1. Using the associativity of matrix multiplication, we have $(BA)B' = B(AB')$. Because B is an inverse of A, the left side is equal to $\mathrm{I}_n B' = B'$. Because B' is an inverse of A, the right side is equal to B. Putting this together, $B = B'$. $\Diamond$

2. (a) If we plug in the entries of A to the formula in Example 18.2.1, we see that A is invertible if $ad \neq 0$, i.e. if $a, d \neq 0$. In that case, the inverse is given by

$$A^{-1} = \frac{1}{ad}\begin{bmatrix} d & 0 \\ 0 & a \end{bmatrix} = \begin{bmatrix} a^{-1} & 0 \\ 0 & d^{-1} \end{bmatrix}.$$

The effect of A on $\mathbf{R}^2$ is to scale by a factor of a in the horizontal direction and d in the vertical direction; the effect of A^{-1} is scale by a factor of $1/a$ in the horizontal direction and $1/d$ in the vertical direction, which exactly undoes the effect of A (as we expect for A^{-1}).

(b) If one of a or d is zero, then either $A\begin{bmatrix} 1 \\ 0 \end{bmatrix} = \mathbf{0}$ or $A\begin{bmatrix} 0 \\ 1 \end{bmatrix} = \mathbf{0}$. On the other hand, $A\mathbf{0} = \mathbf{0}$ always. Thus, if one of our conditions fails, then the "output" $\mathbf{0}$ arises from two different "inputs." Thus A fails condition (a) of Proposition 18.1.5, and cannot be invertible. A more geometric version of the same argument is that if $a = 0$ or $d = 0$ then the effect of A crushes one of the coordinate directions, and this cannot be reversed, so A cannot be invertible.

(c) In part (a), we saw that A^{-1} was obtained from A by inverting the individual diagonal entries. So it makes sense to guess that

$$D^{-1} = \begin{bmatrix} d_1^{-1} & 0 & \cdots & 0 \\ 0 & d_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^{-1} \end{bmatrix}$$

whenever it exists. For this expression to make sense, we need each $d_i \neq 0$, and in such cases this does work: if we multiply these two diagonal matrices in either order then we get I_n:

$$\begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} \cdot \begin{bmatrix} d_1^{-1} & 0 & \cdots & 0 \\ 0 & d_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^{-1} \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}$$

and

$$\begin{bmatrix} d_1^{-1} & 0 & \cdots & 0 \\ 0 & d_2^{-1} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n^{-1} \end{bmatrix} \cdot \begin{bmatrix} d_1 & 0 & \cdots & 0 \\ 0 & d_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & d_n \end{bmatrix} = \begin{bmatrix} 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 1 \end{bmatrix}.$$

On the other hand, if some d_i vanishes then $D\mathbf{e}_i = \mathbf{0}$ since $D\mathbf{e}_i$ is the ith column of D, which vanishes. (This is what we did for $n = 2$ in (b), and the same calculation works in general.) So D cannot be invertible in such cases, exactly as in (b). $\Diamond$

3. (a) We just multiply:

$$M_a M_{-a} = \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a - a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}_3.$$

The same computation (but with a and $-a$ swapping roles) shows that $M_{-a} M_a = \mathrm{I}_3$.

(b) We can again just multiply everything out:

$$N_b N_{-b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b-b \\ 0 & 0 & 1 \end{bmatrix} = \mathrm{I}_3,$$

and $N_{-b}N_b = \mathrm{I}_3$ similarly.

(c) Note that

$$\begin{bmatrix} 1 & a & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & b \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = N_b M_a.$$

(This doesn't work if we multiply N_b and M_a in the other order.) Since N_b and M_a are invertible, so is their product with inverse $(N_b M_a)^{-1} = M_a^{-1} N_b^{-1}$. By parts (a) and (b), this is equal to

$$M_{-a}N_{-b} = \begin{bmatrix} 1 & -a & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & -a & ab \\ 0 & 1 & -b \\ 0 & 0 & 1 \end{bmatrix}. \qquad \Diamond$$

4. (a) We compute partial derivatives of component functions of $\mathbf{f}$ to obtain

$$(D\mathbf{f})(x,y) = \begin{bmatrix} 3x^2 - 3y^2 & -6xy \\ 6xy & 3x^2 - 3y^2 \end{bmatrix}.$$

[Just as an aside, let's check when this is invertible even though we don't need such generality for the rest of the problem. This will be invertible if its determinant is non-zero; that is, if $(3x^2-3y^2)(3x^2-3y^2)+6xy(6xy) \neq 0$. The left side is equal to $9x^4-18x^2y^2+9y^4+36x^2y^2 = 9x^2+18x^2y^2+9y^4 = 9(x^2+y^2)^2$. This vanishes precisely when $x^2+y^2=0$, which happens exactly when $(x,y)=(0,0)$. Hence, $(D\mathbf{f})(x,y)$ is invertible when $(x,y) \neq (0,0)$.]

(b) Using the formula from Example 18.2.1, $(D\mathbf{f})(2/3,1/3) = \begin{bmatrix} 1 & -4/3 \\ 4/3 & 1 \end{bmatrix}$ is invertible with inverse

$$\frac{1}{1(1)-(4/3)(-4/3)} \begin{bmatrix} 1 & 4/3 \\ -4/3 & 1 \end{bmatrix} = \frac{1}{25/9} \begin{bmatrix} 1 & 4/3 \\ -4/3 & 1 \end{bmatrix} = \begin{bmatrix} 9/25 & 12/25 \\ -12/25 & 9/25 \end{bmatrix}.$$

Also, $\mathbf{f}(\mathbf{a}_1) = (-25/27, 11/27)$. Thus,

$$\mathbf{a}_2 = \mathbf{a}_1 - \begin{bmatrix} 9/25 & 12/25 \\ -12/25 & 9/25 \end{bmatrix} \cdot \begin{bmatrix} -25/27 \\ 11/27 \end{bmatrix} = \begin{bmatrix} 2/3 \\ 1/3 \end{bmatrix} - \begin{bmatrix} -31/225 \\ 133/225 \end{bmatrix} = \begin{bmatrix} 181/225 \\ -58/225 \end{bmatrix} \approx \begin{bmatrix} 0.804 \\ -0.258 \end{bmatrix}.$$

(c) The point $\mathbf{a}_2$ differs from $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ by $\begin{bmatrix} 44/225 \\ 58/225 \end{bmatrix} = \frac{2}{225}\begin{bmatrix} 22 \\ 29 \end{bmatrix}$. This difference has length

$$\frac{2}{225}\sqrt{22^2+29^2} = \frac{2}{225}\sqrt{1325} = \frac{2}{45}\sqrt{53} \approx 0.32,$$

whereas the distance from $\mathbf{a}_1$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is $\sqrt{(1-2/3)^2+(-1/3)^2} = \sqrt{2/9} = \sqrt{2}/3 \approx 0.47$. Since the first distance $2\sqrt{53}/45$ is smaller than the second distance $\sqrt{2}/3$ (either by comparing the decimal approximations or by cross-multiplying and squaring both sides), this step of Newton's method does bring us closer to the solution $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$. The points $\mathbf{a}_1$ and $\mathbf{a}_2$ are in yellow near the red point $(1,0)$ in Figure 18.8.1 (showing that $\mathbf{a}_2$ is somewhat off the "direct route" from $\mathbf{a}_1$ to $\begin{bmatrix} 1 \\ 0 \end{bmatrix}$). $\Diamond$

5. (a) We have

$$\mathbf{v}'_j = \mathbf{e}_j - \frac{\mathbf{e}_j \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} = \mathbf{e}_j - \frac{\mathbf{e}_j \cdot \mathbf{v}}{30}\mathbf{v}$$

with $\mathbf{e}_j \cdot \mathbf{v}$ equal to the jth entry in $\mathbf{v}$. Hence, we compute

$$\mathbf{v}'_1 = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{2}{30}\begin{bmatrix}2\\-1\\5\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \frac{1}{15}\begin{bmatrix}2\\-1\\5\end{bmatrix} = \begin{bmatrix}1\\0\\0\end{bmatrix} - \begin{bmatrix}2/15\\-1/15\\1/3\end{bmatrix} = \begin{bmatrix}13/15\\1/15\\-1/3\end{bmatrix} = \frac{1}{15}\begin{bmatrix}13\\1\\-5\end{bmatrix},$$

$$\mathbf{v}'_2 = \begin{bmatrix}0\\1\\0\end{bmatrix} - \frac{-1}{30}\begin{bmatrix}2\\-1\\5\end{bmatrix} = \begin{bmatrix}0\\1\\0\end{bmatrix} + \begin{bmatrix}1/15\\-1/30\\1/6\end{bmatrix} = \begin{bmatrix}1/15\\29/30\\1/6\end{bmatrix} = \frac{1}{30}\begin{bmatrix}2\\29\\5\end{bmatrix},$$

and

$$\mathbf{v}'_3 = \begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{5}{30}\begin{bmatrix}2\\-1\\5\end{bmatrix} = \begin{bmatrix}0\\0\\1\end{bmatrix} - \frac{1}{6}\begin{bmatrix}2\\-1\\5\end{bmatrix} = \begin{bmatrix}-1/3\\1/6\\1/6\end{bmatrix} = \frac{1}{6}\begin{bmatrix}-2\\1\\1\end{bmatrix}.$$

To verify each $\mathbf{v}'_i$ really is orthogonal to $\mathbf{v} = \begin{bmatrix}2\\-1\\5\end{bmatrix}$ (as a safety check on our work), it is harmless to ignore the scalar factors in front and so we just compute the dot products

$$\begin{bmatrix}13\\1\\-5\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\5\end{bmatrix} = 26 - 1 - 25 = 26 - 26 = 0, \quad \begin{bmatrix}2\\29\\5\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\5\end{bmatrix} = 4 - 29 + 25 = 29 - 29 = 0,$$

and

$$\begin{bmatrix}-2\\1\\1\end{bmatrix} \cdot \begin{bmatrix}2\\-1\\5\end{bmatrix} = -4 - 1 + 5 = 0.$$

(b) Since P has dimension 2, any linearly independent set of size 2 in P is a basis. A pair of nonzero vectors is linearly dependent precisely when each is a scalar multiple of the other, so to give a basis of P is the same as to give two nonzero vectors in P for which neither is a scalar multiple of the other.

By inspection, all vectors $\mathbf{v}'_j$ computed in (a) are nonzero. We just need to check that no two are a scalar multiple of each other. To do this, it is harmless first to multiply each by whatever nonzero scalar we like, so we get rid of the fractional factors in the answer to (a) to reduce to considering the three vectors

$$\begin{bmatrix}13\\1\\-5\end{bmatrix}, \quad \begin{bmatrix}2\\29\\5\end{bmatrix}, \quad \begin{bmatrix}-2\\1\\1\end{bmatrix}.$$

None of these is a scalar multiple of the other, by considering ratios among nonzero vector entries (say first entry divided by the second, for example) or various other ways.

(c) Applying Gram-Schmidt to $\{\mathbf{v}'_3, \mathbf{v}'_1\}$ yields

$$\mathbf{w}_1 = \mathbf{v}'_3 = \frac{1}{6}\begin{bmatrix}-2\\1\\1\end{bmatrix}$$

(visibly nonzero) and

$$\mathbf{w}_2 = \mathbf{v}_1' - \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_1') = \mathbf{v}_1' - \frac{\mathbf{v}_1' \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\,\mathbf{w}_1.$$

To compute this final expression it is harmless to replace $\mathbf{w}_1$ with any nonzero scalar multiple we like, so we'll use $\mathbf{w}_1' = 6\mathbf{w}_1 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$:

$$\mathbf{w}_2 = \mathbf{v}_1' - \frac{\mathbf{v}_1' \cdot \mathbf{w}_1'}{\mathbf{w}_1' \cdot \mathbf{w}_1'}\,\mathbf{w}_1' = \mathbf{v}_1' - \frac{-30/15}{6}\,\mathbf{w}_1' = \mathbf{v}_1' + \frac{1}{3}\mathbf{w}_1'.$$

Plugging in $\mathbf{v}_1' = (1/15)\begin{bmatrix} 13 \\ 1 \\ -5 \end{bmatrix}$ and $\mathbf{w}_1' = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$ then yields

$$\mathbf{w}_2 = \frac{1}{15}\left(\begin{bmatrix} 13 \\ 1 \\ -5 \end{bmatrix} + \begin{bmatrix} -10 \\ 5 \\ 5 \end{bmatrix}\right) = \frac{1}{15}\begin{bmatrix} 3 \\ 6 \\ 0 \end{bmatrix} = \frac{1}{5}\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}.$$

As a final safety check, we verify that $\mathbf{w}_1$ and $\mathbf{w}_2$ are orthogonal. For this it is harmless to ignore scalar factors in front, so we compute the dot product

$$\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} = -2 + 2 + 0 = 0$$

as desired. $\diamondsuit$

6. (a) These are direct calculations:

$$3\begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} + 2\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} - 3\begin{bmatrix} 9 \\ 5 \\ -3 \\ 0 \end{bmatrix} - 2\begin{bmatrix} -7 \\ -7 \\ 9 \\ 4 \end{bmatrix} = \begin{bmatrix} 3+10-27+14 \\ -3+4-15+14 \\ 9+0+9-18 \\ 6+2+0-8 \end{bmatrix} = \begin{bmatrix} 27-27 \\ -18+18 \\ 18-18 \\ 8-8 \end{bmatrix} = \mathbf{0}$$

and

$$-5\begin{bmatrix} 1 \\ -1 \\ 3 \\ 2 \end{bmatrix} + 6\begin{bmatrix} 5 \\ 2 \\ 0 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 9 \\ 5 \\ -3 \\ 0 \end{bmatrix} + \begin{bmatrix} -7 \\ -7 \\ 9 \\ 4 \end{bmatrix} = \begin{bmatrix} -5+30-18-7 \\ 5+12-10-7 \\ -15+0+6+9 \\ -10+6+0+4 \end{bmatrix} = \begin{bmatrix} 25-25 \\ 17-17 \\ -15+15 \\ -10+10 \end{bmatrix} = \mathbf{0}.$$

(b) Adding twice the second relation to the first eliminates the $\mathbf{v}_4$-terms, yielding

$$\mathbf{0} = (3\mathbf{v}_1 + 2\mathbf{v}_2 - 3\mathbf{v}_3 - 2\mathbf{v}_4) + 2(-5\mathbf{v}_1 + 6\mathbf{v}_2 - 2\mathbf{v}_3 + \mathbf{v}_4) = -7\mathbf{v}_1 + 14\mathbf{v}_2 - 7\mathbf{v}_3 = 7(-\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3),$$

so $-\mathbf{v}_1 + 2\mathbf{v}_2 - \mathbf{v}_3 = \mathbf{0}$. Bringing $\mathbf{v}_3$ to the other side yields

$$\mathbf{v}_3 = -\mathbf{v}_1 + 2\mathbf{v}_2.$$

Likewise, multiplying the first relation by 2 and subtracting from this 3 times the second relation eliminates the $\mathbf{v}_3$-terms, yielding

$$\mathbf{0} = 2(3\mathbf{v}_1 + 2\mathbf{v}_2 - 3\mathbf{v}_3 - 2\mathbf{v}_4) - 3(-5\mathbf{v}_1 + 6\mathbf{v}_2 - 2\mathbf{v}_3 + \mathbf{v}_4) = 21\mathbf{v}_1 - 14\mathbf{v}_2 - 7\mathbf{v}_4 = 7(3\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_4),$$

so $3\mathbf{v}_1 - 2\mathbf{v}_2 - \mathbf{v}_4 = \mathbf{0}$. Bringing $\mathbf{v}_4$ to the other side yields

$$\mathbf{v}_4 = 3\mathbf{v}_1 - 2\mathbf{v}_2.$$

(c) These are direct calculations:

$$-\mathbf{v}_1 + 2\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ -3 \\ -2 \end{bmatrix} + \begin{bmatrix} 10 \\ 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} 9 \\ 5 \\ -3 \\ 0 \end{bmatrix} = \mathbf{v}_3$$

and

$$3\mathbf{v}_1 - 2\mathbf{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 9 \\ 6 \end{bmatrix} - \begin{bmatrix} 10 \\ 4 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -7 \\ -7 \\ 9 \\ 4 \end{bmatrix} = \mathbf{v}_4. \qquad \diamondsuit$$

7. (a) We expand the left side to rewrite it as $a_1\mathbf{v}_1 - 3t\mathbf{v}_1 + a_2\mathbf{v}_2 + 7t\mathbf{v}_2 + a_3\mathbf{v}_3 - 5t\mathbf{v}_3 + a_4\mathbf{v}_4 + 2t\mathbf{v}_4$ and then collect the t-parts to rewrite this as

$$(a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4) + t(-3\mathbf{v}_1 + 7\mathbf{v}_2 - 5\mathbf{v}_3 + 2\mathbf{v}_4).$$

The assumed linear dependence relation gives that the t-part vanishes, leaving us with $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 + a_3\mathbf{v}_3 + a_4\mathbf{v}_4$, as desired.

(b) By (a) with $a_1 = 2$, $a_2 = 4$, $a_3 = 3$, and $a_4 = -5$ we see that for any t at all,

$$(-2 - 3t)\mathbf{v}_1 + (4 + 7t)\mathbf{v}_2 + (3 - 5t)\mathbf{v}_3 + (-5 + 2t)\mathbf{v}_4 = -2\mathbf{v}_1 + 4\mathbf{v}_2 + 3\mathbf{v}_3 - 5\mathbf{v}_4.$$

By using two different nonzero values for t, we get two different 4-tuples of coefficients yielding the same linear combination of the $\mathbf{v}_i$'s as desired. For instance, by using $t = 1$ and then $t = -2$ we get

$$(b_1, b_2, b_3, b_4) = (-2 - 3, 4 + 7, 3 - 5, -5 + 2) = (-5, 11, -2, -3)$$

and

$$(c_1, c_2, c_3, c_4) = (-2 + 6, 4 - 14, 3 + 10, -5 - 4) = (4, -10, 13, -9).$$

(c) By subtracting the right side from the left side and combining $\mathbf{w}_i$-terms for each i, we get

$$(a_1 - b_1)\mathbf{w}_1 + (a_2 - b_2)\mathbf{w}_2 + (a_3 - b_3)\mathbf{w}_3 + (a_4 - b_4)\mathbf{w}_4 = \mathbf{0}.$$

But one of the features of linear independence is that the *only* way a linear combination can yield the zero vector is when the coefficients all vanish. Hence, each coefficient $a_i - b_i$ must equal 0, which is the same as saying $a_i = b_i$ for all i, as desired. $\diamondsuit$

8. (a) We see that both columns of A are orthogonal and have length 1, so it is orthogonal. Hence,

$$A^{-1} = A^\top = (1/\sqrt{5}) \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}.$$

(b) While the columns of B are nonzero and orthogonal to each other, they are not unit vectors. Hence, this is not an orthogonal matrix.

(c) We have

$$B' = \frac{1}{2} \begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

with the columns pairwise orthogonal (this is inherited from B) and now all unit vectors. Hence, B' is indeed an orthogonal matrix and so

$$B'^{-1} = B^\top = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

(d) We have $B = 2B'$, so (using the inverse of B' found in (c))

$$B^{-1} = \frac{1}{2}B'^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & -1 & 1 & 1 \\ -1 & 1 & -1 & 1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$ ◊

9. (a) Let $B = \frac{1}{2}(A + A^\top)$ and let b_{ij} be the entry in the ith row and jth column of B. Then $b_{ij} = \frac{1}{2}(a_{ij} + a_{ji})$. If we look at b_{ji}, we get $b_{ji} = \frac{1}{2}(a_{ji} + a_{ij}) = \frac{1}{2}(a_{ij} + a_{ji}) = b_{ij}$. Alternatively, we can compute

$$B^\top = \frac{1}{2}(A + A^\top)^\top = \frac{1}{2}(A^\top + A^{\top\top}) = \frac{1}{2}(A^\top + A) = \frac{1}{2}(A + A^\top) = B,$$

so B is symmetric.

(b) We have $\|B\mathbf{x}\|^2 = (B\mathbf{x}) \cdot (B\mathbf{x}) = (B\mathbf{x})^\top (B\mathbf{x}) = \mathbf{x}^\top B^\top B\mathbf{x}$, so we use

$$M = B^\top B$$

(which is clearly $n \times n$). This is symmetric since $M^\top = (B^\top B)^\top = B^\top (B^\top)^\top = B^\top B = M$. ◊

10. (a) It meets the xy-plane in the curve given by setting z to be 0 in the equation of S_+, which is the circle $x^2 + y^2 = 1$ in the xy-plane. It meets the xz-plane in the curve given by setting y to be 0 in the equation of S_+, which is the hyperbola $x^2 - z^2 = 1$ in the xz-plane. Similarly it meets yz-plane in the hyperbola $y^2 - z^2 = 1$. The curves along which S_+ meets each of the coordinate planes is shown in Figure 2.

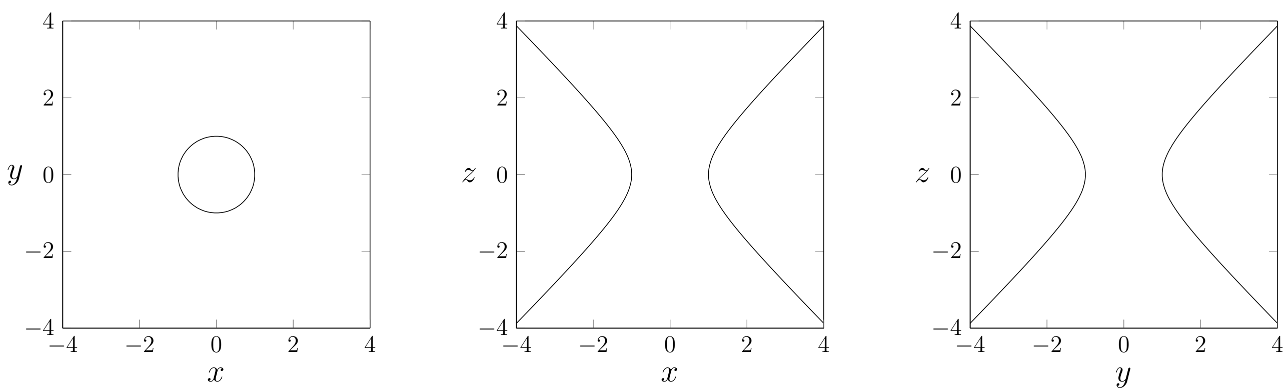

FIGURE 2. The curves where S_+ meets the xy-plane (left), xz-plane (center), and yz-plane (right).

(b) By setting z to be a in the equation of S_+, we get the equation $x^2 + y^2 = 1 + a^2$. This is a circle with radius $\sqrt{1 + a^2}$ having center $(0, 0, a)$ on the z-axis. Rotating around the z-axis has the effect in each plane $z = a$ of rotating around the origin in that plane. But that origin corresponds to the point $(0, 0, a)$ that is the center of the circle $x^2 + y^2 = 1 + a^2$ in that plane, so such rotation carries the circle into itself. Since S_+ is assembled from where it meets each plane (every point in S_+ lies in exactly one such plane, namely the one for which a is the third coordinate of the point), this information for every a tells us that every such rotation carries S_+ into itself.

(c) We claim that every point P in space can be rotated around the z-axis to wind up in the half-plane $x \geq 0$ in the xz-plane. This can be seen by drawing a 3-dimensional picture, but here is another way. If a is the third coordinate of P and we work inside the plane $z = a$ through P that meets the xz-plane along the "x-axis" of points $(x, 0, a)$ in that plane, this becomes the assertion that any point in a plane can be rotated around the origin to wind up on the non-negative x-axis, which we see by staring at points in a plane. Hence, anything in space that is rotationally symmetric around the z-axis is obtained by taking the part of it in the half-plane $x \geq 0$ in the xz-plane and rotating that around the z-axis.

Now consider the region in S_+ where it meets the half-plane $x \geq 0$ in the xz-plane $y = 0$ meets S_+. This is given by setting y to be 0 and requiring $x \geq 0$ in the equation of S_+. The equation is $Q(x, 0, z) = 1$, which is to say the hyperbola $x^2 - z^2 = 1$, and by demanding $x \geq 0$ it is exactly $x = \sqrt{1 + z^2}$. This is one of the two connected parts (or "branches") of the hyperbola $x^2 - z^2 = 1$. Hence, S_+ is obtained by rotating the curve $x = \sqrt{1 + z^2}$ (one "branch" of $x^2 - z^2 = 1$) around the z-axis, as shown on the left in Figure 3.

(d) It meets each plane $z = a$ in the region given by the equation $x^2 + y^2 = a^2 - 1$. This is a circle centered at the origin when $|a| > 1$, it is the point $(0, 0, a)$ when $a = \pm 1$, and it is empty when $|a| < 1$. In all cases, rotation around the origin in the plane $z = a$ carries this into itself, so once again S_- is rotationally symmetric around the z-axis.

(e) The part of S_- in the xz-plane is given by setting y to be 0 in the equation of S_-; this is $Q(x, 0, z) = -1$, or equivalently $x^2 - z^2 = -1$. This is again a hyperbola (written equivalently as $z^2 - x^2 = 1$). Imposing the condition $x \geq 0$ yields the curve $x = \sqrt{z^2 - 1}$, which only makes sense when $|z| \geq 1$, and for $z \geq 1$ and for $z \leq -1$ it yields two curves that *do not touch*: one curve C with $z > 0$ and another curve C' with $z < 0$. Applying rotation around the z-axis to this curve consisting of two parts recovers the entirety of S_- by (d), so this level set is a surface consisting of two parts (each obtained from rotating one of the two curves C, C' around the z-axis). This surface is shown on the right in Figure 3.

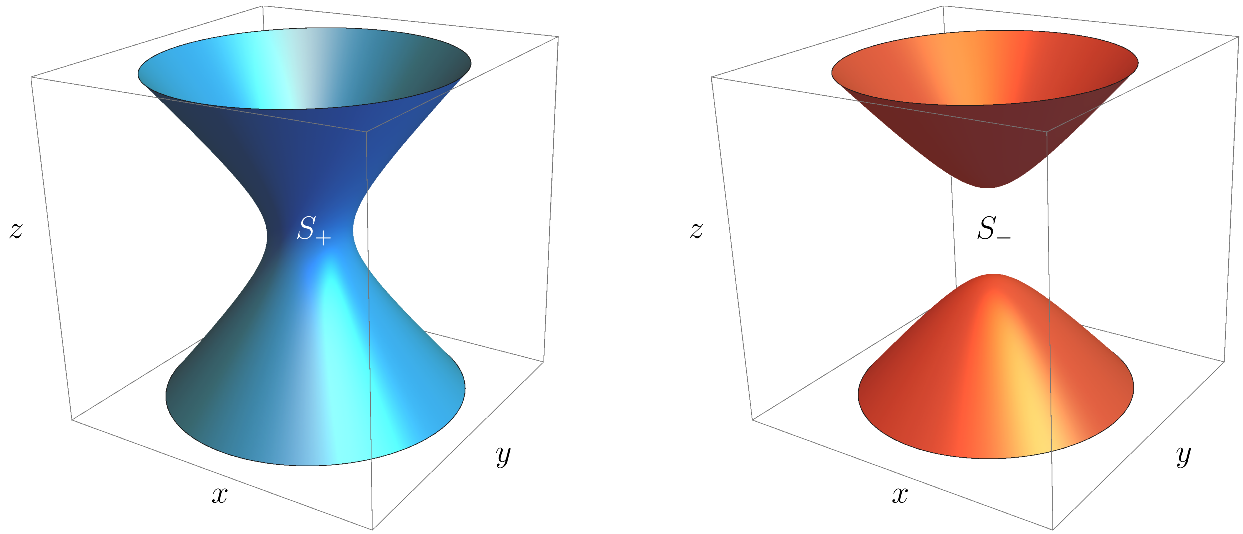

FIGURE 3. The surfaces S_+ (left) and S_- (right).

$\Diamond$

11. (a) We compute

$$\begin{aligned} H_{\mathbf{v}}^\top = (\mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top)^\top = \mathrm{I}_n^\top - \frac{2}{\|\mathbf{v}\|^2}(\mathbf{v}\mathbf{v}^\top)^\top &= \mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}(\mathbf{v}^\top)^\top\mathbf{v}^\top \\ &= \mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top \\ &= H_{\mathbf{v}}. \end{aligned}$$

(b) We compute

$$\begin{aligned} H_{\mathbf{v}}^\top H_{\mathbf{v}} = H_{\mathbf{v}} H_{\mathbf{v}} &= (\mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top)(\mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top) \\ &= \mathrm{I}_n^2 - \frac{4}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top + \frac{4}{\|\mathbf{v}\|^4}\mathbf{v}(\mathbf{v}^\top)\mathbf{v}\mathbf{v}^\top \\ &= \mathrm{I}_n - \frac{4}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top + \frac{4}{\|\mathbf{v}\|^4}\mathbf{v}(\|\mathbf{v}\|^2)\mathbf{v}^\top \\ &= \mathrm{I}_n - \frac{4}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top + \frac{4}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top \\ &= \mathrm{I}_n. \end{aligned}$$

(c) We compute

$$\begin{aligned} H_{\mathbf{v}}(c\mathbf{v}) = (\mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top)c\mathbf{v} = c\mathbf{v} - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}(\mathbf{v}^\top c\mathbf{v}) &= c\mathbf{v} - c\frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\|\mathbf{v}\|^2 \\ &= c\mathbf{v} - 2c\mathbf{v} \\ &= -c\mathbf{v}. \end{aligned}$$

(d) We compute

$$H_{\mathbf{v}}(\mathbf{u}) = (\mathrm{I}_n - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top)\mathbf{u} = \mathbf{u} - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}(\mathbf{v}^\top\mathbf{u}),$$

and the final term being subtracted is $\mathbf{0}$ since $\mathbf{v}^\top \mathbf{u}$ is the 1×1 matrix whose unique entry is $\mathbf{v} \cdot \mathbf{u} = 0$ (by hypothesis). Thus, we are left with just $\mathbf{u}$ (as desired).

(e) A normal vector to the plane is $\mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$, and the corresponding Householder reflection matrix is given by

$$H_{\mathbf{v}} = \mathrm{I}_3 - \frac{2}{\|\mathbf{v}\|^2}\mathbf{v}\mathbf{v}^\top = \mathrm{I}_3 - \frac{2}{9}\begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}\begin{bmatrix} 2 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{9}\begin{bmatrix} 4 & -2 & 4 \\ -2 & 1 & -2 \\ 4 & -2 & 4 \end{bmatrix}$$
$$= \begin{bmatrix} 1/9 & 4/9 & -8/9 \\ 4/9 & 7/9 & 4/9 \\ -8/9 & 4/9 & 1/9 \end{bmatrix}.$$

(By inspection this is symmetric, as it has to be, and it is orthogonal since the columns are seen by inspection to be mutually perpendicular – for that we can ignore the common factor of $1/9$ everywhere – and all have length 1 because the numerators constitute a matrix whose columns all have squared-length equal to $81 = 9^2$.) ♢

12. (a) Each of A and C has two columns, so each of $N(A)$ and $N(C)$ is a subspace of $\mathbf{R}^2$. On the other hand, $N(B)$ is a subspace of $\mathbf{R}^4$.

(b) The null space of A is the set of solutions $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2$ to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Similarly, $N(B)$ and $N(C)$ are the sets of solutions to $B\mathbf{x} = \mathbf{0}$ and $C\mathbf{x} = \mathbf{0}$, respectively. Plugging in the entries of A, B, C, these homogeneous systems are

$$\begin{cases} x - 2y &= 0 \\ 2x - 4y &= 0 \end{cases},$$

$$\begin{cases} x - 3y + 0z + w &= 0 \\ 2x + 5y - z + 0w &= 0 \\ 3x + 2y - z + w &= 0 \end{cases},$$

and

$$\begin{cases} 3x + y &= 0 \\ -4x - y &= 0 \end{cases},$$

respectively.

(c) In the system for $N(A)$, we note that the second equation is obtained from the first by multiplying everything through by 2. Thus, the second equation gives us no new information, so $N(A)$ is the set of vectors $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbf{R}^2$ for which $x = 2y$. We can thus describe $N(A)$ as the line $\left\{\begin{bmatrix} 2y \\ y \end{bmatrix} : y \in \mathbf{R}\right\}$ in $\mathbf{R}^2$, which has $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ as a basis vector.

For $N(C)$, using the first equation we get $y = -3x$. Substituting this into the second equation, we obtain $-4x + 3x = 0$. That is, $-x = 0$. Plugging this back into the first equation implies $y = 0$. Thus, $N(C) = \{(0,0)\} \subset \mathbf{R}^2$. That is, $N(C)$ is the origin (consistent with the fact that C is invertible, since an invertible transformation does not carry distinct inputs to the same output vector such as $\mathbf{0}$).

For $N(B)$, the first two equations respectively say

$$w = -x + 3y, \quad z = 2x + 5y.$$

Hence, these say that $N(B)$ consists of 4-vectors of the form

$$\begin{bmatrix} x \\ y \\ 2x+5y \\ -x+3y \end{bmatrix} = \begin{bmatrix} x \\ 0 \\ 2x \\ -x \end{bmatrix} + \begin{bmatrix} 0 \\ y \\ 5y \\ 3y \end{bmatrix} = x \begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix} + y \begin{bmatrix} 0 \\ 1 \\ 5 \\ 3 \end{bmatrix}$$

(the equation arising from the third row imposes no further condition, since any vector of the form on the left is checked to automatically satisfy the third equation; explicitly, this is due to the third row being the sum of the first two rows, so the corresponding third equation in the system for B in (b) is the sum of the first two equations and hence gives no new information).

This is the span of the two 4-vectors $\begin{bmatrix} 1 \\ 0 \\ 2 \\ -1 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 1 \\ 5 \\ 3 \end{bmatrix}$ that are nonzero and not scalar multiples of each other, so these two vectors constitute a basis for $N(B)$. $\Diamond$

13. (a) Recall that $C(A)$ is the span of the columns of A and $C(A')$ is the span of the columns of A'. Since A' has all of the columns of A, plus one extra, we conclude that $C(A)$ is spanned by $C(A)$ and $\mathbf{v}$. Algebraically, this amounts to the calculations (which you don't need to have done)

$$c(a_1\mathbf{v}_1 + \cdots + a_k\mathbf{v}_k) + c'\mathbf{v} = (ca_1)\mathbf{v}_1 + \cdots + (ca_k)\mathbf{v}_k + c'\mathbf{v}$$

and

$$c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k + c_{k+1}\mathbf{v} = 1(c_1\mathbf{v}_1 + \cdots + c_k\mathbf{v}_k) + c_{k+1}\mathbf{v}_k.$$

To analyze the dimensions of these column spaces, let $\mathbf{v}_1, \dots, \mathbf{v}_k$ be a basis for $C(A)$, so $k = \dim C(A)$. Since $C(A')$ is spanned by these k vectors along with $\mathbf{v}$, we have $\dim C(A') \leq 1 + k = 1 + \dim C(A)$. But also $C(A) \subset C(A')$ forces $\dim C(A) \leq \dim C(A')$. (Concretely, since any linearly independent subset of a linear subspace of $\mathbf{R}^m$ can be extended to a basis, a basis for $C(A)$ can be extended to a basis for $C(A')$ and so the common size of all bases of $C(A')$ is at least as big as those of $C(A)$.)

(b) First, if $C(A) = C(A')$, then certainly $\mathbf{v} \in C(A') = C(A)$. On the other hand, if $\mathbf{v} \in C(A)$ then $\mathbf{v} = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$, where $c_1, \dots, c_n \in \mathbf{R}$ and $\mathbf{a}_i$ is the ith column of A ($1 \leq i \leq n$), so anything in $C(A')$ has the form

$$x_1\mathbf{a}_1 + \cdots + x_n\mathbf{a}_n + x\mathbf{v} = (x_1 + c_1x)\mathbf{a}_1 + \cdots + (x_n + c_nx)\mathbf{a}_n,$$

which means it belongs to $C(A)$, as desired.

(c) If $\mathbf{v} \in C(A)$ then as in the solution for (b) we can write $\mathbf{v}$ as a linear combination of the columns of $C(A)$: $\mathbf{v} = c_1\mathbf{a}_1 + \cdots + c_n\mathbf{a}_n$ with $c_1, \dots, c_n \in \mathbf{R}$ and $\mathbf{a}_i$ is the ith column of A ($1 \leq i \leq n$). But then

$$A' \begin{bmatrix} c_1 \\ \vdots \\ c_n \\ -1 \end{bmatrix} = \mathbf{0}.$$

Since $\begin{bmatrix} c_1 \\ \vdots \\ c_n \\ -1 \end{bmatrix} \neq \mathbf{0}$ (look at the bottom entry of this vector), it is a non-zero vector $N(A')$. $\Diamond$

14. (a) An orthogonal basis is $\{\mathbf{a}_1, \mathbf{a}_2'\}$ for

$$\mathbf{a}_2' = \mathbf{a}_2 - \frac{\mathbf{a}_2 \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1} \mathbf{a}_1 = \mathbf{a}_2 - \frac{44}{11} \mathbf{a}_1 = \mathbf{a}_2 - 4\mathbf{a}_1 = \begin{bmatrix} 2 \\ 13 \\ -3 \end{bmatrix} - \begin{bmatrix} 4 \\ 12 \\ -4 \end{bmatrix} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.$$

The given relations $\mathbf{a}_3 = -3\mathbf{a}_1 + 2\mathbf{a}_2$ and $\mathbf{a}_4 = \mathbf{a}_1 - \mathbf{a}_2$ (easily verified directly) can be written as linear dependence relations among columns:

$$-3\mathbf{a}_1 + 2\mathbf{a}_2 - \mathbf{a}_3 = \mathbf{0}, \quad \mathbf{a}_1 - \mathbf{a}_2 - \mathbf{a}_4 = \mathbf{0},$$

whose left sides are respectively $A \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ via the relationship between $A\mathbf{x}$ and a linear combination of columns of A. This explains why $\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$ belong to $N(A)$.

(b) For any 3-vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$ we have

$$\begin{aligned}
\mathbf{Proj}_{C(A)}(\mathbf{b}) = \mathbf{Proj}_{\mathbf{a}_1}(\mathbf{b}) + \mathbf{Proj}_{\mathbf{a}_2'}(\mathbf{b}) &= \frac{\mathbf{b} \cdot \mathbf{a}_1}{\mathbf{a}_1 \cdot \mathbf{a}_1} \mathbf{a}_1 + \frac{\mathbf{b} \cdot \mathbf{a}_2'}{\mathbf{a}_2' \cdot \mathbf{a}_2'} \mathbf{a}_2' \\
&= \frac{b_1 + 3b_2 - b_3}{11} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + \frac{-2b_1 + b_2 + b_3}{6} \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}.
\end{aligned}$$

Plugging in $\mathbf{b}_1 = \begin{bmatrix} 1 \\ 6 \\ 8 \end{bmatrix}$, $\mathbf{b}_2 = \begin{bmatrix} 4 \\ 1 \\ 7 \end{bmatrix}$, and $\mathbf{b}_3 = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix}$ respectively yields

$$\mathbf{Proj}_{C(A)}(\mathbf{b}_1) = \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -3 \\ 5 \\ 1 \end{bmatrix} \neq \mathbf{b}_1,$$

$$\mathbf{Proj}_{C(A)}(\mathbf{b}_2) = - \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} + 2 \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix} = \mathbf{b}_2,$$

$$\mathbf{Proj}_{C(A)}(\mathbf{b}_3) = - \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} = \mathbf{b}_3.$$

(Of course, these projections could be computed directly without first going through a formula for general $\mathbf{b}$ as we did above.) Hence, $A\mathbf{x} = \mathbf{b}_1$ does not have a solution whereas $A\mathbf{x} = \mathbf{b}_2$ and $A\mathbf{x} = \mathbf{b}_3$ each have a solution.

The work we did to verify that $A\mathbf{x} = \mathbf{b}_2$ has a solution yields an expression for $\mathbf{b}_2$ in terms of the first and second columns of A:

$$\mathbf{b}_2 = -\mathbf{a}_1 + 2\mathbf{a}_2' = -\mathbf{a}_1 + 2(\mathbf{a}_2 - 4\mathbf{a}_1) = -9\mathbf{a}_1 + 2\mathbf{a}_2,$$

so this says $\mathbf{b}_2 = A \begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix}$. In other words, one solution to $A\mathbf{x} = \mathbf{b}_2$ is $\mathbf{x}_0 = \begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix}$. (There are many other solutions, as we will see in part (c).) This is readily verified to really be a solution:

$$A\mathbf{x}_0 = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & 13 & 17 & -10 \\ -1 & -3 & -3 & 2 \end{bmatrix} \begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} -9+4 \\ -27+26 \\ 9-6 \end{bmatrix} = \begin{bmatrix} -5 \\ -1 \\ 3 \end{bmatrix} = \mathbf{b}_2.$$

Likewise, work we did to verify that $A\mathbf{x} = \mathbf{b}_3$ has a solution yields an expression for $\mathbf{b}_3$ in terms of the first and second columns of A:

$$\mathbf{b}_3 = -\mathbf{a}_1 - \mathbf{a}_2' = -\mathbf{a}_1 - (\mathbf{a}_2 - 4\mathbf{a}_1) = 3\mathbf{a}_1 - \mathbf{a}_2,$$

so this says $\mathbf{b}_3 = A \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. In other words, one solution to $A\mathbf{x} = \mathbf{b}_3$ is $\mathbf{x}_0' = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$. (There are many other solutions, as we will see in part (c).) This is readily verified to really be a solution:

$$A\mathbf{x}_0' = \begin{bmatrix} 1 & 2 & 1 & -1 \\ 3 & 13 & 17 & -10 \\ -1 & -3 & -3 & 2 \end{bmatrix} \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 3-2 \\ 9-13 \\ -3+3 \end{bmatrix} = \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} = \mathbf{b}_3.$$

(c) The solutions to $A\mathbf{x} = \mathbf{b}_2$ are the vectors of the form $\mathbf{x}_0 + \mathbf{v}$ for $\mathbf{x}_0 = \begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} \in N(A)$, where we know from (a) that $N(A)$ has as a basis $\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$. Hence, solutions are given by the parametric form

$$\begin{bmatrix} -9 \\ 2 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t' \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} -9-3t+t' \\ 2+2t-t' \\ -t \\ -t' \end{bmatrix}$$

for $t, t' \in \mathbf{R}$.

The solutions to $A\mathbf{x} = \mathbf{b}_3$ are the vectors of the form $\mathbf{x}_0' + \mathbf{v}$ for $\mathbf{x}_0' = \begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix}$ and $\mathbf{v} \in N(A)$, where we know from (a) that $N(A)$ has as a basis $\begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix}$. Hence, solutions are given by the parametric form

$$\begin{bmatrix} 3 \\ -1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 2 \\ -1 \\ 0 \end{bmatrix} + t' \begin{bmatrix} 1 \\ -1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 3 - 3t + t' \\ -1 + 2t - t' \\ -t \\ -t' \end{bmatrix}$$

for $t, t' \in \mathbf{R}$. $\diamondsuit$

15. (a) Since $C(A)$ is a linear subspace of $\mathbf{R}^2$, its dimension is at most 2, hence is either 0, 1, or 2. The same applies to $N(A)$. But $C(A)$ is nonzero since some column is nonzero, so its dimension is 1 or 2; this says that $C(A)$ is either a line through the origin (dimension 1) or the entirety of $\mathbf{R}^2$ (the only n-dimensional linear subspace of $\mathbf{R}^n$ is itself). Likewise, $N(A)$ can't exhaust $\mathbf{R}^2$ since the columns are $A\mathbf{e}_1$ and $A\mathbf{e}_2$, at least one of which is nonzero by assumption. This rules out the case $\dim N(A) = 2$, so $\dim N(A) \leq 1$. The 1-dimensional case is when it is a line through the origin, and the 0-dimensional case is when it consists of just the origin.

    (b) If some column vanishes then the other column is nonzero and $C(A)$ is the line spanned by that nonzero column. Likewise, $N(A)$ is the line corresponding to the coordinate axis for the vanishing column. So the case when some column vanishes checks out fine, and now we may and do suppose both columns are nonzero. In particular, the columns are linearly dependent precisely when each column is a scalar multiple of the other.

    Since A has a nonzero column, the column space is a line exactly when such a column spans the column space (any nonzero vector in a line through the origin spans that line). In such cases the other column (which could be $\mathbf{0}$) also belongs to the same line and so is a scalar multiple of the initial choice of nonzero column. In other words, if $C(A)$ is a line then some column is a scalar multiple of the other, and so the columns are linearly dependent. On the other hand, when the columns $\mathbf{a}_1$ and $\mathbf{a}_2$ are linearly dependent then we have $c_1\mathbf{a}_1 + c_2\mathbf{a}_2 = \mathbf{0}$ with some $c_i \neq 0$, so dividing by the nonzero coefficient expresses one column as a scalar multiple of the other. This forces the latter column to span the entire $C(A)$, but $C(A)$ is nonzero and so (being spanned by one vector) it must be a line.

    Next, we turn our attention to $N(A)$. This is a line exactly when it is nonzero (since its dimension is 0 or 1), and to say a nonzero vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$ lies in $N(A)$ is to say $A\mathbf{x} = \mathbf{0}$. But

    $$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2$$

    where $\mathbf{a}_i$ is the ith column of A, so the vanishing of $A\mathbf{x}$ for a nonzero $\mathbf{x}$ is exactly the vanishing of a linear combination $x_1\mathbf{a}_1 + x_2\mathbf{a}_2$ with $\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \neq \mathbf{0}$, which is to say $x_1 \neq 0$ or $x_2 \neq 0$. But this in turn expresses precisely linear dependence of the columns (i.e., the vanishing of some linear combination using at least one nonzero coefficient).

(c) First suppose $C(A) = \mathbf{R}^2$, so $C(A)$ is not a line; i.e., $\dim C(A) \neq 1$. By (b) it follows that $\dim N(A) \neq 1$. By (a), $N(A)$ is either a line through the origin or just the origin itself, so the only option we have is that $N(A)$ is equal to the origin. Next, suppose $N(A)$ is equal to the origin, so $\dim N(A) \neq 1$. By (b), it follows that $\dim C(A) \neq 1$. But by (a) the only possibilities for $\dim C(A)$ are 1 and 2, so $\dim C(A) = 2$. The only 2-dimensional subspace of $\mathbf{R}^2$ is itself, so $C(A) = \mathbf{R}^2$ as desired. $\Diamond$

