Math 51
Lecture 21 – August 4, 2023

Topic(s): solution(s) to linear systems, column space, null space

Let us consider the system of linear equations

$$\begin{aligned} 2x + y - z &= 1 \\ x + 2y + z &= 4 \\ -x - y + z &= 2 \end{aligned}$$

We have seen that this can be written as $A\mathbf{x} = \mathbf{b}$, where A is the coefficient matrix $\begin{bmatrix} 2 & 1 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 1 \end{bmatrix}$, $\mathbf{x}$ is the variable vector $\begin{bmatrix} x \\ y \\ z \end{bmatrix}$, and $\mathbf{b}$ is the scalar vector $\begin{bmatrix} 1 \\ 4 \\ 2 \end{bmatrix}$.

In general, if you have m linear equations in n variables, the coefficient matrix A is an $m \times n$ matrix. A fundamental question is: given A, for which $\mathbf{b} \in \mathbb{R}^m$ does $A\mathbf{x} = \mathbf{b}$ have a solution $\mathbf{x} \in \mathbb{R}^n$?

Revisiting our example, we see that

$$A\mathbf{x} = \begin{bmatrix} 2x + y - z \\ x + 2y + z \\ -x - y + z \end{bmatrix} = x \begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} + y \begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix} + z \begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix},$$

which is a linear combination of the column vectors of A. So the earlier question is equivalent to: does $\mathbf{b}$ lie in the span of the columns of A?

The **column space** of an $m \times n$ matrix A, denoted $C(A)$, is the span in $\mathbb{R}^m$ of the columns of A. The column space $C(A)$ is a linear subspace of $\mathbb{R}^m$ and the equation $A\mathbf{x} = \mathbf{b}$ has a solution precisely when $\mathbf{b} \in C(A)$.

Example 1. Determine the column space of $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 3 \end{bmatrix}$.

Since $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$ are not scalar multiples of each other, $\text{span}\left(\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix}\right) = \mathbb{R}^2$.

Hence, $C(A) = \mathbb{R}^2$.

Example 2. Determine the column space of $A = \begin{bmatrix} 2 & 1 & 4 \\ 4 & 2 & 3 \\ 6 & 3 & 1 \end{bmatrix}$.

Since $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, $\begin{bmatrix} 2 \\ 4 \\ 6 \end{bmatrix}$ is redundant. Hence, $C(A) = \text{span}\left(\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} 4 \\ 3 \\ 1 \end{bmatrix}\right)$.

For any two sets V and W of objects, the notation $V \subset W$, read as "V is contained in W" or "V is a subset of W" means that every object of the set V belongs to the set W.

Proposition 21.2.5. Let A be a 2×3 matrix whose columns are all nonzero. The subspace $C(A) \subset \mathbb{R}^2$ is a line when all columns are multiples of one another, or equivalently, "have the same slope"; if this is not the case, then $C(A) = \mathbb{R}^2$. For any such matrix A and any $\mathbf{b} \in \mathbb{R}^2$, the linear system $A\mathbf{x} = \mathbf{b}$ of 2 equations in 3 unknowns has a solution precisely in the following circumstances:

- if $C(A)$ is a line, then there is a solution exactly when $\mathbf{b}$ lies on that line (or more concretely, either $\mathbf{b} = \mathbf{0}$ or the slope b_2/b_1 is the same as that of all nonzero vectors in the line $C(A)$);
- if $C(A) = \mathbb{R}^2$, then there is a solution for any $\mathbf{b}$.

Proposition 21.2.6. Let A be a 3×3 matrix whose columns are all nonzero. The subspace $C(A) \subset \mathbb{R}^3$ is a line when all columns are scalar multiples of each other, it is equal to $\mathbb{R}^3$ when the three columns are all linearly independent, and in all other cases, it is a plane.
For such A and any $\mathbf{b} \in \mathbb{R}^3$, the linear system $A\mathbf{x} = \mathbf{b}$ of 3 equations and 3 unknowns has a solution precisely in the following circumstances (depending on $\dim(C(A))$):

- if $C(A)$ is a line, then there is a solution exactly when $\mathbf{b}$ lies in that line (more concretely, $\mathbf{b}$ is a scalar multiple of any one of the three columnes);
- if $C(A)$ is a plane, then there is a solution exactly when $\mathbf{b}$ lies in that plane;
- if $C(A) = \mathbb{R}^3$, then there is a solution for any $\mathbf{b}$.

We now have a better understanding of how to check if $A\mathbf{x} = \mathbf{b}$ has a solution. We first came up with an equivalent question – is $\mathbf{b} \in C(A)$? How can we check if a vector is in a subspace? One way is: we can check whether it is equal to its projection onto the subspace. In other words, is it true that $\mathbf{b} = \mathbf{Proj}_{C(A)}(\mathbf{b})$?
Remember that we can compute $\mathbf{Proj}_{C(A)}(\mathbf{b})$ by using Fourier formula because we can obtain an orthogonal basis for $C(A)$ by using Gram-Schmidt on the columns of A.

Example 3. Consider the matrix $A = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 0 \\ 1 & 0 & 2 & 0 \\ 1 & 0 & 2 & 1 \end{bmatrix}$. Find an orthogonal basis for $C(A)$ by using Gram-Schmidt.

Take $\vec{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Then, $\mathrm{Proj}_{\vec{w}_1}(\vec{v}_2) = \frac{2}{4}\vec{w}_1$, and $\vec{v}_2 - \frac{1}{2}\vec{w}_1 = \begin{bmatrix} \frac{1}{2} \\ \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}$. Let $\vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix} = 2\vec{v}_2 - \vec{w}_1$.

Next, $\mathrm{Proj}_{V_2}(\vec{v}_3) = \mathrm{Proj}_{\vec{w}_1}(\vec{v}_3) + \mathrm{Proj}_{\vec{w}_2}(\vec{v}_3) = \frac{6}{4}\vec{w}_1 + \frac{-2}{4}\vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 2 \end{bmatrix} = \vec{v}_3$, and so, $\vec{v}_3 \in V_2$. Then,

$\mathrm{Proj}_{V_3}(\vec{v}_4) = \mathrm{Proj}_{\vec{w}_1}(\vec{v}_4) + \mathrm{Proj}_{\vec{w}_2}(\vec{v}_4) = \frac{2}{4}\vec{w}_1 + \frac{0}{4}\vec{w}_2$, and $\vec{v}_4 - \mathrm{Proj}_{V_3}(\vec{v}_4) = \vec{v}_4 - \frac{1}{2}\vec{w}_1 = \begin{bmatrix} \frac{1}{2} \\ -\frac{1}{2} \\ -\frac{1}{2} \\ \frac{1}{2} \end{bmatrix}$.

Letting $\vec{w}_3 = \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 2\vec{v}_4 - \vec{w}_1$, $\boxed{\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}}$ is an orthogonal basis for $C(A)$.

Note the relations $\vec{w}_1 = \vec{v}_1$, $\vec{w}_2 = -\vec{v}_1 + 2\vec{v}_2$, and $\vec{w}_3 = -\vec{v}_1 + 2\vec{v}_4$.

Example 3 continued. Gram-Schmidt on the columns of A gives us an orthogonal basis for $C(A)$:

$$\left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ -1 \\ 1 \end{bmatrix} \right\}$$

with the relations $\mathbf{w}_1 = \mathbf{v}_1, \mathbf{w}_2 = -\mathbf{v}_1 + 2\mathbf{v}_2$, and $\mathbf{w}_3 = -\mathbf{v}_1 + 2\mathbf{v}_4$.

Is there a solution to

$$\begin{aligned} x + y + \ z + w &= 10 \\ x + y + \ z \qquad &= 6 \\ x \qquad + 2z \qquad &= 7 \\ x \qquad + 2z + w &= 11 \end{aligned}$$

The question becomes: is $\begin{bmatrix} 10 \\ 6 \\ 7 \\ 11 \end{bmatrix} = \vec{b} \in C(A)$?

We see that

$$\text{Proj}_{C(A)}(\vec{b}) = \text{Proj}_{\vec{w}_1}(\vec{b}) + \text{Proj}_{\vec{w}_2}(\vec{b}) + \text{Proj}_{\vec{w}_3}(\vec{b}) = \frac{34}{4}\vec{w}_1 + \frac{-2}{4}\vec{w}_2 + \frac{8}{4}\vec{w}_3 = \begin{bmatrix} 10 \\ 6 \\ 7 \\ 11 \end{bmatrix} = \vec{b}.$$

Hence, $\vec{b} \in C(A)$, and thus, there is a solution.

In fact, $\vec{b} = \frac{17}{2}\vec{w}_1 - \frac{1}{2}\vec{w}_2 + 2\vec{w}_3 = \frac{17}{2}(\vec{v}_1) - \frac{1}{2}(-\vec{v}_1 + 2\vec{v}_2) + 2(-\vec{v}_1 + 2\vec{v}_4) = 7\vec{v}_1 - \vec{v}_2 + 4\vec{v}_4$.

So, $(7, -1, 0, 4)$ is a solution to the system.

Is there a solution to

$$\begin{aligned} x + y + \ z + w &= 0 \\ x + y + \ z \qquad &= 0 \\ x \qquad + 2z \qquad &= 0 \\ x \qquad + 2z + w &= 1 \end{aligned}$$

Now, we need to check if $\vec{b} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} \in C(A)$. Similarly as above,

$$\text{Proj}_{C(A)}(\vec{b}) = \frac{1}{4}\vec{w}_1 + \frac{-1}{4}\vec{w}_2 + \frac{1}{4}\vec{w}_3 = \begin{bmatrix} \frac{1}{4} \\ -\frac{1}{4} \\ \frac{1}{4} \\ \frac{3}{4} \end{bmatrix} \neq \vec{b}.$$

Hence, $\vec{b} \notin C(A)$, and so, the system has no solution.

Now that we have answered the question "for $\mathbf{b}$, does $A\mathbf{x} = \mathbf{b}$ have a solution?" we can move onto the next question – "if $A\mathbf{x} = \mathbf{b}$ does have a solution, is there more than one?"

Suppose that $A\mathbf{x} = \mathbf{b}$ has two solutions, $\mathbf{x}_1$ and $\mathbf{x}_2$. In other words, $A\mathbf{x}_1 = \mathbf{b}$ and $A\mathbf{x}_2 = \mathbf{b}$; subtracting the second equation from the first, we can get

$$A(\mathbf{x}_2 - \mathbf{x}_1) = A\mathbf{x}_2 - A\mathbf{x}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

Setting $\mathbf{d} = \mathbf{x}_2 - \mathbf{x}_1$, this becomes $A\mathbf{d} = \mathbf{0}$. Note that, for a scalar t, $\mathbf{x}_1 + t\mathbf{d}$ is a solution to $A\mathbf{x} = \mathbf{b}$ since

$$A(\mathbf{x}_1 + t\mathbf{d}) = A\mathbf{x}_1 + tA\mathbf{d} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Hence, whether there is *more than one* solution to $A\mathbf{x} = \mathbf{b}$ or not is closely related to whether there is a *non-zero* solution to $A\mathbf{x} = \mathbf{0}$, which is called the **homogeneous system** associated with A.

The **null space** of A, denoted $N(A)$, is the set of all solutions in $\mathbb{R}^n$ to the homogeneous system $A\mathbf{x} = \mathbf{0}$. Note that, while $C(A)$ is a subset of $\mathbb{R}^m$, $N(A)$ is a subset of $\mathbb{R}^n$.

If we think about the linear transformation $T_A : \mathbb{R}^n \to \mathbb{R}^m$ associated with A, then $N(A)$ consists of everything that T_A sends to $\mathbf{0}$. Hence, if T_A is invertible, then $N(A) = \{\mathbf{0}\}$.

Example 4. Let V be a linear subspace of $\mathbb{R}^n$ and consider $\mathbf{Proj}_V : \mathbb{R}^n \to \mathbb{R}^n$. What is $N(\mathbf{Proj}_V)$?

If $\text{Proj}_V(\vec{x}) = \vec{0}$, then $\vec{x} \perp V$, and so, $\vec{x} \in V^\perp$. On the other hand, if $\vec{y} \in V^\perp$, $\vec{y} \perp V$, and so, $\text{Proj}_V(\vec{y}) = \vec{0}$. Hence,

$$\boxed{N(\text{Proj}_V) = V^\perp}.$$

Proposition 21.3.5. For any $m \times n$ matrix A, the null space $N(A) \subset \mathbb{R}^n$ contains $\mathbf{0}$. Also, if $\mathbf{x}_1, \ldots, \mathbf{x}_k \in N(A)$, then any linear combination $c_1\mathbf{x}_1 + \cdots + c_k\mathbf{x}_k \in N(A)$.

Since $A\vec{0} = \vec{0}$, $\vec{0} \in N(A)$.

If $\vec{x}_1, \ldots, \vec{x}_k \in N(A)$, then $A\vec{x}_1 = \cdots = A\vec{x}_k = \vec{0}$. So,

$$A(c_1\vec{x}_1 + \cdots + c_k\vec{x}_k) = c_1A\vec{x}_1 + \cdots + c_kA\vec{x}_k = \vec{0}.$$

Thus, $c_1\vec{x}_1 + \cdots + c_k\vec{x}_k \in N(A)$. ▨

Proposition 21.3.6. The null space $N(A)$ is a linear subspace of $\mathbb{R}^n$.

Proposition 21.3.10. For any $m \times n$ matrix A and $\mathbf{b} \in \mathbb{R}^m$ for which the vector equation $A\mathbf{x} = \mathbf{b}$ has *some* solution $\mathbf{x}_0 \in \mathbb{R}^n$, the solutions to $A\mathbf{x} = \mathbf{b}$ are precisely the vectors of the form $\mathbf{x}_0 + \mathbf{d}$ for $\mathbf{d} \in N(A)$. There are infinitely many solutions whenever $N(A)$ contains a non-zero vector.

Example 5. How many solutions does the following system of linear equations

$$\begin{aligned} x + \ y \qquad &= 2 \\ x - \ y + z &= 2 \\ 2y - z &= 0 \end{aligned}$$

have?

Starting with $C(A) = \text{span}\left(\begin{array}{ccc} \vec{v}_1 & \vec{v}_2 & \vec{v}_3 \\ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, & \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}, & \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} \end{array}\right)$ and $\vec{b} = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix}$, we note that $\vec{v}_1 \perp \vec{v}_2$. So, Gram-Schmidt yields $\vec{w}_1 = \vec{v}_1$ and $\vec{w}_2 = \vec{v}_2$, and since $\text{Proj}_{V_2}(\vec{v}_3) = \frac{1}{2}\vec{w}_1 + \frac{-3}{6}\vec{w}_2 = \vec{v}_3$, it follows that $\left\{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \\ 2 \end{bmatrix}\right\}$ is an orthogonal basis for $C(A)$.

Now, $\text{Proj}_{C(A)}(\vec{b}) = \frac{4}{2}\vec{w}_1 + \frac{0}{6}\vec{w}_2 = \begin{bmatrix} 2 \\ 2 \\ 0 \end{bmatrix} = \vec{b}$, and so, $\vec{b} \in C(A)$. Thus, the system has a solution; in particular, since $\vec{b} = 2\vec{w}_1 = 2\vec{v}_1$, $(2,0,0)$ is a solution.

Now we consider the homogeneous system

$$\begin{aligned} x + \ y \qquad &= 0 \\ x - \ y + z &= 0 \\ 2y - z &= 0 \end{aligned}$$

Solving for y and z, in terms of x, gives $y = -x$ and $z = -2x$. Hence, if $(x,y,z) \in N(A)$,

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x \\ -x \\ -2x \end{bmatrix} = x\begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

Since $\vec{0} \neq (1,-1,-2) \in N(A)$, there are infinitely many solutions. In particular,

$$\begin{bmatrix} 2+t \\ 2-t \\ -2t \end{bmatrix}$$

is a solution for all $t \in \mathbb{R}$.

Note: By rank-nullity, $\dim(N(A)) = 3 - 2 = 1$, and so, there is a non-zero vector in $N(A)$.

Theorem 21.3.14. Let A be an $m \times n$ matrix. The equation $A\mathbf{x} = \mathbf{b}$ has

(a) no solution if $\mathbf{b} \notin C(A)$.

(b) exactly one solution if $\mathbf{b} \in C(A)$ and $N(A)$ consists of only $\mathbf{0}$.

(c) infinitely many solutions if $\mathbf{b} \in C(A)$ and $N(A)$ contains a non-zero vector.

Theorem 21.3.16 (Rank-Nullity theorem). For every $m \times n$ matrix A,

$$\underset{\text{rank}}{\dim(C(A))} + \underset{\text{nullity}}{\dim(N(A))} = n.$$

A linear system $A\mathbf{x} = \mathbf{b}$ with m equations and n unknowns (i.e., "variables" $x_1, \ldots, x_n$) is called

- **overdetermined** if there are more equations than unknowns ($m > n$);
- **underdetermined** if there are fewer equations than unknowns ($m < n$).

In the overdetermined case, the linear system $A\mathbf{x} = \mathbf{b}$ often fails to have any solutions at all (namely, whenever $\mathbf{b}$ lies outside the linear subspace $C(A) \neq \mathbb{R}^m$). Informally, there are too many equations.
In the underdetermined case, *if* $A\mathbf{x} = \mathbf{b}$ *has a solution*, then it automatically has infinitely many solutions. Informally, there are too few equations, so they do not pin down the values of all of the unknowns, if there is some solution at all.

Example 6. Let A, B, C be matrices satisfying $A = BC$. Assume that A has size $m \times n$.

(a) Explain why $N(C) \subset N(A)$. In other words, explain why if $\mathbf{v}$ is in $N(C)$, $\mathbf{v}$ must also be in $N(A)$.

Suppose $\vec{v} \in N(C)$, i.e., $C\vec{v} = \vec{0}$. Then,

$$A\vec{v} = (BC)\vec{v} = B(C\vec{v}) = B\vec{0} = \vec{0},$$

and so, $\vec{v} \in N(A)$. Thus, $N(C) \subseteq N(A)$.

(b) Explain why $C(A) \subset C(B)$. In other words, explain why if $\mathbf{v}$ is in $C(A)$, $\mathbf{v}$ must also be in $C(B)$.

Let $\vec{v} \in C(A)$ be arbitrary. Then, $\vec{v} = A\vec{w}$ for some $\vec{w}$, and so,

$$\vec{v} = A\vec{w} = (BC)\vec{w} = B(C\vec{w}).$$

Hence, $\vec{v} = T_B(C\vec{w})$, and so, $\vec{v} \in C(B)$. Thus, $C(A) \subseteq C(B)$.