1. (a) The left side is $\begin{bmatrix} t \\ 2t \end{bmatrix} + \begin{bmatrix} 2t' \\ 3t' \end{bmatrix} = \begin{bmatrix} t + 2t' \\ 2t + 3t' \end{bmatrix}$, so the equality with $\begin{bmatrix} 4 \\ 5 \end{bmatrix}$ amounts to the simultaneous pair of conditions

$$\begin{aligned} t + 2t' &= 4 \\ 2t + 3t' &= 5. \end{aligned}$$

Solving this by elimination (e.g., double the first equation and subtract from the second to cancel t and get $-t' = 5 - 8 = -3$, so $t' = 3$) yields $t = -2$, $t' = 3$.

(b) The left side is $\begin{bmatrix} t \\ 2t \end{bmatrix} + \begin{bmatrix} 2t' \\ 3t' \end{bmatrix} = \begin{bmatrix} t + 2t' \\ 2t + 3t' \end{bmatrix}$, so the equality with $\begin{bmatrix} -2 \\ -1 \end{bmatrix}$ amounts to the simultaneous pair of conditions

$$\begin{aligned} t + 2t' &= -2 \\ 2t + 3t' &= -1. \end{aligned}$$

Solving this by elimination (e.g., double the first equation and subtract from the second to cancel t and get $-t' = -1 + 4 = 3$, so $t' = -3$) yields $t = 4$, $t' = -3$.

(c) Proceeding as in the previous part yields the simultaneous pair of conditions

$$\begin{aligned} t + 2t' &= x \\ 2t + 3t' &= y. \end{aligned}$$

To solve for t (in terms of x and y) we must eliminate t', so we multiply the first equation by 3 and subtract from it twice the second to get $3t - 4t = 3x - 2y$, or $-t = 3x - 2y$, so $t = -3x + 2y$. Likewise, to solve for t' (in terms of x and y) we must eliminate t, so we multiply the first equation by 2 and subtract it from the second to get $3t' - 4t' = y - 2x$, or $-t' = y - 2x$, so $t' = 2x - y$.

Since $(t, t') = (-3x + 2y, 2x - y)$, for $x = 4$ and $y = 5$ it is $(-12 + 10, 8 - 5) = (-2, 3)$, exactly the answer to part (a).

Since $(t, t') = (-3x + 2y, 2x - y)$, for $x = -2$ and $y = -1$ it is $(6 - 2, -4 + 1) = (4, -3)$, exactly the answer to part (b). $\diamondsuit$

2. (a) If a course is worth u units then its grade is weighted by $u/20$ (the fraction of units from that course out of all 20 units taken), so the vector of GPA's is

$$\frac{5}{20}\mathbf{v}_1 + \frac{5}{20}\mathbf{v}_2 + \frac{4}{20}\mathbf{v}_3 + \frac{3}{20}\mathbf{v}_4 + \frac{3}{20}\mathbf{v}_5.$$

With common denominator 20, the numerators are exactly the unit-values. The sum of the numerators is 20, which is how we defined the denominator, so the sum of the fractions is the numerator-sum divided by 20, which is 20/20 = 1. Thus, this is a convex combination. In this way of describing it, the only role of 20 is as the sum of the unit values. Hence, we get convexity by the same reasoning, regardless of the total unit sum.

(b) If g_i is the grade a student earns in the ith course then the GPA is

$$\frac{5}{20}(g_1 + g_2) + \frac{4}{20}g_3 + \frac{3}{20}(g_4 + g_5),$$

so the 3-unit classes are weighted "less" than 1/5 = 4/20 wheres the 5-unit classes are weighted "more" than 1/5 = 4/20. This motivates considering the outcome in which student 1 earns their

A+'s in lower-unit courses 3, 4, 5 and student 2 earns their A+'s in the higher-unit courses 1 and 2.

Doing the calculation for these proposed grade outcomes, writing each grade as $g_i = 4.0 + e_i$ where e_i is equal to 0 or .3, by using convexity the GPA of the first student is

$$4.0 + .3(4/20 + 3/20 + 3/20) = 4.0 + .3(10/20)$$

and the GPA for the second student is

$$4.0 + .3(5/20 + 5/20) = 4.0 + .3(10/20).$$

These match! (Since we have distributed the location of A+'s to maximize the GPA of student 2 and to minimize that of student 1 and yet the GPA's came out to be the same, this is the only way to distribute the grades to make the GPA's match.) $\diamondsuit$

3. (a) Consider the parallelogram formed by two 2-vectors $\mathbf{v}$ and $\mathbf{w}$ as shown in Figure 2. We will apply the Law of Cosines to the two triangles shown in Figure 2.

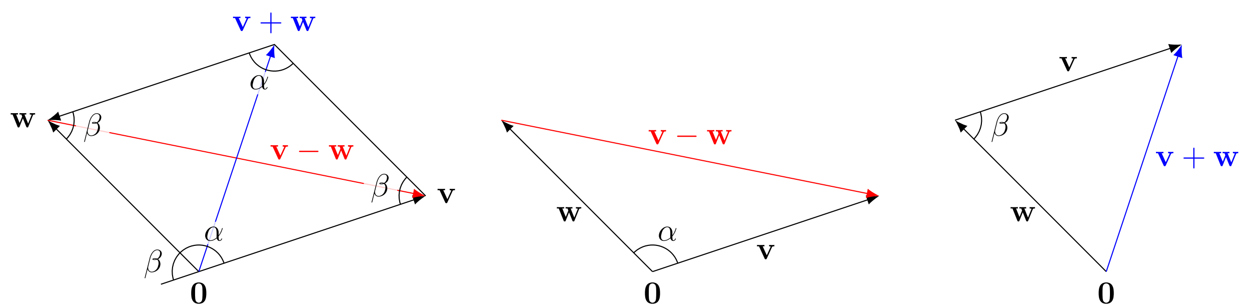

FIGURE 2. The figure used in the geometric proof of the parallelogram law when $n = 2$, using the Law of Cosines.

Applying the Law of Cosines to the first triangle (with the long edge labeled by $\mathbf{v} - \mathbf{w}$) yields

$$\|\mathbf{v} - \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\alpha. \tag{$*$}$$

Applying the Law of Cosines to the second triangle yields

$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\| \cos\beta. \tag{$**$}$$

Adding equations $(*)$ and $(**)$ yields

$$\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2 - 2\|\mathbf{v}\|\|\mathbf{w}\|(\cos\alpha + \cos\beta).$$

Since $\alpha + \beta = \pi$, as shown in the first drawing in Figure 2, we have $\cos\alpha = \cos(\pi - \beta) = -\cos\beta$ (an angle between 0 and π and its supplement mark off points on the unit circle with opposite x-coordinates). Therefore $\cos\alpha + \cos\beta = 0$, so we obtain the equality

$$\|\mathbf{v} + \mathbf{w}\|^2 + \|\mathbf{v} - \mathbf{w}\|^2 = 2\|\mathbf{v}\|^2 + 2\|\mathbf{w}\|^2,$$

as desired.

(b) Writing $\mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}$ and $\mathbf{w} = \begin{bmatrix} w_1 \\ \vdots \\ w_n \end{bmatrix}$, we have

$$\mathbf{v} + \mathbf{w} = \begin{bmatrix} v_1 + w_1 \\ \vdots \\ v_n + w_n \end{bmatrix}, \quad \mathbf{v} - \mathbf{w} = \begin{bmatrix} v_1 - w_1 \\ \vdots \\ v_n - w_n \end{bmatrix},$$

so

$$\begin{aligned}
\|\mathbf{v}+\mathbf{w}\|^2+\|\mathbf{v}-\mathbf{w}\|^2 &= \sum_{i=1}^{n}(v_i+w_i)^2+\sum_{i=1}^{n}(v_i-w_i)^2 \\
&= \sum_{i=1}^{n}(v_i^2+2v_iw_i+w_i^2)+\sum_{i=1}^{n}(v_i^2-2v_iw_i+w_i^2) \\
&= \sum_{i=1}^{n}(v_i^2+2v_iw_i+w_i^2+v_i^2-2v_iw_i+w_i^2) \\
&= \sum_{i=1}^{n}(2v_i^2+2w_i^2) \\
&= 2\sum_{i=1}^{n}v_i^2+2\sum_{i=1}^{n}w_i^2 \\
&= 2\|\mathbf{v}\|^2+2\|\mathbf{w}\|^2
\end{aligned}$$ $\diamondsuit$

4. We calculate each using three ingredients: properties of the dot product, that $\|\mathbf{v}\|^2=\mathbf{v}\cdot\mathbf{v}$ for vectors $\mathbf{v}$, and the assumption that $\mathbf{a},\mathbf{b},\mathbf{c}$ are all unit vectors (so each satisfies $\mathbf{v}\cdot\mathbf{v}=1$).
   (a) $\|\mathbf{a}+\mathbf{b}\|^2=(\mathbf{a}+\mathbf{b})\cdot(\mathbf{a}+\mathbf{b})=\mathbf{a}\cdot\mathbf{a}+\mathbf{a}\cdot\mathbf{b}+\mathbf{b}\cdot\mathbf{a}+\mathbf{b}\cdot\mathbf{b}=1+0+0+1=2.$
   (b) $\|\mathbf{b}-\mathbf{c}\|^2=\mathbf{b}\cdot\mathbf{b}-2(\mathbf{b}\cdot\mathbf{c})+\mathbf{c}\cdot\mathbf{c}=1-\frac{2}{5}+1=\frac{8}{5}.$
   (c) Using (a), we have

$$\|\mathbf{a}+\mathbf{b}+\mathbf{c}\|^2=\|\mathbf{a}+\mathbf{b}\|^2+2(\mathbf{a}+\mathbf{b})\cdot\mathbf{c}+\|\mathbf{c}\|^2=2+2(\mathbf{a}\cdot\mathbf{c}+\mathbf{b}\cdot\mathbf{c})+1=2+2\left(\frac{1}{2}+\frac{1}{5}\right)+1=\frac{22}{5}.$$ $\diamondsuit$

5. (a) The vectors of interest are nonzero $\begin{bmatrix}x\\y\\z\end{bmatrix}$ with $x+2y+2z=0$. One way to make such are by setting $y=0$ or by setting $x=0$. These are $\begin{bmatrix}-2z\\0\\z\end{bmatrix}=z\begin{bmatrix}-2\\0\\1\end{bmatrix}$ and $\begin{bmatrix}0\\-z\\z\end{bmatrix}=z\begin{bmatrix}0\\-1\\1\end{bmatrix}$, so two such vectors not scalar multiples of each other are $\begin{bmatrix}2\\0\\-1\end{bmatrix}$ and $\begin{bmatrix}0\\-1\\1\end{bmatrix}$ (there are many other answers). The collection of vectors orthogonal to $\begin{bmatrix}1\\2\\2\end{bmatrix}$ is a plane.
   (b) These are the unit vectors in the plane found in (a), so this is a circle. One way to get these is to pick anything nonzero in the plane found in (a) and divide by its length to get a unit vector. For example:

$$\frac{1}{\sqrt{5}}\begin{bmatrix}2\\0\\-1\end{bmatrix},\ \frac{1}{\sqrt{2}}\begin{bmatrix}0\\-1\\1\end{bmatrix}$$

are two such (you only need to give one such unit vector; many answers are possible). $\diamondsuit$

6. (a) The perpendicularity of $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ implies

$$0 = (\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2,$$

so $\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2$ and hence $\|\mathbf{v}\| = \|\mathbf{w}\|$; i.e., $\mathbf{v}$ and $\mathbf{w}$ have the same length.

(b) We can run the calculation in the solution to (a) in reverse. More specifically, since $\|\mathbf{v}\| = \|\mathbf{w}\|$ we have $\|\mathbf{v}\|^2 = \|\mathbf{w}\|^2$ and so

$$(\mathbf{v}+\mathbf{w})\cdot(\mathbf{v}-\mathbf{w}) = \mathbf{v}\cdot\mathbf{v} - \mathbf{w}\cdot\mathbf{v} + \mathbf{v}\cdot\mathbf{w} - \mathbf{w}\cdot\mathbf{w} = \|\mathbf{v}\|^2 - \|\mathbf{w}\|^2 = 0.$$

This says that $\mathbf{v}+\mathbf{w}$ and $\mathbf{v}-\mathbf{w}$ are perpendicular. $\Diamond$

7. (a) By definition the correlation coefficient is

$$r = \frac{\mathbf{X}\cdot\mathbf{Y}}{\|\mathbf{X}\|\|\mathbf{Y}\|}.$$

For $k=1$, the numerator is $1+1-1-1=0$. Hence, $r=0$ (regardless of what the denominator is). The picture is as follows (corners of a square):

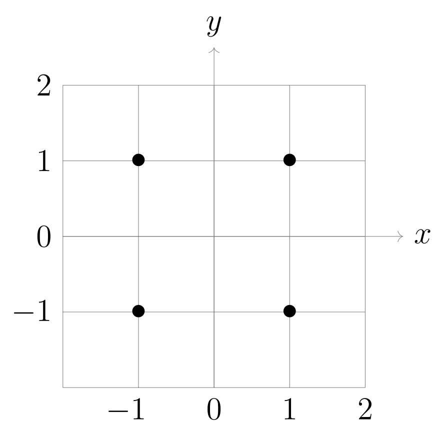

FIGURE 3. The data for $k=1$

(b) In general, using the same expression for r as in the solution to (b) but allowing a general k gives

$$r(k) = \frac{1+1-k^2-k^2}{\sqrt{1+1+k^2+k^2}\sqrt{1+1+k^2+k^2}} = \frac{2-2k^2}{\sqrt{2+2k^2}\sqrt{2+2k^2}} = \frac{2-2k^2}{2+2k^2} = \frac{1-k^2}{1+k^2}.$$

The denominator is positive, so to say $-1 < r(k) < 1$ is to say $-(1+k^2) < 1-k^2 < 1+k^2$. These inequalities always hold because the first says $-1-k^2 < 1-k^2$ or equivalently $-1<1$ (regardless of k) while the second says $1-k^2 < 1+k^2$ or equivalently $-k^2 < k^2$, which holds since we assume k is nonzero.

Geometrically, the cases $r=\pm 1$ (with nonzero data vectors $\mathbf{X}$ and $\mathbf{Y}$) correspond to the data points all lying on a common non-vertical and non-horizontal line, with positive slope for $r=1$ and negative slope for $r=-1$. But the line through the first two data points $(1,1)$ and $(-1,-1)$ is $y=x$ and the line through the last two data points (which are distinct since $k\neq 0$) is $y=-x$, and these lines are *not the same*. Hence, there is no line through all of the data, so $r\neq\pm 1$. $\Diamond$

8. First, we should find a point in the plane. We can do this by setting $x=y=0$, and solving for z. Thus, we see that $(0,0,-7)$ is a point in the plane. Second, we need two nonzero vectors $\mathbf{v}_1$ and $\mathbf{v}_2$

that are not scalar multiples of each other and are pointing along directions parallel to the plane; i.e. perpendicular to the normal vector $\mathbf{n} = \begin{bmatrix} 6 \\ -6 \\ -1 \end{bmatrix}$ and pointing along different directions.

To keep the algebra simple, we'll try vectors of the form $\mathbf{v}_1 = \begin{bmatrix} a \\ b \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} c \\ 0 \\ d \end{bmatrix}$ for which $\mathbf{v}_1 \cdot \mathbf{n} = 0$ and $\mathbf{v}_2 \cdot \mathbf{n} = 0$. Expanding these equations yields $6a - 6b = 0$ and $6c - d = 0$. So we can take $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix}$. Therefore, points in this plane are exactly those of the form

$$\begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix} + t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t' \begin{bmatrix} 1 \\ 0 \\ 6 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ -7 \end{bmatrix} + \begin{bmatrix} t \\ t \\ 0 \end{bmatrix} + \begin{bmatrix} t' \\ 0 \\ 6t' \end{bmatrix} = \begin{bmatrix} t + t' \\ t \\ -7 + 6t' \end{bmatrix},$$

yielding the parametric form

$$\mathcal{P} = \left\{ \begin{bmatrix} t + t' \\ t \\ -7 + 6t' \end{bmatrix} : t, t' \in \mathbf{R} \right\}.$$

Of course, there are different choices of $\mathbf{v}_1$ and $\mathbf{v}_2$ which yield a correct answer. $\Diamond$

9. (a) To find a point in common on these lines, we search for t_1, t_2 for which $\begin{bmatrix} -4 \\ 2 + 2t_1 \\ 7 + 3t_1 \end{bmatrix} = \begin{bmatrix} 1 + 5t_2 \\ 0 \\ 5 + t_2 \end{bmatrix}$. From the second coordinate, we can see that $t_1 = -1$. From the first coordinate, we can see that $t_2 = -1$. Plugging these values in, we observe the third coordinates are equal. In particular, $(-4, 0, 4)$ is a point on both lines.

(b) The parametric form is given by this point and nonzero directions along the *directions* of the two respective lines, such as $(0, 2, 3)$ and $(5, 0, 1)$. Putting everything together, a parametric form for $\mathcal{P}$ is that it consists of points of the form

$$\begin{bmatrix} -4 + 5t' \\ 2t \\ 4 + 3t + t' \end{bmatrix}$$

for scalars t and t'.

(c) A nonzero normal vector $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ to the plane must satisfy $2b + 3c = 0$ and $5a + c = 0$. So $a = -c/5$ and $b = -3c/2$. Setting $c = 10$, we get $\mathbf{n} = \begin{bmatrix} -2 \\ -15 \\ 10 \end{bmatrix}$. Since $(-4, 0, 4)$ is a point on the plane, an equation form is given by $-2(x + 4) - 15y + 10(z - 4) = 0$, or equivalently $-2x - 15y + 10z = 48$. $\Diamond$

10. (a) A normal vector to $\mathcal{P}$ is $\mathbf{n} = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$, and $\mathbf{n}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$ is a normal vector to the xy-plane. Hence, the angle θ between the two satisfies

$$\cos(\theta) = \frac{|\mathbf{n} \cdot \mathbf{n}'|}{\|\mathbf{n}\|\|\mathbf{n}'\|} = \frac{|1|}{\sqrt{9}\sqrt{1}} = \frac{1}{3}.$$

(b) The conditions on $\mathbf{n} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ are orthogonality to difference vectors between one of the 3 points and the other two. That is, $\mathbf{n}$ is orthogonal to

$$\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 3 \end{bmatrix}, \quad \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} - \begin{bmatrix} 3 \\ 1 \\ 2 \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0 \end{bmatrix},$$

so the conditions are $-2a + 3c = 0$ and $-2a - b = 0$. This says $b = -2a$ and $c = 2a/3$. This yields the normal vector $\mathbf{n} = \begin{bmatrix} a \\ -2a \\ 2a/3 \end{bmatrix} = a \begin{bmatrix} 1 \\ -2 \\ 2/3 \end{bmatrix}$ for whatever nonzero a we like.

Setting $a = 3$ to remove the denominator (for convenience, certainly not necessary) gives $\mathbf{n} = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$ and again $\mathbf{n}'$ to be $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$, we have $\mathbf{n} \cdot \mathbf{n}' = 2$, so

$$\cos(\theta) = \frac{|\mathbf{n} \cdot \mathbf{n}'|}{\|\mathbf{n}\| \, \|\mathbf{n}'\|} = \frac{2}{\sqrt{9 + 36 + 4} \cdot 1} = \frac{2}{7}.$$

(c) The plane $\mathcal{P}$ has normal vector $\mathbf{n_1} = \begin{bmatrix} 2 \\ -2 \\ -1 \end{bmatrix}$ and the plane $\mathcal{Q}$ has normal vector $\mathbf{n_2} = \begin{bmatrix} 3 \\ -6 \\ 2 \end{bmatrix}$. These (nonzero) normal vectors are not scalar multiples of each other, so the two planes are not parallel. Let $\mathbf{v}$ be a displacement vector between distinct points in the overlap line L, so $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is perpendicular to both normal vectors $\mathbf{n}_1$ and $\mathbf{n}_2$ (as the line L is in each plane):

$$\mathbf{v} \cdot \mathbf{n}_1 = 0 \text{ and } \mathbf{v} \cdot \mathbf{n}_2 = 0,$$

which is to say

$$2x - 2y - z = 0 \text{ and } 3x - 6y + 2z = 0.$$

There are many non-zero solutions to these two simultaneous equations; we seek just one such to get a $\mathbf{v}$ that we can then use to obtain a parametric form for L.

Letting $z = 1$, we arrive at the pair of equations

$$2x - 2y = 1 \text{ and } 3x - 6y = -2$$

that has the solution $(5/3, 7/6)$, so we obtain a direction vector $\begin{bmatrix} 5/3 \\ 7/6 \\ 1 \end{bmatrix}$. Since any nonzero multiple of $\mathbf{v}$ is also a direction vector along L, for a cleaner final answer let's multiply by 6 to get rid of the denominator and thereby work with $\mathbf{v} = \begin{bmatrix} 10 \\ 7 \\ 6 \end{bmatrix}$. The line L then has the parametric form $\mathbf{x}_0 + t\mathbf{v}$ where $\mathbf{x}_0$ is any point of L, which is to say a common point on the planes $\mathcal{P}$ and $\mathcal{Q}$. We next find such an $\mathbf{x}_0$.

These planes have as their respective equations

$$2x - 2y - z = 3 \text{ and } 3x - 6y + 2z = 7,$$

for which we seek a common solution to get a choice of $\mathbf{x}_0$. For instance, setting z to be 0 gives the pair of equations $2x - 2y = 3$ and $3x - 6y = 7$ whose common solution is $(2/3, -5/6)$, so we obtain the parametric form

$$\begin{bmatrix} 2/3 \\ -5/6 \\ 0 \end{bmatrix} + t\mathbf{v} = \begin{bmatrix} 2/3 + 10t \\ -5/6 + 7t \\ 6t \end{bmatrix}$$

(many others are possible). ◊

11. (a) The point lies in the plane precisely when it satisfies the equation of the plane, so we compute $-2 + 5(1) + 2(-3) = -2 + 5 - 6 = 1$. This is not 3, so the answer is "no".

(b) The values of $-x + 5y + 2z$ at the points $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix}$ respectively are $-1 + 5(2) + 2(3) = -1 + 10 + 6 = 15$ and $-(-1) + 5(2) + 2(3) = 1 + 10 + 6 = 17$. Both values 15 and 17 are on the same side of 3, so these points do lie on the same side of $\mathcal{P}$; i.e., the answer is "yes".

(c) From the given parametric form of $\mathcal{P}$ we can see two such parametric lines immediately: the parametric line ℓ defined by $\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ -1 \\ 3 \end{bmatrix} = \begin{bmatrix} -3 + t \\ -t \\ 3t \end{bmatrix}$ (corresponding to $t' = 0$) and the parametric line ℓ' defined by $\begin{bmatrix} -3 \\ 0 \\ 0 \end{bmatrix} + t' \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 + 2t' \\ 0 \\ t' \end{bmatrix}$ (corresponding to $t = 0$). There are many other such lines too.

For the requested direct verification that the points of the line given do satisfy the equation for $\mathcal{P}$, we give this for both ℓ and ℓ'. We plug the parametric form into $-x + 5y + 2z$ and check that we always get "3":

$$-(-3 + t) + 5(-t) + 2(3t) = 3 - t - 5t + 6t = 3, \quad -(-3 + 2t') + 5(0) + 2(t') = 3 - 2t' + 2t' = 3.$$ ◊

12. (a) Such vectors are those of the form

$$\begin{bmatrix} x \\ y \\ 2x - y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 2x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ -y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix},$$

so this is the span of the vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

(b) This is not preserved under multiplication of anything in it by the scalar 0 (since that yields $\mathbf{0}$, which doesn't satisfy $z = 1 + 2x - y$), or alternatively if we pick some specific vector in here and add it to itself then the output will satisfy $z = 2 + 2x - y$ and so will not lie in this collection of vectors. For example, $\begin{bmatrix}1\\0\\3\end{bmatrix}$ satisfies $z = 1 + 2x - y$ but if we add it to itself then we get $\begin{bmatrix}2\\0\\6\end{bmatrix}$ which does not. (There are many other ways to violate the conclusion of Proposition 4.1.11.)

(c) This contains the vector $\left[\begin{smallmatrix}1\\1\end{smallmatrix}\right]$, but if we multiply this by any scalar $c \neq 0, 1$ (such as $c = 2, -3, \dots$) then we get $\left[\begin{smallmatrix}c\\c\end{smallmatrix}\right]$ which violates $y = x^2$ (since $c \neq c^2$ when $c \neq 0, 1$). There are many other ways to the conclusion of Proposition 4.1.11 (e.g., $\left[\begin{smallmatrix}2\\4\end{smallmatrix}\right] + \left[\begin{smallmatrix}3\\9\end{smallmatrix}\right] = \left[\begin{smallmatrix}5\\13\end{smallmatrix}\right]$ but $13 \neq 5^2$, etc.).

(d) The first condition says $z = y - 3z$, and plugging this into the second condition gives

$$0 = x + y - 4(y - 3x) = 13x - 3y,$$

so $y = (13/3)x$. Hence, $z = y - 3x = (13/3)x - 3x = (4/3)x$, so such vectors have the form

$$\begin{bmatrix}x\\(13/3)x\\(4/3)x\end{bmatrix} = x\begin{bmatrix}1\\13/3\\4/3\end{bmatrix} = (x/3)\begin{bmatrix}3\\13\\4\end{bmatrix}.$$

By varying x, we get exactly the span of $\mathbf{v} = \begin{bmatrix}3\\13\\4\end{bmatrix}$. Any vector in this span is readily checked to satisfy the two vanishing conditions, so this collection of vectors coincides with the span of $\mathbf{v}$. $\Diamond$

13. The condition for $\mathbf{x}$ to belong to V is the pair of equations

$$-2x_1 + 2x_2 + x_3 + x_4 = 0, \quad 3x_1 + 4x_2 + x_4 = 0$$

(expressing that $\mathbf{x} \cdot \mathbf{w} = 0$ and $\mathbf{x} \cdot \mathbf{w}' = 0$). We will use these two equations first to solve for x_3, x_4 in terms of x_1, x_2, and then solve for x_1, x_4 in terms of x_2, x_3.

(a) The two equations solved for x_4 give

$$-3x_1 - 4x_2 = x_4 = 2x_1 - 2x_2 - x_3.$$

Equating the outer terms allows us to solve for x_3 as

$$x_3 = 5x_1 + 2x_2.$$

Plugging this into either of the two preceding expressions for x_4 yields $x_4 = -3x_1 - 4x_2$. Hence,

$$\mathbf{x} = \begin{bmatrix}x_1\\x_2\\5x_1 + 2x_2\\-3x_1 - 4x_2\end{bmatrix} = \begin{bmatrix}x_1\\0\\5x_1\\-3x_1\end{bmatrix} + \begin{bmatrix}0\\x_2\\2x_2\\-4x_2\end{bmatrix} = x_1\begin{bmatrix}1\\0\\5\\-3\end{bmatrix} + x_2\begin{bmatrix}0\\1\\2\\-4\end{bmatrix}.$$

This expresses V as the span of $\begin{bmatrix}1\\0\\5\\-3\end{bmatrix}$ and $\begin{bmatrix}0\\1\\2\\-4\end{bmatrix}$.

(b) The two equations solved for x_1 give

$$x_2 + (1/2)x_3 + (1/2)x_4 = x_1 = -(4/3)x_2 - (1/3)x_4.$$

Equating the outer terms allows us to solve for x_4 via

$$(5/6)x_4 = -(7/3)x_2 - (1/2)x_3,$$

or equivalently

$$x_4 = -(14/5)x_2 - (3/5)x_3.$$

Plugging this into either of the two preceding expressions for x_1 yields $x_1 = -(2/5)x_2 + (1/5)x_3$. Hence,

$$\mathbf{x} = \begin{bmatrix} -(2/5)x_2 + (1/5)x_3 \\ x_2 \\ x_3 \\ -(14/5)x_2 - (3/5)x_3 \end{bmatrix} = \begin{bmatrix} -(2/5)x_2 \\ x_2 \\ 0 \\ -(14/5)x_2 \end{bmatrix} + \begin{bmatrix} (1/5)x_3 \\ 0 \\ x_3 \\ -(3/5)x_3 \end{bmatrix} = x_2 \begin{bmatrix} -(2/5) \\ 1 \\ 0 \\ -(14/5) \end{bmatrix} + x_3 \begin{bmatrix} 1/5 \\ 0 \\ 1 \\ -(3/5) \end{bmatrix}.$$

This expresses V as the span of $\begin{bmatrix} -(2/5) \\ 1 \\ 0 \\ -(14/5) \end{bmatrix}$ and $\begin{bmatrix} 1/5 \\ 0 \\ 1 \\ -(3/5) \end{bmatrix}$. ◊

14. The given condition on x, y, z says $z = -x + (3/2)y$, so

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ y \\ -x + (3/2)y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ (3/2)y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 3/2 \end{bmatrix}.$$

Running the same calculation in reverse, linear combinations as on the right are equal to vectors as on the left with $z = -x + (3/2)y$, which is exactly the desired condition $2x - 3y + 2z = 0$. Hence,

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad \mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ 3/2 \end{bmatrix}$$

works (as do a variety of other pairs of 3-vectors). ◊

15. (a) To show that the span or 1 or 2 nonzero 3-vectors has a nonzero orthogonal vector, it suffices to show that any single nonzero 3-vector $\mathbf{v}$ or pair of nonzero 3-vectors $\mathbf{v}, \mathbf{w}$ has a nonzero orthogonal vector $\mathbf{n}$ (since then anything in their span is also orthogonal to $\mathbf{n}$, since being orthogonal to $\mathbf{n}$ is preserved under passage to linear combinations, due to the good behavior of dot products: $(c\mathbf{v}) \cdot \mathbf{n} = c(\mathbf{v} \cdot \mathbf{n})$ and $(a\mathbf{v} + b\mathbf{w}) \cdot \mathbf{n} = a(\mathbf{v} \cdot \mathbf{n}) + b(\mathbf{w} \cdot \mathbf{n})$).

    Arguing geometrically: we can see that any line has lots of perpendicular lines in $\mathbf{R}^3$, and likewise for any plane.

    Arguing algebraically: if $\mathbf{v} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ is nonzero then an orthogonal vector is any $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying $ax + by + cz = 0$. At least one of x, y, z is nonzero. If $z \neq 0$ then we can use $b = 1$, let a be whatever we like (such as 0, $\sqrt{2}$, or whatever), and $c = (-ax - y)/z$. If instead $x \neq 0$ or $y \neq 0$ then we can do similarly. That shows algebraically that the span of a nonzero vector has a nonzero orthogonal vector. Likewise, for the span of 2 vectors we can also argue algebraically that there is a nonzero orthogonal vector, since the span is either a line (in which case we have already settled that) or it is a plane (in which case our method of parametrizing planes through the origin yields a nonzero normal vector).

    (b) The conditions on $\mathbf{n}$ amount to the simultaneous conditions

$$a - 2b + 3c = 0, \quad 2a + c = 0, \quad 3a - 2b + c = 0.$$

The second condition gives that $c = -2a$, so plugging this into the first and third conditions yields

$$0 = a - 2b + 3(-2a) = -5a - 2b, \;\; 0 = 3a - 2b - 2a = a - 2b,$$

so $a = (2/5)b$ and $a = 2b$. These latter two conditions force $a, b = 0$, so also $c = 0$.

(c) If the dimension of the span V of the three given nonzero 3-vectors isn't equal to 3 then it must be 1 or 2. But then by (a) there would be a *nonzero* 3-vector $\mathbf{n}$ orthogonal to V, and by (b) no such $\mathbf{n}$ exists. $\diamondsuit$

