Math 51
Lecture 9 – July 13, 2023

Topic(s): partial derivatives

Consider a function $f(x_1, x_2)$ of two variables.
The **partial derivative of f with respect to x_1 at** (a, b), denoted

$$\frac{\partial f}{\partial x_1}(a,b), \qquad \left.\frac{\partial f}{\partial x_1}\right|_{(a,b)}, \qquad \text{or} \qquad f_{x_1}(a,b)$$

means the derivative of the function $f(x_1, b)$ at $x_1 = a$.
The partial derivative of f with respect to x_1 at (a, b) is the instantaneous rate of change of f at the point (a, b) if we *only move in the x_1-direction* (x_2 is held constant, at the value b). Mathematically,

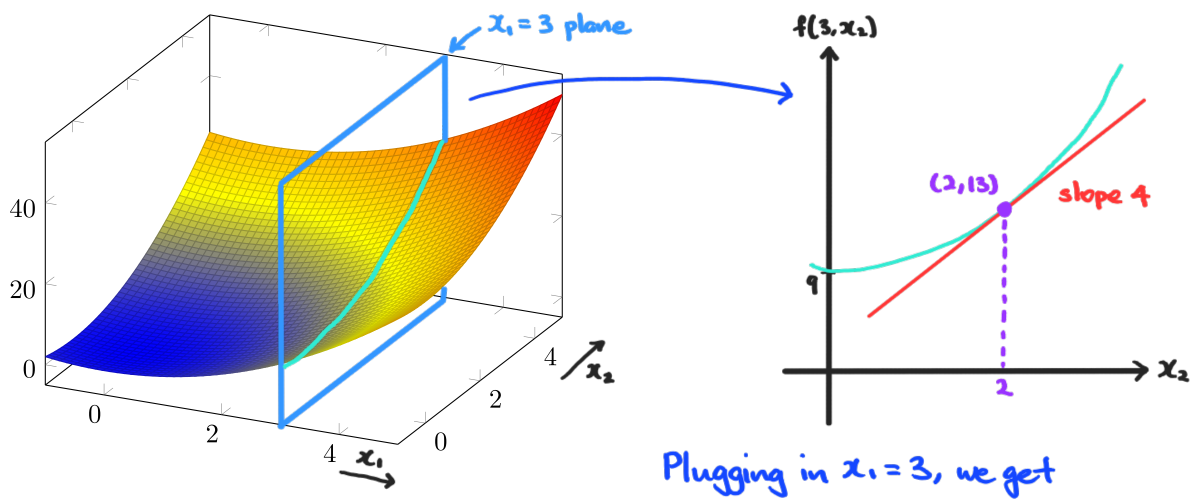
$$\frac{\partial f}{\partial x_1}(a,b) = \lim_{h \to 0} \frac{f(a+h,b) - f(a,b)}{h}.$$

Similarly,

$$\frac{\partial f}{\partial x_2}(a,b) = \left.\frac{\partial f}{\partial x_2}\right|_{(a,b)} = f_{x_2}(a,b) = \lim_{h \to 0} \frac{f(a,b+h) - f(a,b)}{h}.$$

Example 1. Consider $f(x_1, x_2) = x_1^2 + x_2^2$. Compute $f_{x_1}(3, 2)$ and $f_{x_2}(3, 2)$.

Plot of $f(x_1, x_2) = x_1^2 + x_2^2$



Plugging in $x_1 = 3$, we get

$$f(3, x_2) = x_2^2 + 9.$$

So,

$$\frac{\partial f}{\partial x_2}(3,2) = \lim_{h \to 0} \frac{f(3,2+h) - f(3,2)}{h}$$
$$= \lim_{h \to 0} \frac{(2+h)^2 + 9 - (2^2 + 9)}{h}$$
$$= \lim_{h \to 0} \frac{4h + h^2}{h} = \boxed{4}.$$

Similarly, plugging in $x_2 = 2$ gives us

$f(x_1, 2) = x_1^2 + 4$, and so,

$$\frac{\partial f}{\partial x_1}(3,2) = \lim_{h \to 0} \frac{f(3+h,2) - f(3,2)}{h}$$
$$= \lim_{h \to 0} \frac{((3+h)^2 + 4) - (3^2 + 4)}{h}$$
$$= \lim_{h \to 0} \frac{6h + h^2}{h} = \boxed{6}.$$

If $f(x_1, \ldots, x_n)$ is a function of n variables, then its partial derivative with respect to x_i at $(a_1, \ldots, a_n)$ is denoted as

$$f_{x_i}(a_1, \ldots, a_n) \qquad \text{or} \qquad \frac{\partial f}{\partial x_i}(a_1, \ldots, a_n) \qquad \text{or} \qquad \left.\frac{\partial f}{\partial x_i}\right|_{(a_1, \ldots, a_n)},$$

and it can be computed in two ways.

Method 1 (symbolic). We treat the x_j's, where $j \neq i$, as constants and differentiate with respect to x_i as you would with a single-variable function. This will give $f_{x_i}(x_1, \ldots, x_n)$ and you can plug in $(a_1, \ldots, a_n)$.

Method 2 (numerical). We replace x_j with a_j for all $j \neq i$, which gives us $f(a_1, \ldots, a_{i-1}, x_i, a_{i+1}, \ldots, a_n)$. This is a single-variable function of x_i; you can differentiate this and plug in a_i.

The two methods are *identical.* The first is the more practical method of calculating partial derivatives, and the second illustrates the single-variable aspect of partial differentiation.

Example 2. Compute the following derivatives by using both methods.

(a) $\dfrac{\partial f}{\partial x_1}\left(\dfrac{1}{2}, -5\right)$, where $f(x_1, x_2) = x_1 \cos(\pi x_1 x_2)$

Symbolic $f_{x_1}(x_1, x_2) = 1 \cdot \cos(\pi x_1 x_2) + x_1(-\sin(\pi x_1 x_2) \cdot (\pi x_2))$

$= \cos(\pi x_1 x_2) - \pi x_1 x_2 \sin(\pi x_1 x_2)$.

So, $f_{x_1}\left(\frac{1}{2}, -5\right) = \cos\left(-\frac{5\pi}{2}\right) + \frac{5\pi}{2}\sin\left(-\frac{5\pi}{2}\right) = \boxed{-\frac{5\pi}{2}}$.

Numerical Taking $g(x_1) := f(x_1, -5) = x_1\cos(-5\pi x_1)$, we get

$g'(x_1) = \cos(-5\pi x_1) + 5\pi x_1 \sin(-5\pi x_1)$.

Hence, $f_{x_1}\left(\frac{1}{2}, -5\right) = g'\left(\frac{1}{2}\right) = \cos\left(-\frac{5\pi}{2}\right) + \frac{5\pi}{2}\sin\left(-\frac{5\pi}{2}\right) = \boxed{-\frac{5\pi}{2}}$.

(b) $\left.\dfrac{\partial g}{\partial x}\right|_{(5,8)}$, where $g(x, y) = 12e^{\cos y} + y^3$

Symbolic $g_x(x, y) = 0$, and so, $g_x(5, 8) = \boxed{0}$.

Numerical $f(x) := g(x, 8) = 12^{\cos 8} + 8^3$, and $f'(x) = 0$. Hence,

$g_x(5, 8) = f'(5) = \boxed{0}$.

(c) $f_z(1, 2, 3)$, where $f(x, y, z) = z^2 \tan(\pi x/4) + yz$

Symbolic $f_z(x, y, z) = 2z \tan\left(\frac{\pi x}{4}\right) + y$, and so, $f_z(1, 2, 3) = 6\tan\left(\frac{\pi}{4}\right) + 2 = \boxed{8}$.

Numerical $g(z) := f(1, 2, z) = z^2 + 2z$, and $g'(z) = 2z + 2$. Thus,

$f_z(1, 2, 3) = g'(3) = \boxed{8}$.

Example 3. Use partial derivatives to approximate $f(1, 4.1)$, where $f(x, y) = \sqrt{x + 2y}$.

We compute $f_y(x,y) = \frac{1}{2} \cdot \frac{2}{\sqrt{x+2y}} = \frac{1}{\sqrt{x+2y}}$, and $f_y(1,4) = \frac{1}{3}$. Thus,

$$f(1,4.1) \approx f(1,4) + f_y(1,4)(4.1-4) = 3 + \frac{1}{3} \cdot 0.1$$

$$= \frac{91}{30} = 3.03333$$

<u>Note</u> $f(1, 4.1) = \sqrt{9.2} = 3.03315$.

Relating partial derivatives to contour plots. Visualize $f(x, y)$ as the height above (x, y) on the graph $z = f(x, y)$, where x is the east-west coordinate and y is the north-south coordinate. Then,

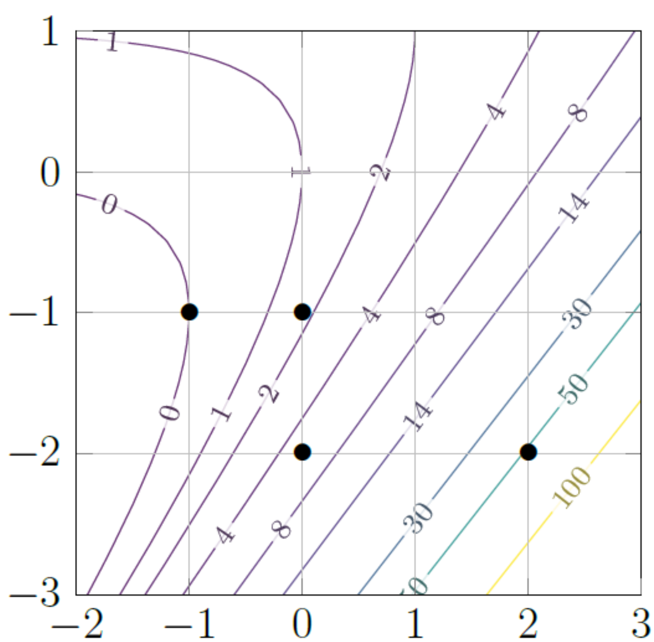
- $f_x(a, b)$ equals the slope experienced by someone walking on the surface just as they go past (a, b) from *west to east*
- $f_y(a, b)$ equals the slope experienced by someone walking on the surface just as they go past (a, b) from *south to north*

Equivalently,

- The sign of $f_x(a, b)$ tell us whether the labels of the contours are increasing or decreasing as we walk through (a, b) from west to east
- The sign of $f_y(a, b)$ tell us whether the labels of the contours are increasing or decreasing as we walk through (a, b) from south to north

If $f_x(a_1, b_1) > f_x(a_2, b_2) > 0$, then means the slope, in the x-direction, at (a_1, b_1) is *steeper* than the slope, in the x-direction, at (a_2, b_2).

Example 4. Consider the contour plot below of a 2-variable function $f(x, y)$.



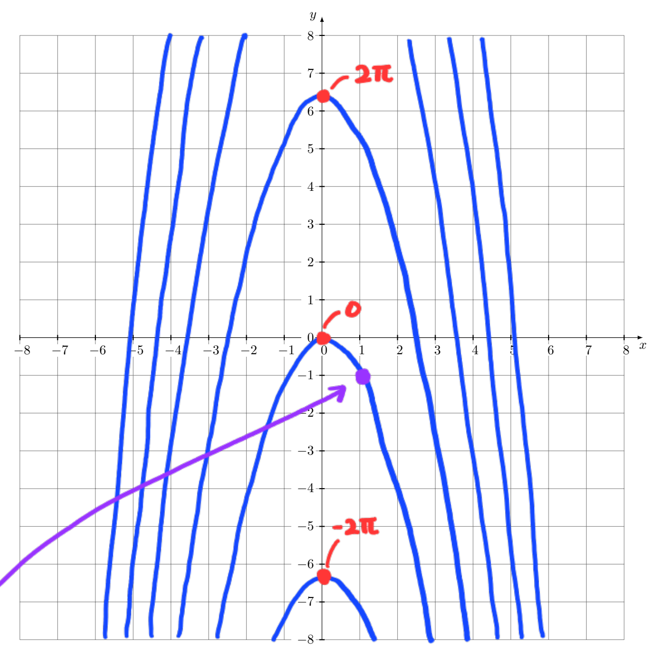
Which is greater? $f_x(-1, -2)$ or $f_x(2, -2)$? $f_x(0, -2)$ or $f_y(0, -2)$? $f_x(1, -1)$ or $f_y(1, -1)$?

Example 5. Consider the function $f : \mathbb{R}^2 \to \mathbb{R}$ given by $f(x, y) = \cos(x^2 + y)$.

(a) Compute $f_x(x, y)$ and $f_y(x, y)$.

$f_x(x,y) = -2x\sin(x^2+y)$ and $f_y(x,y) = -\sin(x^2+y)$.

(b) Since the cosine function takes values between -1 and $+1$, the same is true for f. Plot the level set at $+1$ in the xy-plane below.



$\cos(x^2+y) = 1$

$\Rightarrow x^2 + y = 2n\pi,\ n \in \mathbb{Z}.$

$\Rightarrow y = -x^2 + 2n\pi,\ n \in \mathbb{Z}.$

(c) Give a point in the level set from the previous part whose x-coordinate is 1 (there are many possible choices).

Take the curve $y = -x^2$. Then, $y = -1$ if $x = 1$, and so, $(1, -1)$ is in the level set.

(d) Compute $f_x(1, y)$ and $f_y(1, y)$ at the point you gave in the previous part. Does the answer depend on the point you chose?

No. Since $\cos(x+y^2) = 1$ (level set 1), it follows that $\sin(x+y^2) = 0$. Hence, $f_x(1,y) = 0$ and $f_y(1,y) = 0$ always.

Example 6.

(a) For $f(x, y) = ax^3 + bx^2y + cxy^2 + dy^3$, show that $xf_x + yf_y = 3f$.

We compute

$$f_x = 3ax^2 + 2bxy + cy^2$$
$$f_y = bx^2 + 2cxy + 3dy^2.$$

Thus,

$$xf_x = 3ax^3 + 2bx^2y + cxy^2$$
$$yf_y = bx^2y + 2cxy^2 + 3dy^3,$$

and so,

$$xf_x + yf_y = 3ax^3 + 3bx^2y + 3cxy^2 + 3dy^3 = 3f.$$

(b) For $f(x, y) = \ln\left(\dfrac{x^2}{y^3}\right)$, show that $3xf_x + 2yf_y = 0$.

Noting $\ln\left(\frac{x^2}{y^3}\right) = 2\ln x - 3\ln y$, we get

$$f_x = \frac{2}{x} \quad \text{and} \quad f_y = -\frac{3}{y}.$$

Hence, $3xf_x + 2yf_y = 3x\left(\frac{2}{x}\right) + 2y\left(-\frac{3}{y}\right) = 6 - 6 = 0.$

For a function $f(x_1, \ldots, x_n)$ of n variables that is differentiable in each x_i, the **second partial derivatives** are defined to be

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial}{\partial x_i}\left(\frac{\partial f}{\partial x_j}\right)$$

when they exist. If $j = i$, we denote it by $\dfrac{\partial^2 f}{\partial x_i^2}$.

Theorem 9.6.4. (Clairaut–Schwarz) Consider a function $f(x_1, \ldots, x_n)$ that is continuous. Additionally, suppose that the partial derivatives $\partial f/\partial x_i$ exist for all i, and the second partial derivatives $\partial^2 f/\partial x_i \partial x_j$ exist for all i and j and are continuous. Then, the order of applying $\partial/\partial x_i$ and $\partial/\partial x_j$ to f does not matter:

$$\frac{\partial^2 f}{\partial x_i \partial x_j} = \frac{\partial^2 f}{\partial x_j \partial x_i}.$$

We denote these partial derivatives $f_{x_j x_i}$ and $f_{x_i x_j}$, respectively.

Example 7. One of the most famous equations in physics is **Laplace's equation**: for $f(x, y)$,

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.$$

Functions satisfying this equation are called **harmonic** – these arise in the study of equilibrium configurations in the natural sciences.
Show that the functions $f_1(x, y) = e^x \cos y$ and $f_2(x, y) = x^4 - 6x^2y^2 + y^4$ are harmonic.

We compute $(f_1)_x = e^x \cos y$ and $(f_1)_y = -e^x \sin y$.

So, the second partials are $(f_1)_{xx} = e^x \cos y$ and $(f_1)_{yy} = -e^x \cos y$.

Hence, $(f_1)_{xx} + (f_1)_{yy} = e^x \cos y - e^x \cos y = 0$, and thus, f_1 is harmonic.

Similarly, $(f_2)_x = 4x^3 - 12xy^2$ and $(f_2)_y = -12x^2y + 4y^3$.

Thus, $(f_2)_{xx} = 12x^2 - 12y^2$ and $(f_2)_{yy} = -12x^2 + 12y^2$.

Hence, $(f_2)_{xx} + (f_2)_{yy} = 0$, and so, f_2 is harmonic.

Example 8. Suppose that the temperature of an (one-dimensional) object made from uniform material (such as a solid thin rod) is written as $U(t,x)$ degrees Celsius at position x (in meters) and time t (in seconds). The **heat equation** predicts that the temperature evolves according to

$$\frac{\partial U}{\partial t} = \alpha \frac{\partial^2 U}{\partial x^2}$$

for a positive constant α (called *thermal diffusivity*). Show that

$$U(t,x) = e^{-\frac{c}{\alpha}(x-ct)}$$

satisfies the heat equation for any non-zero constant c.

We compute

$$U_t = \frac{c^2}{\alpha} e^{-\frac{c}{\alpha}(x-ct)} \quad \text{and} \quad U_x = -\frac{c}{\alpha} e^{-\frac{c}{\alpha}(x-ct)}.$$

So, we get

$$U_{xx} = \frac{c^2}{\alpha^2} e^{-\frac{c}{\alpha}(x-ct)},$$

and thus, $U_t = \alpha U_{xx}$.

Example 9. A very famous non-physical situation modeled by a partial differential equation (PDE) occurs in finance with the **Black–Scholes equation** (emerging from a mathematical model whose publication in 1973 earned the 1997 Nobel Prize in Economics for Scholes and Merton; Black passed away in 1995):

$$\frac{\partial V}{\partial t} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 V}{\partial S^2} + rS\frac{\partial V}{\partial S} - rV = 0.$$

Here, $V(t,S)$ is the value of an option as a function of payoff-time t and stock price S, σ is the measure of volatility, and r is the risk-free interest rate. (An *option* is a contract that allows someone to have the right to buy/sell an asset at a specific price on a specific date.) The Black–Scholes is derived by "averaging" out random fluctuations; for more information on PDEs, take Math 53!