Solutions to Math 51 Final Exam (Practice #1)

1. (10 points) Suppose $\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix}$.

   (a) (3 points) For the plane $P = \text{span}(\mathbf{v}, \mathbf{w})$, compute an orthogonal basis containing the vector $\mathbf{v}$ and check that the two vectors in your orthogonal basis are indeed orthogonal. (The other vector in the orthogonal basis aside from $\mathbf{v}$ has integer entries if you follow the algorithm taught in this course.)

   Such a basis is $\{\mathbf{v}, \mathbf{w}'\}$ for

   $$\mathbf{w}' = \mathbf{w} - \mathbf{Proj}_{\mathbf{v}}(\mathbf{w}) = \mathbf{w} - \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \mathbf{w} - \frac{12}{6} \mathbf{v} = \mathbf{w} - 2\mathbf{v}.$$

   Explicitly,

   $$\mathbf{w}' = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \\ 2 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 2 \\ 0 \end{bmatrix}.$$

   By a direct check, $\mathbf{w}' \cdot \mathbf{v} = -2 + 2 + 0 = 0$, as desired.

   (b) (5 points) Let $\mathbf{c} = \begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix}$. Find t_1 and t_2 for which the point $t_1\mathbf{v} + t_2\mathbf{w}$ is the closest point in P to $\mathbf{c}$. (The answers are integers.)

   The point in $\text{span}(\mathbf{v}, \mathbf{w}) = P$ closest to $\mathbf{c} = \begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix}$ corresponds to values of t_1, t_2 minimizing the magnitude of the displacement

   $$\mathbf{c} - (t_1\mathbf{v} + t_2\mathbf{w}),$$

   which can be accomplished when this displacement is perpendicular to every vector in P. Thus, this nearest point is $\mathbf{Proj}_P(\mathbf{c})$, which we calculate using the orthogonal basis $\{\mathbf{v}, \mathbf{w}'\}$ found in (a):

   $$\begin{aligned} \mathbf{Proj}_P(\mathbf{c}) = \frac{\mathbf{c} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} + \frac{\mathbf{c} \cdot \mathbf{w}'}{\mathbf{w}' \cdot \mathbf{w}'} \mathbf{w}' = \frac{-6}{6} \mathbf{v} + \frac{-10}{5} \mathbf{w}' &= -\mathbf{v} - 2\mathbf{w}' \\ &= -\mathbf{v} - 2(\mathbf{w} - 2\mathbf{v}) \\ &= 3\mathbf{v} - 2\mathbf{w}, \end{aligned}$$

   so $t_1 = 3$ and $t_2 = -2$.

   (c) (2 points) For $\mathbf{c} = \begin{bmatrix} 2 \\ -4 \\ -6 \end{bmatrix}$ as in part (b), compute the distance between $\mathbf{c}$ and this point in P at minimal distance. Your answer should have the form $\sqrt{m}$ for a whole number $m \leq 40$.

   The point in P from part (b) is $3\mathbf{v} - 2\mathbf{w} = \begin{bmatrix} 0 \\ -5 \\ -1 \end{bmatrix}$, so the distance between this and $\mathbf{c}$ is the length of the difference $\mathbf{c} - \begin{bmatrix} 0 \\ -5 \\ -1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -5 \end{bmatrix}$, which is $\sqrt{30}$.

2. (10 points) Consider the function $f(x, y) = x^{1/3}y^{2/3}$ on the first quadrant: $x, y > 0$. On the portion

of the ellipse $x^2 + 2y^2 = 15$ *in the first quadrant*, f attains a maximal value at exactly one point $P = (a, b)$, and that value has the form $\sqrt{m}$ for a whole number m.

Find the point P, as well as the value $f(P)$ written as $\sqrt{m}$ for a whole number m (not as some messier expression).

Letting $g(x, y) = x^2 + 2y^2$, the point $P = (a, b)$ of interest is a local maximum for f on the curve $g = 15$, so either $(\nabla g)(P) = \mathbf{0}$ or there is a scalar λ for which $(\nabla f)(P) = \lambda(\nabla g)(P)$. We first compute the gradients:

$$(\nabla g)(x, y) = \begin{bmatrix} 2x \\ 4y \end{bmatrix}, \quad (\nabla f)(x, y) = \begin{bmatrix} (1/3)(y/x)^{2/3} \\ (2/3)(x/y)^{1/3} \end{bmatrix}.$$

Since we are working in the first quadrant, which is to say $x, y > 0$, the expression for ∇g shows that it is non-vanishing on this region. Hence, there must be a scalar λ for which $(\nabla f)(a, b) = \lambda(\nabla g)(a, b)$. In terms of the explicit vector expressions, this says:

$$\begin{bmatrix} (1/3)(b/a)^{2/3} \\ (2/3)(a/b)^{1/3} \end{bmatrix} = \lambda \begin{bmatrix} 2a \\ 4b \end{bmatrix} = \begin{bmatrix} 2\lambda a \\ 4\lambda b \end{bmatrix}.$$

Equating corresponding vector entries, this says:

$$\frac{1}{3}\left(\frac{b}{a}\right)^{2/3} = 2\lambda a, \quad \frac{2}{3}\left(\frac{a}{b}\right)^{1/3} = 4\lambda b.$$

Since $a, b > 0$, there are no division-by-zero issues: we can get two expressions for λ by division in the usual way, obtaining

$$\frac{b^{2/3}}{6a^{5/3}} = \lambda = \frac{2a^{1/3}}{12b^{4/3}} = \frac{a^{1/3}}{6b^{4/3}}.$$

Cross-multiplying yields $6b^2 = 6a^2$, so $b^2 = a^2$. Since $a, b > 0$, this implies $b = a$ (no issue with negative square roots!).

Now we go back to the constraint $15 = g(a, b) = g(a, a) = 3a^2$, so $a = \sqrt{5}$ and hence $b = a = \sqrt{5}$. In other words, $P = (\sqrt{5}, \sqrt{5})$. The value $f(P)$ is $\sqrt{5}^{1/3}\sqrt{5}^{2/3} = \sqrt{5}^{1/3+2/3} = \sqrt{5}$.

3. (10 points) Let $f : \mathbf{R}^3 \to \mathbf{R}$ be a function and let $g(x, y) = f(x, y, \sqrt{1 - x^2 - y^2})$ (assuming $x^2+y^2 < 1$, so $1 - x^2 - y^2 > 0$).

   (a) (5 points) Use the Chain Rule to find functions $h_1(x, y), h_2(x, y)$ having nothing to do with f so that

$$\frac{\partial g}{\partial x} = \frac{\partial f}{\partial x}(x, y, \sqrt{1 - x^2 - y^2}) + h_1(x, y)\,\frac{\partial f}{\partial z}(x, y, \sqrt{1 - x^2 - y^2}),$$
$$\frac{\partial g}{\partial y} = \frac{\partial f}{\partial y}(x, y, \sqrt{1 - x^2 - y^2}) + h_2(x, y)\,\frac{\partial f}{\partial z}(x, y, \sqrt{1 - x^2 - y^2}),$$

   (Hint: $g(x, y) = (f \circ G)(x, y)$ for $G : \mathbf{R}^2 \to \mathbf{R}^3$ defined by $G(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$.)

   The Chain Rule gives that $(Dg)(x, y) = \begin{bmatrix} g_x & g_y \end{bmatrix}$ is equal to

$$(Df)(G(x, y))\,(DG)(x, y) = \begin{bmatrix} f_x & f_y & f_z \end{bmatrix}\Big|_{G(x,y)} \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ -\frac{x}{\sqrt{1-x^2-y^2}} & -\frac{y}{\sqrt{1-x^2-y^2}} \end{bmatrix},$$

   and $(Dg)(x, y) = \begin{bmatrix} g_x & g_y \end{bmatrix}$. Computing the matrix product on the right as a 1×2 matrix and looking at its first entry thereby gives

$$\begin{aligned} g_x &= f_x\Big|_{G(x,y)} + f_z\Big|_{G(x,y)} \cdot \frac{-x}{\sqrt{1-x^2-y^2}} \\ &= \frac{\partial f}{\partial x}(x,y,\sqrt{1-x^2-y^2}) - \frac{x}{\sqrt{1-x^2-y^2}}\frac{\partial f}{\partial z}(x,y,\sqrt{1-x^2-y^2}). \end{aligned}$$

Thus, we use $h_1(x,y) = -x/\sqrt{1-x^2-y^2}$. Likewise, looking at the second entry in the 1×2 matrix $(Dg)(x,y)$ yields

$$\begin{aligned} g_y &= f_y\Big|_{G(x,y)} + f_z\Big|_{G(x,y)} \cdot \frac{-y}{\sqrt{1-x^2-y^2}} \\ &= \frac{\partial f}{\partial y}(x,y,\sqrt{1-x^2-y^2}) - \frac{y}{\sqrt{1-x^2-y^2}}\frac{\partial f}{\partial z}(x,y,\sqrt{1-x^2-y^2}), \end{aligned}$$

so we use $h_2(x,y) = -y/\sqrt{1-x^2-y^2}$.

(b) (5 points) Using $f(x,y,z) = xy^2z$, directly calculate the partial derivative with respect to x and the partial derivative with respect to y of the function $g(x,y) = f(x,y,\sqrt{1-x^2-y^2}) = xy^2\sqrt{1-x^2-y^2}$ and check that each agrees with the expression you obtained for it in part (a).

We have $f_x = y^2z$ and $f_z = xy^2$, so the expression in part (a) for the x-partial is

$$y^2\sqrt{1-x^2-y^2} - \frac{x}{\sqrt{1-x^2-y^2}}xy^2 = y^2\sqrt{1-x^2-y^2} - \frac{x^2y^2}{\sqrt{1-x^2-y^2}}.$$

On the other hand, direct calculation of the x-partial of $g(x,y)$ yields (using the product rule)

$$y^2\sqrt{1-x^2-y^2} + xy^2\frac{-2x}{2\sqrt{1-x^2-y^2}} = y^2\sqrt{1-x^2-y^2} - \frac{x^2y^2}{\sqrt{1-x^2-y^2}}.$$

These agree by inspection.
Turning to the y-partial, since $f_y = 2xyz$ the expression in part (a) for the y-partial is

$$2xy\sqrt{1-x^2-y^2} - \frac{y}{\sqrt{1-x^2-y^2}}xy^2 = 2xy\sqrt{1-x^2-y^2} - \frac{xy^3}{\sqrt{1-x^2-y^2}}.$$

On the other hand, direct calculation of the y-partial of g yields (by the product rule)

$$2xy\sqrt{1-x^2-y^2} + xy^2\frac{-2y}{2\sqrt{1-x^2-y^2}} = 2xy\sqrt{1-x^2-y^2} - \frac{xy^3}{\sqrt{1-x^2-y^2}}.$$

Once again the desired equality holds by inspection.

4. (10 points) The matrix $A = \begin{bmatrix} 2 & -4 & 4 \\ 1 & -3 & -1 \\ 1 & 0 & 10 \end{bmatrix}$ is equal to LU with $L = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix}$.

(a) (4 points) For $\mathbf{b} = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$, use the given LU-decomposition of A to solve $A\mathbf{x} = \mathbf{b}$ via repeated back-substitutions, and then check directly that your solution really is a solution. (All entries in the solution vector $\mathbf{x}$ are integers.)

We first solve $L\mathbf{y} = \mathbf{b}$, and then $U\mathbf{x} = \mathbf{y}$. The lower triangular system of equations is

$$2y_1 = 2, \;\; y_1 - y_2 = 2, \;\; y_1 + 2y_2 + y_3 = -1.$$

This gives $y_1 = 1$, then $1 - y_2 = 2$, so $y_2 = -1$, and finally $1 + 2(-1) + y_3 = -1$, so $y_3 = 0$. Next, the upper triangular system is

$$x_1 - 2x_2 + 2x_3 = 1, \;\; x_2 + 3x_3 = -1, \;\; 2x_3 = 0,$$

so going backwards gives $x_3 = 0$, so $x_2 = -1$, so $x_1 - 2(-1) + 2(0) = 1$, yielding $x_1 = 1 - 2 = -1$. Hence, $\mathbf{x} = \begin{bmatrix} -1 \\ -1 \\ 0 \end{bmatrix}$. This really works, since

$$2(-1) - 4(-1) = -2 + 4 = 2, \;\; 1(-1) - 3(-1) = -1 + 3 = 2, \;\; 1(-1) = -1.$$

(b) (6 points) Use the given LU-decomposition to compute A^{-1} (its entries are integers or fractions with denominator at most 4), and check that what you obtain really is an inverse to A by multiplying it against A in some order (you do not need to compute the matrix product in both orders; It is recommended to check your calculations of U^{-1} and L^{-1} really work before computing A^{-1}).

Since $A = LU$ with L and U each invertible (due to having no 0 in their diagonals), we have $A^{-1} = U^{-1}L^{-1}$. To calculate U^{-1} and L^{-1}, we set them up as

$$U^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1/2 \end{bmatrix}, \;\; L^{-1} = \begin{bmatrix} 1/2 & 0 & 0 \\ a' & -1 & 0 \\ b' & c' & 1 \end{bmatrix}$$

and need to solve for a, b, c and a', b', c'.
For U^{-1}, we use the requirement

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = UU^{-1} = \begin{bmatrix} 1 & -2 & 2 \\ 0 & 1 & 3 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & a-2 & b-2c+1 \\ 0 & 1 & c+3/2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Looking just above the diagonal, we have $a = 2$ and $c = -3/2$. Then looking in the upper-right, we get $0 = b - 2c + 1 = b + 3 + 1 = b + 4$, so $b = -4$.
Next, for L^{-1} we use the requirement

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = LL^{-1} = \begin{bmatrix} 2 & 0 & 0 \\ 1 & -1 & 0 \\ 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ a' & -1 & 0 \\ b' & c' & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/2 - a' & 1 & 0 \\ 1/2 + 2a' + b' & c' - 2 & 1 \end{bmatrix}.$$

Looking just below the diagonal, we have $a' = 1/2$ and $c' = 2$. Going into the lower-left corner, $0 = 1/2 + 2a' + b' = 1/2 + 1 + b'$, so $b' = -3/2$.
Having computed U^{-1} and L^{-1}, we multiply them (in the correct order!) to obtain A^{-1}:

$$A^{-1} = U^{-1}L^{-1} = \begin{bmatrix} 1 & 2 & -4 \\ 0 & 1 & -3/2 \\ 0 & 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1/2 & 0 & 0 \\ 1/2 & -1 & 0 \\ -3/2 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 15/2 & -10 & -4 \\ 11/4 & -4 & -3/2 \\ -3/4 & 1 & 1/2 \end{bmatrix}.$$

To check this works, we compute the matrix product

$$\begin{bmatrix} 2 & -4 & 4 \\ 1 & -3 & -1 \\ 1 & 0 & 10 \end{bmatrix} \begin{bmatrix} 15/2 & -10 & -4 \\ 11/4 & -4 & -3/2 \\ -3/4 & 1 & 1/2 \end{bmatrix} = \begin{bmatrix} 15-11-3 & -20+16+4 & -8+6+2 \\ 15/2-33/4+3/4 & -10+12-1 & -4+9/2-1/2 \\ 15/2-15/2 & -10+10 & -4+5 \end{bmatrix}$$
$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

as desired.

5. (10 points) Consider the matrix $A = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{bmatrix}$ whose columns from left to right are the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad \mathbf{v}_2 = \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix}, \quad \mathbf{v}_3 = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix}.$$

(a) (5 points) Carry out the Gram–Schmidt process for $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ to construct an orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3\}$ of $\mathbf{R}^3$. Verify directly that the $\mathbf{w}_j$'s you compute are perpendicular to each other. (The entries in the $\mathbf{w}_j$'s are all integers.)

We have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and

$$\mathbf{w}_2 = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \mathbf{v}_2 - \frac{6}{3}\mathbf{w}_1 = \mathbf{v}_2 - 2\mathbf{w}_1$$
$$= \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} - \begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}.$$

Finally,

$$\mathbf{w}_3 = \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 = \mathbf{v}_3 - \frac{3}{3}\mathbf{w}_1 - \frac{-2}{2}\mathbf{w}_2 = \mathbf{v}_3 - \mathbf{w}_1 + \mathbf{w}_2.$$
$$= \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$$
$$= \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}.$$

(Visibly the $\mathbf{w}_j$'s are all nonzero, so the $\mathbf{v}_i$'s are linearly independent.) The dot products are $\mathbf{w}_1 \cdot \mathbf{w}_2 = 1 - 1 = 0$, $\mathbf{w}_1 \cdot \mathbf{w}_3 = 1 + 1 - 2 = 0$, and $\mathbf{w}_2 \cdot \mathbf{w}_3 = 1 - 1 = 0$.

(b) (5 points) Using your work in the previous part, express each $\mathbf{v}_j$ as a linear combination of the $\mathbf{w}_i$'s, and use that to compute the QR-decomposition of A. For the orthogonal Q and upper triangular R that you have found, compute the matrix product QR to confirm it is equal to A.

From the work in (a), we have $\mathbf{v}_1 = \mathbf{w}_1$, $\mathbf{v}_2 = 2\mathbf{w}_1 + \mathbf{w}_2$, and $\mathbf{v}_3 = \mathbf{w}_1 - \mathbf{w}_2 + \mathbf{w}_3$. Define the unit vectors $\mathbf{w}'_j = \mathbf{w}_j/\|\mathbf{w}_j\|$, so $\{\mathbf{w}'_1, \mathbf{w}'_2, \mathbf{w}'_3\}$ is an orthonormal basis of $\mathbf{R}^3$. We have $\|\mathbf{w}_1\| = \sqrt{3}$, $\|\mathbf{w}_2\| = \sqrt{2}$, and $\|\mathbf{w}_3\| = \sqrt{6}$, so

$$\mathbf{w}_1 = \sqrt{3}\mathbf{w}'_1, \quad \mathbf{w}_2 = \sqrt{2}\mathbf{w}'_2, \quad \mathbf{w}_3 = \sqrt{6}\mathbf{w}'_3.$$

Substituting these into the expressions for the $\mathbf{v}_i$'s in terms of the $\mathbf{w}_j$'s, we have

$$\mathbf{v}_1 = \sqrt{3}\mathbf{w}'_1, \quad \mathbf{v}_2 = 2\sqrt{3}\mathbf{w}'_1 + \sqrt{2}\mathbf{w}'_2, \quad \mathbf{v}_3 = \sqrt{3}\mathbf{w}'_1 - \sqrt{2}\mathbf{w}'_2 + \sqrt{6}\mathbf{w}'_3.$$

Putting these successive coefficients as the columns of an upper triangular matrix, we obtain

$$R = \begin{bmatrix} \sqrt{3} & 2\sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{6} \end{bmatrix}.$$

Likewise, using $\mathbf{w}'_j$ as the jth column gives an orthogonal matrix

$$Q = \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix}.$$

Finally, we directly multiply matrices to compute that QR is equal to

$$\begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & -1/\sqrt{2} & 1/\sqrt{6} \\ 1/\sqrt{3} & 0 & -2/\sqrt{6} \end{bmatrix} \begin{bmatrix} \sqrt{3} & 2\sqrt{3} & \sqrt{3} \\ 0 & \sqrt{2} & -\sqrt{2} \\ 0 & 0 & \sqrt{6} \end{bmatrix} = \begin{bmatrix} 1 & 2+1 & 1-1+1 \\ 1 & 2-1 & 1+1+1 \\ 1 & 2 & 1-2 \end{bmatrix} = \begin{bmatrix} 1 & 3 & 1 \\ 1 & 1 & 3 \\ 1 & 2 & -1 \end{bmatrix},$$

which is A as desired.

6. (10 points) **True or False**
For each of the following statements, circle either TRUE (meaning, "always true") or FALSE (meaning, "not always true"), and briefly and convincingly justify your answer. 1 point for the correct choice, and the remaining points for convincing justification.

(a) (3 points) Suppose we want to find the extremum of $f(x, y) = -\frac{43}{9}x + \frac{82}{17}y$ on the level set E defined by $g(x, y) = 15x^2 + 72y^2 = 95$. Then there are exactly two points on E for which the gradient of f and the gradient of g are scalar multiples of each other.
Circle one, and justify below: $\boxed{\text{TRUE}}$ FALSE

Always true: The constraint curve given by $g(x, y) = 15x^2 + 72y^2 = 95$ is an ellipse, and the level sets of $f(x, y)$ are lines with constant slope $m = \frac{43}{9}\frac{17}{82}$. Geometrically, there are only two points on the ellipse at which the tangent lines to the ellipse $g(x, y) = 95$ have slope m. At these two points, the ∇f and ∇g must be scalar multiples of each other.

(b) (3 points) There is a 51×51 matrix whose column space $C(A)$ is the same as its null space $N(A)$.
Circle one, and justify below: TRUE $\boxed{\text{FALSE}}$

Not always true: If there were such a matrix, then $\dim(C(A)) + \dim(N(A)) = 2\dim(C(A))$ would be an even number. But the Rank-Nullity Theorem says $\dim(C(A)) + \dim(N(A)) = 51$ which is odd. So no such matrix A exists.

(c) (4 points) Let $A = QR$ be the QR-decomposition of an $n \times n$ matrix A. If $A\mathbf{v} = \mathbf{0}$, then $R\mathbf{v} = \mathbf{0}$.
Circle one, and justify below: $\boxed{\text{TRUE}}$ FALSE

Always true: if $A\mathbf{v} = \mathbf{0}$, since $A = QR$, then $QR\mathbf{v} = \mathbf{0}$. Q has orthonormal columns, so $Q^\top Q = I$. In particular, $Q^\top QR\mathbf{v} = IR\mathbf{v} = R\mathbf{v}$. But $Q^\top QR\mathbf{v} = Q^\top(QR)\mathbf{v} = Q^\top(A\mathbf{v}) = Q^\top\mathbf{0} = \mathbf{0}$. So $R\mathbf{v} = \mathbf{0}$.

7. (8 points) Define $\mathbf{v} = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$.

(a) (2 points) Compute the 3×3 matrix $A = \mathbf{v}(\mathbf{w}^\top)$, and express each column as a multiple of $\mathbf{v}$ and each row as a multiple of $\mathbf{w}^\top$.

The matrix A is

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} \begin{bmatrix} 2 & 5 & -3 \end{bmatrix} = \begin{bmatrix} 6 & 15 & -9 \\ -2 & -5 & 3 \\ 4 & 10 & -6 \end{bmatrix},$$

with the columns from left to right equal to $2\mathbf{v}$, $5\mathbf{v}$, $-3\mathbf{v}$ and the rows from top to bottom equal to $3\mathbf{w}^\top$, $-\mathbf{w}^\top$, $2\mathbf{w}^\top$.

(b) (6 points) Compute bases of $C(A)$ and $N(A)$. (You don't need anything with matrix factorizations to do this, just geometric thinking in $\mathbf{R}^3$.)

Since all columns are scalar multiples of $\mathbf{v}$, any scalar multiple of a column is a multiple of $\mathbf{v}$ and hence $C(A)$ is the span of the nonzero vector $\mathbf{v}$. Hence, $\mathbf{v}$ is a basis of $C(A)$ (as is any nonzero scalar multiple of $\mathbf{v}$). The null space $N(A)$ consists of vectors $\mathbf{x} \in \mathbf{R}^3$ that are orthogonal to each row of A (by the meaning of $A\mathbf{x}$ and of $N(A)$), but those rows are multiples of $\mathbf{w}^\top$, so this is the same as orthogonality to $\mathbf{w}$. In other words, $N(A)$ is the linear subspace of $\mathbf{R}^3$ consisting of those 3-vectors $\mathbf{x}$ that are orthogonal to the nonzero vector $\mathbf{w} = \begin{bmatrix} 2 \\ 5 \\ -3 \end{bmatrix}$. In other words, $N(A)$ is the plane through the origin with $\mathbf{w}$ as its normal vector, which is to say it is defined by the equation

$$2x + 5y - 3z = 0.$$

This equation says $z = (2/3)x + (5/3)y$, so such $\mathbf{x}$ are precisely of the form

$$\begin{bmatrix} x \\ y \\ (2/3)x + (5/3)y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ (2/3)x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ (5/3)y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2/3 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 5/3 \end{bmatrix}.$$

This exhibits a basis of the plane as the pair of vectors $\begin{bmatrix} 1 \\ 0 \\ 2/3 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 5/3 \end{bmatrix}$ (many other answers are possible).

8. (10 points) Let $f(x, y) = x^2 + xy^2 + 10xy + 10x - (9/2)y^2 - 36y$; you can accept (or check if you wish) that $f(3, -2) = 45$.

(a) (4 points) Verify that $(3, -2)$ is a critical point of f, and compute the symmetric Hessian matrix $(\mathrm{H}f)(3, -2)$ as well as the quadratic approximation for f at $(3, -2)$ (i.e., the quadratic approximation to $f(3 + h, -2 + k)$ for $h, k \approx 0$).

We compute symbolically

$$f_x = 2x + y^2 + 10y + 10, \quad f_y = 2xy + 10x - 9y - 36,$$

so $f_x(3,-2) = 6+4-20+10 = 20-20 = 0$ and $f_y(3,-2) = -12+30+18-36 = -12+48-36 = 36-36 = 0$. Hence, $(\nabla f)(3,-2) = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, so indeed $(3,-2)$ is a critical point for f.

The second partials of f are

$$f_{xx} = 2, \quad f_{yy} = 2x - 9, \quad f_{xy} = 2y + 10,$$

so $f_{xx}(3,-2) = 2$, $f_{yy}(3,-2) = 6-9 = -3$, and $f_{xy}(3,-2) = -4+10 = 6$. Hence,

$$(\mathrm{H}f)(3,-2) = \begin{bmatrix} 2 & 6 \\ 6 & -3 \end{bmatrix}$$

and the quadratic approximation is

$$\begin{aligned} f(3+h,-2+k) &\approx f(3,-2) + (\nabla f)(3,-2) \cdot \begin{bmatrix} h \\ k \end{bmatrix} + \frac{1}{2} \begin{bmatrix} h & k \end{bmatrix} (\mathrm{H}f)(3,-2) \begin{bmatrix} h \\ k \end{bmatrix} \\ &= 45 + \frac{1}{2} \begin{bmatrix} h & k \end{bmatrix} \begin{bmatrix} 2 & 6 \\ 6 & -3 \end{bmatrix} \begin{bmatrix} h \\ k \end{bmatrix} \\ &= 45 + \frac{1}{2}(2h^2 + 12hk - 3k^2) \\ &= 45 + h^2 + 6hk - (3/2)k^2. \end{aligned}$$

(The final equality step is not necessary.)

(b) (6 points) For the Hessian matrix $H = (\mathrm{H}f)(3,-2)$ that you found in part (a), compute its eigenvalues and use this to determine if $(3,-2)$ is a local maximum, local minimun, or saddle point. Use the eigenvalues and eigenvectors of H to sketch what the contour plot of f looks like near $(3,-2)$. (The eigenvalues of H are integers.)

The Hessian has trace -1 and determinant $-6-36 = -42$, so it has characteristic polynomial

$$\lambda^2 - (-1)\lambda - 42 = \lambda^2 + \lambda - 42 = (\lambda+7)(\lambda-6),$$

which has as its roots $\lambda_1 = -7$ and $\lambda_2 = 6$ (which can also be found by the quadratic formula, if you didn't notice how it factors). These are integers as promised and have opposite signs, so the quadratic form associated to the Hessian is indefinite and hence $(3,-2)$ is a saddle point.

To sketch the contour plot, we need to work out perpendicular eigenvectors $\mathbf{w}_1$ and $\mathbf{w}_2$ for λ_1 and λ_2 respectively so as to write the quadratic form associated to the Hessian in a more convenient reference frame. The lines for the eigenvalues are the null spaces of $H - (-7)\mathrm{I}_2 = H + 7\mathrm{I}_2$ and $H - 6\mathrm{I}_2$. We compute these matrices to be

$$H + 7\mathrm{I}_2 = \begin{bmatrix} 9 & 6 \\ 6 & 4 \end{bmatrix}, \quad H - 6\mathrm{I}_2 = \begin{bmatrix} -4 & 6 \\ 6 & -9 \end{bmatrix}.$$

The first has null space corresponding to the pair of equations $9x + 6y = 0$ and $6x + 4y = 0$, which are scalar multiples of each other (as they must be for a line of an eigenvalue): this is the line $y = -(3/2)x$, so it is the span of the vector $\mathbf{w}_1 = \begin{bmatrix} 2 \\ -3 \end{bmatrix}$ (or any nonzero scalar multiple of this). The second of these has null space corresponding to the pair of equations $-4x + 6y = 0$ and $6x - 9y = 0$, which are likewise scalar multiples of each other (as they must be for a line of an eigenvalue): this is the line $y = (2/3)x$, so it is the span of the vector $\mathbf{w}_2 = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$ (or any nonzero scalar multiple of this). The quadratic form q_H is given by the formula

$$q_H(u_1\mathbf{w}_1 + u_2\mathbf{w}_2) = \lambda_1(\mathbf{w}_1 \cdot \mathbf{w}_1)u_1^2 + \lambda_2(\mathbf{w}_2 \cdot \mathbf{w}_2)u_2^2 = -7(13)u_1^2 + 6(13)u_2^2.$$

This has level curves that are hyperbolas centered at $(3, -2)$ with symmetry lines through $\mathbf{w}_1$ and $\mathbf{w}_2$ and having asymptotes with slopes $\pm\sqrt{7/6}$ that have absolute value close to (but a bit more than) 1. The sketch is shown below.

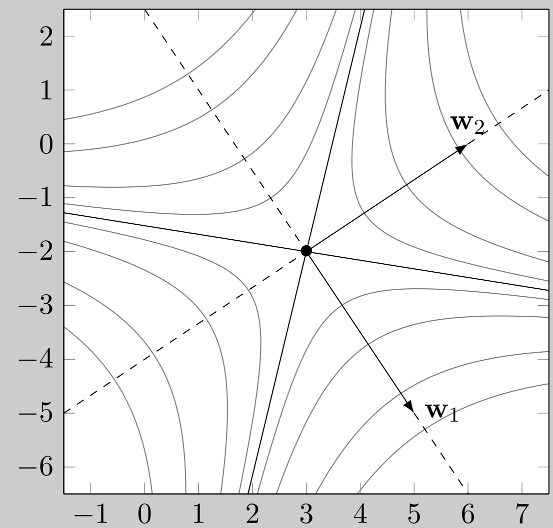

Figure 1: Approximate contour plot of f via its quadratic approximation at $(3, -2)$.

9. (12 points) A population of 60 goats move between three meadows, labelled A, B, and C. Their daily movement is modeled by a symmetric Markov matrix M (whose rows and columns correspond to the meadows A, B, and C, in that order).

$$M = \begin{bmatrix} 2/3 & 2/9 & 1/9 \\ 2/9 & 5/9 & 2/9 \\ 1/9 & 2/9 & 2/3 \end{bmatrix}$$

(a) (2 points) We know that one of the eigenvalues $\lambda_1 = \dfrac{5}{9}$. Find the corresponding eigenvector $\mathbf{v}_1$.

Since $\lambda_1 = \dfrac{5}{9}$ is an eigenvalue, we find the corresponding eigenvector $\mathbf{v}_1$ by finding solution to $\left(M - \dfrac{5}{9}I_3\right)\mathbf{x} = \mathbf{0}$.

$$M - \frac{5}{9}I_3 = \begin{bmatrix} 1/9 & 2/9 & 1/9 \\ 2/9 & 0 & 2/9 \\ 1/9 & 2/9 & 1/9 \end{bmatrix}$$

whose first and third columns are the same, $N\left(M - \dfrac{5}{9}I_3\right)$ is spanned by $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$.

For parts (b) and (c), use the additional fact (which you do *not* have to prove) that for all large k, the matrix $M^k \approx \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{u} & \mathbf{u} \\ | & | & | \end{bmatrix}$ where $\mathbf{u}$ is a certain nonzero vector that does not depend on k.

(b) (8 points) Find the remaining two eigenvalues λ_2 and λ_3 of M as well as their associated eigenvectors $\mathbf{v}_2$ and $\mathbf{v}_3$.
You do not need to know the determinant formula for general 3×3 matrices to solve this problem.

By the given information, for large k we have $M^k \approx \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{u} & \mathbf{u} \\ | & | & | \end{bmatrix}$, and $M^{k+1} \approx M^k$; so we may conclude $M \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{u} & \mathbf{u} \\ | & | & | \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{u} & \mathbf{u} \\ | & | & | \end{bmatrix}$. In particular, $M\mathbf{u} = \mathbf{u}$. So M has 1 as an eigenvalue.

Let $\lambda_2 = 1$, we find $\mathbf{v}_2$ by solving $(M - I_3)\mathbf{x} = \mathbf{0}$,

$$M - I_3 = \begin{bmatrix} -1/3 & 2/9 & 1/9 \\ 2/9 & -4/9 & 2/9 \\ 1/9 & 2/9 & -1/3 \end{bmatrix}$$

Note that the sum of each row of M is also 1, and the sum of each row of $M - I_3$ is 0. So $N(M - I_3)$ is spanned by $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

To find the last eigenvalue and its eigenvector, we use the fact that M is also symmetric, so by the Spectral Theorem, M will have orthogonal eigenvectors. In particular, the eigenvector $\mathbf{v}_3$ must be perpendicular to $\mathbf{v}_1$ and $\mathbf{v}_2$. We need to solve the simultaneous equations

$$\begin{cases} x - z = 0 \\ x + y + z = 0 \end{cases}$$

We have $z = x$, and $y = -2x$. So $\mathbf{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

$$M\mathbf{v}_3 = \begin{bmatrix} 2/3 & 2/9 & 1/9 \\ 2/9 & 5/9 & 2/9 \\ 1/9 & 2/9 & 2/3 \end{bmatrix} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/3 \\ -2/3 \\ 1/3 \end{bmatrix} = \frac{1}{3} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

We conclude that $\lambda_3 = 1/3$.

(c) (2 points) How many goats are in meadow B after k days for large k?

As given, for large k the matrix $M^k \approx \begin{bmatrix} | & | & | \\ \mathbf{u} & \mathbf{u} & \mathbf{u} \\ | & | & | \end{bmatrix}$, which must itself be a Markov matrix; so $\sum_{i=1}^{3} u_i = 1$. By our solution to (b), $\mathbf{u}$ lies in the 1-eigenspace of M, namely span $\left(\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right)$. So

$$\mathbf{u} = \frac{1}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \qquad M^k \approx \frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}.$$

If there are initially x goats in meadow A, y goats in meadow B, and $60 - x - y$ goats in meadow C, then

$$\frac{1}{3} \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ 60 - x - y \end{bmatrix} = \frac{60}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

For large k, after k days, independent of initial distribution of the goats among the three meadows, there are 20 goats in each meadow.