k c^k(s^k) \tag{5}$$

$$s.t. \quad \sum_{k=0}^{K} N^k s^k = D, \tag{6}$$

$$\underline{s}^k \leq s^k \leq \bar{s}^k, \forall k, \tag{7}$$

$$-f^k \leq N^k(s^k - d^k) \leq f^k, \forall k. \tag{8}$$

The above optimization problem is easier to solve than the original social welfare optimization problem (1)–(4). This is because we do not need to solve for the supply at all the nodes. For the equilibrium supply profile, we can solve a similar optimization problem with the same constraints but a different objective function.

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