1. The formula in Theorem 5.3.6 is
$$\mathbf{v} = \sum_{i=1}^{n} \left(\frac{\mathbf{v} \cdot \mathbf{b}_i}{\mathbf{b}_i \cdot \mathbf{b}_i} \right) \mathbf{b}_i$$
(so in case of an orthonormal basis, the denominators $\mathbf{b}_i \cdot \mathbf{b}_i$ are all equal to 1).

   (a) The coefficients are
$$\frac{\mathbf{v} \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1} = \frac{16}{26} = \frac{8}{13}, \quad \frac{\mathbf{v} \cdot \mathbf{b}_2}{\mathbf{b}_2 \cdot \mathbf{b}_2} = \frac{2}{26} = \frac{1}{13},$$
so
$$\begin{bmatrix} 1 \\ 3 \end{bmatrix} = \frac{8}{13} \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \frac{1}{13} \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$
The right side can be readily computed by hand to be equal to the left side since $8+5 = 13$ and $8 \cdot 5 - 1 = 40 - 1 = 39 = 3 \cdot 13$.

   (b) This is an orthonormal basis, so each $\mathbf{b}_i \cdot \mathbf{b}_i = 1$. Hence, the coefficients are
$$\mathbf{v} \cdot \mathbf{b}_1 = \frac{11}{\sqrt{26}}, \quad \mathbf{v} \cdot \mathbf{b}_2 = \frac{3}{\sqrt{26}},$$
so
$$\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \frac{11}{\sqrt{26}} (1/\sqrt{26}) \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \frac{3}{\sqrt{26}} (1/\sqrt{26}) \begin{bmatrix} 5 \\ -1 \end{bmatrix}.$$
To compute the right side explicitly, we first note that each term has in its denominator $\sqrt{26} \cdot \sqrt{26} = 26$, so the right side is the same as
$$\frac{11}{26} \begin{bmatrix} 1 \\ 5 \end{bmatrix} + \frac{3}{26} \begin{bmatrix} 5 \\ -1 \end{bmatrix},$$
which we compute to be equal to the left side since $11 + 3 \cdot 5 = 26$ and $11 \cdot 5 - 3 = 52 = 2 \cdot 26$.

   (c) We calculate the coefficients as
$$\frac{\mathbf{v} \cdot \mathbf{b}_1}{\mathbf{b}_1 \cdot \mathbf{b}_1} = \frac{6}{3} = 2, \quad \frac{\mathbf{v} \cdot \mathbf{b}_2}{\mathbf{b}_2 \cdot \mathbf{b}_2} = \frac{-2}{2} = -1, \quad \frac{\mathbf{v} \cdot \mathbf{b}_3}{\mathbf{b}_3 \cdot \mathbf{b}_3} = \frac{0}{6} = 0,$$
so
$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$
The right side is equal to $\begin{bmatrix} 2 \\ 2 \\ 2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = \begin{bmatrix} 2-1 \\ 2-0 \\ 2-(-1) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, as desired. $\Diamond$

2. Writing $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$, we first rule out the possibility that $a = 0$. If $a = 0$ then $\mathbf{u} = b\mathbf{w}$, but then $b \neq 0$ (since $\mathbf{u}$ is nonzero), so $\mathbf{w} = (1/b)\mathbf{u}$ and so $\mathbf{u}$ and $\mathbf{w}$ span the *same* line: $x\mathbf{u} = (xb)\mathbf{w}$ and $x\mathbf{w} = (x/b)\mathbf{u}$ for any scalar x. But we are given that $\mathbf{u}$ and $\mathbf{w}$ span different lines through the origin. That rules out the possibility $a = 0$ (and arguing similarly with $\mathbf{v}$ in place of $\mathbf{w}$ rules out the possibility $b = 0$). Since each of a and b is nonzero, we can express each of $\mathbf{v}$ and $\mathbf{w}$ in terms of the other two vectors: $a\mathbf{v} = \mathbf{u} - b\mathbf{w}$ implies $\mathbf{v} = (1/a)\mathbf{u} - (b/a)\mathbf{w}$, and likewise $b\mathbf{w} = \mathbf{u} - a\mathbf{v}$ implies $\mathbf{w} = (1/b)\mathbf{u} - (a/b)\mathbf{v}$. $\Diamond$

3. (a) Writing $\mathbf{v}_3 = \begin{bmatrix} a \\ b \\ c \end{bmatrix}$, its orthogonality to $\mathbf{v}_1$ and $\mathbf{v}_2$ amounts to the pair of equations

$$2a + 4c = 0, \quad -2a + b + c = 0,$$

so $c = -a/2$ and $b = 2a - c = 5a/2$. In other words, $\mathbf{v}_3 = \begin{bmatrix} a \\ 5a/2 \\ -a/2 \end{bmatrix}$ for some $a \neq 0$. The cleanest choice is $a = 2$, giving $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}$.

(b) We calculate

$$\begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} = 10, \quad \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} = 7, \quad \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} = 5,$$

so

$$\begin{aligned} \begin{bmatrix} -1 \\ 2 \\ 3 \end{bmatrix} &= \frac{10}{20}\begin{bmatrix} 2 \\ 0 \\ 4 \end{bmatrix} + \frac{7}{6}\begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} + \frac{5}{30}\begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix} \\ &= \frac{1}{2}\mathbf{v}_1 + \frac{7}{6}\mathbf{v}_2 + \frac{1}{6}\mathbf{v}_3. \end{aligned}$$

◊

4. (a) We calculate $\mathbf{b}_1 \cdot \mathbf{b}_1 = 30$, $\mathbf{b}_2 \cdot \mathbf{b}_2 = 5$, $\mathbf{b}_1 \cdot \mathbf{v} = 30$, and $\mathbf{b}_2 \cdot \mathbf{v} = 20$, so

$$\mathbf{v}' = \mathbf{b}_1 + 4\mathbf{b}_2 = \begin{bmatrix} 9 \\ 5 \\ -2 \end{bmatrix}.$$

(b) By inspection $\mathbf{v}' \neq \mathbf{v}$. The reason $\mathbf{v}$ cannot lie in the plane $\mathcal{P}$ is because if it were in this plane, so it is in the span of the orthogonal collection of nonzero vectors $\{\mathbf{b}_1, \mathbf{b}_2\}$, then by Theorem 5.3.6 it would be equal to the expression defining $\mathbf{v}'$. But we observed that $\mathbf{v}' \neq \mathbf{v}$, so this is not possible. ◊

5. (a) We can use

$$\mathbf{v}_2 = \mathbf{w} - \mathbf{Proj}_{\mathbf{v}_1}(\mathbf{w}) = \mathbf{w} - \frac{\mathbf{w} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 = \mathbf{w} - \frac{6}{5}\mathbf{v}_1 = \begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} - \frac{6}{5}\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 3/5 \\ -6/5 \\ 1 \end{bmatrix}.$$

Likewise, we can use

$$\mathbf{w}_2 = \mathbf{v} - \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{w}_1) = \mathbf{v} - \frac{\mathbf{v} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \mathbf{v} - \frac{3}{5}\mathbf{w}_1 = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} - \frac{6}{10}\begin{bmatrix} 3 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1/5 \\ 1 \\ -3/5 \end{bmatrix}.$$

(Any nonzero scalar multiples of these would work just as well.)

(b) By the Fourier formula,

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{u}) = \frac{\mathbf{u} \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 + \frac{\mathbf{u} \cdot \mathbf{v}_2}{\mathbf{v}_2 \cdot \mathbf{v}_2}\mathbf{v}_2 = \frac{1}{5}\mathbf{v}_1 + \frac{4/5}{70/25}\mathbf{v}_2 = \frac{1}{5}\mathbf{v}_1 + \frac{2}{7}\mathbf{v}_2$$

and

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{u}) = \frac{\mathbf{u}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{u}\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\mathbf{w}_2 = \frac{2}{10}\mathbf{w}_1 + \frac{-1/5}{35/25}\mathbf{w}_2 = \frac{1}{5}\mathbf{w}_1 - \frac{1}{7}\mathbf{w}_2.$$

(c) In the solution to (a) we had

$$\mathbf{v}_2 = \mathbf{w} - \frac{6}{5}\mathbf{v}_1 = \mathbf{w} - \frac{6}{5}\mathbf{v}, \quad \mathbf{w}_2 = \mathbf{v} - \frac{3}{5}\mathbf{w}_1 = \mathbf{v} - \frac{3}{5}\mathbf{w},$$

so plugging these into the answers to (b) gives

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{u}) = \frac{1}{5}\mathbf{v}_1 + \frac{2}{7}\mathbf{v}_2 = \frac{1}{5}\mathbf{v} + \frac{2}{7}(\mathbf{w} - \frac{6}{5}\mathbf{v}) = -\frac{1}{7}\mathbf{v} + \frac{2}{7}\mathbf{w}$$

and

$$\mathbf{Proj}_{\mathcal{P}}(\mathbf{u}) = \frac{1}{5}\mathbf{w}_1 - \frac{1}{7}\mathbf{w}_2 = \frac{1}{5}\mathbf{w} - \frac{1}{7}(\mathbf{v} - \frac{3}{5}\mathbf{w}) = -\frac{1}{7}\mathbf{v} + \frac{2}{7}\mathbf{w}.$$

So each way, we get $e = 1/7$ and $f = 2/7$. ♢

6. (a) To show that necessarily $c_1 = 0$ and $c_2 = 0$, we have to rule out the case $c_1 \neq 0$ and the case $c_2 \neq 0$. If $c_1 \neq 0$ then we can divide by c_1 (or equivalently, multiply by $1/c_1$) to get $\mathbf{v}_1 + (c_2/c_1)\mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_1 = -(c_2/c_1)\mathbf{v}_2$. But this is impossible since $\mathbf{v}_1$ isn't a scalar multiple of $\mathbf{v}_2$.

   Likewise, if $c_2 \neq 0$ then we can divide by c_2 (or equivalently, multiply by $1/c_2$) to get $(c_1/c_2)\mathbf{v}_1 + \mathbf{v}_2 = \mathbf{0}$, so $\mathbf{v}_2 = -(c_1/c_2)\mathbf{v}_1$. But this is impossible since $\mathbf{v}_2$ isn't a scalar multiple of $\mathbf{v}_1$.

   (b) If $a_1\mathbf{v}_1 + a_2\mathbf{v}_2 = b_1\mathbf{v}_1 + b_2\mathbf{v}_2$ for some scalar coefficients then by subtracting the right side from the left side we get

$$\mathbf{0} = a_1\mathbf{v}_1 + a_2\mathbf{v}_2 - b_1\mathbf{v}_1 - b_2\mathbf{v}_2 = (a_1 - b_1)\mathbf{v}_1 + (a_2 - b_2)\mathbf{v}_2.$$

   By (a), this can only happen when both coefficients vanish, which is to say $a_1 - b_1 = 0$ and $a_2 - b_2 = 0$, which is exactly the same as saying $a_1 = b_1$ and $a_2 = b_2$. ♢

7. (a) Suppose $c_1 \neq 0$. We then multiply through by $1/c_1$ to get $\mathbf{w}_1 + (c_2/c_1)\mathbf{w}_2 + (c_3/c_1)\mathbf{w}_3 = \mathbf{0}$, so

$$\mathbf{w}_1 = -(c_2/c_1)\mathbf{w}_2 - (c_3/c_1)\mathbf{w}_3.$$

   But then $\mathbf{w}_1$ is "redundant" in the span defining W: any linear combination $a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3$ can be rewritten as

$$\begin{aligned} a(-(c_2/c_1)\mathbf{w}_2 - (c_3/c_1)\mathbf{w}_2) + b\mathbf{w}_2 + c\mathbf{w}_3 &= -(ac_2/c_1)\mathbf{w}_2 - (ac_3/c_1)\mathbf{w}_2 + b\mathbf{w}_2 + c\mathbf{w}_3 \\ &= (b - ac_2/c_1)\mathbf{w}_2 + (c - ac_3/c_1)\mathbf{w}_3. \end{aligned}$$

   This says $W = \mathrm{span}(\mathbf{w}_2, \mathbf{w}_3)$, so $\dim W \leq 2$. But we assumed $\dim W = 3$, so this is impossible. Consequently, the possibility $c_1 \neq 0$ can't occur, so necessarily $c_1 = 0$.

   The exact same reasoning applies equally well to c_2 and c_3 (up to some change in subscripts), so these coefficients also must vanish.

   (b) If $a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + a_3\mathbf{w}_3 = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + b_3\mathbf{w}_3$ for some scalar coefficients, then by subtracting the left side from the right side we get

$$\mathbf{0} = b_1\mathbf{w}_1 + b_2\mathbf{w}_2 + b_3\mathbf{w}_3 - a_1\mathbf{w}_1 - a_2\mathbf{w}_2 - a_3\mathbf{w}_3 = (b_1 - a_1)\mathbf{w}_1 + (b_2 - a_2)\mathbf{w}_2 + (b_3 - a_3)\mathbf{w}_3.$$

   We can apply the conclusion of (a) to this to conclude that all three coefficients must vanish, which is to say that all differences $b_i - a_i$ vanish, or in other words $b_i = a_i$ for all i, as desired. ♢

8. (a) We expand out the right side and then subtract the left side:

$$(a+3t)\mathbf{v}_1 + (b-5t)\mathbf{v}_2 = a\mathbf{v}_1 + 3t\mathbf{v}_1 + b\mathbf{v}_2 - 5t\mathbf{v}_2 = (a\mathbf{v}_1 + b\mathbf{v}_2) + t(3\mathbf{v}_1 - 5\mathbf{v}_2),$$

so subtracting the left side from this leaves us with $t(3\mathbf{v}_1 - 5\mathbf{v}_2)$. But

$$3\mathbf{v}_1 - 5\mathbf{v}_2 = 3((5/3)\mathbf{v}_2) - 5\mathbf{v}_2 = 5\mathbf{v}_2 - 5\mathbf{v}_2 = \mathbf{0},$$

so the t-part vanishes.

(b) Again we expand out the right side and then subtract the left side:

$$\begin{aligned}(a-3t)\mathbf{w}_1 + (b+8t)\mathbf{w}_2 + (c+4t)\mathbf{w}_3 &= a\mathbf{w}_1 - 3t\mathbf{w}_1 + b\mathbf{w}_2 + 8t\mathbf{w}_2 + c\mathbf{w}_3 + 4t\mathbf{w}_3 \\ &= (a\mathbf{w}_1 + b\mathbf{w}_2 + c\mathbf{w}_3) + t(-3\mathbf{w}_1 + 8\mathbf{w}_2 + 4\mathbf{w}_3),\end{aligned}$$

so subtracting the left side from this leaves us with $t(-3\mathbf{w}_1 + 8\mathbf{w}_2 + 4\mathbf{w}_3)$. But

$$-3\mathbf{w}_1 + 8\mathbf{w}_2 + 4\mathbf{w}_3 = -3\mathbf{w}_1 + 8\mathbf{w}_2 + 4((3/4)\mathbf{w}_1 - 2\mathbf{w}_2) = -3\mathbf{w}_1 + 8\mathbf{w}_2 + 3\mathbf{w}_1 - 8\mathbf{w}_2 = \mathbf{0},$$

so the t-part vanishes. $\diamondsuit$

9. (a) Since $\mathbf{v}_1$ and $\mathbf{v}_2$ are visibly nonzero and not scalar multiples of each other, they constitute a basis for their span, so $\dim V = 2$. One orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ consists of $\mathbf{w}_1 = \mathbf{v}_1$ and $\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{Proj}_{\mathbf{v}_1}(\mathbf{v}_2)$. We compute

$$\begin{aligned}\mathbf{w}_2 &= \mathbf{v}_2 - \mathbf{Proj}_{\mathbf{v}_1}(\mathbf{v}_2) \\ &= \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1 \\ &= \mathbf{v}_2 - \frac{12}{6}\mathbf{v}_1 \\ &= \mathbf{v}_2 - 2\mathbf{v}_1 = \begin{bmatrix}0\\5\\2\\1\end{bmatrix} - 2\begin{bmatrix}1\\2\\1\\0\end{bmatrix} = \begin{bmatrix}-2\\1\\0\\1\end{bmatrix},\end{aligned}$$

so an orthogonal basis is given by $\left\{\begin{bmatrix}1\\2\\1\\0\end{bmatrix}, \begin{bmatrix}-2\\1\\0\\1\end{bmatrix}\right\}$. (Safety check: we can verify orthogonality directly since $\mathbf{w}_1 \cdot \mathbf{w}_2 = -2+2+0+0 = 0$, and the spanning property of the 2-dimensional V for the pair of $\mathbf{w}_j$'s holds since $\mathbf{v}_2 = \mathbf{w}_2 + 2\mathbf{w}_1$ by design.)

(b) We use the orthogonal basis $\{\mathbf{w}_1, \mathbf{w}_2\}$ along with the projection formula to compute $\mathbf{Proj}_V(\mathbf{b})$:

$$\begin{aligned}\mathbf{Proj}_V(\mathbf{b}) &= \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{b}) + \mathbf{Proj}_{\mathbf{w}_2}(\mathbf{b}) \\ &= \frac{\mathbf{b} \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 + \frac{\mathbf{b} \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\mathbf{w}_2 \\ &= \frac{6}{6}\mathbf{w}_1 + \frac{-6}{6}\mathbf{w}_2 \\ &= \mathbf{w}_1 - \mathbf{w}_2 = \begin{bmatrix}1\\2\\1\\0\end{bmatrix} - \begin{bmatrix}-2\\1\\0\\1\end{bmatrix} = \begin{bmatrix}3\\1\\1\\-1\end{bmatrix}.\end{aligned}$$

(As a safety check on our work, we can compute that $\mathbf{b} - \begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ -1 \\ 2 \\ 1 \end{bmatrix}$ is indeed orthogonal to both $\mathbf{v}_1$ and $\mathbf{v}_2$, as should be true of $\mathbf{Proj}_V(\mathbf{b})$. It is not necessary to have done this, but checking your work is a good habit.)

(c) In the solution to (b) we saw that $\mathbf{Proj}_V(\mathbf{b}) = \mathbf{w}_1 - \mathbf{w}_2$. By construction, $\mathbf{w}_1 = \mathbf{v}_1$ and $\mathbf{w}_2 = \mathbf{v}_2 - 2\mathbf{v}_1$, so

$$\mathbf{Proj}_V(\mathbf{b}) = \mathbf{v}_1 - (\mathbf{v}_2 - 2\mathbf{v}_1) = 3\mathbf{v}_1 - \mathbf{v}_2.$$

Hence, $r = 3$ and $s = -1$. (As a safety check, which is not necessary but we recommend as a good habit, you can explicitly compute $3\mathbf{v}_1 - \mathbf{v}_2$ to see that it is equal to $\begin{bmatrix} 3 \\ 1 \\ 1 \\ -1 \end{bmatrix}$, the explicit determination of $\mathbf{Proj}_V(\mathbf{b})$ in (b).) ◊

10. There are multiple approaches (and multiple correct answers) to this problem; we will just give one approach. Our strategy will be to find *some* basis for $\mathcal{P}$, then use Theorem 7.1.1 to convert it into an orthogonal basis.

In fact, any two non-zero vectors in $\mathcal{P}$ not on the same line through the origin constitute a basis for $\mathcal{P}$. For example, we can find a basis as follows: let $\mathbf{v}_1$ be the unique point on $\mathcal{P}$ with $x = 1, y = 0$ (so we must solve for z), while $\mathbf{v}_2$ has $x = 0, y = 1$. More specifically, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$ and $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 5/2 \end{bmatrix}$.

As an orthonormal basis, we can take $\mathbf{v}_1$ and $\mathbf{v}_2' = \mathbf{v}_2 - \mathbf{Proj}_{\mathbf{v}_1}(\mathbf{v}_2) = \mathbf{v}_2 - \dfrac{\mathbf{v}_2 \cdot \mathbf{v}_1}{\mathbf{v}_1 \cdot \mathbf{v}_1}\mathbf{v}_1$, so we need to compute some dot products:

$$\mathbf{v}_2 \cdot \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 5/2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = -5, \quad \mathbf{v}_1 \cdot \mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = 5,$$

so

$$\mathbf{v}_2' = \mathbf{v}_2 - \frac{5}{5}\mathbf{v}_1 = \mathbf{v}_2 - \mathbf{v}_1 = \begin{bmatrix} 0 \\ 1 \\ 5/2 \end{bmatrix} - \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 1/2 \end{bmatrix}.$$

We can also rescale $\mathbf{v}_2'$ by 2 without changing the orthogonality (call the result $\mathbf{v}_2''$ for clarity). Summarizing, the orthogonal basis we've found is

$$\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}, \ \mathbf{v}_2'' = \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}.$$ ◊

11. (a) The vectors are

$$\mathbf{X} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \end{bmatrix}, \quad \mathbf{Y} = \begin{bmatrix} -4 \\ -1 \\ 1 \\ 5 \\ 5 \\ 7 \\ 8 \end{bmatrix}.$$

(b) The vector $\mathbf{v}$ can be taken to be $\mathbf{X} - \mathbf{Proj_1}(\mathbf{X}) = \mathbf{X} - \overline{x}\mathbf{1}$ with $\overline{x}$ equal to the average of the entries x_i in $\mathbf{X}$. This average is

$$\frac{1+2+3+4+5+6+7}{7} = \frac{7(1+7)/2}{7} = \frac{28}{7} = 4,$$

so

$$\mathbf{v} = \mathbf{X} - (4)\mathbf{1} = \begin{bmatrix} -3 \\ -2 \\ -1 \\ 0 \\ 1 \\ 2 \\ 3 \end{bmatrix}.$$

(This $\mathbf{v}$ is what is called $\widehat{\mathbf{X}}$ in the main text.) The projection of $\mathbf{Y}$ into V is then given by

$$\begin{aligned} \frac{\mathbf{Y}\cdot\mathbf{v}}{\mathbf{v}\cdot\mathbf{v}}\mathbf{v} + \frac{\mathbf{Y}\cdot\mathbf{1}}{\mathbf{1}\cdot\mathbf{1}}\mathbf{1} &= \frac{12+2-1+0+5+14+24}{28}\mathbf{v} + \frac{-4-1+1+5+5+7+8}{7}\mathbf{1} \\ &= \frac{56}{28}\mathbf{v} + \frac{21}{7}\mathbf{1} \\ &= 2\mathbf{v} + (3)\mathbf{1}. \end{aligned}$$

Hence, $t = 2$ and $s = 3$.

(c) We have $\mathbf{v} = \mathbf{X} - (4)\mathbf{1}$, so

$$\begin{aligned} \mathbf{Proj}_V(\mathbf{Y}) = 2\mathbf{v} + (3)\mathbf{1} &= 2(\mathbf{X} - (4)\mathbf{1}) + (3)\mathbf{1} \\ &= 2\mathbf{X} + (-8+3)\mathbf{1} \\ &= 2\mathbf{X} - (5)\mathbf{1}. \end{aligned}$$

Hence, the line of best fit is $y = 2x - 5$.

(d) Here is the plot of data and the line from (c).

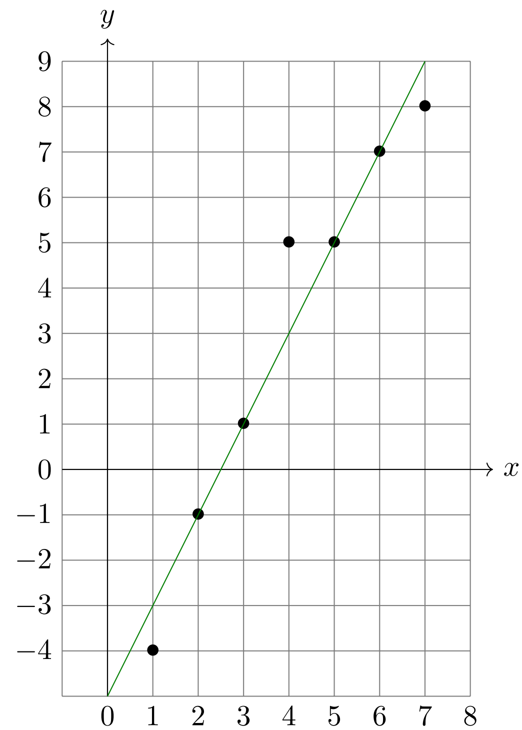

FIGURE 1. The best-fit line $y = 2x - 5$ compared with the data.

It looks like a good fit. $\Diamond$

12. (a) By the parametric form of L, all displacement vectors between points of L are scalar multiples of $\mathbf{x} = \begin{bmatrix} -1 \\ 4 \\ -1 \\ -2 \end{bmatrix}$. We want all of these to belong to $\mathcal{P}$, so it is the same to show $\mathbf{x}$ belongs to $\mathcal{P} = \mathrm{span}(\mathbf{v}, \mathbf{w})$. That is, we seek $c, d \in \mathbf{R}$ for which $\mathbf{x} = c\mathbf{v} + d\mathbf{w}$. That is, we seek c, d satisfying

$$\begin{bmatrix} -1 \\ 4 \\ -1 \\ -2 \end{bmatrix} = \begin{bmatrix} 2c + 11d \\ -c + 5d \\ -c - 10d \\ c + d \end{bmatrix}.$$

By equating coordinates, this is a collection of 4 equations in 2 unknowns:

$$2c + 11d = -1, \quad -c + 5d = 4, \quad -c - 10d = -1, \quad c + d = -2.$$

If we consider the first two equations, we can use the usual method from high-school algebra to find that those two have one simultaneous solution: $c = -7/3$ and $d = 1/3$. Direct evaluation confirms that these values also satisfy the third and fourth equations, so $\mathbf{x} = -(7/3)\mathbf{v} + (1/3)\mathbf{w} \in \mathcal{P}$ as desired.

(b) Based on the parametric form for L, we pick $\mathbf{y} = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -7 \end{bmatrix}$ (corresponding to $t = 0$), though any choice of point on L will work just as well below.

The closest distance from $\mathbf{y}$ to $\mathcal{P}$ is the length of $\mathbf{y} - \mathbf{Proj}_{\mathcal{P}}(\mathbf{y})$: the displacement vector between $\mathbf{y}$ and its closest point $\mathbf{Proj}_{\mathcal{P}}(\mathbf{y})$ on $\mathcal{P}$. In order to compute the projection, we would like an orthogonal basis for $\mathcal{P}$. By Theorem 7.1.1, one such basis is $\mathbf{v}, \mathbf{w}' = \mathbf{w} - \mathbf{Proj}_{\mathbf{v}}(\mathbf{w}) = \mathbf{w} - \dfrac{\mathbf{w} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v}$, so we need to compute some dot products:

$$\mathbf{w} \cdot \mathbf{v} = \begin{bmatrix} 11 \\ 5 \\ -10 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 22 - 5 + 10 + 1 = 28, \quad \mathbf{v} \cdot \mathbf{v} = \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} = 7,$$

so

$$\mathbf{w}' = \mathbf{w} - 4\mathbf{v} = \begin{bmatrix} 11 \\ 5 \\ -10 \\ 1 \end{bmatrix} - 4\begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 5 \\ -10 \\ 1 \end{bmatrix} - \begin{bmatrix} 8 \\ -4 \\ -4 \\ 4 \end{bmatrix} = \begin{bmatrix} 3 \\ 9 \\ -6 \\ -3 \end{bmatrix}.$$

Noting the common factor of 3 in all entries of this final expression, it does us no harm to replace $\mathbf{w}'$ by $\mathbf{w}'' = \dfrac{1}{3}\mathbf{w}' = \begin{bmatrix} 1 \\ 3 \\ -2 \\ -1 \end{bmatrix}$ (though if you don't make this change then it has no impact on the final answer below).

As we explained above, the shortest distance is the length of the vector

$$\mathbf{y} - \mathbf{Proj}_{\mathcal{P}}(\mathbf{y}) = \mathbf{y} - \mathbf{Proj}_{\mathbf{v}}(\mathbf{y}) - \mathbf{Proj}_{\mathbf{w}''}(\mathbf{y}) = \mathbf{y} - \frac{\mathbf{y} \cdot \mathbf{v}}{\mathbf{v} \cdot \mathbf{v}}\mathbf{v} - \frac{\mathbf{y} \cdot \mathbf{w}''}{\mathbf{w}'' \cdot \mathbf{w}''}\mathbf{w}'',$$

which is

$$\mathbf{y} - \left(\frac{4 \cdot 2 + 4 \cdot (-1) + 4 \cdot (-1) + (-7) \cdot 1}{7}\right)\mathbf{v} - \left(\frac{4 \cdot 1 + 4 \cdot 3 + 4 \cdot (-2) + (-7) \cdot (-1)}{1 + 9 + 4 + 1}\right)\mathbf{w}'',$$

that in turn is equal to

$$\mathbf{y} - \frac{-7}{7}\mathbf{v} - \frac{15}{15}\mathbf{w}'' = \begin{bmatrix} 4 \\ 4 \\ 4 \\ -7 \end{bmatrix} + \begin{bmatrix} 2 \\ -1 \\ -1 \\ 1 \end{bmatrix} - \frac{15}{15}\begin{bmatrix} 1 \\ 3 \\ -2 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 + 2 - 1 \\ 4 - 1 - 3 \\ 4 - 1 + 2 \\ -7 + 1 + 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 5 \\ -5 \end{bmatrix}.$$

The length of this vector is $5\sqrt{3} = \sqrt{75}$. $\Diamond$

