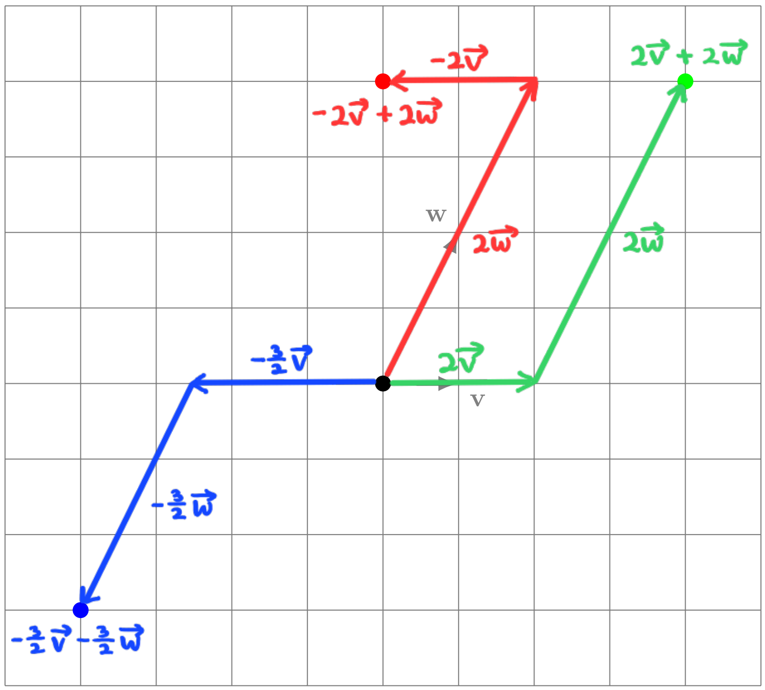
Goal: span of vectors, linear subspaces of $\mathbb{R}^n$

Let $\mathcal{P}$ be a plane in $\mathbb{R}^3$ passing through the origin $\mathbf{0} = (0, 0, 0)$. A plane can be thought of having two degrees of freedom – if you know how to walk in two different directions $\mathbf{v}$ and $\mathbf{w}$, you can reach anywhere on the plane.

Example 1. Consider the plane below. Walking only in the directions of $\mathbf{v} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$, draw how you would get from the black point to each of the colored points. You can walk "backwards," i.e. a negative scalar multiple of a direction vector.



Mathematically, we can write a plane $\mathcal{P}$, containing $\mathbf{0}$, as

$$\mathcal{P} = \{\text{all vectors of the form } a\mathbf{v} + b\mathbf{w}, \text{ for scalars } a \text{ and } b\}.$$

If we take two vectors from the set, say $a_1\mathbf{v} + b_1\mathbf{w}$ and $a_2\mathbf{v} + b_2\mathbf{w}$, the sum of the two vectors is

$$(a_1\mathbf{v} + b_1\mathbf{w}) + (a_2\mathbf{v} + b_2\mathbf{w}) = (a_1 + a_2)\mathbf{v} + (b_1 + b_2)\mathbf{w},$$

which is still in the set since $a_1 + a_2$ and $b_1 + b_2$ are both scalars. Note that the set described above is a parametric form of a plane, with $P = (0, 0, 0)$, since $\mathbf{v}$ and $\mathbf{w}$ are not scalar multiples of each other.

What happens if $\mathbf{v}$ and $\mathbf{w}$ are in the same direction? Then, the set of all $a\mathbf{v} + b\mathbf{w}$ ends up describing a line. Assuming that $\mathbf{v} = c\mathbf{w}$ for some scalar c, any element of the set would look like

$$a\mathbf{v} + b\mathbf{w} = a(c\mathbf{w}) + b\mathbf{w} = (ac + b)\mathbf{w}.$$

Hence, the set is a parametric representation of a line in $\mathbb{R}^3$.

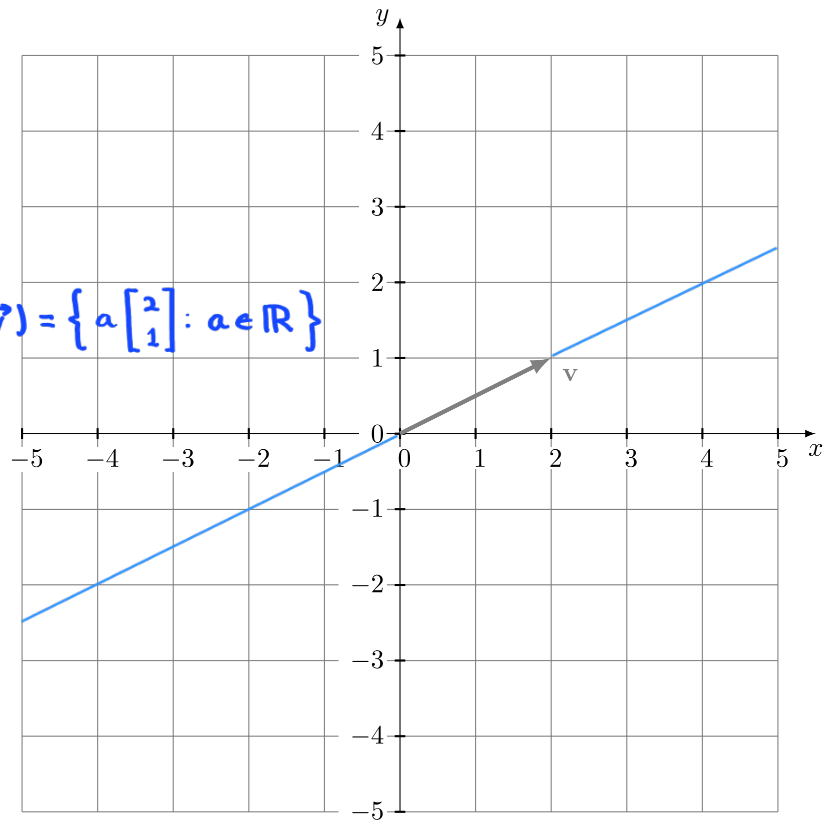
The **span** of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is the collection of all vectors in $\mathbb{R}^n$ that can be expressed as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. In other words,

$$\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) = \{c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k : c_1, \ldots, c_k \text{ are scalars}\}.$$

An important thing to note is that in any span of any vectors, $\mathbf{0}$ is always in the span.

Example 2. What is the span of $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ in $\mathbb{R}^2$? Draw the span in the coordinate plane below.

$\text{span}(\vec{v}) = \left\{ a \begin{bmatrix} 2 \\ 1 \end{bmatrix} : a \in \mathbb{R} \right\}$



Example 3. Consider the set U of 4-vectors $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix}$ that are perpendicular to $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}$. Show that U can be spanned by three 4-vectors.

You can solve for x, z, or w instead.

$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix} = 0 \Rightarrow 2x + y + 5z + w = 0$. Solving for y gives $y = -2x - 5z - w$. Hence, a vector in U looks like

"x-vector" "z-vector" "w-vector"

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ -2x - 5z - w \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ -2x \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -5z \\ z \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -w \\ 0 \\ 0 \end{bmatrix} = x \begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix} + z \begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \end{bmatrix} + w \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

This means that every vector in U can be expressed as a linear combination of

$$\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}, \text{ and } \begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}.$$

A **linear subspace** of $\mathbb{R}^n$ is a subset of $\mathbb{R}^n$ that is the span of a finite collection of vectors in $\mathbb{R}^n$. If V is a linear subspace of $\mathbb{R}^n$, a **spanning set** of V is a collection of n-vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, whose span equals V.

Example 4. Let V be a linear subspace of $\mathbb{R}^n$, and let $\mathbf{x}, \mathbf{y} \in V$. Show that any linear combination of $\mathbf{x}$ and $\mathbf{y}$ is in V.

Since V is a linear subspace of $\mathbb{R}^n$, there exists a spanning set of V - $\{\vec{v}_1, \ldots, \vec{v}_k\}$. Then, we can write

$$\vec{x} = a_1\vec{v}_1 + a_2\vec{v}_2 + \cdots + a_k\vec{v}_k$$
$$\vec{y} = b_1\vec{v}_1 + b_2\vec{v}_2 + \cdots + b_k\vec{v}_k$$

for some scalars $a_1, a_2, \ldots a_k, b_1, b_2, \ldots, b_k$. Then,

$$\alpha\vec{x} + \beta\vec{y} = (\alpha a_1 + \beta b_1)\vec{v}_1 + \cdots + (\alpha a_k + \beta b_k)\vec{v}_k,$$

and so, $\alpha\vec{x} + \beta\vec{y} \in \text{span}(\vec{v}_1, \ldots, \vec{v}_k) = V$. ▨

Proposition 4.1.11. If V is a linear subspace in $\mathbb{R}^n$, then for any vectors $\mathbf{x}_1, \ldots, \mathbf{x}_m \in V$ and scalars $a_1, \ldots, a_m$, the linear combination

$$a_1\mathbf{x}_1 + \cdots + a_m\mathbf{x}_m \in V.$$

Any linear combination of vectors in a linear subspace of $\mathbb{R}^n$ also is in the linear subspace.

Another way of stating the above proposition is: V is a linear subspace of $\mathbb{R}^n$ if and only if

1. $\mathbf{0} \in V$
2. if $\mathbf{v}, \mathbf{w} \in V$, then $a\mathbf{v} + b\mathbf{w} \in V$ for any scalars a and b.

Example 5. Consider the set W of vectors in $\mathbb{R}^4$ that are perpendicular to both $\begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 5 \\ 0 \\ 1 \end{bmatrix}$. Find a spanning set of W.

Suppose $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \in W$. Then, $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \\ -3 \end{bmatrix} = 0$ and $\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} \cdot \begin{bmatrix} -1 \\ 5 \\ 0 \\ 1 \end{bmatrix} = 0$, and so, $x + 2z - 3w = 0$ and $-x + 5y + w = 0$. Solving for y and z in terms of x and w: $y = \frac{1}{5}x - \frac{1}{5}w$ and $z = -\frac{1}{2}x + \frac{3}{2}w$. Hence,

$$\begin{bmatrix} x \\ y \\ z \\ w \end{bmatrix} = \begin{bmatrix} x \\ \frac{1}{5}x - \frac{1}{5}w \\ -\frac{1}{2}x + \frac{3}{2}w \\ w \end{bmatrix} = \begin{bmatrix} x \\ \frac{1}{5}x \\ -\frac{1}{2}x \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ -\frac{1}{5}w \\ \frac{3}{2}w \\ w \end{bmatrix} = \frac{x}{10}\begin{bmatrix} 10 \\ 2 \\ -5 \\ 0 \end{bmatrix} + \frac{w}{10}\begin{bmatrix} 0 \\ -2 \\ 15 \\ 10 \end{bmatrix}.$$

Thus, $W = \text{span}\left(\begin{bmatrix} 10 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 15 \\ 10 \end{bmatrix}\right)$, and so, $\left\{\begin{bmatrix} 10 \\ 2 \\ -5 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -2 \\ 15 \\ 10 \end{bmatrix}\right\}$ is a spanning set for W.

Let V be a nonzero linear subspace of some $\mathbb{R}^n$. The **dimension** of V, denoted $\dim(V)$, is defined to be the smallest number of vectors needed to span V. We define $\dim(\{\mathbf{0}\}) = 0$.

Theorem 4.2.5. For $k \geq 2$, suppose that $\mathbf{v}_1, \ldots, \mathbf{v}_k$ spans V, a linear subspace of $\mathbb{R}^n$. We have $\dim(V) = k$ when "there is no redundancy": each $\mathbf{v}_i$ cannot be expressed as a linear combination of the others. In other words, if removing any $\mathbf{v}_i$ would make the set not be a spanning set for V.
Equivalently, $\dim(V) < k$ if "there is a redundancy": there is a $\mathbf{v}_i$ which can be expressed as a linear combination of the others. In other words, there is a $\mathbf{v}_i$ you can remove and the resulting set would still span V.

Example 6. The 3-vectors $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ span $\mathbb{R}^3$.

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3 \Rightarrow \begin{bmatrix} a \\ b \\ c \end{bmatrix} = a\vec{e_1} + b\vec{e_2} + c\vec{e_3}.$$

Example 7. For $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{w}' = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, show that the collection of vectors

$$V = \left\{ \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in \mathbb{R}^4 : \mathbf{x} \cdot \mathbf{w} = 0, \mathbf{x} \cdot \mathbf{w}' = 0 \right\}$$

$$x_1 + x_2 + x_3 + x_4 = 0$$
$$x_1 + 2x_2 + 3x_3 + 4x_4 = 0$$

is a linear subspace of $\mathbb{R}^4$ in each of the following ways:

(a) for $\mathbf{x} \in V$, solve for each of x_3 and x_4 in terms of x_1 and x_2 to write V as a span of two vectors;

We get $x_3 = -3x_1 - 2x_2$ and $x_4 = 2x_1 + x_2$. Thus,

$$\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ -3x_1 - 2x_2 \\ 2x_1 + x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix},$$

and so, $V = \text{span}\left(\begin{bmatrix} 1 \\ 0 \\ -3 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -2 \\ 1 \end{bmatrix} \right)$.

(b) for $\mathbf{x} \in V$, solve for each of x_1 and x_4 in terms of x_2 and x_3 to write V as a span of two vectors.

We get $x_1 = \frac{2}{3}x_2 - \frac{1}{3}x_3$ and $x_4 = -\frac{1}{3}x_2 - \frac{2}{3}x_3$. Hence,

$$\vec{x} = \begin{bmatrix} -2/3\,x_2 - 1/3\,x_3 \\ x_2 \\ x_3 \\ -1/3\,x_2 - 2/3\,x_3 \end{bmatrix} = \frac{x_2}{3} \begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix} + \frac{x_3}{3} \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix},$$

and so, $V = \text{span}\left(\begin{bmatrix} -2 \\ 3 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ -2 \end{bmatrix} \right)$.

Example 8. Find a nonzero 3-vector $\mathbf{v}$ so that

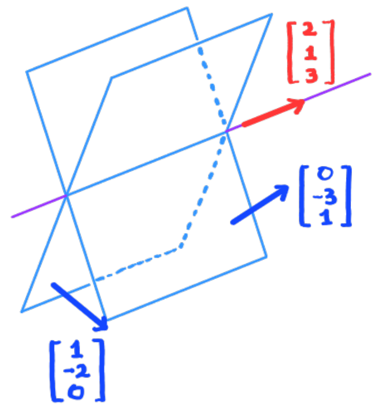
$$\left\{\mathbf{x} \in \mathbb{R}^3 : \mathbf{x} \cdot \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix} = 0, \mathbf{x} \cdot \begin{bmatrix} 0 \\ 3 \\ -1 \end{bmatrix} = 0\right\} = \operatorname{span}(\mathbf{v}).$$

Then, using the *geometric* fact that any two different planes through the origin in $\mathbb{R}^3$ meet along a line through the origin, interpret this algebraic outcome that the left side is the span of a single vector.

If $\vec{x} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ is in the set, then $a - 2b = 0$ and $3b - c = 0$. So, $a = 2b$ and $c = 3b$. Then,

$$\vec{x} = \begin{bmatrix} 2b \\ b \\ 3b \end{bmatrix} = b\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix},$$

and thus, $\boxed{\vec{v} = \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}$ works.



Example 9. Is the span of

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$$

the entirety of $\mathbb{R}^3$?

Since $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = \frac{5}{2}\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + \frac{3}{2}\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ is redundant. Thus, the span has dimension less than 3, and thus, cannot be $\mathbb{R}^3$.