Solutions to Math 51 Midterm Exam (Practice #2)

1. (10 points) Consider the following three distinct points in $\mathbf{R}^3$:

$$\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ -2 \end{bmatrix}, \ \mathbf{v}_2 = \begin{bmatrix} 5 \\ 5 \\ -4 \end{bmatrix}, \ \mathbf{v}_3 = \begin{bmatrix} 6 \\ -2 \\ -1 \end{bmatrix}.$$

(a) (5 points) Verify that these are not all on a common line, so there is a unique plane P in $\mathbf{R}^3$ passing through all of them, and find an equation for P of the form $ax + by + cz = d$ for integers a, b, c, d. (As a safety check on your work, you may want to check that the $\mathbf{v}_i$'s all satisfy the equation you find for P; this is not required.)

To verify these are not on a common line, we just have to check that the nonzero displacements of two of them from the third are not scalar multiples of each other. So we compute the differences

$$\mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 2 \\ 6 \\ -2 \end{bmatrix}, \ \mathbf{v}_3 - \mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 1 \end{bmatrix}.$$

These are not scalar multiples of each other (by inspection, or more precisely by seeing that ratios between corresponding nonzero entries are not all the same), so indeed the three given points $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are not on a common line.

To get an equation for P, we first seek a *nonzero* normal vector $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, which amounts to a nonzero solution to the conditions $\mathbf{n} \cdot (\mathbf{v}_2 - \mathbf{v}_1) = 0$ and $\mathbf{n} \cdot (\mathbf{v}_3 - \mathbf{v}_1) = 0$. These say that we seek a solution other than $(0, 0, 0)$ to

$$2a + 6b - 2c = 0, \ \ 3a - b + c = 0.$$

(Most likely the first solution we find to this pair of equations won't consist entirely of integers, so then we'll scale through to get rid of denominators.)
Setting $c = 1$, this pair of equations becomes $2a+6b = 2$ and $3a-b = -1$. The first of these says $a + 3b = 1$, or equivalently $a = 1 - 3b$. Plugging that into the second gives $3(1 - 3b) - b = -1$, which is to say $3 - 10b = -1$, or equivalently $4 = 10b$, so $b = 2/5$. Hence, $a = 1 - 3b = -1/5$. Thus, we get the normal vector $\begin{bmatrix} -1/5 \\ 2/5 \\ 1 \end{bmatrix}$, so scaling it by 5 gives the normal vector $\begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$.
Hence, an equation for P is $-x + 2y + 5z = d$ for some d to be found, and being satisfied at $\mathbf{v}_1$ tells us $d = -3 + 2(-1) + 5(-2) = -3 - 2 - 10 = -15$. In other words, P has the equation $-x + 2y + 5z = -15$ (and as a safety check one readily verifies that $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$ satisfy this equation).

(b) (5 points) Both P and the plane P' given by the equation $4x + 6y + z = 11$ contain the point $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix}$, so the line L along which P and P' meet has the parametric form $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + t\mathbf{w}$ for a nonzero 3-vector $\mathbf{w}$. Explain with a picture why $\mathbf{w}$ must be perpendicular to normal vectors to both P and P', and use this property to find such a $\mathbf{w}$.
(Note that any nonzero scalar multiple of $\mathbf{w}$ works just as well, by dividing t correspondingly. As a safety check on your calculation, you may want to verify for the $\mathbf{w}$ that you find – or whatever nonzero scalar multiple of it you prefer – that $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \mathbf{w}$ lies in both planes P and P'.)

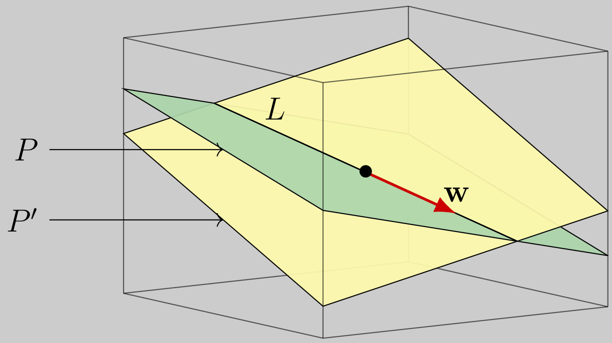

Figure 1: A picture of the given information

The vector $\mathbf{w}$ is a displacement vector for two points on L (such as the difference between the points obtained for $t = 0$ and $t = 1$), as shown in Figure 1. The normal vector to a plane is perpendicular to all displacements among points in the plane, and we have just exhibited $\mathbf{w}$ as a displacement for two distinct points in the line L that lies in both planes, so $\mathbf{w}$ is a displacement vector in each of the two planes (as seen quite visibly in Figure 1. Thus, $\mathbf{w}$ is perpendicular to normal vectors for each of P and P'.

By inspecting the equation for each plane, examples of such normal vectors are given by $\mathbf{n} = \begin{bmatrix} -1 \\ 2 \\ 5 \end{bmatrix}$ for P and $\mathbf{n}' = \begin{bmatrix} 4 \\ 6 \\ 1 \end{bmatrix}$ for P'. Hence, if we write $\mathbf{w} = \begin{bmatrix} r \\ s \\ u \end{bmatrix}$ then the vanishing of $\mathbf{n} \cdot \mathbf{w}$ and $\mathbf{n}' \cdot \mathbf{w}$ expresses the pair of simultaneous equations

$$-r + 2s + 5u = 0, \;\; 4r + 6s + u = 0.$$

To find a nonzero solution, we set $u = 1$ to arrive at the pair of equations

$$-r + 2s = -5, \;\; 4r + 6s = -1.$$

The first says $r = 2s+5$, and plugging this into the second gives $-1 = 4(2s+5)+6s = 14s+20$, or equivalently $14s = -21$, so $s = -3/2$ and hence $r = 2s + 5 = 2$. This has the solution $(r, s) = (2, -3/2)$, so $(r, s, u) = (2, -3/2, 1)$. Multiplying through by 2 for cleanliness (not necessary!), we may use

$$\mathbf{w} = \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix}.$$

As a safety check (not necessary!), the point $\begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \mathbf{w} = \begin{bmatrix} 2 \\ 1 \\ -3 \end{bmatrix} + \begin{bmatrix} 4 \\ -3 \\ 2 \end{bmatrix} = \begin{bmatrix} 6 \\ -2 \\ -1 \end{bmatrix}$ indeed lies in both planes: it satisfies the equation for each since $-6 + 2(-2) + 5(-1) = -6 - 4 - 5 = -15$ and $4(6) + 6(-2) - 1 = 24 - 12 - 1 = 24 - 13 = 11$.

2. (10 points) Let V be the linear subspace of $\mathbf{R}^5$ spanned by the following two nonzero 5-vectors that are not scalar multiples of each other (so $\dim(V) = 2$):

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \\ -4 \\ 3 \end{bmatrix}, \;\; \mathbf{v}_2 = \begin{bmatrix} 0 \\ 5 \\ -1 \\ 2 \\ 2 \end{bmatrix}.$$

(a) (2 points) Let U be the collection of vectors $\mathbf{x} \in \mathbf{R}^5$ that are orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$. Explain why every vector $\mathbf{x} \in U$ is orthogonal to every vector $\mathbf{v} \in V$.

A vector $\mathbf{v} \in V$ has the form $a\mathbf{v}_1 + b\mathbf{v}_2$, so for any $\mathbf{x} \in U$ we have

$$\mathbf{x} \cdot \mathbf{v} = \mathbf{x} \cdot (a\mathbf{v}_1 + b\mathbf{v}_2) = a(\mathbf{x} \cdot \mathbf{v}_1) + b(\mathbf{x} \cdot \mathbf{v}_2) = a(0) + b(0) = 0.$$

(b) (5 points) Find 5-vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ for which $U = \text{span}(\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3)$.

By the definition of V, membership of a 5-vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix}$ in U amounts to the pair of conditions $\mathbf{x} \cdot \mathbf{v}_1 = 0$ and $\mathbf{x} \cdot \mathbf{v}_2 = 0$. Written out explicitly, by the definition of $\mathbf{v}_1$ and $\mathbf{v}_2$ this is the pair of equations

$$x_1 - 2x_2 - 4x_4 + 3x_5 = 0, \quad 5x_2 - x_3 + 2x_4 + 2x_5 = 0.$$

We can rewrite these by "solving" the first for x_1 and solving the second for x_3. Conveniently the first equation doesn't involve x_3 and the second doesn't involve x_1, so we get expressions for each in terms of the same three variables (x_2, x_4, x_5) as follows:

$$x_1 = 2x_2 + 4x_4 - 3x_5, \quad x_3 = 5x_2 + 2x_4 + 2x_5.$$

Hence, this pair of equations is exactly the condition that $\mathbf{x}$ has the form

$$\begin{bmatrix} 2x_2 + 4x_4 - 3x_5 \\ x_2 \\ 5x_2 + 2x_4 + 2x_5 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2x_2 \\ x_2 \\ 5x_2 \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 4x_4 \\ 0 \\ 2x_4 \\ x_4 \\ 0 \end{bmatrix} + \begin{bmatrix} -3x_5 \\ 0 \\ 2x_5 \\ 0 \\ x_5 \end{bmatrix} = x_2 \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

This expresses the vectors in U exactly as the members of the span of the 5-vectors

$$\mathbf{u}_1 = \begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{u}_2 = \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix}, \quad \mathbf{u}_3 = \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix}.$$

(c) (3 points) Using your answer to (b), verify that $\dim(U) = 3$.

By the Dimension Criterion, since the vectors $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ found in (b) are visibly nonzero and not scalar multiples of each other it is equivalent to rule out the possibility $\mathbf{u}_1 = a\mathbf{u}_2 + b\mathbf{u}_3$ for scalars a, b (in particular, we do *not* need to rule out that $\mathbf{u}_2$ and $\mathbf{u}_3$ are each not a linear combination of the others; this comes for free once we check $\mathbf{u}_1$ is not a linear combination of the others since the $\mathbf{u}_i$'s are all nonzero and not scalar multiples of each other).
Assuming such a hypothetical equality $\mathbf{u}_1 = a\mathbf{u}_2 + b\mathbf{u}_3$ does hold for some a and b, this amounts to the vector equation

$$\begin{bmatrix} 2 \\ 1 \\ 5 \\ 0 \\ 0 \end{bmatrix} = a \begin{bmatrix} 4 \\ 0 \\ 2 \\ 1 \\ 0 \end{bmatrix} + b \begin{bmatrix} -3 \\ 0 \\ 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4a \\ 0 \\ 2a \\ a \\ 0 \end{bmatrix} + \begin{bmatrix} -3b \\ 0 \\ 2b \\ 0 \\ b \end{bmatrix} = \begin{bmatrix} 4a - 3b \\ 0 \\ 2a + 2b \\ a \\ b \end{bmatrix}.$$

This is impossible for many reasons, such as: the second entry on the left is 1 and on the right is 0, or the equality of the final two entries forces $a = 0$ and $b = 0$, yet then the equality of first and third entries fails. Hence, no such a and b exist, so indeed $\dim(U) = 3$.

3. (10 points) Consider the 6 data points $(x_1, y_1), \ldots, (x_6, y_6)$ given as follows:

$$(4, -2),\ (2, -1),\ (2, 0),\ (1, 5),\ (-1, 7),\ (-2, 9).$$

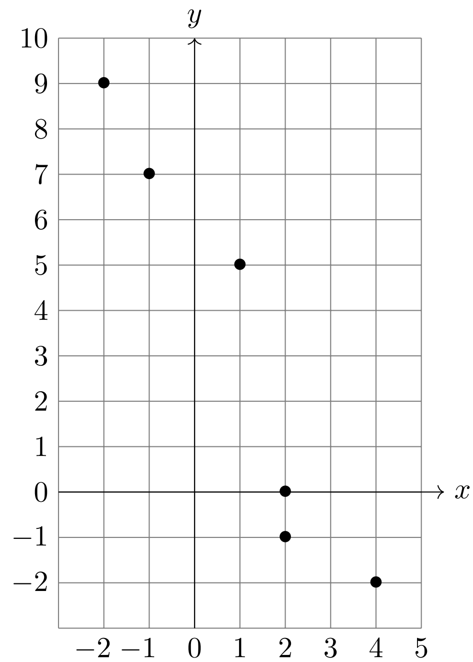

Figure 2: Plot of the 6 given data points

Suppose the line of best fit (in the least squares sense) is written as $y = mx + b$.

(a) (2 points) Write down explicit 6-vectors $\mathbf{X}$ and $\mathbf{Y}$ so that for the 6-vector $\mathbf{1}$ whose entries are all equal to 1, the projection of $\mathbf{Y}$ into the plane $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ in $\mathbf{R}^6$ is $m\mathbf{X} + b\mathbf{1}$. (You are just being asked to write down such $\mathbf{X}$ and $\mathbf{Y}$, nothing more.)

The vectors are

$$\mathbf{X} = \begin{bmatrix} 4 \\ 2 \\ 2 \\ 1 \\ -1 \\ -2 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 5 \\ 7 \\ 9 \end{bmatrix}.$$

(Note: entries can be rearranged, provided that both $\mathbf{X}$ and $\mathbf{Y}$ are rearranged in the same way.)

(b) (4 points) Compute an orthogonal basis of $V = \mathrm{span}(\mathbf{X}, \mathbf{1})$ having the form $\{\mathbf{1}, \mathbf{v}\}$ for a 6-vector $\mathbf{v}$, and find scalars r and s so that $\mathbf{Proj}_V(\mathbf{Y}) = r\mathbf{v} + s\mathbf{1}$. (The values of r and s are integers.)

The vector $\mathbf{v}$ can be taken to be $\mathbf{X} - \mathbf{Proj}_{\mathbf{1}}(\mathbf{X}) = \mathbf{X} - \overline{x}\mathbf{1}$ with $\overline{x}$ equal to the average of the entries x_i in $\mathbf{X}$. This average is

$$\frac{4 + 2 + 2 + 1 - 1 - 2}{6} = \frac{6}{6} = 1,$$

so

$$\mathbf{v} = \mathbf{X} - \mathbf{1} = \begin{bmatrix} 3 \\ 1 \\ 1 \\ 0 \\ -2 \\ -3 \end{bmatrix}.$$

(This $\mathbf{v}$ is what is called $\widehat{\mathbf{X}}$ in the course text.) The projection of $\mathbf{Y}$ into V is then given by

$$\frac{\mathbf{Y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \frac{\mathbf{Y}\cdot\mathbf{1}}{\mathbf{1}\cdot\mathbf{1}}\mathbf{1} = \left(\frac{-48}{24}\right)\mathbf{v} + \left(\frac{18}{6}\right)\mathbf{1} = (-2)\mathbf{v} + (3)\mathbf{1}.$$

Hence, $r = -2$ and $s = 3$.

(c) (4 points) By expressing $\mathbf{v}$ from (b) as a linear combination of $\mathbf{X}$ and $\mathbf{1}$, use your answer to (b) to find m and b so that the equation $y = mx + b$ gives the line of best fit. (The values of m and b are integers.)

We have $\mathbf{v} = \mathbf{X} - \mathbf{1}$, so

$$\begin{aligned} \mathbf{Proj}_V(\mathbf{Y}) = (-2)\mathbf{v} + (3)\mathbf{1} &= (-2)(\mathbf{X} - \mathbf{1}) + (3)\mathbf{1} \\ &= (-2)\mathbf{X} + (-(-2) + 3)\,\mathbf{1} \\ &= (-2)\mathbf{X} + (5)\mathbf{1}. \end{aligned}$$

Hence, the line of best fit is $y = -2x + 5$; so $m = -2$ and $b = 5$.

4. (10 points) Let R be the region $0 \le y \le 9 - x^2$, and define $f(x, y) = x^2y - y^2 - 11x^2$. Draw an approximate picture of R (it need not be to scale), and find the maximal and minimal values of f on R, and the points where they are attained.

The region R lies between the upside-down parabola $y = 9 - x^2$ and the x-axis $y = 0$, as shown in the following picture (which is drawn to scale, but nothing so accurate would be expected on an actual exam; labeling the corners along the x-axis is more important than accuracy of the scale).

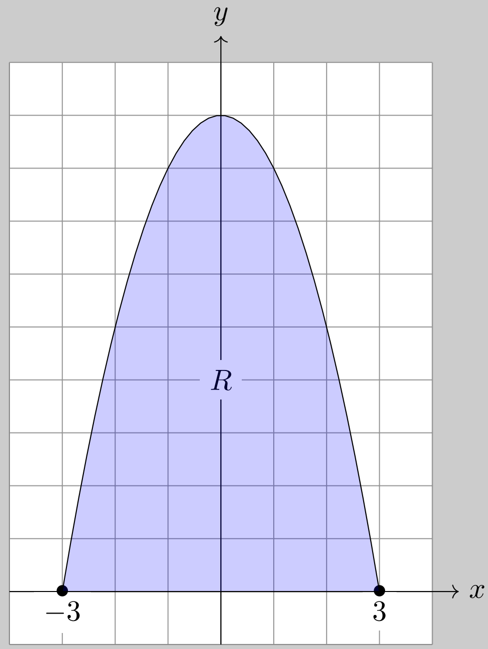

Figure 3: The region R defined by $0 \le y \le 9 - x^2$

We first find the critical points of f, and check if they lie in R. The gradient is

$$\nabla f = \begin{bmatrix} 2xy - 22x \\ x^2 - 2y \end{bmatrix},$$

so its vanishing is the pair of conditions $0 = 2xy - 22x = 2x(y-11)$ and $0 = x^2 - 2y$. The first of these says $x = 0$ or $y = 11$. Inserting these into the second, we get the critical points $(0,0)$ and $(\pm\sqrt{22}, 11)$. The origin lies in R but the other two points *do not* (they violate $y \le 9 - x^2$ since $11 > 9 - 22$). We compute $f(0,0) = 0$. (If you missed that the other critical points are not in R, you would compute $f(\pm\sqrt{22}, 11) = 22(11) - 121 - 11(22) = -121$.)

Next, we look at points on the boundary, which we break into two regions: the segment of points $(x, 0)$ for $-3 \le x \le 3$ and the parabolic arc of points $(x, 9 - x^2)$ for $3 \le x \le 3$. For the former, we have $f(x, 0) = -11x^2$, which by inspection is largest at $x = 0$ and smallest at $x = \pm 3$, with corresponding f-values $f(0,0) = 0$ and $f(\pm 3, 0) = -11(9) = -99$. For the latter, we compute

$$f(x, 9-x^2) = x^2(9-x^2) - (9-x^2)^2 - 11x^2 = 9x^2 - x^4 - 81 + 18x^2 - x^4 - 11x^2 = -2x^4 + 16x^2 - 81.$$

Calling this $g(x)$, we seek its extrema for $-3 \le x \le 3$. We can ignore the endpoints since we have already computed f at $(\pm 3, 0)$. For the critical points of g in this interval, we first compute

$$g'(x) = -8x^3 + 32x = -8x(x^2 - 4),$$

which vanishes for $x = 0, \pm 2$, all of which lie in the interval between -3 and 3. The corresponding points are $(0, 9)$ and $(\pm 2, 9-4) = (\pm 2, 5)$, with $f(0, 9) = -81$ and $f(\pm 2, 5) = 4(5) - 25 - 11(4) = -49$.

Comparing the values, the maximum is 0 attained at the boundary point $(0,0)$ and the minimum is -99 attained at the boundary points $(\pm 3, 0)$. (If you missed that two of the critical points $(\pm\sqrt{22}, 11)$ of f do not lie in the region then you might mistakenly think the minimum of f on R is attained at those points since the value of f there is -121.)

5. (10 points)

(a) (4 points) For $f(x, y, z) = x^y + yz^2$ (with $x > 0$, so x^y makes sense: it means $e^{y\ln(x)}$), compute ∇f and write the best linear approximation to f near $(1, 2, 3)$. (Your approximation should be expressed as a function in x, y, z, involving explicit numbers that you compute.)

The x-partial of $x^y = e^{y\ln(x)}$ is $e^{y\ln(x)}(y\ln(x))' = x^y(y/x) = yx^{y-1}$ and the y-partial of x^y is $e^{y\ln(x)}\ln(x) = \ln(x)x^y$. Hence,

$$\nabla f = \begin{bmatrix} yx^{y-1} \\ \ln(x)x^y + z^2 \\ 2yz \end{bmatrix},$$

so $(\nabla f)(1, 2, 3) = \begin{bmatrix} 2(1^1) \\ 0(1^2) + 9 \\ 12 \end{bmatrix} = \begin{bmatrix} 2 \\ 9 \\ 12 \end{bmatrix}$. Since $f(1, 2, 3) = 1^2 + 2(3^2) = 1 + 18 = 19$, the best linear approximation to f near $(1, 2, 3)$ is

$$19 + 2(x - 1) + 9(y - 2) + 12(z - 3).$$

(b) (4 points) If $g(x, y)$ is some function which satisfies $g(1, 2) = 4$, $g(1.1, 2.1) = 4.1$, and $g(.9, 2.2) = 3.8$, give estimates for $g_x(1, 2)$ and $g_y(1, 2)$. (Hint: define $a = g_x(1, 2)$ and $b = g_y(1, 2)$ and apply linear approximation for g at $(1, 2)$ with the given numerical values to get two simultaneous equations in a and b that you can then solve. This idea underlies how computers approximate partial derivatives via numerical methods.)

The linear approximation for g at $(1, 2)$ is

$$h(x, y) = g(1, 2) + g_x(1, 2)(x - 1) + g_y(1, 2)(y - 2) = 4 + a(x - 1) + b(y - 2)$$

where we see to determine (or at least sensibly estimate!) a and b. The other two numerical values give the approximations

$$4.1 = g(1.1, 2.1) \approx h(1.1, 2.1) = 4 + a(0.1) + b(0.1)$$

and

$$3.8 = g(.9, 2.2) \approx h(.9, 2.2) = 4 + a(-0.1) + b(0.2),$$

so

$$(0.1)a + (0.1)b = 0.1, \quad -(0.1)a + (0.2)b = -0.2.$$

Multiplying through by 10 for each equation turns this into the pair of equations

$$a + b = 1, \quad -a + 2b = -2.$$

This is readily solved: adding the equations gives $3b = -1$, so $b = -1/3$, and then $a = 1-b = 4/3$. To summarize, we have the estimates $g_x(1, 2) \approx 4/3$ and $g_y(1, 2) \approx -1/3$.

(c) (3 points) If $g : \mathbf{R}^2 \to \mathbf{R}^3$ is a vector-valued function which satisfies

$$g(1, 2) = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}, \quad (Dg)(1, 2) = \begin{bmatrix} 0 & 4 \\ 1 & 2 \\ -3 & 6 \end{bmatrix}$$

then use linear approximation for g to estimate $g(1.2, 1.9)$; simplify your answer as much as possible.

For $\begin{bmatrix} h \\ k \end{bmatrix}$ near $\mathbf{0} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ we have

$$g(1+h, 2+k) \approx g(1,2) + (Dg)(1,2)\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 0 & 4 \\ 1 & 2 \\ -3 & 6 \end{bmatrix}\begin{bmatrix} h \\ k \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix} + \begin{bmatrix} 4k \\ h+2k \\ -3h+6k \end{bmatrix}$$
$$= \begin{bmatrix} -1+4k \\ 3+h+2k \\ 1-3h+6k \end{bmatrix}.$$

Setting $h = 0.2$ and $k = -0.1$ then gives the estimate

$$g(1.2, 1.9) \approx \begin{bmatrix} -1-0.4 \\ 3+0.2-0.2 \\ 1-0.6-0.6 \end{bmatrix} = \begin{bmatrix} -1.4 \\ 3 \\ -0.2 \end{bmatrix}.$$

6. (10 points) **True or False**

For each of the following statements, circle either TRUE (meaning, "always true") or FALSE (meaning, "not always true"), and briefly and convincingly justify your answer. Two points each: 0.5 points for the correct choice, and 1.5 points for convincing justification.

(a) Suppose the correlation coefficient of the data set $(f_1, g_1), \ldots, (f_n, g_n)$ is r, where f_i is the temperature on the i-th day of August in Fahrenheit, and g_i is the number of gallons of ice cream consumed on the i-th day of August. If the temperature is instead measured in degrees Celsius $C = \frac{5}{9}(F - 32)$ and the ice cream is instead measured in liters, the correlation can become a different number.

Circle one, and justify below: TRUE $\boxed{\text{FALSE}}$

Not always true: Correlation coefficient tells us how likely the data points can be approximated by a straight line. The effect of scaling all data by some positive factor has the effect of scaling the data vectors and averages by the same factor, so also for the shifted vectors. This common scaling cancels out in the ratio defining r. Likewise, if we add a constant to the data then all entries in the data vector and the average change by the same amount, so this cancels out in the shifted vector. So overall the change of units (for temperature and volume) has no effect.

(b) If $\mathbf{x} \cdot \mathbf{w} = \mathbf{y} \cdot \mathbf{w}$ for some nonzero vector $\mathbf{w}$ in $\mathbf{R}^n$, then $\mathbf{x} = \mathbf{y}$.

Circle one, and justify below: TRUE $\boxed{\text{FALSE}}$

Not always true: For example, $\mathbf{w} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ and $\mathbf{y} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$, then $(\mathbf{x}-\mathbf{y}) \cdot \mathbf{w} = 0$, $\mathbf{x} \cdot \mathbf{w} = \mathbf{y} \cdot \mathbf{w}$, but $\mathbf{x} \neq \mathbf{y}$.

(c) Suppose $\mathbf{u}, \mathbf{v}, \mathbf{w}$ are nonzero vectors in $\mathbf{R}^{51}$. If $\{\mathbf{u}, \mathbf{v}\}$ is a basis for span$(\mathbf{u}, \mathbf{v})$, and if $\mathbf{w}$ is perpendicular to both $\mathbf{u}$ and $\mathbf{v}$, then $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ is a basis for span$(\mathbf{u}, \mathbf{v}, \mathbf{w})$.

Circle one, and justify below: $\boxed{\text{TRUE}}$ FALSE

Always true: We need to show that there is no redundancy among $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$.
Suppose there is redundancy and $\mathbf{w} \in$ span$(\mathbf{u}, \mathbf{v})$, i.e. $\mathbf{w} = a\mathbf{u} + b\mathbf{v}$ for some $a, b \in \mathbf{R}$. Dot both sides of the equation with $\mathbf{w}$, we have

$$\mathbf{w} \cdot \mathbf{w} = (a\mathbf{u} + b\mathbf{v}) \cdot \mathbf{w} = a(\mathbf{u} \cdot \mathbf{w}) + b(\mathbf{v} \cdot \mathbf{w}) = a(0) + b(0) = 0$$

But $\mathbf{w} \cdot \mathbf{w} \neq 0$ since $\mathbf{w}$ is a nonzero vector. We have a contradiction. So $\mathbf{w}$ is not redundant.
You can check that $\mathbf{u}$ is not in span$(\mathbf{v}, \mathbf{w})$ either. If $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$, again dot both sides with $\mathbf{w}$,

we have

$$\mathbf{u} \cdot \mathbf{w} = a\mathbf{v} \cdot \mathbf{w} + b\mathbf{w} \cdot \mathbf{w}, \qquad 0 = b\mathbf{w} \cdot \mathbf{w}$$

Since $\mathbf{w} \cdot \mathbf{w} \neq 0$, $b = 0$ must hold, so $\mathbf{u} = a\mathbf{v}$, this contradicts the fact that $\{\mathbf{u}, \mathbf{v}\}$ is a basis. $\mathbf{v}$ is not in $\text{span}(\mathbf{u}, \mathbf{w})$ can be checked similarly. Since none of the three vectors is redundant, $\{\mathbf{u}, \mathbf{v}, \mathbf{w}\}$ must be a basis.

(d) The critical point $(0, 0)$ of $f(x, y) = (\sin x)^2 + (\cos y)^2$ is a local minimum.

Circle one, and justify below: TRUE $\boxed{\text{FALSE}}$

False: Note that $f(0, 0) = 1$. Along the line $x = 0$, $f(0, y) = (\cos y)^2 < 1$ for nonzero y's near $y = 0$; along the line $y = 0$, $f(x, y) = (\sin x)^2 + 1 > 1$ for nonzero x's near $x = 0$. This makes $(0, 0)$ a saddle, not a local minimum.

(e) If an $n \times n$ matrix A satisfies $A^2 = A$, then $A^{51} = A$.

Circle one, and justify below: $\boxed{\text{TRUE}}$ FALSE

Always true: $A^3 = A(A^2) = AA = A$, $A^4 = A(A^3) = AA = A$. It follows that $A^k = A$ for all $k \geq 2$. In particular, $A^{51} = A$.