Chapter Five

Pure decentralized markets

The semi-centralized OTC market model of the previous chapter is a good benchmark for dealer-intermediated markets where frictions in the inter-dealer market are small relative to frictions between investors and dealers. However, important examples of OTC markets such as the fed funds markets are not dealer-intermediated. Moreover, even if there are dealers it is often the case that the inter-dealer market itself is subject to significant trading frictions.

To accomodate the analysis of such markets we need to construct a model of a two-tiered OTC markets where both dealers and investors are subject to trading frictions. As a first step in this direction, we develop in this chapter a model of a market where participants can trade bilaterally subject to search and bargaining frictions. In the next chapter, we use a version of this market to model the inter-dealer segment of the market in a wider semi-centralized OTC market where investors face search and bargaining friction, thereby obtaining a full fledged model of frictional intermediation.

5.1 THE MODEL

As in previous chapters, we consider a continuous-time, infinite-horizon model of an economy populated by a unit mass of risk-neutral investors who discount the future at rate $r > 0$ and assume that the market consists in a single durable asset with fixed supply equal to $s \in (0, 1)$.

Each investor in the market can hold either 0 or 1 unit of the asset and

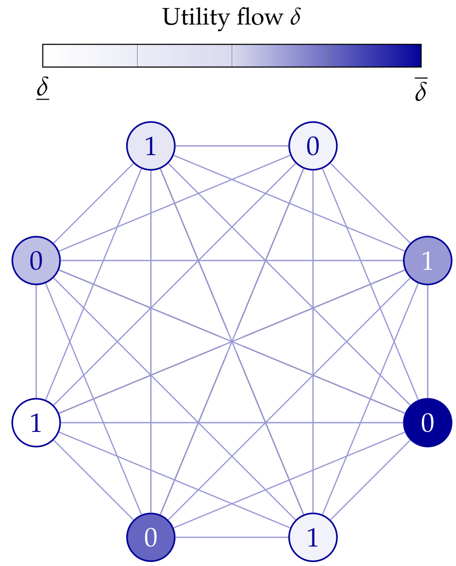

FIGURE 5.1: Decentralized OTC market

This figure illustrates the structure of a decentralized over-the-counter market in which heterogenous investors holding 0 or 1 unit of the asset meet each other randomly over time to bargain over the terms of a possible trade.

enjoys a utility flow $\delta_t dt$ from holding the asset at time $t \geq 0$. The utility flow of each investor is independent from that of any other investor and follows a Markov jump process with arrival rate γ and reset distribution $F(\delta)$ on some open interval $\mathscr{D} = (\underline{\delta}, \overline{\delta})$. As in the previous chapter we assume that preference shocks are iid across investors, so that the cross-sectional distribution of utility flows in the population is equal to $F(\delta)$ at all times.

Unlike the previous chapter, we assume that investors trade bilaterally as illustrated in Figure 5.1. Specifically, we assume that investors participate in a fully decentralized market where each investor is matched with another at the jump times of a Poisson process with intensity λ. Conditional on contact, two investors with complementary holdings determine the terms of a possible trade by solving a Nash bargaining game in which the seller has bargaining power $\theta_1 \in (0, 1)$ and the buyer has bargaining power $\theta_0 = 1 - \theta_1$.

We denote by $V(q, \delta)$ the value function of an investor in state (q, δ) and by $\Phi_q(\delta)$ the measure of investors who hold $q \in \{0,1\}$ units of the asset and have utility flow $\delta' \leq \delta$. These cumulative measures are to be determined endogenously in equilibrium subject to

$$\Phi_0(\delta) + \Phi_1(\delta) = F(\delta), \qquad \forall \delta \in \mathscr{D}, \tag{5.1}$$

and the market clearing condition

$$\Phi_1(\bar{\delta}) \equiv \lim_{\delta \uparrow \bar{\delta}} \Phi_1(\delta) = s \tag{5.2}$$

which ensure, respectively, that the cross-sectional distribution of utility flows is consistent with the population distribution $F(\delta)$ and that the mass of owners equals the asset supply.

5.2 OPTIMAL TRADING BY INVESTORS

5.2.1 Bargaining between investors

Consider a match between an investor with utility flow δ_1 who owns the asset (the *seller*) and an investor with utility flow δ_0 who does not (the *buyer*). If a trade occurs at price P, then the seller receives utility $P + V(0, \delta_1)$. Otherwise, she remains an owner until her next meeting with another investor and receives utility $V(1, \delta_1)$. Hence, the seller's net utility from trading is

$$(P + V(0, \delta_1)) - V(1, \delta_1) = P - R(\delta_1)$$

where, as before, $R(\delta)$ denotes the reservation value function. By the same token, the buyer's net utility from trading is

$$(V(1, \delta_0) - P) - V(0, \delta_0) = R(\delta_0) - P.$$

Therefore, there are gains from trade in the match if and only if the reservation value of the buyer exceeds that of the seller in which case the realized price maximizes the Nash product

$$P(\delta_0, \delta_1) = \underset{P \in [R(\delta_1), R(\delta_0)]}{\operatorname{argmax}} \left(P - R(\delta_1)\right)^{\theta_1} \left(R(\delta_0) - P\right)^{\theta_0}.$$

Since the bargaining powers $\theta_q \in (0,1)$ we have that the objective function is concave in P on the feasible interval. Therefore, the first order condition

$$0 = \frac{d}{dP}\left(P - R(\delta_1)\right)^{\theta_1}\left(R(\delta_0) - P\right)^{\theta_0}$$

is necessary and sufficient for an interior maximum, and solving this equation shows that the bid price is given by a convex combination

$$P(\delta_0, \delta_1) = \theta_0 R(\delta_1) + \theta_1 R(\delta_0). \tag{5.3}$$

of reservation values. If $\theta_1 \approx 1$ then the seller has most of the bargaining power and the price is close to the reservation value of the buyer, which is the best price possible within the match from her point of view. On the contrary, if $\theta_1 \approx 0$ then the seller has almost no bargaining power and, as a result, the price approaches his reservation value.

5.2.2 Reservation values: Basic properties

Fix a pair of cumulative measures $(\Phi_0(\delta), \Phi_1(\delta))$ satisfying the consistency and market clearing conditions (5.1) and (5.2). Proceeding as in Chapter 3.3.1 shows that the reservation value function of an investor who takes these cumulative measures as given satisfies the Bellman equation:

$$\begin{aligned} R(\delta) = \mathbf{E}\Bigg[\int_0^\tau \delta e^{-rt} dt + e^{-r\tau}\Bigg(&\mathbf{1}_{\{\tau=\tau_\gamma\}}\int_{\underline{\delta}}^{\overline{\delta}} R(x)\, dF(x) \\ &+ \mathbf{1}_{\{\tau=\tau_0\}}\int_{\underline{\delta}}^{\overline{\delta}} \max\{R(\delta), P\left(x,\delta\right)\}\frac{d\Phi_0(x)}{1-s} \\ &+ \mathbf{1}_{\{\tau=\tau_1\}}\int_{\underline{\delta}}^{\overline{\delta}} \min\{R(\delta), P\left(\delta,x\right)\}\frac{d\Phi_1(x)}{s}\Bigg)\Bigg] \end{aligned} \tag{5.4}$$

where τ_γ is an exponential random variable with rate γ that represents the time until the arrival of a preference shock, τ_q is an exponential random variable with rate

$$\lambda\Phi_q(\overline{\delta}) = \begin{cases} \lambda s & \text{if } q = 1 \\ \lambda(1-s) & \text{if } q = 0 \end{cases}$$

that counts the time until the next meeting with a randomly selected investor holding q units of the asset, and $\tau = \min\{\tau_0, \tau_1, \tau_\gamma\}$.

Next, substituting in the bargained price defined in (5.3) shows that the Bellman equation can equivalently be written as

$$R(\delta) = \mathbf{E}\Bigg[\int_0^\tau \delta e^{-rt}dt + e^{-r\tau}\Bigg(\mathbf{1}_{\{\tau\neq\tau_\gamma\}}R(\delta) + \mathbf{1}_{\{\tau=\tau_\gamma\}}\int_{\underline{\delta}}^{\overline{\delta}} R(x)dF(x) + \mathbf{1}_{\{\tau=\tau_0\}}\int_{\underline{\delta}}^{\overline{\delta}} \theta_1\left(R(x)-R(\delta)\right)^+ \frac{d\Phi_0(x)}{1-s} - \mathbf{1}_{\{\tau=\tau_1\}}\int_{\underline{\delta}}^{\overline{\delta}} \theta_0\left(R(x)-R(\delta)\right)^- \frac{d\Phi_1(x)}{s}\Bigg)\Bigg]. \tag{5.5}$$

This equation shows that, in contrast with the semi-centralized market of the previous chapter, reservation values in the decentralized market model depend explicitly on the joint cross-sectional distribution of asset holdings and utility flows, as summarized by the measures $\Phi_0(\cdot)$ and $\Phi_1(\cdot)$. To understand this difference, observe that in the semi-centralized market investors trade with a group of homogeneous counterparties (dealers) who all have access to the same outside options through the frictionless inter-dealer market. As a result, both the rate and the price at which an investor trades do not depend explicitly on the measures $\Phi_0(\cdot)$ and $\Phi_1(\cdot)$ in the semi-centralized market. This is no longer true in the decentralized market because, conditional on an investor's current utility flow, heterogeneity in the reservation values of potential counterparties affects both the likelihood and the terms of trades.

A direct implication of the above observation is that the joint distribution of asset holdings and utility flows—which is an infinite-dimensional object—is a state variable for reservation values. In most cases, this would make the analysis considerably more complicated, but not here because, as shown in the following proposition, the optimal trading strategy is purely ordinal.

Proposition 9. *Equation* (5.4) *admits a unique solution and this function is bounded, strictly increasing, and Lipschitz continuous on $\mathscr{D}$. In particular, the optimal strategy of an investor is to buy from any owner whose utility flow is smaller than her own, and to sell to any now owner whose utility flow is larger than her own.*

Proof. First note that this result does not follow from Proposition 3 and Corollary 1 since the trading terms in (5.4) depend on the reservation value function not only at δ but at all utility flow levels. As a result, we do not even know a priori that the reservation value function is bounded and have to establish that property from first principles.

Integrating on both sides of (5.5) against the distribution of the random time τ and simplifying the result shows that $R : \mathscr{D} \to \mathbf{R}$ is a solution to (5.4) if and only if it is a fixed point of the operator defined by

$$T[g](\delta) \equiv \frac{1}{r}\left(\delta + U[g](\delta)\right) = \frac{1}{r}\Bigg[\delta + \gamma \int_{\underline{\delta}}^{\overline{\delta}} (g(x) - g(\delta))\, dF(x) \\ + \lambda\theta_1 \int_{\underline{\delta}}^{\overline{\delta}} (g(x) - g(\delta))^+ \, d\Phi_0(x) - \lambda\theta_0 \int_{\underline{\delta}}^{\overline{\delta}} (g(x) - g(\delta))^- \, d\Phi_1(x)\Bigg].$$

To establish that any such fixed point is strictly increasing, assume towards a contradiction that $R : \mathscr{D} \to \mathbf{R}$ is a fixed point of T such that $R(\delta) \leq R(\delta')$ for some $\delta > \delta'$. Since the maps $y \mapsto \pm (R(x) - y)^{\pm}$ are both weakly decreasing we have that

$$R(\delta) \leq R(\delta') \Rightarrow U[R](\delta) \geq U[R](\delta') \tag{5.6}$$

and therefore

$$R(\delta) = \frac{1}{r}\left(\delta + U[g](\delta)\right) > \frac{1}{r}\left(\delta + U[R](\delta')\right) = R(\delta')$$

where the first and last equalities from the fixed point property. This provides the required contradiction and thus shows that any fixed point of T is strictly increasing. Combining this property with (5.6) we then deduce that any fixed point of T satisfies

$$0 \leq R(\delta) - R(\delta') = \frac{1}{r}\left(\delta - \delta' + U[R](\delta) - U[R](\delta')\right) \leq \frac{1}{r}\left(\delta - \delta'\right)$$

for all $\delta' \leq \delta$ and interchanging the role the roles of δ and δ' establishes that any such function is Lipschitz continuous with modulus no larger than $1/r$. Since $\mathscr{D}$ is bounded this in turn implies that any fixed point of T is bounded and it now only remains to show that such a fixed point is unique.

Assume that $R : \mathscr{D} \to \mathbf{R}$ is a fixed point of T. Since such a function is bounded and strictly increasing we have that

$$U[R](\delta) = V[R](\delta) - R(\delta)V[1](\delta), \qquad \forall \delta \in \mathscr{D},$$

with the operator

$$V[g](\delta) \equiv \int_{\underline{\delta}}^{\overline{\delta}} g(x)\Big(\gamma dF(x) + \lambda\theta_1 \mathbf{1}_{\{x\geq\delta\}} d\Phi_0(x) + \lambda\theta_0 \mathbf{1}_{\{x\leq\delta\}} d\Phi_1(x)\Big),$$

and it follows that R is a bounded fixed point of

$$S[g](\delta) \equiv \frac{1}{r + V[1](\delta)}\left(\delta + V[g](\delta)\right).$$

As is easily seen, S maps the space $\mathscr{B}$ of bounded functions into itself. On the other hand, a direct calculation shows that if $(g,h) \in \mathscr{B}^2$ then

$$\|S[g] - S[h]\| = \left\| \frac{V[g] - V[h]}{r + V[1]} \right\| \leq \left\| \frac{V[1]}{r + V[1]} \right\| \|g - h\|.$$

Since $r > 0$ and $\gamma \leq V[1] \leq \gamma + \lambda$ this shows that S is a contraction on $\mathscr{B}$ and the existence and uniqueness of a fixed point now follows from the Banach fixed point theorem 27. □

5.2.3 Reservation values: Explicit expressions

Building on the necessary conditions of Proposition 9 we now derive explicit expressions for the reservation value function of an investor who takes the measures $\Phi_0(\cdot)$ and $\Phi_1(\cdot)$ as given. These diffrent expressions are convenient for analyzing the effects of changes in the environment on reservation values and the subsequent implications for equilibrium prices. In addition, comparing these expressions to those of the previous chapter delivers key insights about the effects of search frictions.

5.2.3.1 The sequential representation

Using the strict increase of the reservation value function and proceeding as in Chapter 3.4.2 shows that the HJB equation can be written as

$$rR(\delta) = \delta + \gamma \int_{\underline{\delta}}^{\overline{\delta}} (R(x) - R(\delta))\, dF(x) + \lambda\theta_1 \int_{\delta}^{\overline{\delta}} (R(x) - R(\delta))\, d\Phi_0(x) \\ + \lambda\theta_0 \int_{\underline{\delta}}^{\delta} (R(x) - R(\delta))\, d\Phi_1(x). \tag{5.7}$$

This expression highlights once again that in the presence of search frictions the reservation value of an investor is affected by her utility flow, the arrival of preference shocks, and two distinct options: the option of re-selling the asset at a later date, which increases her reservation value; and the foregone option value of buying at a later date, which depresses it.

To derive a sequential (or expected value) representation of reservation values consider the family of conditional distribution defined by

$$\hat{F}(x|\delta) = \frac{\Pi(x,\delta)}{\Pi(\overline{\delta},\delta)}, \qquad (x,\delta) \in \mathscr{D}^2, \tag{5.8}$$

with the function

$$\Pi(x,\delta) \equiv \gamma F(x) + \lambda\theta_1 \left(\Phi_0(x) - \Phi_0(\delta)\right)^+ + \lambda\theta_0 \Phi_1(x \wedge \delta). \tag{5.9}$$

In terms of these objects, the HJB equation for the reservation value function can now be written as

$$rR(\delta) = \delta + \Pi(\overline{\delta},\delta) \int_{\underline{\delta}}^{\overline{\delta}} (R(x) - R(\delta))\, d\hat{F}(x|\delta),$$

and combining this equation with Proposition 25 of Appendix A.5 delivers the following explicit representation of the unique solution.

Proposition 10. *The reservation value function is given by*

$$R(\delta) = \mathbf{E}_\delta \left[\int_0^\infty e^{-rt} \hat{\delta}_t dt \right] \tag{5.10}$$

where the market valuation $(\hat{\delta}_t)_{t \geq 0}$ is a Markov jump process with state dependent jump rate $\hat{\lambda}(\delta) \equiv \Pi(\overline{\delta},\delta)$ and reset distribution $\hat{F}(x|\delta)$.

As in the semi-centralized market environment we refer to $\hat{\delta}_t$ as the *market valuation* process because it takes into account not only the preference shocks that affect an investor, but also her trading opportunities given the search and bargaining friction at play in he market. Specifically, over a small time interval of length h the market valuation in the decentralized market can change for three reasons. First, a preference shock arrives with probability γh, in which case a new utility flow is drawn from $F(x)$. Second, a sale is executed with a probability $\lambda\theta_1(1 - s - \Phi_0(\delta))h$ that combines the physical arrival rate of a buyer with utility flow higher than δ and the share θ_1 of the surplus that the seller retains. Third, a purchase occurs with a probability $\lambda\theta_0\Phi_1(\delta)h$ that combines the physical arrival rate $\lambda\Phi_1(\delta)h$ of a seller with utility flow lower than δ and the share θ_0 of the surplus that the buyer retains.

Comparing the market valuation of Proposition 10 to its semi-centralized market counterpart in Proposition 8 shows that bilateral trading induces serial correlation into the market valuation process. This occurs because, conditional on an investor's current utility flow, the heterogeneity in the utility flows of her potential counterparties in the decentralized market directly influences the likelihood and the terms of possible trades.

5.2.3.2 *The differential representation*

Since the reservation value function is increasing and Lipschitz continuous by Proposition 9 we know that

$$R(x) - R(\delta) = \int_{\delta}^{x} \sigma(x)dx \tag{5.11}$$

for some *bounded*, but not necessarily continuous, function $\sigma(x) \geq 0$ that we interpret as a measure of the *local* trading surplus in a neighbourhood of utility flow x. Differentiating both sides of (5.7) and solving the resulting equation shows that the local surplus is

$$\sigma(\delta) = \frac{1}{r + \gamma + \lambda\theta_1(1 - s - \Phi_0(\delta)) + \lambda\theta_0\Phi_1(\delta)}. \tag{5.12}$$

Observing that the denominator in this expression is the sum of the discount rate r and the jump rate $\Pi(\overline{\delta}, \delta)$ of the market valuation process reveals that the local surplus can be viewed as the present value

$$\sigma(\delta) = \mathbf{E}_\delta \left[\int_0^{\hat{\tau}} e^{-rt} dt \right]$$

of a unit flow of utility to be received until the random time $\hat{\tau}$ of the first jump in the market valuation process. Such a representation of the local surplus also holds in the semi-centralized market of the previous chapter but Proposition 8 shows that in that case the jump rate of the market valuation process is given by $\gamma + \lambda_\theta$ so that the local surplus $1/(r + \gamma + \lambda_\theta)$ is constant. Here again, the difference between the two settings can be traced to the fact that investors in the semi-centralized market trade with an homogenous counterparty and thus only face timing uncertainty, whereas investors in the decentralized market face uncertainty regarding both the timing of trading opportunities and the characteristics of future trading partners.

Substituting (5.11) into (5.7) and changing the order of integration delivers an explicit expression for the reservation value function stated exclusively in terms of exogenous variables and the joint distribution of asset holdings and utility flows that investors take as given. We summarize this expression in the following proposition.

Proposition 11. *The reservation value function is*

$$R(\delta) = \frac{\delta}{r} - \int_{\underline{\delta}}^{\delta} \sigma(x) \Pi(x, \delta) \frac{dx}{r} + \int_{\delta}^{\overline{\delta}} \sigma(x) \left(\Pi\left(\overline{\delta}, \delta\right) - \Pi(x, \delta) \right) \frac{dx}{r}$$

with $\Pi(x, \delta)$ *and* $\sigma(\delta)$ *as in* (5.9) *and* (5.12).

5.3 EQUILIBRIUM

In this section, we characterize the steady state distribution of asset holdings and utility flows in the population of investors and derive the unique steady state equilibrium of the decentralized market model.

5.3.1 Equilibrium distributions

In a steady state equilibrium the cumulative measures $\Phi_0(\cdot)$ and $\Phi_1(\cdot)$ must be consistent with the optimal trading behavior of individual investors and with stationarity in that the inflow to and outflow from any given group of investors must coincide at all times. Proceeding as in Chapter 4.4.2 we will uniquely determine these cumulative measures by expliciting these conditions for a well-chosen groups of investors.

Consider the group of owners with utility flow in $(\underline{\delta}, \delta]$. The measure of this group is $\Phi_1(\delta)$ and stationarity requires that

$$\gamma\Phi_1(\bar{\delta})F(\delta) + \lambda \int_{\underline{\delta}}^{\delta} \Phi_1(x)d\Phi_0(x) = \gamma\Phi_1(\delta) + \lambda \int_{\underline{\delta}}^{\delta} \left(\Phi_0(\bar{\delta}) - \Phi_0(x)\right) d\Phi_1(x). \tag{5.13}$$

The gross inflow on the left consists in two the terms. The first gives the inflow of owners who are hit by a preference shock that resets their utility flow to some new value $x \leq \delta$. The second gives the inflow from trading which equals the flow of non owners with utility flow $x \leq \delta$ who meet an owner with an even lower utility flow and acquire the asset from her. Similarly, the gross outflow on the right is the sum of the flow of investors who exit because they are hit by a preference shock and the flow of owners with utility flow $x \leq \delta$ who meet a non owner with a higher utility flow and sell their asset.

Next, we simplify this equation by changing the order of integration in the last term on the right-hand side of (5.13):

$$\begin{aligned}
\lambda \int_{\underline{\delta}}^{\delta} \left(\Phi_0(\bar{\delta}) - \Phi_0(x)\right) d\Phi_1(x) &= \lambda \int_{\underline{\delta}}^{\delta} \left(\int_{\underline{\delta}}^{\bar{\delta}} \mathbb{I}_{\{x \leq y\}} d\Phi_0(y)\right) d\Phi_1(x) \\
=&\lambda \int_{\underline{\delta}}^{\bar{\delta}} \left(\int_{\underline{\delta}}^{\delta} \mathbb{I}_{\{x \leq y\}} d\Phi_1(x)\right) d\Phi_0(y) = \lambda \int_{\underline{\delta}}^{\bar{\delta}} \Phi_1(\min\{y, \delta\}) d\Phi_0(y) \\
=&\lambda \int_{\underline{\delta}}^{\delta} \Phi_1(y)d\Phi_0(y) + \lambda\Phi_1(\delta)\left(\Phi_0(\bar{\delta}) - \Phi_0(\delta)\right).
\end{aligned}$$

One sees that the first term in the equation above cancels out with the last term on the left-hand side of (5.13). The intuition is that trades between an owner with utility $x \leq \delta$ and a non-owner with utility $x \leq y \leq \delta$ create no net inflow into the set of owners with utility less than δ: the x investor sells and leave the set, while the y investor buys and enter the set. On net, there is only an outflow due to trade, due to owners with utility $x \leq \delta$ who sell to non-owner with utility $y > \delta$. The resulting inflow outflow equation thus simplifies to:

$$\gamma\Phi_1(\bar{\delta})F(\delta) + \lambda \int_{\underline{\delta}}^{\delta} \Phi_1(x)d\Phi_0(x) = \gamma\Phi_1(\delta) + \lambda\Phi_1(\delta)\left(1 - s - \Phi_0(\delta)\right).$$

Substituting the consistency condition (5.1) and the market clearing condition (5.2) shows that in a steady state equilibrium we must have

$$-\lambda\Phi_1(\delta)\left(1 - s - F(\delta) + \Phi_1(\delta)\right) + \gamma\left(sF(\delta) - \Phi_1(\delta)\right) = 0. \tag{5.14}$$

Importantly, this equation is independent from $\Phi_1(x)$ for all $x \neq \delta$, and holds without imposing any regularity conditions on the underlying distribution of utility flows, $F(\delta)$. In particular, this equation is valid for discrete distributions, continuous distributions, or mixture distributions; with or without transient states. By contrast, differentiating (5.14) reveals that the equations for *measures* (instead of the cumulative measures) do exhibit dependences across different utility flows which makes the solution more difficult to derive and analyze in all but the simplest cases, see Exercize 19 for details.

The next proposition shows that (5.14) uniquely pins downs the equilibrium distribution and provides comparative statics.

Proposition 12. *In a steady state equilibrium*

$$\Phi_1(\delta) = F(\delta) - \Phi_0(\delta) = \ell(F(\delta)) \tag{5.15}$$

with the function

$$\ell(x) \equiv -\tfrac{1}{2}\left(1 - s + \phi - x\right) + \sqrt{\phi s x + \tfrac{1}{4}\left(1 - s + \phi - x\right)^2} \tag{5.16}$$

and the constant $\phi \equiv \gamma/\lambda$. In particular, the measure $\Phi_1(\delta)$ is increasing in the asset supply s as well as increasing and concave in ϕ.

Proof. The definition of ϕ and (5.14) imply that for any utility flow $\delta \in \mathscr{D}$ the measure $x = \Phi_1(\delta)$ is a solution to

$$x^2 + x\left(1 - s + \phi - F(\delta)\right) - \phi s F(\delta) = 0.$$

The first part then follows by deriving the unique positive solution to this quadratic equation and the second by differentiating this solution. □

As in the semi-centralized market setting, the steady state distributions of utility flows among owners and non owners only depend on the arrival rates of meetings and type switches through the ratio $\phi = \gamma/\lambda$ that measures the severity of the frictions at play in the market. In line with this interpretation, Proposition 12 shows that as ϕ increases, either because meetings become less frequent or because investors change utility flows more often, the equilibrium allocation becomes less efficient in that owners tend to have lower utility flows. Furthermore, writing (5.14) as

$$\Phi_1(\delta) = \frac{\phi s F(\delta)}{1 - s - \Phi_0(\delta) + \phi}$$

reveals that the distribution of utility flows among asset owners converges to $sF(\delta)$ as $\phi \to \infty$ so that ownership and utility flows become independent. To understand this result observe that in this limit it must be that either $\gamma \to \infty$ so that utility flows become infinitely volatile, or $\lambda \to 0$ so that trading becomes impossible. In either case the market is unable to re-allocate the asset across investors and, as a result, misallocation builds up unchecked until the point where the asset is uniformly distributed across investors.

On the other hand, (5.15) implies that $\Phi_1(\delta)$ converges to its frictionless counterpart in (4.2) as $\phi \to 0$, i.e.

$$\lim_{\phi \to 0} \Phi_1(\delta) = \Psi_1(\delta) = (F(\delta) - (1-s))^+ . \tag{5.17}$$

This asymptotic efficiency result is intuitive. Indeed, in this limit it must be that either $\lambda \to \infty$ so that the search friction vanishes, or that $\gamma \to 0$ so that utility flows become constant in which case the OTC market will always achieve the efficient allocation given enough time.

To illustrate the effect of the search friction on the equilibrium distributions, Figure 5.2 plots the cumulative measures $\Phi_q(\delta)$ as functions of $\delta \in \mathscr{D} = (0,1)$ for different values of ϕ in the setting of Example 11 where the asset supply $s = 0.4$ and the distribution of utility flows is uniform. The figure confirms that the decentralized market becomes more efficient as ϕ decreases in the sense that the distribution of utility flows among owners gradually shifts down and to the right and reaches the frictionless allocation in the limit where investors either

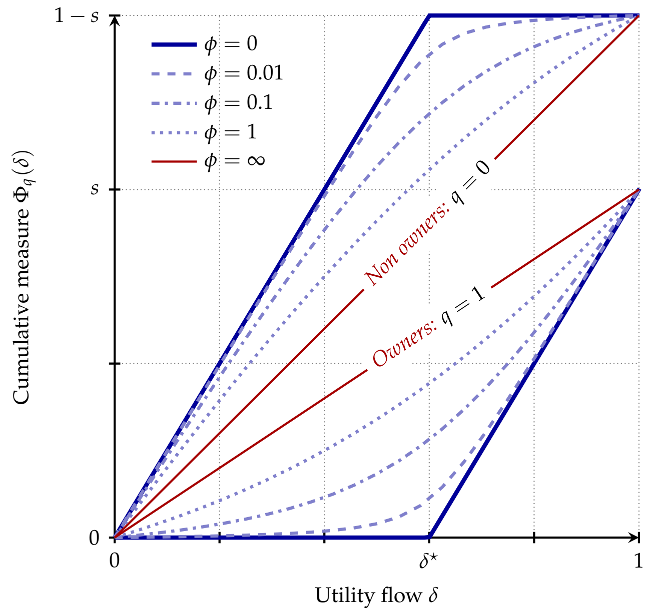

FIGURE 5.2: Equilibrium allocation in the decentralized market

This figure illustrates the equilibrium distribution of utility flows among owners and non owners for different values of the ratio $\phi = \gamma/\lambda$ in a decentralized market model with asset supply $s = 0.4$ and a uniform distribution of utility flows on $\mathscr{D} = (0,1)$. A similar but less lisible picture arises if the asset supply $s \geq 0.4$ because in that case the distributions partially overlap.

trade instantly or have permanent utility flows. From a quantitative point of view, the figure shows that the deviations from the efficient allocation can be sizable. For example, even if $\phi = 1$, so that investors meet and change type at the same frequency (dotted lines), the mere asynchronicity of these events impies that almost half of the asset supply is held by investors who would not be owners absent frictions.

5.3.2 Convergence to the efficient allocation

To further analyze the effect of the friction on the equilibrium allocation it us useful to determine the exact rate at which the equilibrium allocation converges to the efficient allocation. Consider first utility flows $\delta < \delta^\star$ where

$$\delta^\star = \inf\{\delta \in \mathscr{D} : F(\delta) \geq 1 - s\}$$

is the *marginal type* of Section 4.2, i.e. the threshold above which all investors are owners in the frictionless equilibrium. Writing (5.14) as

$$\frac{1}{\phi}\Phi_1(\delta) = \frac{sF(\delta)}{1 - s - \Phi_0(\delta) + \phi}$$

and passing to the limit shows that

$$\Phi_1(\delta) = \Psi_1(\delta) + \frac{sF(\delta)}{1 - s - F(\delta)} \cdot \phi + o(\phi) \tag{5.18a}$$

where the equality follows from the fact that for types $\delta < \delta^\star$ the cumulative measure $\Phi_0(\delta) \to \Psi_0(\delta) = F(\delta)$ as $\phi \to 0$. Consider next utility flows strictly above the marginal type. In that case, writing (5.14) as

$$\frac{1}{\phi}\left(1 - s - F(\delta) + \Phi_1(\delta)\right) = \frac{sF(\delta) - \Phi_1(\delta)}{\Phi_1(\delta)}$$

and passing to the limit shows that

$$\Phi_1(\delta) = \Psi_1(\delta) - \frac{(1-s)(1-F(\delta))}{1 - s - F(\delta)}\phi + o(\phi) \tag{5.18b}$$

where the equality follows from the fact that for types $\delta > \delta^\star$ the cumulative measure $\Phi_1(\delta) \to \Psi_1(\delta) = F(\delta) - (1-s)$ as $\phi \to 0$.

Finally, consider the convergence *at* the marginal type. If the underlying distribution is discontinuous at $\delta^\star$ then $F(\delta^\star) > 1 - s$ and it follows that (5.18b) continues to hold. If instead the underlying distribution is continuous at the marginal type then

$$\frac{1}{\phi}\left(\Phi_1(\delta^\star)\right)^2 = \frac{1}{\phi}\Phi_1(\delta^\star)\left(1 - s - \Phi_0(\delta^\star)\right) = s(1-s) - \Phi_1(\delta^\star)$$

where the second equality follows from (5.14) evaluated at the marginal type, and passing to the limit shows that

$$\Phi_1(\delta^\star) = \sqrt{s(1-s)\phi} + o(\sqrt{\phi}). \tag{5.18c}$$

This expansion reveals that when the underlying distribution is continuous at the marginal type—so that there heterogeneous investors whose utility flow lie in a neighbourhood of $\delta^\star$—then the convergence to the efficient allocation is much slower at the marginal type than away from it. We will see in the next section that this slower convergence arises because of an equilibrium feedback loop between the investors' trading intensities and the distribution of utility flows among owner and non owners.

5.3.3 Summary of the equilibrium

A full characterization of the unique steady state equilibrium readily follows from the above results. In particular, the equilibrium is comprised of the joint distributions $\Phi_q(\delta)$ of utility flows and ownership status in Proposition 12 and of the induced reservation value function $R(\delta)$ in Proposition 11. Note that uniqueness follows from the fact that we established the monotonicity of the reservation values function from first principles taking arbitrary distributions as given, rather than guessing and verifying that such an equilibrium exists, as is often done in the literature.

5.4 EQUILIBRIUM IMPLICATIONS

In the remainder of this chapter, we exploit the explicit characterization of the steady-state equilibrium to explore the model's implications.

As a first step, we study how the trading intensity of an individual investor depend on his utility flow and ownership type. A key result is that investors with more extreme utility flows trade quickly, but less frequently, and most often in the same direction: Low utility flows mostly sell while high utility flows mostly buy. In contrast, investors with moderate utility flows around the marginal type trade more frequently and in both directions.

These individual trading patterns have important aggregate implications. In particular, we show that investors with utility flows near the marginal type of the frictionless benchmark are more likely to be involved in any given trade and, therefore, account for a disproportionate amount of trading volume. As a result, even though the search process that generates meetings is random, the aggregate patterns of trade that emerge in equilibrium are not. Specifically, we establish that a *core-periphery* trading network emerges endogenously despite the fact that investors ex-ante homogeneous.

5.4.1 Trading intensities

Consider a non owner with utility flow process δ_t. In equilibrium, the arrival rate of a (weakly) profitable trading opportunity for this specific investor, i.e. his *buying intensity*, is the random process defined by the product

$$\lambda_0(\delta_t) \equiv \lambda\Phi_1(\delta_t) \tag{5.19a}$$

of the arrival rate of a meeting and the probability that the counterparty is an owner with a lower utility flow. Similarly, the arrival rate of profitable trading opportunities, or *selling intensity*, for an asset owner with utility flow process δ_t is given by the product

$$\lambda_1(\delta_t) \equiv \lambda\left(\Phi_0(\bar{\delta}) - \Phi_0(\delta_t)\right) = \lambda\left(1 - s - \Phi_0(\delta_t)\right) \tag{5.19b}$$

of the arrival rate of a meeting and the probability that the counterparty is a non-owner with a higher utility flow. Since $\Phi_1(\delta)$ is non-decreasing in utility flow, non owners with higher utility flows have a higher buying intensity, and thus tend to remain asset-less for shorter periods. By the same logic, since the measure $1 - s - \Phi_0(\delta)$ is non-increasing in utility flow, we naturally have that asset owners with higher utility flows trade less often, and thus tend to hold the asset over longer periods of time.

The following lemma provides intuite comparative static results for the equilibrium trading intensities.

Lemma 1. *The trading intensity is increasing in s for buyers, decreasing in s for seller, and increasing in λ for both buyers and sellers.*

Proof. A direct calculation using (5.19) and (5.1) shows that

$$\lambda\frac{\partial^2\lambda_q(\delta)}{\partial\lambda^2} = \phi^2\frac{\partial^2\Phi_1(\delta)}{\partial\phi^2},$$

and we know from Proposition 12 that $\Phi_1(\delta)$ is concave in ϕ. Therefore, the trading intensities are concave in λ with

$$\begin{aligned}\frac{\partial \lambda_q(\delta)}{\partial \lambda} &\geq \lim_{\lambda\to\infty} \frac{\partial \lambda_q(\delta)}{\partial \lambda} \\ &= q\,(1-s-F(\delta)) + \lim_{\phi\to 0}\left\{\Phi_1(\delta) - \phi\frac{\partial \Phi_1(\delta)}{\partial \phi}\right\} \\ &\geq q\,(1-s-F(\delta)) + \Psi_1(\delta) \\ &= q\,(1-s-\Psi_0(\delta)) + (1-q)\Psi_1(\delta) \geq 0\end{aligned}$$

where the first equality follows from the definition of ϕ, the second inequality follows from (5.17) and the fact that $\Phi_1(\delta)$ is concave as a function of ϕ, and the last equality follows from the market clearing condition.

The positive relation between the supply and the buying intensity follows from Proposition 12. Turning to the selling intensity we note that

$$\frac{\partial^2 \lambda_1(\delta)}{\partial s^2} = \lambda \frac{\partial^2 \ell(F(\delta))}{\partial s^2} = \frac{2\gamma F(\delta)(1-F(\delta))}{\left((1-s+\phi-F(\delta))^2 + 4s\phi F(\delta)\right)^{3/2}} \geq 0.$$

This shows that the selling intensity is convex in s and combining this property with (5.15) shows that

$$\frac{\partial \lambda_1(\delta)}{\partial s} \leq \left.\frac{\partial \lambda_1(\delta)}{\partial s}\right|_{s=1} = \frac{\gamma(F(\delta)-1)}{\phi+F(\delta)} \leq 0$$

which completes the proof. □

To illustrate the above results, Figure 5.3 plots the trading intensities for different meeting intensities ranging from 4 (quarterly meetings) to 52 (weekly meetings) in the same environment as in Figure 5.2. The figure confirms the monotonicity of the trading intensities and highlights the role of the marginal type. In particular, it shows that, as λ increases, the trading intensities fall more and more sharply for asset owners above the marginal type and for non-owners below. To understand this result, recall from Section 5.3.2 that the equilibrium asset allocation approaches that of frictionless benchmark when the meeting rate is sufficiently large. Since

$$\Psi_1(\delta^\star - \varepsilon) = 1 - s - \Psi_0(\delta^\star + \varepsilon) = 0, \qquad \varepsilon > 0$$

this convergence implies that owners with utility flow $\delta \gg \delta^\star$ and non owners

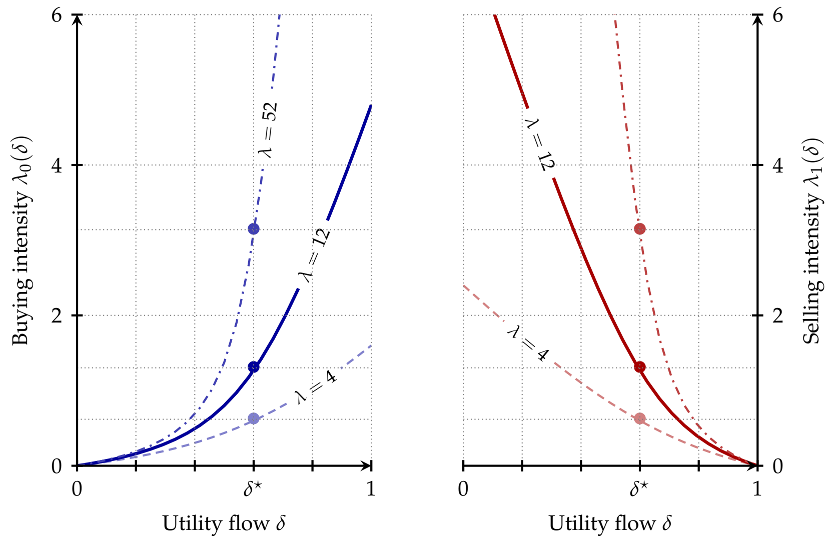

FIGURE 5.3: Equilibrium trading intensities

This figure plots the buying (left panel) and selling (right panel) intensities as functions of utility flow for different values of the meeting intensity λ in a decentralized market model with asset supply $s = 0.4$, arrival rate of preference shocks $\gamma = 1$, and a uniform distribution of utility flows on the set $\mathscr{D} = (0, 1)$. On each curve the $\bullet$ indicates the trading intensity at the marginal type.

with utility flow $\delta \ll \delta^\star$ have essentially no willing counterparties to trade with, and it follows that their trading intensity are very low.

By contrast, the figure shows that owners with utility flow $\delta \ll \delta^\star$ and non owners with utility flows $\delta \gg \delta^\star$ both trade fast because their have many willing counterparties, and also reveals that the intensities cross at the marginal type. Indeed, if the distribution of utility flows is continuous at $\delta^\star$ as in the model of the figure, then $F(\delta^\star) = 1 - s$ and it follows that

$$\begin{aligned}\lambda_0(\delta^\star) = \lambda\Phi_1(\delta^\star) &= \lambda\left(F(\delta^\star) - \Phi_0(\delta^\star)\right) \\ &= \lambda\left(1 - s - \Phi_0(\delta^\star)\right) = \lambda_1(\delta^\star).\end{aligned}$$

Hence, buyers and sellers whose utility flow are close to the marginal type will, on average, trade at similar speeds in equilibrium.

Taken together, these trading patterns show that an investor's current type endogenously determines his role in the market: those with extreme utility flows emerge as natural *customers* who trade infrequently and mostly in the same direction; while those with more moderate utility flows near the marginal type emerge as natural *intermediaries* who buy and sell more frequently and with approximately equal intensities.

5.4.2 Misallocation

The above trading patterns have important implications for *misallocation*, defined as the extent to which the equilibrium asset allocation differs from its frictionless counterpart. To elicit these implications

consider the cumulative measure defined by

$$\begin{aligned} M(\delta) \equiv \int_0^\delta \mathbf{1}_{\{x<\delta^\star\}} d\Phi_1(x) + \int_0^\delta \mathbf{1}_{\{x\geq\delta^\star\}} d\Phi_0(x) \\ = \Phi_1\left(\min\left\{\delta, \delta^\star\right\}\right) + \left(\Phi_0(\delta) - \Phi_0(\delta^\star)\right)^+ \end{aligned}$$

This cumulative measure is the sum of two types of misallocation: the mass of investors with utility flow less than $\min\{\delta, \delta^\star\}$ who would not be allocated an asset in a frictionless environment, but are in the presence of search frictions; and the mass of investors with utility flow less than δ but higher than $\delta^\star$ who would own the asset in a frictionless market environment, but do not in the presence of search frictions.

By Proposition 12 we have that $\Phi_1(\delta)$ is increasing in ϕ for all $\delta \in \mathscr{D}$ and it follows that the total mass of misallocated assets

$$\mathbf{m} \equiv M(\bar{\delta}) = 1 - s - F(\delta^\star) + 2\Phi_1(\delta^\star)$$

is naturally increasing in the severity of the search friction. See the left panel of Figure 5.4 for an illustration in the same environment as in Figure 5.3.

The following result establishes that misallocation concentrates around the marginal type as the search friction vanishes.

Proposition 13. *In equilibrium*

$$\lim_{\phi\to\infty} \frac{M(\delta^\star + a) - M(\delta^\star - b)}{\mathbf{m}} = 1 \tag{5.20}$$

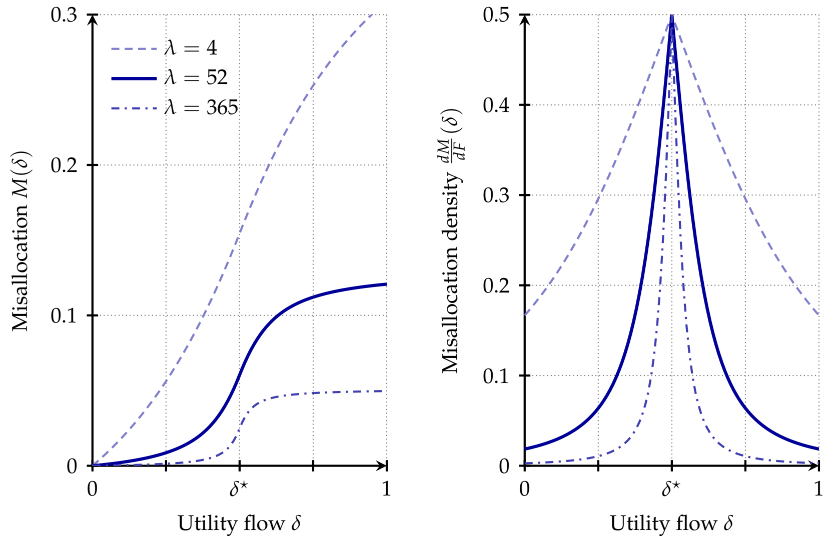

FIGURE 5.4: Equilibrium misallocation

This figure plots the misallocation (left panel) and the misallocation density (right panel) as functions of utility flow for different values of the meeting rate λ in a decentralized market model with asset supply $s = 0.5$, arrival rate of preference shocks $\gamma = 1$, and a uniform distribution of utility flows on the set $\mathscr{D} = (0, 1)$.

for any constants $a, b > 0$ such that $(\delta^\star - b, \delta^\star + a) \subset \mathscr{D}$.

Proof. Equations (5.18a), (5.18b) and (5.18c) imply that as ϕ approaches zero

$$\begin{aligned}
\Phi_0(\delta^\star + a) - \Phi_0(\delta^\star) &\approx C_0\phi + \mathbf{1}_{\{\Delta F(\delta^\star)=0\}}\sqrt{s(1-s)\phi} \\
\Phi_1(\delta^\star - b) - \Phi_1(\delta^\star) &\approx 1 - s - F(\delta^\star) + C_1\phi + \mathbf{1}_{\{\Delta F(\delta^\star)=0\}}\sqrt{s(1-s)\phi} \\
\mathbf{m} &\approx 1 - s - F(\delta^\star) + C_2\phi + \mathbf{1}_{\{\Delta F(\delta^\star)=0\}}\sqrt{s(1-s)\phi}
\end{aligned}$$

for some constants C_i that depend on $(a, b, s, \delta^\star)$ but not on ϕ, and combining these approximations gives (5.20). $\square$

Misallocation clusters around the marginal type due to the presence of an equilibrium feedback loop between the cross-sectional distributions in Figure 5.2 and the trading intensities in Figure 5.3. Indeed, the monotonicity of the trading intensities implies that owners with utility flow $\delta \ll \delta^\star$ and non owners with utility flow $\delta \gg \delta^\star$ are both able to trade quickly. As a result, there are very few such investors in equilibrium and, therefore, little misallocation away from the marginal type. This implies that asset owners just below the marginal type and non owners just above have few compatible counterparties and thus experience a low trading intensity which in turn induces a higher degree of misallocation in the vicinity of the marginal type.

To further illustrate the fact that misallocation tends to cluster around the marginal type we can compute the Radon-Nikodym density

$$m(\delta) \equiv \frac{dM}{dF}(\delta) = \lim_{x \uparrow \delta} \frac{M(\delta) - M(x)}{F(\delta) - F(x)} \tag{5.21}$$

which captures the *fraction* of investors with utility flow δ whose asset holdings with search frictions differ from their frictionless counterpart. Proposition 12 implies that, when the underlying distribution of utility flows is continuous, this density is simply given by

$$m(\delta) = \mathbf{1}_{\{\delta < \delta^\star\}} \ell'(F(\delta)) + \mathbf{1}_{\{\delta \geq \delta^\star\}} \left(1 - \ell'(F(\delta))\right)$$

with the function $\ell(x)$ defined in (5.16). Since this function is convex it is clear that the misallocation density is single peaked with a maximum either at the marginal type or immediately before, which confirms that investors near $\delta^\star$ are those among which misallocation is strongest. In fact, the definition of the function $\ell(x)$ implies that the misallocation density jumps by

$$\Delta m(\delta^\star) = 1 - 2\ell'(1-s) = \frac{(1-2s)\sqrt{\phi}}{\sqrt{\phi + 4s(1-s)}}$$

so that the density is continuous at $\delta^\star$ if and only if $s = 0.5$ as in Figure 5.4 or $\phi = 0$ and reaches its maximum at the marginal type if and only if $s \leq 0.5$.

To conclude this section, note that the above properties of misallocation arise in a decentralized market environment because the equilibrium trading intensities differ across utility flows. Indeed, when all investors trade with equal speed as in the semicentralized market of Chapter **??** the misallocation measure is proportional to the underlying distribution of utility flows and, as a result, its density is constant across utility flows.

5.4.3 Trading volume

The concentration of misallocation translates into a concentration of trading volume among the group of investors with utility flows near the marginal type. To see this, consider the function

$$V(I) \equiv \lambda \int_{I^2} \mathbf{1}_{\{\delta_1 \le \delta_0\}} d\Phi_0(\delta_0) d\Phi_1(\delta_1) \tag{5.22}$$

which represents the flow rate of weakly profitable trades between investors with utility flows in the set $I \subseteq \mathscr{D}$. When the underlying distribution of utility flows is continuous, we can use integration by parts to write the aggregate trading volume as

$$\begin{aligned}\mathbf{v} \equiv V(\mathscr{D}) = \lambda\Phi_1(\delta^\star)(1 - s - \Phi_0(\delta^\star)) + \lambda \int_{\underline{\delta}}^{\delta^\star} (\Phi_0(\delta^\star) - \Phi_0(\delta))\, dM(\delta) \\ + \lambda \int_{\delta^\star}^{\overline{\delta}} (\Phi_1(\delta) - \Phi_1(\delta^\star))\, dM(\delta),\end{aligned}$$

with the cumulative misallocation measure $M(\delta)$ defined in (5.21). The first term on the right capture trades between owners in $[0, \delta^\star]$ and non-owners in $[\delta^\star, 1]$. Absent search frictions these would be the only trades executed. With search frictions, however, there are additional infra-marginal trades, captured by the second and third terms. In particular, the second term accounts for trades between owners with utility flows $\delta < \delta^\star$ and non-owners in $[\delta, \delta^\star]$, while the third term accounts for trades between non-owners with utility flows $\delta > \delta^\star$ and owners in $[\delta^\star, \delta]$. This formula highlights the role of misallocation in generating excess volume and suggests that near-marginal investors, who are subject to greater misallocation, are likely to have a larger contribution to volume. This is confirmed in the next proposition.

Proposition 14. *If the distribution of utility flows is continuous then the trading volume is increasing in λ and explicitly given by*

$$\mathbf{v} = \mathbf{v}^* \equiv \gamma s(1-s) \left[(1+\phi) \log\left(1 + \frac{1}{\phi}\right) - 1 \right].$$

In general, trading volume is such that

$$\lim_{\lambda\to\infty} \mathbf{v} = \infty \quad \textit{and} \quad \lim_{\lambda\to\infty} \frac{V(I)}{\mathbf{v}} = 1$$

for any interval $I = (a,b] \subseteq \mathscr{D}$ that strictly contains $\delta^\star$.

Proof. By Proposition 12 we have that $\Phi_1(\delta) = F(\delta) - \Phi_0(\delta) = \ell(F(\delta))$ for some increasing and continuously differentiable function. Therefore, it follows from (5.22) and the classical change of variable formula for Lebesgue–Stieljes integrals (see, e.g., **?**]) that

$$\mathbf{v} = \lambda \int_{\underline{\delta}}^{\overline{\delta}} \Phi_1(\delta) d\Phi_0(\delta) = \lambda \int_0^1 \ell(F\circ Q(x)) \left(1 - \ell'(x)\right) dx \tag{5.23}$$

where $Q(x) \geq x$ denotes the lowest quantile of the underlying distribution of utility flows at the level $x \in [0,1]$. If the underlying distribution is continuous then $F \circ Q(x) = x$ and substituting into (**??**) shows that

$$\mathbf{v} = \lambda \int_0^1 \ell(x)dx - \lambda \int_0^1 \ell(x)\ell'(x)dx = \lambda \int_0^1 \ell(x)dx - \frac{1}{2}\lambda s^2$$

where the last equality follows from $\ell(0) = s - \ell(1) = 1$. Computing the integral then gives $\mathbf{v} = \mathbf{v}^*$ and the remaining claims in the first part of the statement follow from the definition of $\mathbf{v}^*$.

Let now I be as in the statement and assume that the distribution of utility flows is arbitrary. Since $Q(x) \geq x$ and $\ell \circ F(\delta)$ is non decreasing it follows from (5.23) and the first part of the proof that we have

$$\mathbf{v} \geq \lambda \int_0^1 \ell(x) \left(1 - \ell'(x)\right) dx = \mathbf{v}^*.$$

This shows that $\lim_{\lambda\to\infty} \mathbf{v} = \infty$ independently of the underlying distribution and it now suffices to prove that the volume of trades executed by investors with utility flows in the set $I^c = \mathscr{D}\backslash I$ remains bounded as $\lambda \to \infty$. The assumed structure of I ensures that

$$V(I^c) = \lambda \int_{\underline{\delta}}^{a} \Phi_1(\delta)d\Phi_0(\delta) + \lambda \int_b^{\overline{\delta}} \left(\Phi_1(a) + \Phi_1(\delta) - \Phi_1(b)\right) d\Phi_0(\delta).$$

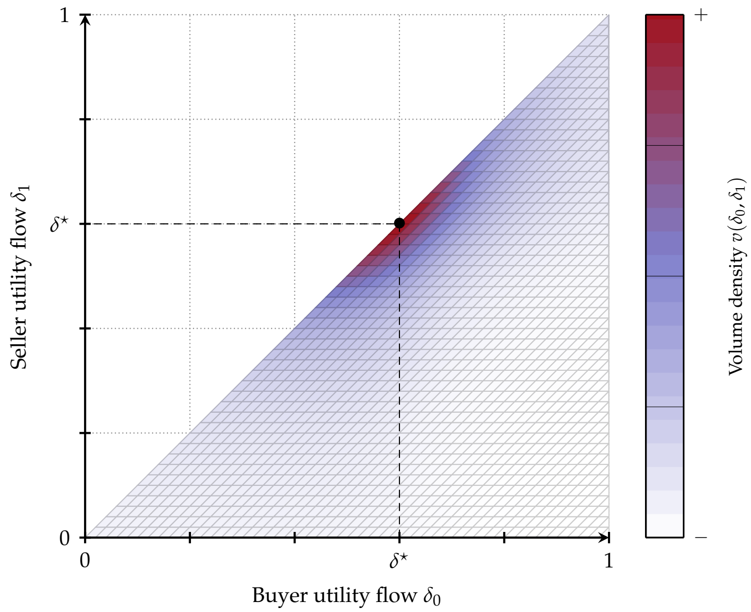

FIGURE 5.5: Contribution to trading volume

This figure plots the volume density as a function of the buyer's and sellers' utility flows in a decentralized market model with $\phi = \gamma/\lambda = 0.05$, asset supply $s = 0.4$, and a uniform distribution of utility flows on the unit interval.

for some constants $a < \delta^\star < b$. This in turn implies that

$$V(I^c) \leq \lambda \Phi_1(a)(1-s) + \lambda(1-s-\Phi_0(b))s$$

and the desired result now follows by letting $\lambda \to \infty$ on both sides and using the asymptotic expansions in (5.18a) and (5.18b) . □

In addition to providing a closed-form expression in the continuous case, Proposition 14 establishes two key results. First, trading volume is unbounded as $\lambda \to \infty$. By contrast, the trading volume per unit of time is finite in the equilibrium of the frictionless benchmark. Therefore, the fully decentralized market can generate arbitrarily large excess volume relative to the frictionless benchmark, as long as search frictions are sufficiently benign.

Second, trading volume is for the most part generated by investors near the marginal type who assume the role of intermediaries and, as a result, the trading network has an endogenous core-periphery structure. To illustrate this phenomenon, Figure 5.5 plots the contribution

$$v(\delta_0, \delta_1) = \mathbf{1}_{\{\delta_1 \leq \delta_0\}} \frac{d\Phi_0}{dF}(\delta_0) \frac{d\Phi_1}{dF}(\delta_1) = \mathbf{1}_{\{\delta_1 \leq \delta_0\}} \ell'(F(\delta_1))(1 - \ell'(F(\delta_0))).$$

of each owner/non-owner pair to the equilibrium trading volume in our base case model. From the figure, one can see that investors with extreme utility flow account for a small fraction of trades and, therefore, lie at the periphery of the trading network. For example, owners with low utility flows may trade quickly, but there are very few such owners in equilibrium and so they contribute little to the trading volume. Likewise, there are many owners with high utility flows, but these investors trade very slowly and thus do not account for many trades in equilibrium. Only in the group of investors with near-marginal utility flows do we find a sufficiently large fraction of individuals who are *both* holding the wrong portfolio and able to meet suitable counterparties at a reasonably high rate, and these are the investors that make up the core of the network.

Remark 17. Our definitions of trading intensities and trading volume in (5.19) and (5.22) assume that meetings in which investors are indifferent between trading and not trading actually result in a trade. This assumption is clearly without loss of generality when the underlying distribution is continuous but may otherwise affect the conclusions of Proposition 14. For example, if $F(\delta)$ is a sum of point masses and we assume that meetings between indifferent investors only lead to trade with some sufficiently small probability then the trading volume no longer diverges as search frictions vanish.

5.5 ASYMPTOTIC PRICE EFFECTS

We close this chapter by briefly studying the convergence of the equilibrium trading prices as the meeting rate $\lambda \to \infty$. This analysis is important for two reasons. First, this fast trading limit is the empirically relevant case in many relevant markets, where trading speeds are finite but steadily becoming larger and larger. Second, this analysis highlights the effect of investor heterogeneity on equilibrium prices. In particular, we show that heterogeneity magnifies the price impact of frictions, and that this impact is more pronounced on price levels than on price dispersion.

Since the proofs of the results in this section are lengthy and technical but rather elementary we omit them and instead refer the interested reader to the appendices of Hugonnier, Lester, and Weill [35] for details.

5.5.1 The frictionless limit

As a first step, we establish that the reservation values of all investors converge to the frictionless equilibrium price as search frictions vanish.

Lemma 2. $\lim_{\lambda\to\infty} R(\delta) = p^\star = \delta^\star/r$ *for all* $\delta \in \mathscr{D}$.

To understand this result, consider the market-valuation process defined in Proposition 10. From the analysis of Section 5.3.2 we know that as frictions vanish it becomes very easy for an investor with utility flow $\delta < \delta^\star$ ($\delta > \delta^\star$) to sell (buy) but a lot more difficult to buy (sell). Specifically, (5.18) shows that the trading intensities are such that

$$\lim_{\lambda\to\infty} \lambda_q(\delta) = \infty \iff (1-2q)\,(\delta-\delta^\star) \geq 0,$$

and combining this property with (5.8) and (5.9) reveals that the jump rate and reset distribution of the market valuation process satisfy

$$\lim_{\lambda\to\infty} \big(1/\hat{\lambda}(\delta)\big) = \lim_{\lambda\to\infty} \Big(\hat{F}(x|\delta) - \mathbf{1}_{\{x\leq\delta^\star\}}\Big) = 0, \quad \forall (x,\delta) \in \mathscr{D}^2.$$

This in turn implies that the market valuation process converges to the constant $\delta^\star$ as frictions vanish, and it now follows from (5.10) that all reservation values converge to the frictionless equilibrium price.

5.5.2 Price level near the frictionless limit

To analyze the behavior of prices near the frictionless limit, we then study the rate at which the market-valuation process converges to the marginal type. This analysis yields the following result.

Proposition 15. *Assume that* $F(\delta)$ *is continuously differentiable in a neighbourhood* *of $\delta^\star$ on which $F'(\delta)$ is bounded away from zero. Then,*

$$R(\delta) = p^\star + \frac{\pi/r}{F'(\delta^\star)}\left(\frac{1}{2} - \theta_0\right)\left(\frac{\gamma s(1-s)}{\theta_0\theta_1}\right)^{\frac{1}{2}}\frac{1}{\sqrt{\lambda}} + o\left(\frac{1}{\sqrt{\lambda}}\right) \tag{5.24}$$

for all utility flows $\delta \in \mathscr{D}$.

The first term in the expansion follows directly from Lemma 2. The main result of the proposition is the second term which determines the deviation of reservation values from the frictionless price. To calculate this term, we center the market valuation process around the marginal type and scale the result by $\sqrt{\lambda}$. This delivers an auxiliary process $\hat{x}_t \equiv \sqrt{\lambda}(\hat{\delta}_t - \delta^\star)$ whose limiting distribution can be derived explicitly, and the second term of the expansion is then obtained by calculating the limit of

$$\sqrt{\lambda}\,(R(\delta) - p^\star) = \mathbf{E}_{\sqrt{\lambda}(\delta-\delta^\star)}\left[\int_0^\infty e^{-rt}\hat{x}_t dt\right]$$

As can be seen from (5.24) the price deviation depends on three features of the decentralized market model. The first is the equilibrium mass of misallocated assets as measured by the term

$$\frac{1}{2}\mathbf{m} = \Phi_1(\delta^\star) \approx \left(\frac{\gamma s(1-s)}{\lambda}\right)^{\frac{1}{2}}.$$

The second is the relative bargaining power of investors which determines the sign of the price deviation: If $\theta_0 > 1/2$ then the asset is traded at a discount and if $\theta_0 < 1/2$ it is traded at a premium. When buyers and sellers have equal bargaining powers, the correction term vanishes so that all reservation values are well approximated by the frictionless equilibrium price.

The third feature of the market that matters for the price deviation is the slope $F'(\delta^\star)$ of the distribution at the marginal type, and we claim that this quantity measures the heterogeneity among investors near the marginal type. To see this, consider a distribution G that is more *more heterogenous* than F in the sense that $G(\delta) \geq F(\delta)$ if and only if $\delta < \delta_0$ for some δ_0. As discussed by **?**], this definition of heterogeneity is very intuitive as it simply requires that G be obtained from F by shifting probability mass to the left in the interval $[0, \delta_0]$ and to the right in the interval $[\delta_0, 1]$. If, in addition, the two distributions induce the same marginal type and satisfy the conditions of Proposition 15, then

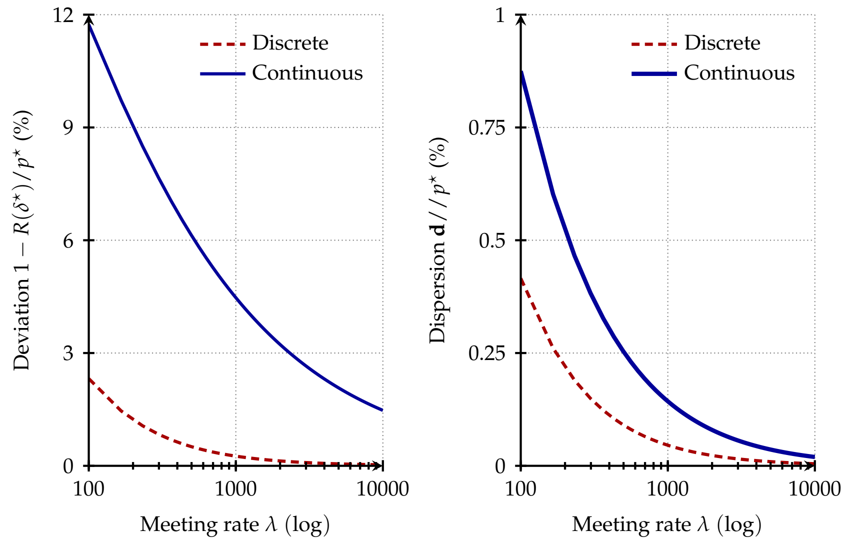

FIGURE 5.6: Equilibrium deviation and dispersion of prices

This figure plots the relative deviation from the frictionless equilibrium price (left panel) and the relative price dispersion (right panel) as functions of the meeting rate for the base case model of Figure 5.3 with bargaining power $\theta_0 = 0.75$ and a model with a two point distribution constructed to have the same mean and the same marginal type as the uniform distribution of the base case model.

$G'(\delta^\star) \leq F'(\delta^\star)$ so that the price deviation implied by the more heterogenous distribution is larger in absolute value.

To further emphasize the role of investor heterogeneity we can also derive the convergence rate of reservation values when the underlying distribution is discontinuous at the marginal type.

Proposition 16. *Assume that $\Delta F(\delta^\star) > 0$. Then*

$$R(\delta) = p^\star + \frac{\xi(\delta)}{\lambda} + o\left(\frac{1}{\lambda}\right)$$

for some function $\xi : \mathscr{D} \to \mathbf{R}$ that is independent of the meeting intensity.

To see that the difference in convergence rates is economically significant, we compare the relative price deviation implied by the uniform distribution of our baseline example to that implied by a two-point distribution chosen to keep the marginal and average investors the same as in the baseline case. The left panel of Figure 5.6 shows that, when investors meet counterparties twice a day on average ($\lambda = 500$), the deviation is 6% for the continuous distribution and only about 0.5% for the corresponding discrete distribution. When meetings occur 20 times per day on average ($\lambda = 10,000$), the deviation is about 1.5% for the continuous distribution but essentially zero for the discrete distribution. Why is there such a quantitatively large difference in the price impact of search frictions? According to our analysis, the difference is driven by a fundamental economic difference between the two classes of distributions: the elasticity of asset demand at the marginal type is infinite with discrete distribution and finite with a continuous one.

5.5.3 Price dispersion near the frictionless limit

An important implication of Proposition 16 is that, to a first-order approximation, there is no price dispersion when the distribution of utility flows is smooth around the marginal type. This can be seen by noting that the correction term in (5.24) does not depend on the investor's utility flows. Hence, in order to obtain results about the impact of frictions on the price dispersion

$$\mathbf{d} \equiv R(\overline{\delta}) - R(\underline{\delta}) = \int_{\underline{\delta}}^{\overline{\delta}} \sigma(x)dx$$

it is necessary to work out higher order terms.

Proposition 17. *If $F(\delta)$ is twice continuously differentiable in a neighbourhood of $\delta^\star$ on which $F'(\delta)$ is bounded away from zero, then*

$$\mathbf{d} = \frac{1}{2\theta_0\theta_1 F'(\delta^\star)} \frac{\log(\lambda)}{\lambda} + O\left(\frac{1}{\lambda}\right).$$

If instead $\Delta F(\delta^\star) > 0$, then

$$\mathbf{d} = \left(\xi(\overline{\delta}) - \xi(\underline{\delta})\right) \frac{1}{\lambda} + o\left(\frac{1}{\lambda}\right)$$

with the same function $\xi(\delta)$ as in Proposition 17.

Remark 18. The asymptotic notation $f(x) \in o(g(x)$ as $x \to \infty$ means that g

grows much faster than f in the precise sense that

$$\forall c > 0 \exists x_0 \text{ such that } |f(x)| \leq c|g(x)| \text{ for all } x \geq x_0 \tag{5.25}$$

By contrast, the asymptotic notation $f(x) \in O(g(x)$ as $x \to \infty$ relaxes (5.25) by replacing the condition $\forall c \exists x_0$ by the less stringent requirement that $\exists(c, x_0)$ and thus results in a weaker asymptotic statement.

Comparing the conclusions of Propositions 15 and 17 shows that, when the distribution of utility flows is sufficiently smooth around the marginal type, the price dispersion induced by search frictions vanishes at a rate $\log(\lambda)/\lambda$ that is much faster than the rate $1/\sqrt{\lambda}$ at which reservation values converge to the frictionless equilibrium price. This finding has important implications for empirical analysis of decentralized markets, as it implies that inferring the impact of search frictions based on the observable level of price dispersion can be misleading. In particular, search frictions can have a very small impact on price dispersion and, yet, have a large impact on the price level.

This finding is illustrated in Figure 5.6. Comparing the two panels, one sees clearly that the price dispersion induced by search frictions vanishes much faster than the price deviation. For instance, when investors meet twice a day on average, the price discount implied by our baseline model is 6% but the corresponding price dispersion is about 20 times smaller. One can also see from the figure that, in accordance with the result of Proposition 17, price dispersion is larger with a continuous distribution than with a discrete distribution.

5.6 EXERCIZES

Exercise 19. Consider the same model as in the text.

1. Show that in a steady state equilibrium $\Phi_1(\delta) = \ell(F(\delta))$ for some smooth function $\ell : [0, 1] \to [0, s]$.
2. Derive a algebraic equation for the function $\ell(x)$.
3. Assume that $F(\delta)$ is absolutely continuous. Derive an integro-differential equation for the density $\phi_1(\delta) = \Phi_1'(\delta)$.

Exercise 20. In a seminal paper, Harrison and Kreps [32] define *speculation* as a situation where the right to resell an asset makes investors willing to pay more for it than they would agree to pay if obliged to hold it forever. This exercize briefly explores conditions under which such a behavior occurs in the decentralized market.

1. Argue that speculation occurs if and only if

$$\min_{(\delta,x)\in\mathscr{D}^2}\{\theta_1 R(\delta)+\theta_0 R(\delta\wedge x)-B(\delta)\}\geq 0$$

   where $B(\delta)$ denotes the buy-and-hold value of the asset. What is the counterpart of this condition in the frictionless benchmark?
2. Provide an explicit expression for $B(\delta)$.
3. Show that speculation arises in the equilibrium of the frictionless benchmark if and only if

$$\delta^\star \geq \frac{r}{r+\gamma}\bar{\delta}+\frac{\gamma}{r+\gamma}\mathbf{E}^F[\delta] \tag{5.26}$$

   and argue that this is equivalent to an upper bound on supply.
4. Show that if (5.26) holds then speculation occurs in the equilibrium of the decentralized market for all sufficiently low values of $\phi=\gamma/\lambda$.

Exercise 21. Denote by $\eta>0$ the random amount of time until the next trade of a given investor and define $g_q(\delta)\equiv\mathbf{E}_{q,\delta}[\eta]$.

1. Use the law of iterated expectations to show that

$$g_q(\delta)=\mathbf{E}_{q,\delta}\left[\mathbf{1}_{\{\eta\leq\tau_\gamma\}}\eta+\mathbf{1}_{\{\eta>\tau_\gamma\}}\left(\tau_\gamma+\mathbf{E}^F[g_q(x)]\right)\right] \tag{5.27}$$

   where τ_γ denotes the exponentially distributed time until the investor's next preference shock.
2. Argue that

$$\mathbf{P}[s\geq\eta\in dt|\tau_\gamma=s|q,\delta]=\mathbf{1}_{\{t\leq s\}}\lambda_q(\delta)e^{-\lambda_q(\delta)t}dt.$$

   Use this expression together with (5.27) and the exponential distribution of the random time τ_γ to show that

$$g_q(\delta)=\frac{1}{\gamma+\lambda_q(\delta)}\mathbf{E}^F\left[1+\frac{\gamma}{\gamma+\lambda_q(x)}\right]^{-1}.$$

3. Use the above result to to show that the expected trading delay is increasing in utility flow for sellers, decreasing in utility flow for buyers, and increasing in $\phi = \gamma/\lambda$ for all investors.

Exercise 22. Assume that the conditions of Proposition 17 hold true and define the welfare cost of search friction as

$$\mathbf{w} \equiv \int_{\underline{\delta}}^{\overline{\delta}} x \left(d\Psi_1(x) - d\Phi_1(x) \right)$$

where $\Psi_1(\delta)$ denote the distribution of utility flows among asset owners in the frictionless benchmark. Show that

$$\mathbf{w} = \frac{\gamma s(1-s)}{F'(\delta^\star)} \frac{\log \lambda}{\lambda} + O\left(\frac{1}{\lambda}\right)$$

and combine this result with Proposition 17 to conclude that, when the meeting rate is sufficiently large, the unobserved welfare cost of frictions may be approximated by the observed amount of price dispersion.

5.7 NOTES AND REFERENCES

TBA