Solutions to Math 51 Final Exam — August 19, 2023

1. (10 points) Consider the curve defined by the equation

$$(x+1)(x^2+y^2) = 4x^2.$$

Find the rightmost point(s) (i.e., the point(s) with the largest x-coordinate) on the curve.

Remark. This curve is called *conchoid of de Sluze.*

We are trying to maximize $f(x,y) = x$ subject to $g(x,y) = (x+1)(x^2+y^2) - 4x^2 = x^3 - 3x^2 + xy^2 + y^2 = 0$. We compute

$$\nabla f = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \nabla g = \begin{bmatrix} 3x^2 - 6x + y^2 \\ 2xy + 2y \end{bmatrix}.$$

We first check where ∇g vanishes. Setting $\nabla g = \mathbf{0}$, the second equation yields $2y(x+1) = 0$.

- $x = -1$ is impossible since the first equation becomes $y^2 = -9$ which has no solutions.
- If $y = 0$, we get $(0,0)$ and $(2,0)$; however $(2,0)$ is not on the constraint.

Hence, $(0,0)$ is the only point of interest from checking $\nabla g = \mathbf{0}$.

The Lagrange multiplier equation $\nabla f = \lambda \nabla g$ gives us

$$\begin{aligned} 1 &= \lambda(3x^2 - 6x + y^2) \\ 0 &= \lambda 2y(x+1) \end{aligned}$$

The first equation implies $\lambda \neq 0$. From the second equation, we get either $y = 0$ or $x = -1$.

- $x = -1$ is impossible since plugging this into the constraint gives us $-4 = 0$.
- $y = 0$ leads to $(0,0)$ and $(3,0)$. But $(0,0)$ gives us $1 = 0$ for the first of the two equations, so the only point of interest here is $(3,0)$.

Between the two points of interest $(0,0)$ and $(3,0)$, we see that $f(3,0) = 3$ is the larger one, and so, $(3,0)$ is the rightmost point on the curve.

2. (6 points) Find the line of best fit (that minimizes SSE (sum of square errors)) for the data

$$(0,1), \qquad (1,0), \qquad (2,1), \qquad (3,1), \qquad (4,-1).$$

Setting $\mathbf{X} = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ and $\mathbf{Y} = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, we get $\hat{\mathbf{X}} = \mathbf{X} - 2\mathbf{1} = \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

Now we compute

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{Y}) = -\frac{3}{10}\hat{\mathbf{X}} + \frac{2}{5}\mathbf{1} = -\frac{3}{10}(\mathbf{X} - 2\mathbf{1}) + \frac{2}{5}\mathbf{1} = -\frac{3}{10}\mathbf{X} + 1\mathbf{1}.$$

Thus, the line of best fit is

$$y = -\frac{3}{10}x + 1.$$

3. (10 points) Let $A = \begin{bmatrix} 1 & 1 & 4 \\ -1 & 3 & 2 \\ 0 & 1 & -3 \end{bmatrix}$.

(a) (6 points) Compute a QR-decomposition for A.
Remark. The only irrational entries will involve $\sqrt{2}$.

Let us label the columns of A $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_3$.

Setting $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, we compute $\mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_2) = \frac{-2}{2}\mathbf{w}_1$. Hence,

$$\mathbf{w}_2 = \mathbf{v}_2 + \mathbf{w}_1 = \begin{bmatrix} 2 \\ 2 \\ 1 \end{bmatrix}.$$

Next, we compute $\mathbf{Proj}_{V_2}(\mathbf{v}_3) = \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_3) + \mathbf{Proj}_{\mathbf{w}_2}(\mathbf{v}_3) = \frac{2}{2}\mathbf{w}_1 + \frac{9}{9}\mathbf{w}_2$. Thus,

$$\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{w}_1 - \mathbf{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -4 \end{bmatrix}.$$

Taking $\mathbf{w}_1' = \frac{1}{\sqrt{2}}\mathbf{w}_1$, $\mathbf{w}_2' = \frac{1}{3}\mathbf{w}_2$, and $\mathbf{w}_3' = \frac{1}{3\sqrt{2}}\mathbf{w}_3$, we get

$$\begin{aligned} \mathbf{v}_1 &= \mathbf{w}_1 &&= \sqrt{2}\mathbf{w}_1' \\ \mathbf{v}_2 &= -\mathbf{w}_1 + \mathbf{w}_2 &&= -\sqrt{2}\mathbf{w}_1' + 3\mathbf{w}_2' \\ \mathbf{v}_3 &= \mathbf{w}_1 + \mathbf{w}_2 + \mathbf{w}_3 &&= \sqrt{2}\mathbf{w}_1 + 3\mathbf{w}_2 + 3\sqrt{2}\mathbf{w}_3. \end{aligned}$$

Therefore, we get

$$A = QR = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{1}{3\sqrt{2}} \\ -\frac{1}{\sqrt{2}} & \frac{2}{3} & \frac{1}{3\sqrt{2}} \\ 0 & \frac{1}{3} & -\frac{4}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 3 & 3 \\ 0 & 0 & 3\sqrt{2} \end{bmatrix}.$$

(b) (4 points) Using your results from (a), write down an expression for A^{-1}. You may leave your answer as a product of two matrices.

We get $Q^{-1} = Q^{\intercal} = \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \end{bmatrix}$. and R^{-1} must be of the form $\begin{bmatrix} \frac{1}{\sqrt{2}} & \alpha & \beta \\ 0 & \frac{1}{3} & \gamma \\ 0 & 0 & \frac{1}{3\sqrt{2}} \end{bmatrix}$.

$$I_3 = R^{-1}R = \begin{bmatrix} \frac{1}{\sqrt{2}} & \alpha & \beta \\ 0 & \frac{1}{3} & \gamma \\ 0 & 0 & \frac{1}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{2} & -\sqrt{2} & \sqrt{2} \\ 0 & 3 & 3 \\ 0 & 0 & 3\sqrt{2} \end{bmatrix} = \begin{bmatrix} 1 & -1+3\alpha & 1+3\alpha+3\sqrt{2}\beta \\ 0 & 1 & 1+3\sqrt{2}\gamma \\ 0 & 0 & 1 \end{bmatrix}$$

yields $\alpha = \frac{1}{3}$, $\beta = -\frac{2}{3\sqrt{2}}$, and $\gamma = -\frac{1}{3\sqrt{2}}$. Thus,

$$A^{-1} = R^{-1}Q^{-1} = \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{3} & -\frac{2}{3\sqrt{2}} \\ 0 & \frac{1}{3} & -\frac{1}{3\sqrt{2}} \\ 0 & 0 & \frac{1}{3\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ \frac{1}{3\sqrt{2}} & \frac{1}{3\sqrt{2}} & -\frac{4}{3\sqrt{2}} \end{bmatrix} = \frac{1}{18}\begin{bmatrix} 11 & -7 & 10 \\ 3 & 3 & 6 \\ 1 & 1 & -4 \end{bmatrix}.$$

4. (10 points) The kea are alpine parrots native to the South Island of New Zealand. The zoologists observe the following migratory patterns of the kea in three mountain ranges A, B, and C.

Every March,

- 20% of the kea in range A move to range B, 40% to range C, and the rest do not migrate;
- 50% of the kea in range B move to range C, and the rest do not migrate; and
- 30% of the kea in range C move to range A, 20% to range B, and the rest do not migrate.

Every September, for some fixed value of β,

- 40% of the kea in range A move to range C, and the rest do not migrate;
- 30% of the kea in range B move to range A, and the rest do not migrate; and
- 40% of the kea in range C move to range A, β% to range B, and the rest do not migrate.

Suppose that, in **January 2020**, there are 20,000, 10,000, and 0 kea in ranges A, B, and C, respectively; suppose also that we can safely ignore population changes due to any other factors, including births and deaths.

(a) (3 points) How many kea are there in range B in **June 2020**?

By analyzing the dynamics of the spring migration, we get the corresponding Markov matrix

$$S = \begin{bmatrix} 0.4 & 0 & 0.3 \\ 0.2 & 0.5 & 0.2 \\ 0.4 & 0.5 & 0.5 \end{bmatrix}.$$

Computing

$$\begin{bmatrix} 0.4 & 0 & 0.3 \\ 0.2 & 0.5 & 0.2 \\ 0.4 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 20000 \\ 10000 \\ 0 \end{bmatrix} = \begin{bmatrix} 8000 & 9000 & 13000 \end{bmatrix},$$

we see there are 9000 kea in range B in June 2020.

(b) (3 points) If there are 4,500 kea in range C in **January 2021**, what is the value of β?

By analyzing the dynamics of the fall migration, we get the corresponding Markov matrix

$$F = \begin{bmatrix} 0.6 & 0.3 & 0.4 \\ 0 & 0.7 & \frac{\beta}{100} \\ 0.4 & 0 & 0.6 - \frac{\beta}{100} \end{bmatrix}.$$

Computing

$$\begin{bmatrix} 0.6 & 0.3 & 0.4 \\ 0 & 0.7 & \frac{\beta}{100} \\ 0.4 & 0 & 0.6 - \frac{\beta}{100} \end{bmatrix} \begin{bmatrix} 8000 \\ 9000 \\ 13000 \end{bmatrix} = \begin{bmatrix} \cdots \\ \cdots \\ 11000 - 130\beta \end{bmatrix},$$

we get $\beta = 50$ from $11000 - 130\beta = 4500$.

(c) (4 points) Let a_n, b_n, c_n represent the number of kea in mountain ranges A, B, C, respectively, in **June of the nth year after 2020**. (For example, a_1 is the number of kea in mountain range A in **June 2021**, and so on.) Write down a matrix M for which

$$M \begin{bmatrix} a_n \\ b_n \\ c_n \end{bmatrix} = \begin{bmatrix} a_{n+1} \\ b_{n+1} \\ c_{n+1} \end{bmatrix}.$$

If you wish, you may express your matrix M as a product of some fixed number of matrices (each of which contains specific numbers), but without carrying out the actual multiplication.

(*Note:* if you found the value of the constant β in part (b), you may use this value in your answer; if you did not, you may express one or more entries of your answer for M in terms of β.)

From June of a year to June of the next year, spring migration happens after fall migration. Hence,

$$M = SF = \begin{bmatrix} 0.4 & 0 & 0.3 \\ 0.2 & 0.5 & 0.2 \\ 0.4 & 0.5 & 0.5 \end{bmatrix} \begin{bmatrix} 0.6 & 0.3 & 0.4 \\ 0 & 0.7 & 0.5 \\ 0.4 & 0 & 0.1 \end{bmatrix} = \begin{bmatrix} 0.36 & 0.12 & 0.19 \\ 0.20 & 0.41 & 0.35 \\ 0.44 & 0.47 & 0.46 \end{bmatrix}.$$

Note that M is a Markov matrix, as expected.

5. (14 points) Consider the matrix $A = \begin{bmatrix} a & 0 & 0 \\ 0 & b & c \\ 0 & d & e \end{bmatrix}$, where a, b, c, d, e are real numbers. Define the matrix $B = \begin{bmatrix} b & c \\ d & e \end{bmatrix}$.

(a) (3 points) Suppose $\mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is an eigenvector for B with eigenvalue λ. Show that $\mathbf{v}' = \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix}$ and $\mathbf{e}_1$ are eigenvectors for A, and find their respective eigenvalues.

Since $B\mathbf{v} = \lambda\mathbf{v}$, we get $v_1 \begin{bmatrix} b \\ d \end{bmatrix} + v_2 \begin{bmatrix} c \\ e \end{bmatrix} = \lambda \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$. Hence,

$$A\mathbf{v}' = A \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix} = v_1 \begin{bmatrix} 0 \\ b \\ d \end{bmatrix} + v_2 \begin{bmatrix} 0 \\ c \\ e \end{bmatrix} = \begin{bmatrix} 0 \\ \lambda v_1 \\ \lambda v_2 \end{bmatrix} = \lambda \begin{bmatrix} 0 \\ v_1 \\ v_2 \end{bmatrix} = \lambda \mathbf{v}',$$

and $A\mathbf{e}_1 = a\mathbf{e}_1$. Thus, $\mathbf{v}'$ and $\mathbf{e}_1$ are eigenvectors of A with eigenvalues λ and a, respectively.

For parts (b) – (d) of this problem, consider the matrix $A = \begin{bmatrix} -2 & 0 & 0 \\ 0 & 3 & 2 \\ 0 & 2 & 0 \end{bmatrix}$.

(b) (4 points) Using the results of part (a), find all the eigenvalues of A and give an eigenvector associated with each eigenvalue. Make sure to verify your answers.

For $B = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}$, we get $p_B(\lambda) = \lambda^2 - 3\lambda - 4$, and so $\lambda_1 = 4$, $\lambda_2 = -1$. We also find eigenvectors $\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\mathbf{w}_2 = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$.

Using part (a), we see that $\begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, and $\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ are eigenvectors of A with eigenvalues 4, -1, and -2, respectively.

(c) (3 points) Find a matrix Q and a diagonal matrix D for which $A = QDQ^\intercal$.

By the spectral theorem,

$$A = QDQ^{-1} = QDQ^\intercal = \begin{bmatrix} 0 & 0 & 1 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -2 \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \end{bmatrix}.$$

(d) (4 points) Find a scalar α and a 3×3 matrix L, whose entries are all real numbers (i.e., no n in the entries), for which

$$A^n \approx \alpha^n L$$

for large n.

Using our results from part (c), we get

$$\begin{aligned}
A^n = QD^nQ^\intercal &= \begin{bmatrix} 0 & 0 & 1 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 4^n & 0 & 0 \\ 0 & (-1)^n & 0 \\ 0 & 0 & (-2)^n \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \end{bmatrix} \\
&= 4^n \begin{bmatrix} 0 & 0 & 1 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \left(-\frac{1}{4}\right)^n & 0 \\ 0 & 0 & \left(-\frac{2}{4}\right)^n \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \end{bmatrix} \\
&\approx 4^n \begin{bmatrix} 0 & 0 & 1 \\ \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} & 0 \\ \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \\ 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 1 & 0 & 0 \end{bmatrix} \\
&= 4^n \begin{bmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \\
&= 4^n \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{4}{5} & \frac{2}{5} \\ 0 & \frac{2}{5} & \frac{1}{5} \end{bmatrix}.
\end{aligned}$$

6. (10 points) Let $\mathbf{f}\colon \mathbf{R}^2 \to \mathbf{R}^3$ be given by

$$\mathbf{f}(u,v) = \begin{bmatrix} (7+\cos v)\sin u \\ 2u + \sin v \\ (7+\cos v)\cos u \end{bmatrix}.$$

The set $\mathbf{f}(\mathbf{R}^2)$ (i.e. image of $\mathbf{R}^2$ under $\mathbf{f}$) describes a surface in $\mathbf{R}^3$ and is in fact the level set of a function $g\colon \mathbf{R}^3 \to \mathbf{R}$ at level 13. In particular, this means that $g(\mathbf{f}(u,v)) = 13$ for all $(u,v) \in \mathbf{R}^2$.

(a) (6 points) Given that

$$(\nabla g)\left(\mathbf{f}(\pi, \pi/2)\right) = \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix},$$

find the values of α and β, using the Multivariate Chain Rule.

We are given that

$$(g\circ \mathbf{f})(u,v) = 13 \quad \text{for all } (u,v) \in \mathbf{R}^2.$$

The right-hand side is a constant independent of both u and v, so

$$(D(g\circ\mathbf{f}))(u,v) = \begin{bmatrix} \frac{\partial}{\partial u}(13) & \frac{\partial}{\partial v}(13) \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

In particular, $(D(g\circ\mathbf{f}))(\pi,\pi/2) = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

Remark: We must calculate $(D(g\circ\mathbf{f}))(u,v)$ first, before evaluating it at $(\pi,\pi/2)$. It is incorrect to evaluate $g\circ\mathbf{f}$ at $(\pi,\pi/2)$ first before taking the derivative.

We can also calculate $(D(g \circ \mathbf{f}))(\pi, \pi/2)$ using the chain rule:

$$(D(g \circ \mathbf{f}))(\pi, \pi/2) = (Dg)(\mathbf{f}(\pi, \pi/2))(D\mathbf{f})(\pi, \pi/2).$$

We have

$$(Dg)(\mathbf{f}(\pi, \pi/2)) = (\nabla g)\,(\mathbf{f}(\pi, \pi/2))^{\intercal} = \begin{bmatrix} 1 & \alpha & \beta \end{bmatrix}$$

and

$$(D\mathbf{f})(\pi, \pi/2) = \left.\begin{bmatrix} (7+\cos v)\cos u & -\sin v \sin u \\ 2 & \cos v \\ -(7+\cos v)\sin u & -\sin v \cos u \end{bmatrix}\right|_{(\pi,\pi/2)} = \begin{bmatrix} -7 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix},$$

so the chain rule says

$$(D(g \circ \mathbf{f}))(\pi, \pi/2) = \begin{bmatrix} 1 & \alpha & \beta \end{bmatrix} \begin{bmatrix} -7 & 0 \\ 2 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} -7+2\alpha & \beta \end{bmatrix}.$$

Equating the two expressions we obtained for $(D(g \circ \mathbf{f}))(\pi, \pi/2)$, we obtain

$$\begin{bmatrix} -7+2\alpha & \beta \end{bmatrix} = \begin{bmatrix} 0 & 0 \end{bmatrix}.$$

The first entry tells us $-7 + 2\alpha = 0$, so $\alpha = 7/2$. The second entry tells us $\beta = 0$.

(b) (2 points) Find the equation of the tangent plane to the surface $\mathbf{f}(\mathbf{R}^2)$ at the point $\mathbf{f}(\pi, \pi/2)$. You may leave your answer in terms of α and β if you have not found them in part (a).

Since

$$\mathbf{f}(\pi, \pi/2) = \begin{bmatrix} 0 \\ 2\pi+1 \\ -7 \end{bmatrix} \quad \text{and} \quad (\nabla g)\,(\mathbf{f}(\pi, \pi/2)) = \begin{bmatrix} 1 \\ \alpha \\ \beta \end{bmatrix} = \begin{bmatrix} 1 \\ 7/2 \\ 0 \end{bmatrix} \neq \mathbf{0},$$

then the equation of the tangent plane to $\mathbf{f}(\mathbf{R}^2)$ at the point $\mathbf{f}(\pi, \pi/2)$ is

$$\begin{bmatrix} 1 \\ 7/2 \\ 0 \end{bmatrix} \cdot \begin{bmatrix} x-0 \\ y-(2\pi+1) \\ z+7 \end{bmatrix} = 0,$$

i.e.,

$$2x + 7y - 14\pi - 7 = 0,$$

or equivalently, $2x + 7y = 14\pi + 7$.

(c) (2 points) Give, with justification, a unit vector $\mathbf{v} \in \mathbf{R}^3$ for which the *absolute value* of the change in g at the point $\mathbf{f}(\pi, \pi/2)$ in the direction of $\mathbf{v}$ is minimized. You may leave your answer in terms of α and β if you have not found them in part (a).

When we move away from $\mathbf{f}(\pi, \pi/2)$ along $\mathbf{f}(\mathbf{R}^2)$ (which is the level set of g at level 13), the value taken by g stays at 13 and does not change. This means the absolute value of the change in g as we move away in any direction along $\mathbf{f}(\mathbf{R}^2)$ is zero (the smallest possible non-negative number). Therefore, we are seeking a unit vector parallel to the tangent plane to $\mathbf{f}(\mathbf{R}^2)$ at $\mathbf{f}(\pi, \pi/2)$, i.e. a unit vector orthogonal to

$$(\nabla g)\,(\mathbf{f}(\pi, \pi/2)) = \begin{bmatrix} 1 \\ 7/2 \\ 0 \end{bmatrix}.$$

One possible candidate for $\mathbf{v}$ is

$$\mathbf{v} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

In general, any vector $\mathbf{v} = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying $a^2 + b^2 + c^2 = 1$ and $2a = -7b$ would work.

We may also arrive at the same conclusion by seeking to minimize the absolute value of the directional derivative of g in the direction $\mathbf{v}$ (a unit vector) at the point $\mathbf{f}(\pi, \pi/2)$, namely the absolute value of

$$(D_{\mathbf{v}}g)(\mathbf{f}(\pi, \pi/2)) = (\nabla g)\left(\mathbf{f}(\pi, \pi/2)\right) \cdot \mathbf{v} = \begin{bmatrix} 1 \\ 7/2 \\ 0 \end{bmatrix} \cdot \mathbf{v}.$$

The absolute value of this is minimized when $\begin{bmatrix} 1 \\ 7/2 \\ 0 \end{bmatrix} \cdot \mathbf{v} = 0$, and we proceed as above.

Remark: The question asked for a ***unit*** *vector* $\mathbf{v}$*, so your vector needs to have length* 1.

7. (10 points) Consider the function

$$f(x, y) = 17x^2 - 8xy + 2y^2 - 10x - 4y - 2.$$

(a) (2 points) Show that f has a unique critical point.

We compute $\nabla f = \begin{bmatrix} 34x - 8y - 10 \\ -8x + 4y - 4 \end{bmatrix}$ and setting $\nabla f = \mathbf{0}$, we get $x = 1$ and $y = 3$. Thus, the unique critical point is $(1, 3)$.

(b) (3 points) By analyzing the Hessian at the critical point, determine whether it is a local maximum, local minimum, or neither.

We compute $Hf(1,3) = \begin{bmatrix} 34 & -8 \\ -8 & 4 \end{bmatrix}$ and

$$p_{Hf(1,3)}(\lambda) = \lambda^2 - 38\lambda + 72 = (\lambda - 36)(\lambda - 2).$$

Thus, $\lambda_1 = 36$ and $\lambda_2 = 2$, and so, $(1, 3)$ is a local minimum since $Hf(1, 3)$ is positive-definite.

(c) (5 points) Sketch an approximate contour plot of f at the critical point on the coordinate plane provided below.

Sketch qualitatively correct level sets, including justification in terms of the eigenvalues and eigenvectors.

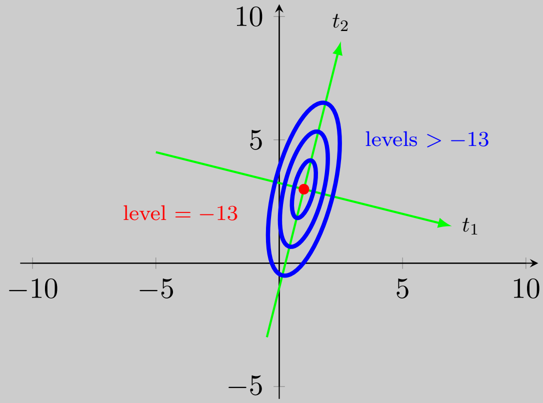
For $\lambda_1 = 36$, $Hf(1,3) - 36I_2 = \begin{bmatrix} -2 & -8 \\ -8 & -32 \end{bmatrix}$, and so, $\mathbf{w}_1 = \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ is a 36-eigenvector.

For $\lambda_2 = 2$, $Hf(1,3) - 2I_2 = \begin{bmatrix} 32 & -8 \\ -8 & 2 \end{bmatrix}$, and so, $\mathbf{w}_2 = \begin{bmatrix} 1 \\ 4 \end{bmatrix}$ is a 2-eigenvector.

Taking $\mathbf{w}_1' = \frac{1}{\sqrt{17}} \begin{bmatrix} 4 \\ -1 \end{bmatrix}$ and $\mathbf{w}_2' = \frac{1}{\sqrt{17}} \begin{bmatrix} 1 \\ 4 \end{bmatrix}$, for $\mathbf{v} = t_1\mathbf{w}_1' + t_2\mathbf{w}_2'$, we get

$$q_{Hf(1,3)}(\mathbf{v}) = 36t_1^2 + 4t_2^2,$$

which is positive-definite. The level curves will be ellipses longer in the t_2 direction. Putting all this together, we get the following approximate contour plot of f at $(1, 3)$:



The green level curve is at level $f(1, 3) = -13$, and the blue level curves are at levels greater than $f(1, 3) = -13$.

8. (10 points) For each of the following statements, circle either TRUE (meaning, "always true") or FALSE (meaning, "not always true"), and briefly and convincingly justify your answer. 1 point for the correct choice, and the rest for convincing justification.

(a) (5 points) Suppose A is an $m \times n$ matrix with the property $A^\intercal A = 0$. Then, A must be the zero matrix.

Circle one, and justify below: $\boxed{\text{TRUE}}$ FALSE

Denoting the columns of A by $\mathbf{v}_1, \cdots, \mathbf{v}_n$, we have

$$0 = (A^\intercal A)_{ii} = \mathbf{v}_i^\intercal \mathbf{v}_i = \mathbf{v}_i \cdot \mathbf{v}_i = \|\mathbf{v}_i\|^2$$

for all i. Hence, $\|\mathbf{v}_i\| = 0$ for all i, and so, $\mathbf{v}_i = \mathbf{0}$ for all i. Thus, $A = 0$.

(b) (5 points) If a 2×2 matrix A has characteristic polynomial $P_A(\lambda) = \lambda^2 - \lambda - 2$, then the characteristic polynomial $P_B(\lambda)$ of $B = A^3 - 4A^2 + 7I_2$ must also be $P_B(\lambda) = \lambda^2 - \lambda - 2$.

Circle one, and justify below: $\boxed{\text{TRUE}}$ FALSE

Since $p_A(\lambda) = (\lambda - 2)(\lambda + 1)$, we get $\lambda_1 = 2$ and $\lambda_2 = -1$. Suppose $\mathbf{w}_1$ and $\mathbf{w}_2$ are the corresponding eigenvectors of λ_1 and λ_2, respectively. Note that

$$B\mathbf{w}_1 = A^3\mathbf{w}_1 - 4A^2\mathbf{w}_1 + 7\mathbf{w}_1 = 8\mathbf{w}_1 - 16\mathbf{w}_1 + 7\mathbf{w}_1 = -\mathbf{w}_1$$

and

$$B\mathbf{w}_2 = A^3\mathbf{w}_2 - 4A^2\mathbf{w}_2 + 7\mathbf{w}_2 = -\mathbf{w}_2 - 4\mathbf{w}_2 + 7\mathbf{w}_2 = 2\mathbf{w}_2.$$

Hence, $\mathbf{w}_1$ and $\mathbf{w}_2$ are eigenvectors of B with eigenvalues -1 and 2, respectively. In particular, -1 and 2 are the eigenvalues of B (which can have at most two eigenvalues, since it is 2×2), and so,

$$p_B(\lambda) = (\lambda + 1)(\lambda - 2) = \lambda^2 - \lambda - 2.$$