Chapter Ten

Asymmetric information: Private values

In previous chapters, we studied frictions that make it difficult for investors to locate a counterparty, i.e., *search frictions*. But a commonly held view is that, in addition to search frictions, OTC markets are also plagued by *information frictions*; that is, OTC markets are often opaque, in the sense that investors have limited information about their counterparties and about prices. Moreover, this information is often asymmetric, in the sense that one counterparty knows payoff-relevant information that the other side does not.

While these canonical frictions have been studied extensively in isolation, the next two chapters introduce frameworks to study the way in which search and information frictions *interact*. The analysis is split across two chapters to treat two conceptually different types of asymmetric information. The first type, asymmetric information over *private values*, applies to situations in which one party cannot observe information relevant to the private payoff of the other party , such as preferences. The second type of asymmetric information, over *common values*, applies to situations in which one side of a trade cannot observe payoff-relevant characteristics of the asset being traded; this case is the basis of the classic literature on adverse selection.

To study asymmetric information over private values, we start from a model of semi-centralized trade similar to that of Chapter 4. However, we modify the model to account for the natural observation that, upon meeting a dealer, an investor may not have incentive to truthfully reveal her trading motives, as this could help the dealer extract more of the surplus. Specifically, we assume that dealers know the joint distribution of utility flows and asset holdings across the population of investors but cannot observe the utility flows of the investors they meet. Clearly, such an information friction only matters if dealers have some bargaining power; otherwise they would offer to trade at the interdealer price regardless of the utility flows of their counterparties.

The main insight that arises from the analysis of this model is that search frictions magnify the effects of asymmetric information over private values. To see why, note that under asymmetric information dealers cannot condition their prices on the utility type of their counterparties. As a result, they offer prices that some investors do not accept, despite the fact that there are gains from trade. In contrast, in Chapter 4, trade always occurred when there were gains from trade. A key insight that arises from the analysis of the present chapter is that search frictions exacerbate this inefficiency. In particular, as search frictions worsen and meetings become less frequent, dealers' optimal price offers are accepted by a smaller fraction of investors. Hence, as search frictions become more severe, misallocation worsens for two reasons: because investors cannot re-balance their portfolio as frequently as they would like; and because dealers optimally set prices that customers reject more often, even though there are positive gains from trade.

10.1 THE MODEL

10.1.1 The semi-centralized market

Consider the semi-centralized market of Chapter 4. There is a unit measure of dealers who participate in a frictionless market, and a unit measure of risk neutral investors who discount the future at rate $r > 0$ and can hold either 0 or 1 unit of an asset in supply $s < 1$. Utility flows are iid across the population: the utility flow of an investor follows an idiosyncratic Markov jump process with intensity $\gamma > 0$ and reset distribution $F(\delta)$ on some interval $\mathcal{D} = (\underline{\delta}, \overline{\delta})$. Each investor gets the opportunity to trade with a randomly selected dealer according to a Poisson process with intensity $\lambda > 0$.

Within this framework, we add two new, crucial assumptions. First, while dealers observe whether the investors they meet want to buy or sell—and know the joint distribution of types across the populations of owners and non owners—we assume that they *cannot* observe the utility flow of the investors they meet. Hence, there is asymmetric information about the *private* valuations of investors. Second, we assume that within each meeting dealers maximize their expected profits by making take-it-or-leave-it offers to investors.

For the sake of simplicity and clarity, in most of the analysis below we will assume that the distribution of utility flows, $F(\delta)$, is continuously differentiable with $F'(\delta) = f(\delta) > 0$ for all $\delta \in \mathcal{D}$. However, most of the results we derive in this section remain unchanged if we allow for more general distributions; we briefly cover an example with a discrete distribution in Section 10.2.

10.1.2 Optimal pricing by dealers

Consider a meeting between a dealer and an investor owner. The dealer knows that the investor is a seller but does not know her utility flow, δ. This makes it harder for the dealer to price discriminate. In particular, he cannot extract all the surplus by offering to buy from the investor at her reservation value, $R(\delta)$, as he would under complete information. Instead, the dealer chooses a price to maximize his expected profit taking as given the interdealer price, P; the reservation value function of investors; and the cumulative distribution of utility flows within the group of investor owners, $\Phi_1(\delta)$.

If the dealer offers to buy at some *bid price* b, then the investor owner weakly prefers to sell his asset if his reservation value $R(\delta) \geq b$. Hence, the dealer's optimization problem can be stated as

$$\max_{b \leq P} \left\{ (P - b) \int_{\underline{\delta}}^{\bar{\delta}} \mathbf{1}_{\{R(\delta) \leq b\}} \frac{d\Phi_1(\delta)}{\Phi_1(\bar{\delta})} \right\},$$

where the first factor is the profit that the dealer makes by buying from the investor at price b and re-selling in the interdealer market at price P, and the second factor is the probability that the offer b is accepted. By the same logic, the optimization problem of a dealer who offers an *ask price* a to sell the asset to an investor can be stated as

$$\max_{a \geq P} \left\{ (a - P) \int_{\underline{\delta}}^{\overline{\delta}} \mathbf{1}_{\{R(\delta) \geq a\}} \frac{d\Phi_0(\delta)}{\Phi_0(\overline{\delta})} \right\},$$

where $\Phi_0(\delta)$ is the cumulative distribution of utility flows within the group of investors who do not hold the asset.

To simplify the formulation of these optimization problems, note that we can assume without loss that the interdealer price P and the offered prices (a, b) lie in the range of the reservation value function

$$\mathcal{R} \equiv \left\{ p \in \mathbf{R} : \underline{R} \equiv \min_{\delta \in \mathcal{D}} R(\delta) \leq p \leq \max_{\delta \in \mathcal{D}} R(\delta) \equiv \overline{R} \right\}.$$

Indeed, if $P \notin \mathcal{R}$ then the interdealer market cannot clear because dealers are willing to trade with investors only in one direction. For example, if $P < \underline{R}$ then dealers are willing to sell to all investors but unwilling to buy from any. On the other hand, if the interdealer price $P \in \mathcal{R}$ then offering a price $p \notin \mathcal{R}$ cannot be optimal for the dealer. For example, bidding $b < \underline{R}$ is clearly weakly dominated by bidding any $b' \in [\underline{R}, P]$.

Using this observation, and conjecturing that the reservation value function of investors is continuous and strictly increasing, we can reduce the dealers' bid and ask selection problems to

$$\max_{\delta \in [\underline{\delta}, \delta^\star]} \Pi_b(\delta) \equiv \max_{\delta \in [\underline{\delta}, \delta^\star]} \left\{ (R(\delta^\star) - R(\delta)) \frac{\Phi_1(\delta)}{\Phi_1(\overline{\delta})} \right\} \tag{10.1}$$

and

$$\max_{\delta \in [\delta^\star, \overline{\delta}]} \Pi_a(\delta) \equiv \max_{\delta \in [\delta^\star, \overline{\delta}]} \left\{ (R(\delta) - R(\delta^\star)) \left(1 - \frac{\Phi_0(\delta)}{\Phi_0(\overline{\delta})} \right) \right\}, \tag{10.2}$$

where the utility flow $\delta^\star \in \mathcal{D}$ associated with the interdealer market price is implicitly defined by the requirement that $P = R(\delta^\star)$. In words, (10.1)–(10.2) show that, taking the reservation value function of investors as given, one can equivalently formulate the dealer's problem as one of choosing a price or as one of choosing a utility flow of the marginal investor-buyer or investor-seller, which in turn determines the ask and bid price, respectively. To simplify the presentation we accordingly say throughout this chapter that a dealer quotes a utility flow δ to mean that he quotes a price equal to the reservation value of an investor with that utility flow.

While the maxima in (10.1) and (10.2) are obviously attained, it is not clear a priori whether the maximizers are unique. In fact, we show below that, in equilibrium, there must be multiple maximizers and investors must randomize amongst them. To allow for such mixed pricing strategies, we assume that, upon meeting an investor, a dealer offers to buy at a bid price $b \geq R(\delta)$ with probability $1 - B(\delta-)$ if the investor is an owner and to sell at an ask price $a \leq R(\delta)$ with probability $A(\delta)$ if the investor is a non-owner.

Since dealers never buy (sell) above (below) the interdealer market price, the distributions (A, B) must satisfy

$$\lim_{\delta \uparrow \delta^\star} A(\delta) = 1 - B(\delta^\star) = 0,$$

and we say that such a pair is optimal if

$$\operatorname{supp}(A) \subseteq \operatorname*{argmax}_{\delta \in [\delta^\star, \bar{\delta}]} \Pi_a(\delta) \tag{10.3}$$

$$\operatorname{supp}(B) \subseteq \operatorname*{argmax}_{\delta \in [\underline{\delta}, \delta^\star]} \Pi_b(\delta) \tag{10.4}$$

where $\operatorname{supp}(A) \subseteq \mathcal{D}$ and $\operatorname{supp}(B) \subseteq \mathcal{D}$ denote the *supports* of the ask and bid distributions, respectively. The interpretation is simply that all the prices that are offered in equilibrium must maximize dealers' profits.

Asymmetric information pricing problems of this kind have been studied by many authors, including Myerson [1981] and Myerson and Satterthwaite [1983], but mostly in one sided, partial equilibrium settings where reservation values and distributions are exogenously specified. In contrast, the model of this chapter considers a two-sided problem and embeds it into a dynamic search equilibrium model where all objects are endogenously determined by the trading behavior that investors optimally adopt in response to the pricing strategies of dealers as in, e.g., Albrecht and Axell [1984].

10.1.3 Equilibrium definition

Proceeding as in Chapter 3 reveals that the reservation value function of an investor satisfies the HJB equation

$$\begin{aligned} rR(\delta) = \delta + \gamma \mathbf{E}^F\left[R(x) - R(\delta)\right] \\ + \lambda \int_\delta^{\overline{\delta}} \left(R(x) - R(\delta)\right) dB(x) - \lambda \int_{\underline{\delta}}^{\delta} \left(R(\delta) - R(x)\right) dA(x), \end{aligned} \tag{10.5}$$

where the terms on the second line account for the fact that, upon meeting a dealer, the investor is offered to trade at price $R(x)$ with probability $dB(x)$ if he is an owner and with probability $dA(x)$ otherwise.

The following lemma establishes the existence of a unique solution to this equation and confirms our conjectures regarding its regularity.

Lemma 10.1. *The unique solution to* (10.5) *satisfies*

$$R(\delta) - R(\underline{\delta}) = \int_{\underline{\delta}}^{\delta} \frac{dx}{r + \gamma + \lambda\left(1 - B(x) + A(x)\right)}. \tag{10.6}$$

In particular, the reservation value function is uniformly bounded, strictly increasing, and absolutely continuous on $\mathcal{D}$.

Proof. Existence, uniqueness, absolute continuity, and strict monotonicity of the solution follow from Proposition 3.6 and Corollary 3.8 in Chapter 3. Since the reservation value function is absolutely continuous, we can change the order of integration to rewrite the integrals in (10.5) as

$$\begin{aligned} \int_\delta^{\overline{\delta}} \big(R(x) - R(\delta)\big) dB(x) &= \int_\delta^{\overline{\delta}} \left(\int_\delta^x R'(y)dy\right) dB(x) \\ &= \int_\delta^{\overline{\delta}} R'(y)\left(\int_y^{\overline{\delta}} dB(x)\right) dy = \int_\delta^{\overline{\delta}} R'(y)(1 - B(y))dy \end{aligned}$$

and

$$\int_{\underline{\delta}}^{\delta} \big(R(\delta) - R(x)\big) dA(x) = \int_{\underline{\delta}}^{\delta} R'(y)A(y)dy$$

so that

$$
\begin{aligned}
rR(\delta) = \delta + \gamma \left(\mathbf{E}^F[R(x)] - R(\delta)\right) \\
+ \lambda \int_\delta^{\overline{\delta}} R'(x)(1 - B(x))dx - \lambda \int_{\underline{\delta}}^{\delta} R'(x)A(x)dx.
\end{aligned}
$$

Differentiating both sides and solving the resulting equation shows that the derivative $R'(x)$ is given by the integrand in (10.6) and the conclusion follows from the fundamental theorem of calculus. □

The distribution of utility flows and asset holdings can be characterized by imposing the usual stationarity conditions, subject to consistency and market clearing. As in Chapter 4, the latter conditions write as

$$F(\delta) = \Phi_0(\delta) + \Phi_1(\delta), \tag{10.7}$$

$$s = \Phi_1(\overline{\delta}), \tag{10.8}$$

whereas the former requires that

$$\int_G (\gamma + \lambda_1(\delta))\, d\Phi_1(\delta) = \int_G \left(\gamma \Phi_1(\overline{\delta}) dF(\delta) + \lambda_0(\delta) d\Phi_0(\delta)\right) \tag{10.9}$$

for any Borel set $G \subseteq \mathcal{D}$, where $\lambda_0(\delta)$ and $\lambda_1(\delta)$ denote the equilibrium trading intensity of investors with utility flow $\delta \in \mathcal{D}$. In the current environment, these trading intensities are given by

$$\lambda_0(\delta) = \lambda A(\delta) \tag{10.10}$$

and

$$\lambda_1(\delta) = \lambda \left(1 - B(\delta)\right). \tag{10.11}$$

Note that the definition of the trading intensities assumes that non-owners trade when they are indifferent while owners do not. This assumption is clearly innocuous given that $F(\delta)$ is continuous but it cannot be upheld in equilibrium if the distribution has atoms. Indeed, in this case some randomization (on either one side or on both sides of the market) may be required to ensure market clearing. See Section 10.2 for an illustration in a model with three types.

Using Lemma A.14, we can drop the integral and the explicit reference to Borel sets to informally restate (10.9) pointwise as:

$$(\gamma + \lambda_1(\delta))\, d\Phi_1(\delta) = \gamma \Phi_1(\bar{\delta}) dF(\delta) + \lambda_0(\delta) d\Phi_0(\delta). \tag{10.12}$$

Solving this equation subject to the consistency condition (10.7) then delivers the following characterization.

Lemma 10.2. *The unique solution to* (10.12) *subject to* (10.7) *is*

$$\Phi_1(\delta) = \Phi_1(\delta|m) \equiv \int_{\underline{\delta}}^{\delta} \frac{(\gamma m + \lambda_0(x))\, dF(x)}{\gamma + \lambda_0(x) + \lambda_1(x)}, \tag{10.13}$$

where m is the unique solution to $\Phi_1(\bar{\delta}|m) = m$ in $[0,1]$ and the trading intensities are defined as in (10.11) *and* (10.10)*.*

Proof. The first part follows by substituting (10.7) into (10.12) and solving the resulting equation for $d\Phi_1(\delta)$. See Lemma A.14 for a formal argument. The second part follows from the first by requiring that m coincides with the total mass of asset owners. □

Expanding the stationarity condition (10.9) for the set $G = \mathcal{D}$ and using the definition of the constant m reveals that

$$\int_{\underline{\delta}}^{\bar{\delta}} \lambda_1(\delta) d\Phi_1(\delta) = \int_{\underline{\delta}}^{\bar{\delta}} \lambda_0(\delta) \Phi_0(\delta),$$

where the left-hand side is the rate at which dealers buy and the right-hand side is the rate at which they sell. This identity shows that stationarity implies interdealer market clearing: the distribution of utility flows among owners and non-owners automatically adjusts to the pricing strategy of dealers in such a way that the interdealer market clears. This, in turn, implies that the mass of assets held by investors is constant over time but it does *not* guarantee that this mass equals the asset supply. To obtain that property one must impose the additional market clearing condition (10.8).

Combining Lemmas 10.1 and 10.2 with the market clearing condition (10.8) and using the fact that the supports of the price distributions are separated by the inderdealer utility flow $\delta^\star$ (see (10.3)–(10.4)) reveals that the expected profits of a dealer who meets an owner can be expressed as

$$\Pi_a(\delta) = \int_{\delta^\star}^{\delta} \frac{dx}{r+\gamma+\lambda_0(x)} \int_{\delta}^{\overline{\delta}} \frac{\gamma dF(x)}{\gamma+\lambda_0(x)},$$

while the expected profits of a dealer who meets a non-owner are

$$\Pi_b(\delta) = \int_{\delta}^{\delta^\star} \frac{dx}{r+\gamma+\lambda_1(x)} \int_{\underline{\delta}}^{\delta} \frac{\gamma dF(x)}{\gamma+\lambda_1(x)}. \tag{10.14}$$

The first integral in each expression represents the dealer's profit per trade as a function of the offered price δ and the second integral captures the probability that this offer is accepted.

With these elements in place, we are now ready to formally define a steady state equilibrium for the semi-centralized OTC market model with asymmetric information over private values.

Definition 10.3. *A steady state equilibrium consists of a pair of pricing strategies (A, B), a pair of cumulative measures (Φ_0, Φ_1), a reservation value function R, and an interdealer price $P = R(\delta^\star)$ such that:*

1. *$R(\delta)$ solves (10.5) given A and B;*
2. *$\Phi_0(\delta)$ and $\Phi_1(\delta)$ satisfy (10.7), (10.8), and (10.12) given A and B;*
3. *A and B satisfy (10.3) and (10.4) given $R(\delta)$, $\Phi_0(\delta)$, $\Phi_1(\delta)$ and the price P.*

An equilibrium is in pure strategies if the supports of A and B both consist of a single point. Otherwise, it is in mixed strategies.

10.2 A DETOUR: THE CASE WITH THREE TYPES

Before proceeding to the analysis of the continuous case, we present a simple discrete example designed to illustrate some of the key insights.

10.2.1 Assumptions and equilibrium conjecture

Assume that the utility flows of investors present in the market are drawn from a discrete distribution with support given by

$$\text{supp}(F) = \{\delta_L, \delta_M, \delta_H\}$$

for some $\delta_L < \delta_M < \delta_H \in \mathcal{D}$. In addition, suppose that the mass of investors with the high utility flow exceeds the asset supply, so that

$$s < f_H \equiv \Delta F(\delta_H). \tag{10.15}$$

The results of Chapter 4 imply that, under complete information, this simple model admits a unique steady state equilibrium in which:

i) The interdealer utility flow is $\delta^\star = \delta_H$
ii) Asset owners with utility flow in $\{\delta_L, \delta_M\}$ sell with probability one upon meeting a dealer.
iii) Non owners with utility flow δ_H buy with some probability $\rho_{0H} \in (0,1)$ upon meeting a dealer, and otherwise do not trade.

The main difference with the case of a continuous distribution is Condition iii): because of atoms in the distribution, market clearing may require some randomization by investors who are indifferent between trading or not.

We conjecture, and later confirm, that an equilibrium with features similar to i)–iii) above continues to exists under private information. In particular, we consider a candidate equilibrium in which only investors with utility flow δ_H buy from dealers, so that there is no uncertainty about the utility type of buyers. As a result, as in the perfect information case, the distribution of ask prices is degenerate:

$$A(\delta) = \mathbf{1}_{\{\delta \geq \delta_H\}}. \tag{10.16}$$

On the other side of the market, however, dealers are unsure of the utility flows of the investors who want to sell. Formally, the optimization problem of a dealer who meets an owner can be stated as

$$\max_{x\in(\underline{\delta},\delta_H]}\left\{(R(\delta_H)-R(x))\left(\psi_L\mathbf{1}_{\{x\geq\delta_L\}}+(1-\psi_L)\mathbf{1}_{\{x\geq\delta_M\}}\right)\right\} \tag{10.17}$$

where ψ_L denotes the fraction of sellers with utility flow δ_L.

Since the utility flows lie $\{\delta_L,\delta_M,\delta_H\}$ and the reservation value function is strictly increasing in utility flow, it is clear that the optimal strategy is to either bid δ_L, bid δ_M, or to randomize between these two bids. Accordingly, the bid distribution offered by dealers can be written

$$B(\delta)=\begin{cases}0, & \delta\in(\underline{\delta},\delta_L),\\ \beta, & \delta\in[\delta_L,\delta_M),\\ 1, & \delta\in[\delta_M,\overline{\delta}),\end{cases} \tag{10.18}$$

for some $\beta\in[0,1]$ that captures the probability of bidding δ_L.

To verify that this candidate is an equilibrium, we proceed in two steps. First, we solve for the reservation value function $R(\delta)$ and the asset allocation $\Phi_1(\delta)$, taking as given that dealers quote prices according to (10.16) and (10.18) for some constant $\beta\in[0,1]$. Second, we determine β as the solution to a fixed point problem, which ensures that given the reservation value and distribution functions $(R(\delta),\Phi_1(\delta))$ induced by β it is optimal for dealers to offer the price associated with the low utility flow with probability β.

10.2.2 Reservation values

Given the offered price distributions, we can easily derive the reservation value of investors. In particular, (10.5), and (10.16)–(10.18) imply that

$$\begin{aligned}(r+\gamma)R(\delta)=\delta+\gamma\mathbf{E}^F\left[R(\delta_i)\right]-\lambda\left(R(\delta)-R(\delta_H)\right)^+\\ +\lambda\beta(R(\delta_L)-R(\delta))^+ +\lambda(1-\beta)(R(\delta_M)-R(\delta))^+.\end{aligned}$$

Solving this equation reveals that the gains from trade of potential sellers are simply given by

$$R(\delta_H)-R(\delta_M)=\frac{\delta_H-\delta_M}{r+\gamma} \tag{10.19}$$

and

$$R(\delta_H) - R(\delta_L) = \frac{\delta_H - \delta_M}{r + \gamma} + \frac{\delta_M - \delta_L}{r + \gamma + \lambda(1 - \beta)}. \tag{10.20}$$

Forδ_L investors, note that β plays a role similar to the dealer bargaining power parameter in earlier chapters, in that it captures the share of the surplus that dealers appropriate when trading with δ_L investors. This reflects the fact that asymmetric information endogenously gives these investors what is commonly called an *information rent*: it allows them to sell at a price $R(\delta_M)$ that is strictly larger than their reservation value $R(\delta_L)$.

10.2.3 Asset allocation

Given the bid and ask distributions it is straightforward to derive the induced asset allocation. To that end, let ϕ_{qi} denote the measure of investors with asset holding $q \in \{0, 1\}$ and utility flow $\delta_i \in \{\delta_L, \delta_M, \delta_H\}$. With this notation, equations (10.9) and (10.16)–(10.18) imply that

$$\lambda\phi_{1L} + \gamma\phi_{1L} = \gamma s f_L, \tag{10.21a}$$

$$\lambda(1 - \beta)\phi_{1M} + \gamma\phi_{1M} = \gamma s f_M, \tag{10.21b}$$

$$\gamma\phi_{1H} = \gamma s f_H + \lambda\rho_{0H}\phi_{0H}, \tag{10.21c}$$

subject to

$$s = \phi_{1L} + \phi_{1M} + \phi_{1H}, \tag{10.21d}$$

$$f_i = \phi_{1i} + \phi_{0i}, \qquad \forall i \in \{L, M, H\}, \tag{10.21e}$$

where ρ_{0H} is the probability that a non-owner with utility flow δ_H trades upon meeting a dealer. The constraints (10.21d) and (10.21e) ensure, respectively, that markets clear and that the distributions of utility flows among owners and non owners are consistent with the distribution in the whole population.

Lemma 10.4. *The unique solution to* (10.21) *is given by*

$$\phi_{1L} = f_L - \phi_{0L} = \frac{\gamma s f_L}{\gamma + \lambda} \tag{10.22}$$

$$\phi_{1M} = f_M - \phi_{0M} = \frac{\gamma s f_M}{\gamma + \lambda(1-\beta)} \tag{10.23}$$

$$\phi_{1H} = f_H - \phi_{0H} = s - \frac{\gamma s f_L}{\gamma + \lambda} - \frac{\gamma s f_M}{\gamma + \lambda(1-\beta)} \tag{10.24}$$

and

$$\rho_{0H} = \frac{\gamma}{\lambda} \frac{\phi_{1H} - s f_H}{\phi_{0H}}. \tag{10.25}$$

and lies in the open set $(0,1)^7$ *for any value of* $\beta \in [0,1]$.

Proof. The solution and the fact that $\phi_{qi} \in (0,1)$ for any $\beta \in [0,1]$ follow from straightforward algebra. To complete the proof it thus suffices to show that $\rho_{0H} \in (0,1)$. Substituting (10.24) into (10.25) shows that

$$\rho_{0H} = \frac{s\gamma^2 \beta f_M - s\gamma(\gamma + \lambda(1-\beta))(1-f_H)}{(\gamma + \lambda(1-\beta))(s\lambda - (\lambda + \gamma(1-s))f_H) - \gamma s \lambda \beta f_M}.$$

The desired conclusion now follows by observing that this expression is strictly decreasing as a function of β with

$$\rho_{0H}\Big|_{\beta=0} = \frac{\gamma s(1-f_H)}{\gamma(1-s)f_H - \lambda(s - f_H))} < 1$$

$$\rho_{0H}\Big|_{\beta=1} = \frac{\gamma s f_L}{\gamma(1-s)f_H + \lambda(f_H - s(1-f_M))} > 0,$$

where the strict inequalities are due to (10.15). □

10.2.4 Equilibrium

Equations (10.17), (10.19)–(10.20), and (10.22)–(10.24) imply that, taking as given the reservation values and the asset allocation induced by some $\beta \in [0,1]$, the dealer's optimal bidding strategy is

$$T(\beta) \equiv \begin{cases} 0, & \text{if } \psi_L(\beta) > \psi_L^*(\beta), \\ [0,1], & \text{if } \psi_L(\beta) = \psi_L^*(\beta), \\ 1, & \text{if } \psi_L(\beta) < \psi_L^*(\beta), \end{cases}$$

where

$$\begin{aligned} \psi_L(\beta) &\equiv \frac{\phi_{1L}}{\phi_{1L} + \phi_{1M}} \\ &= \frac{(\gamma + \lambda(1-\beta))f_L}{(\gamma+\lambda)f_M + (\gamma + \lambda(1-\beta))f_L} \end{aligned}$$

is the fraction of sellers with utility flow δ_L induced by β, and

$$\psi_L^*(\beta) \equiv \frac{R(\delta_H) - R(\delta_M)}{R(\delta_H) - R(\delta_L)} = \frac{(r + \gamma + \lambda(1-\beta))\,(\delta_H - \delta_M)}{(r+\gamma)(\delta_H - \delta_L) + \lambda(1-\beta)(\delta_H - \delta_M)}$$

denotes the fraction of sellers with utility flow δ_L that equates the profits from bidding either δ_L or δ_M. An equilibrium is then simply a fixed point of the operator $T : [0,1] \to [0,1]$ and straightforward manipulations lead to the following characterization of the equilibrium.

Proposition 10.5. *Suppose that* (10.15) *holds and define*

$$\begin{aligned} \Pi^* &\equiv \left(1 + \frac{\lambda}{\gamma}\right)\left(\frac{\delta_H - \delta_M}{\delta_M - \delta_L}\right)\frac{f_M}{f_L} > 0 \\ \Pi_* &\equiv \left(1 + \frac{\lambda}{r+\gamma}\right)\left(\frac{\delta_H - \delta_M}{\delta_M - \delta_L}\right)\frac{f_M}{f_L} \in (0, \Pi^*)\,. \end{aligned}$$

Then, there exists a unique equilibrium. The equilibrium price setting strategy of dealer is given by (10.18) *with the constant*

$$\beta = \begin{cases} 0 & \text{if } 1 \le \Pi_* \\ \frac{(\gamma+\lambda)(1-\Pi_*)}{\gamma(\Pi^*-\Pi_*)+\lambda(1-\Pi_*)} \in (0,1) & \text{if } \Pi_* < 1 < \Pi^* \\ 1 & \text{if } \Pi^* \le 1. \end{cases}$$

Moreover, if $\beta \in (0,1)$ then it is strictly decreasing in λ.

To analyze the combined effects of search and information frictions, let us first consider the different types of equilibria that can arise as we vary the meeting intensity. Given that

$$\frac{\partial \Pi^*}{\partial \lambda} = \left(1 + \frac{r}{\gamma}\right) \frac{\partial \Pi_*}{\partial \lambda} > 0$$

and

$$\lim_{\lambda \to 0} \Pi_* = \lim_{\lambda \to 0} \Pi^* = \Pi_0 \equiv \left(\frac{\delta_H - \delta_M}{\delta_M - \delta_L}\right) \frac{f_M}{f_L},$$

it is clear that two cases may occur. If parameters are such that $\Pi_0 \geq 1$, then dealers quote a single bid equal to δ_M irrespective of the meeting intensity. Alternatively, if $\Pi_0 < 1$ then all three types of equilibria may occur, depending on the meeting intensity: dealers only bid δ_L when λ is sufficiently low; they only bid δ_M when λ is sufficiently high; and otherwise they randomize between these two bids. See the top panels of Figure 10.1 for an illustration.

In all cases, since $\lim_{\lambda \to \infty} \Pi_* = \infty$, the only type of equilibrium that can occur as search friction vanish is one in which dealers bid δ_M with probability one. In this limit, owners with the intermediate utility flow sell with probability one but don't receive any of the gains from trade, and $\beta \to 0$ so that owners with the low utility flow also sell with probability one. Interpreting β as the effective bargaining power of dealers with δ_L investors, this shows that δ_L investors are able to extract ever larger information rents as search frictions vanish. Hence, relative to the symmetric information case of Chapter 4 with monopolistic dealers ($\theta = 1$), the effects of information frictions persist as search frictions vanish.

Information frictions can distort the asset allocation over and beyond the effect of search frictions. In particular, if $\beta > 0$ then at each instant there are owners with intermediate utility flow who receive a bid of δ_L and do not sell. As a result, the mass ϕ_{1M} of such investors is larger than its full information counterpart and, since the mass ϕ_{1L} of owners with low utility flow is the same in both models, it follows that the mass ϕ_{0H} of misallocated non-owners is higher under asymmetric information. If instead $\beta = 0$, then the equilibrium asset allocation under asymmetric information is the same as under complete information because non-owners with low or intermediate utility flows trade with probability one upon meeting a dealer. Together with Proposition 10.5,

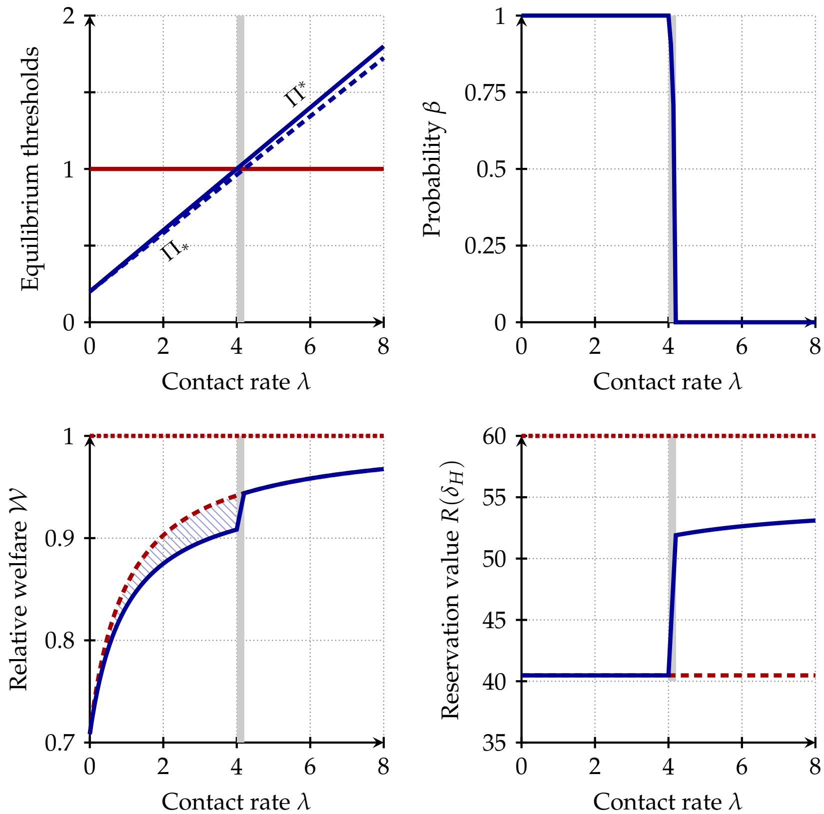

FIGURE 10.1: Search and information frictions

The top left panel plots the tresholds Π_* and Π^* as functions of the meeting intensity. Clockwise from the top right, the other panels plot the low bid probability β, the interdealer price $R(\delta_H)$, and the relative welfare measure $\mathcal{W}$ as functions of the meeting intensity in the semi-centralized market with search and information frictions (solid), the semi-centralized market with only search frictions (dashed), and the frictionless market (dotted). In each panel the shaded region indicates the interval of meeting intensities for which the equilibrium is in mixed strategies. The parameters used to produce the figure satisfy (10.15) and $\Pi_0 < 1$.

these observations imply that asymmetric information magnifies the allocative effects of search frictions only if the meeting intensity is sufficiently low.

Information frictions also distort the prices at which the asset is traded. In particular, the interdealer market price $R(\delta_H)$ under asymmetric information is higher than in the complete information setting of Chapter 4 with dealer bargaining power $\theta = 1$, but lower than its frictionless counterpart δ_H/r, even as $\lambda \to \infty$; see the bottom right panel of Figure 10.1 for an illustration. As we explained in Chapter 4.4.3, the fact that the asset trades at a discount is a consequence of search frictions and the condition $f_H > s$, which implies that the marginal type $\delta^\star = \delta_H$ is higher than the average utility flow. The novel feature in the present chapter is that asymmetric information results in a strictly smaller discount as soon as the meeting intensity is large enough to ensure that $\beta < 1$. This occurs because in an equilibrium with $\beta < 1$, owners with low utility flow extract information rents by being able to sell at δ_M with strictly positive probability. These rents increase the option value of selling and, thereby, increase the reservation value of all investors. Note, however, that this comparative static heavily depends on the assumption that dealers make take-it-or-leave-it offers and, thus, may not be robust to alternative trading protocols in which investors retain a strictly positive share of the surplus.

How do the effects of information frictions depend on the severity of search frictions? To answer this question, we first need to understand how the pricing behavior of dealers depends on the meeting rate, λ. From Proposition 10.5 we know that β is weakly decreasing in λ, so that dealers bid δ_M more often as search frictions weaken, as illustrated in the top right panel of Figure 10.1. This occurs for two reasons. First, an increase in λ skews the distribution of utility flows among owners towards the intermediate utility flow, which provides dealers with an incentive to bid δ_M more frequently. Second, given $\beta < 1$, an increase in λ causes the spread

$$R(\delta_M) - R(\delta_L) = \frac{\delta_M - \delta_L}{r + \gamma + \lambda(1 - \beta)}$$

to decrease, which makes it less attractive for dealers to target potential sellers with the low utility flow.

The fact that the low bid probability β decreases in λ directly implies that misallocation declines as search frictions become less severe. This occurs for two reasons that mirror the frictions at play in the model. First, more frequent meetings allows investors to adjust their holdings more often, which naturally reduces misallocation. Second, private information implies that dealers are more likely to quote δ_M as the meeting intensity increases, which improves the allocation by ensuring that a greater of fraction of positive-surplus matches result in trade. To illustrate this effect, the bottom left panel of Figure 10.1 plots the steady-state welfare relative to the frictionless benchmark

$$\mathcal{W} \equiv \frac{1}{s\delta_H} \sum_i \delta_i \phi_{1i}$$

as a function of the meeting intensity with and without private information. The area between the frictionless benchmark (dotted) and the full information benchmark (dashed) accounts for the efficiency losses due to search frictions, whereas the hatched area between the full and private information welfare (solid) accounts for the additional losses due to the information friction.

10.3 BACK TO THE CONTINUOUSLY DIFFERENTIABLE CASE

After this detour through the discrete case, we now return to the smooth case in which is the distribution of utility flows is continuously differentiable and such that $F'(\delta) = f(\delta) > 0$ for all $\delta \in \mathcal{D}$.

10.3.1 Necessary conditions

Our earlier analysis shows that, when $F(\delta)$ is continuous, the dealer's expected profits from quoting $\delta \in \mathcal{D}$ are respectively given by

$$\Pi_a(\delta) = \int_{\delta^\star}^{\delta} \frac{dx}{r + \gamma + \lambda A(x)} \int_{\delta}^{\overline{\delta}} \frac{\gamma dF(x)}{\gamma + \lambda A(x)}$$

when meeting a buyer, and

$$\Pi_b(\delta) = \int_{\delta}^{\delta^\star} \frac{dx}{r + \gamma + \lambda(1 - B(x))} \int_{\underline{\delta}}^{\delta} \frac{\gamma dF(x)}{\gamma + \lambda(1 - B(x))}.$$

when meeting a seller. To narrow down the search for an equilibrium, we now establish several properties that must hold in any equilibrium.

Proposition 10.6. *In any equilibrium:*

1. *The interdealer utility flow $\delta^\star \in \mathcal{D}$.*

2. *The price distributions are such that* $\text{supp}(B) \subset (\underline{\delta}, \delta^\star)$ *and* $\text{supp}(A) \subset (\delta^\star, \overline{\delta})$.
3. *The price distributions are atomless and thus continuous. In particular, all equilibria are in mixed strategies.*

Proof. To establish the first assertion, assume towards a contradiction that there exists an equilibrium in which $\delta^\star = \underline{\delta}$. In such an equilibrium dealers never buy because the mass of investors with utility flow $\underline{\delta}$ is equal to zero. Since the interdealer market must clear, it follows that dealers never sell either. But this cannot be an equilibrium because, in the absence of trade,

$$\Pi_a(\delta) = \frac{(\delta - \underline{\delta})(1 - F(\delta))}{r + \gamma}$$

is strictly positive on $\mathcal{D}$. Symmetrically, in an equilibrium with $\delta^\star = \overline{\delta}$, dealers never sell and, thus, never buy. However, this cannot be optimal because, absent trade, the expected dealer profit

$$\Pi_b(\delta) = \frac{(\overline{\delta} - \delta)F(\delta)}{r + \gamma} > 0, \qquad \delta \in \mathcal{D}.$$

To prove the second assertion, assume towards a contradiction that there exists an equilibrium in which $\underline{\delta} \in \text{supp}(B)$. Because the expected profit function is by definition constant over the latter set, it must be that

$$\max_{\delta \in [\underline{\delta}, \delta^\star]} \Pi_b(\delta) = \Pi_b(\underline{\delta}) = 0.$$

However, since reservation values are strictly increasing and

$$\Phi_1'(\delta) = \frac{(\gamma s + \lambda A(\delta))f(\delta)}{\gamma + \lambda(1 - B(\delta) + A(\delta))} > 0,$$

we can find $\delta \in (\underline{\delta}, \delta^\star) \neq \emptyset$ such that $\Pi_b(\delta) > 0$ and the required contradiction follows. Similarly, if there exists an equilibrium in which $\delta^\star \in \text{supp}(B)$ then condition (10.4) implies that

$$\max_{\delta \in [\underline{\delta}, \delta^\star]} \Pi_b(\delta) = \Pi_b(\delta^\star) = 0$$

and the same arguments as above deliver a contradiction by showing that we may find $\delta \in (\underline{\delta}, \delta^\star) \neq \emptyset$ such that $\Pi_b(\delta) > 0$.

Let us now turn to the last assertion and assume towards a contradiction that there exists an equilibrium in which $\Delta B(z) > 0$ for some $z \in \text{supp}(B)$. Since B is right-continuous and $f = F'$ is continuous we have that the expected profit in (10.14) is differentiable with

$$\Pi_b'(\delta) = \frac{f(\delta)L(\delta)}{\gamma + \lambda(1 - B(\delta))} - \frac{K(\delta)}{r + \gamma + \lambda(1 - B(\delta))} \tag{10.26}$$

where the functions

$$K(\delta) \equiv \int_{\underline{\delta}}^{\delta} \frac{\gamma f(x)dx}{\gamma + \lambda(1 - B(x))}$$
$$L(\delta) \equiv \int_{\delta}^{\delta^\star} \frac{\gamma dx}{r + \gamma + \lambda(1 - B(x))}$$

are continuous. Using this expression, the first order condition of optimality at a point $z \in \text{supp}(B)$ can be written

$$\Pi_b'(z) \leq 0 \leq \lim_{\delta \uparrow b} \Pi_b'(\delta) = \frac{f(z)L(z)}{\gamma + \lambda(1 - B(z-))} - \frac{K(z)}{r + \gamma + \lambda(1 - B(z-))},$$

where the limit exists due to the fact that B is nondecreasing and thus admits left limits at all points of $\mathcal{D}$. The required contradiction follows by observing that these inequalities cannot both hold if z is an atom of the bid distribution. Indeed, if the second inequality holds then (10.26) implies that

$$\frac{K(z)}{f(z)L(z)} \leq \frac{r + \gamma + \lambda(1 - B(z-))}{\gamma + \lambda(1 - B(z-))} < \frac{r + \gamma + \lambda(1 - B(z))}{\gamma + \lambda(1 - B(z))}$$

where the strict inequality follows from the fact that $\Delta B(z) > 0$. Substituting this inequality into (10.26) yields

$$\Pi_b'(z) = \frac{f(z)L(z)}{r + \gamma + \lambda(1 - B(z))} \left(\frac{r + \gamma + \lambda(1 - B(z))}{\gamma + \lambda(1 - B(z))} - \frac{K(z)}{f(z)L(z)} \right) > 0$$

which contradicts the first inequality. The proof of the corresponding results for the equilibrium ask distribution is similar. □

The proposition offers the striking conclusion that there are no equilibria in pure strategies when the distribution is continuously differentiable; that is, there are no equilibria in which the support of either or both of the price distributions is reduced to a point. To see why, consider an equilibrium in which dealers offer a single bid price b. In such an equilibrium, owners with reservation values $R(\delta) > b$ never sell. As a result, such owners are relatively abundant in equilibrium and this endogenous selection implies that raising the bid is profitable because, when the density is continuous, the induced increase in the mass of accepted offers more than compensates for the higher price that the dealer pays for each asset. Similarly, since asset owners with $R(\delta) \leq b$ always sell upon meeting a dealer, there are relatively few such investors in equilibrium, and this endogenous selection makes it profitable to reduce the bid because the lower price paid for each asset more than compensates the induced decrease in the mass of accepted offers.

The necessary conditions of Proposition 10.6 have important implications for the trading experience of investors in the market. In particular, as illustrated in Figure 10.2, these conditions imply that the equilibrium pricing strategies of dealers endogenously split the population into three groups. Owners with low utility flows $\delta < \underline{\delta}_b$ and non-owners with large utility flows $\delta > \overline{\delta}_a$ have the largest potential gains from trade with dealers and, therefore, are willing to pay a premium to trade fast. With this in mind, dealers optimally offer bid prices in $[\underline{\delta}_b, \overline{\delta}_b]$ and ask prices in $[\underline{\delta}_a, \overline{\delta}_a]$ so that, upon meeting a dealer, these investors trade with probability one.

Next, owners with utility flows $\delta \in [\underline{\delta}_b, \overline{\delta}_b]$ and non-owners with utility flows $\delta \in [\underline{\delta}_a, \overline{\delta}_a]$ have intermediate gains from trade and, thus, may be willing to forego immediacy in the hope of better terms. In equilibrium, dealers exploit this by quoting random prices that offer such investors the possibility of gains of trade but may result in trading delays if the realized price is not accepted. Importantly, and in contrast to the full information case where dealers capture the whole surplus in every match, investors in these first two groups retain a strictly positive share of the surplus in every trade they execute and, thus, enjoy *information rents* that mitigate the market power of dealers within each match. Finally, dealers ensure their profit margin by bounding their bid and ask prices away from the interdealer price so that trade breaks down for owners with utility flows $\delta \in (\overline{\delta}_b, \delta^\star]$ and non-owners with utility flows $\delta \in [\delta^\star, \underline{\delta}_a)$, who offer the smallest potential gains from trade.

If the trade probabilities are as depicted in Figure 10.2, then the equilibrium is fully separating in the sense that all utility flows which are not excluded from trading effectively face different terms. However, Proposition 10.6 does

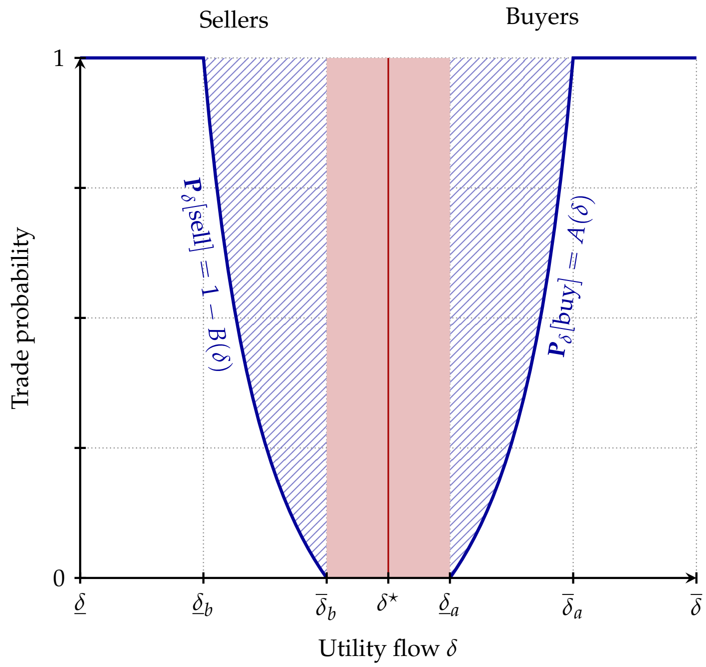

FIGURE 10.2: Equilibrium trade probabilities

This figure illustrates the probability that an owner with utility flow $\delta \leq \delta^\star$ sells upon meeting a dealer (left panel) and the probability that a non-owner with utility flow $\delta \geq \delta^\star$ buys upon meeting a dealer (right panel). The shaded areas indicate utility flows whose trades would occur with probability 1 under complete information but may be delayed or impossible under asymmetric information.

not preclude the possibility of holes in the support of the equilibrium bid and ask distribution, as illustrated in Figure 10.3, where instead of being connected the support of the bid (ask) distribution is the union of two (three) disjoint intervals. In the mechanism design literature, such situations are referred to as *bunching* [e.g. Mussa and Rosen, 1978, Guesnerie and Laffont, 1984, Laffont and Martimort, 2001] because investors whose utility flows fall in a given hole of the distribution are offered equivalent terms of trade and, thus, either all accept or all refuse a given offer under the equilibrium pricing strategy. To exclude bunching a priori, and thus greatly simplify the equilibrium analysis, we need to impose additional conditions on the distribution of utility flows.

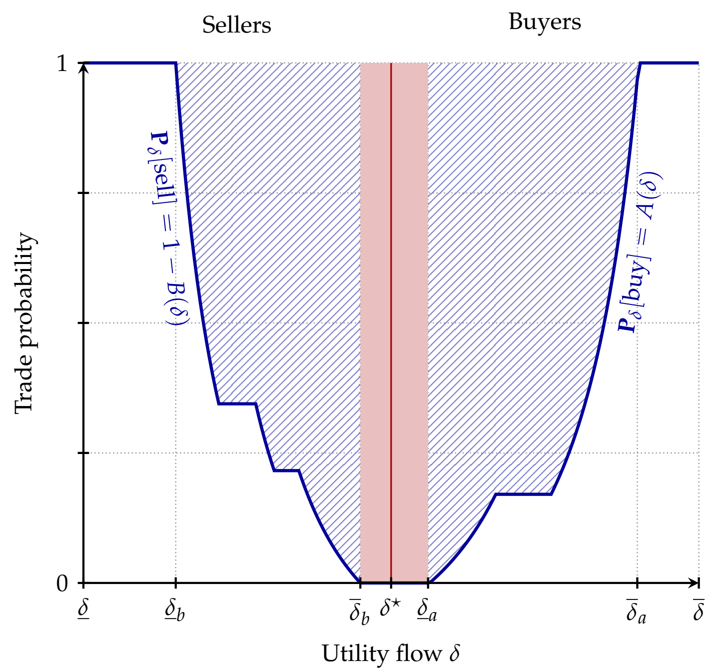

FIGURE 10.3: Equilibrium trade probabilities with bunching

This figure illustrates the probability that an owner with utility flow $\delta \leq \delta^\star$ sells upon meeting a dealer (left panel) and the probability that a non-owner with utility flow $\delta \geq \delta^\star$ buys upon meeting a dealer (right panel). The shaded areas indicate utility flows whose trades would occur with probability 1 under complete information but may be delayed ▨ or impossible ▭ under asymmetric information.

The following result provides a convenient sufficient condition that holds for a large class of distributions.

Proposition 10.7. *If the density $f(\delta)$ is log-concave, then the supports of A and B are connected in equilibrium and, hence, there is no bunching.*

Proof. See Section 10.4.1. □

The class $\mathcal{L}$ of distributions with log-concave densities is broad and has found applications in multiple areas of economics, statistics, and operations research. For example, the uniform distribution, the exponential and double exponential distributions, the normal distribution, the power distribution with

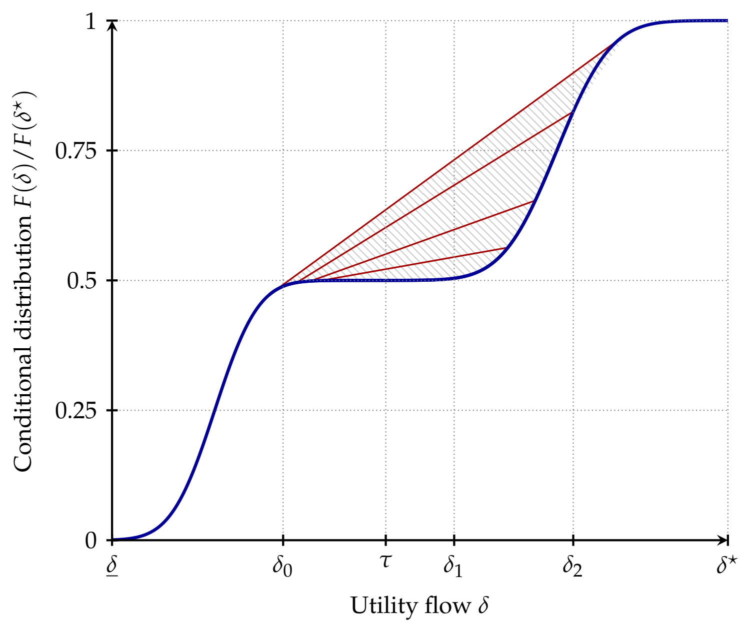

FIGURE 10.4: A distribution of utility flows $F \notin \mathcal{L}$

This figure illustrates a case in which the conditional density $f(\delta)/F(\delta^\star)$ of utility flows among investors with utility flows below $\delta^\star$ is bimodal so that the distribution includes two convex pieces: one starting at the left endpoint $\underline{\delta}$ and another starting at τ. The endpoints of the straight lines represent pairs satisfying condition (10.50).

exponent $p \geq 1$, and the beta distributions with parameters $a, b \geq 1$ (as well as the truncation of any of these distributions to an interval) all belong to the class $\mathcal{L}$. See Saumard and Wellner [2014] for an in-depth review of the theory of log-concave functions, An [1998], Ewerhart [2013] and Bagnoli and Bergstrom [2006] for surveys of applications in economics and statistics, and Appendix A.4 for properties of $\mathcal{L}$ that are relevant for our purposes.

A distinctive feature of the class $\mathcal{L}$ is that its members are *unimodal* in the sense that, for any F in this class there exists a m such that $F'(\delta)$ is increasing below m and decreasing above it. To understand why bunching is likely if this property fails, suppose that the density $f(\delta)$ is bimodal over the interval $(\underline{\delta}, \delta^\star]$ associated with sellers, so that the conditional distribution $F(\delta)/F(\delta^\star)$ is strictly convex on two separate regions, as in Figure 10.4. Assume further that the point $\delta_0 \in \operatorname{supp}(B)$ and consider the possibility of bidding more than δ_0. The fact that the distribution changes from concave to convex at τ implies that there are very few owners with utility flows in the interval $[\delta_0, \delta_1]$. Therefore, offering prices in that interval is unlikely to be optimal, as it would reduce the dealer's profits per trade without significantly increasing the numbers of trades. By contrast, bidding prices to the right of δ_1 may keep the dealer indifferent because the convexity of the distribution over $[\delta_1, \delta_2]$ may generate a sufficiently large increase in the number of trades to offset the higher cost of buying each asset. Combining these observations suggests that the bid distribution is likely to be flat over some interval spanning τ and this intuition is upheld by the straight lines in the figure whose endpoints provide examples of thresholds (α, β) satisfying the bunching condition (10.50).

10.3.2 Fully separating equilibria

To focus on equilibria that do not involve bunching, we assume from now on that the distribution of utility flows $F \in \mathcal{L}$.

The necessary conditions of Propositions 10.6 and 10.7 restrict the sets of bid and ask distributions over which to search for equilibria. However, since these distributions are likely to have kinks on the boundary of their supports, it is more convenient to characterize the equilibrium in terms of the auxiliary functions defined by

$$\begin{aligned}\mathbf{A}(q) &\equiv 1 - \frac{\Phi_0(F^{-1}(q))}{1-s} \\ &= 1 - \frac{q - \Phi_1(F^{-1}(q))}{1-s} = \int_q^1 \frac{\gamma d\theta}{\gamma + \lambda A(F^{-1}(\theta))}\end{aligned} \tag{10.27}$$

and

$$\begin{aligned}\mathbf{B}(q) &\equiv \frac{\Phi_1(F^{-1}(q))}{s} \\ &= \int_{\underline{\delta}}^{F^{-1}(q)} \frac{\gamma F'(x)dx}{\gamma + \lambda(1 - B(x))} = \int_0^q \frac{\gamma d\theta}{\gamma + \lambda(1 - B(F^{-1}(\theta)))}.\end{aligned} \tag{10.28}$$

These two distributions represent, respectively, the fraction of non-owners who agree to buy at an ask price that a fraction $q \geq 1 - F(\delta^\star)$ of the population would accept and the fraction of owners who agree to sell at a bid price that a fraction $q \leq F(\delta^\star)$ of the population would accept. Indeed, the fact that the equilibrium price distributions are continuous implies that these functions must be continuously differentiable and, as we show below, this property opens the way for a differential characterization of equilibrium.

As a first step towards this characterization, we explicit the relationship between (A, B) and $(\mathbf{A}, \mathbf{B})$ by providing necessary and sufficient conditions under which the integrals in (10.27)–(10.28) can be inverted, to recover the former from the latter. To simplify the presentation of this result, we start with a definition. For any given $\delta^\star \in \mathcal{D}$ we denote by $\mathcal{P}(\delta^\star)$ the set of functions $\mathbf{A}, \mathbf{B} : [0,1] \to [0,1]$ with the following properties:

$(\mathcal{P}_1)$ $\mathbf{A}(1) = \mathbf{B}(0) = 0$,
$(\mathcal{P}_2)$ $\mathbf{A}$ and $\mathbf{B}$ are continuously differentiable,
$(\mathcal{P}_3)$ There exist utility flows

$$\underline{\delta}_b < \overline{\delta}_b < \delta^\star < \underline{\delta}_a < \overline{\delta}_a$$

such that $\mathbf{A}$ is strictly convex on the set $F[\underline{\delta}_a, \overline{\delta}_a] \equiv [F(\underline{\delta}_a), F(\overline{\delta}_a)]$, $\mathbf{B}$ is strictly convex on the set $F[\underline{\delta}_b, \overline{\delta}_b]$, and

$$-\mathbf{A}'(q) = \mathbf{B}'(p) = \begin{cases} 1, & (p,q) \in [F(\overline{\delta}_b), 1] \times [0, F(\underline{\delta}_a)], \\ \frac{\gamma}{\gamma+\lambda}, & (p,q) \in [0, F(\underline{\delta}_b)] \times [F(\overline{\delta}_a), 1]. \end{cases} \tag{10.29}$$

With this definition at hand, we have the following result.

Lemma 10.8. *A pair of distributions (A, B) verifies the conditions of Propositions 10.6 and 10.7 for some fixed $\delta^\star$ if and only if*

$$\lambda(A(\delta), 1 - B(\delta)) = \gamma \left(-\frac{1 + \mathbf{A}'(F(\delta))}{\mathbf{A}'(F(\delta))}, \frac{\mathbf{B}'(F(\delta)) - 1}{\mathbf{B}'(F(\delta))} \right) \tag{10.30}$$

for some functions $(\mathbf{A}, \mathbf{B}) \in \mathcal{P}(\delta^\star)$.

Proof. See Exercise 10.15. □

Since $F'(\delta) > 0$ by assumption, it is clear that the quantile map $F^{-1}(q)$ is strictly increasing. Therefore, maximizing the expected bid profit function over utility flows $\delta \in (\underline{\delta}, \delta^\star]$ is equivalent to maximizing

$$\begin{aligned}\hat{\Pi}_b(q) &\equiv \Pi_b(F^{-1}(q)) \qquad (10.31)\\ &= \left[\int_{\underline{\delta}}^{F^{-1}(q)} \frac{\gamma F'(x)dx}{\gamma + \lambda(1 - B(x))}\right] \cdot \left[\int_{F^{-1}(q)}^{\delta^\star} \frac{dx}{r + \gamma + \lambda(1 - B(x))}\right]\end{aligned}$$

over $q \in [0, F(\delta^\star)]$. Changing variables from x to $F^{-1}(\theta)$ in the integrals and substituting (10.30) simplifies this expression to

$$\hat{\Pi}_b(q) = \mathbf{B}(q) \int_q^{F(\delta^\star)} \frac{\ell(\theta)\mathbf{B}'(\theta)}{r\mathbf{B}'(\theta) + \gamma} d\theta,$$

where the function $\ell(q) \equiv 1/F'(F^{-1}(q))$ denotes the derivative of the quantile map. By the same token, the ask problem can be stated as

$$\Pi_a^\star = \max_{\delta \in [\delta^\star, \overline{\delta})} \Pi_a(\delta) = \max_{q \in [F(\delta^\star), 1]} \hat{\Pi}_a(q)$$

with the modified objective

$$\hat{\Pi}_a(q) \equiv \Pi_a(F^{-1}(q)) = \mathbf{A}(q) \int_{F(\delta^\star)}^q \frac{\ell(\theta)\mathbf{A}'(\theta)}{r\mathbf{A}'(\theta) - \gamma} d\theta. \qquad (10.32)$$

In view of (10.27)–(10.28), $\mathbf{A}, \mathbf{B}$ are entirely determined by the intervals $F[\underline{\delta}_a, \overline{\delta}_a]$ and $F[\underline{\delta}_b, \overline{\delta}_b]$ over which they are strictly convex and by their restriction to these sets. Therefore, to obtain an equilibrium characterization it suffices to find conditions on these objects that are necessary and sufficient to ensure that markets clear and that the sets $F[\underline{\delta}_a, \overline{\delta}_a]$ and $F[\underline{\delta}_b, \overline{\delta}_b]$ include the set of maximizers of $\hat{\Pi}_a(q)$ and $\hat{\Pi}_b(q)$, respectively. Such conditions are provided by the following theorem.

Theorem 10.9. *A triple $(A, B; \delta^\star)$ forms an equilibrium if and only if (10.30) holds for some $(\mathbf{A}, \mathbf{B}) \in \mathcal{P}(\delta^\star)$ that solve*

$$r\mathbf{A}'(q) = \gamma - \frac{\ell(q)}{\Pi_a^\star} \mathbf{A}(q)^2, \qquad q \in F[\underline{\delta}_a, \overline{\delta}_a], \qquad (10.33a)$$

$$r\mathbf{B}'(q) = \frac{\ell(q)}{\Pi_b^\star} \mathbf{B}(q)^2 - \gamma, \qquad q \in F[\underline{\delta}_b, \overline{\delta}_b], \qquad (10.33b)$$

subject to

$$\mathbf{A}(F(\underline{\delta}_a)) = F'(\underline{\delta}_a)\left(\underline{\delta}_a - \delta^\star\right) \tag{10.34a}$$

$$\mathbf{B}(F(\overline{\delta}_b)) = F'(\overline{\delta}_b)\left(\delta^\star - \overline{\delta}_b\right) \tag{10.34b}$$

and the market clearing condition

$$s\left(F(\overline{\delta}_b) - \mathbf{B}(F(\overline{\delta}_b))\right) = (1-s)\left(1 - F(\underline{\delta}_a) - \mathbf{A}(F(\underline{\delta}_a))\right), \tag{10.35}$$

where the constants

$$\Pi_a^\star = \frac{\gamma(1-F(\overline{\delta}_a))^2}{(\gamma+\lambda)(r+\gamma+\lambda)F'(\overline{\delta}_a)} = \frac{F'(\underline{\delta}_a)(\underline{\delta}_a - \delta^\star)^2}{r+\gamma} \tag{10.36a}$$

$$\Pi_b^\star = \frac{\gamma F(\underline{\delta}_b)^2}{(\gamma+\lambda)(r+\gamma+\lambda)F'(\underline{\delta}_b)} = \frac{F'(\overline{\delta}_b)(\delta^\star - \overline{\delta}_b)^2}{r+\gamma} \tag{10.36b}$$

give an individual dealer's equilibrium expected profit from a match with a non-owner and an owner, respectively.

Proof. See Section 10.4.2. □

Theorem 10.9 derives necessary and sufficient conditions for an equilibrium but proves neither existence nor uniqueness. For given $(\underline{\delta}_b, \overline{\delta}_a)$, it can be shown that the *Riccati* equations (10.33) subject to (10.36) and the initial conditions (10.34) admit unique solutions that depend continuously on the thresholds. What is a priori less clear is whether these functions are convex, and whether the system of equations obtained by requiring that these functions also satisfy (10.35) and (10.29) admits a solution. The dimension, nonlinearity, and implicit nature of this system makes its analysis quite intricate for a general distribution of utility flows. However, in the next section we show that this system admits an explicit solution when the distribution is uniform. Before turning to that example, we first discuss some implications of Theorem 10.9.

The natural benchmarks against which to gauge the equilibrium effects of private information are the outcomes of an otherwise similar semi-centralized market with full information. From the results of Chapter 4, we know that the equilibrium asset allocation, the reservation values of investors, and the interdealer price in such an environment are given by

$$\Phi_1^{\#}(\delta) = \frac{\gamma}{\gamma+\lambda} sF(\delta) + \frac{\lambda}{\gamma+\lambda} \left(F(\delta) - (1-s)\right)^+$$
$$R^{\#}(\delta) = \frac{r+\lambda(1-\theta)}{r\left(r+\gamma+\lambda(1-\theta)\right)}\delta + \frac{\gamma \mathbf{E}^F[\delta]}{r\left(r+\gamma+\lambda(1-\theta)\right)}$$

and

$$P^{\#} = R^{\#}(\delta^{\#}) = R^{\#}\left(F^{-1}(1-s)\right),$$

where $\theta \in [0,1]$ denotes the bargaining power of dealers relative to investors and the superscripts distinguishes these objects from their private information counterparts. In particular, the case of take-it-or-leave-it offers by dealers can be obtained by letting $\theta = 1$, in which case reservation values are independent of the contact rate and given by the autarky values:

$$R^{\#}(\delta)\Big|_{\theta=1} = \frac{\delta}{r+\gamma} + \frac{\gamma \mathbf{E}^F[\delta]}{r\left(r+\gamma\right)} = \mathbf{E}_\delta\left[\int_0^\infty e^{-rt}\delta_t dt\right]. \tag{10.37}$$

Intuitively, the fact that dealers have full information and all the bargaining power grants them monopoly power within every match and allows them to extract the entire surplus. As a result, the possibility of trading with dealers becomes worthless to investors which, in turn, implies that their reservation values are equal to the autarky values irrespective of the contact rate. This leads to the well-known Diamond [1971] paradox: local monopoly power by a continuum of dealers leads to the fully discriminating monopolistic outcome, even with arbitrarily large contact rates.

By contrast, the results of Proposition 10.6 and Theorem 10.9 show that private information allows some investors to retain a strictly positive share of the surplus generated by their trades and, thereby, provides a simple resolution of the paradox. Indeed, the fact that dealers have incomplete information forces them to screen investors by offering a non zero share of the surplus to investors with sufficiently large gains from trades and excluding all others. Since some matches with strictly positive surpluses no longer result in trade, this pricing strategy by dealers implies that the asset allocation is less efficient than with full information, i.e., that $\Phi_1^{\#}(\delta) \leq \Phi_1(\delta)$. However, it also implies that an investor's expected gain from buying

$$\int_{\underline{\delta}}^{\overline{\delta}} (R(\delta) - R(x))^+ \, dA(x) = \int_{\underline{\delta}_a}^{\max\{\delta, \underline{\delta}_a\}} \frac{A(x)dx}{r + \gamma + \lambda A(x)}$$

and from selling

$$\int_{\underline{\delta}}^{\overline{\delta}} (R(x) - R(\delta))^+ \, dB(x) = \int_{\max\{\delta, \underline{\delta}_b\}}^{\overline{\delta}_b} \frac{(1 - B(x))dx}{r + \gamma + \lambda(1 - B(x))}$$

are strictly positive and depend on the intensity of the search friction over the set of utility flows $\mathcal{D} \backslash [\overline{\delta}_b, \underline{\delta}_a]$ that traded in equilibrium.

Going one step further, the next proposition shows that, as search frictions vanish, private information drives the profits of dealers to zero so that the outcomes of the semi-centralized market with private information converge to those of a frictionless market with full information.

Proposition 10.10. *Assume that $f(\delta)$ is uniformly bounded from above and away from zero. Then, along a sequence of equilibria indexed by λ:*

$$\lim_{\lambda \to \infty} \Pi^\star_{a,b} = 0 \tag{10.38}$$

$$\lim_{\lambda \to \infty} \Phi_1(\delta) = \Psi_1(\delta) = (F(\delta) - (1 - s))^+ \tag{10.39}$$

$$\lim_{\lambda \to \infty} \left(\underline{\delta}_a, \overline{\delta}_b, \delta^\star\right) = \delta^\# \tag{10.40}$$

and the reservation values of all investors converge to $\delta^\# / r$.

Proof. See Section 10.4.3. □

The intuition behind Proposition 10.10 is that when contacts with investors are instantaneous, every dealer effectively competes against all others for each trade. Indeed, in this limit investors can instantaneously contact infinitely many dealers. Therefore, it is clear that a non-owner with utility flow larger than $\delta^\star$ will sample successive dealers until he receives an offer to buy at the lowest price $R(\underline{\delta}_a)$ in the support of the ask price distribution, which occurs with probability one by the law of large numbers. Moreover, it must be that this price is equal to the interdealer price; otherwise an individual dealer could deviate by offering to sell at $a < R(\underline{\delta}_a)$ and make a strictly positive profit.

Similarly, an owner with utility flow below $\delta^\star$ will instantaneously contact dealers until she receives an offer to sell at the largest price in the support of the bid distribution, and this price must equal the interdealer price; otherwise a dealer could outbid all others and still make strictly positive profits. As a result, dealers' profits are driven to zero, which implies that no types are excluded from trade and guarantees that the allocation is efficient.

The intuitive arguments above also apply to the semi-centralized market with full information. However, in that case the equilibrium converges to its frictionless counterpart as search frictions vanish *only if* investors have strictly positive bargaining power $1-\theta > 0$. Indeed, if dealers have full information and all the bargaining power, then reservation values are given by the autarky values (10.37), independently of λ, so that neither the reservation values nor the interdealer price can converge to their frictionless counterparts.

In order to establish the existence and uniqueness of the equilibrium and to discuss comparative statics we now consider the explicitly solvable special case where utility flows are uniformly distributed.

10.3.3 Uniformly distributed utility flows

Assume that utility flows are uniformly distributed on the unit interval, so that $f(\delta) = \ell(q) = 1$. Substituting these functions into Theorem 10.9 shows that the equilibrium requirements simplify to

$$r\mathbf{A}'(\delta) = \gamma - \frac{\mathbf{A}(\delta)^2}{\Pi_a^\star}, \qquad \delta \in [\underline{\delta}_a, \overline{\delta}_a], \tag{10.41a}$$

$$r\mathbf{B}'(\delta) = \frac{\mathbf{B}(\delta)^2}{\Pi_b^\star} - \gamma, \qquad \delta \in [\underline{\delta}_b, \overline{\delta}_b], \tag{10.41b}$$

subject to the boundary conditions

$$\mathbf{A}(\underline{\delta}_a) = \underline{\delta}_a - \delta^\star \tag{10.42a}$$

$$\mathbf{B}(\overline{\delta}_b) = \delta^\star - \overline{\delta}_b \tag{10.42b}$$

$$-\mathbf{A}'(\overline{\delta}_a) = \mathbf{B}'(\underline{\delta}_b) = \tfrac{\gamma}{\gamma+\lambda} \tag{10.42c}$$

and the market clearing condition

$$s(2\overline{\delta}_b - \delta^\star) = (1-s)(\delta^\star + 1 - 2\underline{\delta}_a) \tag{10.43}$$

where the constants

$$\Pi_a^\star = \frac{\gamma(1-\overline{\delta}_a)^2}{(\gamma+\lambda)(r+\gamma+\lambda)} = \frac{(\underline{\delta}_a-\delta^\star)^2}{r+\gamma} \tag{10.44a}$$

$$\Pi_b^\star = \frac{\gamma\underline{\delta}_b^2}{(\gamma+\lambda)(r+\gamma+\lambda)} = \frac{(\delta^\star-\overline{\delta}_b)^2}{r+\gamma} \tag{10.44b}$$

give the expected equilibrium profits of a dealer. Standard results on Riccati equations (see for example Reid [1972]) based on the transformation

$$(\mathbf{A}(q),\mathbf{B}(q)) = -r\left(\Pi_a^\star\frac{\alpha'(q)}{\alpha(q)},\Pi_b^\star\frac{\beta'(q)}{\beta(q)}\right)$$

show that the general solutions to (10.41) are given by

$$\mathbf{B}(\delta) = \left(\frac{2\gamma\underline{\delta}_b}{r\xi_\lambda}\right)\frac{b+e^{\left(\frac{\xi_\lambda}{\underline{\delta}_b}\right)(\delta-\underline{\delta}_b)}}{b-e^{\left(\frac{\xi_\lambda}{\underline{\delta}_b}\right)(\delta-\underline{\delta}_b)}} \tag{10.45}$$

$$\mathbf{A}(\delta) = \left(\frac{2\gamma(1-\overline{\delta}_a)}{r\xi_\lambda}\right)\frac{a+e^{\left(\frac{\xi_\lambda}{1-\overline{\delta}_a}\right)(\overline{\delta}_a-\delta)}}{a-e^{\left(\frac{\xi_\lambda}{1-\overline{\delta}_a}\right)(\overline{\delta}_a-\delta)}} \tag{10.46}$$

with

$$\xi_i \equiv \frac{2}{r}\sqrt{(\gamma+i)(r+\gamma+i)} \tag{10.47}$$

for some constants $a,b\in\mathbf{R}$. Inserting these solutions into (10.42), (10.43), and (10.44) gives a system of seven equations. Solving that system for (a,b) and the equilibrium thresholds $(\underline{\delta}_b,\overline{\delta}_b,\delta^\star,\underline{\delta}_a,\overline{\delta}_b)$ provides an explicit solution for the unique equilibrium. To state the result, we denote by $\cosh(x)\equiv(e^x+e^{-x})/2$ the hyperbolic cosine function.

Proposition 10.11. *If utility flows are uniformly distributed on the unit interval then the equilibrium is uniquely given by*

$$A(\delta) = \begin{cases} 0, & \delta \leq \underline{\delta}_a, \\ \frac{r}{2\lambda}\cosh\left(\eta_\lambda + \xi_\lambda \cdot \frac{\overline{\delta}_a - \delta}{1-\overline{\delta}_a}\right) - \frac{r}{2\lambda}\left(\eta_0 + \xi_0\right), & \delta \in [\underline{\delta}_a, \overline{\delta}_a], \\ 1, & \delta \geq \overline{\delta}_a, \end{cases}$$

$$B(\delta) = \begin{cases} 0, & \delta \leq \underline{\delta}_b, \\ \frac{r}{2\lambda}\left(\eta_\lambda - \xi_\lambda\right) - \frac{r}{2\lambda}\cosh\left(\eta_\lambda - \xi_\lambda \cdot \frac{\delta - \underline{\delta}_b}{\underline{\delta}_b}\right), & \delta \in [\underline{\delta}_b, \overline{\delta}_b], \\ 1, & \delta \geq \overline{\delta}_b, \end{cases}$$

and the thresholds

$$\begin{aligned} \delta^\star &= 1 - s, \\ \underline{\delta}_a &= 1 - \frac{s\left(\eta_0 + \eta_\lambda + \xi_\lambda\right)}{\eta_0 + \eta_\lambda + \xi_0 + \xi_\lambda}, \quad \overline{\delta}_a = 1 - \frac{s\xi_\lambda}{\eta_0 + \eta_\lambda + \xi_0 + \xi_\lambda}, \\ \underline{\delta}_b &= \frac{(1-s)\xi_\lambda}{\eta_0 + \eta_\lambda + \xi_0 + \xi_\lambda}, \qquad \overline{\delta}_b = \frac{(1-s)\left(\eta_0 + \eta_\lambda + \xi_\lambda\right)}{\eta_0 + \eta_\lambda + \xi_0 + \xi_\lambda}, \end{aligned}$$

with the nonnegative constants defined by

$$\eta_i \equiv \log\left(1 + \frac{2(\gamma + i)}{r} - \xi_0 + \mathbf{1}_{\{i>0\}}\left(\xi_0 + \xi_i\right)\right)$$

and (10.47) *for any given* $i \geq 0$.

Proof. Given Theorem 10.9 it suffices to verify that $\mathbf{A}$ and $\mathbf{B}$ are strictly convex on $[\underline{\delta}_a, \overline{\delta}_a]$ and $[\underline{\delta}_b, \overline{\delta}_b]$, respectively. Substituting the solutions for (a, b) and the equilibrium thresholds into (10.45)–(10.46) shows that

$$\mathbf{B}(\delta) = \left(\frac{2\gamma\underline{\delta}_b}{r\xi_\lambda}\right) \frac{1 + e^{-\eta_\lambda + \xi_\lambda\left(\frac{\delta}{\underline{\delta}_b} - 1\right)}}{1 - e^{-\eta_\lambda + \xi_\lambda\left(\frac{\delta}{\underline{\delta}_b} - 1\right)}}, \tag{10.48}$$

$$\mathbf{A}(\delta) = \left(\frac{2\gamma(1 - \overline{\delta}_a)}{r\xi_\lambda}\right) \frac{1 + e^{\eta_\lambda + \xi_\lambda\left(\frac{\delta - \overline{\delta}_b}{1 - \overline{\delta}_b}\right)}}{1 - e^{\eta_\lambda + \xi_\lambda\left(\frac{\delta - \overline{\delta}_b}{1 - \overline{\delta}_b}\right)}}. \tag{10.49}$$

Differentiating (10.49) shows that $\mathbf{A}'(\delta) < 0$. Since $\mathbf{A}(\overline{\delta}_a) > 0$ this implies that $\mathbf{A}(\delta) > 0$ for all $\delta \in [\underline{\delta}_a, \overline{\delta}_a]$ and differentiating a second time gives

$$\mathbf{A}''(\delta) = -\frac{r}{2\gamma}\left(\frac{\xi_\lambda}{1-\overline{\delta}_a}\right)^2 \mathbf{A}(\delta)\mathbf{A}'(\delta) > 0.$$

Similarly, differentiating (10.48) shows that B is strictly positive as well as strictly increasing and differentiating a second time gives

$$\mathbf{B}''(\delta) = \frac{r}{2\gamma}\left(\frac{\xi_\lambda}{\underline{\delta}_b}\right)^2 \mathbf{B}(\delta)\mathbf{B}'(\delta) > 0$$

which completes the proof. □

By symmetry we have that the ratio of the mass of excluded investors to the mass of unaffected investors is the same for buyers and sellers. Furthermore, (10.44) implies that this ratio is given by

$$w \equiv \frac{\delta^\star - \overline{\delta}_b}{\underline{\delta}_b} = \frac{\underline{\delta}_a - \delta^\star}{1-\overline{\delta}_a} = \frac{\sqrt{\gamma(r+\gamma)}}{\sqrt{(\gamma+\lambda)(r+\gamma+\lambda)}}$$

and a direct calculation shows that

$$\frac{\partial w}{\partial \lambda} \leq 0 \leq \frac{\partial w}{\partial \gamma}, \frac{\partial w}{\partial r}.$$

To further illustrate the properties of the equilibrium, Figure 10.5 plots the equilibrium thresholds and the width of the equilibrium price distributions as functions of the arrival rate of preference shocks (left panels) and the contact rate between investors and dealers (right panels).

The top panels of the figure show that dealers tend to offer better trading terms—i.e., lower ask and higher bids—as the contact rate λ increases or the reset rate γ decreases. Intuitively, this occurs because either variation tends to limit the potential gains from trade with investors by making the steady state distributions of asset holding and utility flows more selected which partly offsets the market power of dealers. Dealers also tend to exclude less investors as λ increases or γ decreases but the numbers are significant. For example, the equilibrium pricing strategy of dealers excludes about 16% of the investor

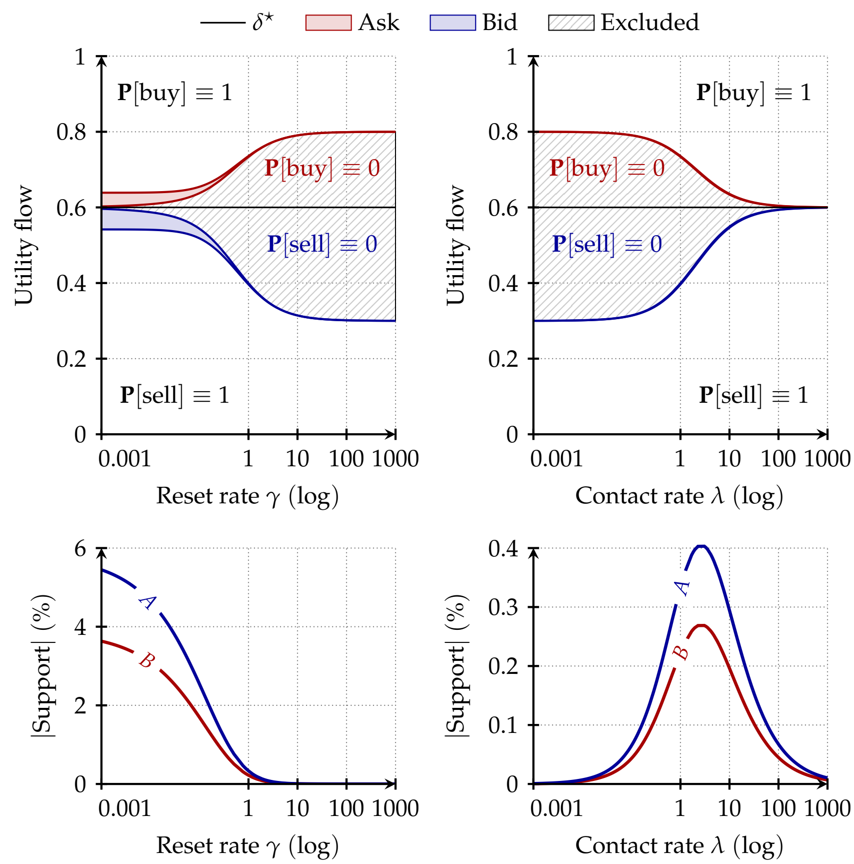

FIGURE 10.5: Equilibrium thresholds

The left panels plot the equilibrium thesholds (top) and the width of support of the equilibrium price distributions (bottom) as functions of the arrival rate γ of preference shock in a model with contact rate $\lambda = 1$, asset supply $s = 0.4$, interest rate $r = 5\%$, and a uniform distribution of utility flows on the unit interval. The right panel plots the same quantities as functions of the contact rate λ between investors and dealers when the arrival rate of preference shocks is set to $\gamma = 1$.

population in the base case and this number grows to 24% in the limit where the reset rate $\gamma \to \infty$ or the contact rate $\lambda \to 0$.

Comparing the top panels of the figure shows that the mass of investors who are subject to trading delays, i.e., those whose utility flows lies in the support of the price distribution, is more affected by the reset rate than by the contact rate. In fact, the top right panel may even give the impression that the supports of the equilibrium price distributions are reduced to a point, but this is not so. As a confirmation, the bottom panels plot the width of the support of the bid and ask distributions expressed in percentage of the population. In both panels the qualitative behavior is the same: The width of the supports are decreasing in the reset rate and bell-shaped in the contact rate, with the ask distribution slightly more spread out than the bid. The latter feature is directly related to the fact that the figure is constructed with an asset supply $s = 0.4$ that favours the position of sellers.

10.4 OMITTED PROOFS

10.4.1 Proof of Proposition 10.7

Assume towards a contradiction that, in some equilibrium, there exist $\alpha, \beta \in \text{supp}(B)$ such that the interval $(\alpha, \beta) \notin \text{supp}(B)$. Since f and B are continuous it is clear that Π_b, defined in (10.14), is continuously differentiable. Combining this property with the fact that $\alpha, \beta \in \text{supp}(B)$, we deduce that

$$\Pi_b(\alpha) - \Pi_b(\beta) = \Pi_b'(\alpha) = \Pi_b'(\beta) = 0.$$

Exercise 10.12 shows that these conditions can only hold if the density of utility flows is such that $f(\alpha) \neq f(\beta)$, in which case they imply that

$$f(\beta)f(\alpha)(\beta - \alpha)^2 = (F(\beta) - F(\alpha))^2.$$

It remains to show that this equality cannot hold under the stated assumptions. To this end, we first observe that since $\beta > \alpha$:

$$\begin{aligned}
\log\left(\tfrac{1}{\beta-\alpha}\int_\alpha^\beta f(t)dt\right) &> \tfrac{1}{\beta-\alpha}\int_\alpha^\beta \log f(t)dt \\
&= \tfrac{1}{\beta-\alpha}\int_\alpha^\beta \log f\left(\tfrac{\beta-t}{\beta-\alpha}\cdot\alpha + \tfrac{t-\alpha}{\beta-\alpha}\cdot\beta\right)dt \\
&\geq \tfrac{1}{\beta-\alpha}\int_\alpha^\beta \left(\tfrac{\beta-t}{\beta-\alpha}\log f(\alpha) + \tfrac{t-\alpha}{\beta-\alpha}\log f(\beta)\right)dt = \log\sqrt{f(\alpha)f(\beta)},
\end{aligned}$$

where the strict inequality follows from Jensen's inequality and the assumption that f is not constant on $[\alpha, \beta]$; the second inequality follows from the log concavity of f; and the equality follows by computing the integral. Multiplying both sides by two and exponentiating the result shows that

$$(F(\beta) - F(\alpha))^2 = \left(\int_\alpha^\beta f(t)dt\right)^2 > (\beta - \alpha)^2 f(\alpha)f(\beta), \tag{10.50}$$

which completes the proof. The corresponding result for the ask distribution can be proved using similar steps. □

10.4.2 Proof of Theorem 10.9

PROOF OF NECESSITY.

Assume that $(A, B; \delta^\star)$ form an equilibrium. By Lemma 10.8 and Propositions 10.6 and 10.7 this implies that condition (10.30) holds for some $(\mathbf{A}, \mathbf{B}) \in \mathcal{P}(\delta^\star)$ with associated thresholds $(\underline{\delta}_b, \overline{\delta}_b, \underline{\delta}_a, \overline{\delta}_a)$. Substituting (10.30) into (10.8) and computing the integrals gives (10.35) and it now remains to show the necessity of (10.33), (10.34), and (10.36).

Since $(A, B; \delta^\star)$ is an equilibrium, we have that $\hat{\Pi}_b(q)$ is maximal and, thus, constant on the interval $F[\underline{\delta}_b, \overline{\delta}_b]$. Because the function $\hat{\Pi}_b(q)$ is continuously differentiable, this property is equivalent to

$$\begin{aligned} &\hat{\Pi}_b'(q) = 0, && q \in F[\underline{\delta}_b, \overline{\delta}_b], \\ &\hat{\Pi}_b(\theta) \leq \Pi_b^\star = \hat{\Pi}_b(q) && q \in F[\underline{\delta}_b, \overline{\delta}_b], \theta \notin F[\underline{\delta}_b, \overline{\delta}_b]. \end{aligned}$$

The latter condition and (10.31) imply that

$$\frac{\Pi_b^\star}{\mathbf{B}(q)} = \int_q^{F(\delta^\star)} \frac{\ell(\theta)\mathbf{B}'(\theta)}{r\mathbf{B}'(\theta) + \gamma} d\theta, \qquad q \in F[\underline{\delta}_b, \overline{\delta}_b].$$

Substituting this identity into $\hat{\Pi}_b'(q)$ gives

$$\begin{aligned} 0 = \hat{\Pi}_b'(q) &= \mathbf{B}'(q) \int_q^{F(\delta^\star)} \frac{\ell(\theta)\mathbf{B}'(\theta)}{r\mathbf{B}'(\theta) + \gamma} d\theta - \frac{\ell(q)\mathbf{B}(q)\mathbf{B}'(q)}{r\mathbf{B}'(q) + \gamma} \\ &= \frac{\mathbf{B}'(q)\Pi_b^\star}{\mathbf{B}(q)} - \frac{\ell(q)\mathbf{B}(q)\mathbf{B}'(q)}{r\mathbf{B}'(q) + \gamma}, \qquad q \in F[\underline{\delta}_b, \overline{\delta}_b], \end{aligned}$$

from which we deduce that $\mathbf{B}$ solves the ODE (10.33b). Evaluating this equation at the endpoints of its domain and using (10.29) then shows that (10.36b) holds and that $\mathbf{B}$ satisfies (10.34b). Similar arguments establish the necessity of the corresponding properties of $\mathbf{A}$. □

PROOF OF SUFFICIENCY.

Assume that $(\mathbf{A}, \mathbf{B}; \delta^\star)$ satisfy the conditions of the theorem and let $(\underline{\delta}_b, \overline{\delta}_b, \underline{\delta}_a, \overline{\delta}_a)$ denote the corresponding thresholds. By Lemma 10.8 we have that the price distributions (A, B) defined by (10.30) satisfy the conditions of Propositions 10.6 and 10.7. On the other hand, substituting (10.30) into (10.13) shows that the market clearing condition (10.8) can be stated as

$$\begin{aligned} s = \Phi_1(\overline{\delta}) &= \int_{\underline{\delta}}^{\overline{\delta}_b} \frac{\gamma s f(x)dx}{\gamma + \lambda(1 - B(x))} \\ &+ \int_{\overline{\delta}_b}^{\underline{\delta}_a} s f(x)dx + \int_{\underline{\delta}_a}^{\overline{\delta}} \frac{(\gamma s + \lambda A(x)) f(x)}{\gamma + \lambda A(x)} dx \\ &= s\mathbf{B}(F(\overline{\delta}_b)) + s\left(F(\underline{\delta}_a) - F(\overline{\delta}_b)\right) + 1 - F(\underline{\delta}_a) - (1-s)\mathbf{A}(F(\underline{\delta}_a)), \end{aligned}$$

which is equivalent (10.35). To complete the proof it now remains to show that the price distributions are optimal, i.e., that $F[\underline{\delta}_a, \overline{\delta}_a]$ and $F[\underline{\delta}_b, \overline{\delta}_b]$ coincide with the set of maximizers of $\hat{\Pi}_a(q)$ and $\hat{\Pi}_b(q)$, respectively. Consider first the case of the bid distribution. Substituting (10.29) into (10.31) and using the differential equation (10.33b) we deduce that

$$\hat{\Pi}_b(q) = \begin{cases} \frac{\gamma q}{\gamma+\lambda}\left(\frac{\underline{\delta}_b - F^{-1}(q)}{r+\gamma+\lambda} + \frac{\Pi_b^\star}{\mathbf{B}(F(\underline{\delta}_b))}\right), & q \leq F(\underline{\delta}_b), \\ \Pi_b^\star & q \in F[\underline{\delta}_b, \overline{\delta}_b], \\ \frac{\delta^\star - F^{-1}(q)}{r+\gamma}\left(q + \mathbf{B}(F(\overline{\delta}_b)) - F(\overline{\delta}_b)\right), & q \geq F(\overline{\delta}_b). \end{cases} \tag{10.51}$$

Differentiating the first line at $q = F(\delta)$ shows that

$$\hat{\Pi}_b'(F(\delta)) \propto \pi_b(\delta) \equiv \underline{\delta}_b + \frac{(r+\gamma+\lambda)\Pi_b^\star}{\mathbf{B}(F(\underline{\delta}_b))} - \delta - \frac{F(\delta)}{f(\delta)}, \qquad \delta \in (\underline{\delta}, \underline{\delta}_b].$$

Since f is log-concave we have that F/f is non decreasing. Therefore, $\pi(q)$ is decreasing on $(\underline{\delta}, \underline{\delta}_b]$ and, since

$$\pi_b(\underline{\delta}_b) = \underline{\delta}_b + \frac{(r+\gamma+\lambda)\Pi_b^\star}{\mathbf{B}(F(\underline{\delta}_b))} - \underline{\delta}_b - \frac{F(\overline{\delta}_b)}{F'(\overline{\delta}_b)} = 0$$

due to (10.29) and (10.36b), we conclude that $\hat{\Pi}_b(q)$ is increasing on $[0, F(\underline{\delta}_b)]$. Next, differentiating the third line of (10.51) and using the boundary condition (10.34b) we obtain that

$$\text{sign}\left(\hat{\Pi}_b'(F(\delta))\right) = \text{sign}\left(\kappa_b(\overline{\delta}_b) - \kappa_b(\delta)\right), \qquad \delta \in [\overline{\delta}_b, \delta^\star],$$

with the function

$$\kappa_b(\delta) \equiv F(\delta) - f(\delta)\,(\delta^\star - \delta) = F(\delta)\left(1 - \frac{f(\delta)}{F(\delta)}(\delta^\star - \delta)\right).$$

It now suffices to show that $\kappa_b(\delta)$ is increasing on $[\overline{\delta}_b, \delta^\star]$. Since f is log concave, we have that f/F is decreasing. This, in turn, implies that the bracketed term in the definition of $\kappa_b(\delta)$ is increasing and, since $F(\delta)$ is strictly increasing, the desired result now follows by observing that

$$\begin{aligned}
\kappa(\overline{\delta}_b) = F(\overline{\delta}_b) - \mathbf{B}(F(\overline{\delta}_b)) &= F(\underline{\delta}_b) - \mathbf{B}(F(\underline{\delta}_b)) + \int_{F(\underline{\delta}_b)}^{F(\overline{\delta}_b)} \left(1 - \mathbf{B}'(\theta)\right) d\theta \\
&\geq F(\underline{\delta}_b) - \mathbf{B}(F(\underline{\delta}_b)) + \left(1 - \mathbf{B}'(\overline{\delta}_b)\right) \int_{F(\underline{\delta}_b)}^{F(\overline{\delta}_b)} d\theta \\
&= F(\underline{\delta}_b) - \mathbf{B}(F(\underline{\delta}_b)) = \frac{\lambda F(\underline{\delta}_b)}{\gamma+\lambda} > 0,
\end{aligned}$$

where the first equality is due to (10.34b), the inequality is due to the convexity of $\mathbf{B}$ on $F[\underline{\delta}_b, \overline{\delta}_b]$, the third equality is due (10.29), and the last equality follows from the combination (10.29) and $(\mathcal{P}_1)$.

Likewise, substituting (10.29) into (10.32) and using (10.33a) we deduce that the expected ask profit satisfies

$$\hat{\Pi}_a(q) = \begin{cases} \frac{F^{-1}(q)-\delta^\star}{r+\gamma}\left(F(\underline{\delta}_a) - q + \frac{(r+\gamma)\Pi^\star_a}{\underline{\delta}_a - \delta^\star}\right), & q \le F(\underline{\delta}_a), \\ \Pi^\star_a & q \in F[\underline{\delta}_a, \overline{\delta}_a], \\ \frac{\gamma(1-q)}{\gamma+\lambda}\left(\frac{F^{-1}(q)-\overline{\delta}_a}{r+\gamma+\lambda} + \frac{\Pi^\star_a}{\mathbf{A}(F(\overline{\delta}_a))}\right), & q \ge F(\overline{\delta}_a). \end{cases} \tag{10.52}$$

Differentiating the third line at $q = F(\delta)$ shows that

$$\hat{\Pi}'_a(F(\delta)) \propto \pi_a(\delta) \equiv \frac{1-F(\delta)}{f(\delta)} - \delta + \overline{\delta}_a - \frac{\Pi^\star_a}{\mathbf{A}(F(\overline{\delta}_a))}, \qquad \delta \in [\overline{\delta}_a, \overline{\delta}).$$

Since f is log-concave we have that $1-F$ is also log-concave and it follows that the ratio $(1-F)/f$ is non decreasing, see Proposition A.21. This implies that $\pi_a(q)$ is decreasing on the interval $[\overline{\delta}_a, \overline{\delta})$ and, since we also have

$$\pi_a(\overline{\delta}_a) = \frac{1-F(\overline{\delta}_a)}{f(\overline{\delta}_a)} - \frac{\Pi^\star_a}{\mathbf{A}(F(\overline{\delta}_a))} = 0$$

as a result of (10.29) and (10.36a), we conclude that $\hat{\Pi}_a(q)$ is decreasing on the interval $[F(\overline{\delta}_a), 1]$. Next, differentiating the first line of (10.52) and using the boundary condition (10.34a) we deduce that

$$\operatorname{sign}\left(\hat{\Pi}'_a(F(\delta))\right) = \operatorname{sign}\left(\kappa_a(\underline{\delta}_a) - \kappa_a(\delta)\right), \qquad \delta \in [\delta^\star, \underline{\delta}_a],$$

with the function

$$\kappa_a(\delta) \equiv F(\delta) + f(\delta)(\delta - \delta^\star) \tag{10.53}$$

and it remains to show that $\kappa_a(\delta)$ is increasing on $[\delta^\star, \underline{\delta}_a]$.

Since f is log-concave and strictly positive we know that it admits a *right* derivative f' that is nonnegative if and only $\delta \in (\underline{\delta}, m]$ for some $m \in \mathrm{cl}(\mathcal{D})$ and such that the ratio f'/f is decreasing. Differentiating (10.53) and using these properties we deduce that

$$\frac{\kappa'_a(\delta)}{f(\delta)} = 2 + \frac{f'(\delta)}{f(\delta)}(\delta - \delta^\star) \ge 2 + \frac{f'(\underline{\delta}_a)}{f(\underline{\delta}_a)}(\delta - \delta^\star), \qquad \delta \in [\delta^\star, \underline{\delta}_a].$$

If $\underline{\delta}_a \leq m$ then $f'(\underline{\delta}_a) \geq 0$ so that the right-hand side of this inequality is strictly positive an the proof is complete. If instead $\underline{\delta}_a > m$ then $f'(\underline{\delta}_a) \leq 0$ in which case the inequality implies that

$$\frac{\kappa_a'(\delta)}{f(\delta)} \geq 2 + \frac{f'(\underline{\delta}_a)}{f(\underline{\delta}_a)} \left(\underline{\delta}_a - \delta^\star\right), \qquad \delta \in [\delta^\star, \underline{\delta}_a].$$

To show that the right-hand side is positive, consider (10.33a). Since $\mathbf{A}(q)$ and $\ell(q)$ are convex on $F[\underline{\delta}_a, \overline{\delta}_a]$, they both admit right derivatives on that interval. Combining these properties with (10.33a) shows that $\mathbf{A}$ admits a second right derivative that satisfies

$$\begin{aligned} 0 \leq r\Pi_a^\star \mathbf{A}''(q) &= -\frac{d}{dq}\left(\ell(q)\mathbf{A}(q)^2\right) \\ &= -\ell'(q)\mathbf{A}(q)^2 - 2\mathbf{A}(q)\mathbf{A}'(q)\ell(q), \qquad \delta \in [\delta^\star, \underline{\delta}_a]. \end{aligned}$$

Evaluating this inequality at the point $q = F(\underline{\delta}_a)$ and substituting conditions (10.29) and (10.34a) we finally obtain that

$$\begin{aligned} 0 \leq r\Pi_a^\star \mathbf{A}''(\underline{\delta}_a) &= \frac{f'(\underline{\delta}_a)\mathbf{A}(F(\underline{\delta}_a))^2}{f(\underline{\delta}_a)^3} - \frac{2\mathbf{A}(F(\underline{\delta}_a))\mathbf{A}'(F(\underline{\delta}_a))}{f(\underline{\delta}_a)} \\ &= (\underline{\delta}_a - \delta^\star)\left(2 + \frac{f'(\underline{\delta}_a)}{f(\underline{\delta}_a)}\left(\underline{\delta}_a - \delta^\star\right)\right) \end{aligned}$$

and the proof is complete. □

10.4.3 Proof of Proposition 10.10

Using (10.36a) we deduce that

$$\begin{aligned} 0 = \lim_{\lambda\to\infty} \frac{\gamma(r+\gamma)}{(\gamma+\lambda)(r+\gamma+\lambda)} &= \lim_{\lambda\to\infty} \frac{f(\underline{\delta}_a)f(\overline{\delta}_a)\left(\underline{\delta}_a - \delta^\star\right)^2}{(1-F(\overline{\delta}_a))^2} \\ &\geq \lim_{\lambda\to\infty} \underline{\varepsilon}^2\left(\underline{\delta}_a - \delta^\star\right)^2 \end{aligned}$$

where $\underline{\varepsilon} > 0$ is a uniform lower bound on $f(\delta)$. This implies that $\underline{\delta}_a - \delta^\star \to 0$ and, circling back to (10.36a), implies that

$$0 \leq \lim_{\lambda\to\infty} \Pi_a^\star = \lim_{\lambda\to\infty} \frac{f(\underline{\delta}_a)\left(\underline{\delta}_a - \delta^\star\right)^2}{r+\gamma} \leq \lim_{\lambda\to\infty} \frac{\overline{\varepsilon}\left(\underline{\delta}_a - \delta^\star\right)^2}{r+\gamma} = 0,$$

where $\bar{\varepsilon} > 0$ is a uniform upper bound on $f(\delta)$. Applying similar arguments to the expected profits from offering a bid price reveals that

$$\lim_{\lambda\to\infty}\left(\delta^\star - \bar{\delta}_b\right) = \lim_{\lambda\to\infty}\Pi_b^\star = 0,$$

and it follows that $\bar{\delta}_b, \underline{\delta}_a$, and $\delta^\star$ converge to a common limit $\hat{\delta}$. To determine this limit, note that the definition of $(\mathbf{A},\mathbf{B})$ and the consistency condition (10.7) imply that

$$\Phi_1(\delta) = \begin{cases} s\mathbf{B}(F(\delta)), & \delta \leq \bar{\delta}_b, \\ s(F(\delta) - F(\bar{\delta}_b) + \mathbf{B}\left(F(\bar{\delta}_b)\right), & \delta \in (\bar{\delta}_b, \underline{\delta}_a), \\ F(\delta) + (1-s)(\mathbf{A}(F(\delta)) - 1), & \delta \geq \underline{\delta}_a. \end{cases}$$

Letting the contact rate $\lambda \to \infty$ on both sides of this equality, and using the fact that $\underline{\delta}_a - \bar{\delta}_b \to 0$, reveals that

$$\begin{aligned}\lim_{\lambda\to\infty}\Phi_1(\delta) &= \lim_{\lambda\to\infty}\mathbf{1}_{\{\delta\leq\bar{\delta}_b\}}s\mathbf{B}(F(\delta)) \\ &\quad + \lim_{\lambda\to\infty}\mathbf{1}_{\{\delta\geq\bar{\delta}_a\}}\left(F(\delta) + (1-s)\left(\mathbf{A}(F(\delta)) - 1\right)\right).\end{aligned}$$

Since the functions $\mathbf{B}$ and $-\mathbf{A}$ are decreasing it follows from (10.34) and the first part of the proof that we have

$$\begin{aligned}0 \leq \lim_{\lambda\to\infty}\mathbf{1}_{\{\delta\leq\bar{\delta}_b\}}\mathbf{B}(F(\delta)) &\leq \lim_{\lambda\to\infty}\mathbf{1}_{\{\delta\leq\bar{\delta}_b\}}\mathbf{B}(F(\bar{\delta}_b)) \\ &\leq \lim_{\lambda\to\infty}\mathbf{1}_{\{\delta\leq\bar{\delta}_b\}}\bar{\varepsilon}\left(\delta^\star - \bar{\delta}_b\right) = 0, \\ 0 \leq \lim_{\lambda\to\infty}\mathbf{1}_{\{\delta\geq\bar{\delta}_a\}}\mathbf{A}(F(\delta)) &\leq \lim_{\lambda\to\infty}\mathbf{1}_{\{\delta\geq\bar{\delta}_a\}}\mathbf{A}(F(\underline{\delta}_a)) \\ &\leq \lim_{\lambda\to\infty}\mathbf{1}_{\{\delta\geq\bar{\delta}_a\}}\bar{\varepsilon}\left(\underline{\delta}_a - \delta^\star\right) = 0,\end{aligned}$$

and combining these inequalities we conclude that

$$\begin{aligned}\lim_{\lambda\to\infty}\Phi_1(\delta) &= \lim_{\lambda\to\infty}\mathbf{1}_{\{\delta\geq\bar{\delta}_a\}}\left(F(\delta) - (1-s)\right) \\ &= \mathbf{1}_{\{\delta\geq\hat{\delta}\}}\left(F(\delta) - (1-s)\right). \end{aligned} \tag{10.54}$$

Since the left-hand side is nonnegative it follows that $\hat{\delta} \geq \delta^{\#}$. Alternatively, combining (10.54) and (10.7) shows that

$$\lim_{\lambda\to\infty} \Phi_0(\delta) = F(\delta) - \lim_{\lambda\to\infty} \Phi_1(\delta) = F(\delta) + \mathbf{1}_{\{\delta \geq \hat{\delta}\}} \left(1 - s - F(\delta)\right).$$

Since the left-hand side is nondecreasing we deduce that $F(\hat{\delta}) \leq 1 - s$ which is equivalent to $\hat{\delta} \leq \delta^{\#}$. Substituting $\hat{\delta} = \delta^{\#}$ into (10.54) gives (10.39) and it remains to establish the claim regarding reservation values.

Integrating by parts in (10.5) and using the results of Lemmas 10.1 and 10.2 we deduce that the smallest and largest reservation values, respectively, satisfy

$$\begin{aligned} rR(\underline{\delta}) &= \underline{\delta} + \int_{\underline{\delta}}^{\overline{\delta}_b} \frac{\gamma(s - F(\delta)\Phi_1'(\delta))}{\gamma s + r\Phi_1'(\delta)} d\delta \\ &\quad + \int_{\overline{\delta}_b}^{\underline{\delta}_a} \underline{\kappa}(\delta) d\delta + \int_{\underline{\delta}_a}^{\overline{\delta}} \frac{\gamma\delta(1 - F(\delta))(1 - \Phi_1'(\delta))}{\gamma(1 - s) + r(1 - \Phi_1'(\delta))} d\delta \end{aligned} \tag{10.55}$$

$$\begin{aligned} rR(\overline{\delta}) &= \overline{\delta} - \int_{\underline{\delta}}^{\overline{\delta}_b} \frac{\gamma F(\delta)\Phi_1'(\delta)}{\gamma s + r\Phi_1'(\delta)} d\delta \\ &\quad - \int_{\overline{\delta}_b}^{\underline{\delta}_a} \overline{\kappa}(\delta) d\delta - \int_{\underline{\delta}_a}^{\overline{\delta}} \frac{\gamma(1 - F(\delta))\Phi_1'(\delta) + \gamma(F(\delta) - s)}{\gamma(1 - s) + r(1 - \Phi_1'(\delta))} d\delta \end{aligned} \tag{10.56}$$

for some uniformly bounded functions $\underline{\kappa}(\delta)$ and $\overline{\kappa}(\delta)$. Before passing to the limit in this expression we claim that

$$\lim_{\lambda\to\infty} \Phi_1'(\delta) = \Psi_1'(\delta) \equiv \mathbf{1}_{\{\delta \geq \delta^{\#}\}} f(\delta). \tag{10.57}$$

Indeed, (10.39) implies that

$$0 = \lim_{\lambda\to\infty} \left(\Phi_1(\delta) - \Psi_1(\delta)\right) = \lim_{\lambda\to\infty} \int_{\underline{\delta}}^{\delta} \left(\Phi_1'(x) - \Psi_1'(x)\right) dx$$

for all $\delta \in \mathcal{D}$ and, since the integrand is uniformly bounded from above and below, we may apply Fatou's lemma twice to obtain

$$\int_{\underline{\delta}}^{\delta} \limsup_{\lambda\to\infty} \left(\Phi_1'(x) - \Psi_1'(x)\right) dx \leq 0 \leq \int_{\underline{\delta}}^{\delta} \liminf_{\lambda\to\infty} \left(\Phi_1'(x) - \Psi_1'(x)\right) dx$$

from which the claim follows. Now, letting $\lambda \to \infty$ in (10.55)–(10.56) and using (10.57) together with (10.40) and the dominated convergence theorem we obtain that

$$rR(\underline{\delta}) = \underline{\delta} + \int_{\underline{\delta}}^{\delta^{\#}} d\delta = \delta^{\#} = \overline{\delta} - \int_{\delta^{\#}}^{\overline{\delta}} d\delta = rR(\overline{\delta}),$$

and the desired result follows by observing that the reservation value function is non decreasing. □

10.5 EXERCISES

Exercise 10.12. Assume that in some equilibrium there exists $\alpha, \beta \in \operatorname{supp}(B)$ such that the interval $(\alpha, \beta) \notin \operatorname{supp}(B)$.

1. Show that there are constants $M_0, N_0 > 0$ such that

   $$\Pi_b(\delta) = (M_0 + M_1(\delta))\,(N_0 + N_1(\delta))$$

   for all $\alpha \in [\alpha, \beta]$ where

   $$(M_1(\delta), N_1(\delta)) \equiv \left(\frac{\gamma\,(F(\delta) - F(\alpha))}{r + \lambda(1 - B(\alpha))}, \frac{\beta - \delta}{r + \gamma + \lambda(1 - B(\alpha)))}\right).$$

2. Show that the condition $\Pi_b(\alpha) = \Pi_b(\beta)$ is equivalent to

   $$M_0 N_1(\alpha) = N_0 M_1(\beta)$$

3. Show that the condition $\Pi_b'(\alpha) = \Pi_b'(\beta) = 0$ is equivalent to the system of equations:

   $$N_1'(\alpha) M_0 + M_1'(\alpha) N_0 = -M_1'(\alpha) N_1(\alpha) \tag{10.58}$$
   $$N_1'(\beta) M_0 + M_1'(\beta) N_0 = -M_1(\beta) N_1'(\beta). \tag{10.59}$$

4. Equations (10.58)–(10.59) can be viewed as a linear system of two equations for the two unknown constants (M_0, N_0). Show that this system admits a solution only if $f(\alpha) \neq f(\beta)$.

5. Apply Cramer's rule to the system (10.58)–(10.59) to calculate M_0/N_0 and use the result to show that

$$f(\alpha)f(\beta)\,(\beta-\alpha)^2 = (F(\beta)-F(\alpha))^2\,.$$

is necessary for $\Pi_b(\alpha)-\Pi_b(\beta)$ and $\Pi_b'(\alpha)=\Pi_b'(\beta)=0$.

Exercise 10.13. Assume that $F(\delta)$ is continuously differentiable. Show that the equilibrium bid and ask distributions converge to single point distributions to be specified as either the contact rate $\lambda \to 0$ or the reset rate $\gamma \to \infty$.

Exercise 10.14. Show that the limits in (10.38), (10.39), and (10.40) also hold as the reset rate $\gamma \to 0$. What happens to reservation values in that limit?

Exercise 10.15. Denote by $B(\delta)$ an arbitrary distribution on the set $\mathcal{D}$ and define the function $\mathbf{B}(q)$ as in equation (10.28).

1. Show that if B is continuous then $\mathbf{B}$ is strictly convex on $\mathrm{supp}(B)$.
2. Show that if B is supported on $[\underline{\delta}_b,\overline{\delta}_b]\subset(\underline{\delta},\delta^\star)$ then $\mathbf{B}$ satisfies (10.29) .
3. Conversely, show that if $\mathbb{B}:[0,1]\to[0,1]$ with $\mathbb{B}(0)=0$ is continuously differentiable, strictly convex on some interval $F[\underline{\delta}_b,\overline{\delta}_b]\subset(0,F(\delta^\star))$, and such that (10.29) is satisfied then

$$\hat{B}(\delta)\equiv 1+\frac{\gamma}{\lambda}\left(1-\frac{1}{\mathbb{B}'(F(\delta))}\right)$$

defines a continuous distribution that is supported on $[\underline{\delta}_b,\overline{\delta}_b]$.

10.6 NOTES AND REFERENCES

The material of this chapter is to karge extent drawn from Hugonnier, Lester, and Weill [2024a], which incorporates the search problem with asymmetric information over private values [first studied by Albrecht and Axell, 1984] into the semi-centralized OTC market model. Other studies that introduce asymmetric information over private values into search theoretic models include Gehrig [1993], Spulber [1996], Zhu [2011], Cujean and Praz [2013], Mäkinen and Palazzo [2017], Garrett, Gomes, and Maestri [2018], Lester et al. [2018], Akın and Platt [2022], and Glebkin, Shen, and Yueshen [2019].

Two closely related papers are are Bethune, Sultanum, and Trachter [2019] and Zhang [2017]. Bethune et al. [2019] study the trading and issuance of assets in a decentralized market, much like the interdealer market of Chapter 7, but assume that participants do not observe each others utility flows. Zhang [2017] studies a semi-centralized market with private information, more similar to the model studied in this chapter, but only considers the non-generic case in which the distribution of utility flows admits a density that is symmetric around the asset supply $s = 1/2$. A key difference in both papers is that, instead of assuming that one party quotes a price upon meeting, Bethune et al. [2019] and Zhang [2017] both adopt a mechanism design approach. Specifically, these authors assume that one of the parties offers a direct mechanism that specifies a price $P(\delta)$ and probability of execution $\pi(\delta)$ as functions of the utility flow reported by the other party and rely on incentive compatibility to elicit truthful revelation. In such an approach, the terms of trade are known in advance and uncertainty affects the trade execution. By contrast, in our formulation the price itself is random, and this randomness generates uncertainty over the trade execution. The fact that investors have quasilinear preferences implies that the two approaches are equivalent despite this difference; Zhang [2017] notes this connection but does not provide a differential characterization of the equilibrium.

Finally, note that the competitive search model of Section 8.3 in Chapter 8 already accommodates asymmetric information about private values. That is, in the equilibrium we constructed, it is not necessary to assume that dealers observe investors' utility flows when they post quotes. Instead, dealers in that model post quotes and have rational expectations about the utility flows of the investors who find it optimal to choose their quote.