Towards understanding the hierarchical learning in multi-layer (over-parameterized) neural networks

Yuanzhi Li

Assistant Professor, Carnegie Mellon University
Consulting Researcher, Microsoft Research

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  - The complexity of the features (as a function of the input) F_i gradually increases for deeper layers.

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  - We assume there is a multi-layer (i.e. hierarchical) target network that generates the labels by composing features layer by layer.
  - We will show how training a neural network via SGD from random initialization can recover this hierarchical composition of the target network gradually (learning higher level features can also improve the quality of lower ones).

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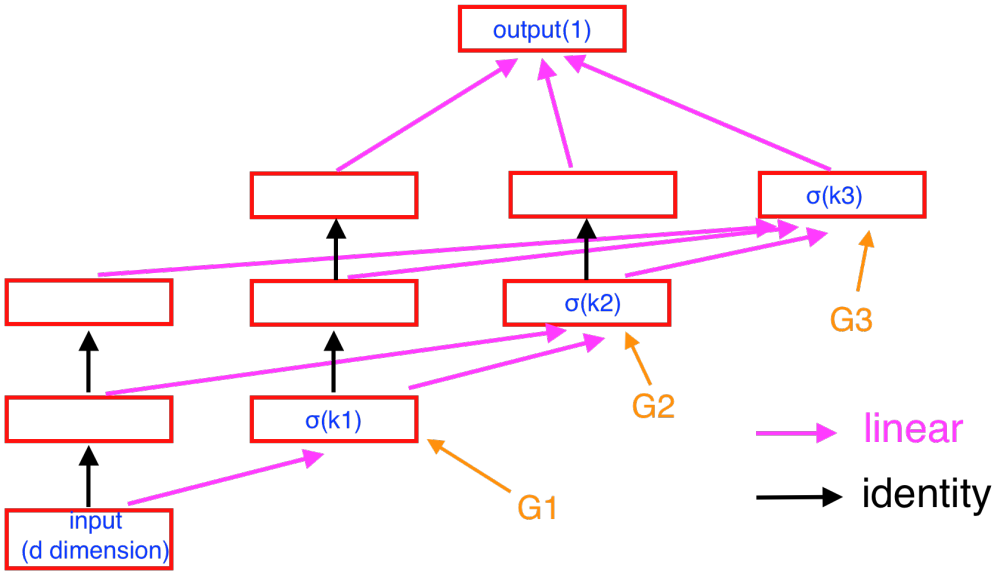

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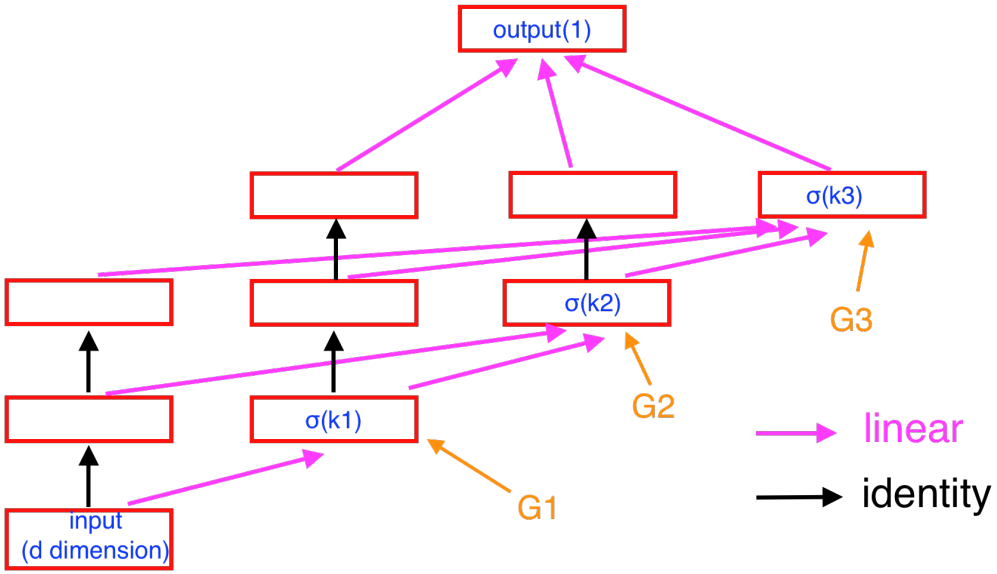

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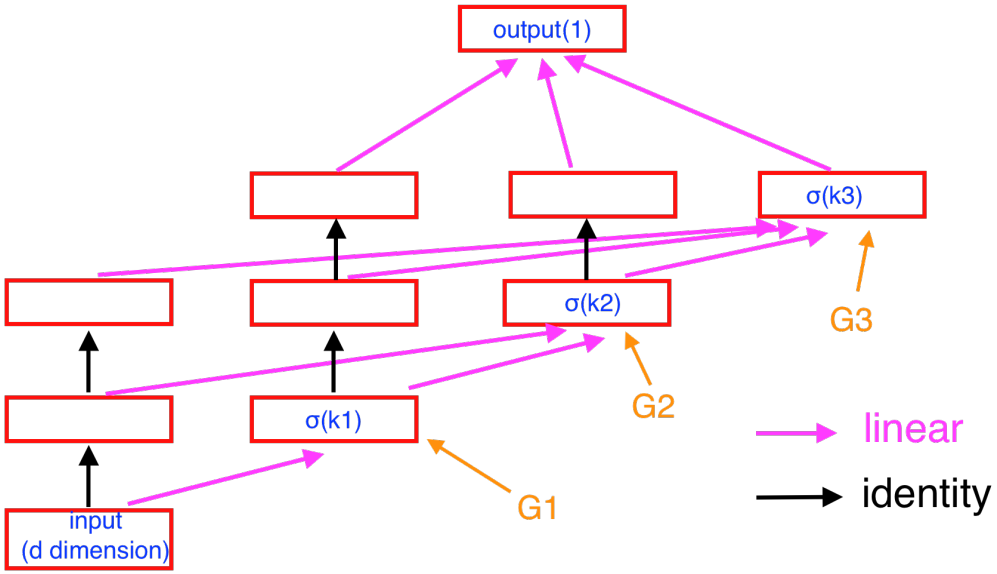

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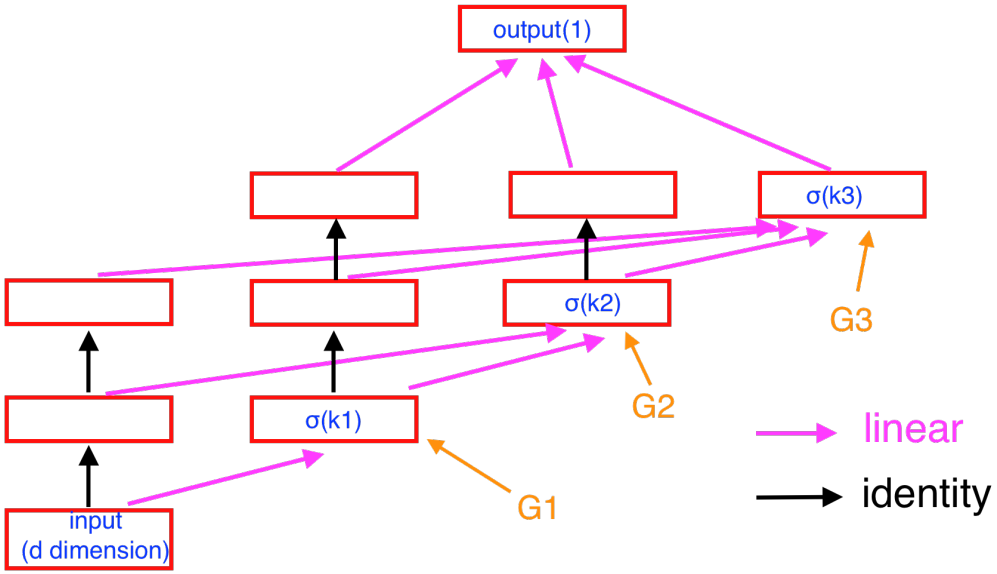

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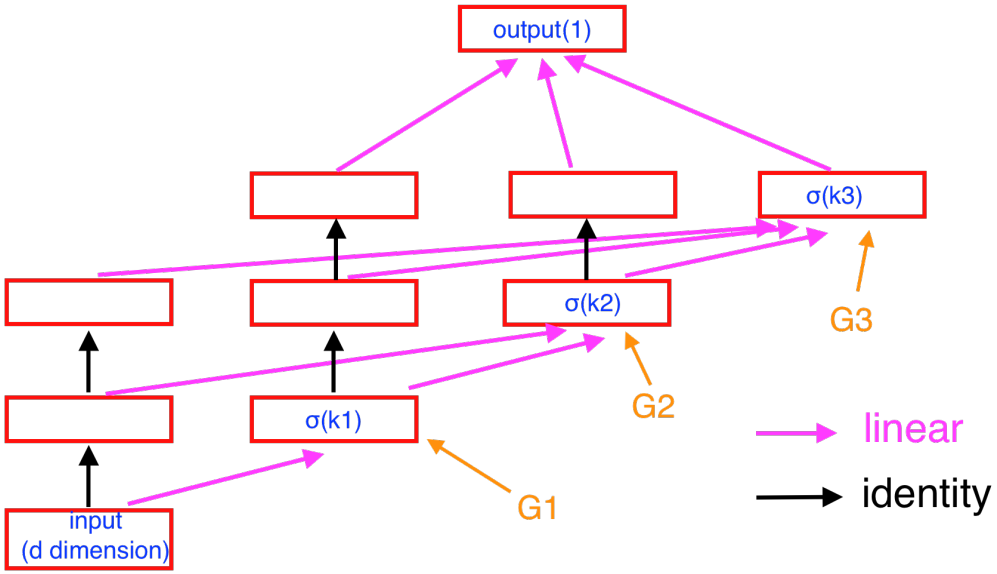

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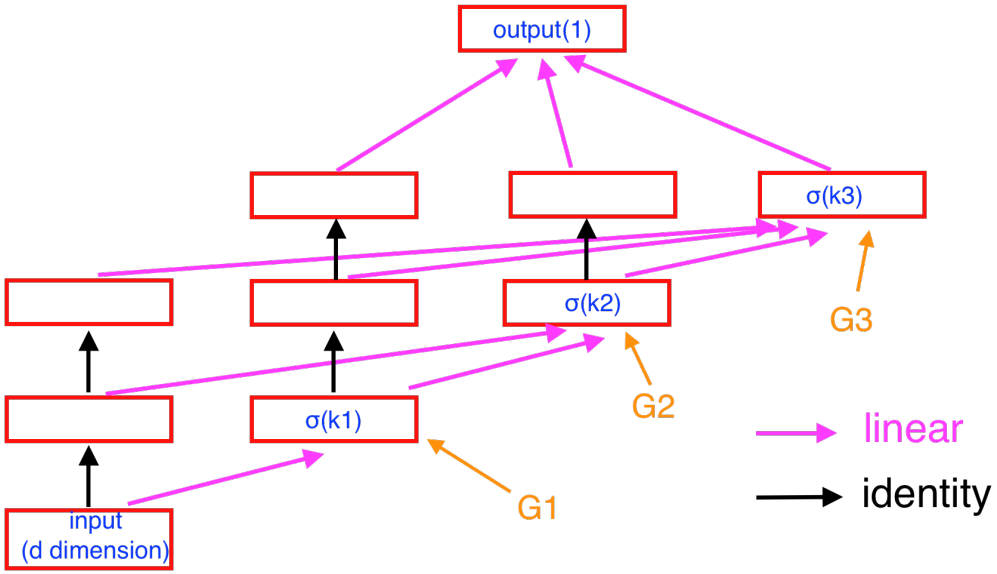

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- $G(x) = \sum_{\ell \in [L]} \alpha_\ell w_\ell^\top G_\ell(x)$. ($\alpha_\ell > 0$ is a scaler)

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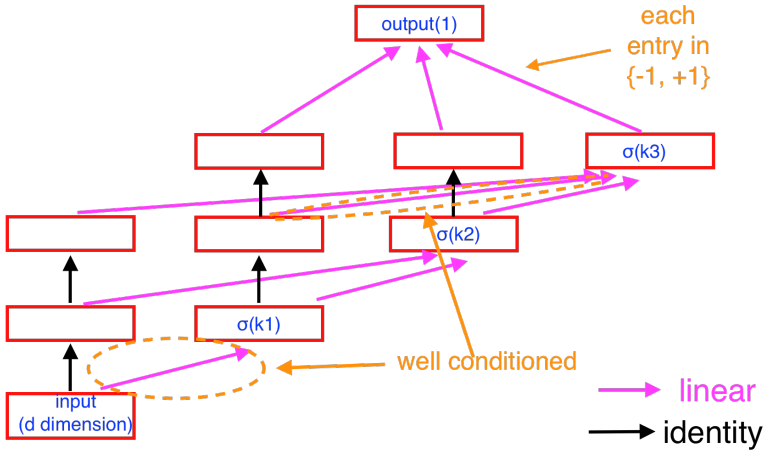

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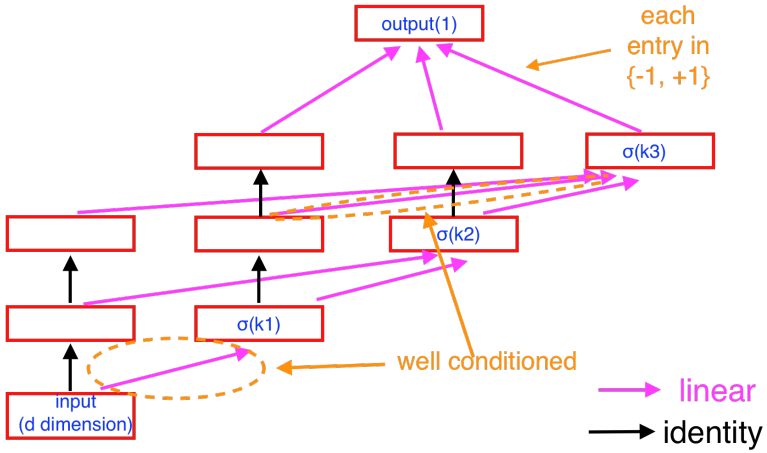

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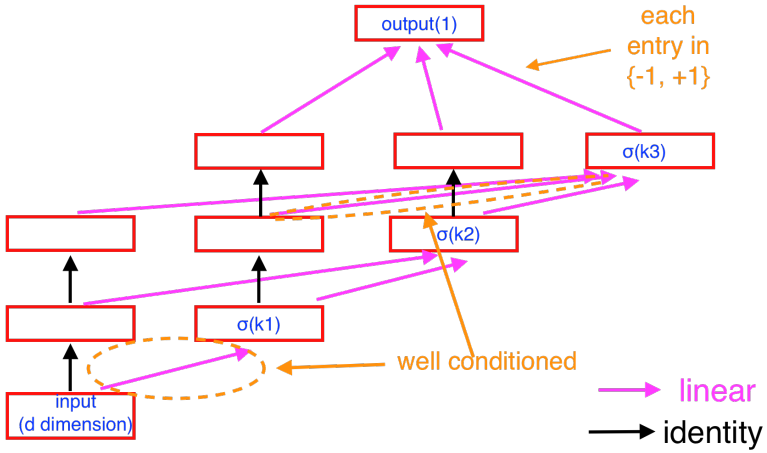

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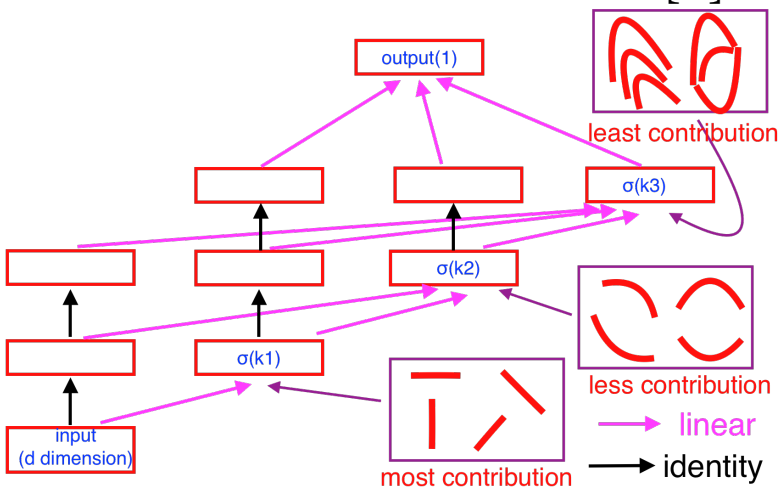

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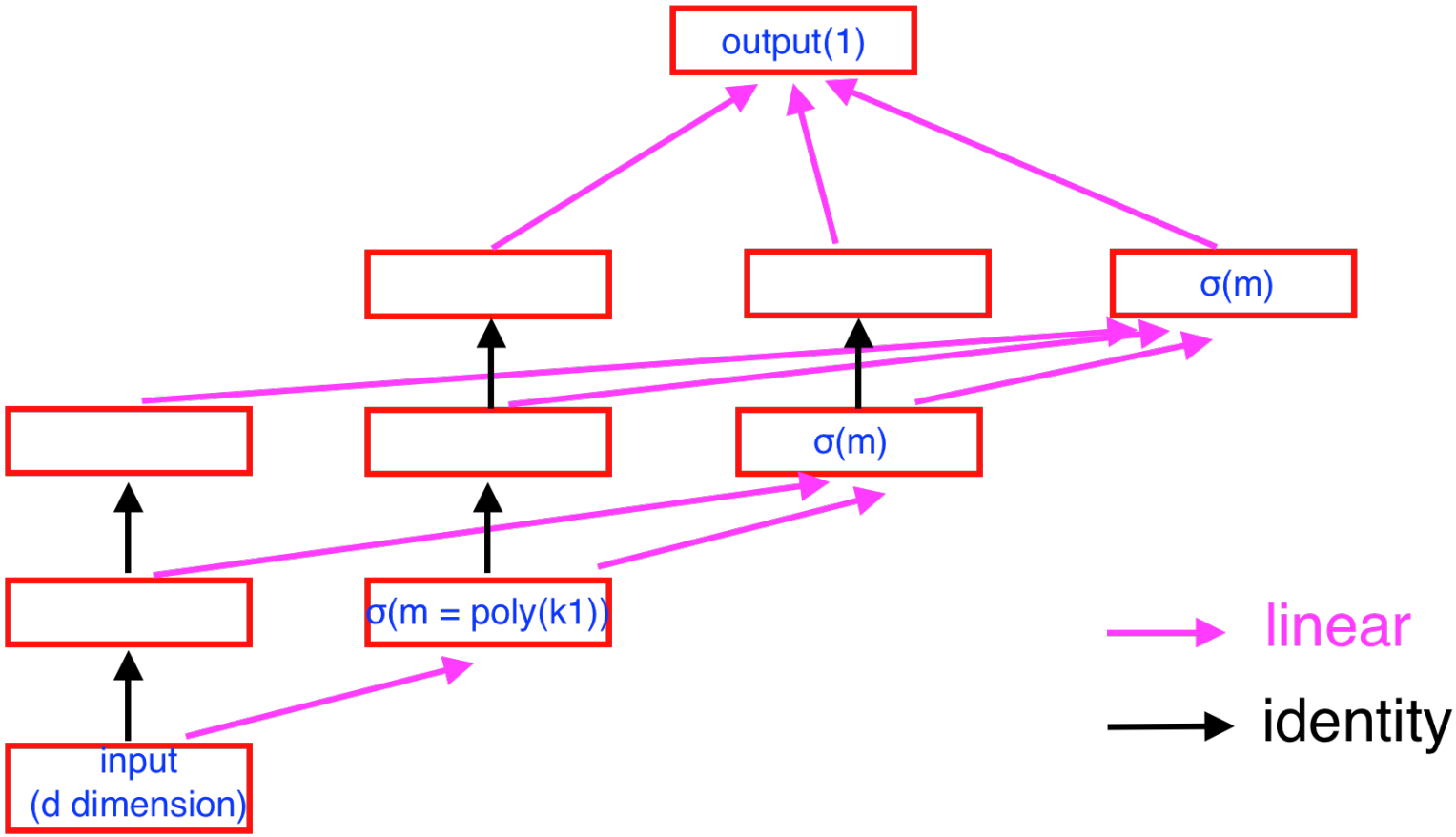

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- $k_\ell \approx d^{\frac{1}{2^\ell}}$ implies that the hierarchical learning must be done for at least $\omega(1)$ times, to avoid learning in one shot a $\omega(1)$ degree polynomial over $d^{\Omega(1)}$ dimension input.

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- But if neural networks are simulating those algorithms, then why do we use deep learning in practice?
- This work: hierarchical learning in multi-layer networks, to the best of my knowledge, only achievable by training a neural network.

Intuition of the proof via a super simple example

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- The rich representation regularizer of over-parameterization: By over-parameterization + gaussian random initialization, It enforces the $m \gg 1$ neurons to learn:
  - $(\alpha_i x_1 + \beta_i x_2)^2_{i \in [m]}$, α_i, β_i looks like independent standard Gaussian: $E_{\alpha_i, \beta_i}(\alpha_i x_1 + \beta_i x_2)^2 = x_1^2 + x_2^2$. This is enough to construct $x_1^4 + x_2^4$.
- The rich representation does not matter for the current layer, but it is crucial for the next one.

Intuition of the "correction" stage in hierarchical learning via a super simple example

- $x \in \mathbb{R}^{1000}$, $\mathbb{E}[x_i] = 1$; $G(x) = x_1^2 + x_2^2 + \alpha((x_1^2 + x_3)^2 + (x_2^2 + x_4)^2)$. Say $\alpha = 0.3$.

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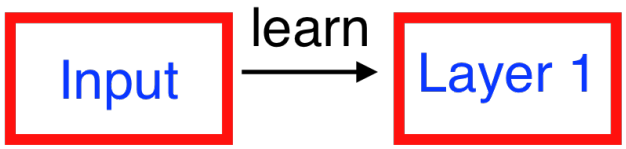
- Keep going...

Multi-layer hierarchical learning

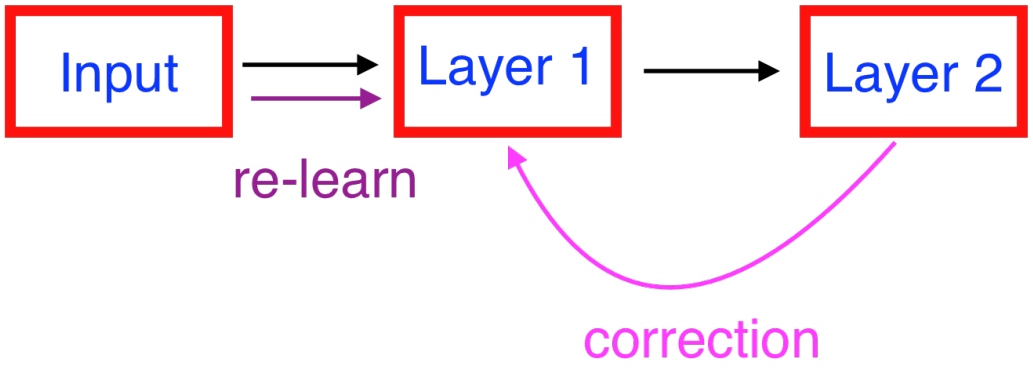


The first layer learns a quadratic approximation of the target function.
Which is not very accurate due to over-fitting to the higher level features.

Multi-layer hierarchical learning

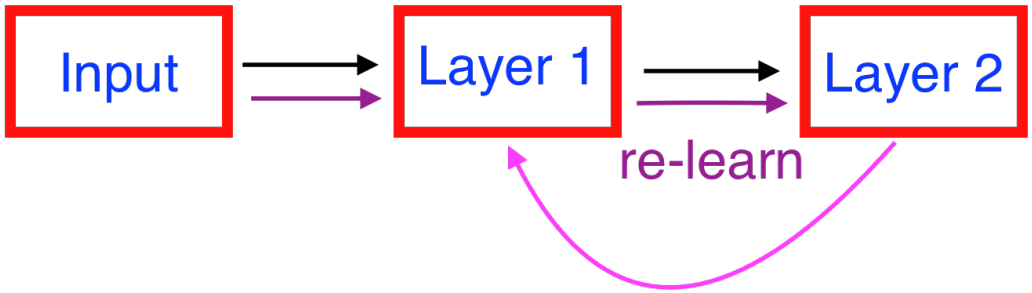
The second layer then learns a quadratic function on top of the first layer (doable via the rich representation given by over-parameterization in the first layer). To approximate the remaining higher degree (3,4) terms in the target function.

Multi-layer hierarchical learning



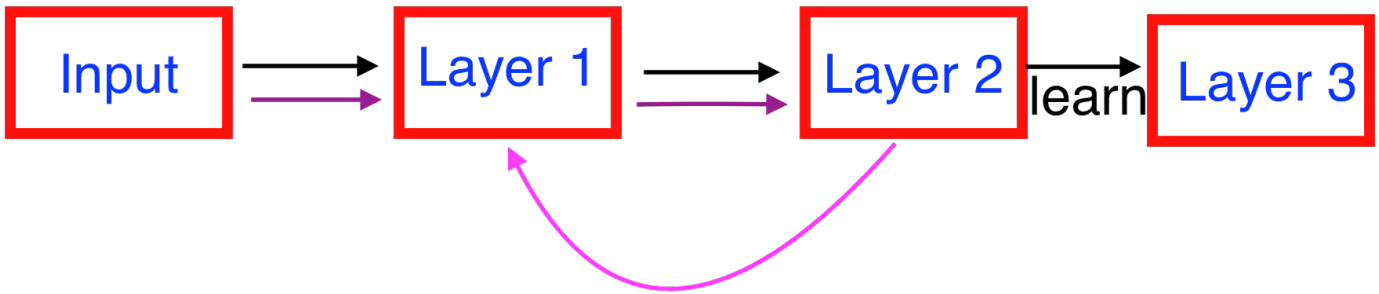
The first layer then "correct" its features due to a reduction of over-fitting in the present of the second layer.

Multi-layer hierarchical learning



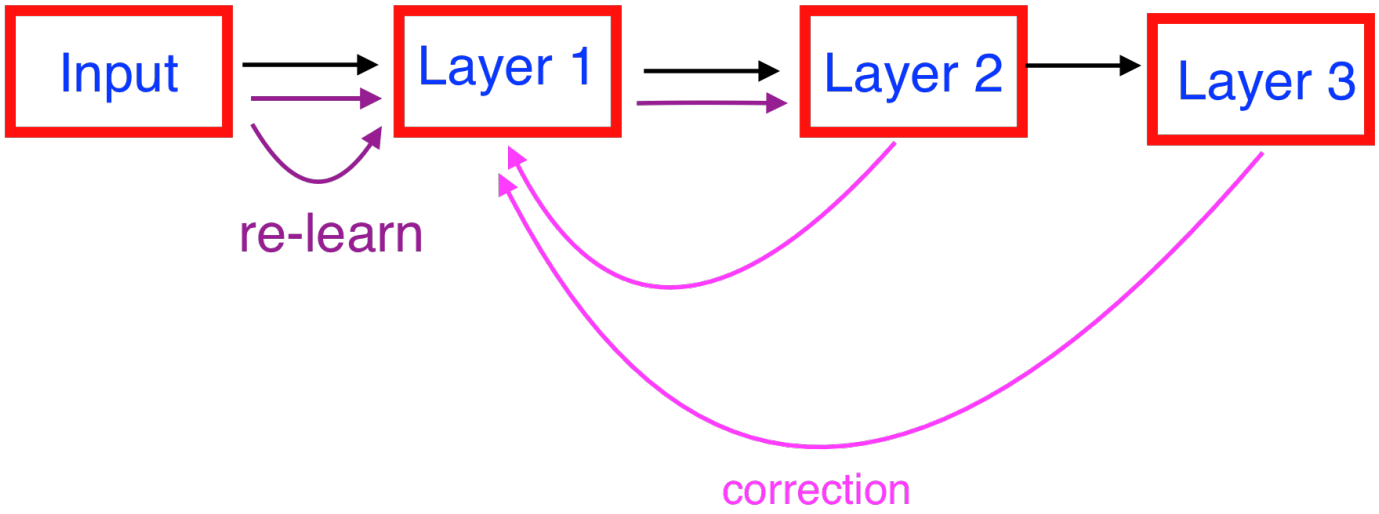
The second layer also learns a more accurate features due to the better input features from the first layer.

Multi-layer hierarchical learning



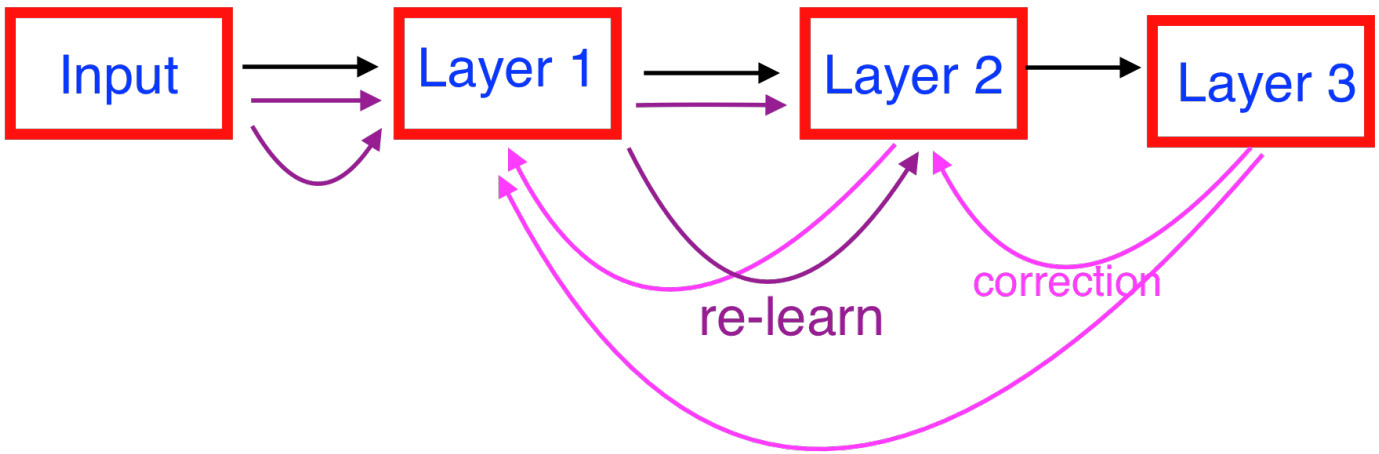
The third layer starts to learn.

Multi-layer hierarchical learning



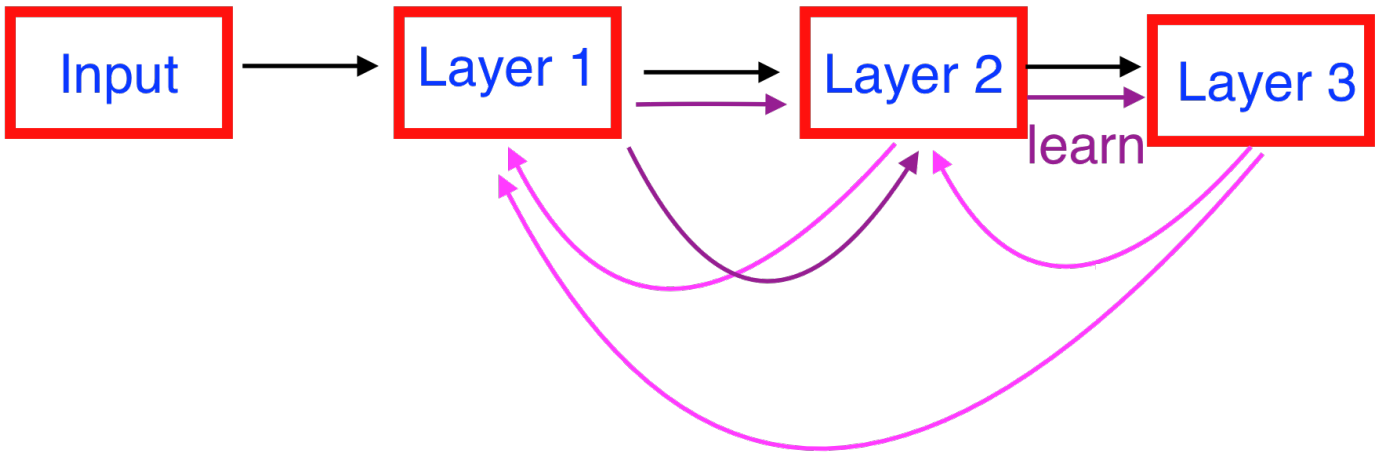
The first layer then further "correct" its features due to a reduction of over-fitting in the present of the third layer.

Multi-layer hierarchical learning



The second layer then further "correct" its features due to a reduction of over-fitting in the present of the third layer and better features from the first layer.

Multi-layer hierarchical learning



KEY MESSAGE: In our work, we do not explicitly train the layers of the network in this order (layers are essentially trained simultaneously), and we can still prove this result (so the hierarchical learning is rather implicit).

Extension to classification and the information gap α

- Our theorem also extends to classification: When the labeling function is given by:

$$y = 1_{\langle w,x\rangle + G(\Pi_{w^\perp} x) \geq 0}$$

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Theorem (AL'19)

When $k_\ell \approx d^{\frac{1}{2^\ell}}$, $\alpha_{\ell+1} \approx \frac{\alpha_\ell}{d^{\frac{1}{2^\ell}}}$, $\mathbb{E}[G(\Pi_{w^\perp} x)] \approx 0, \|w\|_2^2 \approx var(G)$*: Over standard gaussian distribution, for every* $L = o(\log\log d)$*, for every* $\varepsilon > 0$*, given* $N = poly(d, 1/\varepsilon)$ *many training examples, SGD (cross entropy loss + regularizer, starting from random initialization) finds a target network* F *with generalization error* ε *in time* $poly(d, 1/\varepsilon)$

-

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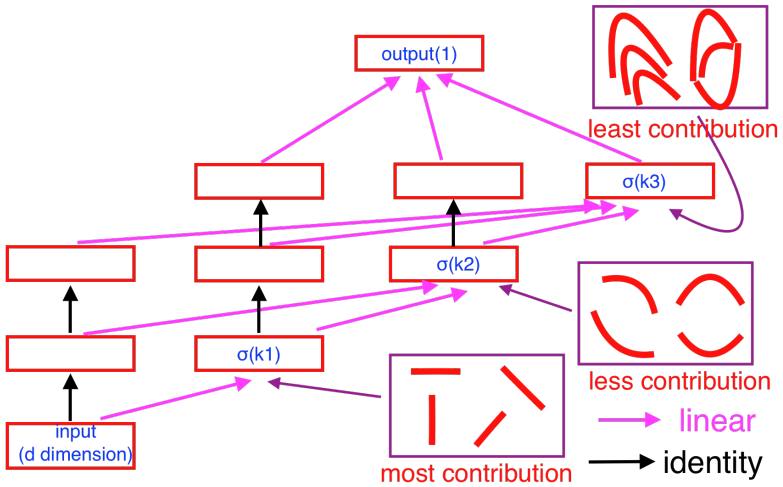

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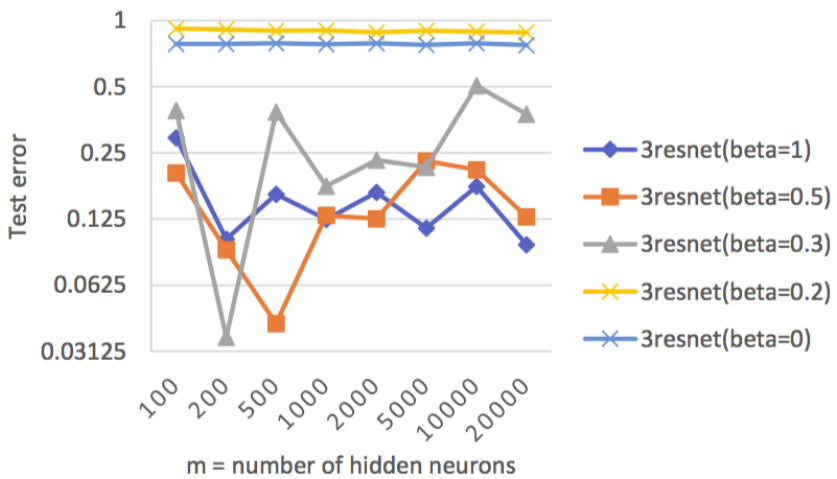
- To the best of my knowledge, no known algorithms (NTK, tensor decomposition, SOS etc) can be directly applied to this setting within the same resource (samples + running time).
- TODO: Generalize to ReLU network? Remove the information gap assumption? – Our earlier paper What can ResNet efficiently learn, Going Beyond Kernels



(b) sensitivity test on $\alpha = 0.3$