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Problem 1: Linear approximation

Consider the function $f : \mathbf{R}^2 \to \mathbf{R}$ defined by $f(x,y) = 4x + y^3 + xy$.

(a) Compute the gradient $(\nabla f)(x,y)$, and then use it to give the linear approximation to f at $(1,1)$.

(b) Using your answer to (a), estimate $f(0.9, 1.2)$. Compare your answer to the exact result on a calculator, and compare the effort in computing the approximation by hand versus the exact answer by hand.

(c) Give the linear approximation to f at $(2,-2)$. Use it to estimate $f(3,-1)$, and then compare this estimate to the exact value using a calculator. Why is the approximation so bad?

Solution:

(a) By computing partial derivatives of f, we have $(\nabla f)(x,y) = \begin{bmatrix} 4+y \\ 3y^2+x \end{bmatrix}$, so $(\nabla f)(1,1) = \begin{bmatrix} 5 \\ 4 \end{bmatrix}$. Hence, for s, t near 0 we have

$$f(1+s, 1+t) \approx f(1,1) + \begin{bmatrix} 5 \\ 4 \end{bmatrix} \cdot \begin{bmatrix} s \\ t \end{bmatrix} = 6 + 5s + 4t,$$

or equivalently

$$f(x,y) \approx 6 + 5(x-1) + 4(y-1) = 5x + 4y - 3$$

for (x,y) near $(1,1)$.

(b) Using $s = -0.1$ and $t = .2$, we have

$$f(0.9, 1.2)) \approx 6 + 5(-0.1) + 4(.2) = 6 - 0.5 + 0.8 = 6.3.$$

The exact value on a calculator is 6.408, so we got a pretty good approximation considering that carrying out the exact calculation by hand is rather more tedious due to the cubing involved.

(c) By (a), $(\nabla f)(2,-2) = \begin{bmatrix} 2 \\ 14 \end{bmatrix}$, so the linear approximation at $(2,-2)$ says that for s, t near 0 we have

$$f(2+s, -2+t) \approx f(2,-2) + \begin{bmatrix} 2 \\ 14 \end{bmatrix} \cdot \begin{bmatrix} s \\ t \end{bmatrix} = -4 + 2s + 14t.$$

If we plug in $s = 1$ and $t = 1$ (which are not close to 0!) then the "approximating" expression is

$$-4 + 2(1) + 14(1) = -4 + 2 + 14 = 12.$$

But $f(3,-1) = 12 - 1 - 3 = 8$, so the value 12 is not a good approximation. The reason it is not surprising that this is a bad approximation is that we are using values s and t that are not close to 0.

Problem 2: Finding tangents to implicit curves and surfaces

For each of the following, find the equation of the tangent line to the given curve or the tangent plane to the given surface at the specified point $\mathbf{a}$.

(a) $x^3 + y^2 = 31$ at $\mathbf{a} = (3,2)$.

(b) $xz^2 + y^2z^5 = 19$ at $\mathbf{a} = (3,4,1)$.

Solution:

(a) For $f(x, y) = x^3 + y^2$ we have $f_x = 3x^2$ and $f_y = 2y$, so $f_x(3, 2) = 27$ and $f_y(3, 2) = 4$. Hence, the equation of the tangent line is $0 = 27(x - 3) + 4(y - 2) = 27x - 81 + 4y - 8 = 27x + 4y - 89$. In other words, this is the line $27x + 4y = 89$.

(b) For $g(x, y, z) = xz^2 + y^2z^5$ we have

$$g_x = z^2, \;\; g_y = 2yz^5, \;\; g_z = 2xz + 5y^2z^4.$$

Hence, the equation of the tangent plane at $(3, 4, 1)$ is

$$\begin{aligned} 0 = g_x(3,4,1)(x-3) + g_y(3,4,1)(y-4) + g_z(3,4,1)(z-1) &= 1(x-3) + 8(y-4) + (6+80)(z-1) \\ &= x - 3 + 8y - 32 + 86z - 86 \\ &= x + 8y + 86z - 121. \end{aligned}$$

In other words, the plane is $x + 8y + 86z = 121$.

Problem 3: Tangent planes: graphs versus level sets

Let S be the sphere of radius 3 centered at the origin in $\mathbf{R}^3$. Let's consider two approaches to finding the equation of the tangent plane to S at the point $(2, 2, 1)$.

(a) For the surface graph $z = f(x, y)$ of a function $f(x, y)$, its tangent plane at a point $(a, b, f(a, b))$ is given by the equation $z = L(x, y)$ where $L(x, y)$ is the linear approximation to f at (a, b). Describe the upper half ($z > 0$) of the sphere S as a graph of a function of x and y, and use this to compute the equation of the tangent plane to S at the point $(2, 2, 1)$ in that upper hemisphere.

(b) The surface S is also a level set of $g(x, y, z) = x^2 + y^2 + z^2$ at a certain level c (what is the value of c?). Use the approach via gradients to compute the tangent plane to S at $(2, 2, 1)$. Verify that this is the same plane as you found in (a). (Note: the equation might not literally be the same as in (a) even though the solution sets to the equations – which are what actually matter – are the same, much as $2x - 2y + 2z = 0$ and $x - y + z = 0$ define the same plane; why?)

(c) Which method was easier? Do you have any thoughts about which method should usually be easier?

Solution:

(a) When $z > 0$ we have $z = \sqrt{9 - x^2 - y^2}$ on the sphere S, so the upper hemisphere is the graph of $f(x, y) = \sqrt{9 - x^2 - y^2}$. We want to compute its linear approximation at $(2, 2)$, so first we calculate

$$(\nabla f)(x, y) = \begin{bmatrix} -x/\sqrt{9 - x^2 - y^2} \\ -y/\sqrt{9 - x^2 - y^2} \end{bmatrix}.$$

Hence, $(\nabla f)(2, 2) = \begin{bmatrix} -2 & -2 \end{bmatrix}$, so the linear approximation at $(2, 2)$ is

$$f(2, 2) + \begin{bmatrix} -2 \\ -2 \end{bmatrix} \cdot \begin{bmatrix} x - 2 \\ y - 2 \end{bmatrix} = 1 - 2(x - 2) - 2(y - 2) = 9 - 2x - 2y.$$

The tangent plane to S at $(2, 2, 1) = (2, 2, f(2, 2))$ is therefore given by $z = 9 - 2x - 2y$, or equivalently

$$2x + 2y + z = 9.$$

(b) By the 3-dimensional Pythagorean theorem, S is the level set $g(x, y, z) = 3^2 = 9$. The equation of its tangent plane at $(2, 2, 1)$ is given by the vanishing of

$$(\nabla g)(x, y, z) \cdot \begin{bmatrix} x - 2 \\ y - 2 \\ z - 1 \end{bmatrix},$$

and we calculate

$$(\nabla g)(x,y,z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix},$$

so $(\nabla g)(2,2,1) = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix}$. Hence, the equation of the tangent plane is

$$0 = \begin{bmatrix} 4 \\ 4 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} x-2 \\ y-2 \\ z-1 \end{bmatrix} = 4(x-2) + 4(y-2) + 2(z-1) = 4x + 4y + 2z - 18,$$

which is to say

$$4x + 4y + 2z = 18.$$

This is not literally the same equation as obtained in (a), but rather is related to it via multiplying both sides by 2. Multiplying both sides of an equation by a nonzero scalar has no impact on where the equation holds, so the two equations do indeed define the same plane (as we know they must).

(c) The method based on the gradient of a level set is probably easier in general since to use the graph description we usually have to "solve for z" in some implicit equation $h(x,y,z) = c$, and such implicit solutions are usually algebraically a bit of a mess (involving inverse functions such as square roots, cube roots, inverse trig functions, and so on).

Problem 4: Constrained optimization

In what follows, you may accept that $f(x,y) = xy$ attains maximal and minimal values on the curve $x^2 - xy + y^2 = 3$.

(a) Use the method of Lagrange multipliers to find these extreme values and the point(s) where they are attained.

(b) The quadratic formula allows you to solve for y in terms of x on the curve: $y(x) = (x \pm \sqrt{x^2 - 4(x^2-3)})/2 = (x \pm \sqrt{12 - 3x^2})/2$ (with $|x| \leq 2$ so that the square root makes sense). Hence, we could instead try to find the extreme values for $f(x, y(x)) = x \cdot y(x) = (x^2 \pm x\sqrt{12 - 3x^2})/2$ for $-2 \leq x \leq 2$ via single-variable calculus. Is that more or less appetizing than the method in (a)?

Solution:

(a) Let $g(x,y) = x^2 - xy + y^2$, so we seek the local extrema for f on the level set $g = 3$ and then will compare the values of f at those points to see which are largest and smallest.

The method of Lagrange multipliers says that if (a,b) is a point on the level curve where f attains a local extremum then $(\nabla f)(a,b)$ is a scalar multiple of $(\nabla g)(a,b)$ provided that $(\nabla g)(a,b) \neq \mathbf{0}$. So our first step is to compute the gradients of f and g at a general point (x,y) and to figure out where (if anywhere) ∇g vanishes on the level set $g = 3$ (such bad points will have to be analyzed separately).

We compute $\nabla f = \begin{bmatrix} y \\ x \end{bmatrix}$ and $\nabla g = \begin{bmatrix} 2x - y \\ -x + 2y \end{bmatrix}$. Where does ∇g vanish? This is the pair of simultaneous conditions $2x - y = 0$ and $-x + 2y = 0$, the only solution to which is $(0,0)$ (this is "2 equations in 2 unknowns" as considered in high school algebra). Since $(0,0)$ isn't on the level set $g(x,y) = 3$ (since $g(0,0) = 0$), there are no bad points to worry about. So at any local extremum (x,y) for f on $g = 3$ we must have $(\nabla f)(x,y) = \lambda(\nabla g)(x,y)$ for some scalar λ, which is to say

$$\begin{bmatrix} y \\ x \end{bmatrix} = \lambda \begin{bmatrix} 2x - y \\ -x + 2y \end{bmatrix}.$$

Hence, we want to find all solutions to the combined system of equations

$$y = \lambda(2x - y), \quad x = \lambda(-x + 2y), \quad 3 = x^2 - xy + y^2$$

(where λ is also unknown, and we don't really care so much about its value); the final condition in this combined list of three equations is what records that we are finding a point on the level set $g = 3$ of interest.

We will use the first two equations in our combined system to get two different expressions for λ that we then compare to get another condition on x and y that we exploit in conjunction with the third equation. But please always remember: *never divide by* 0. In other words, the first two equations in the combined system give rise to two expressions for λ as

$$\frac{y}{2x-y} = \lambda = \frac{x}{-x+2y},$$

but to write these requires knowing that the denominators don't vanish: $2x - y \neq 0$ and $-x + 2y \neq 0$.

So let's first handle the problematic cases where $2x - y = 0$ **or** $-x + 2y = 0$ (this is not "and", but "or": make sure you understand the distinction, and why it matters for a systematic analysis of all possibilities!). If $2x - y = 0$ then the first equation forces $y = 0$ (regardless of what λ is), but then the condition $2x - y = 0$ forces $x = 0$ too, and we've already discussed that $(0, 0)$ isn't on the level set (i.e., it violates the third equation in our combined system).

Hence, for our purposes we can assume $2x - y \neq 0$. The exact same reasoning works in the case $-x + 2y = 0$ via the second equation in our combined system. Hence, we can also assume $-x + 2y \neq 0$, so our two expressions for λ are both valid. Equating them says

$$\frac{y}{2x-y} = \frac{x}{-x+2y},$$

and cross-multiplying gives $y(-x+2y) = x(2x-y)$, or equivalently $-xy+2y^2 = 2x^2-xy$, which says $2x^2 = 2y^2$, or in other words $x = \pm y$.

Hence, our conditions are $x = \pm y$ and $3 = x^2 - xy + y^2$ (and then we can solve for λ using our expressions for it, but we actually don't care what λ is for our purposes). Thus, we separately treat each of the cases $x = y$ and $x = -y$ along with the equation $3 = x^2 - xy + y^2$ that is the "level set" condition.

If $x = y$ then $3 = x^2 - xy + y^2 = y^2 - y^2 + y^2 = y^2$, which says $y = \pm\sqrt{3}$. Since $x = y$, we arrive at the candidates $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$.

Next, if $x = -y$ then $3 = (-y)^2 - (-y)y + y^2 = y^2 + y^2 + y^2 = 3y^2$, or equivalently $1 = y^2$, so $y = \pm 1$ and then $x = -y = \mp 1$. In other words, we get two more candidates: $(1, -1)$ and $(-1, 1)$.

Finally, we compute the function $f(x, y) = xy$ at each of these 4 candidates: $f(\sqrt{3}, \sqrt{3}) = 3$, $f(-\sqrt{3}, -\sqrt{3}) = 3$, $f(1, -1) = -1$ and $f(-1, 1) = -1$. Inspecting which of these values are biggest and smallest, we conclude that the maximum for xy on $g = 3$ is 3, attained at the points $(\sqrt{3}, \sqrt{3})$ and $(-\sqrt{3}, -\sqrt{3})$, and the minimum for xy on $g = 3$ is -1, attained at the points $(1, -1)$ and $(-1, 1)$.

(b) Definitely less appetizing!

Problem 5: Optimization review (what technique(s) would you use?)

(a) Given the function $f(x, y) = x + y$, find the maximum and minimum values of f on the domain

$$D_1 = \{(x, y)\colon 0 \leq y \leq x^2 \text{ and } -1 \leq x \leq 1\}.$$

(b) Find the maximum and minimum values of $G(x, y) = 3x^2 + 4xy$ on the region

$$D_2 = \{(x, y)\colon y \geq 0 \text{ and } x^2 + y^2 \leq 9\}.$$

(When doing this, one part of the boundary will be a mess via single-variable calculus, so employ Lagrange multipliers there with the boundary curve as a constraint condition. You may encounter the expression $2x^2 - 3xy - 2y^2$, in which case it will be useful to then observe that this factors as $(2x + y)(x - 2y)$.)

(c) (**Extra**) Let C be the curve in $\mathbf{R}^2$ defined by the equation

$$y^2 = x^3 - 4x^2 + 5x$$

Determine all points on C at minimal distance to $(5/2, 0)$.

Solution:

(a) We'll search the following regions:

$$\begin{array}{ccccc} \text{interior} & \text{top edge} & \text{bottom edge} & \text{left edge} & \text{right edge} \\ \begin{bmatrix} 0 < y < x^2 \\ -1 < x < 1 \end{bmatrix} & \begin{bmatrix} y = x^2 \\ -1 \le x \le 1 \end{bmatrix} & \begin{bmatrix} y = 0 \\ -1 \le x \le 1 \end{bmatrix} & \begin{bmatrix} x = -1 \\ 0 \le y \le 1 \end{bmatrix} & \begin{bmatrix} x = 1 \\ 0 \le y \le 1 \end{bmatrix} \end{array}$$

interior. First we compute that $\nabla f = \begin{bmatrix} 1 \\ 1 \end{bmatrix} \neq \mathbf{0}$, so f has no critical points in the interior.

For potential extrema on the boundary, we use single-variable calculus and treat the boundary parts one at a time.

top edge. Here we have $f = x + x^2$ across the interval $-1 \le x \le 1$. We have $\dfrac{d}{dx}(x + x^2) = 1 + 2x$, which equals zero when $x = -1/2$, a value in the interval in question. Thus, this point $(x, y) = (-1/2, 1/4)$ and the endpoints $(-1, 1)$ and $(1, 1)$ are put on the list of candidates for where the extrema may be attained.

bottom edge. Here we have $f = x$ across the interval $-1 \le x \le 1$. Although $\dfrac{d}{dx}(x) = 1$ never vanishes, we must put the endpoints $(-1, 0)$ and $(1, 0)$ on the list of candidates.

left edge. Here we have $f = -1 + y$ across the interval $\{0 \le y \le 1\}$. Since the single-variable derivative of this expression never vanishes, we just need to include the endpoints $(-1, 0)$ and $(-1, 1)$ among the candidates.

right edge. Here we have $f = 1 + y$ across the interval $\{0 \le y \le 1\}$. As before, therefore we just need to put the endpoints $(1, 0)$ and $(1, 1)$ are among the candidates.

Our overall list of candidates for locations of where f attains extreme values on D_1 are: $(-1/2, 1/4)$, $(-1, 1)$, $(1, 1)$, $(-1, 0)$, $(1, 0)$. By computing the value of f at each of these points, we see that the maximum value is $f(1, 1) = 2$ and the minimum value is $f(-1, 0) = -1$.

(b) We'll search the following regions for candidate extrema:

$$\begin{array}{ccc} \text{interior} & \text{bottom edge} & \text{top semicircle} \\ \begin{bmatrix} x^2 + y^2 < 9 \\ y > 0 \end{bmatrix} & \begin{bmatrix} y = 0 \\ -3 \le x \le 3 \end{bmatrix} & \begin{bmatrix} x^2 + y^2 = 9 \\ y \ge 0 \end{bmatrix} \end{array}$$

interior First we compute that $\nabla G = \begin{bmatrix} 6x + 4y \\ 4x \end{bmatrix}$, which equals $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ only for $(x, y) = (0, 0)$, which lies in D_2, so it has to be on the list of candidates where extreme values might be attained. (This critical point is not actually in the interior — it's on the bottom boundary — so we will see it crop up as a potential extremum along that edge below. But it never hurts to add this point to the list now!)

For potential extrema on the boundary, we use single-variable calculus and treat the boundary parts (bottom edge and top semicircle) one at a time.

bottom edge. Here we have $G = 3x^2$ across the interval $-3 \le x \le 3$. Since $\dfrac{d}{dx}(3x^2) = 6x$ vanishes when $x = 0$, which lies in this interval, we have three points to include in the list of candidates: $(0, 0)$ and the endpoints $(-3, 0)$ and $(3, 0)$.

top semicircle. This is a circular arc rather than a line segment as for the bottom boundary, and trying to use single-variable calculus leads to a mess for the resulting single-variable expression for G. So away from the endpoints of this semicircle we'll employ Lagrange multipliers by viewing that curve as a constraint condition! Here the constraint is $g = x^2 + y^2 = 9$, for which $\nabla g = \begin{bmatrix} 2x \\ 2y \end{bmatrix}$, which never vanishes on the constraint region (it only vanishes at the origin, which is not on $g = 9$).

Thus, there are no "problematic" points for the Lagrange multiplier method and hence we seek points on this semicircle away from its endpoints where $\nabla G = \lambda(\nabla g)$ for some λ. This condition, along with the constraint condition $g = 9$, is expressed by the combined system of equations

$$6x + 4y = 2\lambda x, \quad 4x = 2\lambda y, \quad x^2 + y^2 = 9.$$

As usual, we "solve for λ" in each of the first two equations and set the resulting expressions equal to each other to get a new condition on x and y (that can be combined with the constraint equation). But as always, we must be careful about cases that would lead to division by 0. So we first "solve for λ" in each of the first two equations in the combined system to figure out what are the potentially problematic division-by-zero cases:

$$\frac{6x+4y}{2x} = \lambda = \frac{4x}{2y}.$$

Hence, we first deal with situations where either $x = 0$ or $y = 0$. If $x = 0$ then the first equation in our combined system says $0 + 4y = 0$ (regardless of λ), so $y = 0$, but we have already noted that $(0,0)$ violates the constraint condition $g = 9$, so this possibility cannot occur. Likewise, if $y = 0$ then the second equation in our combined system says $4x = 0$ (regardless of λ), forcing $x = 0$, so again we are at the point $(0,0)$ which has aleady been ruled out. So there is no division-by-zero problem.

Coming back to our two expressions for λ, setting them equal to each other says

$$\frac{6x+4y}{2x} = \frac{4x}{2y},$$

and cross-multiplying converts this into the equation

$$2y(6x+4y) = (2x)(4x),$$

or equivalently $12xy + 8y^2 = 8x^2$, which is the same as $3xy + 2y^2 = 2x^2$. Remembering the constraint equation, we have arrived at two equations on x and y without reference to λ:

$$x^2 + y^2 = 9, \;\; 2x^2 - 3xy - 2y^2 = 0$$

with $x, y \neq 0$. Ah, but we were given the hint to use that $2x^2 - 3xy - 2y^2 = (2x+y)(x-2y)$, whose vanishing happens precisely when $x = 2y$ or $y = -2x$.

So we now treat the possibilities $x = 2y$ or $y = -2x$ separately. If $x = 2y$ then the constraint $x^2 + y^2 = 9$ says $(2y)^2 + y^2 = 9$, or equivalently $5y^2 = 9$, so $y = \pm 3/\sqrt{5}$ and then $x = 2y = \pm 6/\sqrt{5}$ (same sign as for y). If instead $y = -2x$ then $9 = x^2 + y^2 = x^2 + (-2x)^2 = 5x^2$, so $x = \pm 3/\sqrt{5}$ and then $y = -2x = \mp 6/\sqrt{5}$. But recall that we are on the upper semicircle with $y \geq 0$, so we're left with just two points obtained in this way: $(6/\sqrt{5}, 3/\sqrt{5})$ and $(-3/\sqrt{5}, 6/\sqrt{5})$.

Thus, our final list of candidate points for extrema of $G(x,y) = 3x^2 + 4xy$ on D_2 is:

$$(0,0), \;\; (-3,0), \;\; (3,0), \;\; (6/\sqrt{5}, 3/\sqrt{5}), \;\; (-3/\sqrt{5}, 6/\sqrt{5}).$$

We have $G(0,0) = 0$, $G(-3,0) = 27$, $G(3,0) = 27$, $G(6/\sqrt{5}, 3/\sqrt{5}) = 36$, and $G(-3/\sqrt{5}, 6/\sqrt{5}) = 27/5 - 72/5 = -45/5 = -9$. Hence, the maximal value of G on this region is 36 (attained only at $(6/\sqrt{5}, 3/\sqrt{5})$) and the minimal value is -9 (attained only at $(-3/\sqrt{5}, 6/\sqrt{5})$).

(c) We work with the squared distance function to $(5/2, 0)$, namely $f(x,y) = (x - 5/2)^2 + y^2$. The Lagrange Multiplier Theorem states that for $g(x,y) = y^2 - x^3 + 4x^2 - 5x$, the points where the function $f(x,y)$ is minimized on the level set $C = \{g(x,y) = 0\}$ are among the points where either $\nabla f = \lambda \nabla g$, for some real number λ, or where $\nabla g = \mathbf{0}$.

The vanishing of $\nabla g = \begin{bmatrix} -3x^2 + 8x - 5 \\ 2y \end{bmatrix}$ happens precisely when $y = 0$ and $-3x^2 + 8x - 5 = 0$. By the quadratic formula, the latter condition on x has as its solutions exactly $x = 1, 5/3$, so this singles out the points $(1,0)$ and $(5/3, 0)$ for separate treatment. But these points can't occur because neither of them satisfies the constraint condition $g(x,y) = 0$ (indeed, $g(1,0) = -1 + 4 - 5 = -2 \neq 0$ and $g(5/3, 0) = -125/27 + 100/9 - 25/3 = -50/27 \neq 0$).

So now we may focus on the multiplier condition $\nabla f = \lambda(\nabla g)$, which says

$$\begin{bmatrix} 2x - 5 \\ 2y \end{bmatrix} = \lambda \begin{bmatrix} -3x^2 + 8x - 5 \\ 2y \end{bmatrix}.$$

Expressing this as a pair of scalar equations and bringing in the constraint equation $g = 0$, we arrive at the combined system of three equations

$$2x - 5 = \lambda(-3x^2 + 8x - 5), \;\; 2y = \lambda(2y), \;\; y^2 - x^3 + 4x^2 - 5x = 0.$$

As usual, we "solve for λ" in each of the first two equations, assuming denominators don't vanish: we obtain

$$\frac{2x-5}{-3x^2+8x-5} = \lambda = \frac{2y}{2y}.$$

This works as long as both denominators are nonzero, so first we dispoose of the problematic case where some denominator vanishes: either $y = 0$ or $-3x^2 + 8x - 5 = 0$.

Case 1: Suppose $y = 0$. The second multiplier equation then tells us nothing, and we also learn nothing from the first: it has nothing to do with y (and we haven't learned anything about λ yet). But going back to the constraint equation with $y = 0$, we have $0 = g(x, 0) = -x^3 + 4x^2 - 5x = -x(x^2 - 4x + 5)$. $x^2 - 4x + 5 = (x-2)^2 + 1 = 0$ has no real solutions, so $x = 0$ is the only solution. In other words, we have singled out the point $(0, 0)$ as needing separate treatment. We put this into the fridge and will come back to them later.

Case 2: Suppose $-3x^2 + 8x - 5 = 0$, which (by the quadratic formula) says $x = 1, 5/3$. The first multiplier equation then gives $2x - 5 = 0$ (regardless of λ), so $x = 5/2$; this is absurd since we are only allowing $x = 1, 5/3$, so this situation cannot occur.

Now returning to the original combined system (away from the problematic points already identified), we have (with non-vanishing denominators) the two fraction expressions for λ given above. Setting them equal to each other and cross-multiplying gives

$$2y(2x-5) = 2y(-3x^2+8x-5).$$

But we have $y \neq 0$ (since we are avoiding the problematic points for now), so cancelling $2y \neq 0$ from both sides gives $2x - 5 = -3x^2 + 8x - 5$, or equivalently $3x^2 - 6x = 0$, which says $3x(x-2) = 0$, so $x = 0$ or $x = 2$. Then the constraint equation $g(x, y) = 0$ says $y^2 = x^3 - 4x^2 + 5x$ is equal to 0 when $x = 0$ and is equal to $8 - 16 + 10 = 2$ when $x = 2$. In other words, if $x = 0$ then $y = 0$ and if $x = 2$ then $y = \pm\sqrt{2}$. Hence, we obtain the further candidate points $(0, 0)$ and $(2, \pm\sqrt{2})$.

Putting it all together, all local minima for the squared distance $f(x, y)$ on the constraint curve $g = 0$ must occur among the following points on the constraint curve: $(0, 0)$, $(2, \pm\sqrt{2})$. Evaluating $f(x, y) = (x - 5/2)^2 + y^2$ at these points gives the values $f(0, 0) = 25/4$, $f(2, \pm\sqrt{2}) = 1/4 + 2 = 9/4$. The least value among these is $9/4$, attained only at the points $(2, \pm\sqrt{2})$, so these two points on C attain the least distance to $(5/2, 0)$, with minimal distance equal to $\sqrt{9/4} = 3/2$.

Problem 6: Identifying linear functions

In each case below, is $\mathbf{f} : \mathbf{R}^2 \to \mathbf{R}^3$ linear? If it is, find the matrix representing it. If not, explain why not, by exhibiting the failure of $\mathbf{f}$ to behave well with respect to vector addition or scalar multiplication on some specific inputs.

(a) $\mathbf{f}(x_1, x_2) = (x_1, x_2^2, 2x_1 + x_2)$

(b) $\mathbf{f}(x_1, x_2) = (1, x_2, 2x_1 + x_2)$

(c) $\mathbf{f}(x_1, x_2) = (0, x_2, 2x_1 + x_2)$

(d) $\mathbf{f}(x_1, x_2) = (0, x_1 x_2, 2x_1 + x_2)$

(e) $\mathbf{f}(x_1, x_2) = (ax_1 + bx_2, cx_1 + dx_2, ex_1 + fx_2)$

Solution: (c) is linear, coming from the matrix $\begin{bmatrix} 0 & 0 \\ 0 & 1 \\ 2 & 1 \end{bmatrix}$; (e) is linear, coming from the matrix $\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix}$.

The rest are not linear, since some vector entry in the output is not a linear combination of the input variables and so cannot arise as a matrix-vector product against $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$. More concretely, for (a) we have bad behavior for addition due to the appearance of x_2^2, leading us to try

$$\mathbf{f}(0,1) + \mathbf{f}(0,1) = (0,1,1) + (0,1,1) = (0,2,2), \quad \mathbf{f}((0,1)+(0,1)) = \mathbf{f}(0,2) = (0,4,2),$$

which are not equal (so linearity fails for (a)). Likewise, for (b) we have $\mathbf{f}(0,0) = (1,0,0) \neq (0,0,0)$, so it cannot be linear (as linear functions always carry the origin to the origin). Finally, for (d) we have addition issues with the entry x_1x_2, so will proceed similarly to (a) but using (1,1):

$$\mathbf{f}(1,1) + \mathbf{f}(1,1) = (0,1,3) + (0,1,3) = (0,2,6), \quad \mathbf{f}((1,1)+(1,1)) = \mathbf{f}(2,2) = (0,4,6),$$

which are not equal (so linearity fails for (d)).

Problem 7: Derivative matrix and numerical linear approximation

Consider the function $f : \mathbf{R}^2 \to \mathbf{R}^2$ given by

$$f(x,y) = (x^3y^2, 4x + y^3 + xy).$$

(a) Compute the derivative matrix $(Df)(x,y)$, and then use it to give the linear approximation to f at $(1,1)$.

(b) Use your answer to (a) to estimate the 2-vector $f(0.8, 1.1)$, and then compare it with an exact calculation using a calculator. Is it a good approximation?

(c) Give the linear approximation to f at $(2,-2)$ and use it to estimate the 2-vector $f(2.1,-1.9)$ and then compare this to the exact 2-vector using a calculator. Is the approximation good or bad?

Solution:

(a) By computing partial derivatives of the component functions of f, we have $(Df)(x,y) = \begin{bmatrix} 3x^2y^2 & 2x^3y \\ 4+y & 3y^2+x \end{bmatrix}$, so $(Df)(1,1) = \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix}$. Hence, for s, t near 0 we have

$$f(1+s, 1+t) \approx f(1,1) + \begin{bmatrix} 3 & 2 \\ 5 & 4 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 1 \\ 6 \end{bmatrix} + \begin{bmatrix} 3s+2t \\ 5s+4t \end{bmatrix} = \begin{bmatrix} 1+3s+2t \\ 6+5s+4t \end{bmatrix}.$$

(b) We set $s = -0.2$ and $t = 0.1$ in the linear approximation in (a) to get the estimate

$$f(0.8, 1.1) \approx \begin{bmatrix} 1 - 0.6 + 0.2 \\ 6 - 1.0 + 0.4 \end{bmatrix} = \begin{bmatrix} 0.6 \\ 5.4 \end{bmatrix}.$$

An exact calculation with a calculator gives that $f(0.8, 1.1) = \begin{bmatrix} .61952 \\ 5.411 \end{bmatrix}$, so the approximation turned out quite well!

(c) By (a), $(Df)(2,-2) = \begin{bmatrix} 48 & -32 \\ 2 & 14 \end{bmatrix}$, so the linear approximation at $(2,-2)$ says that for s,t near 0 we have

$$f(2+s, -2+t) \approx f(2,-2) + \begin{bmatrix} 48 & -32 \\ 2 & 14 \end{bmatrix} \begin{bmatrix} s \\ t \end{bmatrix} = \begin{bmatrix} 32 \\ -4 \end{bmatrix} + \begin{bmatrix} 48s - 32t \\ 2s + 14t \end{bmatrix} = \begin{bmatrix} 32 + 48s - 32t \\ -4 + 2s + 14t \end{bmatrix}.$$

Hence, with $s = 0.1$ and $t = 0.1$ we get

$$f(2.1, -1.9) \approx \begin{bmatrix} 32 + 48(0.1) - 32(0.1) \\ -4 + 2(0.1) + 14(0.1) \end{bmatrix} = \begin{bmatrix} 32 + 4.8 - 3.2 \\ -4 + 0.2 + 1.4 \end{bmatrix} = \begin{bmatrix} 33.6 \\ -2.4 \end{bmatrix}.$$

Using a calculator, we have the exact result $f(2.1, -1.9) = (33.43221, -2.449)$. So in this case the linear approximation is still not bad.

Problem 8: Checking for Linearity

For the following functions F, analyze their interaction with vector addition and scalar multiplication to determine if they are linear or not. If not linear, give an explicit pair of vectors $\mathbf{v}, \mathbf{w}$ for which $F(\mathbf{v}+\mathbf{w}) \neq F(\mathbf{v}) + F(\mathbf{w})$ or an explicit vector $\mathbf{v}$ and scalar c for which $F(c\mathbf{v}) \neq c\,F(\mathbf{v})$.

(a) $f(x) = \begin{bmatrix} 2x \\ 2x+3 \end{bmatrix}$ (b) $g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x + y.$ (c) $h\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \\ x^2 \end{bmatrix}$ (d) $k\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 2x \\ y \\ 0 \end{bmatrix}$

Solution:

(a) We have $f(0) = \begin{bmatrix} 0 \\ 3 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. So f is not linear (e.g., $f(0 \cdot 0) \neq 0\,f(0)$). Alternatively, if you compute $f(x_1 + x_2)$ and $f(x_1) + f(x_2)$ algebraically then you'll see they are *never* the same:

$$f(x_1+x_2) = \begin{bmatrix} 2(x_1+x_2) \\ 2(x_1+x_2)+3 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ 2x_1+2x_2+3 \end{bmatrix}, \quad f(x_1)+f(x_2) = \begin{bmatrix} 2x_1 \\ 2x_1+3 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ 2x_2+3 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ 2x_1+2x_2+6 \end{bmatrix},$$

with the second entries always differing by 3.

(b) We check

$$g\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = g\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = (x_1+x_2) + (y_1+y_2)$$

and

$$g\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + g\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = (x_1+y_1) + (x_2+y_2),$$

whose right sides are equal. Hence, $g(\mathbf{v}+\mathbf{w}) = g(\mathbf{v}) + g(\mathbf{w})$ for every $\mathbf{v}, \mathbf{w} \in \mathbf{R}^2$. Likewise, for every scalar c we have $g(c\begin{bmatrix} x \\ y \end{bmatrix}) = g(\begin{bmatrix} cx \\ cy \end{bmatrix}) = cx + cy$ and $c\,g\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = c(x+y)$, so again the right sides coincide and we conclude that $g(c\mathbf{v}) = c\,g(\mathbf{v})$ for every $\mathbf{v} \in \mathbf{R}^2$.

Hence, g is linear (as we can also see from the component functions, but the point is to understand linearity through the interaction with vector operations).

(c) This does not behave well for addition (nor for scalar multiplication). If we work it out explicitly,

$$h\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = h\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(x_1+x_2) \\ y_1+y_2 \\ (x_1+x_2)^2 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ y_1+y_2 \\ x_1^2+2x_1x_2+x_2^2 \end{bmatrix}$$

whereas

$$h\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + h\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ y_1 \\ x_1^2 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ y_2 \\ x_2^2 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ y_1+y_2 \\ x_1^2+x_2^2 \end{bmatrix},$$

which differ by $2x_1x_2$ in the final entry. To make an explicit counterexample, we just make sure $x_1, x_2 \neq 0$, such as:

$$h\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} + \begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = h\left(\begin{bmatrix} 2 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ 0 \\ 4 \end{bmatrix}, \quad h\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) + h\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 2 \end{bmatrix}.$$

(d) This goes similarly to (b):

$$k\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix} + \begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = k\left(\begin{bmatrix} x_1+x_2 \\ y_1+y_2 \end{bmatrix}\right) = \begin{bmatrix} 2(x_1+x_2) \\ y_1+y_2 \\ 0 \end{bmatrix}$$

and

$$k\left(\begin{bmatrix} x_1 \\ y_1 \end{bmatrix}\right) + k\left(\begin{bmatrix} x_2 \\ y_2 \end{bmatrix}\right) = \begin{bmatrix} 2x_1 \\ y_2 \\ 0 \end{bmatrix} + \begin{bmatrix} 2x_2 \\ y_2 \\ 0 \end{bmatrix} = \begin{bmatrix} 2x_1+2x_2 \\ y_1+y_2 \\ 0 \end{bmatrix};$$

the right sides are equal, so $k(\mathbf{v}+\mathbf{w}) = k(\mathbf{v}) + k(\mathbf{w})$ for every $\mathbf{v}, \mathbf{w} \in \mathbf{R}^2$.

Likewise, for every scalar c we have

$$k\left(c\begin{bmatrix}x\\y\end{bmatrix}\right) = k\left(\begin{bmatrix}cx\\cy\end{bmatrix}\right) = \begin{bmatrix}2(cx)\\cy\\0\end{bmatrix}, \quad c\,k\left(\begin{bmatrix}x\\y\end{bmatrix}\right) = c\begin{bmatrix}2x\\y\\0\end{bmatrix} = \begin{bmatrix}c(2x)\\cy\\0\end{bmatrix},$$

and the outcomes of both calculations agree. Hence, $k(c\mathbf{v}) = c\,k(\mathbf{v})$ for every $\mathbf{v} \in \mathbf{R}^2$ and every scalar c.

We conclude that k is linear (as can also be seen from the component functions).

Problem 9: Composition of linear maps and matrix multiplication

(a) Let $R : \mathbf{R}^2 \to \mathbf{R}^2$ be the operation that carries any vector $\mathbf{v}$ to its reflection across the y-axis. Explain in words or with a picture why $R\left(\begin{bmatrix}a\\b\end{bmatrix}\right) = \begin{bmatrix}-a\\b\end{bmatrix}$ for any a, b, so R is linear. Then calculate the 2×2 matrix A for which $R(\mathbf{v}) = A\mathbf{v}$ for every $\mathbf{v} \in \mathbf{R}^2$.

(b) Find the matrix corresponding to the linear transformation that first rotates a vector in $\mathbf{R}^2$ by an angle of α counterclockwise and then reflects across the y-axis. Then find the matrix corresponding to the opposite order of these operations (it is a different matrix when $\alpha \neq 0, \pi$, which is to say when $\sin\alpha \neq 0$).

(c) Draw a picture to illustrate visually (so without any appeal to matrices) why the composition of the two operations in (b) (rotation and reflection) depends on the order in which they are carried out when $\alpha \neq 0, \pi$.

Solution:

(a) The reflection of $\mathbf{v}$ across the y-axis corresponds to drawing a perpendicular line segment from the tip of $\mathbf{v}$ to the y-axis, so really a horizontal segment, and then continuing horizontally across the y-axis for the same distance on the other side. This process stays on the same horitzonal line, so the y-coordinate does not change. However, the x-coordinate is flipped across 0 with the same magnitude, so it is negated. These two conclusions express together exactly the desired formula for R. From this formula, the matrix is $A = \begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}$ (or separately calculate $R(\mathbf{e}_1)$ and $R(\mathbf{e}_2)$ to compute the columns of A).

(b) We compute the matrix product $\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix}\begin{bmatrix}\cos\alpha & -\sin\alpha\\\sin\alpha & \cos\alpha\end{bmatrix} = \begin{bmatrix}-\cos\alpha & \sin\alpha\\\sin\alpha & \cos\alpha\end{bmatrix}$. Composing the operations in the other order corresponds to the matrix product in the other order: $\begin{bmatrix}\cos\alpha & -\sin\alpha\\\sin\alpha & \cos\alpha\end{bmatrix}\begin{bmatrix}-1 & 0\\0 & 1\end{bmatrix} = \begin{bmatrix}-\cos\alpha & -\sin\alpha\\-\sin\alpha & \cos\alpha\end{bmatrix}$. By inspection of the upper right and lower left entries, these two matrix products are different precisely when $\sin\alpha \neq 0$, which is to say $\alpha \neq 0, \pi$.

(c) Figure 1 demonstrates the transformation of rotating first and then reflecting:

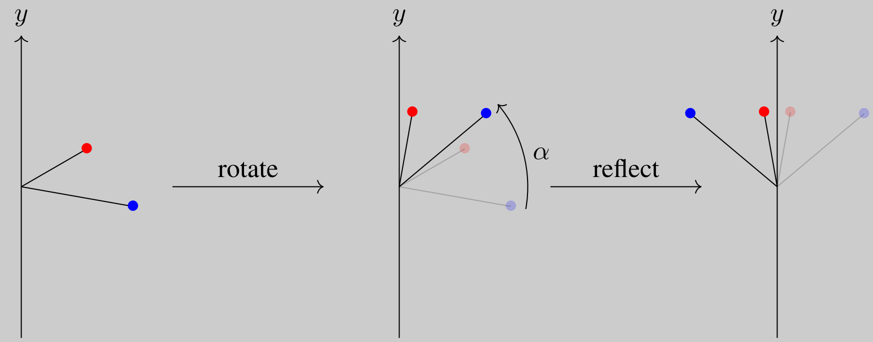

Figure 1: First rotating by α, and then reflecting.

On the other hand, Figure 2 demonstrates the transformation of *reflecting* first and then rotating:

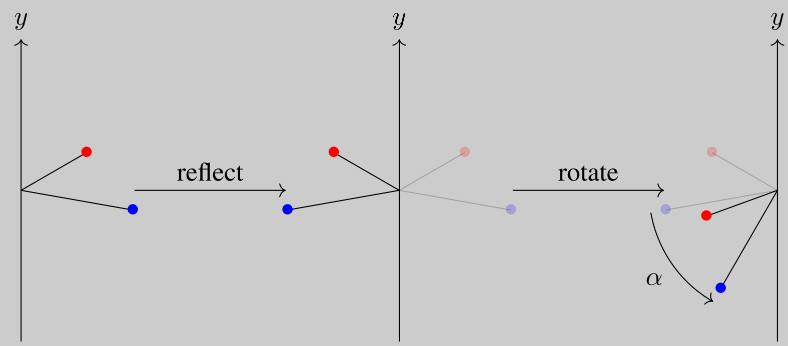

Figure 2: First reflecting, and then rotating by α.

It is clear from the figures that the two operations are not the same.

Problem 10: Matrix multiplication

(a) Compute the following matrix products.

$$\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 9 & 11 \\ 2 & 5 & 6 \end{bmatrix} = \boxed{\begin{bmatrix} 1 & 9 & 11 \\ 0 & -13 & -16 \end{bmatrix}}$$

$$\begin{bmatrix} 1 & 9 \\ 2 & 5 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 90 \\ 4 & 50 \end{bmatrix}}$$

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 9 & 11 \\ 2 & 5 & 6 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 5 & 6 \\ 1 & 9 & 11 \end{bmatrix}}$$

$$\begin{bmatrix} 2 & 0 \\ 0 & 10 \end{bmatrix} \begin{bmatrix} 1 & 9 \\ 2 & 5 \end{bmatrix} = \boxed{\begin{bmatrix} 2 & 18 \\ 20 & 50 \end{bmatrix}}$$

$$\left(\begin{matrix}\text{for } \mathbf{v}, \mathbf{w} \text{ two} \\ n\text{-vectors}\end{matrix}\right) \quad \begin{bmatrix} v_1 & \cdots & v_n \end{bmatrix} \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix} = \boxed{[\mathbf{v} \cdot \mathbf{w}]}$$

$$\begin{bmatrix} 2 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 3 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \boxed{\begin{bmatrix} 2a & 2b & 2c \\ -d & -e & -f \\ 3g & 3h & 3i \end{bmatrix}}$$

(b) Let $q(x, y, z) = x^2 + 2y^2 - z^2 - 3xy + 4xz + yz$. Find values of a, b, c, d, e, f that satisfy

$$\begin{bmatrix} x & y & z \end{bmatrix} \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = q(x, y, z)$$

for every x, y, z. Strictly speaking, the left side multiplies out to be a 1×1 matrix and the equality means that the scalar $q(x, y, z)$ on the right side is the unique entry in that matrix. (Hint: multiply the left side fully, and compare coefficients on the two sides, such as for x^2, yz, etc.)

(c) (**Extra**) Is there a version of (b) for any $q(x, y, z) = Ax^2 + By^2 + Cz^2 + Dxy + Exz + Fyz$ in general?

Solution:

(a) See boxed answers above.

(b) Carrying out the matrix multiplication of the firsr two matrices on the left, we get

$$\begin{bmatrix} ax + dy + ez & dx + by + fz & ex + fy + cz \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix},$$

and this is the 1×1 matrix whose entry is

$$(ax + dy + ez)x + (dx + by + fz)y + (ex + fy + cz)z = ax^2 + dyx + ezx + dxy + by^2 + fzy + exz + fyz + cz^2.$$

But on the right side each of xy, xz, and yz appears twice with the same coefficient, so combining terms yields

$$ax^2 + by^2 + cz^2 + 2dxy + 2exz + 2fyz.$$

Equating this to the given $q(x, y, z)$ and comparing coefficients yields $a = 1$, $b = 2$, $c = -1$, $d = -3/2$, $e = 2$, $f = 1/2$.

(c) $a = A$, $b = B$, $c = C$, $d = D/2$, $e = E/2$, $f = F/2$.

Problem 11: Algebra and geometry with hyperbolas

This problem studies hyperbolas, using some algebra and thinking (more instructive than a computer) to determine the basic geometry of a hyperbola from an equation. This will be useful later when relating contour plots to the multivariable second derivative test. You do *not* need a calculator for this exercise; human brainpower is sufficient!

(a) For $A, B, C > 0$, the equations $Ax^2 - By^2 = \pm C$ (one for each sign) are hyperbolas with asymptotes $y = \pm\sqrt{A/B}\,x$. For the pair of hyperbolas $H_\pm$ in each of (i) and (iii) below (treating each $\pm$ case separately), or else in each of (ii) and (iv) below (treating each $\pm$ case separately), compute:

- which of the coordinate axes each crosses (the x-axis consists of points $(u, 0)$, the y-axis consists of points $(0, v)$),
- the slopes $\pm m$ of the asymptotes,
- which coordinate axis is "nearer" to the asymptotes (note: if the slope c of a line $y = cx$ satisfies $|c| < 1$ then the line is "closer" to the x-axis, whereas if $|c| > 1$ then the line is "closer" to the y-axis; draw the cases $c = \pm 2, \pm 1/2$ to see why this is reasonable).

(i) $x^2 - 6y^2 = \pm 10$ (ii) $3x^2 - 5y^2 = \pm 13$ (iii) $-7x^2 + 2y^2 = \pm 18$ (iv) $-5x^2 + y^2 = \pm 21$

(b) Use the information found in (a) to approximately draw each pair of hyperbolas (both signs) on a common coordinate grid (one grid for each pair), indicating the approximate axis crossings and the asymptotes drawn "closer" to the appropriate coordinate axis (a qualitatively correct picture is sufficient).

(c) Based on your work, for a general hyperbola $Ax^2 - By^2 = C$ with $C \neq 0$ and both A and B with the same sign, how does the coordinate axis (x or y) to which the asymptotes are "closer" related to how $|A|$ compares to $|B|$ (in terms of which is bigger and which is smaller)?

Solution:

(a) We treat each pair of hyperbolas in turn.

(i) The x-intercept corresponds to setting $y = 0$, so it is $x^2 = \pm 10$, which has no solution for -10. Thus, H_- does not meet the x-axis, whereas H_+ meets the x-axis at solutions of $x^2 = 10$, which is to say $\pm\sqrt{10}$. The y-intercept corresponds to setting $x = 0$, so it is $-6y^2 = \pm 10$. This has no solution for 10, so H_+ does not meet the y-axis, whereas H_- meets the y-axis at solutions of $-6y^2 = -10$, which is to say $y^2 = (-10)/(-6) = 5/3$, or equivalently $y = \pm\sqrt{5/3}$.

Since $9 < 10 < 16$ and $1 < 5/3 < 4$ (as $5/3$ is between 1 and 2), we have $\pm\sqrt{10}$ lies between ± 3 and ± 4 and likewise $\pm\sqrt{5/3}$ lies between ± 1 and ± 2. The asymptotes are $x^2 - 6y^2 = 0$, which is to say $y^2 = (1/6)x^2$, so $y = \pm\sqrt{1/6}\,x$. These asymptotes have slopes $\pm\sqrt{1/6}$ with absolute value < 1, so they are "closer" to the x-axis.

(ii) The x-intercept corresponds to setting $y = 0$, so it is $3x^2 = \pm 13$, which has no solution for -13. Thus, H_- does not meet the x-axis, whereas H_+ meets the x-axis at solutions of $3x^2 = 13$, which is to say $x = \pm\sqrt{13/3}$. The y-intercept corresponds to setting $x = 0$, so it is $-5y^2 = \pm 13$. This has no solution for 13, so H_+ does not meet the y-axis, whereas H_- meets the y-axis at solutions of $-5y^2 = -13$, which is to say $y^2 = 13/5$; i.e., $y = \pm\sqrt{13/5}$.

Since $4 < 13/3 < 9$ (as $13/3$ lies between 4 and 5) and $1 < 13/5 < 4$ (as $13/5$ lies between 2 and 3), we have $\pm\sqrt{13/3}$ lies between ± 2 and ± 3, and likewise $\pm\sqrt{13/5}$ lies between ± 1 and ± 2. The asymptotes are $3x^2 - 5y^2 = 0$, which is to say $y^2 = (3/5)x^2$, so $y = \pm\sqrt{3/5}\,x$. These asymptotes have slopes $\pm\sqrt{3/5}$ with absolute value < 1, so "closer" to the x-axis.

(iii) The x-intercept corresponds to setting $y = 0$, so it is $-7x^2 = \pm 18$, which has no solution for 18. Thus, H_+ does not meet the x-axis, whereas H_- meets the x-axis at solutions of $-7x^2 = -18$; i.e., $x = \pm\sqrt{18/7}$. The y-intercept corresponds to setting $x = 0$, so it is $2y^2 = \pm 18$. This has no solution for -18, so H_- does not meet the y-axis, whereas H_+ meets the y-axis at solutions of $2y^2 = 18$, which is to say $y^2 = 9$; i.e., $y = \pm 3$.

Since $1 < 18/7 < 4$ (as $18/7$ lies between 2 and 3), we have $\pm\sqrt{18/7}$ lies between ± 1 and ± 2. The asymptotes are $-7x^2 + 2y^2 = 0$, which is to say $y^2 = (7/2)x^2$, so $y = \pm\sqrt{7/2}\,x$. These asymptotes have slopes $\pm\sqrt{7/2}$ with absolute value > 1, so "closer" to the y-axis.

(iv) The x-intercept corresponds to setting $y = 0$, so it is $-5x^2 = \pm 21$, which has no solution for 21. Thus, H_+ does not meet the x-axis, whereas H_- meets the x-axis at solutions of $-5x^2 = -21$, which is to say $x^2 = (-21)/(-5) = 21/5$; i.e., $x = \pm\sqrt{21/5}$. The y-intercept corresponds to setting $x = 0$, so it is $y^2 = \pm 21$. This has no solution for -21, so H_- does not meet the y-axis, whereas H_+ meets the y-axis at solutions of $y^2 = 21$, which is to say $y = \pm\sqrt{21}$.

Since $4 < 21/5 < 9$ (as $21/5$ lies between 4 and 5), we have $\pm\sqrt{21/5}$ lies between ± 2 and ± 3. Since $16 < 21 < 25$, we have $\pm\sqrt{21}$ lies between ± 4 and ± 5. The asymptotes are $-5x^2 + y^2 = 0$, which is to say $y^2 = 5x^2$, so $y = \pm\sqrt{5}\,x$. These asymptotes have slopes $\pm\sqrt{5}$ with absolute value > 1, so "closer" to the y-axis.

(b) The pictures for (i) and (ii) are respectively on the left and right in Figure 3 (all that is expected is approximate accuracy in the axis crossings and qualitative accuracy with the asymptotes being "closer" to the correct coordinate axis).

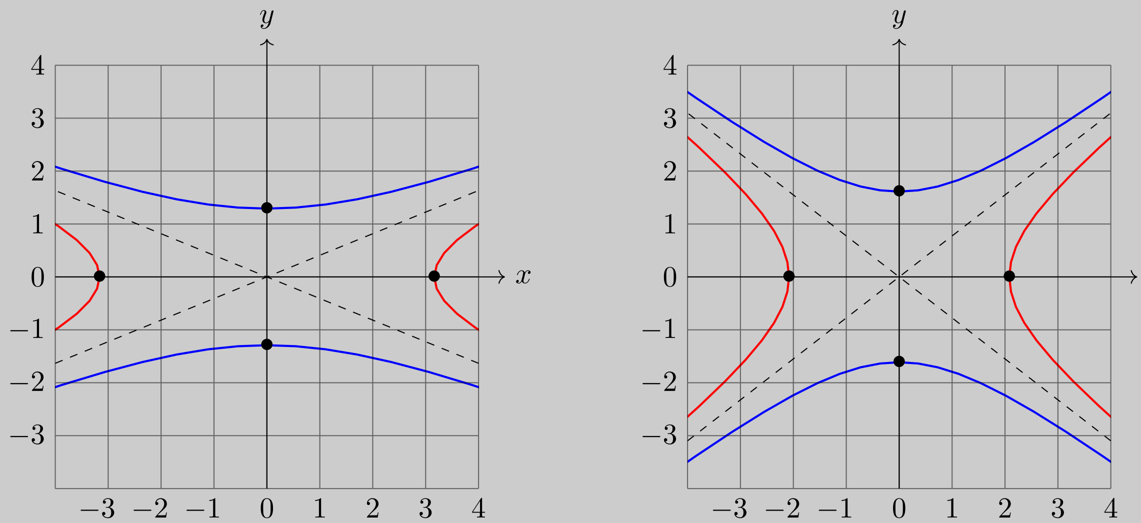

Figure 3: The hyperbolas $x^2 - 6y^2 = 10$ (red) and $x^2 - 6y^2 = -10$ (blue) on the left, and $3x^2 - 5y^2 = 13$ (red) and $3x^2 - 5y^2 = -13$ (blue) on the right

The picture for (iii) and (iv) are respectively on the left and right in Figure 4 (all that is expected is approximate accuracy in the axis crossings and qualitative accuracy with the asymptotes being "closer" to the correct coordinate axis).

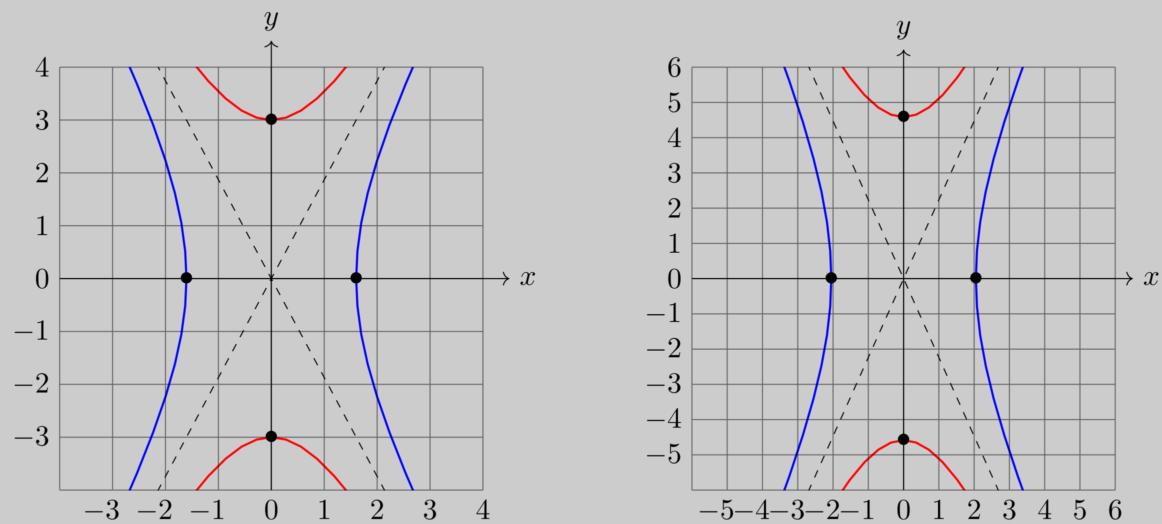

Figure 4: The hyperbolas $-7x^2 + 2y^2 = 18$ (red) and $-7x^2 + 2y^2 = -18$ (blue) on the left, and $-5x^2 + y^2 = 21$ (red) and $-5x^2 + y^2 = -21$ (blue) on the right.

(c) The pattern is that in all cases, whichever of x^2 or y^2 has its coefficient with the *smaller* absolute value corresponds to the coordinate axis to which the asymptotes are *closer*.