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Problem 1: Computations with inverses

(a) Which of the following matrices are invertible? For each of the invertible ones, write down the inverse and check that it works by multiplying in both orders to confirm that you get I_2 each time:

$$A = \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix}, \quad B = \begin{bmatrix} 2 & 3 \\ 4 & 6 \end{bmatrix}, \quad C = \begin{bmatrix} -1 & -4 \\ 5 & -3 \end{bmatrix}.$$

(b) For the matrices

$$A = \begin{bmatrix} 4 & 1 & -1 \\ 0 & 5 & 1 \\ 0 & 0 & 6 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & -1 & 2 \\ 3 & -3 & 5 \\ 3 & -2 & 2 \end{bmatrix}$$

check that the matrices

$$A' = \begin{bmatrix} 1/4 & -1/20 & 1/20 \\ 0 & 1/5 & -1/30 \\ 0 & 0 & 1/6 \end{bmatrix}, \quad B' = \begin{bmatrix} 4 & -2 & 1 \\ 9 & -4 & 1 \\ 3 & -1 & 0 \end{bmatrix}$$

are respective inverses by multiplying to see that $A'A$ and AA' both equal I_3 and likewise for $B'B$ and BB'.

Solution:

(a) In the 2×2 case, invertibility of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is exactly the condition that the $ad - bc \neq 0$, in which case the inverse is

$$\frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

For A, we have $1 \cdot 8 - 3 \cdot 2 = 2$, so the inverse is $\frac{1}{2} \begin{bmatrix} 8 & -3 \\ -2 & 1 \end{bmatrix}$. To see that this works, we multiply

$$\frac{1}{2} \begin{bmatrix} 8 & -3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 2 & 8 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 8-6 & 24-24 \\ -2+2 & -6+8 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The product in the other order is an extremely similar calculation even down to the level of the arithmetic for finding individual matrix entries (this is special to the 2×2 case).

For B, we have $2 \cdot 6 - 3 \cdot 4 = 0$, so B is not invertible.

For C, we have $(-1) \cdot (-3) - (-4) \cdot 5 = 23$, so the inverse is $\frac{1}{23} \begin{bmatrix} -3 & 4 \\ -5 & -1 \end{bmatrix}$. To see that this works, we multiply

$$\frac{1}{23} \begin{bmatrix} -3 & 4 \\ -5 & -1 \end{bmatrix} \begin{bmatrix} -1 & -4 \\ 5 & -3 \end{bmatrix} = \frac{1}{23} \begin{bmatrix} 3+20 & 12-12 \\ 5-5 & 20+3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

The product in the other order works out very similarly, as it did for A.

(b) These are all direct matrix multiplications, the main point of which is to see that computing products such as $A'A$ and AA' in opposite orders involves *very different* calculations at the level of matrix entries, in contrast with the 2×2 cases in (a).

Problem 2: Using inversion

(a) Consider the system of equations

$$4x + y - z = 7, \quad 5y + z = -3, \quad 6z = 2.$$

Explain why this is the same as $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$, and A as in Problem 1(b). Then use your knowledge of A^{-1} from there to "solve for $\mathbf{x}$" (hint: multiply both sides of the vector formulation by A^{-1}), and check that the solution you obtained really works. Does the method work if the constants on the right side of these equations change?

(b) Consider the system of equations

$$3x + 2y = 7, \ \ 2x + y = -3, \ \ x + y = 2.$$

Explain why this is the same as $M\mathbf{x} = \mathbf{b}$ for $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix}$, and $M = \begin{bmatrix} 3 & 2 \\ 2 & 1 \\ 1 & 1 \end{bmatrix}$.

Verify that $M'M = \mathrm{I}_2$ for $M' = \begin{bmatrix} 0 & 1 & -1 \\ 2 & -3 & 0 \end{bmatrix}$, and (via multiplication on the left by M') that if $M\mathbf{x} = \mathbf{b}$ then $\mathbf{x} = M'\mathbf{b}$. But compute $M'\mathbf{b}$ explicitly and check that it does *not* actually satisfy the given system of equations. What went wrong?

Solution:

(a) Multiplying the matrix-vector product $A\mathbf{x}$ produces the given system of equations entry by entry, so $A\mathbf{x} = \mathbf{b}$ at the level of equating corresponding entries is indeed the given system of equations. Multiplying both sides by A^{-1} gives $A^{-1}(A\mathbf{x}) = A^{-1}\mathbf{b}$, with the left side equal to $(A^{-1}A)\mathbf{x} = \mathrm{I}_3\mathbf{x} = \mathbf{x}$, so we get $\mathbf{x} = A^{-1}\mathbf{b}$. Explicitly, this is

$$\begin{bmatrix} 1/4 & -1/20 & 1/20 \\ 0 & 1/5 & -1/30 \\ 0 & 0 & 1/6 \end{bmatrix} \begin{bmatrix} 7 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ -2/3 \\ 1/3 \end{bmatrix}.$$

Plugging this into the given system of equations indeed works out:

$$8 - 2/3 - 1/3 = 7, \ \ 5(-2/3) + 1/3 = -9/3 = -3, \ \ 6/3 = 2.$$

The same method works in general, because for $\mathbf{x} = A^{-1}\mathbf{b}$ for any $\mathbf{b} \in \mathbf{R}^3$ we have $A\mathbf{x} = A(A^{-1}\mathbf{b}) = (AA^{-1})\mathbf{b} = \mathrm{I}_3\mathbf{b} = \mathbf{b}$. Note that to "derive" the solution we used that $A^{-1}A = \mathrm{I}_3$ but to check it works we used the equation $AA^{-1} = \mathrm{I}_3$ involving multiplication of A and A^{-1} in the *other* order.

(b) The translation into matrix language goes as in (a), and that $M'M = \mathrm{I}_2$ is a direct calculation. We compute $M'\mathbf{b} = \begin{bmatrix} -5 \\ 23 \end{bmatrix}$, which fails *all* of the equations in the system we're given.

The problem is that from the condition $\mathbf{x} = M'\mathbf{b}$ we calculate $M\mathbf{x} = M(M'\mathbf{b}) = (MM')\mathbf{b}$ and MM' might not equal I_3 (so $(MM')\mathbf{b}$ has no reason to equal $\mathbf{b}$). In general, when proceeding from an assumption that there is a solution to deduce what the solution must be, unless all steps were reversible (which isn't the case here: multiplying by M' on the left winds up not being undone by multiplication by M on the left, for example: $MM' \neq \mathrm{I}_3$) we *have to check* if what we obtain actually works. It could fail, as happened here: ultimately it was our *assumption* that there is a solution that was false.

Explicit calculation shows MM' is very different from I_3:

$$MM' = \begin{bmatrix} 4 & -3 & -3 \\ 2 & -1 & -2 \\ 2 & -2 & -1 \end{bmatrix}.$$

Problem 3: Determining linear independence

For each of the following collections of vectors, determine if it is linearly independent (think in terms of expressing a vector as a linear combination of others or studying if "$\sum c_j \mathbf{v}_j = \mathbf{0}$" can happen with some nonzero c_j, not by using Gram–Schmidt), and give a basis of their span in each case.

(a) $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 3 \\ 4 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 5 \\ 5 \\ 6 \end{bmatrix}.$

(b) $\mathbf{u} = \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 5 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}.$

(c) $\mathbf{u} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 3 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 11 \\ 6 \end{bmatrix}.$

(d) $\mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 1 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 1 \end{bmatrix}$

Solution:

(a) They satisfy the linear relation $\mathbf{u} - 2\mathbf{v} + \mathbf{w} = \mathbf{0}$. To find such a relation, note that all of these three vectors are nonzero and are not scalar multiples of each other. Hence, if any one of them were to be a linear combination of the others then all coefficients in the linear combination would have to be nonzero and so each of them would be a linear combination of the others. Thus, we lose nothing by just focusing on trying to write $\mathbf{u}$ in terms of $\mathbf{v}$ and $\mathbf{w}$ (i.e., if any linear dependence relation exists among the three then this approach must succeed). Trying to write $\mathbf{u} = x\mathbf{v} + y\mathbf{w}$ is a system of 3 equations in 2 unknowns, and solving two of those equations gives one solution that is also checked to satisfy the third equation, leading to the expression $2\mathbf{v} - \mathbf{w}$ for $\mathbf{u}$, hence the linear relation mentioned at the start.

A basis for the span is given by any two of the three vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ since none of them are scalar multiples of another and each can be written as a linear combination of the other two (as can be seen from their linear relation $\mathbf{u} - 2\mathbf{v} + \mathbf{w} = \mathbf{0}$).

(b) These are linearly independent. If we express the condition $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} = \mathbf{0}$ in terms of scalar equations we obtain $\begin{cases} c_2 + 2c_3 = 0 \\ c_1 - c_3 = 0 \\ c_1 + 5c_2 + c_3 = 0 \end{cases}$ from where we immediately see $c_3 = c_1$ and $c_2 = -2c_1$. Substituting these into the last equation gives $c_1 - 10c_1 + c_1 = 0$ so $c_1 = 0$ and hence $c_2 = c_3 = 0$. This confirms linear independence. (Alternatively, since the vectors are all nonzero and none is a scalar multiple of another, if there is any linear dependence relation among the vectors then all coefficients must be nonzero and so we could write each of them as a linear combination of the other two. Hence, it is equivalent to show that the situation $\mathbf{u} = x\mathbf{v} + y\mathbf{w}$ is impossible: this is a system of 3 equations in x and y, and one can check that it has no simultaneous solution.)

Since these vectors are linearly independent, they are a basis of their span.

(c) This is three vectors in $\mathbf{R}^2$, so they cannot be linearly independent since $\dim \mathbf{R}^2 = 2$. But the first two vectors are nonzero and neither is a scalar multiple of the other, so their span is 2-dimensional. Being inside $\mathbf{R}^2$, they span $\mathbf{R}^2$ and so are a basis of that span. In particular, the third vector must be a linear combination of the first two. (The specific linear combination is found by solving the pair of equations $11 = a + 3b$ and $6 = -2a + b$, which is $(a, b) = (-1, 4)$.)

In fact, any two of the three vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{w}$ will form a basis for the span (which is $\mathbf{R}^2$) since all are nonzero and none of them is a scalar multiple of any of the others.

(d) These are linearly independent. If we express $c_1\mathbf{u} + c_2\mathbf{v} + c_3\mathbf{w} + c_4\mathbf{x} = \mathbf{0}$ in terms of 4 scalar equations via each vector entry, we get $\begin{cases} c_1 + c_2 + c_3 = 0 \\ c_1 + c_2 + c_4 = 0 \\ c_1 + c_3 + c_4 = 0 \\ c_2 + c_3 + c_4 = 0 \end{cases}$

Comparing the equations pairwise, we conclude that $c_1 = c_2 = c_3 = c_4$ (e.g., the difference of the first and third gives $c_2 = c_4$, the difference of the first and second gives $c_3 = c_4$, and the difference of the third and fourth gives $c_1 = c_2$, so all c_i's are equal to each other). Now substituting this into any of the equations gives that all c_i's vanish.

Consequently, these four vectors form a basis for their span, which must be $\mathbf{R}^4$ for dimension reasons ($\mathbf{R}^n$ has only itself as an n-dimensional subspace).

Problem 4: Matrix of a quadratic form

(a) Consider the quadratic form $q(x, y, z) = x^2 + 2y^2 - z^2 + 4xy + 6xz - 2yz$. Find the symmetric matrix A so that $q(x, y, z) = \begin{bmatrix} x & y & z \end{bmatrix} A \begin{bmatrix} x \\ y \\ z \end{bmatrix}$.

(b) Consider the quadratic form $q(w, x, y, z) = 2x^2 + y^2 + 3xy + 4yw - zw$ (note that w is the first coordinate, x is the second coordinate, etc.). Find the symmetric matrix A so that $q(w, x, y, z) = \begin{bmatrix} w & x & y & z \end{bmatrix} A \begin{bmatrix} w \\ x \\ y \\ z \end{bmatrix}$.

Solution:

(a) We can read off

$$A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 2 & -1 \\ 3 & -1 & -1 \end{bmatrix}.$$

(b) Keeping in mind that this has been set up with x as the second coordinate and not the first, we can read off

$$A = \begin{bmatrix} 0 & 0 & 2 & -1/2 \\ 0 & 2 & 3/2 & 0 \\ 2 & 3/2 & 1 & 0 \\ -1/2 & 0 & 0 & 0 \end{bmatrix}.$$

Problem 5: Computing the Gram–Schmidt process

Run the Gram–Schmidt process on the following collection of vectors, and obtain an orthogonal basis for its span.

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} -4 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} -2 \\ 1 \\ 2 \\ 3 \end{bmatrix}, \quad \mathbf{v}_4 = \begin{bmatrix} -4 \\ 5 \\ -2 \\ -3 \end{bmatrix}.$$

Using the outcome of your calculations, also compute the dimension of the span and if you encounter any $\mathbf{w}_i$ equal to $\mathbf{0}$ then use this to produce a linear dependence relation among the $\mathbf{v}_j$'s (in which case, and as a safety check, confirm such a linear dependence relation by direct computation once you have found one).

As a safety check on your work, make sure at each step that each $\mathbf{w}_i$ is orthogonal to the previous $\mathbf{w}_j$'s. (The $\mathbf{v}_i$'s have been designed so that you only have to work with integers throughout, and in particular the $\mathbf{w}_i$'s have integer entries. If you find yourself at any step grappling with things like $-5/3$ or $11/4$ and so on, you have made a mistake.)

Solution: We start with

$$\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}.$$

The second vector is given by

$$\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{Proj}_{\mathbf{w}_1} \mathbf{v}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 = \mathbf{v}_2 - \frac{-6}{6}\mathbf{w}_1 = \mathbf{v}_2 + \mathbf{w}_1 = \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix},$$

which (as a safety check) is directly checked to be orthogonal to $\mathbf{w}_1$. The third vector is

$$\begin{aligned}
\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_1} \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_2} \mathbf{v}_3 &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} \mathbf{w}_2 \\
&= \mathbf{v}_3 - \frac{6}{6}\mathbf{w}_1 - \frac{12}{12}\mathbf{w}_2 \\
&= \mathbf{v}_3 - \mathbf{w}_1 - \mathbf{w}_2 \\
&= \begin{bmatrix} -2 \\ 1 \\ 2 \\ 3 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.
\end{aligned}$$

This means that $\mathbf{v}_3$ can be written as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$. In fact, the vanishing of $\mathbf{w}_3$ gives that $\mathbf{v}_3 = \mathbf{w}_1 + \mathbf{w}_2 = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{w}_1) = \mathbf{v}_1 + (\mathbf{v}_2 + \mathbf{v}_1) = 2\mathbf{v}_1 + \mathbf{v}_2$ (as can also be confirmed directly once we have discovered it).

For the last step, we disregard the vanishing $\mathbf{w}_3$ and compute

$$\begin{aligned}
\mathbf{w}_4 = \mathbf{v}_4 - \mathbf{Proj}_{\mathbf{w}_1} \mathbf{v}_4 - \mathbf{Proj}_{\mathbf{w}_2} \mathbf{v}_4 = \mathbf{v}_4 - \frac{\mathbf{v}_4 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1} \mathbf{w}_1 - \frac{\mathbf{v}_4 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2} \mathbf{w}_2 &= \mathbf{v}_4 - \frac{-12}{6}\mathbf{w}_1 - \frac{12}{12}\mathbf{w}_2 \\
&= \mathbf{v}_4 + 2\mathbf{w}_1 - \mathbf{w}_2 \\
&= \begin{bmatrix} -4 \\ 5 \\ -2 \\ -3 \end{bmatrix} + 2\begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} -3 \\ 1 \\ 1 \\ 1 \end{bmatrix} \\
&= \begin{bmatrix} 1 \\ 4 \\ -1 \\ 0 \end{bmatrix}.
\end{aligned}$$

This can be directly checked to be orthogonal to $\mathbf{w}_1$ and $\mathbf{w}_2$ (as a safety check).

So the nonzero $\mathbf{w}_1, \mathbf{w}_2$, and $\mathbf{w}_4$ are an orthogonal basis for the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and $\mathbf{v}_4$, and hence this span is therefore 3-dimensional.

Problem 6: Normal matrices (Extra)

An $n \times n$ matrix A is called *normal* if A commutes with its transpose $A^\top$ (i.e., $AA^\top = A^\top A$). For example, every symmetric matrix (and in particular every diagonal matrix) is normal, since it is even equal to its own transpose.

(a) Check that every matrix of the form $A = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$ (which is not symmetric when $b \neq 0$) is normal, and that $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ is normal. (When computing $MM^\top$ and $M^\top M$ you only need to compute the part on or above the diagonal, since the product is symmetric; saving work in this way assumes one doesn't make a miscalculation.)

(b) Explain why any orthogonal $n \times n$ matrix is normal, and that if M is normal then so are M^2 and M^3. (How about M^r for general $r \geq 1$?)

(c) In (a) we give a type of non-symmetric normal 2×2 matrix (with $b \neq 0$). Show by explicit example that for a symmetric (hence normal) 2×2 matrix M and normal 2×2 matrix M' of the type in (a), the product $A = MM'$ can fail to be normal. (Nearly anything you try for M and M' should work, as long as you avoid too many matrix entries equal to 0.)

In connection with (c), it is a fact that if M and M' are normal $n \times n$ matrices that *commute* (i.e., $MM' = M'M$) then the product MM' is normal. However, this has no short explanation; it requires a big generalization of an upcoming important result called the Spectral Theorem.

Solution:

(a) Direct computation shows $AA^\top$ and $A^\top A$ both equal $\begin{bmatrix} a^2+b^2 & 0 \\ 0 & a^2+b^2 \end{bmatrix}$. Direct computation also shows that $BB^\top$ and $B^\top B$ are both equal to $\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}$.

(b) For an orthogonal $n \times n$ matrix A, by definition A is invertible with $A^\top = A^{-1}$. Hence, normality for such A amounts to saying A commutes with A^{-1}. But in the definition of invertibiity we have that both products AA^{-1} and $A^{-1}A$ are equal to I_n.

We know that $(M^r)^\top = (M^\top)^r$ for any $r \geq 1$, so normality of M^r for normal M means that M^r and $(M^\top)^r$ commute when M and $M^\top$ commutes. For $r = 2$ such commuting of those rth powers for normal M is seen directly: since $MM^\top = M^\top M$ we can slide $M^\top$'s all the way to the left:

$$(M^2)(M^\top)^2 = MMM^\top M^\top = MM^\top MM^\top = M^\top MMM^\top = M^\top MM^\top M = M^\top M^\top MM = (M^\top)^2 M^2.$$

The same sliding pattern works for $r = 3$ (with more sliding), and for any r. (The same argument even shows that if $n \times n$ matrices M and M' commute then M^r and ${M'}^r$ commute for every $r \geq 1$.)

(c) Nearly anything will work, but here is an example. Take M to be $\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}$ and $M' = \begin{bmatrix} 2 & 1 \\ -1 & 2 \end{bmatrix}$. Then $A = MM' = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}$, and we compute

$$AA^\top = \begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix}\begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 10 & -10 \\ -10 & 10 \end{bmatrix}, \quad A^\top A = \begin{bmatrix} 3 & -3 \\ -1 & 1 \end{bmatrix}\begin{bmatrix} 3 & -1 \\ -3 & 1 \end{bmatrix} = \begin{bmatrix} 18 & -6 \\ -6 & 2 \end{bmatrix}.$$

Problem 7: Determining independence with vector algebra (Extra)

If $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbf{R}^{1000}$ are linearly independent, show that $\mathbf{u}+\mathbf{v}, 2\mathbf{u}+\mathbf{v}+\mathbf{w}, -\mathbf{u}+\mathbf{v}+\mathbf{w}$ are linearly independent. (Hint: don't think in terms of vector entries! Think in terms of the formulation of linear independence as: "$\sum c_i \mathbf{v}_i = \mathbf{0}$ implies all c_i vanish".)

Solution: Assuming $c_1(\mathbf{u}+\mathbf{v}) + c_2(2\mathbf{u}+\mathbf{v}+\mathbf{w}) + c_3(-\mathbf{u}+\mathbf{v}+\mathbf{w}) = \mathbf{0}$, we want to show that the c_i's all vanish. Let's collect common terms to rewrite this as

$$(c_1 + 2c_2 - c_3)\mathbf{u} + (c_1 + c_2 + c_3)\mathbf{v} + (c_2 + c_3)\mathbf{w} = \mathbf{0}.$$

But $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are linearly independent, so such vanishing is equivalent to the vanishing of all three coefficients. In other words, this amounts to the system of three simulataneous equations

$$c_1 + 2c_2 - c_3 = 0, \;\; c_1 + c_2 + c_3 = 0, \;\; c_2 + c_3 = 0.$$

The last equation says $c_3 = -c_2$, so substituting this into each of the first two equations turns those into the conditions

$$c_1 + 2c_2 + c_2 = 0, \;\; c_1 + c_2 - c_2 = 0,$$

or equivalently

$$c_1 + 3c_2 = 0, \;\; c_1 = 0.$$

The second of these says $c_1 = 0$, so then the first of these says $c_2 = 0$, so also $c_3 = -c_2 = 0$. Hence, all c_i's must vanish, which establishes the desired linear independence.

(This particular example was designed so that one could attack the system of 3 equations in 3 unknowns by bare hands. For a more complicated situation this wouldn't have been feasible. We will see soon how to systematically attack the problem of solving large linear systems by using results in matrix algebra.)