Goal: basis and dimension for linear subspaces of $\mathbb{R}^2$ and $\mathbb{R}^3$, Fourier formula, Dimension Criterion

A **basis** for a nonzero linear subspace V in $\mathbb{R}^n$ is a spanning set for V consisting of exactly $\dim(V)$ vectors.

Dimension Criterion. For two nonzero vectors $\mathbf{v}, \mathbf{w} \in \mathbb{R}^n$, we have $\dim(\mathrm{span}(\mathbf{v}, \mathbf{w})) = 2$ except for precisely when the vectors are scalar multiples of each other, in which case the dimension is 1.
For three nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3 \in \mathbb{R}^n$ with span V, we have $\dim(V) = 3$ except for precisely the following cases:

1. all three vectors are scalar multiples of each other, in which case $\dim(V) = 1$;
2. exactly two of the vectors are scalar multiples of each other, in which case $\dim(V) = 2$;
3. no $\mathbf{v}_i$ is a scalar multiple of another $\mathbf{v}_j$ but some $\mathbf{v}_i$ is a linear combination of the other two, in which case $\dim(V) = 2$ and *every* $\mathbf{v}_i$ is a linear combination of the other two.

Example 1. Consider the following pairs of nonzero vectors. What is the dimension of the span of each pair?

(a) $\mathbf{v} = \begin{bmatrix} 3/2 \\ -2 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} -2 \\ 8/3 \end{bmatrix}$ $\quad \vec{w} = -\frac{4}{3}\vec{v} \Rightarrow \dim = 1.$

(b) $\mathbf{v} = \begin{bmatrix} 2 \\ -4 \\ 5 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 3 \\ -6 \\ 15 \end{bmatrix}$ $\quad \vec{w}$ and $\vec{v}$ are not scalar multiples of each other $\Rightarrow \dim = 2$.

(c) $\mathbf{v} = \begin{bmatrix} 3 \\ 6 \\ -15 \end{bmatrix}, \mathbf{w} = \begin{bmatrix} 2 \\ 4 \\ -10 \end{bmatrix}$ $\quad \vec{w} = \frac{2}{3}\vec{v} \Rightarrow \dim = 1.$

Example 2. Recall Example 3 from Lecture 4, where we showed that the set U of 4-vectors perpendicular to $\begin{bmatrix} 2 \\ 1 \\ 5 \\ 1 \end{bmatrix}$ can be spanned by $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}$, and $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$. What is the dimension of U?

The three vectors are not scalar multiples of each other. Also, $\begin{bmatrix} 0 \\ -1 \\ 0 \\ 1 \end{bmatrix}$ cannot be expressed as a linear combination of $\begin{bmatrix} 1 \\ -2 \\ 0 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ -5 \\ 1 \\ 0 \end{bmatrix}$ because any linear combination of the two would have 0 as its fourth component. Thus, $\dim U = 3$.

Example 3. Consider the linear subspace V of $\mathbb{R}^3$ that is spanned by $\mathbf{v}_1 = \begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix}$. In Example 9 from Lecture 4, we showed that $\text{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$ is not $\mathbb{R}^3$. Show that $\dim(V) < 3$ by using the Dimension Criterion.

$\vec{v_1}, \vec{v_2}, \vec{v_3}$ are not scalar multiples of each other. Thus, $\dim V \geq 2$. Now we check if one of them can be written as a linear combination of the other two.

$$\begin{bmatrix} 3 \\ -1 \\ 2 \end{bmatrix} = a\begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix} + b\begin{bmatrix} 2 \\ -4 \\ 3 \end{bmatrix} = \begin{bmatrix} 2b \\ 2a-4b \\ -a+3b \end{bmatrix} \Rightarrow \begin{cases} 2b = 3 \\ 2a-4b=-1 \\ -a+3b=2 \end{cases}$$

Solving the system, we get $a = \frac{5}{2}$ and $b = \frac{3}{2}$, and thus, $\vec{v_1} = \frac{5}{2}\vec{v_2} + \frac{3}{2}\vec{v_3}$. Hence, by dimension criterion, $\dim V = 2$.

A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is said to be **orthogonal** if all the vectors are perpendicular to others. In other words, for any different i and j, $\mathbf{v}_i \cdot \mathbf{v}_j = 0$.

Theorem 5.2.2. If $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is an orthogonal collection of *nonzero* vectors in $\mathbb{R}^n$, then it is a basis for $\text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$. In particular, the span has dimension k, and we call $\mathbf{v}_1, \ldots, \mathbf{v}_k$ an **orthogonal basis** for the span. Note that a single nonzero vector is always an orthogonal basis for its span.

Example 4. Let us revisit Example 3, where we showed that $\dim(V) = 2$. Find an orthogonal basis for V.

We need to find $\vec{w} \in V$ that is perpendicular to $\vec{v_2}$, say $\vec{w} = a\vec{v_2} + b\vec{v_3}$. Then,

$$0 = \vec{v_2} \cdot \vec{w} = a\vec{v_2} \cdot \vec{v_2} + b\,\vec{v_2} \cdot \vec{v_3} = 5a - 11b,$$

and so, $b = \frac{5}{11}a$. So,

$$\vec{w} = a\vec{v_2} + \frac{5}{11}a\vec{v_3} = \frac{a}{11}(11\vec{v_2} + 5\vec{v_3}) = \frac{a}{11}\begin{bmatrix} 10 \\ 2 \\ 4 \end{bmatrix} = \frac{2a}{11}\begin{bmatrix} 5 \\ 2 \\ 1 \end{bmatrix}.$$

We found $\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \in V$ and $\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix} \perp \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}$, and so, $\left\{\begin{bmatrix} 5 \\ 1 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \end{bmatrix}\right\}$ is an orthogonal basis for V.

Note Try to find an orthogonal basis containing $\vec{v_3}$.

Theorem 5.2.5. Every nonzero linear subspace of $\mathbb{R}^n$ has an orthogonal basis.

A collection of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ is called **orthonormal** if they are orthogonal and they are all unit vectors. Note that Theorem 5.2.2 implies that any orthonormal collection of vectors is a basis of its span.

Example 5. The **standard basis** for $\mathbb{R}^n$

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \cdots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}$$

is an orthonormal basis.

Example 6. Which of the following are orthogonal/orthonormal bases of $\mathbb{R}^3$?

(a) $\left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \right\}$ orthogonal basis of $\mathbb{R}^3$

(b) $\left\{ \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} \right\}$ orthogonal, but not a basis of $\mathbb{R}^3$

(c) $\left\{ \begin{bmatrix} 1 \\ 2 \\ -3 \end{bmatrix}, \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$ non-orthogonal basis of $\mathbb{R}^3$

Fourier formula. (Theorem 5.3.6) For any orthogonal collection of nonzero vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$ and $\mathbf{v} \in \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$,

$$\mathbf{v} = \sum_{i=1}^{k} \left(\frac{\mathbf{v} \cdot \mathbf{v}_i}{\mathbf{v}_i \cdot \mathbf{v}_i} \right) \mathbf{v}_i.$$

scalar

In particular, if the $\mathbf{v}_i$'s are all unit vectors, then $\mathbf{v} = \sum (\mathbf{v} \cdot \mathbf{v}_i) \mathbf{v}_i$.

Example 7. Let $\begin{bmatrix} x \\ y \\ z \end{bmatrix} \in \mathbb{R}^3$. Apply the Fourier formula with the standard basis.

Setting $\vec{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$, we get

$$\left(\frac{\vec{v} \cdot \vec{e}_1}{\vec{e}_1 \cdot \vec{e}_1} \right) \vec{e}_1 = x \vec{e}_1, \quad \left(\frac{\vec{v} \cdot \vec{e}_2}{\vec{e}_2 \cdot \vec{e}_2} \right) \vec{e}_2 = y \vec{e}_2, \quad \left(\frac{\vec{v} \cdot \vec{e}_3}{\vec{e}_3 \cdot \vec{e}_3} \right) \vec{e}_3 = z \vec{e}_3.$$

So, by Fourier formula, $\vec{v} = x\vec{e}_1 + y\vec{e}_2 + z\vec{e}_3$.

Example 8. Consider the orthogonal basis $\left\{ \overset{\vec{v}_1}{\begin{bmatrix}1\\1\\0\end{bmatrix}}, \overset{\vec{v}_2}{\begin{bmatrix}1\\-1\\0\end{bmatrix}}, \overset{\vec{v}_3}{\begin{bmatrix}0\\0\\1\end{bmatrix}} \right\}$ from Example 6. Express $\overset{\vec{u}}{\begin{bmatrix}3\\-1\\5\end{bmatrix}}$ and $\overset{\vec{w}}{\begin{bmatrix}5\\1\\3\end{bmatrix}}$ as linear combinations of the basis vectors by using Fourier formula.

We see that $\left(\frac{\vec{u}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1}\right)\vec{v}_1 = \frac{2}{2}\vec{v}_1 = \vec{v}_1$, $\left(\frac{\vec{u}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2}\right)\vec{v}_2 = \frac{4}{2}\vec{v}_2 = 2\vec{v}_2$, $\left(\frac{\vec{u}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3}\right)\vec{v}_3 = \frac{5}{1}\vec{v}_3 = 5\vec{v}_3$.

By Fourier formula, $\vec{u} = \vec{v}_1 + 2\vec{v}_2 + 5\vec{v}_3$.

Indeed, $\begin{bmatrix}3\\-1\\5\end{bmatrix} = \begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}1\\-1\\0\end{bmatrix} + 5\begin{bmatrix}0\\0\\1\end{bmatrix}$.

Similarly, $\frac{\vec{w}\cdot\vec{v}_1}{\vec{v}_1\cdot\vec{v}_1} = 3$, $\frac{\vec{w}\cdot\vec{v}_2}{\vec{v}_2\cdot\vec{v}_2} = 2$, and $\frac{\vec{w}\cdot\vec{v}_3}{\vec{v}_3\cdot\vec{v}_3} = 3$, and so, by Fourier formula, we have $\vec{w} = 3\vec{v}_1 + 2\vec{v}_2 + 3\vec{v}_3$.

Indeed, $\begin{bmatrix}5\\1\\3\end{bmatrix} = 3\begin{bmatrix}1\\1\\0\end{bmatrix} + 2\begin{bmatrix}1\\-1\\0\end{bmatrix} + 3\begin{bmatrix}0\\0\\1\end{bmatrix}$.

Example 9. Find an orthogonal basis for the plane in $\mathbb{R}^3$ defined by the equation $2x - 3y - z = 0$. (There are many possible answers.)

One vector on the plane is $\begin{bmatrix}1\\0\\2\end{bmatrix}$ (obtained by picking $x = 1$ and $y = 0$). To round out an orthogonal basis, we need a vector on the plane that is orthogonal to $\begin{bmatrix}1\\0\\2\end{bmatrix}$, say $\begin{bmatrix}a\\b\\c\end{bmatrix}$. This needs to be perpendicular to $\begin{bmatrix}2\\-3\\-1\end{bmatrix}$ as well. Hence,

$$\begin{cases} a \qquad\; + 2c = 0 \\ 2a - 3b - c = 0 \end{cases} \Rightarrow a = -2c,\ b = -\frac{5}{3}c \Rightarrow \begin{bmatrix}-2c\\-\frac{5}{3}c\\c\end{bmatrix} = \frac{c}{3}\begin{bmatrix}-6\\-5\\3\end{bmatrix}.$$

Thus, $\boxed{\left\{\begin{bmatrix}1\\0\\2\end{bmatrix}, \begin{bmatrix}-6\\-5\\3\end{bmatrix}\right\}}$ is an orthogonal basis for the plane.

Example 10. This exercise illustrates the important general fact (to be discussed in detail in Chapter 19) that every nonzero subspace of $\mathbb{R}^n$ has an orthogonal basis. Consider the collection W of vectors in $\mathbb{R}^4$ orthogonal to $\mathbf{v} = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 0 \end{bmatrix}$ and $\mathbf{v}' = \begin{bmatrix} 0 \\ 2 \\ 1 \\ -1 \end{bmatrix}$; that is,

$$W = \left\{ \mathbf{x} \in \mathbb{R}^4 : x_1 + x_2 + 2x_3 = 0, 2x_2 + x_3 - x_4 = 0 \right\}.$$

(a) Show that W is a linear subspace of $\mathbb{R}^4$ by expressing it as the span of two vectors.

If $\vec{w} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} \in W$, then $x_1 = -x_2 - 2x_3$ and $x_4 = 2x_2 + x_3$. So,

$$\vec{w} = \begin{bmatrix} -x_2 + 2x_3 \\ x_2 \\ x_3 \\ 2x_2 + x_3 \end{bmatrix} = x_2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + x_3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$$

Thus, $W = \text{span}\left(\begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} \right)$.

Note $\dim(W) = 2$.

(b) Find an orthogonal pair of nonzero vectors $\{\mathbf{w}_1, \mathbf{w}_2\}$ in W. (Hint: take $\mathbf{w}_1$ to be one of the vectors that you found in (a), and then $\mathbf{w}_2$ must be orthogonal to that *and* satisfy the two equations defining W).

Take $\vec{w}_1 = \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}$. We need $\vec{w}_2$ to satisfy

① $\vec{w}_2 \in W$, i.e., $\vec{w}_2 = \alpha \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$ for some α and β.

② $\vec{w}_1 \perp \vec{w}_2$.

Then, $\vec{w}_1 \cdot \vec{w}_2 = 6\alpha + 4\beta = 0$, and so, $\alpha = -\frac{2}{3}\beta$. Thus,

$$\vec{w}_2 = -\frac{2}{3}\beta \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + \beta \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix} = \frac{\beta}{3} \begin{bmatrix} -4 \\ -2 \\ 3 \\ -1 \end{bmatrix}.$$

Hence, $\left\{ \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} -4 \\ -2 \\ 3 \\ -1 \end{bmatrix} \right\}$ is an orthogonal basis for W.

$-2 \begin{bmatrix} -1 \\ 1 \\ 0 \\ 2 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 0 \\ 1 \\ 1 \end{bmatrix}$