Topic(s): using partial derivatives to find local extrema

A function $f(x,y)$ achieves a **local maximum** at (a,b) if $f(a,b) \geq f(x,y)$ for all (x,y) that are sufficiently near (a,b). In other words, if we move in any direction from (a,b), $f(x,y)$ decreases or stays the same. Similarly, a function $f(x,y)$ achieves a **local minimum** at (a,b) if $f(a,b) \leq f(x,y)$ for all (x,y) that are sufficiently near (a,b). In other words, if we move in any direction from (a,b), $f(x,y)$ increases or stays the same.

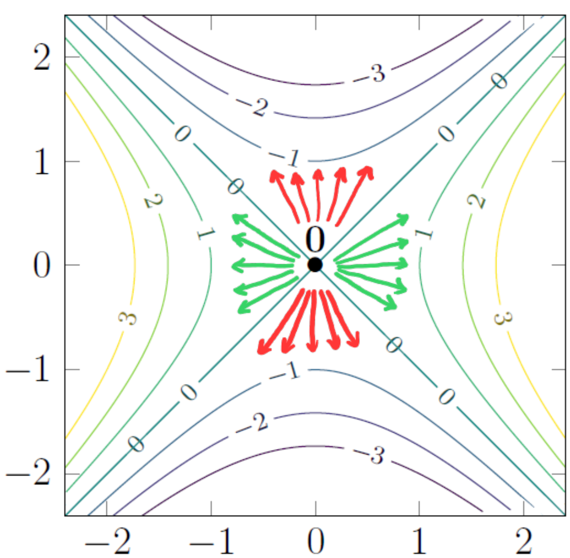
Theorem 10.2.2. Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. Suppose that a point $\mathbf{a} \in \mathbb{R}^n$ is either a local maximum or a local minimum of f. Then, all partial derivatives of f vanish at $\mathbf{x} = \mathbf{a}$; i.e. $\dfrac{\partial f}{\partial x_i}(\mathbf{a}) = 0$ for all i.

Let $f : \mathbb{R}^n \to \mathbb{R}$ be a function. If $\dfrac{\partial f}{\partial x_i}(\mathbf{a}) = 0$ for all i, we say that $\mathbf{a}$ is a **critical point** of f. In particular, every local maximum and local minimum of f is a critical point. However, the converse is not true – a critical point need not be a local extremum.

Example 1. Find the critical points of $f(x,y) = (x-2)^2 + 2(y-3)^2$.

We see that $f_x = 2(x-2)$ and $f_y = 2(y-3)$. Setting $f_x = 0$ and $f_y = 0$ gives us $x = 2$ and $y = 3$. Hence, $(2,3)$ is the only critical point.

Example 2. Find the critical points of $f(x,y) = x^2 - y^2$.



$f_x = 2x = 0 \Rightarrow x = 0$

$f_y = -2y = 0 \Rightarrow y = 0$

$(0,0)$ is the only critical point.

In the green directions away from $\vec{0}$, f is increasing.

In the red directions away from $\vec{0}$, f is decreasing.

$\vec{0}$ is neither a local maximum nor a local minimum.

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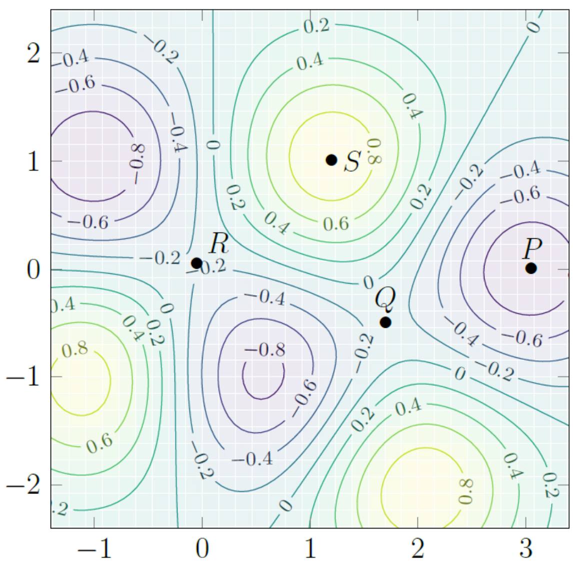
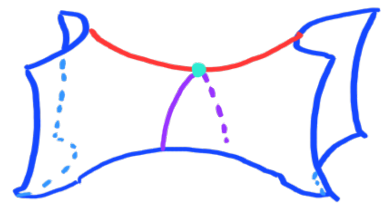
A critical point $\mathbf{a} \in \mathbb{R}^n$ of $f : \mathbb{R}^n \to \mathbb{R}$ is a **saddle point** if

1. as we move away from $\mathbf{a}$ in one direction, then f increases nearby ($\mathbf{a}$ would look like a local minimum along the line), **AND**
2. as we move away from $\mathbf{a}$ in some other direction, then f decreases nearby ($\mathbf{a}$ would look like a local maximum along the line.

Example 3. Some of the critical points are labeled in the contour map below. Determine if any of P, Q, R, and S are local extrema or saddle points.





P: local minimum

Q: saddle point

R: saddle point

S: local maximum

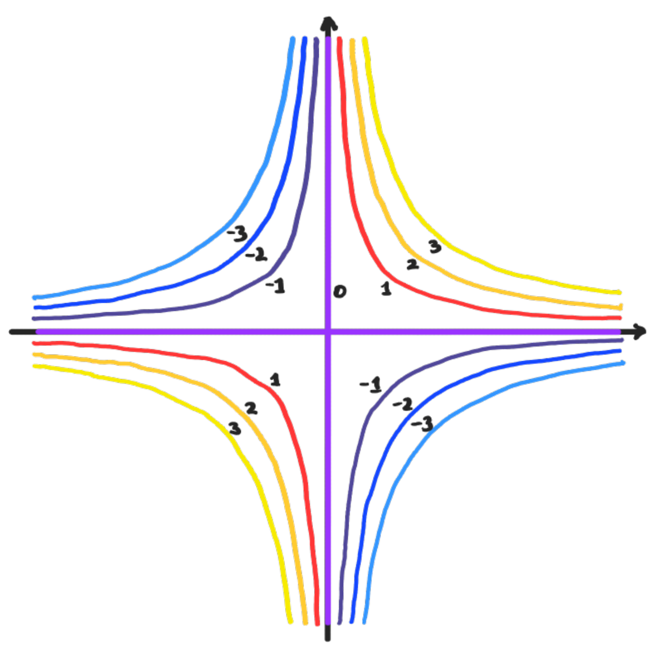
Example 4. Find the critical points of $f(x, y) = xy$. Draw a contour map to determine what is going on at the critical point(s).

$f_x = y = 0 \Rightarrow y = 0$

$f_y = x = 0 \Rightarrow x = 0$

$(0,0)$ is the only critical point.

$(0,0)$ is a saddle point.



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Theorem 10.4.6. For a function $f : \mathbb{R}^n \to \mathbb{R}$ and a region D inside $\mathbb{R}^n$, suppose $f : D \to \mathbb{R}$ considered on D has a local extremum at $\mathbf{a} \in D$. In other words, $f(\mathbf{x}) \geq f(\mathbf{a})$ for all $\mathbf{x} \in D$ near $\mathbf{a}$, or $f(\mathbf{x}) \leq f(\mathbf{a})$ for all $\mathbf{x} \in D$ near $\mathbf{a}$.
Then, $\mathbf{a}$ must be a critical point of f if $\mathbf{a}$ is in the interior of D. In particular, any local extremum of $f : D \to \mathbb{R}$ is either a critical point on the interior of D or is a boundary point of D.

Finding extrema for $f : D \to \mathbb{R}$, where $D \subset \mathbb{R}^2$.

1. Identify possible extrema in the interior of D by using the first partial derivatives
2. Look at the restriction of the function to the boundary curves. On each boundary curve, the function can be reduced to a single variable function – you can optimize these as you would in Math 19/AP Calculus.

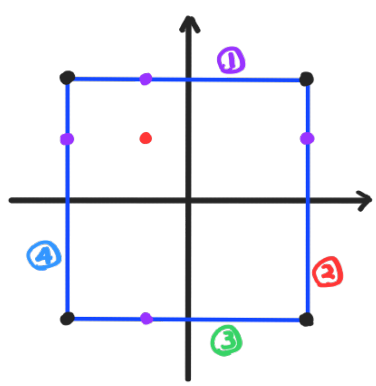
Example 5. Find the maximum and minimum of $f(x, y) = 3x^2 - 2y^2 + 2x + 2y - 1$ on the square $D = \{(x, y) \in \mathbb{R}^2 : -1 \leq x \leq 1, -1 \leq y \leq 1\}$.

<u>Interior critical point</u>

$f_x = 6x + 2 = 0 \Rightarrow x = -\frac{1}{3}$

$f_y = -4y + 2 = 0 \Rightarrow y = \frac{1}{2}$

$\left(-\frac{1}{3}, \frac{1}{2}\right)$ critical point.



(x, y)	$f(x, y)$
$(-1/3, 1/2)$	$-5/6$
$(-1, -1)$	-4
$(-1, 1)$	0
$(1, -1)$	0
$(1, 1)$	4
$(-1/3, -1)$	$-16/3$
$(-1/3, 1)$	$-4/3$
$(-1, 1/2)$	$1/2$
$(1, 1/2)$	$9/2$

<u>Boundary</u>

Edge ①: $y = 1, -1 \leq x \leq 1$.

$g(x) = f(x, 1) = 3x^2 + 2x - 1 \Rightarrow g'(x) = 6x + 2 = 0$

Critical point $\left(-\frac{1}{3}, 1\right)$, endpoints $(-1, 1), (1, 1)$

Edge ②: $x = 1, -1 \leq y \leq 1$.

$g(y) = f(1, y) = -2y^2 + 2y + 4 \Rightarrow g'(y) = -4y + 2 = 0$

Critical point $\left(1, \frac{1}{2}\right)$, endpoints $(1, 1), (1, -1)$

Edge ③: $y = -1, -1 \leq x \leq 1$.

$g(x) = f(x, -1) = 3x^2 + 2x - 5 \Rightarrow g'(x) = 6x + 2 = 0$

Critical point $\left(-\frac{1}{3}, -1\right)$, endpoints $(-1, -1), (1, -1)$

Edge ④: $x = -1, -1 \leq y \leq 1$.

$g(y) = f(-1, y) = -2y^2 + 2y \Rightarrow g'(y) = -4y + 2 = 0$

Critical point $\left(-1, \frac{1}{2}\right)$, endpoints $(-1, -1), (-1, 1)$

On D, f obtains a maximum of $\frac{9}{2}$ and a minimum of $-\frac{16}{3}$.

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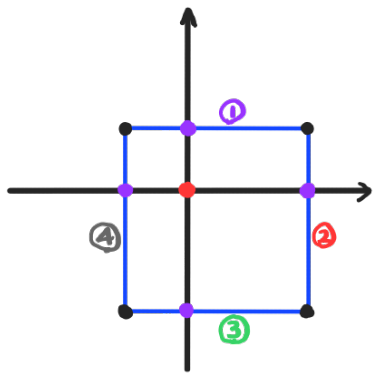
Example 6. Find the maximum and minimum of $f(x,y) = x^2 + y^2$ on the rectangle $D = \{(x,y) \in \mathbb{R}^2 : -1 \le x \le 2, -2 \le y \le 1\}$.

<u>Interior critical point</u>

$f_x = 2x = 0 \Rightarrow x = 0$

$f_y = 2y = 0 \Rightarrow y = 0$

$(0,0)$ critical point



(x,y)	$f(x,y)$
$(0,0)$	0
$(-1,0)$	1
$(2,0)$	4
$(0,-2)$	4
$(0,1)$	1
$(-1,-2)$	5
$(-1,1)$	2
$(2,-2)$	8
$(2,1)$	5

<u>Boundary</u>

Edge ①: $y = 1, -1 \le x \le 2$.

$g(x) = f(x,1) = x^2 + 1 \Rightarrow g'(x) = 2x = 0$

Critical point $(0,1)$, endpoints $(-1,1), (2,1)$

Edge ②: $x = 2, -2 \le y \le 1$.

$g(y) = f(2,y) = y^2 + 4 \Rightarrow g'(y) = 2y = 0$

Critical point $(2,0)$, endpoints $(2,-2), (2,1)$

Edge ③: $y = -2, -1 \le x \le 2$.

$g(x) = f(x,-2) = x^2 + 4 \Rightarrow g'(x) = 2x = 0$

Critical point $(0,-2)$, endpoints $(-1,-2), (2,-2)$

Edge ④: $x = -1, -2 \le y \le 1$.

$g(y) = f(-1,y) = y^2 + 1 \Rightarrow g'(y) = 2y = 0$

Critical point $(-1,0)$, endpoints $(-1,-2), (-1,1)$

On D, f obtains a maximum of 8 and a minimum of 0.

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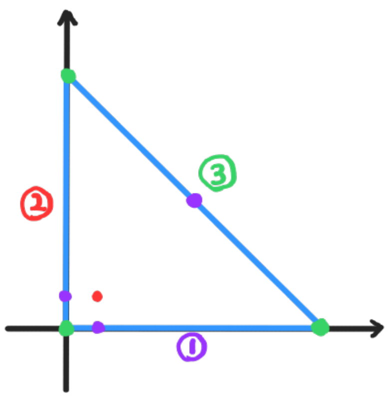
Example 7. Find the maximum and minimum of $f(x,y) = 2 + 2x + 2y - x^2 - y^2$ on the triangular plate in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

Interior critical point

$f_x = -2x + 2 = 0 \Rightarrow x = 1$

$f_y = -2y + 2 = 0 \Rightarrow y = 1$

$(1, 1)$ critical point



Boundary

Edge ①: $y = 0,\ 0 \leq x \leq 9$.

$g(x) = f(x,0) = -x^2 + 2x + 2 \Rightarrow g'(x) = -2x + 2 = 0$

Critical point $(1,0)$, endpoints $(0,0), (9,0)$

Edge ②: $x = 0,\ 0 \leq y \leq 9$.

$g(y) = f(0,y) = -y^2 + 2y + 2 \Rightarrow g'(y) = -2y + 2 = 0$

Critical point $(0,1)$, endpoints $(0,0), (0,9)$

Edge ③: $x = 9 - y,\ 0 \leq y \leq 9$.

$g(y) = f(9-y, y) = -2y^2 + 18y - 61 \Rightarrow g'(y) = -4y + 18 = 0$

Critical point $\left(\frac{9}{2}, \frac{9}{2}\right)$, endpoints $(0,9), (9,0)$

(x,y)	$f(x,y)$
$(1,1)$	$\boxed{4}$
$(1,0)$	3
$(0,1)$	3
$(9/2, 9/2)$	$-41/2$
$(0,0)$	2
$(0,9)$	$\boxed{-61}$
$(9,0)$	$\boxed{-61}$

On the triangular plate, f obtains a maximum of 4 and a minimum of -61.

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Example 8. Find the maximum and minimum of $f(x, y) = x^2 - y^2$ on the circle $D = \{(x, y) : x^2 + y^2 \leq 4\}$.

Interior critical point

$f_x = 2x = 0 \Rightarrow x = 0$

$f_y = -2y = 0 \Rightarrow y = 0$

$(0,0)$ critical point

Boundary

Top semicircle: $y = \sqrt{4-x^2}$, $-2 \leq x \leq 2$.

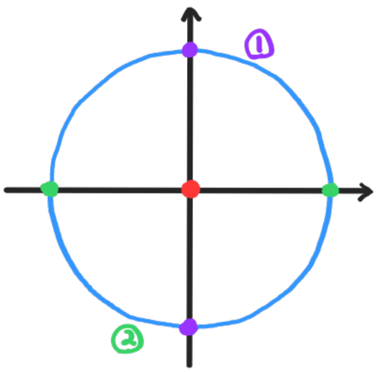
$g(x) = f(x, \sqrt{4-x^2}) = 2x^2 - 4 \Rightarrow g'(x) = 4x = 0$

Critical point $(0,2)$, endpoints $(-2,0)$, $(2,0)$

Bottom semicircle: $y = -\sqrt{4-x^2}$, $-2 \leq x \leq 2$.

$g(x) = f(x, -\sqrt{4-x^2}) = 2x^2 - 4 \Rightarrow g'(x) = 4x = 0$

Critical point $(0,-2)$, endpoints $(-2,0)$, $(2,0)$



(x, y)	$f(x, y)$
$(0,0)$	0
$(0,-2)$	-4
$(0,2)$	-4
$(-2,0)$	4
$(2,0)$	4

On D, f obtains a maximum of 4 and a minimum of -4.

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