Topic(s): Gram-Schmidt process

For $k > 1$, a collection of n-vectors $\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is said to be **linearly dependent** if some $\mathbf{v}_i$ belongs to the span of the others. Otherwise, it is said to be **linearly independent**; in other words, no $\mathbf{v}_i$ belongs to the span of the others. A collection $\{\mathbf{v}\}$ consisting of a single vector is linearly dependent if and only if $\mathbf{v} = \mathbf{0}$ and linearly independent if $\mathbf{v} \neq \mathbf{0}$.

Theorem 19.1.5. A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is linearly independent precisely when the only collection of scalars $a_1, \ldots, a_k$ for which

$$a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + \cdots + a_k\mathbf{v}_k = \mathbf{0}$$

is $a_1 = a_2 = \cdots = a_k = 0$. Equivalently, the collection of $\mathbf{v}_i$'s is linearly dependent precisely when there is some collection of scalars $a_1, \ldots, a_k$ *not all* zero for which $\sum_{i=1}^{k} a_i\mathbf{v}_i = \mathbf{0}$.

Example 1. Consider the vectors $\mathbf{v}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \mathbf{v}_3 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v}_4 = \begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \end{bmatrix}$.

(a) Show that $\mathbf{v}_4 \in \operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

Solving

$$\begin{bmatrix} 1 \\ -2 \\ 2 \\ 1 \end{bmatrix} = a\begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + b\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} + c\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} b+c \\ 2c \\ a+b \\ -c \end{bmatrix}$$

gives us $a = 0$, $b = 2$, $c = -1$. Hence, $\vec{v}_4 = 2\vec{v}_2 - \vec{v}_3 \in \operatorname{span}(\vec{v}_1, \vec{v}_2, \vec{v}_3)$.

(b) Use your answer from part (a) to find scalars a_1, a_2, a_3, a_4 for which $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4 = \mathbf{0}$, showing that the collection $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$ is linearly dependent.

From (a),

$$0\vec{v}_1 + 2\vec{v}_2 + (-1)\vec{v}_3 + (-1)\vec{v}_4 = \vec{0}.$$

not all zero

Hence, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3, \vec{v}_4\}$ is linearly dependant.

Example 2. Show that $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ is linearly independent by solving the equation $a_1\mathbf{e}_1 + a_2\mathbf{e}_2 + a_3\mathbf{e}_3 = \mathbf{0}$ for a_1, a_2, a_3.

Solving

$$a_1\vec{e}_1 + a_2\vec{e}_2 + a_3\vec{e}_3 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

forces $a_1 = a_2 = a_3 = 0$. Hence, $\{\vec{e}_1, \vec{e}_2, \vec{e}_3\}$ is linearly independent.

The Gram-Schmidt process is an algorithm to find an orthogonal basis $\mathbf{w}_1, \mathbf{w}_2, \ldots$ of a non-zero subspace V of $\mathbb{R}^n$ when given a spanning set $\mathbf{v}_1, \ldots, \mathbf{v}_k$ of non-zero vectors in V.

An orthogonal basis has many uses; one is the ability to use Fourier formula – given an orthogonal basis $\{\mathbf{w}_1, \ldots, \mathbf{w}_m\}$ for a subspace V of $\mathbb{R}^n$, if

$$\mathbf{v} = c_1\mathbf{w}_1 + \cdots + c_m\mathbf{w}_m,$$

then $c_i = \dfrac{\mathbf{v} \cdot \mathbf{w}_i}{\mathbf{w}_i \cdot \mathbf{w}_i}$.

Gram-Schmidt process. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be non-zero n-vectors that span a linear subspace V or $\mathbb{R}^n$. Define

$$V_1 = \text{span}(\mathbf{v}_1), \qquad V_2 = \text{span}(\mathbf{v}_1, \mathbf{v}_2), \qquad V_3 = \text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3), \qquad \cdots.$$

The following algorithm given an orthogonal basis for V.

- Let $\mathbf{w}_1 := \mathbf{v}_1$ and define $\mathcal{B}_1 := \{\mathbf{w}_1\}$. Note that $\mathcal{B}_1$ is an orthogonal basis for V_1.
- Let $\mathbf{w}_2 := \mathbf{v}_2 - \mathbf{Proj}_{V_1}(\mathbf{v}_2)$.
  1. if $\mathbf{w}_2 \neq \mathbf{0}$, then $\mathcal{B}_2 := \{\mathbf{w}_1, \mathbf{w}_2\}$ is an orthogonal basis for V_2.
  2. if $\mathbf{w}_2 = \mathbf{0}$, then $\mathbf{v}_2 \in V_1$, and so, $\mathcal{B}_2 := \mathcal{B}_1$ is an orthogonal basis for V_2.
- Let $\mathbf{w}_3 := \mathbf{v}_3 - \mathbf{Proj}_{V_2}(\mathbf{v}_3)$.
  1. if $\mathbf{w}_3 \neq \mathbf{0}$, then $\mathcal{B}_3 := \mathcal{B}_2 \cup \{\mathbf{w}_3\}$ is an orthogonal basis for V_3.
  2. if $\mathbf{w}_3 = \mathbf{0}$, then $\mathbf{v}_3 \in V_2$, and so, $\mathcal{B}_3 := \mathcal{B}_2$ is an orthogonal basis for V_3.
- Continue this process of considering $\mathbf{w}_j := \mathbf{v}_j - \mathbf{Proj}_{V_{j-1}}(\mathbf{v}_j)$ and whether to add $\mathbf{w}_j$ to $\mathcal{B}_{j-1}$ or not.

After k steps, $\mathcal{B}_k$ will be an orthogonal basis for V.

Example 3. Find a vector in $\mathbb{R}^2$ that is orthogonal to $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

If we compute an orthogonal basis $\left\{\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \vec{v}\right\}$, $\vec{v}$ is orthogonal to $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Consider $\left\{\begin{bmatrix} 5 \\ 1 \end{bmatrix}, \vec{e_1}, \vec{e_2}\right\}$, a spanning set for $\mathbb{R}^2$.

First vector gets preserved by G-S

$\mathcal{B}_1 = \left\{\underbrace{\begin{bmatrix} 5 \\ 1 \end{bmatrix}}_{\vec{w_1}}\right\}$. We compute $\text{Proj}_{\vec{w_1}}(\vec{e_1}) = \frac{5}{26}\vec{w_1}$, and so,

$$\vec{w_2} = \vec{e_1} - \frac{5}{26}\vec{w_1} = \begin{bmatrix} \frac{1}{26} \\ -\frac{5}{26} \end{bmatrix} = \frac{1}{26}\begin{bmatrix} 1 \\ -5 \end{bmatrix}.$$

Hence, $\begin{bmatrix} 1 \\ -5 \end{bmatrix}$ is perpendicular to $\begin{bmatrix} 5 \\ 1 \end{bmatrix}$.

Example 4. Apply the Gram-Schmidt process to the vectors

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 3 \\ 2 \\ -1 \end{bmatrix},$$

and use this to explain why the $\mathbf{v}_i$'s are linearly independent. Also, use your output to make an orthonormal basis of $\mathbb{R}^3$.

We start with $\mathcal{B}_1 = \{\vec{v}_1\}$. We compute

$$\mathrm{Proj}_{V_1}(\vec{v}_2) = \mathrm{Proj}_{\vec{v}_1}(\vec{v}_2) = \frac{2}{2}\vec{v}_1 = \vec{v}_1.$$

So, $\vec{w}_2' = \vec{v}_2 - \vec{v}_1 = \begin{bmatrix} 0 \\ -2 \\ 0 \end{bmatrix}$ and letting $\vec{w}_2 = \frac{1}{2}\vec{w}_2' = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$, $\mathcal{B}_2 = \{\vec{v}_1, \vec{w}_2\}$. Next,

$$\mathrm{Proj}_{V_2}(\vec{v}_3) = \mathrm{Proj}_{\vec{v}_1}(\vec{v}_3) + \mathrm{Proj}_{\vec{w}_2}(\vec{v}_3) = \frac{2}{2}\vec{v}_1 + \frac{-2}{1}\vec{w}_2 = \vec{v}_1 - 2\vec{w}_2.$$

Thus, $\vec{w}_3' = \vec{v}_3 - (\vec{v}_1 - 2\vec{w}_2) = \begin{bmatrix} 2 \\ 0 \\ -2 \end{bmatrix}$ and letting $\vec{w}_3 = \frac{1}{2}\vec{w}_3'$,

$$\mathcal{B}_3 = \left\{ \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$$

is an orthogonal basis of $\mathbb{R}^3$.

Since the output has 3 vectors (same as input), $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is linearly independent.

An orthonormal basis for $\mathbb{R}^3$ is $\left\{ \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} \right\}$.

Example 5. Consider the linear subspace V spanned by $\underset{\vec{v}_1}{\begin{bmatrix}1\\0\\2\\-3\end{bmatrix}}$ and $\underset{\vec{v}_2}{\begin{bmatrix}-1\\5\\0\\1\end{bmatrix}}$.

(a) Find an orthogonal basis for V.

We get $\text{Proj}_{\vec{v}_1}(\vec{v}_2) = \frac{-4}{14}\vec{v}_1$, and

$$\vec{v}_2 - \text{Proj}_{\vec{v}_1}(\vec{v}_2) = \vec{v}_2 + \frac{2}{7}\vec{v}_1 = \frac{1}{7}\begin{bmatrix}-5\\35\\4\\1\end{bmatrix}.$$

Hence, $\left\{\begin{bmatrix}1\\0\\2\\-3\end{bmatrix}, \begin{bmatrix}-5\\35\\4\\1\end{bmatrix}\right\}$ is an orthogonal basis for V.

(b) Let W be the set of vector in $\mathbb{R}^4$ orthogonal to every vector in V. In Lecture 4, Example 5, we saw that W is spanned by $\underset{\vec{w}_1}{\begin{bmatrix}10\\2\\-5\\0\end{bmatrix}}$ and $\underset{\vec{w}_2}{\begin{bmatrix}0\\-2\\15\\10\end{bmatrix}}$. Find an orthogonal basis for W.

We get $\text{Proj}_{\vec{w}_1}(\vec{w}_2) = \frac{-79}{129}\vec{w}_1$, and

$$\vec{w}_2 - \text{Proj}_{\vec{w}_1}(\vec{w}_2) = \vec{w}_2 + \frac{79}{129}\vec{w}_1 = \frac{10}{129}\begin{bmatrix}79\\-10\\154\\129\end{bmatrix}.$$

Thus, $\left\{\begin{bmatrix}10\\2\\-5\\0\end{bmatrix}, \begin{bmatrix}79\\-10\\154\\129\end{bmatrix}\right\}$ is an orthogonal basis for W.

Note that $\left\{\begin{bmatrix}1\\0\\2\\-3\end{bmatrix}, \begin{bmatrix}-5\\35\\4\\1\end{bmatrix}, \begin{bmatrix}10\\2\\-5\\0\end{bmatrix}, \begin{bmatrix}79\\-10\\154\\129\end{bmatrix}\right\}$ is an orthogonal set.

Example 6. Find a vector perpendicular to both $\underset{\vec{v}_1}{\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}}$ and $\underset{\vec{v}_2}{\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}}$.

Consider $\{\vec{v}_1, \vec{v}_2, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$, a spanning set of $\mathbb{R}^3$.

The output will be $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$, where $\vec{v}_2 \in \text{span}(\vec{w}_1, \vec{w}_2)$ and $\vec{w}_3 \perp \text{span}(\vec{w}_1, \vec{w}_2)$.

In particular, $\vec{w}_3 \perp \vec{v}_1, \vec{v}_2$.

Letting $\vec{w}_1 = \vec{v}_1$ and computing $\text{Proj}_{\vec{w}_1}(\vec{v}_2) = \frac{5}{5}\vec{w}_1 = \vec{w}_1$. So, $\vec{w}_2 = \vec{v}_2 - \vec{w}_1 = \begin{bmatrix} 2 \\ -1 \\ -2 \end{bmatrix}$.

Next, $\text{Proj}_{V_2}(\vec{e}_1) = \text{Proj}_{\vec{v}_1}(\vec{e}) + \text{Proj}_{\vec{w}_2}(\vec{e}_1) = \frac{0}{5}\vec{v}_1 + \frac{2}{9}\vec{w}_2 = \frac{2}{9}\vec{w}_2$. Hence,

$$\vec{e}_1 - \frac{2}{9}\vec{w}_2 = \frac{1}{9}\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix},$$

and $\boxed{\vec{w}_3} = 9\left(\vec{e}_1 - \frac{2}{9}\vec{w}_2\right) = \boxed{9\vec{e}_1 - 2\vec{w}_2} = \boxed{\begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}}$ is perpendicular to both $\vec{v}_1$ and $\vec{v}_2$.

Example 7. Find an orthogonal basis for the plane $x - 2y + z = 0$.

Letting $\vec{v} = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, we run $\{\vec{v}, \vec{e}_1, \vec{e}_2, \vec{e}_3\}$ through Graham-Schmidt.

We get $\text{Proj}_{\vec{v}}(\vec{e}_1) = \frac{1}{6}\vec{v}$ and $\vec{e}_1 - \frac{1}{6}\vec{v} = \frac{1}{6}\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$. Let $\vec{w}_2 = 6\vec{e}_1 - \vec{v} = \begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}$.

Next, $\text{Proj}_{V_2}(\vec{e}_2) = \text{Proj}_{\vec{v}}(\vec{e}_2) + \text{Proj}_{\vec{w}_2}(\vec{e}_2) = \frac{-2}{6}\vec{v} + \frac{2}{30}\vec{w}_2$, and

$$\vec{e}_2 - \left(-\frac{1}{3}\vec{v} + \frac{1}{15}\vec{w}_2\right) = \frac{1}{5}\begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}.$$

Letting $\vec{w}_3 = 5\vec{e}_2 + \frac{5}{3}\vec{v} - \frac{1}{3}\vec{w}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}$, we see that $\boxed{\left\{\begin{bmatrix} 5 \\ 2 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix}\right\}}$ is an orthogonal basis for $x - 2y + z = 0$.

Theorem 19.2.5. If V is a linear subspace of $\mathbb{R}^n$, then the collection $V^\perp$ of n-vectors orthogonal to everything in V ($V^\perp$ is called the **orthogonal complement** of V) is a linear subspace of $\mathbb{R}^n$ and

$$\dim(V^\perp) = n - \dim(V).$$

Example 8. Show that $\dim(V^\perp) = n - \dim(V)$.

Let $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ be an orthogonal basis for V ($\dim(V) = k$). Then, we can apply Gram-Schmidt to $\{\vec{v}_1, \ldots, \vec{v}_k, \vec{e}_1, \ldots, \vec{e}_n\}$, which is a spanning set for $\mathbb{R}^n$. The output will be

$$\{\underbrace{\vec{v}_1, \ldots, \vec{v}_k}_{\text{why are these unaffected?}}, \vec{w}_1, \vec{w}_2, \ldots, \vec{w}_{\underbrace{n-k}_{\text{because } \dim(\mathbb{R}^n)=n}}\},$$

an orthogonal basis for $\mathbb{R}^n$. Then, $\{\vec{w}_1, \ldots, \vec{w}_{n-k}\}$ $\overset{\text{WHY?}}{\text{is}}$ an orthogonal basis for $V^\perp$. Hence,

$$\dim(V^\perp) = n - k = n - \dim V. \qquad \blacksquare$$

Example 9. Let $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 3 \\ 1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$. Perform Gram-Schmidt on $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and use the results to write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

We get $\mathrm{Proj}_{\vec{v}_1}(\vec{v}_2) = \frac{-2}{2}\vec{v}_1$, and $\vec{w}_2 = \vec{v}_2 + \vec{v}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}$.

Next, $\mathrm{Proj}_{V_2}(\vec{v}_3) = \mathrm{Proj}_{\vec{v}_1}(\vec{v}_3) + \mathrm{Proj}_{w_2}(\vec{v}_3) = \frac{2}{2}\vec{v}_1 + \frac{18}{9}\vec{w}_2 = \vec{v}_1 + 2\vec{w}_2 = \vec{v}_3$.

Hence, $\vec{v}_3 = \vec{v}_1 + 2\vec{w}_2 = \vec{v}_1 + 2(\vec{v}_2 + \vec{v}_1) = 3\vec{v}_1 + 2\vec{v}_2$.