SUPPLY FUNCTION EQUILIBRIUM IN POWER MARKETS: MESH NETWORKS

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ABSTRACT

We study decentralized power markets with strategic power generators. In decentralized markets, each generator submits its supply function (i.e., the amount of electricity it is willing to produce at various unit prices) to the independent system operator (ISO), who takes the submitted supply functions as the true marginal cost functions, and dispatches the generators to clear the market. If all generators reported their true marginal cost functions, the market outcome would be efficient (i.e., the total generation cost is minimized). However, when generators are strategic and aim to maximize their own profits, the reported supply functions are not necessarily the marginal cost functions, and the resulting market outcome may be inefficient. The efficiency loss depends on the topology of the underlying transmission network, because the topology sets constraints on the feasible power supply from generators. This paper provides an analytical upper bound of the efficiency loss due to strategic generators. Our upper bound sheds light on how the efficiency loss depends on the (mesh) transmission network topology (e.g., the degrees of buses, the admittances and flow limits of transmission lines).

Index Terms— Power markets, supply function bidding, price of anarchy

1. INTRODUCTION

A special feature of the power system is that the supply and the demand must be balanced at any time, because imbalance may cause serious consequences such as blackout [1]. Therefore, due to the lack of large-scale energy storage, electricity markets become the major instrument in balancing supply and demand and maintaining the stability of power systems.

In decentralized electricity markets, the power generators submit bids to the independent system operator (ISO). A bid, also called a supply function, specifies the amounts of electricity a generator is willing to produce at different prices. After receiving the bids from the generators, the ISO considers the bids as their marginal cost functions, calculates the cost functions (by integration), and dispatches the generators such that the demand is met and the total generation cost is minimized. This procedure is called economic dispatch.

If the generators submitted their true marginal cost functions as their bids, the economic dispatch would result in the socially optimal outcome that minimizes the true total cost. However, the generators aim to maximize their own profits, and for this purpose, may choose bids that are different from their true marginal cost functions. In this case, the outcome of the economic dispatch is inefficient. The goal of this paper is to analytically quantify this inefficiency due to strategic behavior of generators.

To analyze the efficiency loss in decentralized electricity markets, we first characterize the supply function equilibrium (SFE) of the market. We show that the supply profile (i.e., the amounts of electricity produced by each generator) at the equilibrium is unique. Following the game theory literature, we define the efficiency loss as price of anarchy (PoA), namely the ratio of the total generation cost at the SFE to the total cost at the social optimum. Then we provide an analytical upper bound of the PoA. This upper bound depends on the capacity limits of generators, and more importantly, the topology of the underlying transmission networks (e.g., the numbers of outgoing lines from each generator, the admittances and flow limits of transmission lines). Our results provide insights on how to optimize the transmission network in order to reduce the efficiency loss caused by strategic generators. Finally, we will show that our bound generalizes and is tighter than existing bounds in prior work.

The rest of this paper is organized as follows. We discuss related works in Section 2. In Section 3, we will describe our model of electricity markets and define the supply function equilibrium. We analyze the equilibrium in general electricity networks in Section 4. Finally, Section 5 concludes the paper.

2. RELATION TO PRIOR WORK

Two models have been used in most of the works that study strategic behavior in decentralized electricity markets. The first model is the Cournot competition model, where each generator submits the amount of electricity to produce (i.e., a quantity) [2][3]. These works suggest that the network topology plays an important role in the efficiency loss. However, in the Cournot competition model, the generators act quite differently from the way they bid in reality. Hence, we want to analyze the efficiency loss under a more realistic model.

The second model commonly used in the literature is the supply function equilibrium model, where each generator submits the amounts of electricity to produce at different prices (i.e., a curve of price versus quantity) [4][5][6][7]. The SFE model is closer to the real bidding formats in electricity markets. However, most existing works using the SFE model do not study the impact of the transmission network topology on the efficiency loss [4][5][6]. Their upper bounds of the PoA depend on the number of generators only [4], or on the number and the capacity limits of generators [5][6].

The work that is most closed to this paper is our prior work [7], where we analyze the efficiency loss using the SFE model, and quantify the impact of *certain aspects* of the transmission network topology. Specifically, in the analysis in [7], we treated the transmission network as a *radial* network by ignoring the cycles. Consequently, the upper bound of the PoA in [7] does not depend on the admittances of transmission lines, and is not tight when there are cycles in the transmission network. However, it is well known that the transmission network is a mesh network with cycles. Hence, we will obtain a tighter upper bound of the PoA in this paper by considering the cycles in the network.

3. SYSTEM MODEL

We model a power system as a graph $(\mathcal{N}, \mathcal{E})$, where each node in $\mathcal{N}$ is a bus[1] with a generator or a load or both, and each edge in $\mathcal{E}$ is a transmission line connecting two buses. A representative power system, called IEEE 14-bus system, is shown in Fig. 1 [11]. Denote the set of buses that have a generator by $\mathcal{N}_g \subseteq \mathcal{N}$ (For the IEEE 14-bus system, we have $\mathcal{N}_g = \{1, 2, 3, 6, 8\}$). Since the majority of the load in the electricity market is inelastic [12], we assume that the load is inelastic, and denote the inelastic load profile by $\boldsymbol{d} = (d_1, \ldots, d_{|\mathcal{N}|})$. The total demand is then $D \triangleq \sum_{j \in \mathcal{N}} d_j$.

Each generator $n \in \mathcal{N}_g$ has a cost of $c_n(s_n)$ in providing s_n unit of electricity. We make the following standard assumption about cost functions.

Assumption 1 *For each generator n, the cost function $c_n(s_n)$ is strictly convex, increasing, and continuously differentiable in $s_n \in [0, +\infty)$.*

Due to physical constraints, each generator n's supply s_n must be in a range $[\underline{s}_n, \bar{s}_n]$. In addition, the supply profile $\boldsymbol{s} = (s_n)_{n \in \mathcal{N}_g}$ must satisfy physical constraints of the electrical network. First, in a power system, it is crucial to balance the supply and the demand at all time for the stability of the system [1]. Hence, we need to have

$$\textstyle\sum_{n \in \mathcal{N}_g} s_n = D. \tag{1}$$

Second, the flow on each transmission line, which depends on the supply profile, cannot exceed the flow limit of the line.

[1] We will use "node" and "bus" interchangeably.

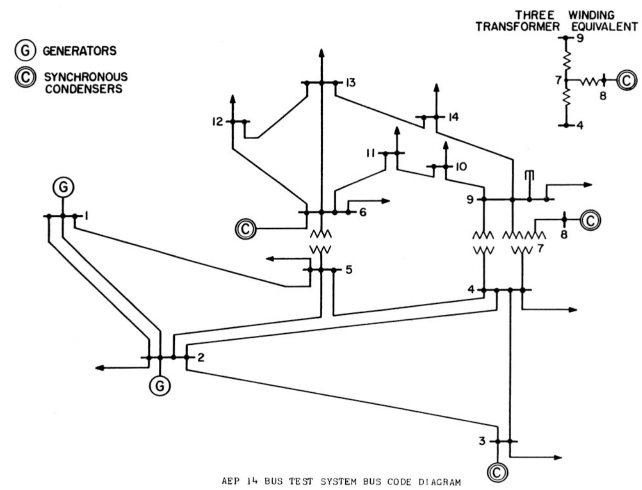

Fig. 1. Illustration of the IEEE 14-bus system, which will serve as a running example and be used in the simulation.

In economic dispatch, the ISO uses the linearized power flow model, where the flow on each line is the linear combination of injections from each node [1][3]. Hence, the line flow constraints can be written as follows:

$$-\boldsymbol{f} \leq \mathbf{A}_g \cdot \boldsymbol{s} + \mathbf{A}_\ell \cdot \boldsymbol{d} \leq \boldsymbol{f}, \tag{2}$$

where $\boldsymbol{f} \in \mathbb{R}^{|\mathcal{E}|}$ is the vector of flow limits, $\mathbf{A}_g \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{N}_g|}$ and $\mathbf{A}_\ell \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{N}|}$ are shift-factor matrices. The shift-factor matrices $\mathbf{A}_g$ and $\mathbf{A}_\ell$ depend on the underlying transmission network topology (e.g., the degrees of nodes and the admittance of transmission lines).

3.1. Benchmark - Social Optimum

Suppose that the ISO knows the true cost functions of each generator. Then it determines the optimal supply profile $\boldsymbol{s}^*$ that minimizes the total generation cost subject to the aforementioned constraints. We summarize the optimization problem to solve as follows:

$$\begin{array}{ll} \max_{\boldsymbol{s}} & \sum_{n \in \mathcal{N}_g} c_n(s_n) \\ s.t. & \sum_{n \in \mathcal{N}_g} s_n = D, \\ & \underline{s}_n \leq s_n \leq \bar{s}_n, \quad \forall n \in \mathcal{N}_g, \\ & -\boldsymbol{f} \leq \mathbf{A}_g \cdot \boldsymbol{s} + \mathbf{A}_\ell \cdot \boldsymbol{d} \leq \boldsymbol{f}. \end{array} \tag{3}$$

To avoid triviality, we assume that the feasible set of power generation is non-empty and is not a singleton.

Assumption 2 *There exists a strictly feasible allocation of power generation $\boldsymbol{s}$.*

Since the cost functions are strictly convex, the optimization (3) has a unique solution. We write this solution as $\boldsymbol{s}^* = (s_n^*)_{n \in \mathcal{N}_g}$, and call it the socially optimal supply profile.

3.2. Deregulated Markets and Supply Function Bidding

In practice, each generator submits a supply function (i.e., a bid) to the ISO. A supply function is a mapping from the unit selling price of electricity to the amount of electricity produced by a generator. In practice, the supply function is usually a step function. For analytical tractability, we assume that each generator n submits a parametrized supply function of the following form: [4][5][7]

$$S_n(p, w_n) = D - \frac{w_n}{p},$$

where $w_n \in \mathbb{R}_+$ is generator n's strategic action, and $p \in \mathbb{R}_+$ is the unit price of electricity. To clear the market, (i.e., to find the price p satisfies the condition $\sum_{n\in\mathcal{N}_g} S_n(p, w_n) = D$), the ISO sets the price p as follows:

$$p(\boldsymbol{w}) = \frac{\sum_{n\in\mathcal{N}_g} w_n}{(|\mathcal{N}_g| - 1)\, D}.$$

where $\boldsymbol{w} = (w_n)_{n\in\mathcal{N}_g} \in \mathbb{R}_+^{|\mathcal{N}_g|}$ is the bidding profile.

Each generator n aims to maximizes its profit, written as $u_n(w_n, \boldsymbol{w}_{-n})$. Here, we write $\boldsymbol{w}_{-n}$ as the action profile of all the generators other than generator n. We can calculate generator n's profit as follows:

$$\begin{aligned} u_n(w_n, \boldsymbol{w}_{-n}) &= p(w_n, \boldsymbol{w}_{-n}) \cdot S_n[p(w_n, \boldsymbol{w}_{-n}), w_n] \\ &- c_n(S_n[p(w_n, \boldsymbol{w}_{-n}), w_n]). \end{aligned}$$

Now we formally define the supply function equilibrium.

Definition 1 *An action profile $\boldsymbol{w}^{**}$ is a supply function equilibrium, if each generator n's action w_n^{**} is a solution to the following profit maximizing problem:*[2]

$$\begin{aligned} \max_{w_n} \quad & u_n\left(w_n, \boldsymbol{w}_{-n}^{**}\right) \\ s.t. \quad & \underline{s}_n \le S_n\left[p\left(w_n, \boldsymbol{w}_{-n}^{**}\right), w_n\right] \le \bar{s}_n, \\ & -\boldsymbol{f} \le [\mathbf{A}_g]_{*n} \cdot S_n\left[p(w_n, \boldsymbol{w}_{-n}^{**}), w_n\right] \\ & + \sum_{\substack{m\in\mathcal{N}_g \\ m\neq n}} \left\{[\mathbf{A}_g]_{*m} \cdot S_m\left[p(w_n, \boldsymbol{w}_{-n}^{**}), w_m^{**}\right]\right\} \\ & + \mathbf{A}_\ell \cdot \boldsymbol{d} \le \boldsymbol{f}. \end{aligned}$$

In a SFE, each generator's action maximizes its own profit given the others' actions. Note that the set of feasible actions of each generator depends on the others' actions. Therefore, the SFE is a generalized Nash equilibrium [13].

4. EFFICIENCY LOSS AT SUPPLY FUNCTION EQUILIBRIUM

4.1. Uniqueness of Equilibrium Supply Profile

Now we will show that the SFE exists and that there is a unique equilibrium supply profile at any SFE.

[2]We denote the nth column of a matrix $\mathbf{A}$ by $[\mathbf{A}]_{*n}$.

Proposition 1 *The SFE exists. In addition, any SFE results in an unique equilibrium supply profile $\boldsymbol{s}^{**} = (s_n^{**})_{n\in\mathcal{N}_g}$.*

Proof: Due to space limitation, the proof is in Appendix A of the technical report [14].

Proposition 1 ensures that although there may be multiple equilibrium bidding profiles, the resulting equilibrium supply profile is always unique.

4.2. Analysis of Efficiency Loss

We quantify the efficiency loss by PoA defined below:

Definition 2 *PoA is the ratio of the total cost at SFE to the total cost at social optimum:*

$$\frac{\sum_{n\in\mathcal{N}_g} c_n(s_n^{**})}{\sum_{n\in\mathcal{N}_g} c_n(s_n^{*})}.$$

By definition, the PoA is never smaller than 1. A larger PoA indicates that the efficiency loss at the equilibrium is larger.

We derive an upper bound of the PoA that depends on the topology of the transmission networks. Before detailed analysis, we need to introduce several useful concepts from graph theory.

Definition 3 (Cycle) *A cycle C of a graph is a sequence of nodes $n_1, n_2, \ldots n_k, n_1$ ($k \ge 3$) that satisfies: 1) there is an edge between every consecutive nodes (i.e., n_i and n_{i+1} for $i = 1, \ldots, k-1$) and between n_k and n_1, and 2) the nodes $n_1, \ldots, n_k$ are distinct.*

By definition, the nodes in a cycle need to be distinct. Take the IEEE 14-bus system in Fig. 1 for example. The sequence $1, 2, 5, 1$ forms a cycle, while the sequence $2, 1, 5, 2, 4, 3, 2$ does not (although all consecutive nodes are connected).

Denote the set of node n's neighbors by $\mathcal{N}(n)$. We then define a partition of $\mathcal{N}(n)$, denoted by $\mathcal{P}(n) \subset 2^{\mathcal{N}(n)}$.

Definition 4 (Partition of neighbors) *A partition $\mathcal{P}(n)$ is a set of singletons and duples of nodes that satisfy:*

1. *the sets in $\mathcal{P}(n)$ are mutually exclusive, and the union of all sets in $\mathcal{P}(n)$ is $\mathcal{N}(n)$;*
2. *any duple of nodes $\{i, j\} \in \mathcal{P}(n)$ are in a cycle with node n, namely i, n, j or j, n, i are in a cycle;*
3. *any two singletones $\{i\}, \{j\} \in \mathcal{P}(n)$ are not in the same cycle.*

The partition $\mathcal{P}(n)$ divides node n's neighbors into several subsets. Roughly speaking, we divide the neighbors by their affiliation to the cycles. Since node n appears only once in a cycle, it has exactly two neighbors in the cycle. Therefore, each subset is either a duple of two nodes (in the case these two nodes are in the same cycle with node n), or a singleton (in the case this node is not in a cycle with either node n or node n's remaining neighbors).

Note that the partition is not unique, but any partition can be chosen for the purpose of deriving our upper bound on PoA. In the following, we will assume that a partition $\mathcal{P}(n)$ has been chosen for each node $n \in \mathcal{N}_g$. Given a partition $\mathcal{P}(n)$, we define a mapping $\mathcal{C}_n : \{i,j\} \mapsto C$ for each duple $\{i,j\} \in \mathcal{P}(n)$. The mapping indicates which cycle C nodes i, j, n belong to. In the case that nodes i, j, n belong to multiple cycles, this mapping selects one of them. Again, this mapping is not unique, but any mapping can be chosen for our purpose. Hence, we will fix one mapping $\mathcal{C}_n$ for each node $n \in \mathcal{N}_g$ in the following.

Take the IEEE 14-bus in Fig. 1 for example again. Node 1 has two neighbors: nodes 2 and 5, which are in cycle $1,2,5,1$ with node 1. Hence, the partition of node 1's neighbors is simply $\mathcal{P}(1) = \{\{2,5\}\}$. Since nodes 1, 2 and 5 belong to multiple cycles, the mapping $\mathcal{C}_1$ is not unique. In this case, we can choose $\mathcal{C}_1(\{2,5\})$ to be either $1,2,5,1$ or $1,2,4,5,1$. Node 2 has four neighbors: nodes $1,3,4,5$. For node 2, the partition of its neighbors is not unique. We could choose either $\mathcal{P}(2) = \{\{1,5\},\{3,4\}\}$ or $\mathcal{P}'(2) = \{\{1\},\{3\},\{4,5\}\}$. If we choose $\mathcal{P}(2)$, we can set the mapping $\mathcal{C}_2$ as $\mathcal{C}_2(\{1,5\}) = 1,2,5,1$ and $\mathcal{C}_2(\{3,4\}) = 2,3,4,2$.

For each generator $n \in \mathcal{N}_g$, we need the partition of its neighbors and the association of neighbors with cycles in order to define an "effective flow limit" for each outgoing link from generator n. We write the effective flow limit of the line from node n to node m as $\hat{f}_{nm}$. If node n's neighbor m is a singleton in the partition (i.e., $\{m\} \in \mathcal{P}(n)$), the effective flow limit is the same as the original flow limit, namely $\hat{f}_{nm} = f_{nm}$. If node n's neighbor i is in a duple in the partition (i.e., $\{i,j\} \in \mathcal{P}(n)$), the effective flow limit is as follows:

$$\hat{f}_{ni} = \min \left\{ f_{ni}, \sum_{\substack{l,k \in \mathcal{C}_n(\{i,j\}) \\ \{l,k\} \neq \{n,i\}}} \frac{f_{lk}}{B_{lk}} \cdot B_{ni} \right\}. \tag{4}$$

The effective flow limit is the minimum between the original flow limit and another term that depends on the flow limits and admittances of the other lines in the cycle. The latter term is the flow from n to i when the other lines in the cycle reach flow limits. The flow from n to i cannot exceed this term, even if the flow limit f_{ni} allows.

Note that the effective flow limit depends on the direction of the flow, namely $\hat{f}_{ni} \neq \hat{f}_{in}$ in general. This is because the partition of the neighbors and the associated cycles is different for different nodes n and i. For example, for the IEEE 14-bus system in Fig. 1, we have

$$\hat{f}_{12} = \min \left\{ f_{12}, \left(\frac{f_{25}}{B_{25}} + \frac{f_{51}}{B_{51}} \right) B_{12} \right\}$$

under the partition $\mathcal{P}(1) = \{\{2,5\}\}$, and

$$\hat{f}_{21} = f_{21} = f_{12}$$

under the partition $\mathcal{P}'(2) = \{\{1\},\{3\},\{4,5\}\}$.

Now we give our analytical upper bound of the PoA.

Theorem 1 *The PoA is upper bounded by*

$$1 + \max_{n \in \mathcal{N}_g} \frac{\min \left\{ \bar{s}_n, D - \sum_{\substack{m \in \mathcal{N}_g \\ m \neq n}} \underline{s}_m, d_n + \sum_{m \in \mathcal{N}(n)} \hat{f}_{nm} \right\}}{\sum_{m \in \mathcal{N}_g m \neq n} \bar{s}_m - D}.$$

Proof: Due to space limitation, the proof is in Appendix B of the technical report [14].

The upper bound in Theorem 1 gives us insights on the key factors that influence the efficiency loss. First, the upper bound is higher if one generator has a significantly higher capacity limit than the others. In this case, this generator may have market power, especially when the total capacity from the other generators are barely enough to fulfill the demand. Second, the upper bound is higher if one generator has higher local demand and higher effective flow limits of its outgoing links. In this case, this generator has advantage over the other generators in fulfilling its local demand (because incoming flow limits constrain the import of electricity from the other generators), and can more easily export its electricity generation to other nodes due to higher outgoing effective flow limits. Therefore, this generator has more influence on the market outcome.

Our upper bound in Theorem 1 recovers the bounds in prior work as special cases. When all the generators have the same capacity limits of $\underline{s}_n = 0$ and $\bar{s}_n = D$ for all $n \in \mathcal{N}_g$, and when we ignore the network topology (i.e., ignore the term $d_n + \sum_{m \in \mathcal{N}(n)} \hat{f}_{nm}$), the bound reduces to the one in [4]. If we allow generators to have different capacity limits $\bar{s}_n$, the bound reduces to the one in [5]. If we ignore the cycles in the transmission network, we have $\hat{f}_{nm} = f_{nm}$ for all n and m, and hence recover the bound in [7].

5. CONCLUSION

We analyzed the efficiency loss in decentralized electricity markets. The distinct feature of our work is our consideration of the topology of the mesh transmission network. We show that there exists a unique equilibrium supply profile, and gave an analytical upper bound of the efficiency loss at the equilibrium. Our upper bound suggests that to reduce the efficiency loss, we should evenly distribute the generation capacity and outgoing effective flow limits among the generators.

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