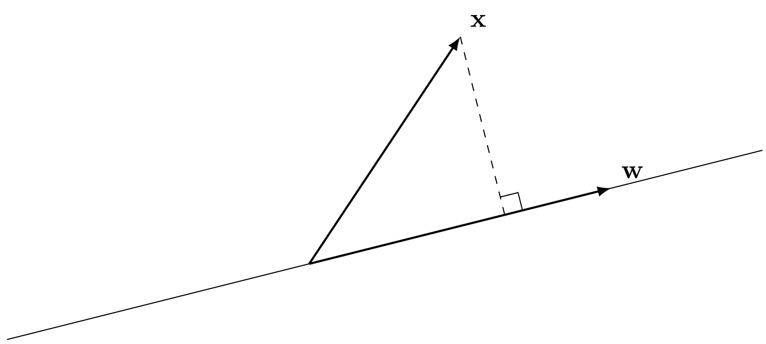
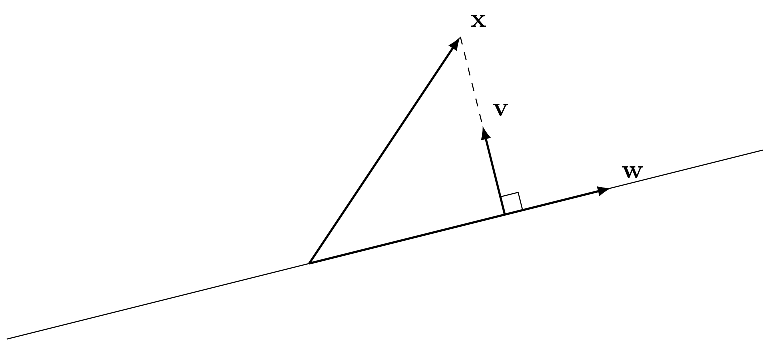
Math 51
Lecture 6 – July 5, 2023

Goal: projections!

Consider the subspace generated by $\mathbf{w}$ in $\mathbb{R}^n$. The subspace will be the line in direction of $\mathbf{w}$ and is of dimension 1. Now consider an arbitrary vector $\mathbf{x} \in \mathbb{R}^n$.



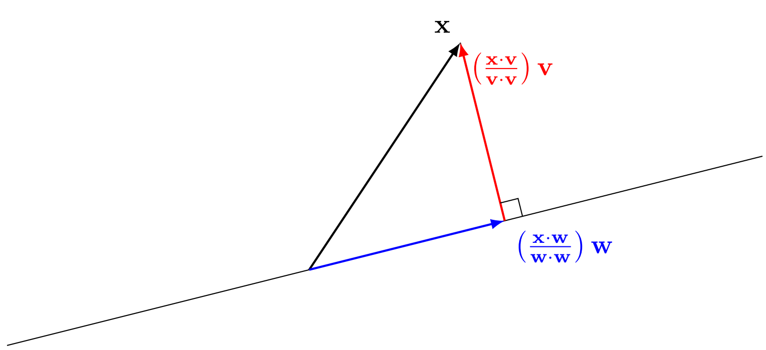
From the picture above, we can see that if we take a vector along the dashed line, say $\mathbf{v}$, then $\mathbf{v} \perp \mathbf{w}$. So, $\{\mathbf{v}, \mathbf{w}\}$ is an orthogonal basis for the plane containing $\mathbf{x}$ and $\mathbf{w}$.



This means we can write $\mathbf{x}$ as a linear combination of $\mathbf{v}$ and $\mathbf{w}$ by using Fourier formula:

$$\mathbf{x} = \left(\frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w} + \left(\frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\right) \mathbf{v}.$$

We can actually visualize what $\left(\frac{\mathbf{x}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w}$ and $\left(\frac{\mathbf{x}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\right)\mathbf{v}$ should look like.

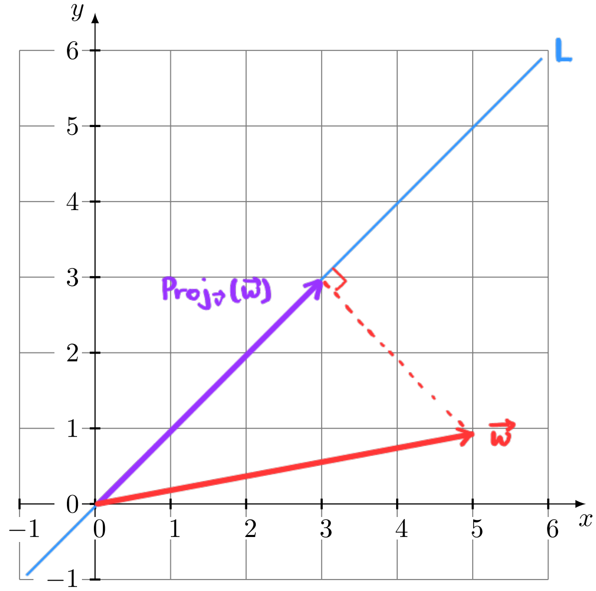


Proposition 6.1.1. Let $L = \text{span}(\mathbf{w}) = \{c\mathbf{w} : c \in \mathbb{R}\}$ be a 1-dimensional linear subspace of $\mathbb{R}^n$, a "line". Choose any point $\mathbf{x} \in \mathbb{R}^n$. There is exactly one point in L closest to $\mathbf{x}$, and it is

$$\left(\frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}}\right) \mathbf{w}.$$

This is denoted by $\mathbf{Proj}_{\mathbf{w}}(\mathbf{x})$ and is read "the projection of $\mathbf{x}$ onto span$(\mathbf{w})$".

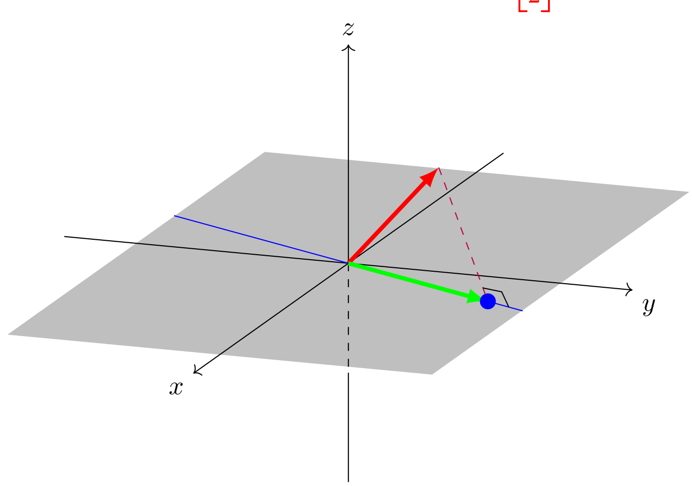
Example 1. Let L be the line represented by $y = x$ in $\mathbb{R}^2$. For $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 5 \\ 1 \end{bmatrix}$, compute $\mathbf{Proj_v}(\mathbf{w})$. Then, draw L, $\mathbf{w}$, and $\mathbf{Proj_v}(\mathbf{w})$ in the coordinate plane below.



$$\text{Proj}_{\vec{v}}(\vec{w}) = \left(\frac{\vec{w}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

$$= \frac{6}{2}\vec{v} = 3\vec{v} = \boxed{\begin{bmatrix} 3 \\ 3 \end{bmatrix}}.$$

Example 2. Find the closest point to $\mathbf{x} = \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$ on the line $\text{span}(\mathbf{v})$ for $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}$



$$\text{Proj}_{\vec{v}}(\vec{x}) = \left(\frac{\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\right)\vec{v}$$

$$= \frac{6}{5}\vec{v} = \boxed{\begin{bmatrix} 6/5 \\ 12/5 \\ 0 \end{bmatrix}}.$$

There is a very useful algebraic property of projections onto lines L through $\mathbf{0}$ that is difficult to see directly in purely geometric terms but is quite convenient when one needs to compute $\mathbf{Proj}_L(\mathbf{v})$ for many different points $\mathbf{v}$. If we calculate the vectors $\mathbf{Proj}_L(\mathbf{e}_i)$ ahead of time, then we can calculate $\mathbf{Proj}_L(\mathbf{v})$ very quickly. We need the following key property of projections of linear combinations:

$$\begin{aligned}
\mathbf{Proj}_{\mathbf{w}}(a\mathbf{u}+b\mathbf{v}) &= \left(\frac{(a\mathbf{u}+b\mathbf{v})\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} \\
&= \left(\frac{a(\mathbf{u}\cdot\mathbf{w})}{\mathbf{w}\cdot\mathbf{w}}+\frac{b(\mathbf{v}\cdot\mathbf{w})}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} \\
&= a\left(\frac{\mathbf{u}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w}+b\left(\frac{\mathbf{v}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w} \\
&= a\mathbf{Proj}_{\mathbf{w}}(\mathbf{u})+b\mathbf{Proj}_{\mathbf{w}}(\mathbf{v}).
\end{aligned}$$

Generalizing this to multiple vectors gives us

$$\mathbf{Proj}_{\mathbf{w}}\left(c_1\mathbf{x}_1+\cdots+c_k\mathbf{x}_k\right) = c_1\mathbf{Proj}_{\mathbf{w}}(\mathbf{x}_1)+\cdots+c_k\mathbf{Proj}_{\mathbf{w}}(\mathbf{x}_k).$$

Example 3. Let $\mathbf{w} = \begin{bmatrix}1\\2\\0\end{bmatrix}$. Compute $\mathbf{Proj}_{\mathbf{w}}(\mathbf{e}_1)$, $\mathbf{Proj}_{\mathbf{w}}(\mathbf{e}_2)$, and $\mathbf{Proj}_{\mathbf{w}}(\mathbf{e}_3)$.

$$\mathrm{Proj}_{\vec{w}}(\vec{e}_1) = \frac{1}{5}\vec{w} = \begin{bmatrix}1/5\\2/5\\0\end{bmatrix},\ \mathrm{Proj}_{\vec{w}}(\vec{e}_2) = \frac{2}{5}\vec{w} = \begin{bmatrix}2/5\\4/5\\0\end{bmatrix},\ \mathrm{Proj}_{\vec{w}}(\vec{e}_3) = \frac{0}{5}\vec{w} = \begin{bmatrix}0\\0\\0\end{bmatrix}.$$

Example 4. Use your answers from Example 3 to compute:

- $\mathbf{Proj}_{\mathbf{w}}\left(\begin{bmatrix}2\\2\\2\end{bmatrix}\right) = 2\mathrm{Proj}_{\vec{w}}(\vec{e}_1)+2\mathrm{Proj}_{\vec{w}}(\vec{e}_2)+2\mathrm{Proj}_{\vec{w}}(\vec{e}_3) = \frac{6}{5}\vec{w} = \boxed{\begin{bmatrix}6/5\\12/5\\0\end{bmatrix}}.$

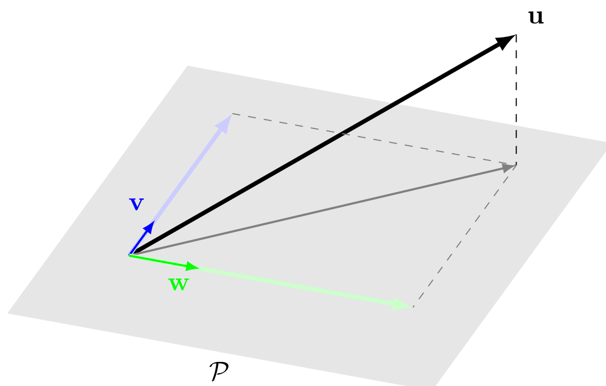
- $\mathbf{Proj}_{\mathbf{w}}\left(\begin{bmatrix}2\\4\\2\end{bmatrix}\right) = 2\left(\frac{1}{5}\vec{w}\right)+4\left(\frac{2}{5}\vec{w}\right)+2(0\vec{w}) = 2\vec{w} = \boxed{\begin{bmatrix}2\\4\\0\end{bmatrix}}.$

- $\mathbf{Proj}_{\mathbf{w}}\left(\begin{bmatrix}-2\\-4\\0\end{bmatrix}\right) = -2\left(\frac{1}{5}\vec{w}\right)-4\left(\frac{2}{5}\vec{w}\right)+0(0\vec{w}) = -2\vec{w} = \boxed{\begin{bmatrix}-2\\-4\\0\end{bmatrix}}.$

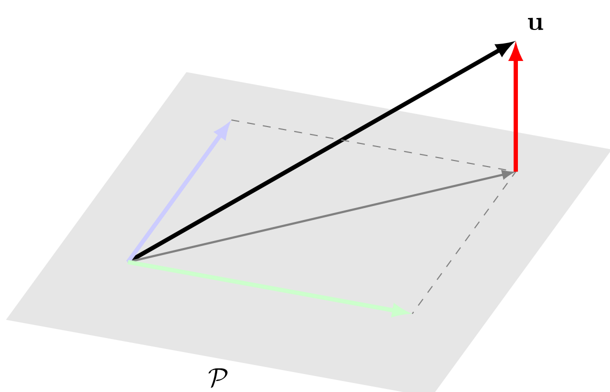
- $\mathbf{Proj}_{\mathbf{w}}\left(\begin{bmatrix}-3\\1\\-2\end{bmatrix}\right) = -3\left(\frac{1}{5}\vec{w}\right)+\left(\frac{2}{5}\vec{w}\right)-2(0\vec{w}) = -\frac{1}{5}\vec{w} = \boxed{\begin{bmatrix}-1/5\\-2/5\\0\end{bmatrix}}.$

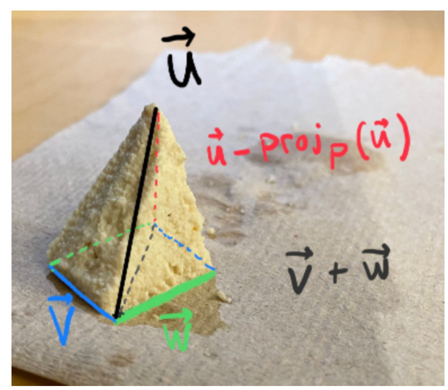
We can also use the same method to find the closest point on a linear subspace to a given point. In other words, let V be a linear subspace of $\mathbb{R}^n$. For $\mathbf{v} \in \mathbb{R}^n$, what is the closest point in V to $\mathbf{v}$?

Let us consider the problem of, given a plane $\mathcal{P}$ in $\mathbb{R}^3$ and $\mathbf{u} \in \mathbb{R}^3$, finding the closest point on $\mathcal{P}$ to $\mathbf{u}$. Suppose $\{\mathbf{v}, \mathbf{w}\}$ is an orthogonal basis for $\mathcal{P}$. In the diagram below, the light blue vector is $\mathbf{Proj}_{\mathbf{v}}(\mathbf{u})$, the light green vector is $\mathbf{Proj}_{\mathbf{w}}(\mathbf{u})$, and the gray vector is the sum of the two vectors.



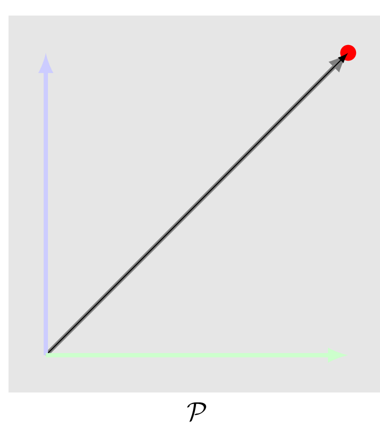
We can think of the gray vector as the "projection of $\mathbf{u}$ onto the plane $\mathcal{P}$," or $\mathbf{Proj}_{\mathcal{P}}(\mathbf{u})$. The red vector, shown below, is $\mathbf{u} - \mathbf{Proj}_{\mathcal{P}}(\mathbf{u})$ and is perpendicular to $\mathcal{P}$.





The picture on the right is a physical model of the picture on the left, made with tofu by Aristotle Marangu '25.

The point at the base of the red vector, $\mathbf{Proj}_{\mathcal{P}}(\mathrm{u})$ is the closest point on $\mathcal{P}$ to $\mathbf{u}$. A bird's eye view is provided in hopes of better visual understanding (the red vector is coming straight out towards you).



Orthogonal projection theorem, version 1 (Theorem 6.2.1). For any $\mathbf{x} \in \mathbb{R}^n$ and linear subspace V of $\mathbb{R}^n$, there is a unique $\mathbf{v} \in V$ that is closest to $\mathbf{x}$. This $\mathbf{v}$ is called the **projection of x onto** V, and is denoted $\mathbf{Proj}_V(\mathbf{x})$.
The projection $\mathbf{Proj}_V(\mathbf{x})$ is the *ONLY* vector $\mathbf{v} \in V$ such that $\mathbf{x} - \mathbf{v}$ is perpendicular to V ($\mathbf{x} - \mathbf{v}$ is perpendicular to *every* vector in V).
If V is nonzero, then for any orthogonal basis $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ of V, we have

$$\mathbf{Proj}_V(\mathbf{x}) = \mathbf{Proj}_{\mathbf{v}_1}(\mathbf{x}) + \cdots + \mathbf{Proj}_{\mathbf{v}_k}(\mathbf{x}).$$

For $\mathbf{x} \in V$, $\mathbf{Proj}_V(\mathbf{x}) = \mathbf{x}$, which is the Fourier formula!

Example 5. Let U be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$. If $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, find the point $\mathbf{u} \in U$ that is closest to $\mathbf{v}$.

Since $\vec{u}_1 \perp \vec{u}_2$, $\{\vec{u}_1, \vec{u}_2\}$ is an orthogonal basis for U. By the orthogonal projection theorem,

$$\vec{u} = \mathrm{Proj}_U(\vec{v}) = \mathrm{Proj}_{\vec{u}_1}(\vec{v}) + \mathrm{Proj}_{\vec{u}_2}(\vec{v}) = -\frac{1}{2}\vec{u}_1 + 2\vec{u}_2 = \boxed{\begin{bmatrix} 3/2 \\ 5/2 \\ 2 \\ 4 \end{bmatrix}}$$

is the point in U closest to $\vec{v}$.

Orthogonal projection theorem, version 2 (Theorem 6.2.4). If V is a linear subspace of $\mathbb{R}^n$, then every vector $\mathbf{x} \in \mathbb{R}^n$ can be *uniquely* expressed as a sum

$$\mathbf{x} = \mathbf{v} + \mathbf{v}',$$

where $\mathbf{v} \in V$ and $\mathbf{v}'$ is orthogonal to every vector in V. Explicitly, $\mathbf{v} = \mathbf{Proj}_V(\mathbf{x})$.

Example 6. Let U be the subspace of $\mathbb{R}^4$ spanned by $\mathbf{u}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\mathbf{u}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 2 \end{bmatrix}$, and $\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}$. If $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$, find the point $\mathbf{u} \in U$ that is closest to $\mathbf{v}$.

Note that $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ is an orthogonal basis for U. Hence,

$$\mathrm{Proj}_U(\vec{v}) = \mathrm{Proj}_{\vec{u}_1}(\vec{v}) + \mathrm{Proj}_{\vec{u}_2}(\vec{v}) + \mathrm{Proj}_{\vec{u}_3}(\vec{v}) = -\tfrac{1}{2}\vec{u}_1 + 2\vec{u}_2 - \tfrac{2}{5}\vec{u}_3$$

$$= \begin{bmatrix} -1/2 \\ 1/2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 2 \\ 2 \\ 2 \\ 4 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 4/5 \\ -2/5 \end{bmatrix} = \boxed{\begin{bmatrix} 3/2 \\ 5/2 \\ 14/5 \\ 18/5 \end{bmatrix}}.$$

Example 7. Let $\{\mathbf{v}_1, \mathbf{v}_2\}$ be an orthogonal basis for a plane $\mathcal{P}$ in $\mathbb{R}^n$ with $\|\mathbf{v}_1\| = 3$ and $\|\mathbf{v}_2\| = 1$. Let $\mathbf{w}$ be an n-vector lying outside $\mathcal{P}$ for which $\mathbf{Proj}_{\mathcal{P}}(\mathbf{w}) = 2\mathbf{v}_1 + \mathbf{v}_2$.

(a) Explain why the span of the nonzero vectors $\mathbf{v}_1$ and $\mathbf{w}$ is a plane $\mathcal{P}'$, and that this plane cannot contain $\mathbf{v}_2$; the visualization is that the planes $\mathcal{P}$ and $\mathcal{P}'$ meet along the line $\mathrm{span}(\mathbf{v}_1)$.

If $\vec{v}_2 \in \mathcal{P}' = \mathrm{span}(\vec{v}_1, \vec{w})$, then $\vec{v}_2 = \alpha\vec{v}_1 + \beta\vec{w}$ for some scalars α and β. Note that $\beta \neq 0$ (if $\beta = 0$, then $\vec{v}_2$ is a scalar multiple of $\vec{v}_1$, which is impossible). Then, $\vec{w} = -\frac{\alpha}{\beta}\vec{v}_1 + \frac{1}{\beta}\vec{v}_2 \in \mathrm{span}(\vec{v}_1, \vec{v}_2) = \mathcal{P}$, which is impossible. Hence, $\vec{v}_2 \notin \mathcal{P}'$.

(b) For any linear subspace V of $\mathbb{R}^n$ and vector $\mathbf{v} \in V$, show that $\mathbf{x} \cdot \mathbf{v} = \mathbf{Proj}_V(\mathbf{x}) \cdot \mathbf{v}$ for every n-vector $\mathbf{x}$.

Let $\vec{x}$ be arbitrary. By the orthogonal projection theorem,

$$\vec{x} = \mathrm{Proj}_V(\vec{x}) + \vec{v}'$$

where $\vec{v}'$ is orthogonal to every vector in V. Thus, for any $\vec{v} \in V$,

$$\vec{x} \cdot \vec{v} = (\mathrm{Proj}_V(\vec{x}) + \vec{v}') \cdot \vec{v} = \mathrm{Proj}_V(\vec{x}) \cdot \vec{v} + \vec{v}' \cdot \vec{v} = \mathrm{Proj}_V(\vec{x}) \cdot \vec{v}.$$

(c) Show that $\mathbf{w} \cdot \mathbf{v}_1 = 18$ and $\mathbf{w} \cdot \mathbf{v}_2 = 1$, and use this to compute $\mathbf{Proj}_{\mathbf{v}_1}(\mathbf{w})$ and $\mathbf{Proj}_{\mathbf{v}_2}(\mathbf{w})$ as explicit scalar multiples of $\mathbf{v}_1$ and $\mathbf{v}_2$, respectively.

By (b), $\vec{w} \cdot \vec{v}_1 = \mathrm{Proj}_{\mathcal{P}}(\vec{w}) \cdot \vec{v}_1 = (2\vec{v}_1 + \vec{v}_2) \cdot \vec{v}_1 = 2\|\vec{v}_1\|^2 + \vec{v}_2 \cdot \vec{v}_1 = 18$. Similarly, $\vec{w} \cdot \vec{v}_2 = (2\vec{v}_1 + \vec{v}_2) \cdot \vec{v}_2 = 2\vec{v}_1 \cdot \vec{v}_2 + \|\vec{v}_2\|^2 = 1$. Hence,

$$\mathrm{Proj}_{\vec{v}_1}(\vec{w}) = \left(\frac{\vec{w} \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\right)\vec{v}_1 = \boxed{2\vec{v}_1} \quad \text{and} \quad \mathrm{Proj}_{\vec{v}_2}(\vec{w}) = \left(\frac{\vec{w} \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\right)\vec{v}_2 = \boxed{\vec{v}_2}.$$