Math 51
Lecture 22 – August 7, 2023

Topic(s): solving linear systems using QR- or LU-decomposition of a matrix, using Gram-Schmidt to construct a QR-decomposition

Example 1. **(back-substitution)** Suppose that A is an upper triangular square matrix; for example consider the following system:

$$\begin{aligned} 3x_1 - 2x_2 + 4x_3 - 8x_4 &= 32 \\ 9x_2 + 5x_3 + 6x_4 &= -3 \\ x_3 + 2x_4 &= -1 \\ -5x_4 &= 25 \end{aligned}$$

Solve the system by working your way up from the bottom.

We get $x_4 = -5$, $x_3 = 9$, $x_2 = -2$, and $x_1 = -16$. Hence, $(-16, -2, 9, 5)$ is the unique solution.

This method is called **back-substitution** – it is quite easy to carry out to uniquely solve $A\mathbf{x} = \mathbf{b}$ for any $n \times n$ upper triangular A and n-vector $\mathbf{b}$ *provided* that the diagonal entries of A are non-zero. Essentially, the same calculations also work for lower triangular matrices.

What happens when some of the diagonal entries of A is/are zero?

There are either no solutions or infinitely many solutions.

Theorem 22.1.3. If A is an upper or lower triangular $n \times n$ matrix with all diagonal entries non-zero, then A is invertible.

Next, suppose A is an $m \times n$ matrix with orthonormal columns. To solve $A\mathbf{x} = \mathbf{b}$, we can multiply both sides by $A^\intercal$ on the left to get

$$\mathbf{x} = I_n\mathbf{x} = (A^\intercal A)\mathbf{x} = A^\intercal\mathbf{b}.$$

So, $\mathbf{x} = A^\intercal\mathbf{b}$ is the only possible solution, and if $m = n$, it really is a solution because

$$A\mathbf{x} = A(A^\intercal\mathbf{b}) = (AA^\intercal)\mathbf{b} = I_n\mathbf{b} = \mathbf{b}.$$

Theorem 22.1.4. For an $n \times n$ orthogonal matrix A, the equation $A\mathbf{x} = \mathbf{b}$ has exactly one solution: $\mathbf{x} = A^\intercal\mathbf{b}$.

In practice, one often encounters non-square linear systems $A\mathbf{x} = \mathbf{b}$. The simplest of these to handle will be A that are upper triangular or lower triangular; there are two cases for each. The underdetermined case ($m < n$: fewer equations than unknowns) are

$$U = \begin{bmatrix} * & * & \cdots & * & * & \cdots & * \\ 0 & * & \cdots & * & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \vdots & \cdots & \vdots \\ 0 & 0 & \cdots & * & * & \cdots & * \\ 0 & 0 & \cdots & 0 & * & \cdots & * \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} * & 0 & \cdots & 0 & 0 & 0 & \cdots & 0 \\ * & * & \cdots & 0 & 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \cdots & \vdots \\ * & * & \cdots & * & 0 & 0 & \cdots & 0 \\ * & * & \cdots & * & * & 0 & \cdots & 0 \end{bmatrix}$$

The overdetermined case ($m > n$: more equations than unknowns) are

$$U = \begin{bmatrix} * & * & \cdots & * & * \\ 0 & * & \cdots & * & * \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & \cdots & * & * \\ 0 & 0 & \cdots & 0 & * \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & \cdots & 0 & 0 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} * & 0 & \cdots & 0 & 0 \\ * & * & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ * & * & \cdots & * & 0 \\ * & * & \cdots & * & * \\ \vdots & \vdots & \cdots & \vdots & \vdots \\ * & * & \cdots & * & * \end{bmatrix}$$

The next four examples will solve systems of each type.

Example 2. (underdetermined upper triangular) Solve

$$\begin{bmatrix} 2 & -3 & 1 & 3 & -1 \\ 0 & 1 & 4 & 4 & -7 \\ 0 & 0 & 3 & 3 & -6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ 3 \end{bmatrix}$$

By back-substitution, we get $x_3 = -x_4 + 2x_5 + 1$, $x_2 = -x_5 + 1$, and $x_1 = -x_4 - 2x_5 + 2$. Hence,

$$\boxed{\begin{bmatrix} -x_4 - 2x_5 + 2 \\ -x_5 + 1 \\ -x_4 + 2x_5 + 1 \\ x_4 \\ x_5 \end{bmatrix} = x_4 \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} -2 \\ -1 \\ 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}$$

is a solution for all $x_4, x_5 \in \mathbb{R}$.

Example 3. (underdetermined lower triangular) Solve

$$\begin{bmatrix} 2 & 0 & 0 & 0 & 0 \\ -3 & 1 & 0 & 0 & 0 \\ 1 & 4 & 3 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 6 \\ -7 \\ 5 \end{bmatrix}$$

By back-substitution, we get $x_1 = 3$, $x_2 = 2$, and $x_3 = -2$. Hence,

$$\boxed{\begin{bmatrix} 3 \\ 2 \\ -2 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \\ -2 \\ 0 \\ 0 \end{bmatrix} + x_4 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ 0 \end{bmatrix} + x_5 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}}$$

is a solution for all $x_4, x_5 \in \mathbb{R}$.

Now, we will consider some overdetermined $m \times n$ triangular systems $A\mathbf{x} = \mathbf{b}$. Since $C(A)$ is the subspace of $\mathbb{R}^m$ spanned by the n columns of A ($\dim C(A) \leq n < m$), for dimension reasons, $C(A)$ cannot coincide with $\mathbb{R}^m$. Thus, a typical $\mathbf{b} \in \mathbb{R}^m$ does not belong to $C(A)$. In other words, $A\mathbf{x} = \mathbf{b}$ has no solution unless $\mathbf{b}$ is quite special, so to organize our work, we will consider a specific A but leave $\mathbf{b}$ in a form with symbolic entries b_j that we regard as numbers for which we seek the *constraints* on them ensuring $A\mathbf{x} = \mathbf{b}$ has a solution (i.e., $\mathbf{b} \in C(A)$).

Example 4. (overdetermined upper triangular) Solve

$$\begin{bmatrix} 2 & -3 & 1 \\ 0 & 1 & 4 \\ 0 & 0 & 3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}.$$

By back-substitution, $x_3 = \frac{1}{3}b_3$, $x_2 = b_2 - \frac{4}{3}b_3$, and $x_1 = \frac{1}{2}b_1 + \frac{3}{2}b_2 - \frac{13}{6}b_3$. In addition, we have $0 = b_4$ and $0 = b_5$. Thus, the system has a unique solution

$$\begin{bmatrix} \frac{1}{2}b_1 + \frac{3}{2}b_2 - \frac{13}{6}b_3 \\ b_2 - \frac{4}{3}b_3 \\ \frac{1}{3}b_3 \end{bmatrix}$$

only if $b_4 = b_5 = 0$. Otherwise, the system is inconsistent and has no solution.

Example 5. (overdetermined lower triangular) Solve

$$\begin{bmatrix} 2 & 0 & 0 \\ -3 & 1 & 0 \\ 1 & -4 & 3 \\ 2 & -7 & 3 \\ 4 & 5 & -1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}.$$

By back-substitution, $x_1 = \frac{1}{2}b_1$, $x_2 = \frac{3}{2}b_1 + b_2$, and $x_3 = \frac{11}{6}b_1 + \frac{4}{3}b_2 + \frac{1}{3}b_3$. We also need $b_4 = 2x_1 - 7x_2 + 3x_3 = -4b_1 - 3b_2 + b_3$ and $b_5 = 4x_1 + 5x_2 - x_3 = \frac{23}{3}b_1 + \frac{11}{3}b_2 - \frac{1}{3}b_3$. Hence, the system has a unique solution

$$\begin{bmatrix} \frac{1}{2}b_1 \\ \frac{3}{2}b_1 + b_2 \\ \frac{11}{6}b_1 + \frac{4}{3}b_2 + \frac{1}{3}b_3 \end{bmatrix}$$

only if $b_4 = -4b_1 - 3b_2 + b_3$ and $b_5 = \frac{23}{3}b_1 + \frac{11}{3}b_2 - \frac{1}{3}b_3$. Otherwise, the system is inconsistent and has no solutions.

Theorem 22.1.9. Let A be an upper or lower triangular $m \times n$ matrix with *all diagonal entries nonzero.*

1. If $m = n$, then $A\mathbf{x} = \mathbf{b}$ always has a unique solution (found via back-substitution).
2. In the underdetermined case ($m < n$), $A\mathbf{x} = \mathbf{b}$ always has a solution (i.e., $C(A) = \mathbb{R}^m$), and $\dim N(A) = n - m > 0$. The solutions are given by uniquely solving for $x_1, \ldots, x_m$ in terms of arbitrarily chosen $x_{m+1}, \ldots, x_n$. For lower triangular A, the linear system $A\mathbf{x} = \mathbf{b}$ is really m equations in $x_1, \ldots, x_m$ with the other x_j's having coefficient 0 in every equation.
3. In the overdetermined case ($m > n$), $\dim C(A) = n < m$ (so $A\mathbf{x} = \mathbf{b}$ only has a solution for special $\mathbf{b}$) and $N(A) = \{\mathbf{0}\}$ (so a solution to $A\mathbf{x} = \mathbf{b}$ is unique when one exists). The column space $C(A)$ is described by expressions for each of $b_{n+1}, \ldots, b_m$ in terms of $b_1, \ldots, b_n$; for upper triangular A, these expressions are simply $b_{n+1} = \cdots = b_m = 0$.

Theorem 22.2.1.

1. (LU-decomposition) "Most" $n \times n$ matrices A have the form $A = LU$, where L is an $n \times n$ lower triangular matrix and U is an $n \times n$ upper triangular matrix. The matrix A is invertible precisely when the diagonal entries of L and U are all non-zero.

2. (QR-decomposition) An invertible $n \times n$ matrix A can be written as $A = QR$, where Q is an $n \times n$ orthogonal matrix and R is an $n \times n$ upper triangular matrix with positive diagonal entries.

Example 6. Suppose that we are given the following LU-decomposition for A:

$$A = \begin{bmatrix} 5 & 0 & 1 \\ 1 & 2 & 2 \\ -1 & 4 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1/5 & 1 & 0 \\ -1/5 & 2 & 1 \end{bmatrix} \begin{bmatrix} 5 & 0 & 1 \\ 0 & 2 & 9/5 \\ 0 & 0 & -7/5 \end{bmatrix}.$$

Solve the following system of linear equations

$$\begin{aligned} 5x \quad\quad + z &= 1 \\ x + 2y + 2z &= 3 \\ -x + 4y + 2z &= 4 \end{aligned}$$

$A\vec{x} = (LU)\vec{x} = L(U\vec{x}) = \vec{b}$

① Solve $L\vec{y} = \vec{b}$.

② Solve $U\vec{x} = \vec{y}$.

① $\begin{bmatrix} 1 & 0 & 0 \\ \frac{1}{5} & 1 & 0 \\ -\frac{1}{5} & 2 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} \Rightarrow \vec{y} = \begin{bmatrix} 1 \\ \frac{14}{5} \\ -\frac{7}{5} \end{bmatrix}.$

② $\begin{bmatrix} 5 & 0 & 1 \\ 0 & 2 & \frac{9}{5} \\ 0 & 0 & -\frac{7}{5} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ \frac{14}{5} \\ -\frac{7}{5} \end{bmatrix} \Rightarrow \boxed{\vec{x} = \begin{bmatrix} 0 \\ \frac{1}{2} \\ 1 \end{bmatrix}}.$

Example 7. Suppose that we are given the following LU-decomposition for A:

$$A = \begin{bmatrix} 2 & -2 & -3 \\ 4 & -3 & -4 \\ -6 & 6 & 14 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ -3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -2 & -3 \\ 0 & -1 & -2 \\ 0 & 0 & 5 \end{bmatrix}.$$

Solve the following system of linear equations

$$\begin{aligned} 2x - 2y - 3z &= 0 \\ 4x - 3y - 4z &= 6 \\ -6x + 6y + 14z &= 10 \end{aligned}$$

① $\begin{bmatrix} 1 & 0 & 0 \\ 2 & -1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 6 \\ 10 \end{bmatrix} \Rightarrow \vec{y} = \begin{bmatrix} 0 \\ -6 \\ 10 \end{bmatrix}.$

② $\begin{bmatrix} 2 & -2 & -3 \\ 0 & -1 & -2 \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 0 \\ -6 \\ 10 \end{bmatrix} \Rightarrow \boxed{\vec{x} = \begin{bmatrix} 5 \\ 2 \\ 2 \end{bmatrix}}.$

If A is an $n \times n$ matrix that has been written as a product of $n \times n$ matrices LU or QR *and* there are no 0's on the diagonal in L and U, or no 0's on the diagonal in R, then it is easy to compute A^{-1} as follows. The key point used is that for *invertible* $n \times n$ matrices M and N, their product MN is invertible: $(MN)^{-1} = N^{-1}M^{-1}$.

- if $A = LU$, then $A^{-1} = U^{-1}L^{-1}$;
- if $A = QR$, then $A^{-1} = R^{-1}Q^{-1} = R^{-1}Q^{\intercal}$.

Since U and R are upper triangular matrices and L is a lower triangular matrix, we only need to know how to compute the inverses of these triangular matrices. In fact, we only need to know how to do that for upper triangular matrices since the method is the same for lower triangular matrices. The key fact here is: if an upper triangular matrix is invertible, its inverse is upper triangular, and the same holds for lower triangular matrices.

Example 8. Compute the inverse of the upper triangular matrix $A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}$.

A^{-1} will have form $\begin{bmatrix} 1 & \alpha \\ 0 & \frac{1}{3} \end{bmatrix}$. Then

$$AA^{-1} = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & \alpha \\ 0 & \frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & \alpha + \frac{2}{3} \\ 0 & 1 \end{bmatrix} = I_2,$$

which implies $\alpha = -\frac{2}{3}$, and so, $\boxed{A^{-1} = \begin{bmatrix} 1 & -\frac{2}{3} \\ 0 & \frac{1}{3} \end{bmatrix}}$.

Example 9. Compute the inverse of the upper triangular matrix $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix}$.

$A^{-1} = \begin{bmatrix} \frac{1}{2} & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}$ for some α, β, γ. Then,

$$AA^{-1} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 3 \\ 0 & 0 & -3 \end{bmatrix}\begin{bmatrix} \frac{1}{2} & \alpha & \beta \\ 0 & 1 & \gamma \\ 0 & 0 & -\frac{1}{3} \end{bmatrix} = \begin{bmatrix} 1 & 2\alpha & 2\beta - \frac{1}{3} \\ 0 & 1 & \gamma - 1 \\ 0 & 0 & 1 \end{bmatrix} = I_3,$$

and so, $\alpha = 0$, $\beta = \frac{1}{6}$, $\gamma = 1$. Hence, $\boxed{A^{-1} = \begin{bmatrix} \frac{1}{2} & 0 & \frac{1}{6} \\ 0 & 1 & 1 \\ 0 & 0 & -\frac{1}{3} \end{bmatrix}}$.

We now explain how the QR-decomposition for a matrix A is just a repackaging of the Gram-Schmidt process for the columns of A.

We will primarily focus on the square $n \times n$ case and additionally that the columns of A are linearly independent (so A is invertible). Since the column vectors are linearly independent, we will never obtain a zero vector during the Gram-Schmidt process.

We make a small but important refinement in the Gram-Schmidt process – at the end, we divide each of the "output" vectors $\mathbf{w}_i$ by its length to obtain a set of vectors that is orthonormal.

Let us see how these unit vectors $\mathbf{w}_i'$'s are related to the $\mathbf{v}_j$'s.

1. In the first step, we have $\mathbf{w}_1 = \mathbf{v}_1$, so $\mathbf{w}_1' = \mathbf{w}_1/\|\mathbf{w}_1\| = \mathbf{v}_1/\|\mathbf{v}_1\|$. In other words, $r_{11}\mathbf{w}_1'$, where r_{11} is the scalar $\|\mathbf{v}_1\| > 0$.

2. At the next step, the orthogonal pair of unit vectors $\mathbf{w}_1', \mathbf{w}_2'$ has the same span as $\mathbf{w}_1, \mathbf{w}_2$, which has the same span as $\mathbf{v}_1, \mathbf{v}_2$, and by design, $\mathbf{w}_2 = \mathbf{v}_2 - t\mathbf{w}_1$ for some scalar t. Hence, $\mathbf{v}_2 = t\mathbf{w}_1 + \mathbf{w}_2 = r_{12}\mathbf{w}_1' + r_{22}\mathbf{w}_2'$ for some scalars r_{12} and r_{22}, with $r_{22} = \|\mathbf{w}_2\| > 0$.

3. Similarly, the orthonormal triple $\mathbf{w}_1', \mathbf{w}_2', \mathbf{w}_3'$ has the same span as $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, with $\mathbf{w}_3 = \mathbf{v}_3 - a\mathbf{v}_1 - b\mathbf{v}_2$ for some scalars a and b, so $\mathbf{v}_3 = a\mathbf{w}_1 + b\mathbf{w}_2 + \mathbf{w}_3 = r_{13}\mathbf{w}_1' + r_{23}\mathbf{w}_2' + r_{33}\mathbf{w}_3'$ for scalars r_{13}, r_{23}, r_{33} with $r_{33} = \|\mathbf{w}_3\| > 0$.

The Gram-Schmidt process gives formula for the scalar coefficients $r_{11}, r_{12}, r_{22}, r_{13}, r_{23}, r_{33}$ in terms of the dot products $\mathbf{v}_i \cdot \mathbf{v}_j$.

In matrix language, let Q be the 3×3 matrix whose columns are $\mathbf{w}_1', \mathbf{w}_2', \mathbf{w}_3'$; i.e.

$$Q = \begin{bmatrix} | & | & | \\ \mathbf{w}_1' & \mathbf{w}_2' & \mathbf{w}_3' \\ | & | & | \end{bmatrix}.$$

By design, Q is an orthogonal matrix. The three identities about the $\mathbf{v}_i$'s correspond to

$$Q\begin{bmatrix} r_{11} \\ 0 \\ 0 \end{bmatrix} = \mathbf{v}_1, \qquad Q\begin{bmatrix} r_{12} \\ r_{22} \\ 0 \end{bmatrix} = \mathbf{v}_2, \qquad Q\begin{bmatrix} r_{13} \\ r_{23} \\ r_{33} \end{bmatrix} = \mathbf{v}_3,$$

which gives us

$$Q\begin{bmatrix} r_{11} & r_{12} & r_{13} \\ 0 & r_{22} & r_{23} \\ 0 & 0 & r_{33} \end{bmatrix} = \begin{bmatrix} | & | & | \\ \mathbf{v}_1 & \mathbf{v}_2 & \mathbf{v}_3 \\ | & | & | \end{bmatrix} = A.$$

Example 10. Find the QR-decomposition of $A = \begin{array}{c} \begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix} \\ \vec{v}_1 \quad \vec{v}_2 \end{array}$.

Applying Gram-Schmidt to $\vec{v}_1$ and $\vec{v}_2$ yields $\vec{w}_1 = \vec{v}_1$ and $\vec{w}_2 = \vec{v}_2 - \frac{1}{5}\vec{v}_1 = \begin{bmatrix} \frac{7}{5} \\ \frac{14}{5} \end{bmatrix}$. Then,

$$\vec{w}_1' = \frac{1}{\sqrt{5}}\vec{w}_1 = \begin{bmatrix} -\frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \quad \text{and} \quad \vec{w}_2' = \frac{\sqrt{5}}{7}\vec{w}_2 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ \frac{2}{\sqrt{5}} \end{bmatrix}.$$

Also, $\vec{v}_1 = \vec{w}_1 = \sqrt{5}\,\vec{w}_1'$ and $\vec{v}_2 = \frac{1}{5}\vec{v}_1 + \vec{w}_2 = \frac{1}{\sqrt{5}}\vec{w}_1' + \frac{7}{\sqrt{5}}\vec{w}_2'$. Therefore,

$$\underset{A}{\begin{bmatrix} -2 & 1 \\ 1 & 3 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} -\frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} \end{bmatrix}} \underset{R}{\begin{bmatrix} \sqrt{5} & \frac{1}{\sqrt{5}} \\ 0 & \frac{7}{\sqrt{5}} \end{bmatrix}}.$$

Example 11. Consider the following matrix A and its column vectors:

$$A = \begin{bmatrix} 2 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}, \qquad \mathbf{v}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \qquad \mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix}.$$

If you run Gram-Schmidt on these, you get the following:

$$\begin{aligned} \mathbf{w}_1 &= \mathbf{v}_1 &&= \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix} \\ \mathbf{w}_2 &= \mathbf{v}_2 - \frac{3}{5}\mathbf{w}_1 &&= \frac{1}{5}\begin{bmatrix} -1 \\ -2 \\ 0 \end{bmatrix} \\ \mathbf{w}_3 &= \mathbf{v}_3 &&= \begin{bmatrix} 0 \\ 0 \\ -2 \end{bmatrix} \end{aligned}$$

Using this information, construct the QR-decomposition of A and check that your answer is correct.

We get

$$\vec{w}_1' = \frac{1}{\sqrt{5}}\vec{w}_1 = \begin{bmatrix} \frac{2}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} \\ 0 \end{bmatrix}, \qquad \vec{w}_2' = \sqrt{5}\,\vec{w}_2 = \begin{bmatrix} -\frac{1}{\sqrt{5}} \\ -\frac{2}{\sqrt{5}} \\ 0 \end{bmatrix}, \qquad \vec{w}_3' = \frac{1}{2}\vec{w}_3 = \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}.$$

Solving for the $\vec{v}_i$'s yields

$$\begin{aligned} \vec{v}_1 &= \vec{w}_1 &&= \sqrt{5}\,\vec{w}_1' \\ \vec{v}_2 &= \frac{3}{5}\vec{w}_1 + \vec{w}_2 &&= \frac{3}{\sqrt{5}}\vec{w}_1' + \frac{1}{\sqrt{5}}\vec{w}_2' \\ \vec{v}_3 &= \vec{w}_3 &&= 2\vec{w}_3' \end{aligned}$$

Hence,

$$\underbrace{\begin{bmatrix} 2 & 1 & 0 \\ -1 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix}}_{A} = \underbrace{\begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ -\frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \\ 0 & 0 & -1 \end{bmatrix}}_{Q} \underbrace{\begin{bmatrix} \sqrt{5} & \frac{3}{\sqrt{5}} & 0 \\ 0 & \frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 2 \end{bmatrix}}_{R}$$

Example 12. Find the QR-decomposition of $A = \begin{array}{c} \begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix} \\ \begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \end{array} \end{array}$.

By Gram-Schmidt, $\vec{w}_1 = \vec{v}_1$, $\vec{w}_2 = 3\vec{v}_1 + 5\vec{v}_2 = \begin{bmatrix} -2 \\ 5 \\ 1 \end{bmatrix}$, and $\vec{w}_3 = \frac{6}{5}\vec{v}_3 - \frac{12}{25}\vec{w}_1 - \frac{11}{25}\vec{w}_2 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$.

Hence,

$$\vec{w}_1{}' = \frac{1}{\sqrt{5}}\vec{w}_1 = \begin{bmatrix} \frac{1}{\sqrt{5}} \\ 0 \\ \frac{2}{\sqrt{5}} \end{bmatrix}, \qquad \vec{w}_2{}' = \frac{1}{\sqrt{30}}\vec{w}_2 = \begin{bmatrix} -\frac{\sqrt{2}}{\sqrt{15}} \\ \frac{\sqrt{5}}{\sqrt{6}} \\ \frac{1}{\sqrt{30}} \end{bmatrix}, \qquad \vec{w}_3{}' = \frac{1}{\sqrt{6}}\vec{w}_3 = \begin{bmatrix} -\frac{\sqrt{2}}{\sqrt{3}} \\ -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \end{bmatrix}.$$

Solving for $\vec{v}_i$'s leads to

$$\begin{aligned} \vec{v}_1 &= \vec{w}_1 &&= \sqrt{5}\,\vec{w}_1{}' \\ \vec{v}_2 &= -\frac{3}{5}\vec{w}_1 + \frac{1}{5}\vec{w}_2 &&= -\frac{3}{\sqrt{5}}\vec{w}_1{}' + \frac{\sqrt{6}}{\sqrt{5}}\vec{w}_2{}' \\ \vec{v}_3 &= \frac{2}{5}\vec{w}_1 + \frac{11}{30}\vec{w}_2 + \frac{5}{6}\vec{w}_3 &&= \frac{2}{\sqrt{5}}\vec{w}_1{}' + \frac{11}{\sqrt{30}}\vec{w}_2{}' + \frac{5}{\sqrt{6}}\vec{w}_3{}' \end{aligned}$$

Thus,

$$\underset{A}{\begin{bmatrix} 1 & -1 & -2 \\ 0 & 1 & 1 \\ 2 & -1 & 2 \end{bmatrix}} = \underset{Q}{\begin{bmatrix} \frac{1}{\sqrt{5}} & -\frac{\sqrt{2}}{\sqrt{15}} & -\frac{\sqrt{2}}{\sqrt{3}} \\ 0 & \frac{\sqrt{5}}{\sqrt{6}} & -\frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{30}} & \frac{1}{\sqrt{6}} \end{bmatrix}} \underset{R}{\begin{bmatrix} \sqrt{5} & -\frac{3}{\sqrt{5}} & \frac{2}{\sqrt{5}} \\ 0 & \frac{\sqrt{6}}{\sqrt{5}} & \frac{11}{\sqrt{30}} \\ 0 & 0 & \frac{5}{\sqrt{6}} \end{bmatrix}}.$$