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## Problem 1: Visualizing vectors and convex combinations in $\mathbb{R}^2$

Let $\mathbf{a} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} -1 \\ -1 \end{bmatrix}$.

(a) Compute $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$. Draw $\mathbf{a}$, $\mathbf{b}$ and $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$ in a coordinate plane, and describe geometrically where the sum lies relative to $\mathbf{a}$ and $\mathbf{b}$. Do you expect such a geometric relationship is true for any 2-vectors $\mathbf{a}, \mathbf{b}$? How about for 3-vectors?

(b) Compute $\frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{b}$ and $\frac{3}{4}\mathbf{a}+\frac{1}{4}\mathbf{b}$, and plot these. Do you notice a pattern that should hold for any 2-vectors $\mathbf{a}$ and $\mathbf{b}$?

(c) Compute $\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}+\frac{1}{3}\mathbf{c}$, plot this point and draw segments joining it to each of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$, and describe geometrically where this lies relative to $\mathbf{a}, \mathbf{b}, \mathbf{c}$.

(d) Find a nonzero vector that is perpendicular to $\mathbf{a}$. (Hint: draw a picture.)

**Solution:**

(a) The vector $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ has both entries halfway between the corresponding entries for $\mathbf{a}$ and $\mathbf{b}$, and when plotted it is the midpoint of the segment joining the tips of $\mathbf{a}$ and $\mathbf{b}$, as shown in Figure 1. This midpoint relation works for any $\mathbf{a}$ and $\mathbf{b}$ since it can be analyzed using each vector entry separately, in which case it becomes the fact that the average of two numbers lies halfway between them. This argument applies equally well to 3-vectors.

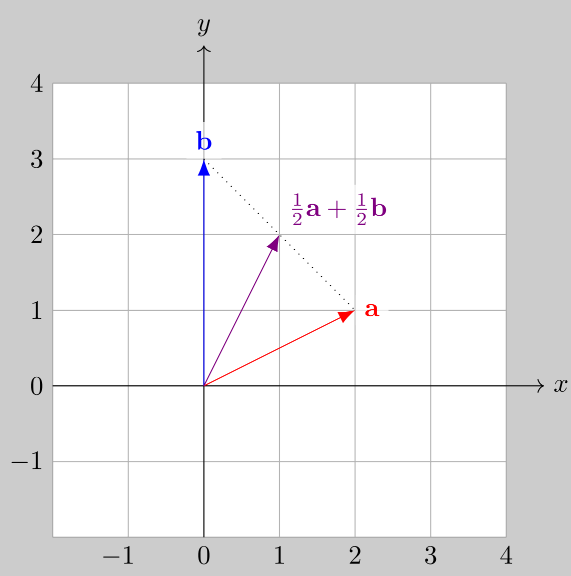


Figure 1: The vectors $\mathbf{a}$, $\mathbf{b}$ and $\frac{1}{2}\mathbf{a}+\frac{1}{2}\mathbf{b}$.

(b) We compute $\frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{b} = \begin{bmatrix} 2/3 \\ 7/3 \end{bmatrix}$ and $\frac{3}{4}\mathbf{a}+\frac{1}{4}\mathbf{b} = \begin{bmatrix} 3/2 \\ 3/2 \end{bmatrix}$, which when plotted are each on the segment joining $\mathbf{a}$ and $\mathbf{b}$, respectively 1/3 and 3/4 of the way from the $\mathbf{a}$ endpoint (as in part (a), we can think about this entry by entry), as shown in Figure 2. By the same reasoning, for any convex combination $t\mathbf{a}+(1-t)\mathbf{b}$ with $0 \le t \le 1$ we get a point on the segment joining $\mathbf{a}$ and $\mathbf{b}$ whose proportion of distance along the segment from the tip of $\mathbf{a}$ is $1-t$.

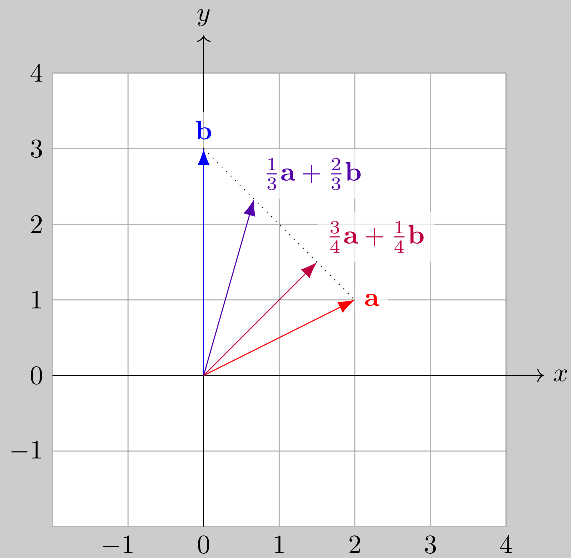


Figure 2: The vectors $\mathbf{a}$, $\mathbf{b}$, $\frac{1}{3}\mathbf{a}+\frac{2}{3}\mathbf{b}$ and $\frac{3}{4}\mathbf{a}+\frac{1}{4}\mathbf{b}$.

(c) The vector $\dfrac{1}{3}\mathbf{a}+\dfrac{1}{3}\mathbf{b}+\dfrac{1}{3}\mathbf{c}$ is equal to $\begin{bmatrix}1/3\\1\end{bmatrix}$, and when plotted it lies inside the triangle whose vertices are the tips of $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$. It is equally balanced from each of the corners as shown in Figure 3 since the coefficient $1/3$ on each vector make each entry of the sum be the average of the corresponding entries. Informally, this vector average is the "center of mass," or balancing point, of the triangle.

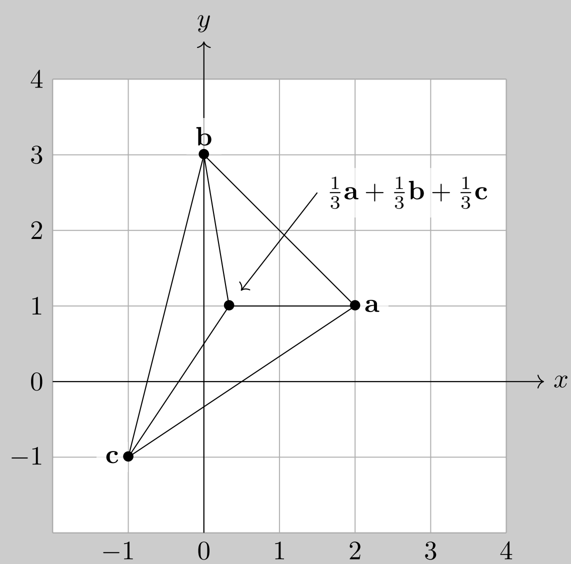


Figure 3: The point $\frac{1}{3}\mathbf{a}+\frac{1}{3}\mathbf{b}+\frac{1}{3}\mathbf{c}$ is the "center of mass" of the triangle formed by vertices $\mathbf{a}$, $\mathbf{b}$, $\mathbf{c}$.

(d) By staring at a grid with $\mathbf{a}$ marked, we see that 90 degrees to the left is $\begin{bmatrix}-1\\2\end{bmatrix}$, so that works (as does any nonzero scalar multiple of it).

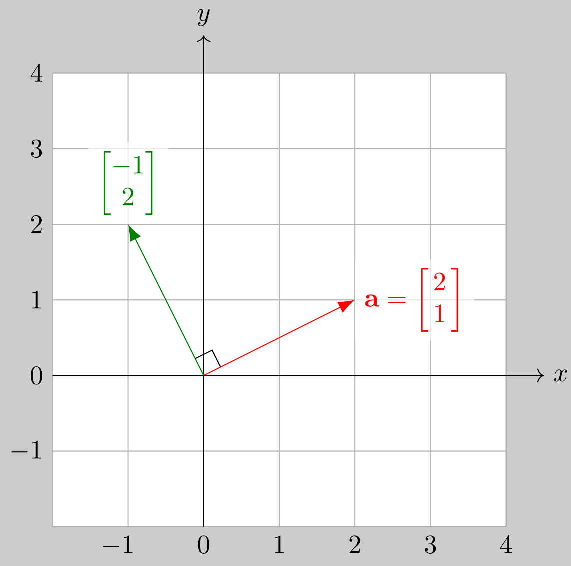


Figure 4: A vector that is perpendicular to $\mathbf{a}$.

## Problem 2: Linear combinations

(a) Express $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$.

(b) Express $\begin{bmatrix} 5 \\ 4 \end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. (This amounts to solving 2 equations in 2 unknowns.)

(c) Write a general 2-vector $\begin{bmatrix} x \\ y \end{bmatrix}$ as a linear combination of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. The coefficients of the linear combination will depend on $x$ and $y$. Make sure that for $x = 5$ and $y = 4$ it agrees with your answer to (b)!

(d) (**Extra**) Draw a picture for each of (a) and (b). Then draw all points $n\mathbf{v} + m\mathbf{w}$ for integers $n$ and $m$ with $-2 \leq n, m \leq 3$, and draw lines through these parallel to each of $\mathbf{v}$ and $\mathbf{w}$, which should yield a tiling of the plane with copies of the parallelogram $P$ whose corners are at the tips of the vectors $\mathbf{0}$, $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v} + \mathbf{w}$.

Interpret geometrically (without any calculations) the meaning of the answer to (b) in terms of these parallelograms, and do the same for the result in (c) that such a linear combination always exists. (Hint: for (c), mark $(6, 5)$ and $(3, 4)$ and compare where they lie among the parallelograms with the general formula in (c) for these two cases.)

**Solution:**

(a) If we write $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = a\mathbf{a} + b\mathbf{b} = \begin{bmatrix} a \\ b \end{bmatrix}$ then we see that $a = 5$ and $b = 4$ works: $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = 5 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 4 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$. This one could also have been done by inspection, thinking about the meaning of coordinates of a point.

(b) If we write $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = a\mathbf{v} + b\mathbf{w} = \begin{bmatrix} 2a + b \\ a + 2b \end{bmatrix}$ then the conditions are $2a + b = 5$ and $a + 2b = 4$, which we solve to get $a = 2$ and $b = 1$: $\begin{bmatrix} 5 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ (which can be directly checked as a safety measure).

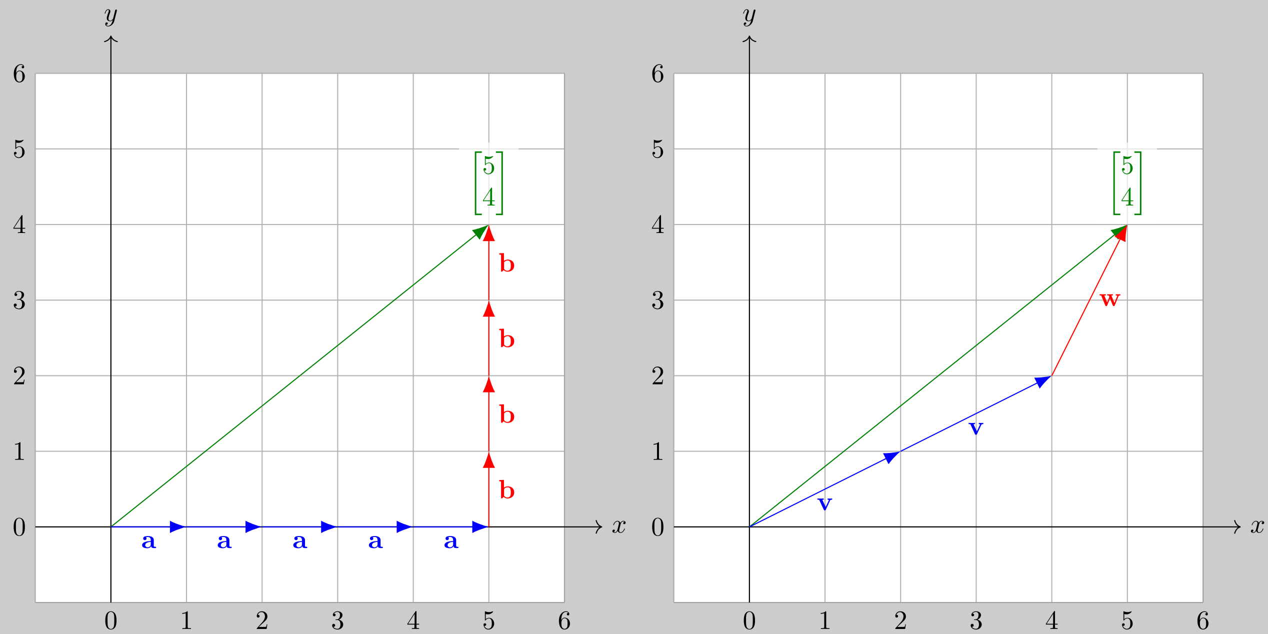


Figure 5: Drawing of solution for (a) (left) and solution for (b) (right).

(c) If we write $\begin{bmatrix}x\\y\end{bmatrix} = a\mathbf{v} + b\mathbf{w} = \begin{bmatrix}2a+b\\a+2b\end{bmatrix}$ then the conditions are $2a + b = x$ and $a + 2b = y$. We can then apply the method from high school algebra to isolate each of $a$ and $b$ on its own (e.g., double the first equation and subtract the second to get rid of $b$), eventually obtaining $a = (2x-y)/3$ and $b = (2y-x)/3$, so $\begin{bmatrix}x\\y\end{bmatrix} = \dfrac{2x-y}{3}\begin{bmatrix}2\\1\end{bmatrix} + \dfrac{2y-x}{3}\begin{bmatrix}1\\2\end{bmatrix}$ (which can be directly checked, as a safety measure). For $x = 5$ and $y = 4$ this indeed agrees with the answer to (b), and in Figure 6 we illustrate the "general case".

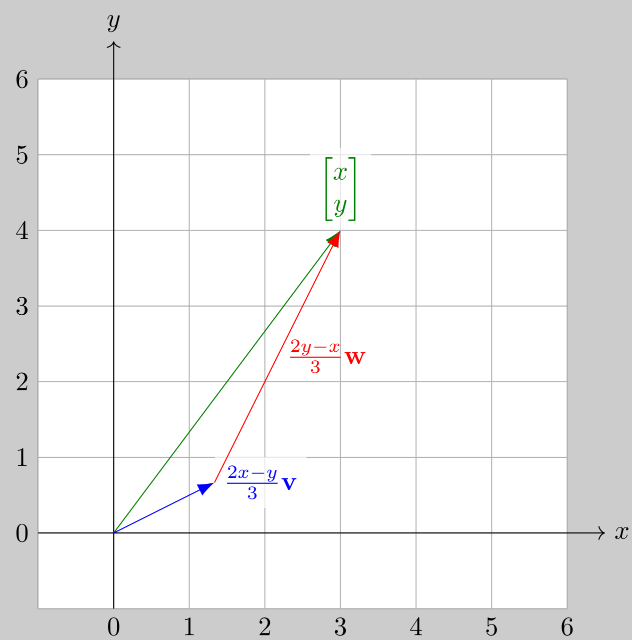


Figure 6: Drawing of solution to (c).

(d) The drawings for (a), (b), (c) are shown in Figures 5 and 6. In each case, the parallelogram $P$ as in the hint can be translated around to exactly tile the plane, with the corners corresponding to the linear combinations $n\mathbf{v} + m\mathbf{w}$ where $n$ and $m$ are *integers* (counting how many steps up/down or left/right we moved the parallelogram), as shown in Figure 7.

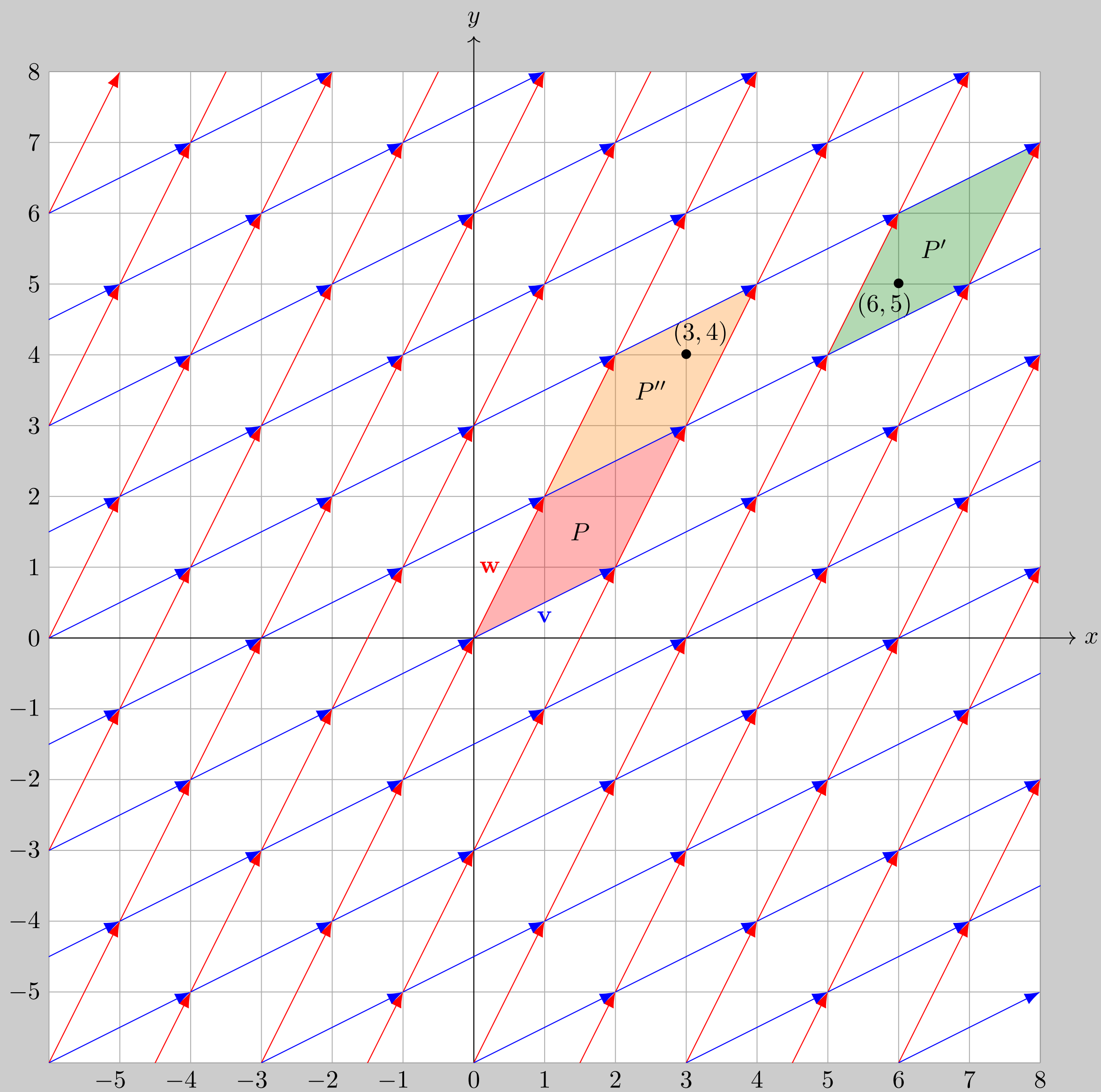


Figure 7: Translated copies of $P$ tile the plane

The points $(6,5)$ and $(3,4)$ are marked in Figure 7. The general formula in (c) yields in these two cases the expressions

$$\begin{bmatrix} 6 \\ 5 \end{bmatrix} = \frac{7}{3}\mathbf{v} + \frac{4}{3}\mathbf{w}, \quad \begin{bmatrix} 3 \\ 4 \end{bmatrix} = \frac{2}{3}\mathbf{v} + \frac{5}{3}\mathbf{w},$$

and in Figure 7 these lie in the parallelograms as shown that are respectively translated by $P$ as follows. The point $(6,5)$ is in the green parallelogram $P'$ obtained from $P$ by moving 2 steps in the direction of $\mathbf{v}$ and 1 step in the direction of $\mathbf{w}$, with $(6,5)$ lying inside $P'$ an amount 1/3 of the way inside in the directions of each of $\mathbf{v}$ and $\mathbf{w}$, encoding that $7/3 = 2 + 1/3$ and $5/3 = 1 + 1/3$. The point $(3,4)$ is in the orange parallelogram $P''$ obtained from $P$ by moving 0 steps in the direction of $\mathbf{v}$ and 1 step in the direction of $\mathbf{w}$, with $(3,4)$ lying inside $P'$ an amount 2/3 of the way inside in the directions of each of $\mathbf{v}$ and $\mathbf{w}$, encoding that $2/3 = 0 + 1/3$ and $5/3 = 1 + 2/3$.

In general, each point $\mathbf{x}$ in the plane lies in *some* parallelogram $\mathcal{P}$ as in Figure 7, with its position inside $\mathcal{P}$ nailing down the specific coefficients $a$ and $b$ for which $a\mathbf{v} + b\mathbf{w} = \mathbf{x}$: the "integer parts" of $a$ and $b$ record the lower-left corner of $\mathcal{P}$, and their "fractional parts" record what fraction of each side direction into $\mathcal{P}$ the point $\mathbf{x}$ lies relative to the lower-left corner (as we worked out for $\mathbf{x} = \begin{bmatrix} 6 \\ 5 \end{bmatrix}$ and $\mathbf{x} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$ above).

## Problem 3: Length and distance

(a) Compute the distance between $\begin{bmatrix} 7 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} -5 \\ 3 \end{bmatrix}$. (The answer is an integer.)

(b) Compute the distance between $\begin{bmatrix} 4 \\ -1 \\ 0 \\ -2 \end{bmatrix}$ and $\begin{bmatrix} 7 \\ -6 \\ 1 \\ -3 \end{bmatrix}$. (The answer is an integer.)

(c) If a nonzero vector $\mathbf{v}$ lies at an angle $30°$ counterclockwise from the positive $x$-axis, what is the unit vector in the same direction as $\mathbf{v}$? (Draw a picture to get an idea.) What if $30°$ is replaced with a general angle $\theta$?

**Solution:**

(a) The difference vector is $\begin{bmatrix} 12 \\ -5 \end{bmatrix}$ (up to a sign), whose length is $\sqrt{12^2 + (-5)^2} = \sqrt{169} = 13$.

(b) The difference vector is $\begin{bmatrix} -3 \\ 5 \\ -1 \\ 1 \end{bmatrix}$ (up to a sign), whose length is $\sqrt{9 + 25 + 1 + 1} = \sqrt{36} = 6$.

(c) If we write $\mathbf{v} = \begin{bmatrix} a \\ b \end{bmatrix}$ then it lies on the line through the origin making an angle of $30°$, so the unit vector of interest is where that line crosses the unit circle in the first quadrant. This is $(\cos(30°), \sin(30°)) = (\sqrt{3}/2, 1/2)$. By the same method, for a general angle the unit vector is $(\cos\theta, \sin\theta)$.

## Problem 4: Vector operations with data

Suppose there are three students in Math 51 with the following components for their course grades:
Student 1: 81/100 on homework, 83/100 on midterm A, 70/100 on midterm B, 75/100 on the final.
Student 2: 73/100 on homework, 75/100 on midterm A, 74/100 on midterm B, 88/100 on the final.
Student 3: 90/100 on homework, 95/100 on midterm A, 88/100 on midterm B, 92/100 on the final.

(a) Write down vectors $\mathbf{v}_{\mathrm{HW}}, \mathbf{v}_{\mathrm{A}}, \mathbf{v}_{\mathrm{B}}, \mathbf{v}_{\mathrm{Final}}$ (all in $\mathbf{R}^3$) representing respectively the grades as percentages on homework, midterm A, midterm B, and the final exam (e.g., for a score of $83/100$, the vector entry should be 83 rather than .83).

(b) Give a general formula as a linear combination of those four vectors in $\mathbf{R}^3$ for a 3-vector $\mathbf{v}_{\mathrm{CG}}$ whose entries are the course grades of the three students in order from student 1 to student 3, assuming the breakdown of the total grade for the course is 16% homework, 36% final, 24% each midterm, and then compute it (you may use a calculator).

**Solution:**

(a) We have

$$\mathbf{v}_{\text{HW}} = \begin{bmatrix}81\\73\\90\end{bmatrix}, \mathbf{v}_{\text{A}} = \begin{bmatrix}83\\75\\95\end{bmatrix}, \mathbf{v}_{\text{B}} = \begin{bmatrix}70\\74\\88\end{bmatrix}, \mathbf{v}_{\text{Final}} = \begin{bmatrix}75\\88\\92\end{bmatrix}.$$

(b) We have

$$\mathbf{v}_{\text{CG}} = \frac{16}{100}\mathbf{v}_{\text{HW}} + \frac{24}{100}\mathbf{v}_{\text{A}} + \frac{24}{100}\mathbf{v}_{\text{B}} + \frac{36}{100}\mathbf{v}_{\text{Final}}$$

since in each individual vector entry (first, second, or third) this expresses precisely the uniform rule for computing the course grade. Using a calculator, we get $\mathbf{v}_{\text{CG}} = \begin{bmatrix}76.68\\79.12\\91.44\end{bmatrix}$.

## Problem 5: Geometry with dot products

(a) Using that perpendicularity is governed by the dot products being equal to 0, find a nonzero vector in $\mathbf{R}^3$ that is perpendicular to $\mathbf{v} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$. Next, find a *unit* vector with the same property. Finally, find a vector perpendicular to $\mathbf{v}$ that is *not* a scalar multiple of the first.

(b) Find an equation in $x, y, z$ that characterizes when $\begin{bmatrix}x\\y\\z\end{bmatrix}$ is perpendicular to $\begin{bmatrix}2\\-1\\1\end{bmatrix}$. What does this collection of vectors look like?

(c) (**Extra**) What does the collection of nonzero vectors $\mathbf{w} = \begin{bmatrix}x\\y\end{bmatrix}$ making an angle of at most $60^\circ$ against $\mathbf{v} = \begin{bmatrix}3\\-4\end{bmatrix}$ look like? Using the relation of dot products and cosines to describe this region with a pair of conditions of the form $ax^2 + bxy + cy^2 \geq 0$ and $y \leq (3/4)x$ (away from the origin).

**Solution:** The perpendicularity of a vector $\begin{bmatrix}x\\y\\z\end{bmatrix}$ against $\mathbf{v} = \begin{bmatrix}2\\-1\\1\end{bmatrix}$ is the condition

$$0 = 2x - y + z; \tag{$\star$}$$

this answers the first part of (b). To visualize the collection of all such vectors, consider the directions perpendicular to the line through $\mathbf{v}$ in $\mathbf{R}^3$; this is a *plane* through the origin (as we'll see formally in Chapter 3). To answer (a), we can pick whatever $x$ and $y$ we like (not both 0) and then set $z$ to be $y - 2x$ to enforce perpendicularity as in ($\star$). So two such vectors are $\mathbf{w}_1 = \begin{bmatrix}1\\0\\-2\end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix}1\\1\\-1\end{bmatrix}$. Notice that $\mathbf{w}_1$ and $\mathbf{w}_2$ are not scalings of each other (since their correponding entries are not related by a common scaling factor), but that they have lengths $\|\mathbf{w}_1\| = \sqrt{5} \neq 1$ and $\|\mathbf{w}_2\| = \sqrt{3} \neq 1$ respectively; so to finish part (a) it remains to find a unit vector perpendicular to $\mathbf{v}$.

We can affect the length (but not the "direction") of a non-zero vector by scaling it by a positive constant; the new vector's length is the original vector's length multiplied by this constant. Meanwhile, notice that scaling does not affect the perpendicularity condition (this can be seen by multiplying both sides of ($\star$) by a constant; it is also consistent with the idea from plane geometry that angle measurements are not affected by the lengths of the angles' sides). So *unit*-length vectors in the direction of $\mathbf{w}_1$ and $\mathbf{w}_2$, namely the vectors multipled by the reciprocals of their lengths:

$$\frac{1}{\|\mathbf{w}_1\|}\mathbf{w}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix}1\\0\\-2\end{bmatrix} \quad \text{and} \quad \frac{1}{\|\mathbf{w}_2\|}\mathbf{w}_2 = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\-1\end{bmatrix},$$

are automatically perpendicular to $\mathbf{v}$.

For (c), it looks like a sector: the region between two half-lines emanating from the origin (at a total angle of $120°$). Since $\cos 60° = 1/2$, the condition (for nonzero $\mathbf{w}$) is

$$\frac{1}{2} \le \frac{\mathbf{v}\cdot\mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{3x-4y}{5\sqrt{x^2+y^2}},$$

or equivalently

$$\sqrt{x^2+y^2} \le (2/5)(3x-4y).$$

This forces $3x-4y \ge 0$, or in other words $y \le (3/4)x$, in which case it is harmless to square both sides to arrive at the equivalent inequality $x^2+y^2 \le (4/25)(9x^2-24xy+16y^2)$, or in other words $25x^2+25y^2 \le 36x^2-96xy+64y^2$, which is to say $11x^2-96xy+39y^2 \ge 0$ (away from the origin since we needed $\mathbf{w} \ne \mathbf{0}$ to make sense of the angle, though allowing the origin for this final inequality simply inserts the vertex of the sector).

## Problem 6: Algebra with dot products

(a) For $\mathbf{a} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix}$ show that $\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = \mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c}$.

(b) Give an example of 2-vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ for which $(\mathbf{a}\cdot\mathbf{b})\mathbf{c} \ne (\mathbf{a}\cdot\mathbf{c})\mathbf{b}$. (Hint: what if $\mathbf{b}$ and $\mathbf{c}$ are not on the same line through the origin?)

(c) (**Extra**) Explain in terms of variables why $\mathbf{v}\cdot(\mathbf{w}_1+\mathbf{w}_2) = \mathbf{v}\cdot\mathbf{w}_1 + \mathbf{v}\cdot\mathbf{w}_2$ for any 3-vectors $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, $\mathbf{w}_1 = \begin{bmatrix} a_1 \\ b_1 \\ c_1 \end{bmatrix}$, and $\mathbf{w}_2 = \begin{bmatrix} a_2 \\ b_2 \\ c_2 \end{bmatrix}$. If you then replace $\mathbf{v}$ with $\mathbf{v}_1+\mathbf{v}_2$ for 3-vectors $\mathbf{v}_1$ and $\mathbf{v}_2$ and apply another instance of the same general identity, why does it follow without any extra work with algebra in vector entries that

$$(\mathbf{v}_1+\mathbf{v}_2)\cdot(\mathbf{w}_1+\mathbf{w}_2) = \mathbf{v}_1\cdot\mathbf{w}_1 + \mathbf{v}_1\cdot\mathbf{w}_2 + \mathbf{v}_2\cdot\mathbf{w}_1 + \mathbf{v}_2\cdot\mathbf{w}_2?$$

(This is showing the analogue for vectors of the fact for numbers that the distributive law $r(s+t) = rs+rt$ is what makes the identity $(a+b)(c+d) = ac+ad+bc+bd$ hold, since $(a+b)(c+d) = (a+b)c+(a+b)d = ac+bc+ad+bd$.)

Do your arguments work for $n$-vectors for any $n$?

(d) For $n$-vectors $\mathbf{w}_1$ and $\mathbf{w}_2$, verify that $\|\mathbf{w}_1+\mathbf{w}_2\|^2 = \|\mathbf{w}_1\|^2 + 2(\mathbf{w}_1\cdot\mathbf{w}_2) + \|\mathbf{w}_2\|^2$ by using the relation $\|\mathbf{w}\|^2 = \mathbf{w}\cdot\mathbf{w}$ and general properties of dot products as stated in (c) (even if you didn't do (c)), *not* by writing out big formulas for lengths and dot products in terms of vector entries.

**Solution:**

(a) We compute

$$\mathbf{a}\cdot(\mathbf{b}-\mathbf{c}) = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -5 \\ 9 \\ -1 \end{bmatrix} = -20-18-3 = -41$$

and

$$\mathbf{a}\cdot\mathbf{b} - \mathbf{a}\cdot\mathbf{c} = \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 5 \\ -2 \end{bmatrix} - \begin{bmatrix} 4 \\ -2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 6 \\ -4 \\ -1 \end{bmatrix} = (4-10-6)-(24+8-3) = -12-29 = -41.$$

(b) There are many options. Take $\mathbf{b} = \begin{bmatrix}1\\0\end{bmatrix}$ and $\mathbf{c} = \begin{bmatrix}0\\1\end{bmatrix}$, so the two sides point along different lines as long as they're nonzero. So take $\mathbf{a}$ not perpendicular to either one, say $\mathbf{a} = \begin{bmatrix}1\\1\end{bmatrix}$. Then the left side is $\mathbf{c}$ and the right side is $\mathbf{b}$.

(c) Writing $\mathbf{v} = \begin{bmatrix}x\\y\\z\end{bmatrix}$, $\mathbf{w}_i = \begin{bmatrix}a_i\\b_i\\c_i\end{bmatrix}$ we have

$$\mathbf{v} \cdot (\mathbf{w}_1 + \mathbf{w}_2) = \begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}a_1 + a_2\\b_1 + b_2\\c_1 + c_2\end{bmatrix} = x(a_1 + a_2) + y(b_1 + b_2) + z(c_1 + c_2)$$

and

$$\mathbf{v}\cdot\mathbf{w}_1 + \mathbf{v}\cdot\mathbf{w}_2 = \begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}a_1\\b_1\\c_1\end{bmatrix} + \begin{bmatrix}x\\y\\z\end{bmatrix} \cdot \begin{bmatrix}a_2\\b_2\\c_2\end{bmatrix} = (xa_1 + yb_1 + zc_1) + (xa_2 + yb_2 + zc_2) = xa_1 + xa_2 + yb_1 + yb_2 + zc_1 + zc_2,$$

and these two outcomes are equal because of the distributive law (i.e., $r(s + t) = rs + rt$). For this reason, the same calculation works for $n$-vectors for every $n$ (each vector entry is treated on its own – no interaction among entries in different positions – so it doesn't matter how many of them there are).

In the case $\mathbf{v} = \mathbf{v}_1 + \mathbf{v}_2$, we then have

$$(\mathbf{v}_1 + \mathbf{v}_2) \cdot (\mathbf{w}_1 + \mathbf{w}_2) = (\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w}_1 + (\mathbf{v}_1 + \mathbf{v}_2) \cdot \mathbf{w}_2 = (\mathbf{v}_1 \cdot \mathbf{w}_1 + \mathbf{v}_2 \cdot \mathbf{w}_1) + (\mathbf{v}_1 \cdot \mathbf{w}_2 + \mathbf{v}_2 \cdot \mathbf{w}_2),$$

and the right side is the same as the desired expression up to rearranging terms in this sum of 4 numbers.

(d) We use the distributive law for dot product over vector addition (as discussed in (b)):

$$\|\mathbf{w}_1 + \mathbf{w}_2\|^2 = (\mathbf{w}_1 + \mathbf{w}_2) \cdot (\mathbf{w}_1 + \mathbf{w}_2) = \mathbf{w}_1 \cdot \mathbf{w}_1 + \mathbf{w}_1 \cdot \mathbf{w}_2 + \mathbf{w}_2 \cdot \mathbf{w}_1 + \mathbf{w}_2 \cdot \mathbf{w}_2,$$

and the right side is what we want since the outer terms are length squared and the two middle terms agree and hence add up to $2(\mathbf{w}_1 \cdot \mathbf{w}_2)$.

## Problem 7: A correlation coefficient

Consider the collection of 5 data points: $(-2, 5)$, $(-1, 3)$, $(0, 0)$, $(1, -2)$, $(2, -6)$.

(a) Plot the points to see if they look close to a line.

(b) Compute the compute correlation coefficient exactly. Plug that into a calculator to approximate it to three decimal digits to see if its nearness to $\pm 1$ fits well with the visual quality of fit of the line to the data plot in (a).

**Solution:**

(a) A plot of the data, as in Figure 8, shows it is reasonably close to a line with negative slope.

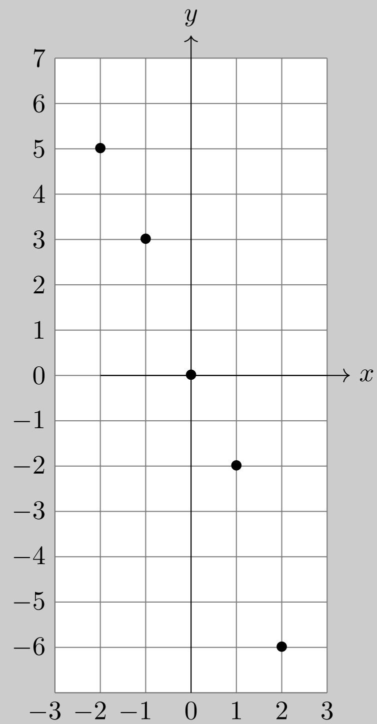


Figure 8: The data appears to fit a line of negative slope quite well.

(b) The initial data vectors are $\mathbf{X} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} 5 \\ 3 \\ 0 \\ -2 \\ -6 \end{bmatrix}$.

The correlation coefficient is

$$r = \frac{\mathbf{X} \cdot \mathbf{Y}}{\|\mathbf{X}\|\|\mathbf{Y}\|},$$

and we have

$$\mathbf{X} \cdot \mathbf{Y} = -27, \quad \|\mathbf{X}\| = \sqrt{10}, \quad \|\mathbf{Y}\| = \sqrt{74},$$

so $r = -27/(\sqrt{10}\sqrt{74}) \approx -0.992$. This is extremely close to $-1$, as we would expect since the data is seen by inspection to look very close to a line of negative slope (though the actual negative slope is not $-1$, as the line is quite steep).

## Problem 8: More convex combinations (Extra)

(a) For the 2-vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, describe the set of all possible vectors $r\mathbf{a} + s\mathbf{b} + t\mathbf{c}$ where $r + s + t = 1$ with $0 \leq r, s, t \leq 1$. Which points in your description correspond to the case $t = 0$. How about $s = 0$? Or $r = 0$? (Hint: plot points for a variety of triples $(r, s, t) = (r, s, 1 - r - s)$ with $0 \leq r, s, 1 - (r + s) \leq 1$.)

(b) Try the same using the 3-vectors $\mathbf{a} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{b} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{c} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$. (Hint: first sketch the points you get with $t = 0$, then with $s = 0$, then with $r = 0$, and finally with $r = s = t = 1/3$.)

(c) Can you explain why your description in (a) applies to any three 2-vectors $\mathbf{a}, \mathbf{b}, \mathbf{c}$ not on a common line? Use whatever physical or mathematical idea comes to mind. (Here is one approach: for $0 \leq t < 1$ check the equality $r\mathbf{a} + s\mathbf{b} +$

$t\mathbf{c} = (1-t)\mathbf{d}_{r,s} + t\mathbf{c}$ with a convex combination on the right where $\mathbf{d}_{r,s}$ is defined to be the convex combination $(r/(r+s))\mathbf{a} + (s/(r+s))\mathbf{b}$; this algebra works because $r + s = 1 - t > 0$. Interpret these convex combinations geometrically.)

(d) Is there a version for a triple of 3-vectors not all on a common line in space? Can you explain why it works?

**Solution:**

(a) This gives all points in the triangle bounded by the coordinate axes and the line $x + y = 1$; setting one of $r, s, t$ to be 0 gives the side of the triangle opposite the corner corresponding to the parameter set to be 0.

(b) Setting one of the parameters to be 0 gives the respective segments $x + y = 1$ in the $xy$-plane ($z = 0$), $x + z = 1$ in the $xz$-plane ($y = 0$), and $y + z = 1$ in the $yz$-plane ($x = 0$). Varying the parameters more generally gives the points in the triangle with those segments as edges (and the points $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ as corners).

(c) We always get the triangle with corners at $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$; when all of $r, s, t$ are positive then we're on the interior and if any of them equal 0 then we're on an edge (and if two of them are 0 – so the third parameter is equal to 1 – then we're at a corner).

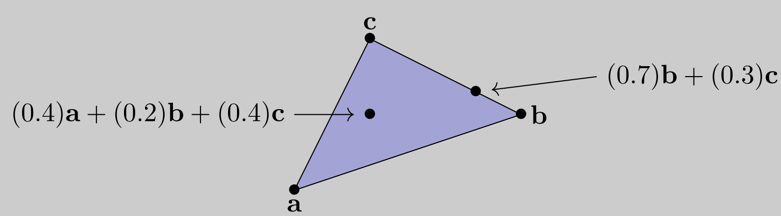


Figure 9: A triangle with vertices $\mathbf{a}, \mathbf{b}, \mathbf{c}$. We have shown two convex combinations of the vertices.

To explain this, given $0 \le t < 1$ we note that the values $r, s \ge 0$ with $r+s = 1-t$ make the ratios $r/(r+s) = r/(1-t)$ and $s/(r+s) = s/(1-t) = 1 - r/(r+s)$ go through exactly the pairs $(q, 1-q)$ with $0 \le q \le 1$. The points $\mathbf{d}_{r,s}$ for such varying $r$ and $s$ therefore account for exactly the points on the edge $E$ joining $\mathbf{a}$ to $\mathbf{b}$ without repetition. Thus, $(1-t)\mathbf{d}_{r,s} + t\mathbf{c}$ is the point on the segment joining $\mathbf{d}_{r,s}$ to the other vertex $\mathbf{c}$ whose distance along the segment from $\mathbf{d}_{r,s}$ at a proportion $t$ of the entire length of that segment (think about $t = 0$). So we are sweeping out the triangle using all of the line segments joining one vertex $\mathbf{c}$ to each of the points on the opposite edge $E$, with $t$ keeping proportional track of where along such a segment a point is located distance-wise from the endpoint on $E$.

(d) The argument in the solution to (c) works without change for 3-vectors, since all of the reasoning in terms of the geometry of a triangle (now in space, rather than in a plane) continues to hold, and likewise for the geometric meaning of a convex combination $t\mathbf{v} + (1-t)\mathbf{w}$ for $\mathbf{v}, \mathbf{w}$ in $\mathbf{R}^3$ rather than in $\mathbf{R}^2$ (much as the interpretation of such convex combinations in terms of being at a point some proportion of the distance along a line segment works for 3-vectors as well as it does for 2-vectors).

## Problem 9: Parametric and equational forms of a plane in $\mathrm{R}^3$

Let $P$ be the plane in $\mathbf{R}^3$ containing the points $(1, 1, 1)$, $(1, 2, 3)$, and $(3, 2, 1)$. (Why is there a unique such plane?)

(a) Find a parametric representation of $P$. (Extra: can you write down many other parametrizations?)

(b) Use the dot product to find a normal vector to $P$. (Hint: Think about why it is the same as a vector perpendicular to two different "directions" within the plane, and then form some displacement vectors.)

(c) Find an equation for $P$ of the form $ax + by + cz = d$ for some $a, b, c, d$ in $\mathbf{R}$.

**Solution:** (a) The difference vectors connecting the point $(1,1,1)$ to the points $(1,2,3)$ and $(3,2,1)$ are $\begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$, respectively. These are clearly not scalar multiples of each other, so indeed there is a unique plane $P$ through the three points. It follows that one parametric representation of the plane $P$ is

$$\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}1\\1\\1\end{bmatrix} + t\begin{bmatrix}0\\1\\2\end{bmatrix} + t'\begin{bmatrix}2\\1\\0\end{bmatrix}, \quad t, t' \in \mathbf{R}.$$

To get other parametrizations of the plane, we can replace the point $\begin{bmatrix}1\\1\\1\end{bmatrix}$ in the above expression by any other point in the plane and then adjust the two difference vectors accordingly, or even leave the above difference vectors unchanged (which corresponds to simply using another reference point from which all motion in the plane is swept out as $t$ and $u$ vary). Or we could go back to the start and use one of the other two given vectors in place of $(1,1,1)$ as the " base point" whose displacement we compute from the others (and there are many other options that could be considered, as we will see later in the course).

(b) We want to find a nonzero vector $\begin{bmatrix}a\\b\\c\end{bmatrix}$ that is perpendicular to the difference vectors $\begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$ that point along different directions in the plane. This amounts to finding a nonzero solution to the system of equations

$$\begin{cases} a\cdot 0 + b\cdot 1 + c\cdot 2 = 0 \\ a\cdot 2 + b\cdot 1 + c\cdot 0 = 0 \end{cases}$$

The first equation means $b = -2c$. The second equation means $b = -2a$. Thus, the condition is $a = c = -\frac{1}{2}b$.

Clearly $a = 1, b = -2, c = 1$ is one such solution. Using the dot product, we can check that the vector $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ is indeed perpendicular to both $\begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$.

One may also notice that this system of equations has many solutions - indeed, for any nonzero $t$, we can take $a = t, b = -2t, c = t$. This reflects the fact that there are many vectors perpendicular to both $\begin{bmatrix}0\\1\\2\end{bmatrix}$ and $\begin{bmatrix}2\\1\\0\end{bmatrix}$: if $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ is perpendicular to those two vectors, then so is every scalar multiple of $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ (the nonzero multiples being the interesting ones).

(c) Since $\begin{bmatrix}1\\-2\\1\end{bmatrix}$ is a normal vector to $P$, it is perpendicular to every vector connecting any two points on $P$. In particular, for any point $\begin{bmatrix}x\\y\\z\end{bmatrix}$ on $P$, the vector $\begin{bmatrix}x-1\\y-1\\z-1\end{bmatrix}$ connecting $\begin{bmatrix}1\\1\\1\end{bmatrix}$ to $\begin{bmatrix}x\\y\\z\end{bmatrix}$ must be perpendicular to $\begin{bmatrix}1\\-2\\1\end{bmatrix}$. This perpendicularity is expressed by a dot product being 0:

$$1\cdot(x-1) + (-2)\cdot(y-1) + 1\cdot(z-1) = 0.$$

Thus, expanding out and combining terms, we obtain that

$$x - 2y + z = 0$$

is an equation for $P$.

## Problem 10: Determining if points are on the same or opposite sides of a plane, and parallel planes

Let $P$ be the plane in $\mathbf{R}^3$ given by the equation $11x - 3y + 2z = 12$, and let $Q$ be the plane parallel to $P$ passing through the point $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$.

Consider the following points: $A = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$, $B = \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}$, $C = \begin{bmatrix} 3 \\ 8 \\ -1 \end{bmatrix}$ (check that none of these lie on $P$!).

(a) For each pair among the three points (i.e. $A$ and $B$, $A$ and $C$, $B$ and $C$), determine whether the two points are on the same side of $P$ or on opposite sides.

(b) Work out an equation for $Q$, and then repeat part (a) but for the plane $Q$.

(c) Note that, because $P$ and $Q$ are parallel to one another and distinct, there is a region of $\mathbf{R}^3$ that is *between* $P$ and $Q$. Determine which (if any) of $A$, $B$, and $C$ lie between $P$ and $Q$.

**Solution:**

(a) The two sides of $P$ are given by the inequalities $11x - 3y + 2z > 12$ and $11x - 3y + 2z < 12$, respectively. We can answer the question by plugging in $A$, $B$, and $C$, and seeing which of these inequalities they verify. For $A$, we obtain $11 \cdot 1 - 3 \cdot 0 + 2 \cdot 1 = 13 > 12$. For $B$, we obtain $11 \cdot 1 - 3 \cdot 2 + 2 \cdot (-1) = 3 < 12$. Finally, for $C$, we obtain $11 \cdot 3 - 3 \cdot 8 + 2 \cdot (-1) = 7 < 12$. Therefore, $B$ and $C$ are on the same side of $P$, and $A$ is on the opposite side of $P$ from both of them.

(b) We would like to use the same strategy as part (a), but we need an equation for the plane $Q$, which is parallel to $P$. The key is that two planes in $\mathbf{R}^3$ are parallel precisely when their normal vectors are scalar multiples of one another (why?); thus, we know that $\begin{bmatrix} 11 \\ -3 \\ 2 \end{bmatrix}$ must be normal to $Q$. It follows that an equation for $Q$ is $11x - 3y + 2z = d$, where $d \in \mathbf{R}$. To determine what $d$ should be, we plug in the point $\mathbf{v}$. That is, we must have $d = 11 \cdot 2 - 3 \cdot 1 + 2 \cdot 0 = 19$. Now we're ready to carry out the strategy as in part (a). In particular, for the points $A$, $B$, and $C$, we have $11 \cdot 1 - 3 \cdot 0 + 2 \cdot 1 = 13 < 19$, $11 \cdot 1 - 3 \cdot 2 + 2 \cdot (-1) = 3 < 19$, and $11 \cdot 3 - 3 \cdot 8 + 2 \cdot (-1) = 7 < 19$, respectively. Thus, $A, B$, and $C$ are all on the same side of $Q$.

(c) The thing we need to realize is that the region between $P$ and $Q$ is characterized by the inequalities

$$12 < 11x - 3y + 2z < 19.$$

One way to think about this is to observe that there are two sides of $P$ and two sides of $Q$, and (by visualizing) among the two sides of each there is only one such overlap that isn't empty! Namely, the two sides of $P$ are

$$11x - 3y + 2z < 12, \quad 11x - 3y + 2z > 12$$

and the two sides of $Q$ are

$$11 - 3y + 2z < 19, \quad 11x - 3y + 2z > 19,$$

with the same expression $11x - 3y + 2z$ on the left side of all four inequalities. Hence, the only way one of the first pair of inequalities and one of the second pair of inequalities can correspond to regions which touch is if it is consistent with the fact that $12 < 19$. That brings us to the description of the region in between that we gave above.

Now we plug in $A, B$, and $C$, and check which of these verify these combined inequalities. The only one that does is $A$, since $12 < 11 \cdot 1 - 3 \cdot 0 + 2 \cdot 1 = 13 < 19$. Thus, $A$ is between $P$ and $Q$, but $B$ and $C$ are not.