Solutions to Math 51 Midterm Exam (Practice #1)

1. (10 points) Let $\mathbf{v}$ and $\mathbf{w}$ be unit vectors in $\mathbf{R}^n$ satisfying $\mathbf{v} \cdot \mathbf{w} = -\dfrac{1}{3}$.

(a) (3 points) Show that $\|3\mathbf{v} + 2\mathbf{w}\| = 3$.

We compute the squared length via dot products:

$$\begin{aligned}(3\mathbf{v} + 2\mathbf{w}) \cdot (3\mathbf{v} + 2\mathbf{w}) &= 9(\mathbf{v} \cdot \mathbf{v}) + 6(\mathbf{v} \cdot \mathbf{w}) + 6(\mathbf{w} \cdot \mathbf{v}) + 4(\mathbf{w} \cdot \mathbf{w}) \\ &= 9\|\mathbf{v}\|^2 + 12(\mathbf{v} \cdot \mathbf{w}) + 4\|\mathbf{w}\|^2 \\ &= 9 + 12(-1/3) + 4 \\ &= 9 - 4 + 4 \\ &= 9,\end{aligned}$$

so taking the square root of both sides yields $\|3\mathbf{v} + 2\mathbf{w}\| = 3$.

(b) (3 points) If θ is the angle (measured in radians) between $\mathbf{v}$ and $\mathbf{w}$, briefly explain why $\theta > \pi/2$ and show the angle between $3\mathbf{v}+2\mathbf{w}$ and $\mathbf{w}$ is $\pi - \theta$. (You may use without explanation that $\pi - \theta$ and θ have cosines negative to each other, which is a general property of supplementary angles.)

Since $0° \leq \theta \leq 180°$, from the definition of the cosine function in terms of the unit circle (centered at the origin) we know that θ is obtuse precisely when $\cos\theta < 0$ (the points (a, b) on the unit circle with $b \geq 0$ whose angle with the positive x-axis is obtuse are precisely those whose x-coordinate a is < 0). But

$$\cos\theta = \frac{\mathbf{v} \cdot \mathbf{w}}{\|\mathbf{v}\|\|\mathbf{w}\|} = \frac{-1/3}{1 \cdot 1} < 0,$$

so indeed θ is obtuse.

Since $\cos(180° - \theta) = -\cos(\theta)$, this is equal to $-(-1/3) = 1/3$, we want to show that the angle between $3\mathbf{v} + 2\mathbf{w}$ and $\mathbf{w}$ has cosine equal to $1/3$. This cosine is equal to

$$\frac{(3\mathbf{v} + 2\mathbf{w}) \cdot \mathbf{w}}{\|3\mathbf{v} + 2\mathbf{w}\|\|\mathbf{w}\|} = \frac{3(\mathbf{v} \cdot \mathbf{w}) + 2(\mathbf{w} \cdot \mathbf{w})}{3 \cdot 1} = \frac{3(-1/3) + 2\|\mathbf{w}\|^2}{3} = \frac{-1+2}{3} = \frac{1}{3},$$

as desired.

(c) (4 points) Verify that

$$\frac{3\mathbf{v} + (1 + 2\sqrt{2})\mathbf{w}}{4}$$

is a unit vector. (Its angle with $\mathbf{w}$ is 45°, which underlies how this vector was built; you don't need to compute this angle.)

We compute the dot product against itself:

$$\begin{aligned}\frac{(3\mathbf{v} + (1 + 2\sqrt{2})\mathbf{w}) \cdot (3\mathbf{v} + (1 + 2\sqrt{2})\mathbf{w})}{16} &= \frac{9(\mathbf{v} \cdot \mathbf{v}) + 2 \cdot 3(1 + 2\sqrt{2})(\mathbf{v} \cdot \mathbf{w}) + (1 + 2\sqrt{2})^2(\mathbf{w} \cdot \mathbf{w})}{16} \\ &= \frac{9 + 6(1 + 2\sqrt{2})(-1/3) + (1 + 4\sqrt{2} + 8)}{16} \\ &= \frac{9 - 2(1 + 2\sqrt{2}) + (9 + 4\sqrt{2})}{16} \\ &= \frac{9 - 2 - 4\sqrt{2} + 9 + 4\sqrt{2}}{16} \\ &= \frac{18 - 2}{16} \\ &= 1.\end{aligned}$$

2. (10 points) Let V be the linear subspace of $\mathbf{R}^4$ spanned by $\mathbf{v}_1 = \begin{bmatrix} 9 \\ -2 \\ 1 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -3 \\ -1 \\ 8 \\ 7 \end{bmatrix}$.

(a) (3 points) Explain why $V = \mathrm{span}(\mathbf{v}_1, \mathbf{v}_2)$ and why this is 2-dimensional.

Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are nonzero and (by inspection) not scalar multiples of each other, their span is 2-dimensional. To show that V is equal to that span, we just have to check that $\mathbf{v}_3$ belongs to it. That is, we seek scalars a and b so that $\mathbf{v}_3 = a\mathbf{v}_1 + b\mathbf{v}_2$. Plugging in the actual 4-vectors, this equality says

$$\begin{bmatrix} -3 \\ -1 \\ 8 \\ 7 \end{bmatrix} = a \begin{bmatrix} 9 \\ -2 \\ 1 \\ -1 \end{bmatrix} + b \begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9a + 3b \\ -2a - b \\ a + 2b \\ -a + b \end{bmatrix},$$

which is the system of 4 simultaneous equations

$$9a + 3b = -3, \quad -2a - b = -1, \quad a + 2b = 8, \quad -a + b = 7$$

in a and b.

We'll solve the second and fourth equations simultaneously (we try this since its coefficients are "smallest") and then check the solution works for the other two equations. Adding the second and fourth gives $-3a = 6$, so $a = -2$, and then from the fourth equation we get $b = 7+a = 5$. The pair $(a, b) = (-2, 5)$ also satisfies the first and third equations ($9(-2) + 3(5) = -18 + 15 = -3$, $-2 + 2(5) = -2 + 10 = 8$). Thus, $\mathbf{v}_3 = -2\mathbf{v}_1 + 5\mathbf{v}_2$.

(b) (3 points) Compute an orthogonal basis of V containing $\mathbf{v}_2$, and verify that the basis you build is orthogonal. (If you use the method from class applied to $\{\mathbf{v}_1, \mathbf{v}_2\}$, the other vector in the orthogonal basis will have integer entries.)

From the basis $\{\mathbf{v}_1, \mathbf{v}_2\}$ for V, we get the orthogonal basis $\{\mathbf{v}_2, \mathbf{v}_1'\}$ with

$$\mathbf{v}_1' = \mathbf{v}_1 - \frac{\mathbf{v}_1 \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \mathbf{v}_1 - \frac{27 + 2 + 2 - 1}{9 + 1 + 4 + 1}\mathbf{v}_2 = \mathbf{v}_1 - \frac{30}{15}\mathbf{v}_2 = \mathbf{v}_1 - 2\mathbf{v}_2.$$

Plugging in the explicit 4-vectors for $\mathbf{v}_1$ and $\mathbf{v}_2$, we get

$$\mathbf{v}_1' = \begin{bmatrix} 9 \\ -2 \\ 1 \\ -1 \end{bmatrix} - 2\begin{bmatrix} 3 \\ -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 9 \\ -2 \\ 1 \\ -1 \end{bmatrix} + \begin{bmatrix} -6 \\ 2 \\ -4 \\ -2 \end{bmatrix} = \begin{bmatrix} 3 \\ 0 \\ -3 \\ -3 \end{bmatrix}.$$

To verify orthogonality, we compute $\mathbf{v}_1' \cdot \mathbf{v}_2 = 3(3) + 0(-1) - 3(2) - 3(1) = 9 - 6 - 3 = 0$.

(c) (4 points) For the 4-vector $\mathbf{x} = \begin{bmatrix} 16 \\ 0 \\ -1 \\ -1 \end{bmatrix}$, compute $\mathbf{Proj}_V(\mathbf{x})$ and find the scalars a and b for which $\mathbf{Proj}_V(\mathbf{x}) = a\mathbf{v}_1 + b\mathbf{v}_2$. (The entries in $\mathbf{Proj}_V(\mathbf{x})$ are integers, and a and b are integers.)

Using the orthogonal basis $\{\mathbf{v}_1', \mathbf{v}_2\}$ that we found in (b), we have

$$\begin{aligned}\mathbf{Proj}_V(\mathbf{x}) = \frac{\mathbf{x}\cdot\mathbf{v}_1'}{\mathbf{v}_1'\cdot\mathbf{v}_1'}\mathbf{v}_1' + \frac{\mathbf{x}\cdot\mathbf{v}_2}{\mathbf{v}_2\cdot\mathbf{v}_2}\mathbf{v}_2 &= \frac{48+3+3}{27}\mathbf{v}_1' + \frac{48-3}{15}\mathbf{v}_2 \\ &= \frac{54}{27}\mathbf{v}_1' + \frac{45}{15}\mathbf{v}_2 \\ &= 2\mathbf{v}_1' + 3\mathbf{v}_2.\end{aligned}$$

Plugging in the explicit 4-vectors for $\mathbf{v}_1'$ and $\mathbf{v}_2$, we find this is equal to

$$2\begin{bmatrix}3\\0\\-3\\-3\end{bmatrix} + 3\begin{bmatrix}3\\-1\\2\\1\end{bmatrix} = \begin{bmatrix}6\\0\\-6\\-6\end{bmatrix} + \begin{bmatrix}9\\-3\\6\\3\end{bmatrix} = \begin{bmatrix}15\\-3\\0\\-3\end{bmatrix}.$$

We seek (a, b) so that $\mathbf{Proj}_V(\mathbf{x}) = a\mathbf{v}_1 + b\mathbf{v}_2$. One way is to use our knowledge of $\mathbf{v}_1', \mathbf{v}_1, \mathbf{v}_2$ as explicit 4-vectors to set this up as a collection of 4 simultaneous equations we then solve (first for two of them, and then checking it works for the other two). A more efficient way is to note that in the solution to (b) we saw that $\mathbf{v}_1' = \mathbf{v}_1 - 2\mathbf{v}_2$, so the projection $2\mathbf{v}_1' + 3\mathbf{v}_2$ of $\mathbf{x}$ is equal to $2(\mathbf{v}_1 - 2\mathbf{v}_2) + 3\mathbf{v}_2 = 2\mathbf{v}_1 - \mathbf{v}_2$. Hence, $(a, b) = (2, -1)$.

3. (10 points) Define the function

$$f(x, y, z) = \frac{e^{xz}}{x^2 + y^2}$$

(away from points $(0, 0, z)$, where the denominator vanishes).

(a) (5 points) Compute $\dfrac{\partial f}{\partial x}, \dfrac{\partial f}{\partial y}, \dfrac{\partial f}{\partial z}, \dfrac{\partial^2 f}{\partial z^2}$, and $\dfrac{\partial^2 f}{\partial y \partial z}$.

For the x-partial we use the quotient rule (regarding y and z as "constant") to get

$$\frac{\partial f}{\partial x} = \frac{-2xe^{xz}}{(x^2+y^2)^2} + \frac{ze^{xz}}{x^2+y^2}.$$

The other two first-order partials are direct calculations, again using the quotient rule for the y-partial:

$$\frac{\partial f}{\partial y} = \frac{-2ye^{xz}}{(x^2+y^2)^2}, \quad \frac{\partial f}{\partial z} = \frac{xe^{xz}}{x^2+y^2}.$$

For the second-order partial derivatives, regarding x and y as "constant", we get

$$\frac{\partial^2 f}{\partial z^2} = \frac{\partial f}{\partial z}\left(\frac{\partial f}{\partial z}\right) = \frac{x^2 e^{xz}}{x^2+y^2}$$

We can compute, using the quotient rule for the y-partial:

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial f}{\partial y}\left(\frac{\partial f}{\partial z}\right) = \frac{-2xye^{xz}}{(x^2+y^2)^2}$$

Alternatively, recalling the equality of mixed partials, we can compute

$$\frac{\partial^2 f}{\partial y \partial z} = \frac{\partial^2 f}{\partial z \partial y} = \frac{\partial f}{\partial z}\left(\frac{\partial f}{\partial y}\right) = \frac{-2xye^{xz}}{(x^2+y^2)^2}$$

(b) (5 points) Below is a contour plot of a function $g(x, y)$ over the square of points (x, y) with $-3 \le x \le 1$ and $-2 \le y \le 2$, with the dashed line $y = 0$ drawn over it.

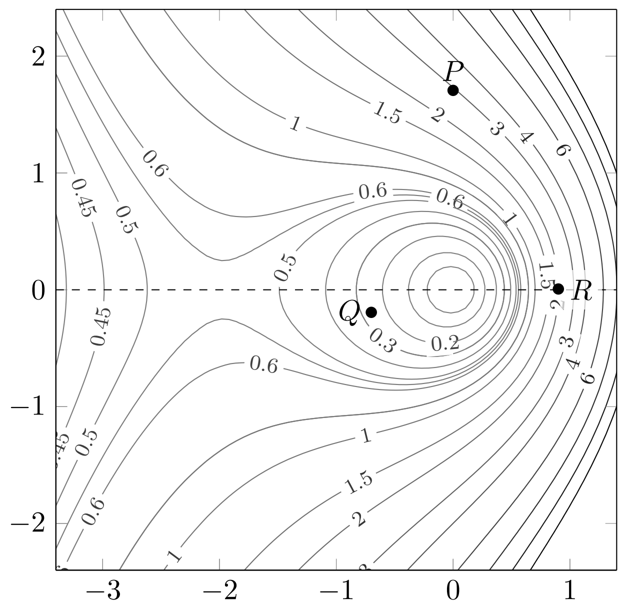

Figure 1: A contour plot for a function $g(x, y)$

On the dashed line $y = 0$, indicate approximately where $g_x < 0$ (don't worry about endpoints) and for the points labeled P, Q, R determine for each of $g_y(P), g_x(Q), g_x(R), g_y(R)$ if it is positive, negative, or 0.

On the dashed line, the contour labels are increasing up to around $x = -2$ (where contour lines seem to approximately "pinch"), then decrease up to around $x = 0$ (the center of the nested collection of approximate circles), and then increase again. So the part of the line $y = 0$ where $-2 < x < 0$ is where $g_x < 0$.

As we move through Q horizontally from left to right, the numerical labels on the contour lines are decreasing, so $g_x(Q) < 0$. As we move through R horizontally from left to right, the numerical labels on the contour lines are increasing, so $g_x(R) > 0$. Likewise, as we move up through P vertically the numerical labels on the contour lines are increasing, so $g_y(P) > 0$.

Finally, as we move up through R vertically the line of motion is tangent to the level curve through $R = (a, 0)$, and this indicates that $g_y(R) = 0$; more visually, just below and just above R the function values go down to $g(R) = 2$ and then back up again, so $g(a, y)$ has a local minimum at $y = 0$ and hence its derivative vanishes at $y = 0$. But this derivative is $g_y(a, 0) = g_y(R)$, so $g_y(R) = 0$.

4. (10 points) Let $f(x, y, z) = 6xy + z^3$.

(a) (3 points) Determine the tangent plane to the level surface $f(x, y, z) = 2$ through $\mathbf{a} = (1, -1, 2)$. Express your answer in two ways: write an *equation* involving x, y, z; and additionally give a parametric form (this has many possible answers).

Since $f_x = 6y$, $f_y = 6x$, and $f_z = 3z^2$, the tangent plane

$$f_x(\mathbf{a})(x - 1) + f_y(\mathbf{a})(y + 1) + f_z(\mathbf{a})(z - 2) = 0$$

is

$$-6(x - 1) + 6(y + 1) + 12(z - 2) = 0.$$

Equivalently (upon cancelling 6 throughout to clean it up — *not* necessary), this is

$$-(x - 1) + (y + 1) + 2(z - 2) = 0.$$

Upon combining constant terms (also not necessary), this says

$$-x + y + 2z = 2.$$

A parametric form is $\mathbf{a} + s\mathbf{v} + t\mathbf{w}$ for any two non-collinear displacement vectors. For example, $Q = (-2, 0, 0)$ and $R = (0, 2, 0)$ are two points in the plane other than $\mathbf{a}$ (we just picked two "easy" solutions to the equation of the plane), and their displacements from $\mathbf{a}$ are

$$\mathbf{v} = Q - \mathbf{a} = \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix}, \quad \mathbf{w} = R - \mathbf{a} = \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix},$$

so a parametric form is

$$\begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix} + r \begin{bmatrix} -3 \\ 1 \\ -2 \end{bmatrix} + t \begin{bmatrix} -1 \\ 3 \\ -2 \end{bmatrix} = \begin{bmatrix} 1 - 3r - t \\ -1 + r + 3t \\ 2 - 2r - 2t \end{bmatrix}.$$

(b) (7 points) Find the maximal and minimal values of f on the sphere $x^2 + y^2 + z^2 = 36$, and the points at which those extremal values are attained.

The sphere is $g(x, y, z) = 36$ for $g(x, y, z) = x^2 + y^2 + z^2$, and

$$\nabla g = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}$$

only vanishes at the origin, which is not on the sphere. Hence, ∇g is non-vanishing on the sphere, so by the theorem of Lagrange multipliers any extremum $\mathbf{a}$ for f on the sphere must satisfy $(\nabla f)(\mathbf{a}) = \lambda(\nabla g)(\mathbf{a})$ for some scalar λ. Written out explicitly for $\mathbf{a} = (x, y, z)$, this vector equality says

$$\begin{bmatrix} 6y \\ 6x \\ 3z^2 \end{bmatrix} = \lambda \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix} = \begin{bmatrix} 2\lambda x \\ 2\lambda y \\ 2\lambda z \end{bmatrix}.$$

Equating corresponding entries, we arrive at three scalar equations

$$6y = 2\lambda x, \;\; 6x = 2\lambda y, \;\; 3z^2 = 2\lambda z$$

along with the constraint equation $x^2 + y^2 + z^2 = 36$.

Solving for λ in each of those equations, we get

$$\frac{3y}{x} = \lambda, \;\; \frac{3x}{y} = \lambda, \;\; \frac{3z^2}{2z} = \lambda$$

with the understanding that each such equation only makes sense when its denominator is *nonzero*. So first we address cases with a vanishing denominator. If $x = 0$ then $6y = 2\lambda x = 0$, so $y = 0$; likewise, if $y = 0$ then $6x = 2\lambda y = 0$, so $x = 0$. In such cases with $x = 0$ and $y = 0$, the constraint $g = 36$ forces $z = \pm 6$, so we have the candidate points $(0, 0, \pm 6)$. Setting that aside for now, we may assume $x, y \neq 0$, so

$$\frac{3y}{x} = \lambda = \frac{3x}{y},$$

and cross-multiplying (and cancelling 3) gives $y^2 = x^2$, so $y = \pm x$. Hence, $\lambda = 3y/x = \pm 3$ with same sign as for the relation $y = \pm x$. The final multiplier equation $3z^2 = 2\lambda z = \pm 6z$ then gives that either $z = 0$ or (upon cancelling a *nonzero* z) $3z = \pm 6$, so $z = \pm 2$ with the same sign as for the relation $y = \pm x$. To summarize, when $x, y \neq 0$ the point has the form $(x, \pm x, 0)$ or $(x, \pm x, \pm 2)$ where in the latter case the signs are *the same*. We need to figure out the possibilities for x.

Bringing in the constraint equation $g = 36$, the point $(x, \pm x, 0)$ must satisfy $2x^2 = 36$, so $x^2 = 18$ or equivalently $x = \pm 3\sqrt{2}$. In other words, we get the four points $(3\sqrt{2}, \pm 3\sqrt{2}, 0)$ and $(-3\sqrt{2}, \pm 3\sqrt{2}, 0)$. If instead we're at $(x, \pm x, \pm 2)$ (with the same sign) then the constraint equation $g = 36$ forces $2x^2 + 4 = 36$, or equivalently $x^2 = 16$, so $x = \pm 4$. Hence, we get the points $(4, 4, 2), (4, -4, -2), (-4, 4, -2), (-4, -4, 2)$.

Finally, we evaluate $f(x, y, z) = 6xy + z^3$ at each of our candidates and thereby identity the biggest and smallest values. We have $f(0, 0, \pm 6) = \pm 6^3 = \pm 216$, and with unrelated signs $f(\pm 3\sqrt{2}, \pm 3\sqrt{2}, 0) = \pm 6(3\sqrt{2})^2 = \pm 108$. Finally,

$$f(4,4,2) = 6(16) + 8 = 96 + 8 = 104, \quad f(-4,-4,2) = 96 + 8 = 104,$$

$$f(4,-4,-2) = -96 - 8 = -104, \quad f(-4,4,-2) = -96 - 8 = -104.$$

Inspecting these results, the largest and smallest values are 216 and -216 respectively, attained at the points $(0, 0, 6)$ and $(0, 0, -6)$ respectively.

5. (10 points) Let V be the plane $x + y + z = 0$ in $\mathbf{R}^3$ through the origin, so V has an orthogonal basis $\{\mathbf{v}, \mathbf{w}\}$ for $\mathbf{v} = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix}$. Let $L : \mathbf{R}^3 \to \mathbf{R}^3$ be the function $L(\mathbf{x}) = \mathbf{Proj}_V(\mathbf{x})$.

(a) (4 points) Compute the 3×3 matrix A for L; the entries should be fractions with denominator 3. (Hint: what is the meaning of each column?)

The jth column is $L(\mathbf{e}_j)$, and in general for any $\mathbf{x} \in \mathbf{R}^3$ we have

$$\mathbf{Proj}_V(\mathbf{x}) = \frac{\mathbf{x} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}} \mathbf{v} + \frac{\mathbf{x} \cdot \mathbf{w}}{\mathbf{w} \cdot \mathbf{w}} \mathbf{w}.$$

Setting $\mathbf{x} = \mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3$ respectively, we obtain

$$\mathbf{Proj}_V(\mathbf{e}_1) = \frac{1}{2}\mathbf{v} + \frac{1}{6}\mathbf{w} = \begin{bmatrix} 1/2 + 1/6 \\ -1/2 + 1/6 \\ -2/6 \end{bmatrix} = \begin{bmatrix} 2/3 \\ -1/3 \\ -1/3 \end{bmatrix},$$

$$\mathbf{Proj}_V(\mathbf{e}_2) = \frac{-1}{2}\mathbf{v} + \frac{1}{6}\mathbf{w} = \begin{bmatrix} -1/2 + 1/6 \\ 1/2 + 1/6 \\ -2/6 \end{bmatrix} = \begin{bmatrix} -1/3 \\ 2/3 \\ -1/3 \end{bmatrix},$$

$$\mathbf{Proj}_V(\mathbf{e}_3) = 0\mathbf{v} + \frac{-2}{6}\mathbf{w} = \begin{bmatrix} -1/3 \\ -1/3 \\ 2/3 \end{bmatrix}.$$

Assembling these together,

$$A = \begin{bmatrix} 2/3 & -1/3 & -1/3 \\ -1/3 & 2/3 & -1/3 \\ -1/3 & -1/3 & 2/3 \end{bmatrix}.$$

(b) (3 points) For $\mathbf{a} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$, compute $\mathbf{Proj}_V(\mathbf{a})$ in two ways: using the orthogonal basis $\{\mathbf{v}, \mathbf{w}\}$ for

V, and using the matrix-vector product against your answer in (a). (You should get the same answer both ways, a vector with integer entries.)

The general formula gives the projection as

$$\frac{\mathbf{a}\cdot\mathbf{v}}{2}\mathbf{v}+\frac{\mathbf{a}\cdot\mathbf{w}}{6}\mathbf{w}=-\mathbf{v}-\mathbf{w}=\begin{bmatrix}-2\\0\\2\end{bmatrix}.$$

The matrix-vector product is

$$A\mathbf{a}=\begin{bmatrix}2/3&-1/3&-1/3\\-1/3&2/3&-1/3\\-1/3&-1/3&2/3\end{bmatrix}\begin{bmatrix}1\\3\\5\end{bmatrix}=\begin{bmatrix}2/3-1-5/3\\-1/3+2-5/3\\-1/3-1+10/3\end{bmatrix}=\begin{bmatrix}-3/3-1\\2-6/3\\9/3-1\end{bmatrix}=\begin{bmatrix}-2\\0\\2\end{bmatrix},$$

the same as the output of the direct calculation.

(c) (3 points) The geometric definition of $\mathbf{Proj}_V$ gives that its output lies in V, on which $\mathbf{Proj}_V$ has no effect, so $\mathbf{Proj}_V\circ\mathbf{Proj}_V=\mathbf{Proj}_V$. Check that your answer A in (a) satisfies the corresponding matrix equality $A^2=A$. (Hint: if you write $A=(1/3)B$ for a matrix B with integer entries then the calculation will be cleaner.)

We have $A=(1/3)B$ for the matrix

$$B=\begin{bmatrix}2&-1&-1\\-1&2&-1\\-1&-1&2\end{bmatrix},$$

so $A^2=(1/9)B^2$. By direct matrix multiplication we have

$$B^2=\begin{bmatrix}4+1+1&-2-2+1&-2+1-2\\-2-2+1&1+4+1&1-2-2\\-2+1-2&1-2-2&1+1+4\end{bmatrix}=\begin{bmatrix}6&-3&-3\\-3&6&-3\\-3&-3&6\end{bmatrix},$$

and by factoring out the factor of 3 we see this is $3B$. Hence, dividing by 9 gives that $(1/9)B^2$ is equal to $(1/9)(3B)=(1/3)B=A$, as desired.

6. (10 points) For each of the 30 weeks of the academic year at a certain university, first-year undergraduates are in one of two types: those who plan to major in computer science, and everyone else. Let's call these two types of students "CS" and "non-CS", and assume that during each week a student changes their type at most once (and may change again in subsequent weeks). Let $\mathbf{p}_n=\begin{bmatrix}x_n\\y_n\end{bmatrix}$ be the 2-vector whose entries x_n and y_n are the proportions of such students of type non-CS and CS respectively at the end of the nth week (so $x_n+y_n=1$ always).

(a) (3 points) Among those of type CS at the start of each week suppose 90% remain that way at the end of the week but 10% switch to non-CS. Among those of type non-CS at the start of each week suppose 85% remain that way at the end of the week but 15% switch to CS. Write down an explicit 2×2 Markov matrix M for which $\mathbf{p}_{n+1}=M\mathbf{p}_n$ for all n.

Since the first entry of $\mathbf{p}_n$ corresponds to non-CS and the second entry corresponds to CS, if we write

$$M=\begin{bmatrix}p&1-q\\1-p&q\end{bmatrix}$$

then p is the frequency with which type non-CS stays that way during a week and q is likewise for type CS. Hence,

$$M = \begin{bmatrix} .85 & .1 \\ .15 & .9 \end{bmatrix}.$$

(b) (3 points) Using your answer to (a), what proportion of students who are type CS at the end of a given week are also type CS two weeks later (they may have switched to non-CS and back in the meantime)?

We need to compute the $(2, 2)$-entry of

$$M^2 = \begin{bmatrix} .85 & .1 \\ .15 & .9 \end{bmatrix} \begin{bmatrix} .85 & .1 \\ .15 & .9 \end{bmatrix},$$

which is $(.15)(.1) + (.9)(.9) = .015 + .81 = .825$; i.e, it is 82.5% (lower than 90%!).

(c) (4 points) If you computed M correctly then it turns out that to an accuracy of two decimal digits for all $m \geq 17$ we have

$$M^m \approx \begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix}.$$

Interpret in words what this means, and also interpret in words the fact (verified by direct calculation) that for *any* $0 \leq x \leq 1$ we have

$$\begin{bmatrix} .4 & .4 \\ .6 & .6 \end{bmatrix} \begin{bmatrix} x \\ 1-x \end{bmatrix} = \begin{bmatrix} .4 \\ .6 \end{bmatrix}.$$

The power M^m has entries that are the transition probabilities between two (possibly equal) states after m weeks have passed (from a given week). Hence, the (approximate) equality given for M^m for all $m \geq 17$ says that after 17 or more weeks have passed, students of *each* type have a 60% chance of winding up in type CS (this corresponds to *both* entries in the bottom row being .6) and a 40% chance of winding up in type non-CS.

The second equality expresses the same fact because we can interpret $\begin{bmatrix} x \\ 1-x \end{bmatrix}$ as a "population vector" recording the propertions of each type (x of type non-CS, and $1 - x$ of type CS) at the end of some week and then applying the matrix gives use the proportions after 17 or more weeks have passed: it will be 40% non-CS and 60% CS.