8.3 COMPETITIVE SEARCH

Up to this point, we have considered market environments in which prices between investors and dealers are determined *ex post*, i.e., after the parties have made contact. Such environments are arguably appropriate for studying opaque markets in which the trading interests of counterparties are not known in advance and investors must search for dealers to bargain over prices.

However, in several prominent OTC markets, new regulations and trading technologies have been implemented to promote greater price transparency. One convenient way to model a transparent price-setting protocol is to utilize models of competitive (or directed) search, in which investors can direct their orders towards specific prices posted by dealers. In particular, to capture a quote-driven OTC market, in this section we consider a model in which dealers commit to prices that are disseminated to all investors in the market, and investors direct their search towards the prices they find most attractive.

This theoretical setting captures an important aspect of many real world OTC markets—namely, that dealers have *some* ability to attract order flow by offering better terms of trade. This ability could, for example, arise from the so-called dealer runs utilized in certain fixed-income markets, where dealers post indicative (but not binding) quotes reflecting their willingness to buy or sell. It could also arise from less formal channels, whereby some dealers have a reputation for offering better or worse terms. In either case, a key feature of the competitive search model is that investors face a trade-off between trading at a good price and trading quickly, as is often the case in real world OTC markets (see, e.g., Li and Schürhoff [2019]).

Methodologically, we follow the seminal contribution of Moen [1997] and the large literature on competitive search that followed. As is well known from this literature, competitive search models have many attractive features. First, these models do not rely on adhoc assumptions about agents' bargaining powers. Instead, the surplus-sharing rule is endogenously determined and, thus, depends on fundamentals. Second, in most settings, the surplus-sharing rule satisfies the Hosios [1990] condition, so that the number of dealers who post prices in equilibrium is efficient. Third, competitive search models tend to be highly tractable and amenable to a number of extensions, including the introduction of rich heterogeneity and alternative trading mechanisms. Lastly, as discussed above, such models offer a framework where equilibrium prices play an allocative role and investors face a trade-off between execution speed and trading costs.

8.3.1 The market setting

Our starting point is the semi-centralized model introduced in Chapter 4. There is a continuum of investors, with mass normalized to one, who can each hold 0 or 1 unit of an asset and have utility flows $\delta \in \mathcal{D}$. Shocks to utility flows arrive at Poisson rate γ and trigger the draw of a new utility flow from a distribution F with support contained in $\mathcal{D}$. Investors search to trade through dealers, who can trade with each other instantly in a competitive inter-dealer market. We add three new features to this basic model.

First, we assume that dealers post and commit to a price at which they're willing to trade; for simplicity, we require that each dealer posts at most one quote that can be either to buy or to sell one unit of the asset. Dealers incur a cost κ to post a price quote and the size of the dealer sector is determined by *free entry* (unlike in earlier chapters where it was exogenously fixed). The assumption that dealers have a limited capacity to post prices captures various costs, including the cost of time, the opportunity cost of mobilizing capital, and the costs of complying with regulation; see, e.g., Yueshen and Zou [2022] for an in-depth discussion of these costs.

As will become clear below, it is convenient to express the quotes posted by dealers as absolute markups or markdowns relative to the interdealer price, P. Specifically, we denote by m_1 a bid quote to purchase the asset from an investor owner at price $P - m_1$. Likewise, we denote by m_0 an ask quote to sell the asset to a investor non owner at price $P + m_0$.

Our second departure from the semi-centralized market model is that we assume the matching process between dealers' quotes and investors exhibits *congestion*. Namely, if a mass μ_d of dealers post a quote m and a mass μ_i of investors direct their order to this quote, then we assume that transactions at this quote are executed at rate $\Xi(\mu_d, \mu_i)$. The function Ξ plays the same role as the matching function in other economic applications [see, e.g., Pissarides, 1985]. It is assumed to be homogenous of degree one, strictly increasing in both arguments, strictly concave, twice continuously differentiable, and to satisfy the Inada conditions given by

$$\lim_{\mu_j \to 0} \frac{\partial \Xi}{\partial \mu_j} = \infty \qquad \lim_{\mu_j \to \infty} \frac{\partial \Xi}{\partial \mu_j} = 0, \qquad \text{for all } \mu_{-j} > 0.$$

A classical example of a matching function satisfying those assumptions is the constant elasticity function given by

$$\Xi(\mu_d, \mu_i) = \Lambda \mu_d^{\alpha} \mu_i^{1-\alpha}$$

for some $\alpha \in (0,1)$. We will use this highly tractable specification to illustrate the main features of the equilibrium later in this section.

If orders are executed at random, then our assumptions imply that investors who direct their order to a quote posted by a mass μ_d of dealers and sought by a mass μ_i of investors effectively trades at Poisson rate

$$\frac{\Xi(\mu_d, \mu_i)}{\mu_i} = \Xi\left(\frac{\mu_d}{\mu_i}, 1\right) \equiv \lambda(\omega),$$

where $\omega \equiv \frac{\mu_d}{\mu_i}$ is the *market tightness* at this quote. On the other side of the market, a dealer who posts this quote trades at Poisson rate

$$\frac{\Xi(\mu_d, \mu_i)}{\mu_d} = \frac{\mu_i}{\mu_d} \Xi\left(\frac{\mu_d}{\mu_i}, 1\right) = \frac{\lambda(\omega)}{\omega}.$$

Note for later use that our assumptions on the matching function imply that the functions $\lambda_i(\omega)$ and $-\frac{\lambda(\omega)}{\omega}$ are twice continuously differentiable, strictly increasing, strictly concave with a derivative satisfying the Inada conditions, and map $[0, \infty]$ onto, respectively, $[0, \infty]$ and $[-\infty, 0]$.

8.3.2 Optimal behavior and equilibrium definition

NOTATION

Let $\mathcal{M}_q = \mathbb{R}_+$ denote the feasible set of quotes a dealer can post to trade with investors holding $q \in \{0, 1\}$ units of the asset and $\mathcal{M}_q^{\star} \subset \mathcal{M}_q$ denote the set of quotes posted by dealers in equilibrium, where we simply take $\mathcal{M}_q^{\star}$ to be the empty if dealers choose not to post any quote.

In addition, let $\Omega_q(m)$ denote the tightness of the market at quote m or, equivalently, the inverse of the length of the queue to trade at that quote. This function is to be determined in equilibrium and, importantly, needs to be defined for all possible quotes $m \in \mathcal{M}_q$, and not only for those in $\mathcal{M}_q^\star$. This is crucial to pin down the expectations of dealers about market tightness when considering deviations from the set of equilibrium quotes.

DEALERS' OPTIMAL QUOTES AND ZERO PROFITS.

Taking as given the market tightness function, $\Omega_q(m)$, the problem of a dealer is to choose a quote $m \in \mathcal{M}_q$ to offer investors who hold $q \in \{0,1\}$ units of the asset in order to maximize the profit function:

$$\frac{\lambda(\Omega_q(m))}{\Omega_q(m)} m - \kappa.$$

The first term in this expression is the expected flow profit from posting quote m, whereas the second term is the flow cost. Given free entry, the equilibrium profits of dealers must be at most zero, so that

$$\frac{\lambda(\Omega_q(m))}{\Omega_q(m)} m \leq \kappa \text{ for all } m \in \mathcal{M}_q \text{ with equality if } m \in \mathcal{M}_q^\star. \tag{8.12}$$

INVESTORS' RESERVATION VALUES.

Next, we characterize the reservation values of investors taking as given the market tightness $\Omega_q(m)$ at all quotes $m \in \mathcal{M}_q^\star$ for $q \in \{0,1\}$. Proceeding as in previous chapters shows that the reservation value $R(\delta)$ of an investor with current utility flow δ satisfies

$$\begin{aligned}(r+\gamma)\, R(\delta) = \delta + \gamma \mathbf{E}^F\left[R(x)\right] &+ \max_{\mu \in \mathcal{M}_1^\star \cup \{n\}} \lambda(\Omega_1(\mu))\,(P - \mu - R(\delta)) \\ &- \max_{\mu \in \mathcal{M}_0^\star \cup \{n\}} \lambda(\Omega_0(\mu))\,(R(\delta) - P - \mu)\,,\end{aligned} \tag{8.13}$$

with the convention that $\Omega_q(n) \equiv 0$ where n denotes the option of not seeking to trade. This equation for reservation values is similar to that which holds in the semi-centralized market model of Chapter 4 but there is one important difference: instead of continuously searching for dealers with whom to bargain over a trade, investors now choose whether or not to search and can direct their search efforts towards the best quotes subject to a congestion friction.

Market tightness function.

Given the reservation values of investors, we follow Chang [2018] and state the equilibrium conditions for the tightness functions as

$$\Omega_0(m) = \inf\Big\{\omega \geq 0 : \exists\delta \in \mathrm{supp}(F) \text{ such that} \\ \lambda(\omega)\,(R(\delta) - P - m) > \max_{\mu\in\mathcal{M}_0^\star\cup\{n\}} \lambda(\Omega_0(\mu))\,(R(\delta) - P - \mu)\Big\} \tag{8.14}$$

and

$$\Omega_1(m) = \inf\Big\{\omega \geq 0 : \exists\delta \in \mathrm{supp}(F) \text{ such that} \\ \lambda(\omega)\,(P - m - R(\delta)) > \max_{\mu\in\mathcal{M}_1^\star\cup\{n\}} \lambda(\Omega_0(\mu))\,(R(\delta) - P - \mu)\Big\}, \tag{8.15}$$

with the convention that $\inf\emptyset = \infty$. Acemoglu and Shimer [1999] interpret (8.14) and (8.15) in the spirit of subgame perfection. Namely, suppose that a dealer considers the possibility of deviating by posting a quote $m \notin \mathcal{M}_0^\star$ (the reasoning for bids is analogous). If the new quote is such that

$$R(\delta) - P - m \leq 0, \qquad \forall\delta \in \mathrm{supp}(F)$$

then no investor will seek to trade at m and, as a result, $\Omega_0(m) = \infty$; intuitively, if an investor were to seek that quote, then he would be alone in doing so and, thus, would obtain immediate execution. On the contrary, if the quote is attractive to some investors, in the sense that

$$R(\delta) - P - m > 0$$

for some $\delta \in \mathrm{supp}(F)$, then these investors will seek to trade at m and the market tightness at that quote will fall—or the queue of investors will continue to grow—until the expected flow profits of all such investor from seeking to trade at m no longer exceed those they can obtain by seeking an optimally selected equilibrium quote $\mu \in \mathcal{M}_0^\star \cup \{n\}$.

Notice that we define the tightness functions $\Omega_0(m)$ and $\Omega_1(m)$ based on strict gains from trade, and not weak gains from trade: conditions (8.14) and (8.15) are stated in terms of strict inequalities and not weak inequalities. If, instead, they were defined in terms of weak inequalities, then the tightness functions would be identically zero. Indeed, investors who have just traded with dealers and have not yet changed utility flow have no gains from trade at equilibrium quotes. Hence, these investors have weak incentive to seek any quote such that $\omega = 0$. This implies that, if conditions (8.14) and (8.15) were based on weak inequalities, then $\omega = 0$ would belong to the sets, and evidently the market tightness functions would be zero as well.

The market tightness functions are a crucial input into the characterisation of a competitive search equilibrium. Intuitively, suppose we break down each small interval into a two stage game in which dealers first post quotes, and then investors direct their search to the desired quotes. Conditions (8.14) and (8.15) characterize a Nash equilibrium in the second stage game: they require that the search strategies of investors are optimal, given the behavior of other investors and the quotes posted in the first stage. Moving to the first stage game, conditions (8.14) and (8.15) are akin to demand and supply functions, respectively: taking as given the behavior of other dealers, they determine the flow of buy or sell orders from investors (that emerge in the second stage game) from posting any quote. See Burdett, Shi, and Wright [2001] for explicit microfoundations of this two-stage game.

ASSET ALLOCATION AND MARKET CLEARING.

A common result in competitive search models with heterogeneous agents is that the equilibrium features full separation. In our context this means that each quote $m \in \mathcal{M}_q^\star$ is optimally chosen by investors with $q \in \{0, 1\}$ units of the asset and a unique utility flow δ. Technically, this occurs because the $(m, \omega)-$indifference curves of investors, which capture their willingness to trade off execution speed and trading costs, satisfy a single crossing property; see Moen [1997] and Acemoglu and Shimer [1999] for early instances of this result. Anticipating this property, we let $m_q(\delta)$ denote the quote $m \in \mathcal{M}_q^\star \cup \{n\}$

chosen by investors of type (q, δ) with the convention that investors who do not seek to trade choose the default option $m_q(\delta) = n$.

As before, we denote by $\Phi_q(\delta)$ the measure of investors with asset holdings $q \in \{0, 1\}$ and utility flow less than or equal to $\delta \in \mathcal{D}$. Again, these measures must satisfy three conditions. First, they must add up to the distribution of utility flows across the whole population:

$$\Phi_0(\delta) + \Phi_1(\delta) = F(\delta), \qquad \delta \in \mathcal{D}. \tag{8.16}$$

Second, they must be consistent with market clearing in the sense that the total measure of asset owners must equal the asset supply:

$$\Phi_1(\bar{\delta}) \equiv \lim_{\delta \uparrow \bar{\delta}} \Phi_1(\delta) = s.$$

Third, these cumulative measures must be consistent with stationarity, which requires that the inflow to and outflow from any given group of investors be equalized in the sense that:

$$\begin{aligned}\int_B \gamma s dF(\delta) + \int_B \lambda(\Omega_0(m_0(\delta))) d\Phi_0(\delta) \\ = \int_B \gamma d\Phi_1(\delta) + \int_B \lambda(\Omega_1(m_1(\delta)))\, d\Phi_1(\delta),\end{aligned} \tag{8.17}$$

for any Borel set $B \subseteq \mathcal{D}$. Lemma A.14 in Appendix A.2 allows us to drop the explicit reference to Borel sets and heuristically write that the relationship holds at all utility flows:

$$\gamma s dF(\delta) + \lambda(\Omega_0(m_0(\delta)))\, d\Phi_0(\delta) = \gamma d\Phi_1(\delta) + \lambda(\Omega_1(m_1(\delta)))\, d\Phi_1(\delta).$$

Substituting (8.16) and solving yields

$$d\Phi_1(\delta) = \frac{\gamma s + \lambda(\Omega_0(m_0(\delta))}{\gamma + \lambda(\Omega_1(m_1(\delta))) + \lambda(\Omega_0(m_0(\delta)))}\, dF(\delta). \tag{8.18}$$

Note that condition (8.17) implies that the interdealer market automatically clears. In particular, evaluating this condition at $B = \mathcal{D}$ reveals that

$$\int_{\mathcal{D}} \lambda(\Omega_0(m_0(\delta))) d\Phi_0(\delta) = \int_{\mathcal{D}} \lambda(\Omega_1(m_1(\delta)))\, d\Phi_1(\delta),$$

where the left-hand side represents the flow of assets that dealers sell and the right-hand side represents the flow of assets that they buy. As in our earlier analysis, this result attains because market clearing is an implication of stationarity: the only way that the mass of assets in the hand of investors remains constant is if the flow of purchases and the flow of sales are equalized.

DEFINITION OF EQUILIBRIUM.

A *competitive search equilibrium* comprises an interdealer price, P, an investor reservation value function, $R(\delta)$, a pair of functions $m_0(\delta)$ and $m_1(\delta)$ for the quotes that an investor of current type (q, δ) finds optimal to seek, a pair of sets $\mathcal{M}_0^\star$ and $\mathcal{M}_1^\star$ for the ask and bid quotes that dealers find optimal to post, a pair of market tightness functions $\Omega_0(m)$ and $\Omega_1(m)$, and a pair of cumulative measures $\Phi_0(\delta)$ and $\Phi_0(\delta)$ that jointly solve equations (8.12)–(8.18).

8.3.3 An alternative representation of reservation values

To facilitate the equilibrium construction we start by deriving an alternative representation of reservation values. Specifically, we show that competition among dealers leads to quotes that maximize investors' option value of search, subject to the constraint that dealers make zero profit. As we show below, if this were not the case, dealers would be able to post quotes that are more attractive to investors and, thus, generate strictly positive profits.

Proposition 8.4. *In a competitive search equilibrium the reservation value function of investors satisfies*

$$(r+\gamma)\, R(\delta) = \delta + \gamma \mathbf{E}^F\left[R(x)\right] + \max_{\omega \geq 0}\left\{\lambda(\omega)\,(P - R(\delta)) - \kappa\omega\right\} - \max_{\omega \geq 0}\left\{\lambda(\omega)\,(R(\delta) - P) - \kappa\omega\right\} \tag{8.19}$$

for all utility flows $\delta \in \operatorname{supp}(F)$.

Proof. To establish the result it suffices to show that, for any $\delta \in \operatorname{supp}(F)$, the last two terms in (8.19) are equal to the last two terms of (8.13):

$$\max_{\mu \in \mathcal{M}_1^\star \cup \{n\}} \lambda(\Omega_1(\mu))\,(P - \mu - R(\delta)) = \max_{\omega \geq 0} \{\lambda(\omega)\,(P - R(\delta)) - \kappa\omega\}$$
$$\max_{\mu \in \mathcal{M}_0^\star \cup \{n\}} \lambda(\Omega_0(\mu))\,(R(\delta) - P - \mu) = \max_{\omega \geq 0} \{\lambda(\omega)\,(R(\delta) - P) - \kappa\omega\}\,.$$

Let us focus on the first equality, as the proof is the same for the latter. From the zero profit condition (8.12) we deduce that

$$\begin{aligned}\lambda(\Omega_1(\mu))\,(P - \mu - R(\delta)) &= \lambda(\Omega_1(\mu))\,(P - R(\delta)) - \kappa\Omega_1(\mu) \\ &\leq \max_{\omega \geq 0} \{\lambda(\omega)\,(P - R(\delta)) - \kappa\omega\}\end{aligned}$$

for all $\mu \in \mathcal{M}_1^\star \cup \{n\}$ and, therefore,

$$\max_{\mu \in \mathcal{M}_1^\star \cup \{n\}} \lambda(\Omega_1(\mu))\,(P - \mu - R(\delta)) \leq \max_{\omega \geq 0} \{\lambda(\omega)\,(P - R(\delta)) - \kappa\omega\}$$

for any $\delta \in \operatorname{supp}(F)$. Now suppose towards a contradiction that this inequality is not an equality so that there exist $\hat{\omega}$ and $\hat{\delta} \in \operatorname{supp}(F)$ such that

$$\max_{\mu \in \mathcal{M}_1^\star \cup \{n\}} \lambda(\Omega_1(\mu))\,\big(P - \mu - R(\hat{\delta})\big) < \lambda(\hat{\omega})\,\big(P - R(\hat{\delta})\big) - \kappa\hat{\omega}. \tag{8.20}$$

Consider the quote $\hat{m}$ such that $\frac{\lambda(\hat{\omega})}{\hat{\omega}}\hat{m} = \kappa$, that is, the quote that achieves zero profits when the market tightness is exogenously set to $\hat{\omega}$. The definition of the tightness function (8.15) and (8.20) imply that we have

$$\lambda(\hat{\omega})\,\big(P - \hat{m} - R(\hat{\delta})\big) > \max_{\mu \in \mathcal{M}_1^\star \cup \{n\}} \lambda(\Omega_1(\mu))\,\big(P - \mu - R(\hat{\delta})\big)$$

and, therefore, $\Omega_1(\hat{\omega}) < \hat{\omega}$. Since the trading intensity of dealers is strictly increasing in the market tightness this in turn implies

$$\frac{\lambda(\Omega_1(\hat{\omega}))}{\Omega_1(\hat{\omega})}\,\hat{m} - \kappa > \frac{\lambda(\hat{\omega})}{\hat{\omega}}\,\hat{m} - \kappa = 0$$

which shows that quoting $\hat{m}$ generates strictly positive dealer profits and thus contradicts the zero profit condition. □

The proposition establishes an equivalence between the competitive search model and an alternative model in which investors can choose the intensity with which they contact a centralized market where the asset is traded at the constant price P. More specifically, in a competitive search model, equilibrium search times are determined as if investors directly chose the market tightness subject to a linear cost—or, equivalently, as if they directly chose the intensity of contact with the centralized market, $\Lambda \equiv \lambda(\omega)$, subject to a convex cost given by $\kappa\lambda^{-1}(\Lambda)$. Of course, the competitive search equilibrium has a much richer structure than its search-intensity counterpart, since it also makes predictions about the equilibrium markups and markdown charged by dealers, and their relationships with search times.

Note that since (8.19) holds at all points of $\text{supp}(F)$, we can replace the reservation value function by the solution to (8.19) posed over the whole set of utility flows without altering the reservation value of any actual investor in the market, at least if this solution is unique. Exercise 8.10 shows that this is indeed the case and also establishes that, as in previous chapters, the reservation value function is strictly increasing and absolutely continuous in utility flow.

8.3.4 Solving for equilibrium

Proposition 8.4 provides the basis for the construction of equilibrium. As a first step, we note that the optimization problems in (8.19) are both strictly concave and thus admit unique optimal solutions that we denote by $\omega_1(\delta)$ and $\omega_0(\delta)$, respectively. This confirms our earlier conjecture that the equilibrium features full separation of utility types, so that each investor follows a *pure strategy* that directs her search towards a single quote at a time.

Differentiating (8.19) with respect to utility flow and using the envelope theorem shows that

$$R'(\delta) = \frac{1}{r + \gamma + \lambda(\omega_1(\delta)) + \lambda(\omega_0(\delta))} > 0. \tag{8.21}$$

This establishes that reservation values are strictly increasing in utility flow and implies that the optimizers $\omega_1(\delta)$ and $\omega_0(\delta)$ are strictly increasing and strictly decreasing, respectively. The interpretation of this result is clear: more extreme types have stronger incentives to trade, and thus optimally choose higher search intensities, $\lambda(\omega)$. However, with faster trading times come larger transaction costs that are explicitly given by

$$m_q(\delta) \equiv \frac{\kappa\, \omega_q(\delta)}{\lambda(\omega_q(\delta))}$$

as a result of the dealers' zero profit condition.

Next, we observe that, as in previous chapters, the interdealer price P must lie in $(R(\underline{\delta}), R(\overline{\delta}))$, as otherwise the interdealer market would not clear. Since the reservation value function is continuous, there exists a marginal type $\delta^\star$ such that $R(\delta^\star) = P$ and it immediately follows that

$$\omega_0(\delta_0) = \omega_1(\delta_1) = 0, \qquad \forall \delta_0 \leq \delta^\star \leq \delta_1. \tag{8.22}$$

In other words, just as in the semi-centralized market of Chapter 4, owners (non-owners) do not trade if $\delta_1 \geq \delta^\star$ ($\delta_0 \leq \delta^\star$). This is intuitive: given that there is a cost to post a quote—or, under our alternative interpretation, a search intensity cost—investors never find it optimal to search on both sides of the market at the same time.

To determine the equilibrium tightness functions, we start by taking first order conditions with respect to ω in (8.19). This gives

$$\begin{aligned}
\kappa &= \lambda'\left(\omega_1(\delta)\right)\left(P - R(\delta)\right) = \lambda'\left(\omega_1(\delta)\right)\int_\delta^{\delta^\star} R'(x)dx, \qquad \delta \in (\underline{\delta}, \delta^\star], \\
&= \lambda'\left(\omega_0(\delta)\right)\left(R(\delta) - P\right) = \lambda'\left(\omega_0(\delta)\right)\int_{\delta^\star}^{\delta} R'(x)dx, \qquad \delta \in [\delta^\star, \overline{\delta}),
\end{aligned}$$

where the second equality follows the fundamental theorem of calculus and the fact that $P = R(\delta^\star)$. Next, using (8.21) together with (8.22) to compute the two integrals we obtain that the functions

$$\lambda'(\omega_0(\delta)) \int_{\delta^\star}^{\delta} \frac{dx}{r + \gamma + \lambda(\omega_0(x))} \tag{8.23}$$

and

$$\lambda'(\omega_1(\delta)) \int_{\delta}^{\delta^\star} \frac{dx}{r + \gamma + \lambda(\omega_1(x))} \tag{8.24}$$

are constant and equal to κ on $(\underline{\delta}, \delta^\star]$ and $[\delta^\star, \overline{\delta})$, respectively. Combining the first-order conditions and their respective derivatives with respect to utility flows, we obtain that

$$\omega_q(\delta) = T\left[((2q-1)(\delta^\star - \delta))^+\right] \tag{8.25}$$

for some strictly increasing function $T : \mathbf{R}_+ \to \mathbf{R}_+$ with $T(0) = 0$ that it described in the following lemma.

Lemma 8.5. *The function $T(x)$ is the inverse of*

$$S(y) \equiv \frac{\kappa\,(r + \gamma + \lambda(y) - y\lambda'(y))}{\lambda'(y)}. \tag{8.26}$$

In particular, the equilibrium tightness functions are decreasing in (r, γ, κ).

Proof. Setting the derivatives of (8.23) and (8.24) to zero and using (8.25) shows that the function $T(x)$ must satisfy

$$T'(x) = -\frac{\lambda'(T(x))^2}{\kappa\lambda''(T(x))\,(r + \gamma + \lambda(T(x)))} \tag{8.27}$$

subject to $T(0) = 0$. If the function $T(x)$ is a strictly increasing solution to this ODE then it admits a well defined inverse $S(y)$ that satisfies

$$S'(y) = \frac{1}{T'(S(y))} = -\frac{\kappa\lambda''(y)\,(r + \gamma + \lambda(y))}{\lambda'(y)^2} \tag{8.28}$$

subject to the initial condition $S(0) = 0$. Integrating on both sides shows that $S(y)$ is given by (8.26). Conversely, the function $S(y)$ defined by (8.26) is strictly increasing since the intensity function $\lambda(\omega)$ is strictly increasing and strictly concave. Therefore, $S(y)$ admits a well defined inverse $T(x)$ and a direct calculation using (8.28) shows that this inverse is a solution to the ODE (8.27) subject to $T(0) = 0$. The remaining claims in the statement follow from the implicit function theorem since $S(y)$ is increasing in y, r, γ, and κ. □

Having constructed the tightness functions in terms of the marginal type, we can plug them into (8.18) evaluated at the upper end point of $\mathcal{D}$ to state the market clearing condition as

$$\int_{\underline{\delta}}^{\delta^\star} \frac{\gamma s}{\gamma + \lambda\left(T(\delta^\star - \delta)\right)} dF(\delta) + \int_{\delta^\star}^{\overline{\delta}} \frac{\gamma s + \lambda\left(T(\delta - \delta^\star)\right)}{\gamma + \lambda\left(T(\delta - \delta^\star)\right)} dF(\delta) = s. \tag{8.29}$$

The following proposition completes the equilibrium construction by showing that this equation uniquely pins down the marginal type.

Proposition 8.6. *Equation (8.29) admits unique solution $\delta^\star \in \mathcal{D}$ and, as a result, there exists a unique competitive search equilibrium.*

Proof. Equation (8.29) can be written compactly as

$$L(\delta^\star) \equiv \int_{\underline{\delta}}^{\overline{\delta}} \frac{\gamma s + \lambda \circ T\left((\delta - \delta^\star)^+\right)}{\gamma + \lambda \circ T\left(|\delta - \delta^\star|\right)} dF(\delta) = s.$$

Direct calculations show that the integrand is continuous in $(\delta, \delta^\star)$ and strictly decreasing in $\delta^\star$ for each fixed δ. Therefore, $L(\delta^\star)$ is continuous as well as strictly decreasing, continuous function and the result now follows from the intermediate value theorem by observing that $L(\overline{\delta}) < s < L(\underline{\pi})$. □

Example 8.7. To illustrate the features of the competitive search equilibrium, consider the case where the distribution of utility flows is uniform over the unit interval $\mathcal{D} = [0, 1]$ and assume that

$$\lambda(\omega) = \Lambda \omega^\alpha$$

for some $\Lambda > 0$ and $\alpha \in [0, 1]$. In this case, the inversion of $S(\theta)$ boils down to solving the nonlinear equation

$$x = S(T(x)) = \frac{\kappa}{\alpha\Lambda}\left(r + \gamma\right) T(x)^{1-\alpha} + \left(\frac{1}{\alpha} - 1\right) \kappa T(x)$$

for each $x \in [0, 1]$. This equation admits an explicit solution for specific parameter values. For example, if $\alpha = \frac{1}{2}$ then

$$T(x) = \left\{ \sqrt{\left(\frac{r+\gamma}{\Lambda}\right)^2 + \frac{x}{\kappa}} - \frac{r+\gamma}{\Lambda} \right\}^2$$

which implies that the equilibrium tightness functions $\omega_0(\delta)$ and $\omega_1(\delta)$ are explicitly given by

$$\omega_q(\delta) = T\left[(\delta - \delta^\star)^+ + q\,(\delta^\star - \delta)\right].$$

On the other hand, the fact that the distribution is uniform implies that the market clearing condition (8.29) can be written as

$$s = \int_0^{\delta^\star} \frac{\gamma s}{\gamma + \lambda\,(T(x))} dx + \int_0^{1-\delta^\star} \frac{\gamma s + \lambda\,(T(x))}{\gamma + \lambda\,(T(x))} dx.$$

In particular, if $s = \frac{1}{2}$ then a straightforward, albeit tedious, calculation shows that the marginal type is given by $\delta^\star = 1 - s = \frac{1}{2}$ independently of all the other parameters of the model. Taking advantage of this simplification, we plot in Figure 8.3 the tightness functions $\omega_q(\delta)$, the distribution

$$\Phi_1(\delta) = \int_0^{\delta} \frac{\gamma s + \lambda(\omega_0(x))}{\gamma + \lambda(\omega_0(x)) + \lambda(\omega_1(x))} dx,$$

and the equilibrium trading intensities $\lambda(\omega_q(\delta))$ and $\frac{\lambda(\omega_q(\delta))}{\omega_q(\delta)}$ of investors and dealers for various values of the quote posting cost parameter κ in a model with parameters $\alpha = s = \frac{1}{2}$ and $\Lambda = 3$.

The interpretation of Figure 8.3 is intuitive. Starting from the top left, the figure shows that the equilibrium tightness is larger for less favorable quotes and decreases in all open markets as the quote posting cost κ decreases. This is natural since more attractive quotes are sought by more investors and because a lower cost implies that more dealers find it optimal to quote any given $m \in \mathcal{M}_q^\star = [0,1]$. As shown in the right panels of the figure, these features imply that investors with larger gains from trade obtain faster execution and that all investors trade faster as the quote posting cost decreases, which in turn improves the overall efficiency of the equilibrium allocation. However, the bottom right panel shows that these efficiency gains are made at the expense of active dealers who experience longer individual waiting times as a result of increased competition.

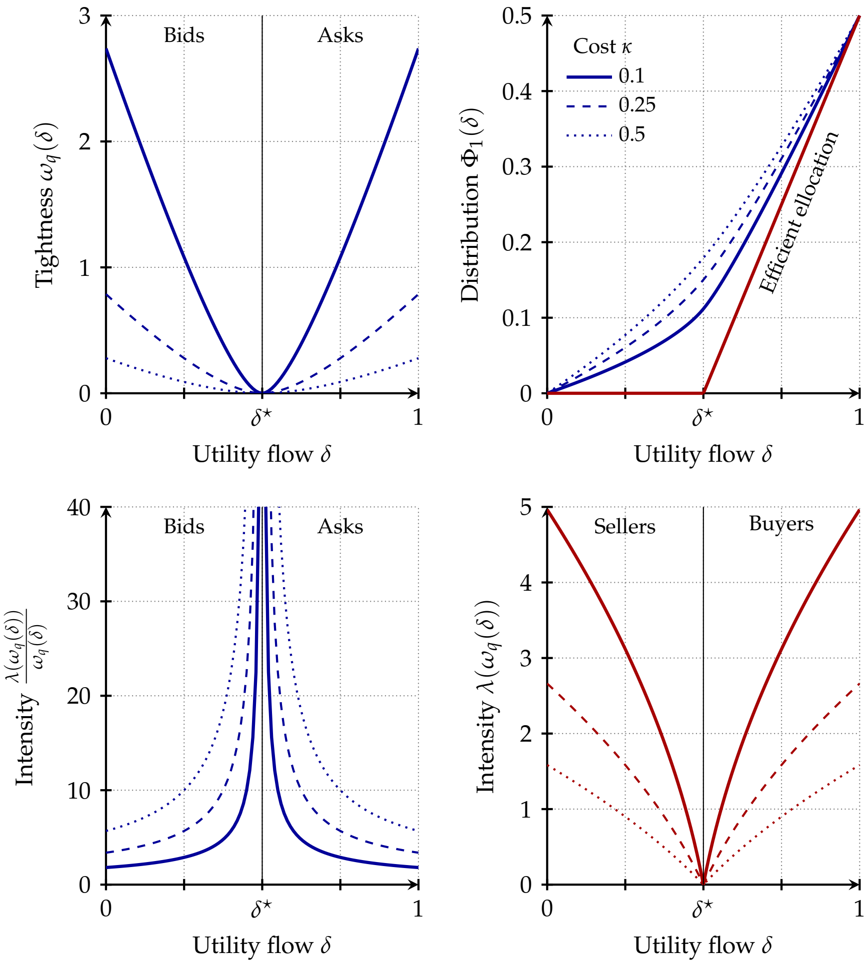

FIGURE 8.3: Competitive search equilibrium

This figure plots (clockwise from top left) the tightness functions, the distribution of utility flows among owners, and the equilibrium trading intensities of investors and dealers in a model with $F(\delta) = \delta$ on $\mathcal{D} = [0, 1]$, $s = \frac{1}{2}$, and $\lambda(\omega) = 3\sqrt{\omega}$ for different values of the quote posting cost parameter κ.

8.4 EXERCISES

Exercise 8.8. Consider a meeting between an investor of type $(1, \delta_H)$ and a dealer in the bargaining game of Section 8.1. The goal of this exercise is to derive parameter restrictions such that, in a stationary SPNE, when a receiver, the investor is indifferent between accepting and quitting the bargaining table, while the dealer is indifferent between accepting and waiting.

1. Write the indifference condition of the investor and of the dealer and solve for $p_I(\delta_L)$ and $p_D(\delta_L)$.
2. Show that, if these indifference conditions hold, $p_I(\delta_L)$ and $p_D(\delta_L)$ converge to $R(\delta_L)$ as $h \to 0$. Provide intuition.
3. State the inequalities ensuring that the investor strictly prefer quitting over waiting, and the dealer waiting over quitting.
4. Show that these inequalities hold for all h small enough if:

$$\frac{\pi_D c_I - \pi_I c_D}{r + \gamma f_H + \lambda} > \frac{\pi_I (\delta_H - \delta_L)}{r + \gamma}.$$

Exercise 8.9. Consider the model of RFQs developed in Section 8.2 and recall that in equilibrium the average share of the surplus that a dealer captures when selected to trade is

$$\psi_1 = \frac{1-\pi}{\pi} \frac{n\pi^n}{1-\pi^n}.$$

where $n \in \mathbf{N}$ is the number of dealers selected in each RFQ and $\pi \in (0,1)$ is the probability that a dealer fails to respond and

1. Show that

$$\frac{\partial \psi_1}{\partial \pi} = \frac{(1-\pi^n)^2}{n\pi^{n-2}} f(\pi)$$

for some decreasing function to be determined and use this result to prove that ψ_1 is increasing in π.

2. Regarding n as a continuous variable show that

$$\frac{\partial \psi_1}{\partial n} = \frac{(1-\pi^n)^2}{\pi^n} g(n)$$

for some decreasing function to be determined. Use this result to prove that ψ_1 is decreasing in n.

Exercise 8.10. The goal of this exercise is to derive existence, uniqueness, and basic properties of the solution to the reservation value equation (8.19) in the competitive search model. Let

$$M_n(x) \equiv \sup_{\omega \le n} \{\lambda(\omega)x - \kappa\omega\}, \qquad n \in [0, \infty],$$

and consider the parameterised family of equations

$$\begin{aligned}(r+\gamma)R(\delta) = S_n[R](\delta) \equiv \delta + \gamma \mathbf{E}^F[R(x)] \\ + M_n(P - R(\delta)) - M_n(R(\delta) - P)\end{aligned} \tag{8.30}$$

so that (8.19) corresponds to the case $n = \infty$.

1. Show that $M_n(x)$ is nondecreasing in (n, x) and convex in x with a derivative that takes values in $[0, \lambda(n)]$.
2. Show that $M_\infty(x) = \lambda(\omega^*(x))x - \kappa\omega^*(x)$ for some function $\omega^*(x) \ge 0$ that is equal to zero on $[-\infty, 0]$ and strictly increasing otherwise.
3. Show that if $R(x)$ is a solution to (8.30) for some $n \le \infty$ then it is strictly increasing and such that

$$\underline{\delta} + M_\infty\left(\frac{\overline{\delta}}{r} - P\right) \le rR(\delta) \le \overline{\delta} + M_\infty\left(P - \frac{\underline{\delta}}{r}\right)$$

   Conclude that there exists $n^* < \infty$ such that $R(\delta)$ is a solution to (8.30) when $n = \infty$ if and only if it also a solution when $n = n^*$.
4. Show that for any $n < \infty$ the operator

$$T_n[R](\delta) \equiv \frac{2\lambda(n)R(\delta) + S_n[R](\delta)}{r + \gamma + 2\lambda(n)}$$

   satisfies Blackwell's conditions and maps

$$\left\{R \in \mathcal{B}(\mathcal{D}) : \sup_{\delta \neq \delta'} \left|\frac{R(\delta) - R(\delta')}{\delta - \delta'}\right| \le \frac{1}{r + \gamma + 2\lambda(n)}\right\}$$

   into itself. Argue that (8.30) with $n = \infty$ admits a unique solution that is bounded, strictly increasing, and Lipschitz continuous.