Math 51
Lecture 1 – June 25, 2023

Goal: vectors, linear combinations, lengths

$\mathbb{R}$: the set of all real numbers

For an integer n, an n-**vector** is a list of n real numbers. We denote the set of all n-vectors by $\mathbb{R}^n$.

For example, $(2, 5, 3)$ is a 3-vector; in this class we will write vectors vertically – $\begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$.

A real number is also called a **scalar**. The most common ways to denote vectors is either a bold-face letter or a letter with an arrow over it – $\vec{x}$ or $\mathbf{x}$. In textbooks, bold-face is usually used, and when we write things, we cannot really write in bold, so we use the arrow-on-top notation.

Vector operations. Let $\mathbf{v}$ and $\mathbf{w}$ be two n-vectors, and let c be a scalar.

have to be the same n ("dimension")

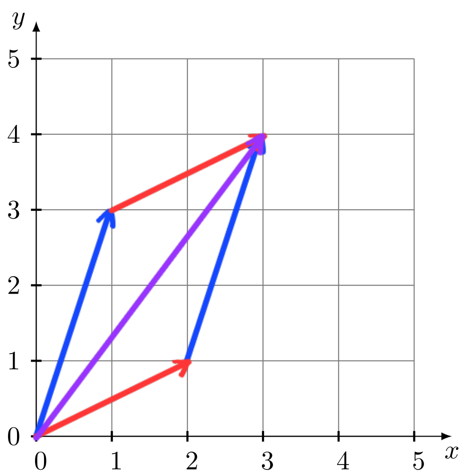
- (addition) the **sum** of $\mathbf{v}$ and $\mathbf{w}$ is

$$\begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} + \begin{bmatrix} w_1 \\ w_2 \\ \vdots \\ w_n \end{bmatrix} = \begin{bmatrix} v_1 + w_1 \\ v_2 + w_2 \\ \vdots \\ v_n + w_n \end{bmatrix}.$$

- (scalar multiplication) $c\mathbf{v}$ is defined as

$$c \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} = \begin{bmatrix} cv_1 \\ cv_2 \\ \vdots \\ cv_n \end{bmatrix}.$$

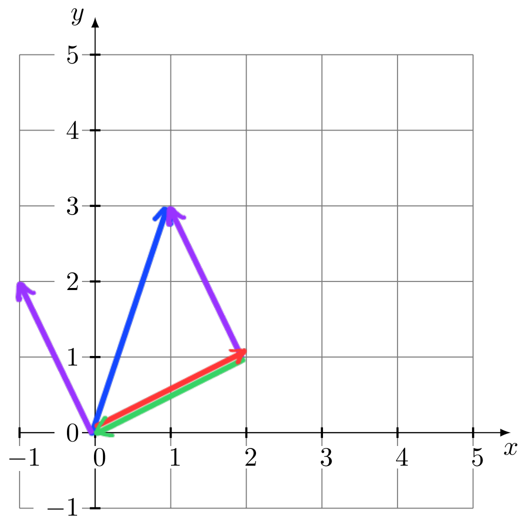
Example 1. If $\mathbf{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$, calculate $\mathbf{v} + \mathbf{w}$. Try to plot all three vectors on the coordinate plane and determine the relationship.



$$\vec{v} + \vec{w} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$$

The vector $\mathbf{v}+\mathbf{w}$ is represented by the diagonal arrow in the parallelogram with one vertex at the origin and two edges given by $\mathbf{v}$ and $\mathbf{w}$. The description of vector addition is called the *parallelogram law.*

Example 2. With the same $\mathbf{v}$ and $\mathbf{w}$ as in Example 1, try to plot $\mathbf{v}$, $\mathbf{w}$, and $\mathbf{v}-\mathbf{w}$ in the same coordinate plane.



$\vec{v} = \begin{bmatrix} 1 \\ 3 \end{bmatrix} \quad \vec{w} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$

$\vec{v} - \vec{w} = \vec{v} + (-1\vec{w}) = \begin{bmatrix} 1 \\ 3 \end{bmatrix} + \begin{bmatrix} -2 \\ -1 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \end{bmatrix}$

$\vec{v} - \vec{w}$ starts at the head of $\vec{w}$ and goes to the head of $\vec{v}$.

A **linear combination** of two n-vectors $\mathbf{v}$ and $\mathbf{w}$ is an n-vector $a\mathbf{v}+b\mathbf{w}$ for scalars a and b. More generally, a linear combination of k n-vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k,$$

where $a_1, a_2, \ldots, a_k$ are scalars.

Example 3. Suppose $\mathbf{u} = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$, $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ -3 \\ 4 \end{bmatrix}$, and $\mathbf{w} = \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix}$. Find:

(a) $\mathbf{u}+\mathbf{v}+\mathbf{w}$

$$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ -3 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 + 2 + 0 \\ 0 + 1 + 2 \\ 1 + (-3) + (-2) \\ 2 + 4 + 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ -4 \\ 7 \end{bmatrix}$$

(b) $\mathbf{u}-\mathbf{v}+\mathbf{w}$

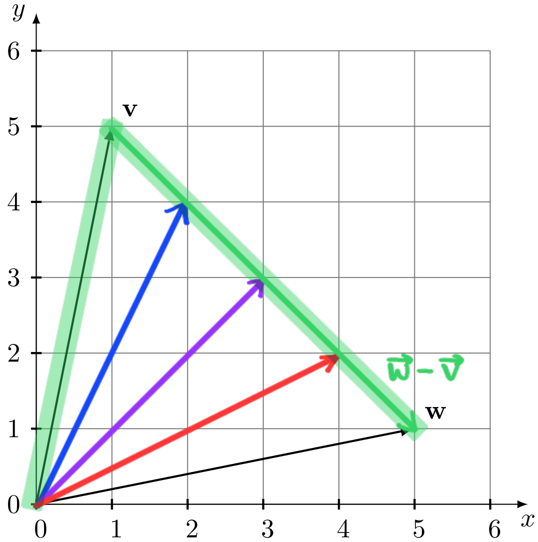
$$\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 1 \\ -3 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -1 - 2 + 0 \\ 0 - 1 + 2 \\ 1 - (-3) + (-2) \\ 2 - 4 + 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 1 \\ 2 \\ -1 \end{bmatrix}$$

(c) $3\mathbf{u}-2\mathbf{w}$

$$3\begin{bmatrix} -1 \\ 0 \\ 1 \\ 2 \end{bmatrix} - 2\begin{bmatrix} 0 \\ 2 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 0 \\ 3 \\ 6 \end{bmatrix} - \begin{bmatrix} 0 \\ 4 \\ -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -3 \\ -4 \\ 7 \\ 4 \end{bmatrix}$$

A special type of linear combination that arises in applications such as linear programming and probability theory is a **convex linear combination**. For two n-vectors $\mathbf{v}$ and $\mathbf{w}$, a convex linear combination is $(1-t)\mathbf{v}+t\mathbf{w}=\mathbf{v}+t(\mathbf{w}-\mathbf{v})$, where $0 \le t \le 1$.

Example 4. Suppose $\mathbf{v}=\begin{bmatrix}1\\5\end{bmatrix}$ and $\mathbf{w}=\begin{bmatrix}5\\1\end{bmatrix}$ as shown in the coordinate plane below. For $t=\frac{1}{4},\frac{1}{2},\frac{3}{4}$, compute $(1-t)\mathbf{v}+t\mathbf{w}$ and graph the three vectors on the coordinate plane.



$t=\frac{1}{4}$: $\frac{3}{4}\begin{bmatrix}1\\5\end{bmatrix}+\frac{1}{4}\begin{bmatrix}5\\1\end{bmatrix}=\begin{bmatrix}2\\4\end{bmatrix}$

$t=\frac{1}{2}$: $\frac{1}{2}\begin{bmatrix}1\\5\end{bmatrix}+\frac{1}{2}\begin{bmatrix}5\\1\end{bmatrix}=\begin{bmatrix}3\\3\end{bmatrix}$

$t=\frac{3}{4}$: $\frac{1}{4}\begin{bmatrix}1\\5\end{bmatrix}+\frac{3}{4}\begin{bmatrix}5\\1\end{bmatrix}=\begin{bmatrix}4\\2\end{bmatrix}$

All three lie on the segment between $\vec{v}$ and $\vec{w}$. In fact, each convex linear combination is "t down $\vec{w}-\vec{v}$ after going up $\vec{v}$." This sheds geometric meaning on $\vec{v}+t(\vec{w}-\vec{v})$.

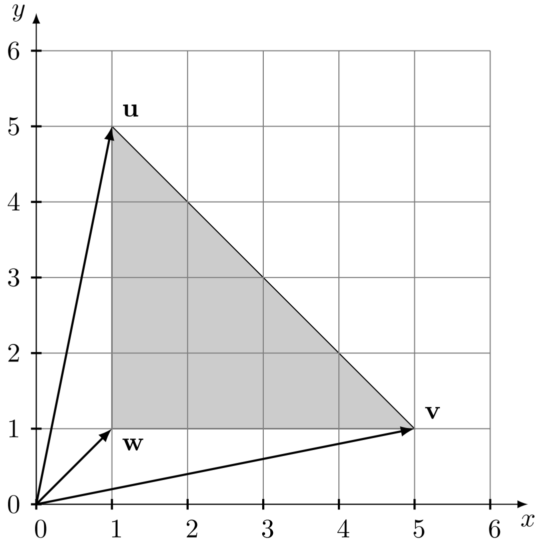
<u>Exercise</u> Try computing $(1-t)\vec{v}+t\vec{w}$ for $t<0$ or $t>1$.

More generally, for any n-vectors $\mathbf{v}_1,\ldots,\mathbf{v}_k$, a convex linear combination is of the form

$$t_1\mathbf{v}_1+\cdots+t_k\mathbf{v}_k,$$

where $t_i \ge 0$ for all i and $\sum t_i = 1$.

Example 5. Suppose $\mathbf{u}=\begin{bmatrix}1\\5\end{bmatrix}$, $\mathbf{v}=\begin{bmatrix}5\\1\end{bmatrix}$, and $\mathbf{w}=\begin{bmatrix}1\\1\end{bmatrix}$. Then, every convex linear combination is contained in the shaded triangle below.



This is an example of a "convex hull." Its Wikipedia has a list of many applications.

Basic vector operations (Theorem 1.5.2). For $\mathbf{u}, \mathbf{v}, \mathbf{w} \in \mathbb{R}^n$ and scalars $c, c' \in \mathbb{R}$, we have

- $\mathbf{v} + \mathbf{w} = \mathbf{w} + \mathbf{v}$
- $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$
- $(c + c')\mathbf{v} = c\mathbf{v} + c'\mathbf{v}$
- $c(c'\mathbf{v}) = (cc')\mathbf{v}$
- $c(\mathbf{v} + \mathbf{w}) = c\mathbf{v} + c\mathbf{w}$

The **length** or **magnitude** of an n-vector $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, denoted $\|\mathbf{v}\|$, is

$$\|\mathbf{v}\| = \sqrt{v_1^2 + v_2^2 + \cdots + v_n^2}.$$

have the same length

Note that the length of a vector is always a non-negative scalar, and $\|-\mathbf{v}\| = \|\mathbf{v}\|$ for any $\mathbf{v}$.

Example 6. Find the lengths of the vectors $\mathbf{u} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}$, $2\mathbf{u}$, and $\mathbf{v} = \begin{bmatrix} 2 \\ 3 \\ 4 \end{bmatrix}$.

$2\vec{u} = \begin{bmatrix} 6 \\ 8 \end{bmatrix}$

$\|\vec{u}\| = \sqrt{3^2 + 4^2} = 5$

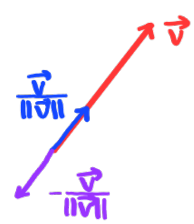
$\|2\vec{u}\| = \sqrt{6^2 + 8^2} = 10$

$\|\vec{v}\| = \sqrt{2^2 + 3^2 + 4^2} = \sqrt{29}$

Note that $\|2\vec{u}\| = 2\|\vec{u}\|$. In general, $\|c\vec{u}\| = |c|\,\|\vec{u}\|$ for any scalar c.

The **distance** between two n-vectors $\mathbf{u}$ and $\mathbf{v}$ is defined to be $\|\mathbf{u} - \mathbf{v}\|$.

The **zero vector** in $\mathbb{R}^n$ is $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}$, and a **unit vector** is a vector with length 1.



Example 7. Find all unit vectors that are scalar multiples of $\begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$.

For any nonzero vector, there are two unit vectors that are scalar multiples of it. How do we find them? Divide by its length to get the unit vector in the same direction and multiply by -1 to get the unit vector in the opposite direction.

Since $\left\| \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix} \right\| = 3$, the two unit vectors are $\begin{bmatrix} -\frac{2}{3} \\ \frac{1}{3} \\ \frac{2}{3} \end{bmatrix}$ and $\begin{bmatrix} \frac{2}{3} \\ -\frac{1}{3} \\ -\frac{2}{3} \end{bmatrix}$.