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Problem 1: Visualizing a span

For each collection of vectors in $\mathbf{R}^2$, sketch its span: is it a point, a line, or all of $\mathbf{R}^2$?

(a) $\begin{bmatrix}1\\0\end{bmatrix}$ (b) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}0\\1\end{bmatrix}$ (c) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}1\\0\end{bmatrix}$ (d) $\begin{bmatrix}1\\0\end{bmatrix}, \begin{bmatrix}2\\0\end{bmatrix}$ (e) $\begin{bmatrix}0\\0\end{bmatrix}$

For each collection of vectors in $\mathbf{R}^3$ sketch its span: is it a point, a line, a plane, or all of $\mathbf{R}^3$?

(f) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}$ (g) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix}$ (h) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\end{bmatrix}, \begin{bmatrix}1\\1\\0\end{bmatrix}$ (i) $\begin{bmatrix}1\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\0\end{bmatrix}$

Solution:

(a) This is the x-axis.

(b) The span is the entirety of $\mathbf{R}^2$ since $x\begin{bmatrix}1\\0\end{bmatrix} + y\begin{bmatrix}0\\1\end{bmatrix} = \begin{bmatrix}x\\y\end{bmatrix}$.

(c) This again is the x-axis.

(d) Since $\begin{bmatrix}2\\0\end{bmatrix} = 2\begin{bmatrix}1\\0\end{bmatrix}$ this is the x-axis.

(e) This is just the origin.

(f) This is the xy-plane.

(g) This is the entirety of $\mathbf{R}^3$, by an analogue of (b): $x\begin{bmatrix}1\\0\\0\end{bmatrix} + y\begin{bmatrix}0\\1\\0\end{bmatrix} + z\begin{bmatrix}0\\0\\1\end{bmatrix} = \begin{bmatrix}x\\y\\z\end{bmatrix}$.

(h) Since $\begin{bmatrix}1\\1\\0\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} + \begin{bmatrix}0\\1\\0\end{bmatrix}$, so this third vector contributes nothing new to the span, we again get the xy-plane. This is an instance of the observation that for any vectors $\mathbf{v}, \mathbf{w}$ and scalars a, b, c, we have that

$$a\mathbf{v} + b\mathbf{w} + c(\mathbf{v}+\mathbf{w}) = a\mathbf{v} + b\mathbf{w} + c\mathbf{v} + c\mathbf{w} = (a+c)\mathbf{v} + (b+c)\mathbf{w}$$

belongs to the span of $\mathbf{v}$ and $\mathbf{w}$.

(i) This is just the x-axis (including the origin in the collection of vectors contributes nothing to the span).

Problem 2: What sets can be linear subspaces, and what cannot?

For each of the following subsets of $\mathbf{R}^2$ or $\mathbf{R}^3$, write down a collection of finitely many vectors whose span is that set or explain why there is no such collection.

(a) The line $x + y = 1$ (b) The line $x + y = 0$ (c) The unit disk $x^2 + y^2 \leq 1$ (d) $\{\mathbf{0}\}$ (e) The plane $x + y + z = 0$

Solution:

(a) Since $\mathbf{0}$ is not on that line ($0 + 0 \neq 1$), it cannot be a span.

(b) This is a line through the origin. The vector $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ is on the line $(1 + (-1) = 0)$ and hence this vector spans the line. Any nonzero multiple of it will also span the line.

(c) Any non-zero vector $\mathbf{v}$ in the unit disk has a multiple $a \cdot \mathbf{v}$ (where a might have to be very large) that is not in the unit disk. Hence the unit disk cannot be the span of anything, since for anything in a span every scalar multiple is also in the span $(c(a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k) = (ca)\mathbf{v}_1 + \cdots + (ca)\mathbf{v}_k)$.

(d) This is the span of $\{\mathbf{0}\}$.

(e) These are $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$ satisfying $x + y + z = 0$, so $z = -x - y$, so they are the vectors $\begin{bmatrix} x \\ y \\ -x-y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$. This exhibits it as the span of two vectors, namely $\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Problem 3: Multiple descriptions as a span (Extra)

Let $\mathbf{v}, \mathbf{w}$ be two vectors in $\mathbf{R}^{12}$. Show that $\mathrm{span}(\mathbf{v}, \mathbf{w}) = \mathrm{span}(\mathbf{v} + \mathbf{w}, \mathbf{v} - \mathbf{w})$. (Hint: You can show that two sets S and T are equal in two steps: everything belong to S also belong to T, and everything belonging to T also belongs to S.)

Solution: Anything in the second span can be written as

$$a(\mathbf{v} + \mathbf{w}) + b(\mathbf{v} - \mathbf{w}) = a\mathbf{v} + a\mathbf{w} + b\mathbf{v} - b\mathbf{w} = (a + b)\mathbf{v} + (a - b)\mathbf{w},$$

so it lies in the first span. In the other direction, we want to show that anything in the first span lies in the second: can we rewrite any $a\mathbf{v} + b\mathbf{w}$ in the form $x(\mathbf{v} + \mathbf{w}) + y(\mathbf{v} - \mathbf{w})$? We have seen that expressions of the latter sort are exactly $(x + y)\mathbf{v} + (x - y)\mathbf{w}$, so by comparing coefficients of $\mathbf{v}$ and of $\mathbf{w}$ it is enough to show for any given a, b that we can find x, y so that $x + y = a$ and $x - y = b$.

But this is "solving 2 equations in 2 unknowns" of a sort done in high school algebra: we add the equations to get $2x = a + b$, so $x = (a + b)/2$, and we subtract the second equation from the first to get $2y = a - b$, so $y = (a - b)/2$. The pair $(x, y) = ((a + b)/2, (a - b)/2)$ is readily seen to be an actual solution to that pair of equations, so we have

$$a\mathbf{v} + b\mathbf{w} = \frac{a + b}{2}(\mathbf{v} + \mathbf{w}) + \frac{a - b}{2}(\mathbf{v} - \mathbf{w})$$

(as is readily verified directly, if we want to do so; this is entirely unnecessary). This shows that the first span is contained in the second, so the two spans coincide.

Problem 4: Determining the nature of a span

For each collection of 3-vectors, determine whether its span is a point, a line, a plane, or all of $\mathbf{R}^3$. Give a basis of the span in each case. (Keep in mind that if a vector in the collection is a linear combination of others then it can be dropped without affecting the span.)

(a) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$

(b) $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ -1 \end{bmatrix}$

(c) $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$

Solution:

(a) Neither of those vectors is a scalar multiple of the other one, so the span is a plane and the two given vectors are a basis for it.

(b) We see $\begin{bmatrix}-1\\1\\-1\end{bmatrix} = \begin{bmatrix}1\\2\\1\end{bmatrix} - \begin{bmatrix}2\\1\\2\end{bmatrix}$, so the span of all three given vectors coincides with the span of the first two. Neither of those two is a scalar multiple of the other, so their span is a plane and those first two vectors are a basis of this plane (as are any two of the three vectors, if you'd like to think through why that should hold).

(c) None of these three nonzero vectors can be written as a linear combination of the other two, as is checked by setting up a hypothetical such expression with unknown coefficients x and y and checking that the resulting 3 equations on x and y (one per vector entry) has no solution. Since none of these given vectors is a scalar multiple of the other, if any one can be expressed as a linear combination of the others then each can be so expressed. So to rule it out, we just need to rule out one of them having such an expression.

For example, if we try to express the third vector as a linear combination of the other two then we are seeking scalars x and y satisfying

$$\begin{bmatrix}-1\\1\\0\end{bmatrix} = x\begin{bmatrix}1\\2\\3\end{bmatrix} + y\begin{bmatrix}2\\1\\2\end{bmatrix} = \begin{bmatrix}x+2y\\2x+y\\3x+2y\end{bmatrix},$$

which amounts to *three simultaneous equations* (one per vector entry) in these two unknowns:

$$x+2y=-1,\ \ 2x+y=1,\ \ 3x+2y=0.$$

The first two equations together are a situation from high school algebra, which one works out to have exactly one solution (corresponding to two non-parallel lines crossing at a point): $(x,y)=(1,-1)$. But this violates the third equation, as $3(1)+2(-1)=1\neq 0$. So there is no solution, and hence the third vector in the given collection cannot be expressed as a linear combination of the other two.

So the span is all of $\mathbf{R}^3$, and hence it cannot be spanned by less than three nonzero vectors (a span of less than three would be a plane or line through the origin, so it would not coincide with $\mathbf{R}^3$). Thus, any three spanning vectors are a basis, such as the three that are given.

Problem 5: Linear subspaces and orthogonality (computations)

Let V be the set of vectors in $\mathbf{R}^4$ orthogonal to both $\begin{bmatrix}1\\0\\4\\2\end{bmatrix}$ and $\begin{bmatrix}0\\1\\1\\1\end{bmatrix}$. Find a pair of vectors that span V, so it is a linear subspace.

Solution: Membership of a 4-vector $\begin{bmatrix}x\\y\\z\\w\end{bmatrix}$ in V amounts to 2 equations in these 4 variables (one equation for each orthogonality condition):

$$x+4z+2w=0,\ \ y+z+w=0.$$

We can rewrite the first as expressing x in terms of z and w, and rewrite the second as expressing y in terms of z and w: $x=-4z-2w$ and $y=-z-w$. So these are vectors of the form

$$\begin{bmatrix}-4z-2w\\-z-w\\z\\w\end{bmatrix} = \begin{bmatrix}-4z\\-z\\z\\0\end{bmatrix} + \begin{bmatrix}-2w\\-w\\0\\w\end{bmatrix} = z\begin{bmatrix}-4\\-1\\1\\0\end{bmatrix} + w\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix},$$

which is exactly the span of $\begin{bmatrix}-4\\-1\\1\\0\end{bmatrix}$ and $\begin{bmatrix}-2\\-1\\0\\1\end{bmatrix}$.

Problem 6: More recognizing and describing linear subspaces

Which of the following subsets S of $\mathbf{R}^3$ are linear subspaces? If a set S is a linear subspace, exhibit it as a span. If it is not a linear subspace, describe it geometrically and explain why it is not a linear subspace.

(a) The set S_1 of points (x, y, z) in $\mathbf{R}^3$ with both $z = x + 2y$ and $z = 5x$.

(b) The set S_2 of points (x, y, z) in $\mathbf{R}^3$ with either $z = x + 2y$ or $z = 5x$.

(c) The set S_3 of points (x, y, z) in $\mathbf{R}^3$ of the form $t\begin{bmatrix}1\\2\\3\end{bmatrix} + t'\begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}3\\3\\3\end{bmatrix}$ for some scalars t and t' (which are allowed to be anything, depending on the point (x, y, z)).

Solution: Points in S_1 are those satisfying $z = 5x$ and $5x = x + 2y$, where the second equation says $4x = 2y$ or equivalently $y = 2x$. Hence, these are points of the form $\begin{bmatrix}x\\2x\\5x\end{bmatrix} = x\begin{bmatrix}1\\2\\5\end{bmatrix}$, so this is the span of the vector $\begin{bmatrix}1\\2\\5\end{bmatrix}$. Geometrically, this is the overlap of two planes through the origin (meeting in a line through the origin).

The subset S_2 not a subspace. Informally, it is the collection of points on either of two planes through the origin, and this cannot be a span: any span is "closed" under forming linear combinations, but if we add a point in one plane to a point in the other plane then the vector sum (in terms of the parallelogram law) is generally outside both planes (apart from the special case when we begin with vectors on the line from (a) along which the planes meet).

Finally, S_3 is obtained by shifting in space by a point $\begin{bmatrix}3\\3\\3\end{bmatrix}$ some plane through the origin. But this point by which we shift is in the initial plane through the origin:

$$\begin{bmatrix}3\\3\\3\end{bmatrix} = \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}2\\1\\0\end{bmatrix}.$$

In other words, we can rewrite the expressions giving S_3 as vectors of the form

$$t\begin{bmatrix}1\\2\\3\end{bmatrix} + t'\begin{bmatrix}2\\1\\0\end{bmatrix} + \begin{bmatrix}1\\2\\3\end{bmatrix} + \begin{bmatrix}2\\1\\0\end{bmatrix} = (t+1)\begin{bmatrix}1\\2\\3\end{bmatrix} + (t'+1)\begin{bmatrix}2\\1\\0\end{bmatrix}$$

with arbitrary t and t' (so equivalently, arbitrary $t + 1$ and $t' + 1$). Hence, this is a span of two vectors in $\mathbf{R}^3$ that aren't scalar multiples of each other, so it is a plane through the origin. In particular, we have exhibited it as a linear subspace.

Problem 7: Building another orthogonal vector (Extra)

If $\{\mathbf{v}, \mathbf{w}\}$ is a pair of nonzero orthogonal vectors in $\mathbf{R}^3$ then we can always enlarge it to an orthogonal basis $\{\mathbf{v}, \mathbf{w}, \mathbf{u}\}$ of $\mathbf{R}^3$ by taking $\mathbf{u}$ to be a nonzero normal vector to the plane $\mathrm{span}(\mathbf{v}, \mathbf{w})$. If $n > 3$ and $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ are mutually orthogonal nonzero vectors in $\mathbf{R}^n$ then can we always find a nonzero $\mathbf{v}_n$ orthogonal to those (so $\{\mathbf{v}_1, \ldots, \mathbf{v}_n\}$ is an orthogonal basis of $\mathbf{R}^n$)?

Solution: It feels like this should always be possible, though it may be very challenging to figure out how to actually do it at the present point of the course. First, we observe that there *are* n-vectors outside $V = \mathrm{span}(\mathbf{v}_1, \ldots, \mathbf{v}_{n-1})$; i.e., V does not exhaust $\mathbf{R}^n$. Indeed, V has dimension $n - 1$ (as any orthogonal collection of nonzero vectors is a basis for its span) whereas $\dim \mathbf{R}^n = n$, so definitely $V \neq \mathbf{R}^n$. Pick a vector $\mathbf{x} \in \mathbf{R}^n$ outside V. Any vector

$$\mathbf{x}' = \mathbf{x} - c_1\mathbf{v}_1 - c_2\mathbf{v}_2 - \cdots - c_{n-1}\mathbf{v}_{n-1}$$

obtained from $\mathbf{x}$ by adjusting it by a linear combinations of the $\mathbf{v}_j$'s is also outside V: by inspection $\mathbf{x}' = \mathbf{x} - \mathbf{v}$ for $\mathbf{v} = c_1\mathbf{v}_1 + \cdots + c_{n-1}\mathbf{v}_{n-1} \in V$, so if we were to have $\mathbf{x}' \in V$ then also $\mathbf{x} = \mathbf{x}' + \mathbf{v}$ would belong to V, contrary to how

$\mathbf{x}$ was chosen. Hence, for any scalars $c_1, \ldots, c_{n-1}$ the resulting vector $\mathbf{x}'$ is not in V (in particular, $\mathbf{x}'$ is nonzero!). We will find such $c_1, \ldots, c_{n-1}$ making $\mathbf{x}'$ orthogonal to $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$; such an $\mathbf{x}'$ can be taken as $\mathbf{v}_n$.

We want to make each $\mathbf{x}' \cdot \mathbf{v}_j$ vanish (for $1 \leq j \leq n-1$). To ease the notation, let's first consider $\mathbf{x}' \cdot \mathbf{v}_1$. Since $\mathbf{v}_1 \cdot \mathbf{v}_i = 0$ for all $i > 1$, we have

$$\mathbf{x}' \cdot \mathbf{v}_1 = \mathbf{x} \cdot \mathbf{v}_1 - c_1(\mathbf{v}_1 \cdot \mathbf{v}_1) - c_2(\mathbf{v}_2 \cdot \mathbf{v}_1) - \cdots - c_{n-1}(\mathbf{v}_{n-1} \cdot \mathbf{v}_1) = \mathbf{x} \cdot \mathbf{v}_1 - c_1(\mathbf{v}_1 \cdot \mathbf{v}_1).$$

To make this vanish, there is exactly one choice for c_1 that does the job:

$$c_1 = \frac{\mathbf{x} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}$$

(the division makes sense since $\mathbf{v}_1 \cdot \mathbf{v}_1 = \|\mathbf{v}_1\|^2 > 0$ since $\mathbf{v}_1$ is nonzero). By exactly the same type of calculation, using that each $\mathbf{v}_j$ is orthogonal to the rest and is nonzero, for each $1 \leq j \leq n-1$ there is exactly one value of c_j which makes $\mathbf{x}'$ orthogonal to $\mathbf{v}_j$:

$$c_j = \frac{\mathbf{x} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j}.$$

Thus, we are led to define

$$\mathbf{v}_n = \mathbf{x} - \sum_{j=1}^{n-1} \left(\frac{\mathbf{x} \cdot \mathbf{v}_j}{\mathbf{v}_j \cdot \mathbf{v}_j} \right) \mathbf{v}_j;$$

the preceding work shows that $\mathbf{v}_n$ is orthogonal to $\mathbf{v}_1, \ldots, \mathbf{v}_{n-1}$ and moreover $\mathbf{v}_n \notin V$, so $\mathbf{v}_n$ is nonzero. This does the job.

Problem 8: Orthogonality and projections

(a) In the span of $\begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix}$ find a non-zero vector $\mathbf{v}$ orthogonal to $\begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix}$.

(b) Here is a geometric analogue to the algebra in (a): for a plane P through the origin in $\mathbf{R}^3$ and a nonzero 3-vector $\mathbf{w}$ not orthogonal to P, why should there always be nonzero vectors in P orthogonal to $\mathbf{w}$? (Hint: visualize the plane W through $\mathbf{0}$ with normal vector $\mathbf{w}$, and think about how it meets the plane P).

(c) Find a nonzero vector $\mathbf{u} \in \mathbf{R}^3$ for which the projections of $\mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 4 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$ onto $\mathbf{u}$ are equal. (Recall that the projection of $\mathbf{x}$ onto a nonzero vector $\mathbf{u}$ is given by the formula $\left(\frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \right) \mathbf{u}$.) There are many answers. Informally, the condition says that $\mathbf{v}$ and $\mathbf{w}$ make the same "shadow" onto the line spanned by $\mathbf{u}$.

Solution:

(a) We seek a vector of the form

$$\mathbf{v} = c_1 \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix} + c_2 \begin{bmatrix} 1 \\ -2 \\ 3 \\ -4 \end{bmatrix} = \begin{bmatrix} c_1 + c_2 \\ 2c_1 - 2c_2 \\ 3c_1 + 3c_2 \\ 4c_1 - 4c_2 \end{bmatrix}$$

satisfying $\mathbf{v} \cdot \begin{bmatrix} 1 \\ 1 \\ 1 \\ -1 \end{bmatrix} = 0$, which says $c_1 + c_2 + 2c_1 - 2c_2 + 3c_1 + 3c_2 - 4c_1 + 4c_2 = 0$, or equivalently $2c_1 + 6c_2 = 0$.

We can now choose $c_1 = 3, c_2 = -1$ to obtain $\mathbf{v} = 3\begin{bmatrix}1\\2\\3\\4\end{bmatrix} - \begin{bmatrix}1\\-2\\3\\-4\end{bmatrix} = \begin{bmatrix}2\\8\\6\\16\end{bmatrix}$. (Any nonzero scalar multiple of this works just as well.)

(b) The visualization is that P and W are two planes through the origin that are *distinct* (since $\mathbf{w}$ is on the normal line to the second of these planes, but not the first). By drawing a picture we see that any two such planes meet exactly along a line through the origin. Anything nonzero in that line does the job: it lies in P and in W, so it is orthogonal to $\mathbf{w}$.

(c) From the projection formula, we seek a nonzero vector $\mathbf{u}$ for which $\mathbf{v} \cdot \mathbf{u} = \mathbf{w} \cdot \mathbf{u}$, or equivalently $(\mathbf{v} - \mathbf{w}) \cdot \mathbf{u} = 0$. Writing $\mathbf{u} = \begin{bmatrix}x\\y\\z\end{bmatrix}$, since $\mathbf{v} - \mathbf{w} = \begin{bmatrix}1\\-2\\3\end{bmatrix}$ this says

$$0 = (\mathbf{v} - \mathbf{w}) \cdot \mathbf{u} = x - 2y + 3z.$$

We can solve for any one among x, y, z in terms of the other two, so we can choose two of coordinates to be whatever we want that isn't both 0 and then uniquely fill in the third coordinate to get a nonzero $\mathbf{u}$ as desired.

For example, one nonzero triple (x, y, z) that works is $x = -1, y = 1, z = 1$, which is to say $\mathbf{u} = \begin{bmatrix}-1\\1\\1\end{bmatrix}$. (There are *many* nonzero solutions: the condition of being orthogonal to $\mathbf{v} - \mathbf{w}$ defines an entire plane through the origin, and the $\mathbf{u}$ we found is but one nonzero point in that plane.)

Problem 9: An orthogonal basis

Let V be the set of vectors $\mathbf{v} \in \mathbf{R}^3$ satisfying $\mathbf{v} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \mathbf{v} \cdot \begin{bmatrix}2\\3\\4\end{bmatrix}$ (this says that both of these explicit 3-vectors have the same projection onto $\mathbf{v}$, or in other words make the same "shadow" onto the line spanned by $\mathbf{v}$).

(a) Express V as the collection of 3-vectors orthogonal to a single nonzero 3-vector.

(b) By fiddling with orthogonality equations, build an orthogonal basis of V. There are many possible answers.

(c) Use your answer to (b) to give an orthonormal basis for V.

Solution:

(a) Some vector algebra simplifies the condition for belonging to V: the given condition

$$\mathbf{v} \cdot \begin{bmatrix}1\\2\\3\end{bmatrix} = \mathbf{v} \cdot \begin{bmatrix}2\\3\\4\end{bmatrix}$$

says exactly that $\mathbf{v} \cdot \left(\begin{bmatrix}2\\3\\4\end{bmatrix} - \begin{bmatrix}1\\2\\3\end{bmatrix}\right) = 0$, which is to say

$$\mathbf{v} \cdot \begin{bmatrix}1\\1\\1\end{bmatrix} = 0.$$

That is, V is the collection of 3-vectors orthogonal to $\begin{bmatrix}1\\1\\1\end{bmatrix}$.

(b) The vector $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ belongs to V precisely when

$$x + y + z = 0,$$

which is the equation defining a plane through the origin (i.e., its dimension is 2) having normal vector $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. So to find an orthogonal basis for V, we need to find two nonzero vectors $\mathbf{v}_1, \mathbf{v}_2$ perpendicular to $\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ *and* perpendicular to each other. For $\mathbf{v}_1$, we can pick any nonzero vector in the given plane; that is, any solution to the defining equation other than $(0, 0, 0)$. One such vector is $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ (and again, there are many other choices). Having chosen $\mathbf{v}_1$, the condition on $\mathbf{v}_2 = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is that it is a solution to the pair of equations

$$\begin{aligned} x + y + z &= 0 \\ y - z &= 0 \end{aligned}$$

other than the zero-vector solution $(0, 0, 0)$. We have $y = z$, and $x = -y - z$, so $x = -z - z = -2z$. One solution is given by choosing $z = 1$, so then $y = 1$ and $x = -2$. That is, we can use $\mathbf{v}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ (and any nonzero scalar multiple of this $\mathbf{v}_2$ works just as well, given the choice we made for $\mathbf{v}_1$). In other words, the pair of vectors

$$\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

is an orthogonal basis for the plane V.

(c) Now dividing by lengths gives an orthonormal basis:

$$\begin{bmatrix} 0 \\ 1/\sqrt{2} \\ -1/\sqrt{2} \end{bmatrix}, \begin{bmatrix} -2/\sqrt{6} \\ 1/\sqrt{6} \\ 1/\sqrt{6} \end{bmatrix}.$$

Problem 10: Subspaces defined by orthogonality, orthogonal bases, and shortest distances in $\mathbf{R}^3$

(a) For each linear subspace V_i in $\mathbf{R}^3$ given below, exhibit the set

$$V_i' = \{\mathbf{x} \in \mathbf{R}^3 \mid \mathbf{x} \text{ is orthogonal to every vector in } V_i\}$$

as the span of a finite collection of vectors (so, as a linear subspace), and give a basis for V_i'.

(i) $V_1 = \operatorname{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}\right)$

(ii) V_2 is the set of solutions in $\mathbf{R}^3$ to the pair of equations $\begin{cases} x_1 + 2x_2 + 3x_3 = 0, \\ 4x_1 + 5x_2 + 6x_3 = 0. \end{cases}$ (*Hint:* relate this to V_1 and think geometrically.)

(b) For each of the two V_i's given above, compute an orthogonal basis for it and *set up* how you'd find the distance from the point $\begin{bmatrix}7\\0\\0\end{bmatrix}$ to V_i (i.e. the minimal distance from $\begin{bmatrix}7\\0\\0\end{bmatrix}$ to a point in V_i) using such a basis. Finally, compute each distance.

(*Hint for computation:* first treat the case of V_2. For the case of the plane V_1, use projections to compute an orthogonal basis and to give an expression for a vector whose length is the distance you want. It gets cumbersome to carry out that distance calculation by hand, so instead compute the distance to V_1 by relating it to the distance to V_2. Try drawing a picture of an orthogonal line and plane to get an idea.)

Solution:

(a)(i) A vector $\mathbf{x}$ in V_1' must be orthogonal to both $\begin{bmatrix}1\\2\\3\end{bmatrix}$ and $\begin{bmatrix}4\\5\\6\end{bmatrix}$, and if it is perpendicular to those two vectors then it is perpendicular to everything in their span:

$$\mathbf{x}\cdot\left(a\begin{bmatrix}1\\2\\3\end{bmatrix}+b\begin{bmatrix}4\\5\\6\end{bmatrix}\right)=a\left(\mathbf{x}\cdot\begin{bmatrix}1\\2\\3\end{bmatrix}\right)+b\left(\mathbf{x}\cdot\begin{bmatrix}4\\5\\6\end{bmatrix}\right)=0a+0b=0.$$

So V_1' consists of exactly those $\mathbf{x}$ that are orthogonal to these two vectors, which is expressed by satisfying the pair of equations

$$\begin{cases}x_1+2x_2+3x_3=0,\\4x_1+5x_2+6x_3=0.\end{cases}$$

Since the two given 3-vectors spanning V_1 are nonzero and not scalar multiples of each other, V_1 is a plane through the origin in $\mathbf{R}^3$. Since the vectors perpendicular to V_1 make up its normal line, the solutions to this system constitute a line through the origin: the multiples of a single nonzero solution.

To make a nonzero solution, we try setting one of the x_i's to be 1 and solve for the others in the resulting pair of equations in the two remaining variables. Setting $x_3=1$, for example, we get the equations $x_1+2x_2=-3$ and $4x_1+5x_2=-6$, for which the solution is found to be $(1,-2)$. Hence, V_1' is the span of $\begin{bmatrix}1\\-2\\1\end{bmatrix}$, so that single vector is a basis for V_1'.

(a)(ii) By inspection, V_2 is defined by exactly the orthogonality equations found in (i) that defined V_1', so $V_2=V_1'$ is the normal line to the plane V_1 through the origin. By visualization in $\mathbf{R}^3$, the vectors perpendicular to the normal line to a plane through the origin is that very same plane. Hence, $V_2'=V_1$. (Later in the course we will discuss the general fact that for any linear subspace V of any $\mathbf{R}^n$, those vectors perpendicular to V' are precisely those in V. In the present case we can "see" this for $V=V_1$ inside $\mathbf{R}^3$.) Thus, a basis for V_2' is the same thing as a basis for V_1, such as the pair of nonzero vectors $\begin{bmatrix}1\\2\\3\end{bmatrix}$, $\begin{bmatrix}4\\5\\6\end{bmatrix}$ whose span defined V_1 (and which aren't scalar multiples of each other).

(b) **Distance to V_2 (setup and computation):** we have seen that V_2 is a line spanned by $\mathbf{u}=\begin{bmatrix}1\\-2\\1\end{bmatrix}$ (which by itself constitutes an orthogonal basis for V_2). The distance from a point $\mathbf{x}$ to a 1-dimensional subspace L of $\mathbf{R}^n$ is the length of the difference vector vector $\mathbf{x}-\mathbf{Proj}_L(\mathbf{x})$, since $\mathbf{Proj}_L(\mathbf{x})$ is the point in L closest to $\mathbf{x}$. Thus, the distance from

$\mathbf{x} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$ to V_2 is the length of the difference vector

$$\begin{aligned} \mathbf{x} - \mathbf{Proj}_{V_2}(\mathbf{x}) = \mathbf{x} - \mathbf{Proj}_{\mathbf{u}}(\mathbf{x}) &= \mathbf{x} - \frac{\mathbf{x} \cdot \mathbf{u}}{\mathbf{u} \cdot \mathbf{u}} \mathbf{u} \\ &= \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix} - \frac{7}{6} \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix} \\ &= \begin{bmatrix} 7 - 7/6 \\ 7/3 \\ -7/6 \end{bmatrix} \\ &= \frac{7}{6} \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}, \end{aligned}$$

whose length is $(7/6)\sqrt{25+4+1} = 7\sqrt{30}/6$. (This answer can be written in various other ways, such as $7\sqrt{5/6}$.)

Distance to V_1 (setup using orthogonal basis): The distance from a point $\mathbf{x}$ to the plane V_1 is the length of the difference $\mathbf{x} - \mathbf{Proj}_{V_1}(\mathbf{x})$ between $\mathbf{x}$ and the unique point $\mathbf{Proj}_{V_1}(\mathbf{x})$ on V_1 closest to $\mathbf{x}$. This latter projection can be computed using an orthogonal basis of V_1, which can be computed from a basis of V_1 (such as the two vectors that span V_1 in its definition). More specifically, if we define $\mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 4 \\ 5 \\ 6 \end{bmatrix}$ then an orthogonal basis of V_1 is given by $\mathbf{v}$ and

$$\mathbf{w}' = \mathbf{w} - \mathbf{Proj}_{\mathbf{v}}(\mathbf{w}) = \mathbf{w} - \frac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} = \mathbf{w} - \frac{32}{14} \mathbf{v} = \begin{bmatrix} 24/14 \\ 6/14 \\ -12/14 \end{bmatrix} = \frac{6}{14} \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix} = \frac{3}{7} \mathbf{w}''$$

for $\mathbf{w}'' = \begin{bmatrix} 4 \\ 1 \\ -2 \end{bmatrix}$. Hence, for any $\mathbf{x}$ (such as $\begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$) we can then compute the difference

$$\mathbf{x} - \mathbf{Proj}_{V_1}(\mathbf{x}) = \mathbf{x} - \mathbf{Proj}_{\mathbf{v}}(\mathbf{x}) - \mathbf{Proj}_{\mathbf{w}''}(\mathbf{x})$$

and its length is the desired minimal distance. (The final vector and its length are written below, at the bottom of the following simpler alternate solution.)

Distance to V_1 (simpler computation using $V_2 = V_1'$): Since the computation we've set up above can get rather cumbersome to do by hand (though a computer can do it rapidly), here is a nice alternative that takes advantage of knowing that $V_1' = V_2$.

Observe that by the Orthogonal Projection Theorem applied to $\mathbf{x}$ and V_1, $\mathbf{x}$ can be expressed uniquely as a sum

$$\mathbf{x} = \mathbf{y} + \mathbf{y}'$$

where $\mathbf{y}$ lies in V_1 and $\mathbf{y}'$ lies in $V_1' = V_2$; and furthermore $\mathbf{y} = \mathbf{Proj}_{V_1}(\mathbf{x})$. But also observe that by the same theorem as applied to $\mathbf{x}$ and V_2, $\mathbf{x}$ can be expressed uniquely as a sum of vectors

$$\mathbf{x} = \mathbf{z} + \mathbf{z}'$$

where $\mathbf{z}$ lies in V_2 and $\mathbf{z}'$ lies in $V_2' = V_1$; and furthermore $\mathbf{z} = \mathbf{Proj}_{V_2}(\mathbf{x})$.

Now, the uniqueness condition of the Orthogonal Projection Theorem means that these *two ways* of expressing $\mathbf{x}$ as a sum of vectors from each of V_1 and V_2 is actually *the same way*! We finally conclude that

$$\mathbf{x} = \mathbf{Proj}_{V_1}(\mathbf{x}) + \mathbf{Proj}_{V_2}(\mathbf{x}), \text{ crucially using the facts that } V_1' = V_2 \text{ and } V_2' = V_1.$$

(Since V_1 is a plane in $\mathbf{R}^3$ and V_2 is its normal line, you can draw a picture to confirm this!) Thus, the vector whose length is the distance we need is simply

$$\mathbf{x} - \mathbf{Proj}_{V_1}(\mathbf{x}) = \mathbf{Proj}_{V_2}(\mathbf{x}) = \mathbf{Proj}_{\mathbf{u}}(\mathbf{x}).$$

For $\mathbf{x} = \begin{bmatrix} 7 \\ 0 \\ 0 \end{bmatrix}$, this projection was computed in our work on the distance to V_2: it is $(7/6)\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. This has length $(7/6)\sqrt{6} = 7/\sqrt{6}$, so this is the distance from $\mathbf{x}$ to V_1.

Problem 11: A linear mathematical model via closest vector and dot products (Extra)

A researcher measures the basal metabolic rate[1], height, and weight for 100 people and expresses the result as vectors:

$$\mathbf{B}, \mathbf{W}, \mathbf{H} \in \mathbf{R}^{100}$$

Here, the ith entry of $\mathbf{H}$ is the height of the ith person in inches, and similarly for $\mathbf{B}$ (basal metabolic rate in kilocalories per day) and $\mathbf{W}$ (weight in pounds).

The researcher would like to work out a linear formula to estimate the basal metabolic rate in terms of height and weight. In mathematical terms, she would like to find $a, b \in \mathbf{R}$ for which

$$a\mathbf{H} + b\mathbf{W} \text{ is as close to } \mathbf{B} \text{ as possible.}$$

(a) Suppose that the vectors were in $\mathbf{R}^3$ rather than $\mathbf{R}^{100}$. Draw a picture to explain why the a, b we are looking for must satisfy

$$\mathbf{B} - (a\mathbf{H} + b\mathbf{W}) \text{ is perpendicular to } \mathbf{H}, \mathbf{W}.$$

(We know this is true in $\mathbf{R}^{100}$ by the Orthogonal Projection Theorem; the point is to understand it intuitively with a picture in $\mathbf{R}^3$.)

(b) Use the orthogonality as discussed in (a) (which must hold for 100-vectors) and dot products to write down a system of linear equations for a, b (whose coefficients involve dot products among 100-vectors).

(c) The researcher computes that $\mathbf{H} \cdot \mathbf{H} = 1/2, \mathbf{W} \cdot \mathbf{W} = 3$ and $\mathbf{H} \cdot \mathbf{W} = 3/2$; also $\mathbf{B} \cdot \mathbf{W} = 3$ and $\mathbf{B} \cdot \mathbf{H} = 2$. Using the vanishing of dot products against $\mathbf{H}$ and $\mathbf{W}$ arising from (a), solve for a and b. (In the real world, such dot products would usually be "ugly" numbers; we made them clean, as we do on exams, so the answer comes out cleanly without using a calculator.)

Observe that the solution did not require knowledge of the 100-element vectors– just knowledge about their dot products! (Of course, to *compute* those dot products one has to know the 100-vectors, but the point is that the *only* way the knowledge of the 100-vectors is relevant is solely to compute those dot products.)

Solution:

(a) In order for a point in the plane (or line) through the origin spanned by $\mathbf{H}$ and $\mathbf{W}$ to be as close to the point $\mathbf{B}$ as possible, such a point should be the foot of the perpendicular drawn from $\mathbf{B}$ to this plane (or line) through the origin — since geometry tells us that the shortest distance from a point to a plane (or line) through the origin is the perpendicular distance. Thus, we want a and b so that the displacement vector $\mathbf{B} - (a\mathbf{H} + b\mathbf{W})$ is perpendicular to the plane (or line) through the origin spanned by $\mathbf{H}$ and $\mathbf{W}$; i.e., is perpendicular to both $\mathbf{H}$ and $\mathbf{W}$.

(b) $\mathbf{B} - (a\mathbf{H} + b\mathbf{W})$ is perpendicular to $\mathbf{H}, \mathbf{W}$ precisely when

$$(\mathbf{B} - (a\mathbf{H} + b\mathbf{W})) \cdot \mathbf{H} = 0, \quad (\mathbf{B} - (a\mathbf{H} + b\mathbf{W})) \cdot \mathbf{W} = 0,$$

which we can rewrite (by properties of dot products) as

$$(\mathbf{H} \cdot \mathbf{H})a + (\mathbf{W} \cdot \mathbf{H})b = \mathbf{B} \cdot \mathbf{H}, \quad (\mathbf{H} \cdot \mathbf{W})a + (\mathbf{W} \cdot \mathbf{W})b = \mathbf{B} \cdot \mathbf{W}.$$

[1] rate at which the body uses energy, measured in kilocalories per day, if the person is at rest

(c) The system of equations is

$$(1/2)a + (3/2)b = 2, \quad (3/2)a + 3b = 3.$$

This is solved by the method from high school algebra to give $a = -2$ and $b = 2$.