**Goal:** various forms of a plane in $\mathbb{R}^3$, parametric form of a line in $\mathbb{R}^3$

The collection of points $(x, y, z)$ in $\mathbb{R}^3$ satisfying an equation of the form

$$ax + by + cz = d,$$

2D analog: $ax+by=C$ is a line in $\mathbb{R}^2$.

where at least one of $a$, $b$, or $c$ is non-zero, is a **plane** in $\mathbb{R}^3$. This is called the *equational form* of a plane. There are three other forms of expressing a plane:

1. (three points on a plane) Three points, that do not all lie on a single line (not collinear), determine a plane. You can think of the triangle formed by the points, and the plane is an "extension" of the triangle.

**Example 1**. Graph the plane $x + 2y + 3z = 6$ in the $xyz$-coordinate system.

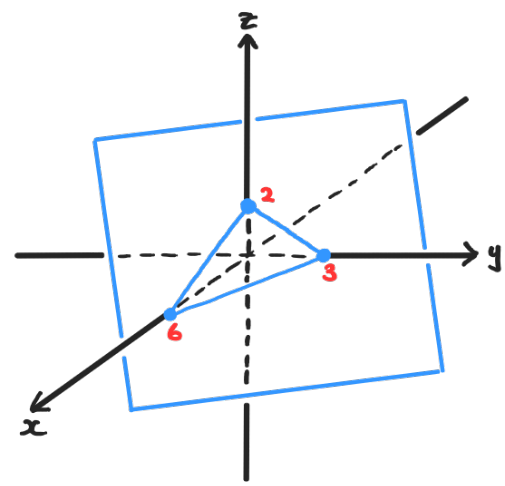
We need three points on the plane.

Commonly used points are the intercepts.

x-int.: $(6, 0, 0)$

y-int.: $(0, 3, 0)$

z-int.: $(0, 0, 2)$




2. (point and normal vector form) You can also determine a plane by a point $P$ the plane passes through and a normal vector $\mathbf{n}$ (called the *normal vector*) that the plane is perpendicular to.

   It actually turns out that if the normal vector is $\mathbf{n} = \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$, then the equational form of the plane is given by $n_1 x + n_2 y + n_3 z = C$, where the constant $C$ depends on the point $P$.

**Example 2**. Find the point and normal vector form to the plane in Example 1.

Reading off the coefficients of $x + 2y + 3z = 6$, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ is normal to $P$.

One point on $P$ is $(1, 1, 1)$.

**Example 3**. Find the equation form of the plane that passes through $(1, -2, 3)$ and is normal to $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$.

Since the plane has $\begin{bmatrix} 3 \\ 1 \\ -2 \end{bmatrix}$ as a normal vector, its equational form is

$$3x + y - 2z = C.$$

In order to find $C$, we plug in $(1,-2,3)$ to get $C = 3(1) + (-2) - 2(3) = -5$. Thus, an equation for the plane is $\boxed{3x + y - 2z = -5.}$

3. (parametric form) Given two *displacement vectors* $\mathbf{e}$ and $\mathbf{e}'$, that are not scalar multiples of each other, and a point $P$, we can also define a plane by

$$P + t\mathbf{e} + t'\mathbf{e}',$$

where $t$ and $t'$ are scalars. You can think of $\mathbf{e}$ and $\mathbf{e}'$ being the directions of the "$t$-axis" and "$t'$-axis", respectively, and the plane is formed by these axes.

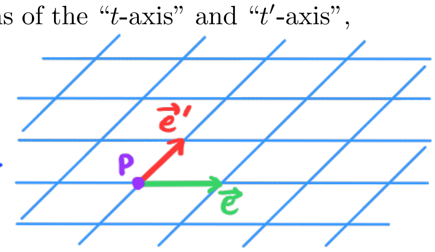
**Example 4**. Find a parametric form of the plane in Examples 1 and 2.

From Example 1, $(6,0,0)$, $(0,3,0)$, and $(0,0,2)$ are on $\mathcal{P}$.

Taking $(0,0,2)$ as $P$, we get $\vec{e} = \begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix}$ and $\vec{e}' = \begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix}$. So,

a parametric form for $\mathcal{P}$ is

$$\boxed{\begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} + t\begin{bmatrix} 6 \\ 0 \\ -2 \end{bmatrix} + t'\begin{bmatrix} 0 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 6t \\ 3t' \\ 2 - 2t - 2t' \end{bmatrix}.}$$




We can think of this plane with "center" P and grid formed by $\vec{e}$ and $\vec{e}'$. I think of this as a "skewed graph paper."

Note Depending on your choice of $P, \vec{e}, \vec{e}'$, the parametrization will be different.

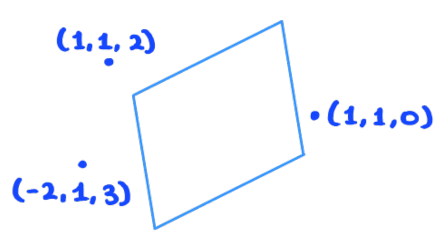
**Example 5**. Do the points $(1, 1, 2)$ and $(-2, 1, 3)$ line on the same side of $x + 2y + 3z = 4$? How about $(1, 1, 2)$ and $(1, 1, 0)$?

| | $x+2y+3z$ |
|---|---|
| $(1,1,2)$ | 9 |
| $(-2,1,3)$ | 9 |

Both are greater than 4, so they lie on the same side of the plane.

| | $x+2y+3z$ |
|---|---|
| $(1,1,2)$ | 9 |
| $(1,1,0)$ | 3 |

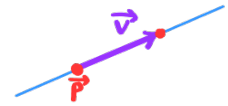
One is greater than and one is less than 4. So, they lie on opposite sides.



We can also use parameters to define a line – we actually only need one. We can think of a plane needing two parameters $t$ and $t'$ to keep track of where you are on the plane with respect to the "$t$-axis" and "$t'$-axis". For the line, we only need one parameter $t$ to keep track of how far down the line you are from a starting point.

Consider a line $l$ that passes through $\mathbf{p}$ in the direction of $\mathbf{v}$. A way to look at this is, if a $\mathbf{x}$ is on $l$, then $\mathbf{x} - \mathbf{p}$ should be a scalar multiple of $\mathbf{v}$. In other words, $\mathbf{x} - \mathbf{p} = t\mathbf{v}$, or equivalently,

$$\mathbf{x} = \mathbf{p} + t\mathbf{v}.$$




Note that if $t$ is positive, you are along the line in the $\mathbf{v}$-direction from $\mathbf{p}$, and if $t$ is negative, you are along the line in the $-\mathbf{v}$-direction from $\mathbf{p}$.

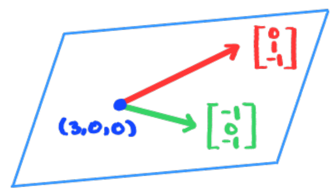
**Example 6**. Find the equation of a line that passes through the points $(1, 3, -2)$ and $(2, 0, 1)$.

Picking $(1, 3, -2)$ as $\vec{p}$, a direction vector is $\vec{v} = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix}$.

$$l: \boxed{\begin{bmatrix} 1 \\ 3 \\ -2 \end{bmatrix} + t\begin{bmatrix} 1 \\ -3 \\ 3 \end{bmatrix} = \begin{bmatrix} 1+t \\ 3-3t \\ -2+3t \end{bmatrix}}.$$

**Example 7**. Find the equation of a plane $\mathcal{P}$ given by the parametric form

$$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t'\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}.$$




Since $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix}$ both lie on $P$, the normal vector $\vec{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ has to be orthogonal to both.

$0 = 0 \cdot a + 1 \cdot b - 1 \cdot c = b - c \Rightarrow b = c$

$0 = -1 \cdot a + 0 \cdot b - 1 \cdot c = -a - c \Rightarrow a = -c$

Picking $c = 1$, we get $a = -1, b = 1$, and so, $\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$ is normal to $P$. Hence,

$$-x + y + z = C$$

is an equational form of $P$. Since $(3,0,0)$ lies on $P$, $C = -3$.

$$\boxed{-x + y + z = -3}$$

$$\begin{bmatrix} 3 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + t'\begin{bmatrix} -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 - t' \\ t \\ -t - t' \end{bmatrix}$$

We plug this vector into $ax + by + cz = d$:

$$d = a(3 - t') + b(t) + c(-t - t')$$
$$= (3a) + (b - c)t + (-a - c)t'$$

Since this equation must hold for all $t, t'$,

$$b - c = 0, \quad -a - c = 0, \quad 3a = d.$$

Hence, $b = c$, $a = -c$, and $d = -3c$.

$$ax + by + cz = d$$
$$\Rightarrow -cx + cy + cz = -3c$$
$$\Rightarrow \boxed{-x + y + z = -3}$$

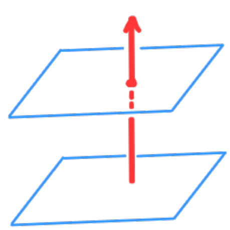
**Example 8**. Find a parametric form for the plane in $\mathbb{R}^3$ given by the equation $x - 2y + 5z = 40$.

The intercepts of $\mathcal{P}$ are $(40,0,0)$, $(0,-20,0)$, and $(0,0,8)$. Picking $(0,-20,0)$ as P, we get displacement vectors $\vec{e} = \begin{bmatrix} 40 \\ 20 \\ 0 \end{bmatrix}$ and $\vec{e}\,' = \begin{bmatrix} 0 \\ 20 \\ 8 \end{bmatrix}$ on $\mathcal{P}$. Hence,

$$\boxed{\begin{bmatrix} 0 \\ -20 \\ 0 \end{bmatrix} + t \begin{bmatrix} 40 \\ 20 \\ 0 \end{bmatrix} + t' \begin{bmatrix} 0 \\ 20 \\ 8 \end{bmatrix} = \begin{bmatrix} 40t \\ -20 + 20t + 20t' \\ 8t' \end{bmatrix}}$$

is a parametric form of $\mathcal{P}$.

**Example 9**. Find an equation for the plane parallel to $2x - 3y + 5z = -3$ and passing through the point $(1, 0, -4)$.



If two planes are parallel, they have the same normal vector.

The plane has equational form

$$2x - 3y + 5z = C.$$

Since $(1,0,-4)$ lies on the plane,

$$C = 2(1) - 3(0) + 5(-4) = -18.$$

Thus, an equation for the plane is

$$\boxed{2x - 3y + 5z = -18}.$$

**Example 10**. Consider the plane $\mathcal{P}$ defined by $x - 2y + z = -4$; this is also described by the parametric form

$$\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + t' \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Using whichever of the two descriptions you find more convenient in each case, answer the following.

(a) Is the point $(5, 3, -2)$ on the plane $\mathcal{P}$?

Plugging $(5, 3, -2)$ into the equation for $P$, we see that it is inconsistent:

$$(5) - 2(3) + (-2) = -3 \neq -4.$$

Hence, $(5, 3, -2)$ is NOT on $P$.

(b) Do the points $(5, 3, -2)$ and $(-3, 0, 1)$ lie on the same side of the plane $\mathcal{P}$?

| | $x - 2y + z$ |
|---|---|
| $(5, 3, -2)$ | $-3$ |
| $(-3, 0, 1)$ | $-2$ |

Since both values are greater than $-4$, the points lie on the same side of $P$.

(c) Give an example of a parametric form of some line contained in the plane $\mathcal{P}$ (this has many possible answers), and as a safety check, verify that all points on this parametric line satisfy the equation for $\mathcal{P}$.

Since $(1, 0, -5)$ is on $P$ and $\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix}$ lies on $P$,

$$\boxed{\begin{bmatrix} 1 \\ 0 \\ -5 \end{bmatrix} + t \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1+2t \\ t \\ -5 \end{bmatrix}}$$

We can check by plugging this into the equation for $P$:

$$x - 2y + z = (1 + 2t) - 2(t) + (-5)$$

$$= 1 + 2t - 2t - 5 = -4. \checkmark$$