Solutions to Math 51 Midterm Exam — July 28, 2023

1. (10 points) Consider the plane $\mathcal{P}$ defined by

$$\begin{bmatrix} 2 \\ -3 \\ 1 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t' \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

for $t, t' \in \mathbf{R}$.

(a) (7 points) Find the point on $\mathcal{P}$ closest to $\mathbf{v} = (4, -7, 4)$.
Remark. Note that $\mathbf{0}$ is not on $\mathcal{P}$.

In order to project onto a plane, we need the plane to be a linear subspace; i.e., $\mathbf{0}$ needs to be on it. Hence, we shift the whole problem by $(2, -3, 1)$.
We need to find the point on $\mathcal{P}'$, defined by

$$\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t' \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}$$

closest to $\mathbf{v}' = (2, -4, 3)$. Note that $\mathbf{a} = (1, 1, 0)$ and $\mathbf{b} = (1, -1, 2)$ form an orthogonal basis for $\mathcal{P}'$. Hence, the point we are looking for is

$$\mathbf{Proj}_{\mathcal{P}'}(\mathbf{v}') = \mathbf{Proj}_{\mathbf{a}}(\mathbf{v}') + \mathbf{Proj}_{\mathbf{b}}(\mathbf{v}') = \frac{-2}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + \frac{12}{6} \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 4 \end{bmatrix}.$$

Shifting back to the original problem gives us the desired point as $(3, -6, 5)$.

(b) (3 points) Give an equational form for $\mathcal{P}$.

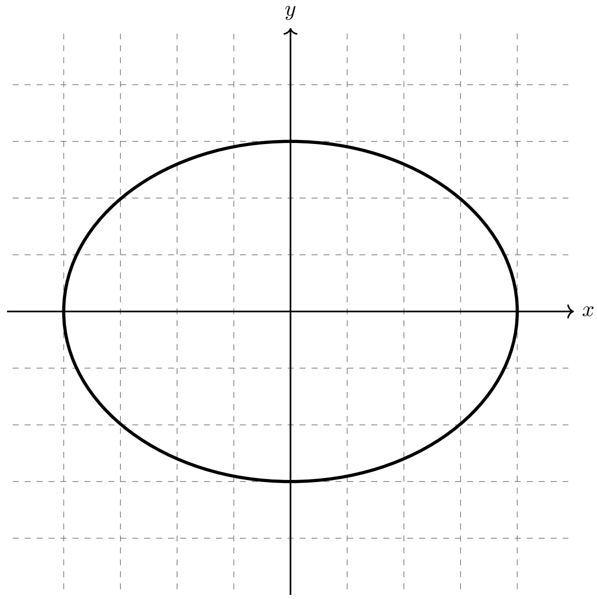
If $\mathbf{n} = (a, b, c)$ is normal to the plane, $\mathbf{n}$ must be perpendicular to $(1, 1, 0)$ and $(1, -1, 2)$. Hence,

$$\begin{aligned} a + b \qquad &= 0 \\ a - b + 2c &= 0 \end{aligned}$$

from which, we get $a = -c$ and $b = c$. Hence, $\mathbf{n}$ is a scalar multiple of $(-1, 1, 1)$, and so, $\mathcal{P}$ has equational form $-x + y + z = d$.
In order to find out d, we plug in the point $(2, -3, 1)$ to get $d = -4$, and so, an equational form of $\mathcal{P}$ is

$$-x + y + z = -4.$$

2. (10 points) Find the dimensions of the rectangle of greatest area that can be inscribed in the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ (shown below) with sides parallel to the coordinate axes. *[Note: We say that a rectangle is inscribed in an ellipse if the corners of the rectangle lie on the ellipse.]*



Let (x, y) be the vertex of the rectangle in the first quadrant; then the rectangle has width $2x$ and height $2y$. We are trying to maximize $f(x, y) = 4xy$ subject to $g(x, y) = 9x^2 + 16y^2 = 144$ (the ellipse equation multiplied by 144).

The gradients are $\nabla f = \begin{bmatrix} 4y \\ 4x \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 18x \\ 32y \end{bmatrix}$.

∇g vanishes at $(0, 0)$, which is not on the ellipse.

The Lagrange multiplier equations are

$$4y = 18\lambda x \qquad \text{and} \qquad 4x = 32\lambda y.$$

The second equation yields $x = 8\lambda y$ and plugging this into the first equation gives us

$$4y = 144\lambda^2 y.$$

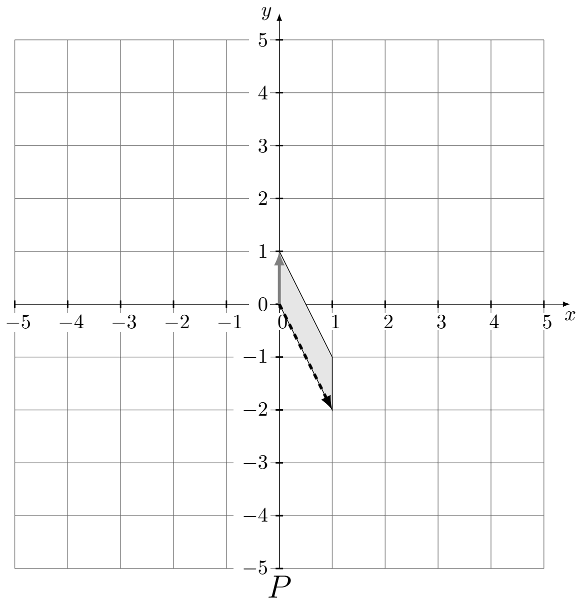
Hence, we get $y(1 - 36\lambda^2) = 0$.

- $y = 0$: Plugging this into the constraint leads us to $(4, 0)$ and $(-4, 0)$; neither is in the first quadrant.
- $\lambda = \frac{1}{6}$: We get $x = \frac{4}{3}y$ and plugging this into the constraint gives us $\left(2\sqrt{2}, \frac{3}{\sqrt{2}}\right)$.
- $\lambda = -\frac{1}{6}$: This is impossible since both $x, y > 0$ and $x = 8\lambda y$.

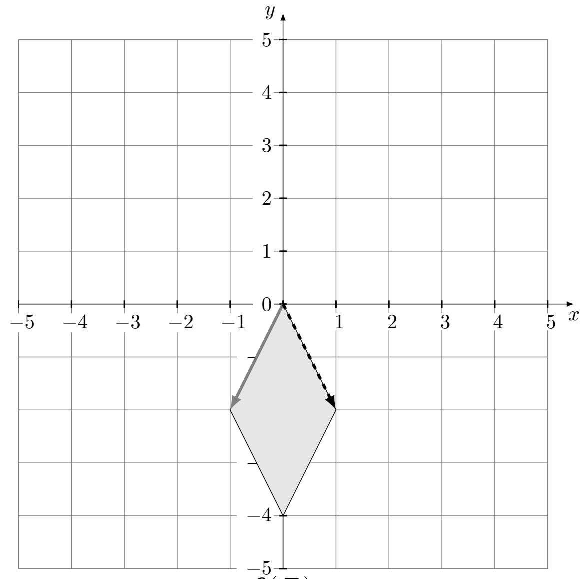
Hence, the only point of interest is $\left(2\sqrt{2}, \frac{3}{\sqrt{2}}\right)$ and comparing the value of f to another point on the ellipse, say $\left(2, \frac{3\sqrt{3}}{2}\right)$ confirms that it is the maximum. Thus, the required dimensions are

$$4\sqrt{2} \times 3\sqrt{2}.$$

3. (10 points) Suppose $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^2$ is a linear function that transforms P, the parallelogram on the left, to $\mathbf{f}(P)$, the parallelogram on the right.



P



$\mathbf{f}(P)$

Note that $\mathbf{f}$ transforms the light gray vector on the left into the light gray vector on the right, and that $\mathbf{f}$ transforms the dotted black vector on the left into the dotted black vector on the right.

(a) (2 points) Express $\begin{bmatrix}1\\0\end{bmatrix}$ as a linear combination of $\begin{bmatrix}1\\-2\end{bmatrix}$ and $\begin{bmatrix}0\\1\end{bmatrix}$.

Setting up $\begin{bmatrix}1\\0\end{bmatrix} = \alpha\begin{bmatrix}1\\-2\end{bmatrix} + \beta\begin{bmatrix}0\\1\end{bmatrix}$, we get two equations $1 = \alpha$ and $0 = -2\alpha + \beta$. Hence, $\alpha = 1$, $\beta = 2$, and

$$\begin{bmatrix}1\\0\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix} + 2\begin{bmatrix}0\\1\end{bmatrix}.$$

(b) (4 points) Determine A, the matrix associated to $\mathbf{f}$; i.e., for which $\mathbf{f}(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x}$ in $\mathbf{R}^2$.

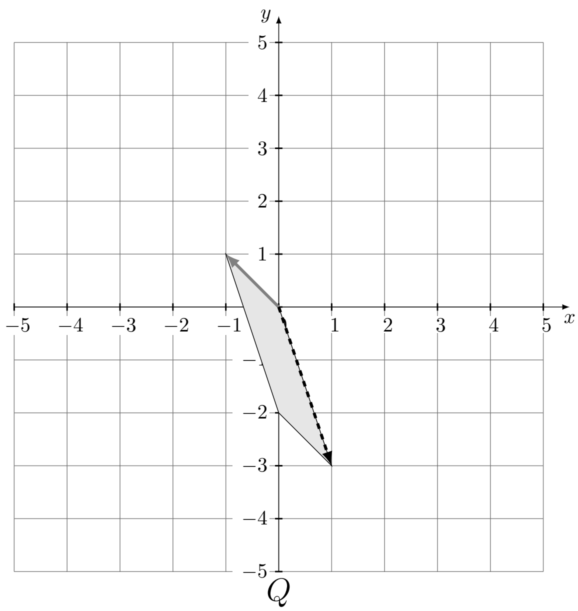
From the given pictures, we know that $A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix}$. From part (a), we know that

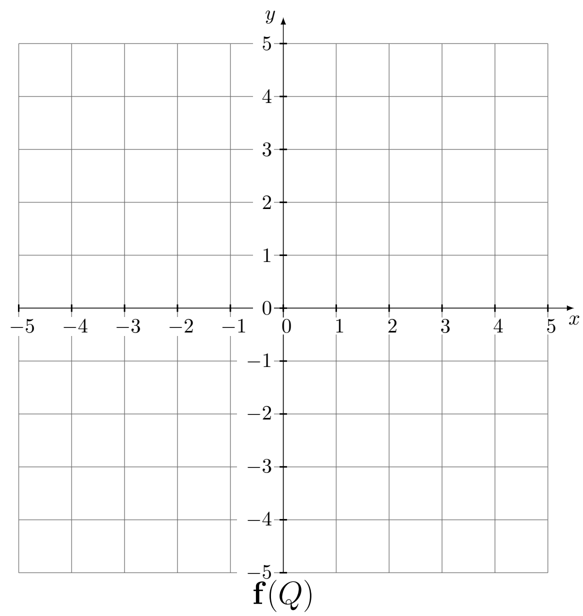
$$A\begin{bmatrix}1\\0\end{bmatrix} = A\begin{bmatrix}1\\-2\end{bmatrix} + 2A\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}1\\-2\end{bmatrix} + 2\begin{bmatrix}-1\\-2\end{bmatrix} = \begin{bmatrix}-1\\-6\end{bmatrix}.$$

Therefore,

$$A = \begin{bmatrix}-1 & -1\\-6 & -2\end{bmatrix}.$$

(c) (4 points) Given Q, the parallelogram on the left below, provide a sketch of $\mathbf{f}(Q)$ on the blank coordinate plane on the right below. (*Note:* you do *not* need to indicate a "light gray" and "dotted black" vector in your sketch.)

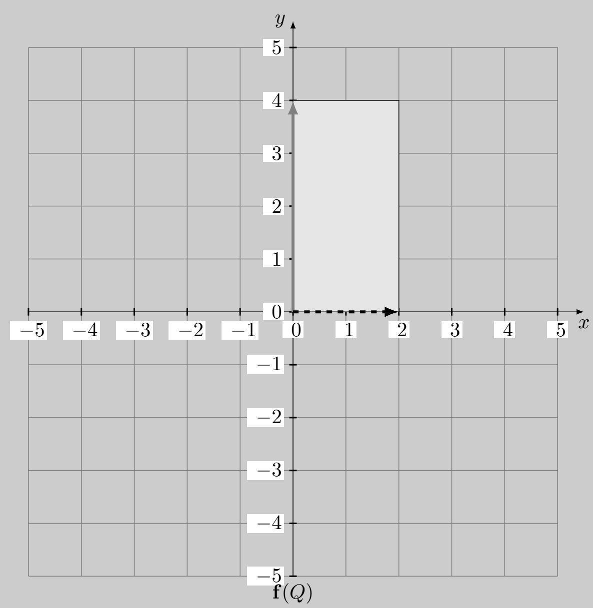




Using our work from part (b), we get

$$A\begin{bmatrix}-1\\1\end{bmatrix}=\begin{bmatrix}0\\4\end{bmatrix} \qquad \text{and} \qquad A\begin{bmatrix}1\\-3\end{bmatrix}=\begin{bmatrix}2\\0\end{bmatrix}.$$

Hence, $\mathbf{f}(Q)$ looks like



4. (10 points) Suppose that Derek and Gene each have \$2, and they play a game with a fair coin (i.e., it comes up heads 50% of the time and tails 50% of the time). If the coin comes up heads, Derek gives Gene \$1, and if the coin comes up tails, Gene gives Derek \$1.

   The game ends once someone has all \$4.

   (a) (5 points) Define the probability vector

$$\mathbf{p}_n = \begin{bmatrix} \text{probability that Derek has \$4 after } n \text{ turns} \\ \text{probability that Derek has \$3 after } n \text{ turns} \\ \text{probability that Derek has \$2 after } n \text{ turns} \\ \text{probability that Derek has \$1 after } n \text{ turns} \\ \text{probability that Derek has \$0 after } n \text{ turns} \end{bmatrix}.$$

   Compute the 5×5 matrix M for which $\mathbf{p}_{n+1} = M\mathbf{p}_n$. Justify your answer.

We can figure out the columns of M by analyzing what happens to the states corresponding to standard basis vectors.

- $\mathbf{e}_1$ corresponds to Derek having \$4 and Gene having \$0.

  Since the game is over, it stays over; this corresponds to $(1, 0, 0, 0, 0)$.

- $\mathbf{e}_2$ corresponds to Derek having \$3 and Gene having \$1.

  After one turn, there is $1/2$ probability that Derek has \$4 and $1/2$ probability that Derek has \$2; this corresponds to $(1/2, 0, 1/2, 0, 0)$.

- $\mathbf{e}_3$ corresponds to Derek having \$2 and Gene having \$2.

  After one turn, there is $1/2$ probability that Derek has \$3 and $1/2$ probability that Derek has \$1; this corresponds to $(0, 1/2, 0, 1/2, 0)$.

- $\mathbf{e}_4$ corresponds to Derek having \$1 and Gene having \$3.

  After one turn, there is $1/2$ probability that Derek has \$2 and $1/2$ probability that Derek has \$0; this corresponds to $(0, 0, 1/2, 0, 1/2)$.

- $\mathbf{e}_5$ corresponds to Derek having \$0 and Gene having \$4.

  Since the game is over, it stays over; this corresponds to $(0, 0, 0, 0, 1)$.

Putting everything together,

$$M = \begin{bmatrix} 1 & \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 \\ 0 & 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} & 1 \end{bmatrix}.$$

(b) (5 points) It is a known result that if player A has \$$a$ and player B has \$$b$, where a and b are integers, the probability of player A eventually getting all \$$(a+b)$ is $\dfrac{a}{a+b}$ and the probability of player B eventually getting all \$$(a+b)$ is $\dfrac{b}{a+b}$.

For example, if Derek has \$20 and Gene has \$10, the probability of Derek winning all the money is $\dfrac{20}{20+10} = \dfrac{2}{3}$.

For the matrix M you found in part (a), compute

$$\lim_{n\to\infty} M^n,$$

and justify your answer.

Hint. It may be helpful to think about what each column of $\lim_{n\to\infty} M^n$ represents.

Similarly as in the previous part, we can analyze what happens to the states corresponding to standard basis vectors.

- $\mathbf{e}_1$ corresponds to Derek starting with \$4 and Gene starting with \$0.

  Derek eventually wins with probability $\frac{4}{4+0} = 1$ and Gene eventually wins with probability $\frac{0}{4+0} = 0$.
- $\mathbf{e}_2$ corresponds to Derek starting with \$3 and Gene starting with \$1.

  Derek eventually wins with probability $\frac{3}{3+1} = \frac{3}{4}$ and Gene eventually wins with probability $\frac{1}{3+1} = \frac{1}{4}$.
- $\mathbf{e}_3$ corresponds to Derek starting with \$2 and Gene starting with \$2.

  Derek eventually wins with probability $\frac{2}{2+2} = \frac{1}{2}$ and Gene eventually wins with probability $\frac{2}{2+2} = \frac{1}{2}$.
- $\mathbf{e}_4$ corresponds to Derek starting with \$1 and Gene starting with \$3.

  Derek eventually wins with probability $\frac{1}{1+3} = \frac{1}{4}$ and Gene eventually wins with probability $\frac{3}{1+3} = \frac{3}{4}$.
- $\mathbf{e}_5$ corresponds to Derek starting with \$0 and Gene starting with \$4.

  Derek eventually wins with probability $\frac{0}{0+4} = 0$ and Gene eventually wins with probability $\frac{4}{0+4} = 1$.

Putting everything together,

$$\lim_{n\to\infty} M^n = \begin{bmatrix} 1 & \frac{3}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{4} & \frac{1}{2} & \frac{3}{4} & 1 \end{bmatrix}.$$

5. (10 points) (a) (4 points) Let

$$S = \{(x,y,z) \in \mathbf{R}^3 \mid xz - y^2 = 3\},$$

and $\mathcal{P}$ be the plane $x+y+z=3$. Find the point Q on S where the plane $\mathcal{P}$ is tangent to S at Q.

Let $F(x,y,z) = xz - y^2$, S is the 3-level set of $F(x,y,z)$. The plane tangent to S at a point (x,y,z) must have normal vector

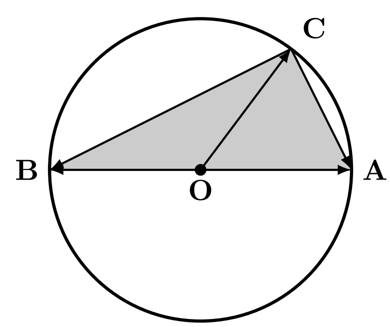
$$\nabla F(x,y,z) = \begin{bmatrix} z \\ -2y \\ x \end{bmatrix}.$$

For the plane $\mathcal{P}$ $x+y+z=3$ with normal vector $\mathbf{n} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ to be tangent to S at a point (x,y,z), we must have $\nabla F(x,y,z) = k\mathbf{n}$ for some non-zero scalar k. So

$$\begin{bmatrix} z \\ -2y \\ x \end{bmatrix} = k \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad x = z = -2y = k.$$

Since S is the 3-level set of $F(x,y,z)$, we look for point $(-2y, y, -2y)$ where $3 = F(-2y, y-2y) = 4y^2 - y^2 = 3y^2$, so $y^2 = 1$, and $y = \pm 1$. The points on S where the tangent plane is parallel to $\mathcal{P}$ are $(-2,1,-2)$ and $(2,-1,2)$, but only the point $(2,-1,2)$ is on $\mathcal{P}$ and S, so $Q = (2,-1,2)$, and $\mathcal{P}$ is tangent to S at Q.

For parts (b) and (c), consider the circle below where O is the center of the circle, points A, B, C are on the cirle, and AB is a diameter of the circle.



Suppose $\overrightarrow{OA} = \mathbf{u}$, and $\overrightarrow{OC} = \mathbf{v}$.

(b) (2 points) Express the vectors $\overrightarrow{OB}$, $\overrightarrow{CA}$, and $\overrightarrow{CB}$ as linear combinations of $\mathbf{u}$ and $\mathbf{v}$.

Since AB is a diameter of the circle,

$$\overrightarrow{OB} = -\overrightarrow{OA} = -\mathbf{u}.$$

$$\overrightarrow{CA} = \overrightarrow{OA} - \overrightarrow{OC} = \mathbf{u} - \mathbf{v}.$$
$$\overrightarrow{CB} = \overrightarrow{OB} - \overrightarrow{OC} = -\mathbf{u} - \mathbf{v}.$$

(c) (4 points) Show that $\overrightarrow{CA}$ and $\overrightarrow{CB}$ are perpendicular to each other.

Note that the vectors $\mathbf{u}$ and $\mathbf{v}$ both have the same length as the radius of the circle, since O is the center of the circle, A, C are on the circle. In particular, $\|\mathbf{u}\| = \|\mathbf{v}\|$.

We compute

$$\begin{aligned}
\overrightarrow{CA} \cdot \overrightarrow{CB} &= (\mathbf{u} - \mathbf{v}) \cdot (-\mathbf{u} - \mathbf{v}) \\
&= -\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{u} - \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{v} \\
&= -\mathbf{u} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\
&= \|\mathbf{v}\|^2 - \|\mathbf{u}\|^2 \\
&= 0
\end{aligned}$$

6. (10 points) For each of the following statements, circle either TRUE (meaning, "always true") or FALSE (meaning, "not always true"), and briefly and convincingly justify your answer. 1 point for the correct choice, and the rest for convincing justification.

(a) (5 points) There is a continuous function $f : \mathbf{R}^2 \to \mathbf{R}$ with

$$f_x = e^{x^2} \sin y \qquad \text{and} \qquad f_y = e^{x^2} \cos y.$$

Circle one, and justify below: TRUE $\boxed{\text{FALSE}}$

Since f is continuous, if f does have the given f_x and f_y, f satisfies the hypotheses of Clairaut–Schwarz theorem (equality of mixed partials). The mixed second partial derivatives are

$$(f_y)_x = 2xe^{x^2} \cos y \qquad \text{and}$$
$$(f_x)_y = e^{x^2} \cos y.$$

Hence, $f_{xy} \neq f_{yx}$, contradicting the Clairaut–Schwarz theorem. It follows that no such f exists.

(b) (5 points) The set $S = \{(x, y, z) \in \mathbf{R}^3 : x^2 + y^2 - z^2 = -1\}$ is the graph of some function $f\colon \mathbf{R}^2 \to \mathbf{R}$.

Circle one, and justify below: TRUE $\boxed{\text{FALSE}}$

The graph of a function $f\colon \mathbf{R}^2 \to \mathbf{R}$ is the subset of $\mathbf{R}^3$ consisting of all points (x, y, z) such that $z = f(x, y)$. In particular, for any pair (x, y), there is exactly one point (x, y, z) in Graph(f), namely $(x, y, f(x, y))$. (If we allow $f(x, y)$ to be undefined at some points, then for any pair (x, y), there is *at most* one point (x, y, z) in Graph(f), namely $(x, y, f(x, y))$ whenever $f(x, y)$ is defined.) However, the set S contains both $(0, 0, 1)$ and $(0, 0, -1)$, so it cannot be the graph of a function.