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Problem 1: Linear system solution set, in parametric form

Let L be the line determined by the overlap of two planes in $\mathbf{R}^3$ (not passing through the origin):

$$x + y + z = 4 \quad \text{and} \quad 2x + 3y + z = 9.$$

(a) Find a point on L whose z-coordinate is 5, and another whose z-coordinate is 6.

(b) Find general formulas for x and y in terms of z for all points (x, y, z) on the line L; this should recover your answers to (a) upon plugging in $z = 5$ and $z = 6$ respectively. (Hint: think about z as a "constant" and x and y as "variables" to guide your algebraic work in the style of what you did for (a).) Express points on L in parametric form: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{p} + t\mathbf{v}$.

(c) For the matrix $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \end{bmatrix}$ and vector $\mathbf{b} = \begin{bmatrix} 4 \\ 9 \end{bmatrix}$, check that the vectors $\mathbf{p}, \mathbf{v}$ you found in (b) satisfy $A\mathbf{p} = \mathbf{b}$ and $A\mathbf{v} = \mathbf{0}$. State what these imply about specific vectors that belong to the column space and null space of A, and about the number of solutions to the linear system $A \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{b}$.

Solution:

(a) First, we plug $z = 5$ into both of the given equations, to get the new simulaneous system just in terms of x and y: $\left\{ \begin{matrix} x + y = -1 \\ 2x + 3y = 4 \end{matrix} \right\}$. This can be solved by the method of high school algebra to get $x = -7$, $y = 6$. Hence, there is one such point: $(-7, 6, 5)$.

Next, we plug $z = 6$ into both of the given equations, to get the new simulaneous system just in terms of x and y: $\left\{ \begin{matrix} x + y = -2 \\ 2x + 3y = 3 \end{matrix} \right\}$. This can be solved as before to get $x = -9$, $y = 7$. Hence, there is one such point: $(-9, 7, 6)$.

(b) We seek to solve the system of equations $\left\{ \begin{matrix} x + y = 4 - z \\ 2x + 3y = 9 - z \end{matrix} \right\}$ for x and y in terms of z, which can be done by the high school algebra method of "elimination" as is implicit in the work done to solve either system of two equations in two unknowns from (a). First subtract twice the first equation from the second equation to eliminate x and obtain $y = 9 - z - 2(4 - z) = 1 + z$. Plugging this back into the first equation, we obtain $x = 4 - z - (1 + z) = 3 - 2z$, so

$$\begin{cases} x = 3 - 2z \\ y = 1 + z \end{cases}$$

is the answer. (We could have instead eliminated y; in the end we would obtain the same final answer.) Note that if we plug in $z = 5$ and $z = 6$ in turn, we get the points we found in (a). In parametric form, the solutions can be written

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 - 2z \\ 1 + z \\ z \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix},$$

where z can be any scalar; thus, here z plays the role of the parameter t.

(c) From (b), we have $\mathbf{p} = \begin{bmatrix} 3 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v} = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$; the given matrix-vector multiplication relations are readily verified.

- The fact that $A\mathbf{p} = \mathbf{b}$ means that $\mathbf{b}$ is a linear combination of the columns of A (namely, it equals 3 times the first column plus the second column), as prescribed by the entries of $\mathbf{p}$; that is, $\mathbf{b}$ lies in the column space of A.

- The fact that $A\mathbf{v} = \mathbf{0}$ means that $\mathbf{v}$ lies in the null space of A (and so does every scalar multiple of $\mathbf{v}$, since $A(t\mathbf{v}) = tA\mathbf{v} = t\mathbf{0} = \mathbf{0}$).
- Finally, taken together, these facts confirm that the linear system $A\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \mathbf{b}$ has infinitely many solutions (namely, the infinitely many points on L), since we may add any scalar multiple of $\mathbf{v}$ to $\mathbf{b}$ to obtain another solution: $A(\mathbf{p} + t\mathbf{v}) = A\mathbf{p} + t(A\mathbf{v}) = \mathbf{b} + t(\mathbf{0}) = \mathbf{b}$. (In geometric terms, we are tracing out the points on the line L through $\mathbf{p}$ by moving parallel to $\mathrm{span}(\mathbf{v})$.)

Problem 2: Recognizing sets as null spaces (or not)

Often in linear algebra one builds a collection of n-vectors satisfying a variety of conditions, and it can be useful to know if the collection is the null space of an $m \times n$ matrix for some m. (in effect: is the collection describable as the simultaneous solution set for a system of m linear equations in n unknowns?)

(a) Explain why the subset $S = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbf{R}^3 : v_1 + v_2 = v_3 + 1 \right\} \subset \mathbf{R}^3$ cannot be the null space of a matrix A (such an A would have to be $m \times 3$ for some m). Hint: try to find some general property of null spaces that S violates.

(b) Find a 2×3 matrix A for which $N(A)$ is equal to the set $S = \left\{ \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} \in \mathbf{R}^3 : v_1 + v_2 - v_3 = 0,\ v_2 + v_3 = 0 \right\} \subset \mathbf{R}^3$.

Solution:

(a) The nullspace $N(A)$ always contains the zero vector but the given set S does not, so we can never have $S = N(A)$. More generally, if we form linear combinations of solutions to the given equation for S then we generally don't get another solution since the constant term 1 is affected. This shows that S is not "closed" under the formation of linear combinations, whereas a null space always is, so that is another way to understand why S cannot be a null space.

(b) We are looking for a matrix A for which $A\mathbf{v} = \mathbf{0}$ precisely when $\left\{ \begin{array}{ccc} v_1 + v_2 - v_3 & = & 0 \\ v_2 + v_3 & = & 0 \end{array} \right.$. We can think of each equation as an entry of the vector $A\mathbf{v}$ that we want to vanish, and "read off" the entries of A from the coefficients of v_1, v_2, v_3 in these equations. In particular the first row of A will be given by the coefficients in the first equation and the second row by the coefficients in the second equation (where the second equation involve v_1 with a coefficient of 0: $0v_1 + v_2 + v_3$ on the left side). We thereby obtain $A = \begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 1 \end{bmatrix}$, and indeed the vanishing of $A\mathbf{v}$ for this A expresses exactly the two vanishing conditions that define S, so $S = N(A)$.

Problem 3: Column space of a matrix: basis and dimension

Consider the matrix

$$A = \begin{bmatrix} 1 & 3 & 5 \\ 2 & -1 & 3 \\ 1 & 0 & 2 \end{bmatrix}.$$

(a) Write the third column of A as a linear combination of the first two columns. Use this to find a basis for $C(A)$. What is $\dim(C(A))$?

(b) Find a basis for $C(A)$ that contains $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$.

(c) (**Extra**) How can having a basis for the column space $C(M)$ of an $m \times n$ matrix M help in figuring out for a given m-vector $\mathbf{b}$ if $M\mathbf{x} = \mathbf{b}$ has a solution? (Hint: think about $\mathbf{Proj}_{C(M)}$.)

Solution:

(a) (a) We want $a, b \in \mathbf{R}$ for which $a \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + b \begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$. That is, we want to simultaneously solve the system

$$\begin{cases} a + 3b & = 5 \\ 2a - b & = 3 \\ a & = 2 \end{cases}.$$

The last condtion says $a = 2$, so the second condition tells us $b = 1$. The resulting pair $(a, b) = (2, 1)$ is checked to satisfy the first condition. The upshot is that the first two columns, $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 3 \\ -1 \\ 0 \end{bmatrix}$, span $C(A)$ and so are a basis since neither is a scalar multiple of the other. It follows that $\dim(C(A)) = 2$.

(b) For a basis for the 2-dimensional subspace $C(A)$, we merely need a pair of non-zero vectors in $C(A)$ that are not scalar multiples of each other, since this pair will automatically span $C(A)$ for dimension reasons (and therefore form a basis). If we want the basis to contain the third column $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$, we note that (say) the first column $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ is not a scalar multiple of this (by inspection), so $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 5 \\ 3 \\ 2 \end{bmatrix}$ is a linearly independent spanning set for $C(A)$ and hence is a basis. (The second and third vectors also consistute a basis, for the same reasons.)

(c) We want to be able to figure out if $\mathbf{b} \in C(M)$. This happens exactly when $\mathbf{b} = \mathbf{Proj}_{C(M)}(\mathbf{b})$ (quite generally, for a linear subspace $V \subset \mathbf{R}^m$ and $\mathbf{w} \in \mathbf{R}^m$, we have $\mathbf{w} \in V$ precisely when $\mathbf{w} = \mathbf{Proj}_V(\mathbf{w})$; why?). If we have a basis for the column space $C(M)$ then we can use Gram–Schmidt to extract from that an orthogonal basis and so can actually compute $\mathbf{Proj}_{C(M)}(\mathbf{b})$ (and so can check if it is or is not equal to $\mathbf{b}$, thereby determining if $\mathbf{b} \in C(M)$).

Problem 4: Column spaces and an overdetermined linear system

Let $A = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix}$, and let the 5-vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ denote the columns of A (from left to right). For a 5-vector $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix}$, the vector equation $A\mathbf{x} = \mathbf{b}$ for $\mathbf{x} \in \mathbf{R}^3$ encodes a system of 5 scalar linear equations in 3 unknowns (the entries of $\mathbf{x}$), so it is overdetermined (more equations than unknowns) and hence the rule of thumb is that only for rather special $\mathbf{b}$ should a solution exist. This problem works out a description of those special $\mathbf{b}$ for this specific A by thinking in terms of column spaces.

(a) If a vector $\mathbf{b} \in \mathbf{R}^5$ belongs to the column space of A (i.e., $\mathbf{b} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_3\mathbf{a}_3$ for some scalars x_1, x_2, x_3, or equivalently $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \in \mathbf{R}^3$) then by turning this vector condition into a collection of 5 simultaneous scalar conditions explain why $b_3 - b_1 - b_2 = 0$, $b_4 + b_1 - b_2 = 0$, and $b_5 - 2b_2 + b_1 = 0$. (Hint: use that each b_i can be expressed in a specific way in terms of the x_j's).

(b) The three equations in the b_i's at the end of (a) can be rewritten to say $b_3 = b_1 + b_2$, $b_4 = -b_1 + b_2$, and $b_5 = -b_1 + 2b_2$, so collectively they say exactly that $\mathbf{b}$ is a 5-vector of the special form $\begin{bmatrix} b_1 \\ b_2 \\ b_1+b_2 \\ -b_1+b_2 \\ -b_1+2b_2 \end{bmatrix}$ with $b_1, b_2 \in \mathbf{R}$. Check explicitly that any $\mathbf{b}$ of this form actually is in $C(A)$. (Hint: for $\mathbf{b}$ of this special form in terms of b_1 and b_2, adapt the type of argument in Problem 1(b) to find an $\mathbf{x} \in \mathbf{R}^3$ with $x_3 = 0$ for which the first two entries of $A\mathbf{x}$ are b_1, b_2 respectively, and then use the assumed three equations on the b_i's to check that actually $A\mathbf{x} = \mathbf{b}$.). This says that three conditions on $\mathbf{b}$ at the end of (a) are not merely a consequence of membership in $C(A)$ but even exactly characterize it.

(c) Give an explicit 5-vector $\mathbf{b}$ for which the linear system "$A\mathbf{x} = \mathbf{b}$" does *not* have a solution. (There are many answers.)

Solution:

(a) A vector $\mathbf{b}$ is in the column space $C(A)$ precisely when it can be written in the form $A\mathbf{x} = A\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$, where $x_1, x_2, x_3 \in \mathbf{R}$. Working out $A\mathbf{x}$ explicitly, this says

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{bmatrix} = \mathbf{b} = A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1+2x_2+3x_3 \\ 2x_1+3x_2+4x_3 \\ 3x_1+5x_2+7x_3 \\ x_1+x_2+x_3 \\ 3x_1+4x_2+5x_3 \end{bmatrix}.$$

This single vector equation amounts to says that (x_1, x_2, x_3) is a solution to the system of five simultaneous scalar equations given by equating corresponding entries in the 5-vectors on the far left and the far right. This expresses each b_i as a linear combination of x_1, x_2, x_3. We are going to substitute those expressions for the b_i's into each of $b_3 - b_1 - b_2$, $b_4 + b_1 - b_2$, and $b_5 - 2b_2 + b_1$ to see that we get 0 for all of them!

If we substitute the expressions for b_1, b_2, and b_3 into $b_3 - b_1 - b_2$ we get

$$b_3 - b_1 - b_2 = (3x_1+5x_2+7x_3) - (x_1+2x_2+3x_3) - (2x_1+3x_2+4x_3) = 0x_1 + 0x_2 + 0x_3 = 0,$$

as desired. Likewise, substituting the expressions for b_1, b_2, and b_4 into $b_4 + b_1 - b_2$ yields

$$b_4 + b_1 - b_2 = (x_1+x_2+x_3) + (x_1+2x_2+3x_3) - (2x_1+3x_2+4x_3) = 0x_1 + 0x_2 + 0x_3 = 0.$$

Finally, substituting in the expressions for b_1, b_2, b_5 gives

$$b_5 - 2b_2 + b_1 = (3x_1+4x_2+5x_3) - 2(2x_1+3x_2+4x_3) + (x_1+2x_2+3x_3) = 0x_1 + 0x_2 + 0x_3 = 0,$$

as desired.

(b) As suggested in the hint, for $\mathbf{b}$ of the given special form (in terms of b_1 and b_2), rather than aim to find straight away a solution to the equality of 5-vectors $A\mathbf{x} = \mathbf{b}$ (a system of 5 scalar conditions), we first seek a solution to the more limited system of two scalar equations

$$\begin{cases} x_1+2x_2+3x_3 &= b_1 \\ 2x_1+3x_2+4x_3 &= b_2 \end{cases}$$

arising from equating the corresponding first two vector entries of $A\mathbf{x}$ and $\mathbf{b}$, subject to the further condition $x_3 = 0$. Under this vanishing condition on x_3, the pair of equations becomes two simultaneous equations

$$x_1 + 2x_2 = b_1, \;\; 2x_1 + 3x_2 = b_2$$

in two unknowns x_1 and x_2.

We now solve this for x_1 and x_2 in terms of b_1 and b_2 by the elimination method from high school algebra, in the spirit of what we did in Problem 1(c). By subtracting twice the first equation from the second to eliminate x_1 we get

$-x_2 = b_2 - 2b_1$, so $x_2 = 2b_1 - b_2$. Then by plugging this back into the first equation, we get $x_1 = -3b_1 + 2b_2$. Alternatively, this pair of equations in x_1 and x_2 can be written in vector language as $\begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \end{bmatrix}$ and so can be solved by multiplying both sides on the left by the inverse of the indicated 2×2 matrix:

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 3 \end{bmatrix}^{-1} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -3 & 2 \\ 2 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \end{bmatrix} = \begin{bmatrix} -3b_1 + 2b_2 \\ 2b_1 - b_2 \end{bmatrix},$$

yielding the same expressions for x_1 and x_2 in terms of b_1 and b_2 that we found when working by hand with the elimination method.

Now trying the vector $\mathbf{x} = \begin{bmatrix} -3b_1 + 2b_2 \\ 2b_1 - b_2 \\ 0 \end{bmatrix}$ as suggested in the hint, we see it works as desired:

$$A\mathbf{x} = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 & 7 \\ 1 & 1 & 1 \\ 3 & 4 & 5 \end{bmatrix} \begin{bmatrix} -3b_1 + 2b_2 \\ 2b_1 - b_2 \\ 0 \end{bmatrix} = \begin{bmatrix} -3b_1 + 2b_2 + 4b_1 - 2b_2 \\ -6b_1 + 4b_2 + 6b_1 - 3b_2 \\ -9b_1 + 6b_2 + 10b_1 - 5b_2 \\ -3b_1 + 2b_2 + 2b_1 - b_2 \\ -9b_1 + 6b_2 + 8b_1 - 4b_2 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_1 + b_2 \\ -b_1 + b_2 \\ -b_1 + 2b_2 \end{bmatrix} = \mathbf{b},$$

the final equality being exactly our assumption on the special form of $\mathbf{b}$. Hence, all such $\mathbf{b}$ really do belong to $C(A)$.

(c) By (a) and (b), the vector equation $A\mathbf{x} = \mathbf{b}$ (which says exactly that the 5-vector $\mathbf{b}$ belongs to $C(A)$) has a solution $\mathbf{x} \in \mathbf{R}^3$ precisely when the 5-vector $\mathbf{b}$ has the special form considered in (b).

Hence, any 5-vector $\mathbf{b}$ not of that special form must lie outside $C(A)$ and so $A\mathbf{x} = \mathbf{b}$ has no solution for such $\mathbf{b}$. So we just need to pick a 5-vector $\mathbf{b}$ whose third, fourth, or fifth entry violates the corresponding special form. That is, we just need to have at least one of the following: $b_3 \neq b_1 + b_2$, $b_4 \neq -b_1 + b_2$, or $b_5 \neq b_1 + 2b_2$ (we don't need *all* of these violations to happen: just one violation is enough!). There are many such vectors (even violating all three conditions on (b_3, b_4, b_5), but that is more than we need). An example is $\mathbf{b} = \begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}$, which violates the conditions on b_3 and b_5 (though it satisfies the condition on b_4).

Problem 5: LU-decomposition

Let $A = \begin{bmatrix} 12 & 9 & 3 \\ -4 & 1 & 7 \\ 4 & 3 & 2 \end{bmatrix}$ and $L = \begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ and $U = \begin{bmatrix} 4 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix}$.

(a) Verify that $LU = A$, so this is an LU-decomposition of A.

(b) Let $\mathbf{b} = \begin{bmatrix} 6 \\ 2 \\ 1 \end{bmatrix}$. Find all solutions to $L\mathbf{y} = \mathbf{b}$. (You should get that $\mathbf{y}_0 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$ is the only solution.)

(c) Find all solutions to $A\mathbf{x} = \mathbf{b}$ with $\mathbf{b}$ as in (b). (Hint: This means solving $LU\mathbf{x} = \mathbf{b}$, which is the same as $U\mathbf{x} = \mathbf{y}_0$. Why?)

Solution:

(a) We calculate

$$\begin{bmatrix} 3 & 0 & 0 \\ -1 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 & 3 & 1 \\ 0 & 2 & 4 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 12 & 9 & 3 \\ -4 & 1 & 7 \\ 4 & 3 & 2 \end{bmatrix}.$$

(b) We solve the system

$$\begin{aligned} 3y_1 \quad\quad\quad &= 6 \\ -y_1 + 2y_2 \quad\quad &= 2 \\ y_1 + \quad\quad y_3 &= 1 \end{aligned}$$

Forward substitution gives $y_1 = 2$, then $y_2 = 2$, and finally $y_3 = -1$.

(c) We already know that $L\mathbf{y} = \mathbf{b}$ has exactly one solution, $\mathbf{y}_0 = \begin{bmatrix} 2 \\ 2 \\ -1 \end{bmatrix}$. Hence, the system $\mathbf{b} = A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x})$ says exactly that $U\mathbf{x} = \mathbf{y}_0$. We express the latter as a system of linear equations:

$$\begin{aligned} 4x_1 + 3x_2 + x_3 &= 2 \\ 2x_2 + 4x_3 &= 2 \\ x_3 &= -1 \end{aligned}$$

Backward substitution gives $x_3 = -1$, then $x_2 = 3$, and finally $x_1 = -\frac{3}{2}$.

Problem 6: QR-decomposition

Let $A = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 0 & 5 \\ 1 & 5 & 3 \end{bmatrix}$, and define $\mathbf{v}_i$ to be the ith column of A.

(a) Apply the Gram–Schmidt process to $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$. The output vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ that you obtain should all be nonzero, and as a check on your work make sure that they are pairwise orthogonal.

(b) Examine your calculations from (a) to express each $\mathbf{v}_i$ as a linear combination of the orthogonal basis of $\mathbf{w}_j$'s. (This should be found from the work already done in (a); do *not* directly compute the projections of $\mathbf{v}_i$ onto each $\mathbf{w}_j$, as that would be defeating the point of the work in (a).) Then compute the unit vectors $\mathbf{w}_j' = \mathbf{w}_j/\|\mathbf{w}_j\|$ and express $\mathbf{v}_i$ as a linear combination of the $\mathbf{w}_j'$'s.

(c) Use (b) to find a decomposition $A = QR$ where Q is an orthogonal matrix and R is an upper triangular matrix. Check your answer is correct by computing the product QR of the Q and R that you find.

(d) Use (c) to find A^{-1} as an explicit 3×3 matrix (with entries that are fractions with denominator that is a factor of 10, no $\sqrt{5}$ anywhere), and check that its product against A on the left or the right is equal to I_3; it is fine to compute just one of those products.

Hint: when computing R^{-1}, you may find it convenient to first extract $\sqrt{5}$ as a factor from every entry of R (i.e., write $R = \sqrt{5}R'$ for an upper triangular matrix R', so $R^{-1} = (1/\sqrt{5}){R'}^{-1}$; it is easier to find ${R'}^{-1}$.)

Solution:

(a) We have $\mathbf{w}_1 = \mathbf{v}_1 = \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix}$. Then

$$\mathbf{w}_2 = \mathbf{v}_2 - \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_2) = \mathbf{v}_2 - \frac{\mathbf{v}_2 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\mathbf{w}_1 = \mathbf{v}_2 - \frac{5}{5}\mathbf{w}_1 = \mathbf{v}_2 - \mathbf{w}_1 = \begin{bmatrix} 0 \\ 0 \\ 5 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix}.$$

Finally,

$$\begin{aligned}
\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_1}(\mathbf{v}_3) - \mathbf{Proj}_{\mathbf{w}_2}(\mathbf{v}_3) &= \mathbf{v}_3 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_1}{\mathbf{w}_1 \cdot \mathbf{w}_1}\,\mathbf{w}_1 - \frac{\mathbf{v}_3 \cdot \mathbf{w}_2}{\mathbf{w}_2 \cdot \mathbf{w}_2}\,\mathbf{w}_2 \\
&= \mathbf{v}_3 - \frac{5}{5}\mathbf{w}_1 - \frac{10}{20}\mathbf{w}_2 \\
&= \mathbf{v}_3 - \mathbf{w}_1 - \frac{1}{2}\mathbf{w}_2 \\
&= \begin{bmatrix} 1 \\ 5 \\ 3 \end{bmatrix} - \begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ 5 \\ 0 \end{bmatrix}.
\end{aligned}$$

The dot products $\mathbf{w}_i \cdot \mathbf{w}_j$ for $i \neq j$ are all checked to be 0 by direct calculation,

(b) We see from the work in (a) that

$$\mathbf{v}_1 = \mathbf{w}_1,\ \ \mathbf{v}_2 = \mathbf{w}_2 + \mathbf{w}_1 = \mathbf{w}_1 + \mathbf{w}_2,\ \ \mathbf{v}_3 = \mathbf{w}_3 + \mathbf{w}_1 + (1/2)\mathbf{w}_2 = \mathbf{w}_1 + (1/2)\mathbf{w}_2 + \mathbf{w}_3.$$

Dividing each $\mathbf{w}_i$ by its length gives that

$$\mathbf{w}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 2 \\ 0 \\ 1 \end{bmatrix},\ \ \mathbf{w}_2' = \frac{1}{\sqrt{20}}\begin{bmatrix} -2 \\ 0 \\ 4 \end{bmatrix} = \frac{1}{\sqrt{5}}\begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix},\ \ \mathbf{w}_3' = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix},$$

and substituting $\mathbf{w}_i = \|\mathbf{w}_1\|\mathbf{w}_i'$ into the expression for each $\mathbf{v}_i$ in terms of the $\mathbf{w}_j$'s yields

$$\mathbf{v}_1 = \sqrt{5}\,\mathbf{w}_1',\ \ \mathbf{v}_2 = \sqrt{5}\,\mathbf{w}_1' + \sqrt{20}\,\mathbf{w}_2',\ \ \mathbf{v}_3 = \sqrt{5}\,\mathbf{w}_1' + \sqrt{5}\,\mathbf{w}_2' + 5\mathbf{w}_3'.$$

(c) From the work in (b) we can read off Q from the $\mathbf{w}_j'$'s and R from the coefficients of each $\mathbf{v}_i$ in terms of the $\mathbf{w}_j'$'s:

$$Q = \begin{bmatrix} \frac{2}{\sqrt{5}} & -\frac{1}{\sqrt{5}} & 0 \\ 0 & 0 & 1 \\ \frac{1}{\sqrt{5}} & \frac{2}{\sqrt{5}} & 0 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} \sqrt{5} & \sqrt{5} & \sqrt{5} \\ 0 & \sqrt{20} & \sqrt{5} \\ 0 & 0 & 5 \end{bmatrix}.$$

(You can write the middle entry in R as $2\sqrt{5}$ if you wish, but this is entirely unnecessary.) Direct multiplication confirms that QR is indeed equal to A.

(d) We know $A^{-1} = R^{-1}Q^{-1}$. We have

$$Q^{-1} = Q^\top = \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix}.$$

To find R^{-1}, we write $R = \sqrt{5}R'$ for

$$R' = \begin{bmatrix} 1 & 1 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & \sqrt{5} \end{bmatrix},$$

so $R^{-1} = (1/\sqrt{5})R'^{-1}$. We seek numbers a, b, c for which

$$R'^{-1} = \begin{bmatrix} 1 & a & b \\ 0 & \frac{1}{2} & c \\ 0 & 0 & \frac{1}{\sqrt{5}} \end{bmatrix}.$$

To find these numbers we calculate

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = R'^{-1}R' = \begin{bmatrix} 1 & 1+2a & 1+a+\sqrt{5}\,b \\ 0 & 1 & (1/2)+\sqrt{5}\,c \\ 0 & 0 & 1 \end{bmatrix}.$$

So we have $a = -\dfrac{1}{2}, b = -\dfrac{1}{2\sqrt{5}}$, and $c = -\dfrac{1}{2\sqrt{5}}$. Hence,

$$R^{-1} = \frac{1}{\sqrt{5}}R'^{-1} = \begin{bmatrix} 1/\sqrt{5} & -1/(2\sqrt{5}) & -1/10 \\ 0 & 1/(2\sqrt{5}) & -1/10 \\ 0 & 0 & 1/5 \end{bmatrix}.$$

Altogether this gives

$$\begin{aligned} A^{-1} = R^{-1}Q^{\top} &= \begin{bmatrix} 1/\sqrt{5} & -1/(2\sqrt{5}) & -1/10 \\ 0 & 1/(2\sqrt{5}) & -1/10 \\ 0 & 0 & 1/5 \end{bmatrix} \begin{bmatrix} \frac{2}{\sqrt{5}} & 0 & \frac{1}{\sqrt{5}} \\ -\frac{1}{\sqrt{5}} & 0 & \frac{2}{\sqrt{5}} \\ 0 & 1 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 2/5+1/10 & -1/10 & 1/5-1/5 \\ -1/10 & -1/10 & 1/5 \\ 0 & 1/5 & 0 \end{bmatrix} \\ &= \begin{bmatrix} 1/2 & -1/10 & 0 \\ -1/10 & -1/10 & 1/5 \\ 0 & 1/5 & 0 \end{bmatrix} \end{aligned}$$

Direct multiplication of this against A on either the left or the right (take your pick) is seen to yield I_3, as desired.

Problem 7: Recognizing Eigenvectors

For the following matrices A and nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, verify that the vectors are eigenvectors for A and find their corresponding eigenvalues.

(a) $A = \begin{bmatrix} 1 & 3 & 0 \\ 3 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 5 & 8 & 16 \\ 4 & 1 & 8 \\ -4 & -4 & -11 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$, and $\mathbf{v}_3 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$.

Solution:

(a) We calculate

$$A\mathbf{v}_1 = \begin{bmatrix} 4 \\ 4 \\ 0 \end{bmatrix} = 4\mathbf{v}_1,$$

so $\mathbf{v}_1$ is an eigenvector with eigenvalue 4. Furthermore

$$A\mathbf{v}_2 = \begin{bmatrix} -2 \\ 2 \\ 0 \end{bmatrix} = -2\mathbf{v}_2,$$

so $\mathbf{v}_2$ is an eigenvector with eigenvalue -2. Lastly

$$A\mathbf{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = 1\mathbf{v}_3,$$

so $\mathbf{v}_3$ is an eigenvector with eigenvalue 1.

(a) We calculate

$$A\mathbf{v}_1 = \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix} = 1\mathbf{v}_1,$$

so $\mathbf{v}_1$ is an eigenvector with eigenvalue 1. Furthermore

$$A\mathbf{v}_2 = \begin{bmatrix} 3 \\ -3 \\ 0 \end{bmatrix} = -3\mathbf{v}_2,$$

so $\mathbf{v}_2$ is an eigenvector with eigenvalue -3. Lastly

$$A\mathbf{v}_3 = \begin{bmatrix} 6 \\ 0 \\ -3 \end{bmatrix} = -3\mathbf{v}_3,$$

so $\mathbf{v}_3$ is an eigenvector with eigenvalue -3.

Problem 8: Geometric meaning of eigenvalues

Identify the eigenvalues of the following linear transformations $\mathbf{R}^3 \to \mathbf{R}^3$, and find *all* eigenvectors for each eigenvalue (expressed in terms of the given geometric data). The eigenvalues are explicit numbers, not depending on the given line or plane.

Hint: Think geometrically by looking for lines carried onto themselves (or crushed into $\{\mathbf{0}\}$: don't overlook the possibility of 0 as an eigenvalue!). In particular, if a line is not carried onto itself or crushed into the origin, it cannot provide any eigenvectors.

(a) The reflection across a plane $V \subset \mathbf{R}^3$ through the origin.

(b) The projection onto a plane $V \subset \mathbf{R}^3$ through the origin.

(c) The rotation by $90°$ around a line $L \subset \mathbf{R}^3$ through the origin.

(d) The rotation by $180°$ around the line $L \subset \mathbf{R}^3$ through the origin.

Solution:

(a) Each nonzero vector in V is carried onto itself, so they are all eigenvectors with eigenvalue 1. Nonzero vectors in its orthogonal complement are reflected exactly onto their negative, so they are eigenvector with eigenvalue -1. Visually we see that lines through the origin not contained in V nor perpendicular to V (in other words, the angle that the line makes with V is strictly between $0°$ and $90°$) are not carried into themselves under the reflection, so there are no other eigenvectors or eigenvalues than what we have found.

(b) Each nonzero vector in V is carried onto itself, so they are all eigenvectors to eigenvalue 1. Nonzero vectors in its orthogonal complement line are projected onto $\mathbf{0}$, so they are eigenvectors with eigenvalue 0. A line not perpendicular to V projects onto a line in V (rather than being crushed into the origin), so such a line cannot be the same as its projection unless it is already inside the plane. Hence, there are no eigenvectors outside the plane and its normal line, so there are no other eigenvectors or eigenvalues than what we have found

(c) The nonzero vectors on the line L will not be rotated at all, being kept where they are, so they are eigenvectors to eigenvalue 1. There are no other lines through the origin carried onto themselves (or crushed into the origin), so there are no other eigenvalues or eigenvectors.

(d) The nonzero vectors on the line L will not be rotated at all, so they are eigenvectors with eigenvalue 1 as in (d). Nonzero vectors in its orthogonal complement get rotated exactly to their negative, so they are eigenvectors to eigenvalue -1. If we visualize lines through the origin distinct from L and not in the plane perpendicular to L the we see that such lines are never carried into themselves under such a rotation, so there are no eigenvectors outside L and $L^{\perp}$. Hence, we have found all of the eigenvectors and eigenvalues.

Problem 9: Eigenvalues of 2×2 matrices

For each of the following 2×2 matrices, find all the eigenvalues and an eigenvector for each eigenvalue.

(a) $A = \begin{bmatrix} 0 & 1 \\ 2 & 1 \end{bmatrix}$.

(b) $B = \begin{bmatrix} 1 & 4 \\ 4 & 1 \end{bmatrix}$.

(c) $C = \begin{bmatrix} 1 & a \\ a & 1 \end{bmatrix}$ for a general number $a \neq 0$. (Your answer may depend on a; for $a = 4$ it should recover the answer to (b).)

Solution:

(a) We calculate the characteristic polynomial as

$$\lambda^2 - \lambda - 2 = (\lambda - 2)(\lambda + 1).$$

So the eigenvalues of A are 2 and -1.

To find an eigenvector with eigenvalue 2, we write the vector equation $A\mathbf{x} = 2\mathbf{x}$ as a pair of scalar equations (by equating vector entries)

$$x_2 = 2x_1 \quad \text{and} \quad 2x_1 + x_2 = 2x_2;$$

these two equations are nonzero scalar multiples of each other (as we know must happen: two nonzero linear equations in two unknowns with no constant term have a nonzero simultaneous solution precisely when the equations are scalar multiples of each other). By inspecting either of these, an eigenvector is given by $\begin{bmatrix} 1 \\ 2 \end{bmatrix}$; any nonzero scalar multiple also works.

To find an eigenvector with eigenvalue -1 we solve the equations

$$x_2 = -x_1 \quad \text{and} \quad 2x_1 + x_2 = -x_2.$$

By inspecting either of these equations, an eigenvector is given by $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$; any nonzero scalar multiple also works (such as its negative).

(b) We calculate the characteristic polynomial as

$$\lambda^2 - 2\lambda - 15 = (\lambda - 5)(\lambda + 3).$$

So the eigenvalues of B are 5 and -3.

To find an eigenvector with eigenvalue 5, we solve the equations

$$x_1 + 4x_2 = 5x_1 \quad \text{and} \quad 4x_1 + x_2 = 5x_2,$$

which both are equivalent to $x_1 = x_2$. So an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (and any nonzero scalar multiple also works).

To find the eigenvectors with eigenvalue -3, we solve the equations

$$x_1 + 4x_2 = -3x_1 \quad \text{and} \quad 4x_1 + x_2 = -3x_2,$$

which both are equivalent to $x_1 = -x_2$. So an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (and any nonzero scalar multiple also works, such as its negative).

(c) We calculate the characteristic polynomial as

$$\lambda^2 - 2\lambda + (1 - a^2) = \lambda^2 - 2\lambda + ((1-a)(1+a)) = (\lambda - (1+a))(\lambda - (1-a)).$$

So the eigenvalues of C are $1 + a$ and $1 - a$ (*not* $a - 1$; watch out for sign errors).

To find an eigenvector with eigenvalue $1 + a$, we solve the equations

$$x_1 + ax_2 = (1+a)x_1, \;\; ax_1 + x_2 = (1+a)x_2,$$

which each say $ax_2 = ax_1$, or equivalently (since $a \neq 0$) $x_2 = x_1$. So an eigenvector is $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$ (which curiously doesn't depend on a; this is a coincidence for this particular class of matrices).

To find an eigenvector with eigenvalue $1 - a$, we solve the equations

$$x_1 + ax_2 = (1-a)x_1, \;\; ax_1 + x_2 = (1-a)x_2,$$

which each say $ax_2 = -ax_1$, or equivalently (since $a \neq 0$) $x_2 = -x_1$. So an eigenvector is $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ (which curiously again doesn't depend on a; this is a coincidence for this particular class of matrices).

Problem 10: Additional practice with eigenvalues and eigenvectors (triangular examples)

For each eigenvalue λ of the given matrix A, compute a basis for the nonzero linear subspace $N(A - \lambda \mathrm{I}_3)$ in $\mathbf{R}^3$ (the "λ-eigenspace"), and as a check on your work verify directly that each vector in that basis is an eigenvector for A with eigenvalue λ.

(a) $A = \begin{bmatrix} 1 & 6 & 3 \\ 0 & -2 & 2 \\ 0 & 0 & 1 \end{bmatrix}$.

(b) $A = \begin{bmatrix} 2 & 0 & 0 \\ -3 & 3 & 0 \\ 6 & -2 & 2 \end{bmatrix}$.

Solution:

(a) As we have seen in the textbook, the eigenvalues for an upper triangular matrix (as well as lower triangular) are its diagonal entries. So this matrix has as its eigenvalues $\lambda_1 = 1$ and $\lambda_2 = -2$. We need to compute a basis for each of the null spaces $N(A - \mathrm{I}_3)$ and $N(A + 2\mathrm{I}_3)$.

For the first of these, we have

$$A - \mathrm{I}_3 = \begin{bmatrix} 0 & 6 & 3 \\ 0 & -3 & 2 \\ 0 & 0 & 0 \end{bmatrix},$$

so a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in its null space precisely when it satisfies the equations

$$6x_2 + 3x_3 = 0, \;\; -3x_2 + 2x_3 = 0, \;\; 0 = 0.$$

The third equation gives nothing, and the first and second equations are two homogeneous equations in two unknowns for which the equations are not scalar multiples of each other and hence has as its only solution $(0,0)$ (as can be checked directly: the second equation says $x_3 = -2x_2$, and plugging this into the third equation forces x_2 and x_3 to vanish). Thus, altogether this null space consists of vectors of the form

$$\begin{bmatrix} x_1 \\ 0 \\ 0 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

Hence, by inspection we see that a basis of this null space is given by the vector

$$\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

direct matrix-vector multiplication gives

$$A\mathbf{v} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \mathbf{v},$$

as desired.

Turning to the null space of $A + 2\mathrm{I}_3$, we have

$$A + 2\mathrm{I}_3 = \begin{bmatrix} 3 & 6 & 3 \\ 0 & 0 & 2 \\ 0 & 0 & 3 \end{bmatrix}.$$

Hence, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ lies in the null space of this precisely when it satisfies the equations

$$3x_1 + 6x_2 + 3x_3 = 0, \;\; 2x_3 = 0, \;\; 3x_3 = 0.$$

The last two equations both say $x_3 = 0$, and plugging this into the first turns it into the condition $3x_1 + 6x_2 = 0$, which says $x_1 = -2x_2$. Hence,

$$\mathbf{x} = \begin{bmatrix} -2x_2 \\ x_2 \\ 0 \end{bmatrix} = x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix},$$

so a basis for this null space is given by the vector $\mathbf{w} = \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix}$, or any nonzero scalar multiple of that. Direct matrix-vector multiplication establishes the eigenvector condition:

$$A\mathbf{w} = \begin{bmatrix} -2+6 \\ -2 \\ 0 \end{bmatrix} = \begin{bmatrix} 4 \\ -2 \\ 0 \end{bmatrix} = -2\mathbf{w}.$$

(b) Since this matrix is lower-triangular, we may read the diagonal entries of A (similarly as in (a)) to conclude that its eigenvalues are $\lambda_1 = 2$ and $\lambda_2 = 3$. We need to compute a basis for each of the null spaces $N(A - 2\mathrm{I}_3)$ and $N(A - 3\mathrm{I}_3)$.

For the first of these, we have

$$A - 2\mathrm{I}_3 = \begin{bmatrix} 0 & 0 & 0 \\ -3 & 1 & 0 \\ 6 & -2 & 0 \end{bmatrix},$$

so a vector $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ is in its null space precisely when it satisfies the equations

$$0 = 0, \;\; -3x_1 + x_2 = 0, \;\; 6x_1 - 2x_2 = 0.$$

The first equation gives nothing, and the second and third equations both say $x_2 = 3x_1$. Thus, the null space consists of vectors of the form

$$\begin{bmatrix} x_1 \\ 3x_1 \\ x_3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where x_1 and x_3 can each be any scalar. Hence, we see that a basis of this null space is given by the linearly independent vectors

$$\mathbf{v} = \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix}, \quad \mathbf{v}' = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Direct matrix-vector multiplication gives

$$A\mathbf{v} = \begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix} = 2\mathbf{v}, \qquad A\mathbf{v}' = \begin{bmatrix} 0 \\ 0 \\ 2 \end{bmatrix} = 2\mathbf{v}',$$

as desired.

Turning to the null space of $A - 3\mathrm{I}_3$, we have

$$A - 3\mathrm{I}_3 = \begin{bmatrix} -1 & 0 & 0 \\ -3 & 0 & 0 \\ 6 & -2 & -1 \end{bmatrix}.$$

Hence, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ lies in the null space of this precisely when it satisfies the equations

$$-x_1 = 0, \;\; -3x_1 = 0, \;\; 6x_1 - 2x_2 - x_3 = 0.$$

The first two equations both say $x_1 = 0$, and plugging this into the last turns it into the condition $-2x_2 - x_3 = 0$, which says $x_3 = -2x_2$. Hence,

$$\mathbf{x} = \begin{bmatrix} 0 \\ x_2 \\ -2x_2 \end{bmatrix} = x_2 \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix},$$

so a basis for this null space is given by the vector $\mathbf{w} = \begin{bmatrix} 0 \\ 1 \\ -2 \end{bmatrix}$, or any nonzero scalar multiple of that. Direct matrix-vector multiplication establishes the eigenvector condition:

$$A\mathbf{w} = \begin{bmatrix} 0 \\ 3 \\ -2-4 \end{bmatrix} = \begin{bmatrix} 0 \\ 3 \\ -6 \end{bmatrix} = 3\mathbf{w}.$$

Problem 11: Quadratic forms and definiteness I

(a) For each of the following 2×2 symmetric matrices M, compute the quadratic form $q_M(x, y) = \begin{bmatrix} x & y \end{bmatrix} M \begin{bmatrix} x \\ y \end{bmatrix}$:

$$A = \begin{bmatrix} 3 & 2 \\ 2 & 0 \end{bmatrix}, \;\; B = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}, \;\; C = \begin{bmatrix} 17 & 4 \\ 4 & 2 \end{bmatrix}, D = \begin{bmatrix} -6 & 2 \\ 2 & -3 \end{bmatrix}.$$

(b) For each M in (a), use its characteristic polynomial to find its eigenvalues (they are all integers in these cases), and from that determine if $q_M(x, y)$ is positive-definite, negative-definite, or indefinite.

Solution:

(a) We compute $q_A(x,y) = 3x^2 + 4xy$, $q_B(x,y) = 2x^2 - 2xy + 2y^2$, $q_C(x,y) = 17x^2 + 8xy + 2y^2$, and $q_D(x,y) = -6x^2 + 4xy - 3y^2$.

(b) The characteristic polynomials $p_M(\lambda) = \lambda^2 - \mathrm{tr}(M)\lambda + \det(M)$ are given by

$$p_A(\lambda) = \lambda^2 - 3\lambda - 4,\ \ p_B(\lambda) = \lambda^2 - 4\lambda + 3,\ \ p_C(\lambda) = \lambda^2 - 19\lambda + 18,\ \ p_D(\lambda) = \lambda^2 + 9\lambda + 14.$$

These all factor, enabling us to see the roots without needing to haul out the quadratic formula: $p_A(\lambda) = (\lambda-4)(\lambda+1)$, $p_B(\lambda) = (\lambda - 1)(\lambda - 3)$, $p_C(\lambda) = (\lambda - 1)(\lambda - 18)$, and $p_D(\lambda) = (\lambda + 2)(\lambda + 7)$.

Hence, the respective eigenvalues are $\{4, -1\}$ for A, $\{1, 3\}$ for B, $\{1, 18\}$ for C, and $\{-2, -7\}$ for D. Thus, q_B and q_C are positive-definite, q_D is negative-definite, and q_A is indefinite.

Problem 12: Quadratic forms and definiteness II

For each of the following symmetric 3×3 matrices M and given nonzero vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, carry out two tasks:

(i) Compute the associated quadratic form $q_M(x,y,z)$, and verify that $\mathbf{v}_i$'s are pairwise orthogonal and eigenvectors, determining the eigenvalue for each.

(ii) Use your answer to (i) to write down the quadratic form $q(u,v,w) = q_M(u\mathbf{v}_1' + v\mathbf{v}_2' + w\mathbf{v}_3')$ when everything is described in terms of the basis of orthonormal eigenvectors $\mathbf{v}_i' = \mathbf{v}_i/\|\mathbf{v}_i\|$, from which you should determine if q_M is positive-definite, negative-definite, indefinite, positive-semidefinite (but not positive-definite), or negative-semidefinite (but not negative-definite). You *do not* need to compute the lengths $\|\mathbf{v}_i\|$.

(a) $\begin{bmatrix} 5 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & 2 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}$.

(b) $\begin{bmatrix} 3 & 1 & -1 \\ 1 & 3 & -1 \\ -1 & -1 & 5 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ -1 \\ 2 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$.

(c) $\begin{bmatrix} 1 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(d) $\begin{bmatrix} 3 & 2 & 4 \\ 2 & 0 & 2 \\ 4 & 2 & 3 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -2 \\ 0 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} 4 \\ 2 \\ -5 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix}$.

(e) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$, $\mathbf{v}_1 = \begin{bmatrix} -1 \\ 2 \\ -1 \end{bmatrix}$, $\mathbf{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\mathbf{v}_3 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

Solution: In all cases the dot products $\mathbf{v}_i \cdot \mathbf{v}_j$ for $i \neq j$ are checked to vanish.

(a) We have $q_M(x,y,z) = 5x^2 - 2y^2 + 2z^2 - 4xz$. One calculates that $M\mathbf{v}_1 = 6\mathbf{v}_1$, $M\mathbf{v}_2 = -2\mathbf{v}_2$, and $M\mathbf{v}_3 = \mathbf{v}_3$, so the corresponding eigenvalues are $6, -2, 1$ and in terms of the associated orthonormal basis of eigenvectors we have $q(u,v,w) = 6u^2 - 2v^2 + w^2$. This is indefinite.

(b) We have

$$q_M(x,y,z) = 3x^2 + 3y^2 + 5z^2 + 2xy - 2xz - 2yz.$$

One calculates that $M\mathbf{v}_1 = 3\mathbf{v}_1$, $M\mathbf{v}_2 = 6\mathbf{v}_2$, and $M\mathbf{v}_3 = 2\mathbf{v}_3$, so the corresponding eigenvalues are $3, 6, 2$ and in terms of the associated orthonormal basis of eigenvectors we have $q(u, v, w) = 3u^2 + 6v^2 + 2w^2$. This is positive-definite.

(c) We have $q_M(x, y, z) = x^2 + 2y^2 + z^2 - 2xy - 2yz$. One calculates that $M\mathbf{v}_1 = \mathbf{v}_1$, $M\mathbf{v}_2 = 3\mathbf{v}_2$, and $M\mathbf{v}_3 = \mathbf{0} = 0\mathbf{v}_3$, so the corresponding eigenvalues are $1, 3, 0$ and in terms of the associated orthonormal basis of eigenvectors we have $q(u, v, w) = u^2 + 3v^2$. This is positive-semidefinite but not positive-definite.

(d) We have $q_M(x, y, z) = 3x^2 + 3z^2 + 4xy + 8xz + 4yz$. One calculates that $M\mathbf{v}_1 = -\mathbf{v}_1$, $M\mathbf{v}_2 = -\mathbf{v}_2$, and $M\mathbf{v}_3 = 8\mathbf{v}_3$, so the corresponding eigenvalues are $-1, -1, 8$ and in terms of the associated orthonormal basis of eigenvectors we have $q(u, v, w) = -u^2 - v^2 + 8w^2$. This is indefinite.

(e) We have $q_M(x, y, z) = 2xy + 2yz + 2xz$. One calculates that $M\mathbf{v}_1 = -\mathbf{v}_1$, $M\mathbf{v}_2 = -\mathbf{v}_2$, and $M\mathbf{v}_3 = 2\mathbf{v}_3$, so the corresponding eigenvalues are $-1, -1, 2$ and in terms of the associated orthonormal basis of eigenvectors we have $q(u, v, w) = -u^2 - v^2 + 2w^2$. This is indefinite.

Problem 13: Large powers of symmetric matrices (a 3×3 example)

Consider the matrix

$$M = \begin{bmatrix} 3/5 & 1/5 & 1/5 \\ 1/5 & 3/5 & 1/5 \\ 1/5 & 1/5 & 3/5 \end{bmatrix}$$

Since M is symmetric, the Spectral Theorem implies that there is an orthogonal basis for $\mathbf{R}^3$ consisting of eigenvectors for M. For this problem, assume that we are also given that the eigenvalues of M are $\lambda_1 = 1$ and $\lambda_2 = \dfrac{2}{5}$.

(a) Let V_1 be the λ_1-eigenspace. Verify that $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ lies in V_1, and explain why it spans that space.

(b) Find the λ_2-eigenspace V_2, and write it as the span of two orthogonal vectors $\mathbf{w}_2, \mathbf{w}_3$.

(c) Let $\mathbf{w}_1', \mathbf{w}_2', \mathbf{w}_3'$ be unit vectors obtained from $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. Let W be the matrix whose columns are $\mathbf{w}_1', \mathbf{w}_2', \mathbf{w}_3'$. Find the diagonal matrix D where $M = WDW^\top$.

(d) Using the fact that $(2/5)^{100} \approx 0$ to over thirty-five decimal places, calculate M^{100} explicitly.

Solution:

(a) We compute $M \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, so $\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$ is a 1-eigenvector. The 1-eigenspace is the null space of $M - I_3 = \begin{bmatrix} -2/5 & 1/5 & 1/5 \\ 1/5 & -2/5 & 1/5 \\ 1/5 & 1/5 & -2/5 \end{bmatrix}$. By the Rank-Nullity Theorem, this is 1-dimensional if the rank is 2 (i.e., 2-dimensional column space). The first two columns of $M - \mathrm{I}_3$ are independent by inspection, and the sum of the columns is $\mathbf{0}$ (since $\mathbf{w}_1$ is in the null space), so the 3rd column is in the span of the other two.

Alternative solution: The λ_1-eigenspace of M is the null space of $M - I_3 = \begin{bmatrix} -2/5 & 1/5 & 1/5 \\ 1/5 & -2/5 & 1/5 \\ 1/5 & 1/5 & -2/5 \end{bmatrix}$, which is the collection of all vectors $\mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$ for which:

$$\begin{cases} -\frac{2}{5}x + \frac{1}{5}y + \frac{1}{5}z = 0 \\ \frac{1}{5}x - \frac{2}{5}y + \frac{1}{5}z = 0 \\ \frac{1}{5}x + \frac{1}{5}y - \frac{2}{5}z = 0 \end{cases}.$$

Subtracting the second line from the first yields $\frac{3}{5}x = \frac{3}{5}y$, that is, $x = y$. Substituting in any of the remaining equations gives $z = x = y$. So $\mathbf{x} = \begin{bmatrix} x \\ x \\ x \end{bmatrix} = x \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, and $V_1 = \text{span} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$.

(b) We solve the system $(M - \lambda_2 I_3)\mathbf{x} = \mathbf{0}$, that is

$$\begin{cases} \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z = 0 \\ \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z = 0 \\ \frac{1}{5}x + \frac{1}{5}y + \frac{1}{5}z = 0 \end{cases},$$

which is the same equation three times. We have $x = -(y+z)$, that is $\mathbf{x} = \begin{bmatrix} -(y+z) \\ y \\ z \end{bmatrix} = y \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$.

Let $\mathbf{w}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$ and $\mathbf{v}_3 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, then we get an orthogonal vector $\mathbf{w}_3 = \mathbf{v}_3 - \mathbf{Proj}_{\mathbf{w}_2} \mathbf{v}_3 = \begin{bmatrix} -1/2 \\ -1/2 \\ 1 \end{bmatrix}$.

(c) The matrix interpretation of the Spectral Theorem is that $M = WDW^\top$ where W has i-th column $\mathbf{w}'_i$ and D is diagonal with i-th entry the eigenvalue for $\mathbf{w}'_i$. Hence, $D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2/5 & 0 \\ 0 & 0 & 2/5 \end{bmatrix}$.

(d) We have seen that $M^k = WD^kW^{-1} = WD^kW^\top$ for any integer $k > 0$. So for $k = 100$, we have $M^{100} = WD^{100}W^\top$. But D is diagonal, so D^{100} is obtained by raising diagonal entries to the 100-th power. Since $(2/5)^{100} \approx 0$, we have $D^{100} \approx \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Therefore,

$$\begin{aligned} M^{100} &\approx \begin{bmatrix} | & | & | \\ \mathbf{w}'_1 & \mathbf{w}'_2 & \mathbf{w}'_3 \\ | & | & | \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} - & \mathbf{w}'^\top_1 & - \\ - & \mathbf{w}'^\top_2 & - \\ - & \mathbf{w}'^\top_3 & - \end{bmatrix} \\ &= \begin{bmatrix} | & | & | \\ \mathbf{w}'_1 & \mathbf{0} & \mathbf{0} \\ | & | & | \end{bmatrix} \begin{bmatrix} - & \mathbf{w}'^\top_1 & - \\ - & \mathbf{w}'^\top_2 & - \\ - & \mathbf{w}'^\top_3 & - \end{bmatrix} \\ &= \begin{bmatrix} 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \\ 1/\sqrt{3} & 0 & 0 \end{bmatrix} \begin{bmatrix} 1/\sqrt{3} & 1/\sqrt{3} & 1/\sqrt{3} \\ - & \mathbf{w}'^\top_2 & - \\ - & \mathbf{w}'^\top_3 & - \end{bmatrix} \\ &= \begin{bmatrix} 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \\ 1/3 & 1/3 & 1/3 \end{bmatrix}. \end{aligned}$$

Problem 14: Large powers of symmetric matrices (a 2×2 example)

Let $A = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}$, a symmetric matrix.

(a) Compute the eigenvalues $\lambda_1 > \lambda_2$ of A and find eigenvectors $\mathbf{v}_1, \mathbf{v}_2$ for λ_1, λ_2, respectively. Check that $\mathbf{v}_1, \mathbf{v}_2$ are orthogonal.

(b) Write $\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $\mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ as linear combinations of the orthogonal basis $\{\mathbf{v}_1, \mathbf{v}_2\}$.

(c) Use your expressions from part (b) to give an exact expression for A^{100}. (Hint: note that the first column of A^{100} is equal to $A^{100}\mathbf{e}_1$, and similarly for the second column. Use (b) to compute $A^{100}\mathbf{e}_i$.)

(d) Using the (very accurate!) approximation $(\lambda_2/\lambda_1)^{100} \approx 0$, give a much simpler approximate expression for A^{100}.

Solution:

(a) The characteristic polynomial of A is $P_A(\lambda) = \lambda^2 - 4\lambda + 3 = (\lambda - 3)(\lambda - 1)$. So the eigenvalues of A are $1, 3$. The eigenvectors for $\lambda_1 = 3$ are those $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$ for which $A\mathbf{x} - 3\mathbf{x} = 0$. That is, $\begin{bmatrix} 2x - y \\ -x + 2y \end{bmatrix} - \begin{bmatrix} 3x \\ 3y \end{bmatrix} = \mathbf{0}$. This amounts to the equation $-x - y = 0$, i.e. $x = -y$. So an eigenvector is $\mathbf{v}_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$.

For $\lambda_2 = 1$, the same setup yields the equations $\begin{bmatrix} x - y \\ -x + y \end{bmatrix} = \mathbf{0}$, which is to say $x = y$, so $\mathbf{v}_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector. We directly compute $\mathbf{v}_1 \cdot \mathbf{v}_2 = 0$, as claimed.

(b) We can either see "by inspection" that $\mathbf{e}_1 = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$ and $\mathbf{e}_2 = \frac{1}{2}(-\mathbf{v}_1 + \mathbf{v}_2)$, or set up the problems $\mathbf{e}_1 = a\mathbf{v}_1 + b\mathbf{v}_2$ and $\mathbf{e}_2 = a'\mathbf{v}_1 + b'\mathbf{v}_2$ are systems of 2 equations in 2 unknowns that one can solve via high school algebra.

(c) We apply the linearity of A^{100} to compute $A^{100}\mathbf{e}_1$ and $A^{100}\mathbf{e}_2$:

$$A^{100}\mathbf{e}_1 = \frac{1}{2}(A^{100}\mathbf{v}_1 + A^{100}\mathbf{v}_2) = \frac{1}{2}(3^{100}\mathbf{v}_1 + 1^{100}\mathbf{v}_2) = \frac{1}{2}\left(\begin{bmatrix} 3^{100} + 1 \\ -3^{100} + 1 \end{bmatrix}\right)$$

and

$$A^{100}\mathbf{e}_2 = \frac{1}{2}(-A^{100}\mathbf{v}_1 + A^{100}\mathbf{v}_2) = \frac{1}{2}(-3^{100}\mathbf{v}_1 + \mathbf{v}_2) = \frac{1}{2}\left(\begin{bmatrix} -3^{100} + 1 \\ 3^{100} + 1 \end{bmatrix}\right).$$

Therefore,

$$A^{100} = \frac{1}{2}\begin{bmatrix} 1 + 3^{100} & 1 - 3^{100} \\ 1 - 3^{100} & 1 + 3^{100} \end{bmatrix}.$$

(d) If we divide the expression from (c) through by 3^{100}, we obtain

$$\frac{A^{100}}{3^{100}} = \frac{1}{2}\begin{bmatrix} 3^{-100} + 1 & 3^{-100} - 1 \\ 3^{-100} - 1 & 3^{-100} + 1 \end{bmatrix}.$$

But $3^{-100} \approx 0$, so this gives us the approximate equality

$$A^{100} \approx \frac{3^{100}}{2}\begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$