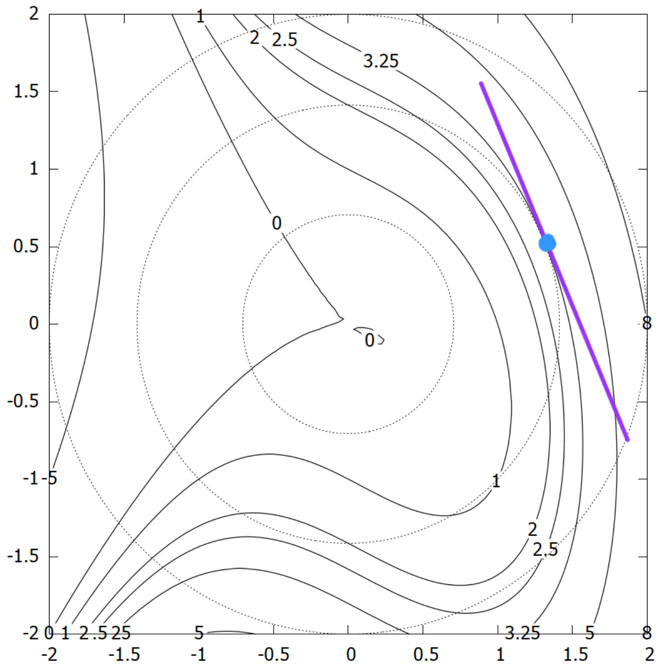
Topic(s): Lagrange multipliers!

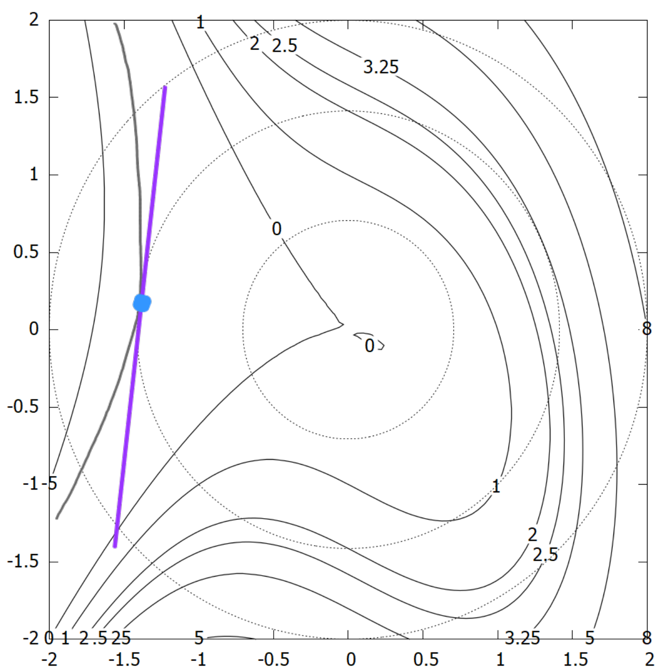
A **constrained optimization problem** is a situation where we want to optimize some function $f(\mathbf{x})$, subject to a constraint $g(\mathbf{x}) = c$.

Example 1. Suppose we want to find the maximum of $f(x, y) = x^3 + xy + y^2$, subject to the constraint $g(x, y) = x^2 + y^2 = 2$. The picture below shows the contour plots of f and g. Look at the level sets of f that intersect the level set of g at 2. What is the maximum of f on this level set? What do you notice about the gradients of f and g at that point?



The maximum of f on g(x,y) = 2 is 3.25. At the point, the level curves f(x,y) = 3.25 and g(x,y) = 2 have the same tangent line. Thus, the gradients must be scalar multiples of each other.

Now, try to find the minimum of f, subject to the constraint $g(x, y) = 2$.

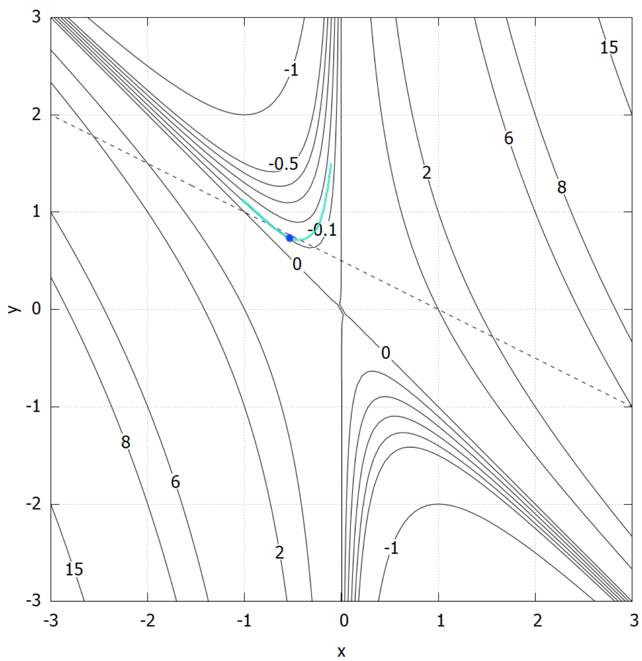


At the point on g(x,y) = 2 at which f obtains its minimum, it seems that the level curves again have the same tangent line.

Theorem 12.2.1. Suppose $f : \mathbb{R}^n \to \mathbb{R}$ and $g : \mathbb{R}^n \to \mathbb{R}$ are functions, and consider the problem of optimizing f on the region where $g(\mathbf{x}) = c$. If a local extremum for f on the region $\{\mathbf{x} : g(\mathbf{x}) = c\}$ occurs at $\mathbf{a}$, then either

- $\nabla g(\mathbf{a}) = \mathbf{0}$, or
- $\nabla f(\mathbf{a}) = \lambda \nabla g(\mathbf{a})$ for some scalar λ that depends on $\mathbf{a}$.

Example 2. We want to find points on the line $\overset{g(x,y)=3}{6y + 3x = 3}$ that minimize the function $x^2 + xy$. The picture below shows the contour lines of $x^2 + xy$ with solid curves. On the picture, indicate the rough location of the minimum and estimate its value. Compute ∇f and ∇g. Does ∇g vanish anywhere?



$$\nabla f = \begin{bmatrix} 2x + y \\ x \end{bmatrix} \text{ and } \nabla g = \begin{bmatrix} 3 \\ 6 \end{bmatrix}.$$

∇g does not vanish anywhere.

Set up the system of equations $\nabla f(a, b) = \lambda \nabla g(a, b)$ and solve for different expressions of λ.

$$\begin{bmatrix} 2x + y \\ x \end{bmatrix} = \lambda \begin{bmatrix} 3 \\ 6 \end{bmatrix} \Rightarrow \begin{cases} 2x + y = 3\lambda \\ x = 6\lambda \end{cases}$$

$$\Rightarrow \lambda = \frac{2x + y}{3}, \quad \lambda = \frac{x}{6}.$$

Determine the values of a and b.

Setting the different expressions for λ equal to each other gives $3x = -2y$. Plugging this into the constraint $6y + 3x = 3$ gives $x = -\frac{1}{2}$, $y = \frac{3}{4}$. Hence, the point of interest is $\left(-\frac{1}{2}, \frac{3}{4}\right)$ which matches with the picture above.

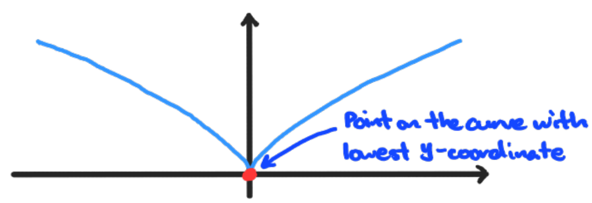
Example 3. Minimize $f(x, y) = y$, subject to $\underbrace{x^2 - y^3 = 0}_{g(x,y)=0}$.

We compute $\nabla f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x \\ -3y^2 \end{bmatrix}$. ∇g vanishes at $(0,0)$.

$$\text{Lagrange: } \nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ -3y^2 \end{bmatrix} \Rightarrow \begin{cases} 0 = 2\lambda x \\ 1 = -3\lambda y^2 \end{cases}$$

Since $0 = 2\lambda x$, either $\lambda = 0$ or $x = 0$. However, $\lambda \neq 0$ since $1 = -3\lambda y^2$. Hence, $x = 0$. Plugging this into $g(x,y) = 0$ gives us $y = 0$. But, $\nabla f = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, and thus, there is no λ for which $\nabla f = \lambda \nabla g$.

The only point of interest is $(0,0)$. To see whether this is a maximum or a minimum, pick another point on $x^2 - y^3 = 0$, say $(1,1)$. Since $f(0,0) = 0$ and $f(1,1) = 1$, $\boxed{0}$ is the minimum value.



Example 4. Using Lagrange multipliers, find the maximum value of the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = -2x + y$ on the curve $\underbrace{x^2 + y^2 = 9}_{g(x,y)=9}$.

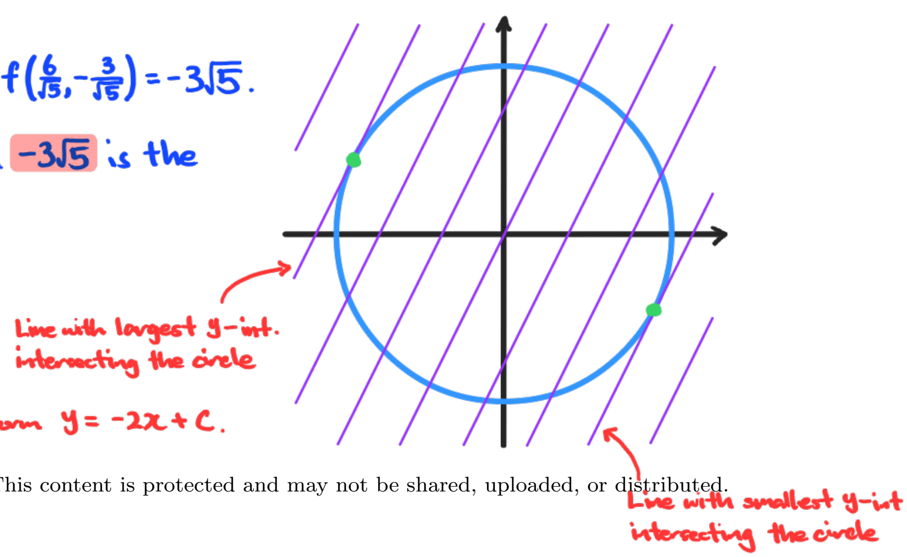
We see that $\nabla f = \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. ∇f vanishes nowhere.

$$\text{Lagrange: } \nabla f = \lambda \nabla g \Rightarrow \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow \begin{cases} -2 = \lambda 2x \\ 1 = \lambda 2y \end{cases}$$

So, $\lambda x = -2\lambda y$ and since $\lambda \neq 0$, we get $x = -2y$. Plugging this into $x^2 + y^2 = 9$ gives $y = \pm \frac{3}{\sqrt{5}}$. The points of interest are $\left(-\frac{6}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right)$ and $\left(\frac{6}{\sqrt{5}}, -\frac{3}{\sqrt{5}}\right)$.

We see $f\left(-\frac{6}{\sqrt{5}}, \frac{3}{\sqrt{5}}\right) = 3\sqrt{5}$ and $f\left(\frac{6}{\sqrt{5}}, -\frac{3}{\sqrt{5}}\right) = -3\sqrt{5}$.

So, $\boxed{3\sqrt{5}}$ is the maximum, and $\boxed{-3\sqrt{5}}$ is the minimum.



The purple lines are of the form $y = -2x + C$.

Example 5. (Lecture 10, Example 8) Find the maximum and minimum of $f(x,y) = x^2 - y^2$ on the circle $D = \{(x,y) : x^2 + y^2 \leq 4\}$.

We get $\nabla f = \begin{bmatrix} 2x \\ -2y \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$. ∇g vanishes at $(0,0)$, which is not on the boundary constraint $x^2 + y^2 = 4$. It is an interior critical point.

$$\text{Lagrange: } \begin{bmatrix} 2x \\ -2y \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \end{bmatrix} \Rightarrow \begin{array}{l} 2x = \lambda 2x \\ -2y = \lambda 2y \end{array} \rightarrow 2x(\lambda - 1) = 0.$$

① If $x = 0$, then $y = \pm 2$ which gives us $(0,2)$ and $(0,-2)$.

② If $\lambda = 1$, then $y = 0$, and so, $x = \pm 2$, which gives us $(2,0)$ and $(-2,0)$.

(x,y)	$f(x,y)$
$(0,0)$	0
$(0,-2)$	-4
$(0,2)$	-4
$(-2,0)$	4
$(2,0)$	4

On D, f obtains a maximum of 4 and a minimum of -4.

Example 6. Find the point(s) on the curve $\{(x,y) : \underbrace{8y^2 - 4x^3 + x^4 = 0}_{g(x,y)=0}\}$ closest to the point $P(3,0)$, and compute the minimal distance.

Minimizing $d = \sqrt{(x-3)^2 + y^2}$ is equivalent to minimizing $f(x,y) = d^2 = (x-3)^2 + y^2$.

We see that $\nabla f = \begin{bmatrix} 2x-6 \\ 2y \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 4x^3 - 12x^2 \\ 16y \end{bmatrix}$. ∇g vanishes at $(0,0)$ and $(3,0)$, of which the former lies on $g(x,y) = 0$.

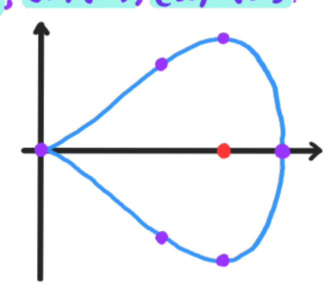
$$\text{Lagrange: } \begin{bmatrix} 2x-6 \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} 4x^3 - 12x^2 \\ 16y \end{bmatrix} \Rightarrow \begin{array}{l} 2x - 6 = \lambda(4x^3 - 12x^2) \\ 2y = \lambda(16y) \end{array}$$

The second equation becomes $2y(8\lambda - 1) = 0$.

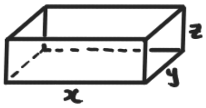
① If $y = 0$, then $-4x^3 + x^4 = 0$. So, $x = 0, 4$, and points are $(0,0)$ and $(4,0)$.

② If $\lambda = \frac{1}{8}$, the first equation from Lagrange becomes $2x - 6 = \frac{1}{8}(4x^3 - 12x^2)$, and so, $x = 3, \pm 2$. The relevant points are $\left(3, \sqrt{\frac{27}{8}}\right)$, $\left(3, -\sqrt{\frac{27}{8}}\right)$, $(2, \sqrt{2})$, $(2, -\sqrt{2})$.

By computing f on the six points, we see that $(4,0)$ is the closest point on the curve from $(3,0)$ at distance 1. Additionally, $(0,0)$ is farthest from $(3,0)$.



Example 7. Make an open-topped rectangular box with volume 4, using as little material as possible.



Minimize $f(x,y,z) = xy + 2xz + 2yz$ subject to $g(x,y,z) = xyz = 4$.

We get $\nabla f = \begin{bmatrix} y+2z \\ x+2z \\ 2x+2y \end{bmatrix}$ and $\nabla g = \begin{bmatrix} yz \\ xz \\ xy \end{bmatrix}$. ∇g does not vanish on $xyz=4$.

$$\nabla f = \lambda \nabla g: \quad \begin{array}{ll} ① & y + 2z = \lambda yz \\ ② & x + 2z = \lambda xz \\ ③ & 2x + 2y = \lambda xy \end{array}$$

Note that none of x, y, z can be zero.

①+②: $x(y+2z) = \lambda xyz = y(x+2z) \Rightarrow x = y$

①+③: $x(y+2z) = \lambda xyz = z(2x+2y) \Rightarrow x = 2z$

Plugging these into $xyz=4$ gives us $x=2,\ y=2,\ z=1 \Rightarrow (2,2,1)$

To see whether $(2,2,1)$ is a maximum or a minimum, we compare with another point on $xyz=4$, say $(4,1,1)$. Since $f(2,2,1) = 12 < 14 = f(4,1,1)$, $2\times 2\times 1$ are the dimensions of an open-topped box of volume 4 with minimal material.

Example 8. Find the largest and smallest values of $x+y+z$ among points on the sphere $\underbrace{x^2+y^2+z^2 = 12}_{g(x,y,z)=12}$.

Taking $f(x,y,z) = x+y+z$, we get $\nabla f = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$. ∇g vanishes at $\vec{0}$ which is not on $g(x,y,z)=12$.

$$\nabla f = \lambda \nabla g: \quad \begin{array}{l} 1 = \lambda 2x \\ 1 = \lambda 2y \\ 1 = \lambda 2z \end{array}$$

Since $\lambda \neq 0$, $x = y = z = \frac{1}{2\lambda}$. Plugging this into $x^2+y^2+z^2=12$ gives us $x=\pm 2$, which leads to $(2,2,2)$ and $(-2,-2,-2)$.

We see $f(2,2,2) = 6$ and $f(-2,-2,-2) = -6$, and thus, the maximum and minimum of $x+y+z$ subject to $x^2+y^2+z^2 = 12$ are 6 and -6, respectively.