1. (a) The region S is exactly the level set $f(x, y, z) = 0$ for
$$f(x, y, z) = x^3 + z^3 + 3y^2z^3 + 5xy.$$
It is also the level set $h(x, y, z) = 7$ for $h(x, y, z) = x^3 + z^3 + 3y^2z^3 + 5xy + 7$, and the level set $F(x, y, z) = 0$ for $F = 13f$ (or with any other nonzero scalar in place of 13).

(b) Computing, we see that S is described alternatively by the equation
$$z = \sqrt[3]{-\frac{x^3 + 5xy}{3y^2 + 1}},$$
so if $g(x, y)$ denotes the expression on the right side then S is the graph of this g. $\Diamond$

2. (a) Computing, we see that
$$\left(\frac{e^t + e^{-t}}{2}\right)^2 - \left(\frac{e^t - e^{-t}}{2}\right)^2 = \frac{e^{2t} + 2 + e^{-2t}}{4} - \frac{e^{2t} - 2 + e^{-2t}}{4} = \frac{2 - (-2)}{4} = 1,$$
so every point in the output of g lies on the hyperbola $x^2 - y^2 = 1$.

(b) Not every point on this hyperbola is of the form $g(t)$, because the x-coordinate $(e^t + e^{-t})/2$ of the formula for $g(t)$ is always positive. For example, $(-1, 0)$ lies on the hyperbola but is not in the output of g (and likewise for any point $(a, \pm\sqrt{a^2 - 1})$ for any $a \leq -1$). $\Diamond$

3. (a) We calculate
$$\begin{aligned}(g \circ f)(\theta, \phi) &= (\cos\theta \sin\phi)^2 + (\sin\theta \sin\phi)^2 + (\cos\phi)^2 \\ &= \sin^2\phi(\cos^2\theta + \sin^2\theta) + \cos^2\phi \\ &= \sin^2\phi + \cos^2\phi \\ &= 1.\end{aligned}$$

(b) By the formula for distance from the origin in $\mathbf{R}^3$, the unit sphere is exactly the level surface $g = 1$. Since the composition $g \circ f$ always has value 1, all output of f lies in that level set. $\Diamond$

4. Each part of the solution below will refer to part of the following figure:

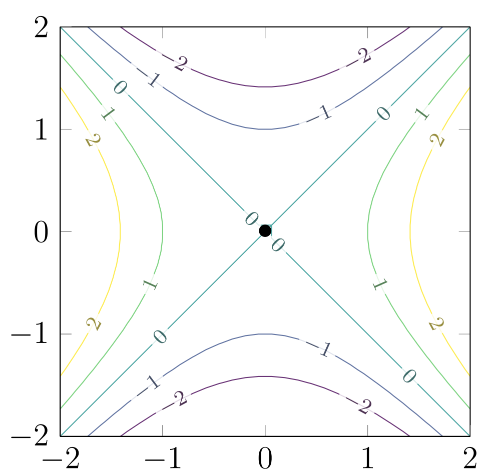

FIGURE 1. A contour plot of $g(v, w) = v^2 - w^2$. In this figure the v-axis is horizontal and the w-axis is vertical.

(a) The level set $g = 0$ is given by $v^2 = w^2$ which is exactly the condition $w = \pm v$. This consists of the diagonal lines $v = w$ and $v = -w$ shown in Figure 1.

(b) The level set $v^2 - w^2 = 1$ is a hyperbola that cuts the line $w = 0$ at $v = \pm 1$, which is to say a hyperbola passing through the points $(\pm 1, 0)$ on the v-axis. (It doesn't cut the line $v = 0$, since $-w^2 = 1$ has no solution in $\mathbf{R}$.) Since $v^2 = w^2 + 1$, if either of $|v|$ or $|w|$ is large then both are large (their squares only differ by 1), and then $v^2 \approx w^2$ (large numbers differing by 1), so passing to square roots gives $|v| \approx |w|$, which is to say $v \approx \pm w$. This explains the "asymptotic behavior" towards the lines $v = \pm w$ on the level set far from the origin, shown in green in Figure 1.

(c) The level set $v^2 - w^2 = -1$ is a hyperbola that cuts the line $v = 0$ at $w = \pm 1$, which is to say a hyperbola passing through the points $(0, \pm 1)$ on the w-axis. (It doesn't cut the line $w = 0$, since $v^2 = -1$ has no solution in $\mathbf{R}$.) The exact same reasoning in (b) applies concerning the asymptotes. Alternatively: $v^2 - w^2 = -1$ is the same as $w^2 - v^2 = 1$, so this literally *is* the situation of (b) except with the roles of v and w swapped, which amounts to reflecting the vw-plane across the line $v = w$. Hence, we get the picture of (b) but flipped across the line $v = w$, shown in blue in Figure 1.

(d) For any $c > 0$ we have $g(v/\sqrt{c}, w/\sqrt{c}) = (1/c)(v^2 - w^2)$, so this is equal to 1 exactly when $v^2 - w^2 = c$. Hence, the level set $g = c$ is obtained from the level set $g = 1$ by multiplying both coordinates by $\sqrt{c}$ (that transforms $(v/\sqrt{c}, w/\sqrt{c})$ into (v, w)). So it is a scaled version of the hyperbola in (b); an example is shown in yellow in Figure 1.

(e) For any $c < 0$ we have $\sqrt{|c|}^2 = |c| = -c$, so $g(v/\sqrt{|c|}, w/\sqrt{|c|}) = (-1/c)(v^2 - w^2)$, so this is equal to -1 precisely when $v^2 - w^2 = c$. Hence, the level set $g = c$ is obtained from the level set $g = -1$ by multiplying both coordinates by $\sqrt{|c|}$ (for reasons similar to the solution of (c)); an example is shown in purple in Figure 1.

(f) By (c) and (d), the level sets $g = c$ for $c \neq 0$ are hyperbolas in specific quadrants. Putting these all together, we obtain the contour plot in Figure 1 in which we have labeled each level set by the value of g on that level set. $\diamondsuit$

5. (a) We rewrite $f(x, y) = 12xy(1 + 4y^2)^{-1}$. Then we can treat x as a constant and use the product rule, followed by the Chain Rule:

$$\begin{aligned}\frac{\partial f}{\partial y}(x, y) &= 12x(1 + 4y^2)^{-1} + 12xy \cdot \left(\frac{d}{dy}(1 + 4y^2)^{-1}\right)\\ &= 12x(1 + 4y^2)^{-1} - 12xy(1 + 4y^2)^{-2} \cdot 8y.\end{aligned}$$

Cleaning up the answer by putting everything over a common denominator, we obtain

$$\frac{\partial f}{\partial y} = \frac{12x(1 + 4y^2) - 96xy^2}{(1 + 4y^2)^2} = \frac{12x - 48xy^2}{(1 + 4y^2)^2} = \frac{12x(1 - 4y^2)}{(1 + 4y^2)^2}.$$

We evaluate $\frac{\partial f}{\partial y}(0, 0) = 0/(1^2) = 0$ and $\frac{\partial f}{\partial y}(2, 0) = (24 \cdot (1))/(1^2) = 24$. Hence, the latter is greater.

(b) We treat y as a constant and use the Chain Rule to differentiate with respect to x:

$$\begin{aligned}g_x(x, y) = e^{-x^2-2xy-2y^2} \cdot \frac{\partial}{\partial x}\left(-x^2 - 2xy - 2y^2\right) &= e^{-x^2-2xy-2y^2} \cdot (-2x - 2y)\\ &= -2(x + y)e^{-x^2-2xy-2y^2}.\end{aligned}$$

Now, we evaluate $g_x(1,1) = -2\cdot(2)e^{-1-2-2} = -4e^{-5}$ and $g_x(-1,-1) = -2\cdot(-2)e^{-1-2-2} = 4e^{-5}$. The former is negative, and the latter positive, so the latter is greater (although they have the same magnitude). $\diamondsuit$

6. (a) The partial derivative $f_{x_1}(-1,5)$ is the derivative of $f(x_1,5)$ (as a function of just x_1) at $x_1 = -1$. In particular, for all h near 0, the change $f(-1+h,5) - f(-1,5)$ is approximately equal to $h \cdot f_{x_1}(-1,5)$. Thus, we can approximate $f(-1.2,5)$ by computing $f_{x_1}(-1,5)$ and setting $h = -0.2$:

$$f_{x_1}(x_1,x_2)\Big|_{(-1,5)} = \left(\frac{1}{2\sqrt{x_1 + 2x_2^2}}\right)\Big|_{(-1,5)} = \frac{1}{2\sqrt{49}} = \frac{1}{14}.$$

Therefore,

$$f(-1.2,5) \approx f(-1,5) + (-0.2)\cdot f_{x_1}(-1,5) = 7 - \frac{1}{5\cdot 14} = 7 - \frac{1}{70} \approx 6.99.$$

(b) We will use the approximation

$$g(2,1,-0.9) \approx g(2,1,-1) + (0.1)\cdot\frac{\partial g}{\partial z}(2,1,-1).$$

To compute the right side, we compute $g(2,1,-1) = e^{-2} + 1$ and

$$\frac{\partial g}{\partial z}(2,1,-1) = (xe^{xz})\Big|_{(2,1,-1)} = 2e^{-2}.$$

Putting this together, $g(2,1,-0.9) \approx e^{-2} + 1 + (0.1)\cdot(2e^{-2}) = (1.2)e^{-2} + 1 \approx 1.162.$ $\diamondsuit$

7. (a) If we move to the right (west-to-east) past the point $(1,1)$ while being on the line $y = 1$ then we cross lines going from larger values to smaller values (the shading is getting darker). Thus, $f_x(1,1) < 0$.

(b) Traveling vertically near $(2,0)$, our path of motion appears to be tangent to the contour lines. That is, if we fix $x = 2$ and vary y near 0, the value of f will change very little. In fact, if the vertical line is exactly tangent to the contour line through that point then the instantaneous change there will be zero. In any case, we can say that $\dfrac{\partial f}{\partial y}(2,0) \approx 0$.

(c) If we move north very near the point $(1,2)$ then we cross contour lines going from smaller values to larger values. Alternatively, the shading is getting lighter. Thus, $f_y(1,2) > 0$.

(d) Near the point $(-2,-1)$, if we move in the x-direction (staying on the line $y = -1$) then the function's values increase (the shading is getting lighter). Therefore, $\dfrac{\partial f}{\partial x}(-2,-1) > 0$. $\diamondsuit$

8. (a) $f_x = (2ax + by)/(ax^2 + bxy + cy^2)$ and $f_y = (bx + 2cy)/(ax^2 + bxy + cy^2)$, so

$$xf_x + yf_y = 2\frac{ax^2 + bxy + cy^2}{ax^2 + bxy + cy^2} = 2.$$

(b) Since the factor $y - z$ doesn't involve x, we have

$$\begin{aligned} f_x = (y-z)\frac{\partial}{\partial x}((x-y)(z-x)) &= (y-z)((z-x) - (x-y)) \\ &= (y-z)(-2x + y + z) \\ &= -2x(y-z) + y^2 - z^2. \end{aligned}$$

Likewise, $f_y = (z-x)(-(y-z)+(x-y)) = (z-x)(x-2y+z) = -2y(z-x)+z^2-x^2$ and $f_z = (x-y)((y-z)-(z-x)) = (x-y)(x+y-2z) = -2z(x-y)+x^2-y^2$. Adding them up, the "square difference" parts all cancel out, leaving us with $f_x + f_y + f_z = -2x(y-z)-2y(z-x)-2z(x-y) = -2xy+2xz-2yz+2xy-2xz+2yz = 0$ as desired. $\diamondsuit$

9. (a) We compute first that $f_x = B + 2Dx + Ey + 3Gx^2 + 2Hxy + Iy^2$, and then computing its y-partial yields

$$f_{xy} = E + 2Hx + 2Iy.$$

This vanishes identically precisely when $E = H = I = 0$, as desired.

(b) We compute that $f_x = g'(x)$ since $h(y)$ is independent of x. But this in turn is independent of y, so its y-partial f_{xy} vanishes. $\diamondsuit$

10. (a) We directly compute

$$f_x = y^2 - 4y + x, \;\; f_y = 2xy - 4x.$$

A critical point is therefore a point (x, y) that satisfies the pair of conditions

$$y^2 - 4y + x = 0, \;\; 2xy - 4x = 0.$$

The second equation says $x(2y-4) = 0$, so $x = 0$ or $y = 2$. If $x = 0$, the first equation becomes $y^2 - 4y = 0$, or $y = 0, 4$. If $y = 2$, the first equation becomes $4 - 8 + x = 0$, so $x = 4$. Hence, the critical points are $(0,0), (0,4), (4,2)$ as claimed.

(b) On the line $y = 0$ the function f is $f(x, 0) = (1/2)x^2 + 1$ which has a local minimum at $x = 0$ (it is parabola pointing up). On the line $y = x$ we have $g(x) = f(x,x) = x^3 - (7/2)x^2 + 1$ with $g'(x) = 3x^2 - 7x$ vanishing at $x = 0$ but $g''(x) = 6x - 7$ negative at $x = 0$, so g has a local maximum at $x = 0$. Hence, we have found 2 lines through the origin, on one of which f has a local maximum at the origin and on the other of which f has a local minimum at the origin. Hence, it is a saddle point.

(c) The only critical point on the interior is $(0,0)$, since the other two found in (a) don't lie inside the square. The restriction of f to the boundary segments are the functions

$$f(x,-1) = (1/2)x^2 + 5x + 1, \;\; f(x,1) = (1/2)x^2 - 3x + 1,$$
$$f(-1,y) = -y^2 + 4y + 3/2, \;\; f(1,y) = y^2 - 4y + 3/2$$

with $-1 \le x, y \le 1$. The derivatives are respectively $x+5$, $x-3$, $-2y+4$, and $2y-4$, none of which vanish on $[-1,1]$, so all have constant sign (positive or negative). Consequently, along each edge the extrema occur at the endpoints, so the extrema of f on the entire boundary must be at a corner.

Evaluating at the corners gives

$$f(-1,-1) = -7/2, f(1,-1) = 13/2, f(-1,1) = 9/2, f(1,1) = -3/2.$$

Since $f(0,0) = 1$, we conclude that on the entirely of S, the maximum is $f(1,-1) = 13/2$ and the minimum is $f(-1,-1) = -7/2$. $\diamondsuit$

11. (a) The partial derivatives are

$$f_x = 6xy + 6y, \;\; f_y = 3x^2 + 3y^2 + 6x.$$

The vanishing of the first says $0 = 6xy + 6y = 6y(x+1)$, so $y = 0$ or $x = -1$. The vanishing of the second says (after cancelling 3 throughout) $x^2 + y^2 + 2x = 0$. In case $y = 0$, this second condition says $x^2 + 2x = 0$, or equivalently $x(x+2) = 0$, so $x = 0$ or $x = -2$. In case $x = -1$, this second condition says $1 + y^2 - 2 = 0$, or equivalently $y^2 = 1$, so $y = 1$ or $y = -1$.

Putting it all together, we have obtained 4 points: $(0,0), (-2,0), (-1,1), (-1,-1)$.

(b) Looking at $f(x,y)$ on the lines $y = x$ and $y = -x$ amounts to looking at the functions $f(x,x)$ and $f(x,-x)$. The first of these is $g_1(x) = f(x,x) = 3x^3 + x^3 + 6x^2 = 4x^3 + 6x^2$ and the second is $g_2(x) = f(x,-x) = -3x^3 - x^3 - 6x^2 = -4x^3 - 6x^2$ (which is the negative of $f(x,x)$, a coincidence for this particular f).

We will show that g_1 has a local minimum at $x = 0$ (corresponding to the point $(0,0)$) and that g_2 has a local maximum at $x = 0$ (corresponding to the point $(0,-0) = (0,0)$), so the saddle point property will then be established. Since $g_2(x) = -g_1(x)$, once we have settled the case of g_1 then the case of g_2 will follow immediately since its behavior is exactly the negative of g_1 (and when a function of x is negated, local minima are turned into local maxima because the graph flips upside-down); one could also just repeat for g_2 the calculations we are about to do for g_1, but we have explained why this isn't necessary to do.

Now for the calculations with g_1 at $x = 0$: we have $g_1'(x) = 12x^2 + 12x$, which vanishes at $x = 0$, and $g_1''(x) = 24x + 12$. Since $g_1''(0) = 12$ is positive, it follows that $x = 0$ is a local minimum for g_1 as claimed. ♢

12. (a) Here is a picture of the domain D; note the corners at $(\pm 3, 1)$.

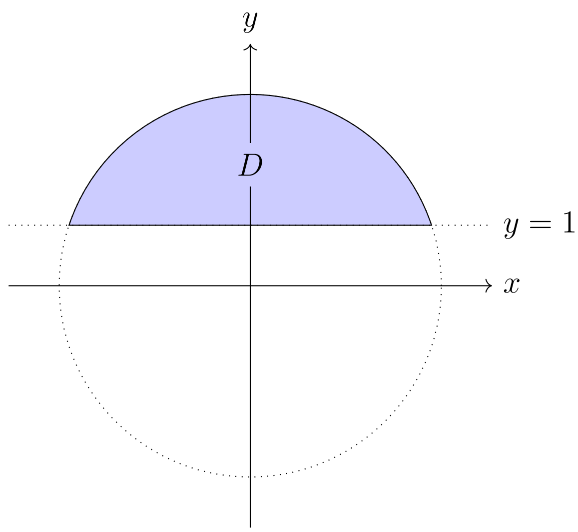

FIGURE 3. The portion of a disk to one side of the line $y = 1$

(b) The partial derivatives are $f_x = 2x + 3y$ and $f_y = 3x + 2y$, and they simultaneously vanish only at $(0,0)$. But this point is not inside D, so there are no critical points in the interior of D.

(c) The boundary consists of the bottom segment $-3 \le x \le 3$ with $y = 1$ and the part of $x^2 + y^2 = 10$ with $y \ge 1$. Along the bottom segment, f becomes $f(x,1) = x^2 + 3x + 1$ and we seek its extrema for $-3 \le x \le 3$. The derivative of $f(x,1)$ is $2x + 3$ that vanishes at $x = -3/2$ at which there is a local minimum, so the candidates for extrema of f along the bottom edge are $(-3/2, 1)$ and the endpoints $(\pm 3, 1)$. The values of f here are $f(-3/2, 1) = -5/4$ (after some algebra) and $f(3,1) = 19$, $f(-3,1) = 1$.

Next, let's look along the circular part of the boundary. On here we have $x^2 + y^2 = 10$, so $f(x,y) = 10 + 3xy = 10 + 3x\sqrt{10 - x^2}$ for $-3 \le x \le 3$. To find its extrema, we compute its x-derivative to be

$$3\sqrt{10 - x^2} + \frac{3x(-x)}{\sqrt{10 - x^2}} = \frac{3(10 - x^2) - 3x^2}{\sqrt{10 - x^2}} = \frac{30 - 6x^2}{\sqrt{10 - x^2}}.$$

This vanishes precisely when $x^2 = 5$, which is to say $x = \pm\sqrt{5}$ (which certainly lies in the interval $[-3, 3]$). For such points on the circular boundary $y^2 = 10 - x^2 = 5$, so $y = \sqrt{5}$ (the negative square root is impossible since $y \geq 1$ now).

The candidates for the extrema are $(-3/2, 1)$, $(3, 1)$, $(-3, 1)$, $(\pm\sqrt{5}, \sqrt{5})$. We evaluated f at all of these except $(\pm\sqrt{5}, \sqrt{5})$. Direct calculation gives that $f(-\sqrt{5}, \sqrt{5}) = -5$ and $f(\sqrt{5}, \sqrt{5}) = 25$. Comparing these against the other values $-5/4$, 1, and 19, the extreme values are $f(-\sqrt{5}, \sqrt{5}) = -5$ as the global minimum and $f(\sqrt{5}, \sqrt{5}) = 25$ as the global maximum. ◊

13. (a) Here is a picture of the domain D.

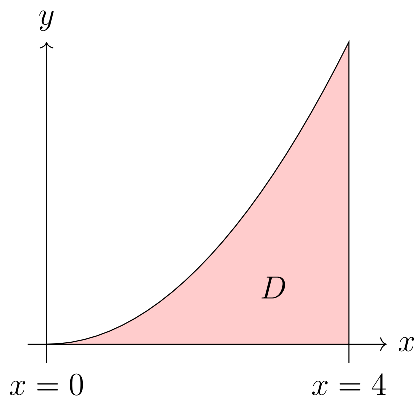

FIGURE 4. A region defined by a parabola and two lines

The partial derivatives are $f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$, so the vanishing of both says $x^2 = y$ and $y^2 = x$. But on the interior we have $y < x^2$, so the vanishing of f_x is impossible here.

(b) The boundary consists of three parts: the parabolic arc $C_1 = \{(x, x^2) : 0 \leq x \leq 4\}$, the segment $C_2 = \{(x, 0) : 0 \leq x \leq 4\}$ in the x-axis, and the segment $C_3 = \{(4, y) : 0 \leq y \leq 4^2 = 16\}$ inside the line $x = 4$. We analyze the behavior of f on each of these in turn, finding extrema on each part and then comparing the values of f at all of them.

On C_1 we have $g_1(x) = f(x, x^2) = x^3 + x^6 - 3x^3 = x^6 - 2x^3$ with $0 \leq x \leq 4$, on C_2 we have $g_2(x) = f(x, 0) = x^3$ with $0 \leq x \leq 4$, and on C_3 we have $g_3(y) = f(4, y) = 64 + y^3 - 12y$ with $0 \leq y \leq 16$. We need to find the extrema for each function on the indicated closed interval, which is a single-variable calculus problem (always remembering to check the endpoints too).

For C_1 we compute $g_1'(x) = 6x^5 - 6x^2 = 6x^2(x^3 - 1)$, which vanishes at $x = 0$ and $x = 1$. So on C_1 we need to check for extrema at the endpoints $x = 0, 4$ and $x = 1$. The values are $g_1(0) = 0$, $g_1(1) = -1$, and $g_1(4) = 3968$. For C_2 we either observe by inspection of x^3 (as an increasing function) or by computing the derivative $3x^2$ noting it doesn't vanish on $(0, 4)$ that the extrema are at the endpoints $x = 0, 4$, with $g_2(0) = 0$ and $g_2(4) = 64$. Finally, for C_3 we compute $g_3'(y) = 3y^2 - 12 = 3(y^2 - 4)$ which vanishes at $y = \pm 2$. Working on the interval $[0, 4]$, we only need to consider its behavior at the interior point $y = 2$ since the endpoints correspond to endpoints of C_1 and C_2 that we have already looked at. The value is $g_3(2) = 48$.

(c) Comparing these values of the g_j's and looking for the biggest and smallest, the biggest value is 3968 attained at $(4, 16)$ and the smallest is -1 attained at $(1, 1)$. ◊

14. The partial derivatives of f are

$$f_x = 3x^2y^2(6 - x - y) - x^3y^2, \qquad f_y = 2x^3y(6 - x - y) - x^3y^2.$$

Setting both to 0, we have

$$3x^2y^2(6-x-y) = x^3y^2, \qquad 2x^3y(6-x-y) = x^3y^2.$$

Since $x, y > 0$, we can cancel x^2y^2 and x^3y respectively in the two equations and obtain

$$3(6-x-y) = x, \qquad 2(6-x-y) = y.$$

This expresses the pair of simultaneous equations

$$4x + 3y = 18, \qquad 2x + 3y = 12,$$

whose only common solution is $(3, 2)$. $\lozenge$

15. (a) Since we are only taking the x-derivative we can fix y. Then this is exactly the single-variable calculus product rule.

    (b) This is a 2-vector equality, so to show it we want to show equality entry-by-entry. For the first entry of the 2-vectors on each side, it is exactly the equality in part (a). For the second entry of the 2-vectors on each side, it is the analogue of part (a) for the partial derivatives in the y-direction:

    $$\frac{\partial h}{\partial y}(x,y) = \frac{\partial f}{\partial y}(x,y)g(x,y) + f(x,y)\frac{\partial g}{\partial y}(x,y),$$

    which holds because for each value of x this is exactly the single-variable calculus product rule (applied to derivatives of functions of y).

    (c) Let $f(x,y) = \sin^2(xy) + \ln(x+y) + y$ and $g(x,y) = e^{xy} + \cos x^3 y$. We calculate $f(0,1) = 1$, $f(x,1) = \sin^2(x) + \ln(x+1) + 1$, and $f(0,y) = \ln(y) + y$, whose respective derivatives with respect to x and y are $2\cos(x)\sin(x) + 1/(x+1)$ and $1 + 1/y$, so $f_x(0,1) = 1$ and $f_y(0,1) = 2$. Hence, $(\nabla f)(0,1) = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$. Likewise, $g(0,1) = 1+1 = 2$, $g(0,y) = e^0 + \cos(0^3 y) = 1+1 = 2$ and $g(x,1) = e^x + \cos(x^3)$, whose respective derivatives with respect to y and x are 0 and $e^x - 3x^2\sin(x^3)\cos(x^3)$, so $g_y(0,1) = 0$ and $g_x(0,1) = 1$. Hence, $(\nabla g)(0,1) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$. Putting this all together via (b), we get

    $$(\nabla h)(0,1) = 2\begin{bmatrix} 1 \\ 2 \end{bmatrix} + 1\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}.$$ $\lozenge$

16. (a) The gradient directions at the points P, Q, R, S, T are shown in the picture below. There is no gradient shown at T because T is a critical point.

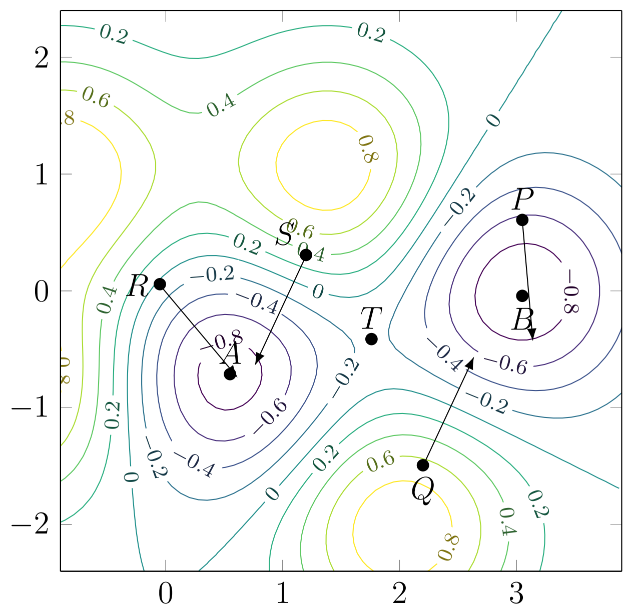

FIGURE 6. The negative gradient directions at P, Q, R, S, T. Note that T is a critical (saddle) point, and so the gradient at T vanishes. The points A and B are local minima.

(b) Gradient descent starting at R or S likely will end at the local minimum A. The gradient descent starting at P or Q likely will end at the local minimum B. The gradient descent starting at T will stay at T forever (since T is a critical point). $\Diamond$

17. The tangent at P to C is parallel to the y-axis precisely when the normal vector to it is parallel to the x-axis, which is to say it has vanishing y-component and nonvanishing x-component. The gradient at P (when nonzero) specifies the direction of the normal vector at P, so in particular we want

$$0 = \frac{\partial}{\partial y}(3yx^2 + 3xy^2 + y^3 + 2x^3) = 3(x^2 + 2xy + y^2) = 3(x + y)^2.$$

This means that at such points P, we must have $x = -y$. We also need these points P on C to satisfy

$$0 \neq \frac{\partial}{\partial x}(3yx^2 + 3xy^2 + y^3 + 2x^3) = 6yx + 3y^2 + 6x^2.$$

This indeed holds if $x = -y$, because then $6yx + 3y^2 + 6x^2 = -6y^2 + 3y^2 + 6y^2 = 3y^2$, which cannot vanish unless both y and x $(= -y)$ vanish; yet the origin doesn't satisfy the equation defining C.

To find the points P on C of the form $(-y, y)$, we plug $x = -y$ into the equation defining C to get $3y^3 - 3y^3 + y^3 - 2y^3 = 27$, which says $y^3 = -27$ or equivalently $y = -3$, so $x = 3$. Hence, there is exactly one such point: $(3, -3)$. $\Diamond$

18. (a) We first find the points of intersection by equating the defining equations:

$$x^2 + y^2 = 1 = (x - 1)^2 + y^2.$$

The quadratic terms cancel, leaving us with $0 = -2x + 1$ so $x = 1/2$. We solve one of the two equations for these circles to find $y = \pm\sqrt{3}/2$, so the points are $(1/2, \pm\sqrt{3}/2)$. This is illustrated in the picture below, where the picture conveys that the crossing points have the form $(1/2, \pm c)$ for some $0 < c < 1$ (in fact $c = \sqrt{3}/2$, but that isn't part of the picture).

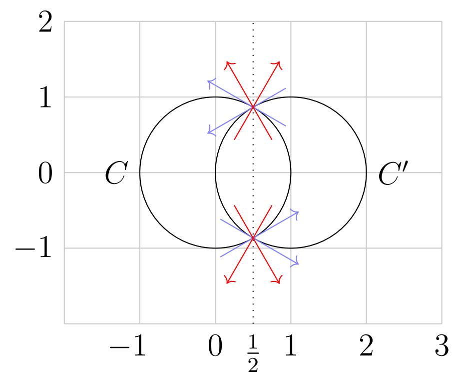

FIGURE 7. Two circles crossing at a pair of points

(b) Let $f(x,y) = x^2 + y^2$ and $g(x,y) = (x-1)^2 + y^2$. The angle between the tangent lines to C and C' at such a point is the *same* as the angle between normal directions. This is because the angle that each normal direction makes with any line through the point is exactly $90°$ plus the angle that the tangent makes with that line.

But the normal directions are given by the gradients of f and g at such a point. Hence we calculate

$$(\nabla f)(1/2, \pm\sqrt{3}/2) = \begin{bmatrix} 1 \\ \pm\sqrt{3} \end{bmatrix}$$

and

$$(\nabla g)(1/2, \pm\sqrt{3}/2) = \begin{bmatrix} -1 \\ \pm\sqrt{3} \end{bmatrix}.$$

These vectors have length 2 (using either sign), so the angle θ between them satisfies

$$\cos\theta = (1/4)\begin{bmatrix} 1 \\ \pm\sqrt{3} \end{bmatrix} \cdot \begin{bmatrix} -1 \\ \pm\sqrt{3} \end{bmatrix} = 1/2.$$

Hence, $\theta = 60°$ (which is acute). $\Diamond$

19. Equality of tangent planes through a common point P is the same as having a common normal direction (by the point-normal form for a plane). The normal direction is given by the gradient of the defining equation (when the gradient is nonzero), so we want the gradients to be nonzero and scalar multiples of each other.

We have

$$(\nabla f)(x,y,z) = \begin{bmatrix} 2x \\ 2y \\ 2z \end{bmatrix}, \quad (\nabla g)(x,y,z) = \begin{bmatrix} 2x \\ 2y \\ 1 \end{bmatrix}.$$

The latter is never zero since its final entry is always 1, and the former only vanishes at the origin, which is not on S (so it cannot be a point in common with S and a level set of f). So both gradients are nonzero at any possible point in common, so the condition of a common tangent plane is the same as saying that ∇f is a scalar multiple of ∇g at such a point.

From inspecting the first two entries in each of the vectors $(\nabla f)(x,y,z)$ and $(\nabla g)(x,y,z)$, as long as one of x or y is nonzero we see that the scalar multiplier must be 1, and then comparing third entries in these gradient vectors forces $z = 1/2$. The equation of S then says $x^2 + y^2 = 1/2$, in which case

$$f(x,y,z) = x^2 + y^2 + z^2 = (x^2 + y^2) + 1/4 = 1/2 + 1/4 = 3/4.$$

The collection of such points is a circle of radius $1/\sqrt{2}$ in the plane $z = 1/2$ centered at the "origin" $(0, 0, 1/2)$ in that plane.

Suppose instead that $x = y = 0$, so the equation of S forces $z = 1$, and hence $P = (0, 0, 1)$. But we forbade this possibility for P in the initial setup. $\Diamond$