Chapter Nine

OTC markets under stress

OTC markets have taken center stage in recent crises. For instance, the markets for various types of derivatives and asset-backed securities have often been identified as ground zero for the Global Financial Crisis of 2007–08 [see, for example, Brunnermeier, 2009]. More recently, at the onset of the COVID–19 crisis, the first important cracks in the US financial system appeared in the OTC markets for Treasury, corporate, and municipal bonds [see, e.g., O'Hara and Zhou, 2022, for an overview.]. Given the central role that OTC markets play in the allocation of capital and the implementation of monetary policy, these events have naturally raised concerns among policymakers about the fragility of OTC markets and their ability to withstand stress. In particular, during these crises, many have argued that dealers should absorb more of the selling pressure in order to maintain a liquid market for investors.

To understand the incentives of dealers to provide liquidity in times of stress—and the subsequent resilience of OTC markets—this chapter studies trading activity in OTC markets in the aftermath of negative aggregate shock that triggers a surge in selling pressure. From a methodological point of view, this requires us to step away from our usual focus on steady state in order to study the temporal dynamics of the model *outside of the steady state*. Building on earlier work by Weill [2007] and Lagos, Rocheteau, and Weill [2011], we focus on several positive implications.

First, we derive conditions under which dealers find it optimal to provide liquidity by accumulating inventories during the stress event. To start, we show that both fluctuations in aggregate demand for assets and search frictions are necessary: when both of these ingredients are present, dealers optimally accumulate assets because they anticipate being able to resell them quickly to the investors who value them the most soon after the stress event subsides. In contrast, if either search frictions or fluctuations in aggregate asset demand are absent, then dealers have no incentive to provide liquidity to investors by accumulating inventories. While these ingredients are necessary, they are not sufficient. For example, if the stress period is expected to be sufficiently long-lived, then dealers have little incentive to accumulate inventory and thus may not provide liquidity to market participants in equilibrium.

Second, we show that dealers accumulate assets in a falling market, i.e., that the asset price continues to fall as dealers purchase more and more assets. This seemingly counterintuitive outcome, where liquidity provision fails to stabilize prices, is a necessary characteristic of equilibrium as it is precisely this feature that incentivizes dealers to build larger inventories, fully expecting that it will take longer to unwind these inventories.

9.1 A NEGATIVE DEMAND SHOCK AND A RANDOM RECOVERY

We consider the semi-centralized OTC market model of Chapter 4, modified to allow for aggregate fluctuations in demand. Specifically, we fix a set of utility flows $\mathcal{D} \subseteq \mathbf{R}_+$ and assume that at time $t = 0$, the market experiences a stress event: *all investors* in the market are hit by a negative preference shock that temporarily scales down all of their individual utility flows by a constant factor $\alpha \in [0, 1)$. This shock could represent, for example, a aggregate housing price shock that decreases the payoffs of all mortgage-backed securities, or an aggregate liquidity event that forces mutual funds to sell corporate bonds to redeem the shares of panicked investors.

After the initial shock, we assume that preferences revert to their pre-shock values at some exponentially distributed time τ_ρ with intensity $\rho > 0$. Figure 9.1 illustrates. Apart from the scaling down by α, the idiosyncratic utility flow process δ_t of an investor evolves as before: it is reset with intensity γ and new values are drawn from a distribution $F(\delta)$ on $\mathcal{D}$, which we assume for simplicity is strictly increasing and continuous.

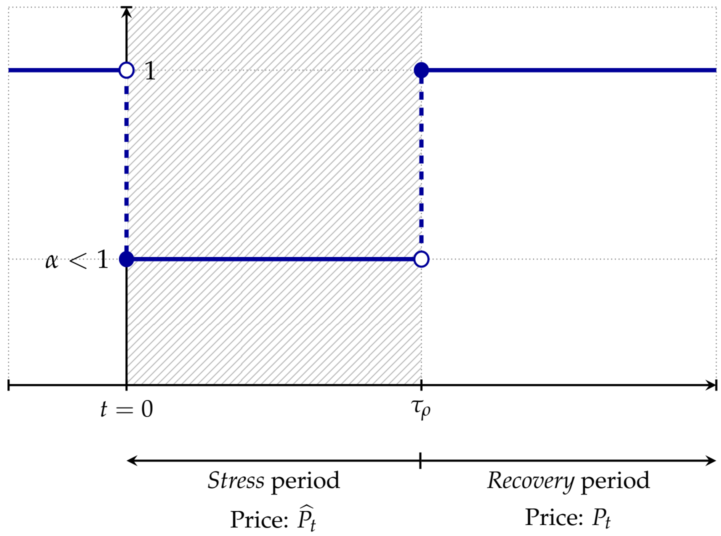

FIGURE 9.1: The timeline of the model

As we show below, the occurrence of the stress event moves the market away from the steady state, which makes the reservation values of investors and the asset allocation change over time. Hence, whenever appropriate, we index all endogenous variables by time $t \geq 0$. Endogenous variables also depend on the state of market stress, i.e., on whether the preferences are still in the stressed state ($t < \tau_\rho$) or have recovered ($t \geq \tau_\rho$). Correspondingly, we use a hat (no hat) to denote endogenous variables during (after) the stress event. For example, we let $\hat{P}_t$ denote the inter-dealer price at time $t < \tau_\rho$ and P_t denote the inter-dealer price at time $t \geq \tau_\rho$. Although endogenous variables post recovery are also functions of the realized recovery time, τ_ρ, we will keep this dependence implicit to simplify notations.

9.1.1 Reservation values in the recovery period

Following the same steps as in previous chapters shows that after the recovery time the reservation value of investors satisfies

$$(r + \gamma) R_t(\delta) = \delta + \gamma \mathbf{E}^F [R_t(x)] + \lambda_\theta (P_t - R_t(\delta)) + \dot{R}_t(\delta) \tag{9.1}$$

where $\lambda_\theta = \lambda(1-\theta)$ denotes the bargaining-adjusted contact rate. The main difference between this equation and the one we derived in the steady state environment of Chapter 4 is that the inter-dealer price, P_t, now depends on time. As a result, the reservation value changes over time as well and, hence, the equation is augmented by a new term, $\dot{R}_t(\delta)$, that represents the partial derivative of the reservation value with respect to time. See Section 3.4.3 in Chapter 3 for a discussion of time-dependent HJB equations.

A key observation is that the time variation induced by a changing price affects the reservation values of all investors in the same way, regardless of their utility flow. In particular, subtracting equation (9.1) evaluated at δ' from itself evaluated at $\delta \neq \delta'$ shows that

$$\Delta R_t(\delta,\delta') \equiv R_t(\delta) - R_t(\delta')$$

is a bounded solution to

$$(r+\gamma+\lambda_\theta)\,\Delta R_t(\delta,\delta') = \delta - \delta' + \Delta\dot{R}_t(\delta,\delta')$$

and it follows that

$$\Delta R_t(\delta,\delta') = \frac{\delta-\delta'}{r+\gamma+\lambda_\theta}, \tag{9.2}$$

since any other other solution to the ODE diverges as $t \to \infty$. Dividing both sides by $\delta - \delta'$ shows that the reservation value function is differentiable in utility flow and that the local surplus is constant:

$$\sigma_t(\delta) = \lim_{\delta'\to\delta} \Delta R_t(\delta,\delta') = \sigma \equiv \frac{1}{r+\gamma+\lambda_\theta}. \tag{9.3}$$

Given this result we can now derive an explicit formula for reservation values. Indeed, the fact the local surplus is constant implies that

$$\mathbf{E}^F\left[R_t(x)\right] - R_t(\delta) = \mathbf{E}^F\left[\sigma\left(x-\delta\right)\right] = \sigma\left(\mathbf{E}^F\left[x\right] - \delta\right)$$

and substituting back into (9.1) shows that for each fixed $\delta \in \mathcal{D}$ the reservation value function satisfies

$$(r+\lambda_\theta)\,R_t(\delta) - \dot{R}_t(\delta) = \delta + \gamma\sigma\left(\mathbf{E}^F\left[x\right] - \delta\right) + \lambda_\theta P_t.$$

If the price path is uniformly bounded, then a direct calculation shows that the unique bounded solution to this equation is

$$R_t(\delta) = \frac{r}{r+\lambda_\theta}D(\delta) + \int_t^\infty \lambda_\theta e^{-(r+\lambda_\theta)(u-t)}P_u\,du,$$

where, as in Chapter 4,

$$rD(\delta) \equiv \frac{r+\lambda_\theta}{r+\gamma+\lambda_\theta}\delta + \frac{\gamma}{r+\gamma+\lambda_\theta}\mathbf{E}^F\left[x\right].$$

This expression confirms that the reservation value function is continuous, differentiable, and strictly increasing in utility flow, as before. Moreover, under the maintained assumptions that the price path is bounded, the reservation value is also an absolutely continuous function of time.

9.1.2 Reservation values in the stress period

Similar steps show that during the stress period the reservation value function of investors satisfies

$$\begin{aligned}(r+\gamma)\,\hat{R}_t(\delta) = \alpha\delta + \gamma\mathbf{E}^F\left[\hat{R}_t(x)\right] + \lambda_\theta\left(\hat{P}_t - \hat{R}_t(\delta)\right)& \\ + \rho\left(R_t(\delta) - \hat{R}_t(\delta)\right) + \dot{\hat{R}}_t(\delta).&\end{aligned} \tag{9.4}$$

There are two differences relative to the post-recovery equation (9.1). First, the utility flow is scaled down by $\alpha < 1$ because of the demand shock. Second, the investor anticipates that the market will recover with intensity ρ at which point her reservation value will jump from $\hat{R}_t(\delta)$ to $R_t(\delta)$.

Combining (9.3) with the same arguments as above reveals that the local surplus is also constant during the stress event, and equal to

$$\hat{R}'_t(\delta) = \hat{\sigma} \equiv \frac{\alpha+\rho\sigma}{r+\gamma+\rho+\lambda_\theta} = \sigma - \frac{1-\alpha}{r+\gamma+\rho+\lambda_\theta} < \sigma. \tag{9.5}$$

This shows that the marginal value of the asset jumps down for all investors upon the occurence the stress event and implies that the market experiences a surge in selling pressure.

9.2 WHEN DEALERS CANNOT ACCUMULATE INVENTORIES

In this section, we analyze the model under the assumption that dealers must keep their individual asset position equal to zero at all times and, thus, are unable to provide liquidity to investors by holding inventories. As in Chapter 4, we assume that they can continue to facilitate trades by instantaneously matching the buyers and sellers who contact them.

We characterize the equilibrium and find that dealers may have incentives to hold inventory if they were allowed to do so. In particular, we show that such incentives exist only if there are both search frictions ($\lambda_\theta < \infty$) and aggregate fluctuations in demand ($\alpha < 1$). Thus, the model predicts that liquidity provision arises in response to stress events precisely because of the existence of OTC market frictions.

9.2.1 Equilibrium

When dealers cannot hold inventories, the same arguments as in Chapter 4 can be used to show that the marginal utility flow $\delta^\star$ remains constant over time and satisfies

$$s = 1 - F(\delta^\star).$$

This simple result obtains for two reasons. First, although investors derive lower utility flows during the stress event, they remain ranked in exactly the same way in the cross section. Second, since dealers do not hold inventories, the supply held by investors is equal to s at all times.

Evaluating (9.1) and (9.4) at $\delta^\star$ reveals that the inter-dealer price remains constant during the stress period at some level $\hat{P} = \hat{R}(\delta^\star)$ and then jumps to its steady-state level $P = R(\delta^\star)$ once the market recovers. These constant price levels satisfy the linear system

$$rP = \gamma\sigma\left(\mathbf{E}^F[x] - \delta^\star\right) + \delta^\star, \tag{9.6}$$

$$r\hat{P} = \gamma\hat{\sigma}\left(\mathbf{E}^F[x] - \delta^\star\right) + \alpha\delta^\star + \rho\left(P - \hat{P}\right), \tag{9.7}$$

and solving yields

$$P - \hat{P} = (\sigma - \hat{\sigma})\left[\left(\frac{\gamma}{r+\rho}\right)\mathbf{E}^F[x] + \left(1 + \frac{\lambda_\theta}{r+\rho}\right)\delta^\star\right].$$

Since the local surplus $\hat{\sigma} < \sigma$ this identity shows that, consistent with a surge in selling pressure, the stress event triggers an unambiguously negative jump in the interdealer price of the asset.

9.2.2 Incentives for inventory accumulation

The fact that $\hat{P} < P$ suggests that, if they could, dealers may have incentives to hold assets in inventory in order to earn the capital gain $P - \hat{P}$. However, it is not completely obvious that this strategy is profitable because dealers cannot perfectly time the recovery: since they view τ_ρ as unpredictably random, they cannot buy the asset just before τ_ρ and sell it just after.

To better understand these incentives, assume that a dealer buys an asset at time zero at the price $\hat{P}$ and holds it until time $\tau = \min\{\tau_\rho, \tau_\ell\}$ where τ_ρ is the time of the recovery and τ_ℓ is an independent exponential time with some intensity $\ell > 0$. If $\tau = \tau_\ell$ then the dealer is hit by a liquidity shock that forces him to sell his position at the interdealer price $\hat{P}$ before the market recovers. Otherwise, if $\tau = \tau_\rho$ then the dealer manages to hold on to his position until the recovery time where he sells at the post-recovery price P. The expected discounted profit of this strategy is:

$$\begin{aligned}\mathbf{E}\left[\mathbf{1}_{\{\tau=\tau_\rho\}}e^{-r\tau}P\right] + \mathbf{E}\left[\mathbf{1}_{\{\tau=\tau_\ell\}}e^{-r\tau}\hat{P}\right] - \hat{P}& \\ = \frac{\rho P}{r+\rho+\ell} + \frac{\ell\hat{P}}{r+\rho+\ell} - \hat{P} = \frac{\rho P - (r+\rho)\hat{P}}{r+\rho+\ell}&\end{aligned} \tag{9.8}$$

where the first equality follows from Lemma A.18. Hence, we obtain that dealers have incentive to hold inventories if and only if:

$$\rho\left(P-\hat{P}\right)>r\hat{P}. \tag{9.9}$$

The left-hand side is the expected growth rate of the price: with intensity ρ, the economy recovers and the price jumps from $\hat{P}$ to P. The right-hand side is the opportunity cost of buying the asset prior to the recovery. Indeed, if dealers were explicitly allowed to borrow and save at the rate r in some bank account, one would see that r is the rate that dealers pay to finance their asset purchase, or the rate that they forgo when investing in the asset instead of their bank account. Equation (9.9) is a classical conditions for speculative demand to be positive in inventory problems [see, e.g., Deaton and Laroque, 1992].

Fluctuations in aggregate asset demand are clearly necessary for condition (9.9) to hold. Otherwise, if $\alpha=1$, then evidently $\hat{P}=P$ so that dealers never profit from buying, holding, and re-selling their inventories. One can also verify that condition (9.9) cannot hold absent frictions. Indeed, as frictions vanish we have that the local surplus $\hat{\sigma}\to 0$ and therefore

$$r\hat{P}=\alpha\delta^{\star}+\rho\left(P-\hat{P}\right)\geq\rho\left(P-\hat{P}\right)$$

as a result of (9.7). Next, we show that the expected profits of the dealer in (9.8) can be positive. To do so, let $\tau_{\ell}\equiv\tau_{\theta}$ denote the bargaining-adjusted contact time and observe that the price initially paid by the dealer is the reservation value of a marginal investor $\hat{P}=\hat{R}(\delta^{\star})$. Proceeding as in Section 4.3.2 of Chapter 4 shows that this reservation value can be written as

$$\hat{R}(\delta^{\star})=\mathbf{E}_{\delta^{\star}}\left[\int_0^{\tau}e^{-rt}\alpha\delta_t dt+e^{-r\tau}\left(\mathbf{1}_{\{\tau=\tau_{\theta}\}}\hat{P}+\mathbf{1}_{\left\{\tau=\tau_{\rho}\right\}}R(\delta_{\tau_{\rho}})\right)\right]$$

and substituting this expression into the last term of (9.8) allows to write the expected profits of the dealer as:

$$-\mathbf{E}_{\delta^{\star}}\left[\int_0^{\tau}e^{-rt}\alpha\delta_t dt\right]+\mathbf{E}_{\delta^{\star}}\left[\mathbf{1}_{\left\{\tau=\tau_{\rho}\right\}}e^{-r\tau}\left(P-R(\delta_{\tau_{\rho}})\right)\right]. \tag{9.10}$$

This decomposition shows that the expected profits consist in two terms. The first term is negative and arises because the dealer does not enjoy the flow utility $\alpha\delta_t$ when holding the asset in inventory. Since these positive utility flows are capitalized in the interdealer price they reduce the expected profits of dealer. The second term can have either sign. It arises because the dealer can instantly resell the asset on the interdealer market at price P at the recovery time whereas the investor must hold on to it with a continuation value equal to $R(\delta_{\tau_\rho})$ that is in general not equal to $P = R(\delta^\star)$ because an investor who was initially marginal may have changed utility flow by time τ_ρ. A direct calculation using (9.2) shows that

$$\mathbf{E}\left[\mathbf{1}_{\{\tau=\tau_\rho\}}e^{-r\tau}\left(P - R(\delta_{\tau_\rho})\right)\right] \propto \sigma\left(\delta^\star - \mathbf{E}^F\left[x\right]\right).$$

Hence the second term in the sequential representation (9.10) is positive if and only if $\mathbf{E}^F\left[x\right] < \delta^\star$ so that a marginal investor expects to switch to a lower utility flow upon his next preference shock, in which case the dealer's profits are positive if the loss fraction $1-\alpha$ is large enough.

We can go one step further and explicitly characterize the region of the parameter space for which dealers have strict incentives to provide liquidity. Indeed, (9.7) implies that

$$\rho P - (r+\rho)\hat{P} = -\alpha\delta^\star + \gamma\hat{\sigma}\left(\mathbf{E}^F\left[x\right] - \delta^\star\right)$$

and plugging in the explicit formula (9.5) for the local surplus $\hat{\sigma}$ during the stress period yields the following result.

Lemma 9.1. *Let*

$$\Psi \equiv \frac{\gamma}{r+\gamma+\lambda_\theta}\left(1 + \frac{\rho\left(\frac{1}{\alpha}-1\right)}{r+\gamma+\lambda_\theta+\rho}\right)\left(1 - \mathbf{E}^F\left[\frac{x}{\delta^\star}\right]\right).$$

Then dealers have strict incentives to accumulate inventories during the stress period if and only if the constant $\Psi > 1$.

The condition in Lemma 9.1 is easy to interpret. Dealers have incentive to accumulate inventories if the interest rate r is small, because the cost of committing capital to buy assets is low; if the recovery rate ρ is large because the stress event is expected to be shorter lived; if the contact rate λ_θ is small, because the bargaining-adjusted frictions faced by investors are large; and if the loss fraction $1-\alpha$ is large, because investors derive less value from holding the asset during the stress period which in turn makes them more willing to sell the asset at low prices that benefit dealers.

9.3 WHEN DEALERS CAN ACCUMULATE INVENTORIES

We now solve for the equilibrium when dealers can hold inventories. To ensure that dealers have incentives to hold inventories, we suppose that $\mathbf{E}^F[x] < \delta^\star$ and make the simplifying assumption that $\alpha \to 0$. Technically, this means that, although the investor does not derive any utility from holding the asset during the stress period, her utility flow δ_t continues to evolve as before and determines her utility flow at the time of the recovery.

We solve for an equilibrium by working backwards, starting from after the recovery and then turning our attention to behavior and payoffs during the period of the stress event.

9.3.1 After the recovery

Assume that, post-recovery, the interdealer price is a bounded and piecewise continuously differentiable function P_t. Consider a dealer who starts out at the recovery time τ_ρ with some inventory $I_{\tau_\rho} > 0$ accumulated during the stress period. The problem of this dealer is to choose a bounded variation path of inventories to maximize

$$-\int_{\tau_\rho}^{\infty} e^{-rt} P_t \, dI_t = I_{\tau_\rho} + \int_{\tau_\rho}^{\infty} e^{-rt} I_t \left(\dot{P}_t - rP_t\right) \, dt, \tag{9.11}$$

where the equality follows from integration by parts, just as in Section 4.2 of Chapter 4. The integrand on the right-hand side of (9.11) can be interpreted as the profit that a dealer makes during a small interval of time. Indeed, when the representative dealer buys inventory I_t at time t and holds it until the later time $t+h$, she makes discounted profits equal to

$$I_t \left(e^{-r(t+h)} P_{t+h} - e^{-rt} P_t\right) \simeq e^{-rt} I_t \left(\dot{P}_t - rP_t\right) h.$$

The integrand in (9.11) can be maximized pointwise, leading to the optimal inventory policy given by

$$I_t = \begin{cases} +\infty & \text{if } \dot{P}_t - rP_t > 0, \\ \in \mathbf{R}_+ & \text{if } \dot{P}_t - rP_t = 0, \\ 0 & \text{if } \dot{P}_t - rP_t < 0. \end{cases}$$

In words, this equation shows that dealers want to hold infinite inventories if the price grows strictly faster than their discount rate, r, and no inventory if it grows strictly slower. In equilibrium, since the asset supply is finite, the asset demand must be finite as well and so we must have $\dot{P}_t \leq rP_t$.

If the interdealer price were to adjust immediately to its steady state level (9.6) at the recovery time, then we would have $\dot{P}_t = 0$ for all $t \geq \tau_\rho$ and the optimality condition above would imply that dealers want to instantly sell all their inventories at time τ_ρ. But this cannot be an equilibrium. Indeed, since investors must search for dealers to trade they cannot instantly buy all of their inventories at the recovery time. As a result, the price will have to gradually adjust so that dealers are willing to sell their asset slowly and the above optimality conditions shows this can only happen if $\dot{P}_t = rP_t$ so that dealers have incentives to keep some assets in inventory.

Let us guess that dealers gradually liquidate their inventory between the recovery time τ_ρ and some time $\tau_\rho + T$ where the length of the liquidation interval will be determined endogenously. After time $\tau_\rho + T$ dealers hold no inventories. Therefore, going through the same steps as in Section 9.2, shows that the interdealer price must have reached the post-recovery steady state leve and combining this boundary condition with the fact that the price grows at rate r we deduce that

$$P_t = e^{-r(T+\tau_\rho - t)}P, \qquad \forall t \in [\tau_\rho, \tau_\rho + T],$$

where the constant P is the steady state price in (9.6).

Since the reservation value function is continuous and strictly increasing in utility flows there is, as usual, a marginal investor who is indifferent between buying or not. However, because the price must continuously adjust to reflect the liquidation strategy of dealers, the utility flow $\delta_t^\star$ of this marginal investor is now changing over time. Specifically, the equation for reservation values evaluated at the point $\delta_t^\star$ implies that:

$$rP_t = rR_t(\delta_t^\star) = \delta_t^\star + \gamma\sigma\left(\mathbf{E}^F[x] - \delta_t^\star\right) + \dot{R}_t(\delta_t^\star). \tag{9.12}$$

An application of the chain rule shows that

$$\dot{P}_t = \dot{R}_t(\delta_t^\star) + R_t'(\delta_t^\star)\dot{\delta}_t^\star = \dot{R}_t(\delta_t^\star) + \sigma\dot{\delta}_t^\star.$$

Plugging this expression back into (9.12) and using the fact that the price grows at rate r when dealers hold inventories we deduce that

$$\dot{\delta}_t^\star = \gamma\mathbf{E}^F[x] + (r+\lambda_\theta)\,\delta_t^\star;$$

and solving this ODE subject to the terminal condition $\delta^\star_{\tau_\rho+T} = \delta^\star$ delivers an explicit solution for the marginal utility flow:

$$\delta_t^\star = \Delta\left(\tau_\rho + T - t\right) \equiv -\frac{\gamma\mathbf{E}^F[x]}{r+\lambda_\theta} + \left(\delta^\star + \frac{\gamma\mathbf{E}^F[x]}{r+\lambda_\theta}\right)e^{-(r+\lambda_\theta)(\tau_\rho+T-t)}. \tag{9.13}$$

Since the function $\Delta(z)$ is strictly decreasing, this expression shows that the marginal utility flow gradually increases as dealers sell inventories and reaches its steady state level $\delta^\star$ once dealers have liquidated all their holdings. This is very natural. Indeed, since dealers sell inventories, investors must absorb more assets than in the steady-state and so the utility flow of the marginal investor must be lower than in the steady state.

The last step in the characterization of equilibrium after the recovery is to use the market clearing condition to determine T. The logic is similar to that in Section 4.4.1 of Chapter 4. However, some additional care is required, since some assets are held by dealers instead of investors. As such, we must consider demand originating from *both* investors and dealers. Indeed, during a small time interval of length h, the gross supply of asset is

$$\lambda h\,(s - I_t) + I_t.$$

The first term is the gross supply of assets originating from investors who collectively own $s - I_t$ assets and contact the market at rate λ. The second term is the gross supply originating from dealers. On the other side of the market, the gross demand is

$$\lambda h \left(1 - F(\delta_t^\star)\right) + I_{t+h}.$$

Equating gross supply and gross demand, dividing by h and letting $h \to 0$, we obtain that inventories evolve according to

$$\dot{I}_t = \lambda \left(s - I_t - 1 + F(\delta_t^\star)\right).$$

Solving this ODE subject to the assumed initial condition and using (9.13) together with the change of variable $u = t - \tau_\rho$ we obtain that

$$I_{\tau_\rho + T} = 0 = I_{\tau_\rho} - \lambda \int_0^T \left(1 - s - F \circ \Delta(T - u)\right) e^{\lambda u} du \tag{9.14}$$

where the first equality follows from the fact that inventories must reach zero by the end of the liquidation period. Therefore, the above expression can be viewed as a single equation to determine the time T that dealers take to unwind their asset positions, as a function of the initial level of inventories I_{τ_ρ}.

Since $\Delta(z) \leq \delta^\star$ is decreasing and the integrand is continuous we have that the right handside of (9.14) is continuous as well as strictly decreasing in T and goes to minus infinity as T goes to infinity. Therefore, (9.14) has a unique solution $\mathcal{T}(I_{\tau_\rho})$ and it is easily checked that this solution is increasing in I_{τ_ρ} so that dealers naturally take longer to unwind a larger asset position. As we will see next, the function $\mathcal{T}(I_{\tau_\rho})$ plays a key role in the characterization of the equilibrium during the stress period because it determines how long dealers hold inventories after the market recovers and, correspondingly, how depressed prices will be relative to the steady state level.

9.3.2 During the stress period

We now solve for equilibrium during the stress period, taking as given our characterization of the equilibrium in the recovery period. The main difference relative to the previous section is that over that period the equilibrium does not evolve on a deterministic path because market participants anticipate the recovery to arrive at rate ρ.

If the market recovers at time $\tau_\rho = s$ then the interdealer price of the asset immediately jumps up to

$$\mathcal{P}(I_s) \equiv e^{-r\mathcal{T}(I_s)}P, \tag{9.15}$$

where P is the steady state price level of equation (9.6), and subsequently grows at the constant rate r until the inventories of dealers are depleted. Accordingly, if a dealer holds a unit of inventory during the time interval $[t, t+h]$ then her expected discounted profits can be written as

$$\rho h e^{-r(t+h)}\mathcal{P}(I_t) + (1-\rho h)e^{-r(t+h)}\hat{P}_{t+h} - e^{-rt}\hat{P}_t$$

where the first term accounts for the fact that the dealer resells at $\mathcal{P}(I_t)$ if the market recovers and at $\hat{P}_{t+h}$ otherwise; and the second term accounts for the cost of acquiring the asset. Dealers are willing to hold a positive but finite inventory if the above expected profit is zero. Dividing both sides by h and letting $h \to 0$ allows to write this condition as

$$r\hat{P}_t = \dot{\hat{P}}_t + \rho\left(\mathcal{P}(I_t) - \hat{P}_t\right). \tag{9.16}$$

As in the analysis above, the price must grow at an expected rate of r. However, the expected growth rate now has two components: the deterministic price change $\dot{\hat{P}}_t$ during the stress event and the upward jump $\mathcal{P}(I_t) - \hat{P}_t$ that occurs upon recovery. Solving this ODE subject to (9.15) and using the results of Appendix A.3 shows that

$$\hat{P}_t = \int_t^\infty e^{-(r+\rho)(s-t)}\mathcal{P}(I_s)ds = \mathbf{E}_t\left[e^{-r(\tau_\rho - t)}\mathcal{P}(I_{\tau_\rho})\right], \qquad \forall t \in [0, \tau_\rho].$$

Hence, the price during the stress period is simply equal to the present value of re-selling at time τ_ρ when the recovery occurs. Since $\mathcal{P}(i)$ is decreasing, it follows that if dealers accumulate inventories throughout the stress event in that $\dot{I}_t > 0$ then the expected price at the time of the recovery must fall, and the price during the stress period must fall as well, $\dot{\hat{P}}_t < 0$. The intuition is that dealers anticipate that their collective inventory accumulation will end up putting downward pressure on the price after the recovery and thus must be compensated by a lower price during the stress period.

To verify that dealers want to accumulate inventories, one must solve for the joint dynamics of I_t and the marginal utility flow $\hat{\delta}_t^\star$ that prevails during the stress period. Evaluating (9.4) at $\hat{\delta}_t^\star$ yields

$$\begin{aligned} r\hat{R}_t(\hat{\delta}_t^\star) - \dot{\hat{R}}_t(\hat{\delta}_t^\star) &= \gamma\hat{\sigma}\left(\mathbf{E}^F[x] - \hat{\delta}_t^\star\right) + \rho\left(R_t(\hat{\delta}_t^\star) - \hat{R}_t(\hat{\delta}_t^\star)\right) \\ &= \gamma\hat{\sigma}\left(\mathbf{E}^F[x] - \hat{\delta}_t^\star\right) + \rho\left(R_t(\hat{\delta}_t^\star) - \mathcal{P}(I_t)\right) + \rho\left(\mathcal{P}(I_t) - \hat{R}_t(\hat{\delta}_t^\star)\right), \end{aligned}$$

where the second line obtains by adding and subtracting $\mathcal{P}(I_t)$. On the other hand, combining (9.5), (9.16), and

$$\mathcal{P}(I_t) = R_t(\delta_t^\star) = R_t(\Delta \circ \mathcal{T}(I_t))$$

with the pre-recovery counterpart of (9.12) we obtain that prior to the recovery, the marginal utility flow $\hat{\delta}_t^\star$ evolves according to

$$\dot{\hat{\delta}}_t^\star = \gamma\mathbf{E}^F[x] + (r+\rho+\lambda_\theta)\,\hat{\delta}_t^\star - (r+\rho+\lambda_\theta+\gamma)\,\Delta\circ\mathcal{T}(I_t). \tag{9.17}$$

Finally, as in the analysis above, the market-clearing condition implies that the inventories of a representative dealer satisfy

$$\dot{I}_t = \lambda\left(s - I_t - 1 + F(\hat{\delta}_t^\star)\right). \tag{9.18}$$

Taken together, these ODEs can be studied in a phase diagram, as illustrated in Figure 9.2. The downward slopping dashed curve labeled *Isocline:* $\dot{\hat{\delta}}_t^\star = 0$ traces the locus of points $(\hat{\delta}_t^*, I_t)$ such that $\dot{\hat{\delta}}_t^\star = 0$ in equation (9.17). Likewise, the upward sloping solid curve labeled *Isocline:* $\dot{I}_t = 0$ depicts the locus of points such that $\dot{I}_t = 0$ in equation (9.18). The intersection of the isoclines, when it exists, is a steady state of the dynamic system (9.17)–(9.18). The arrows summarize inventory and marginal investor dynamics in each region of the diagram: for example, a horizontal arrow pointing to the right (left) indicates that inventory is increasing (decreasing). Vertical arrows have a similar interpretation.

Lagos, Rocheteau, and Weill [2011] provide a complete analysis of a closely related diagram. In particular, they show that the dynamic system has a unique solution, traditionally called the *saddle path*, that converges to the steady state. This saddle path gives the unique equilibrium during the stress period while the steady-state describes the asymptotic behavior of the market in the event

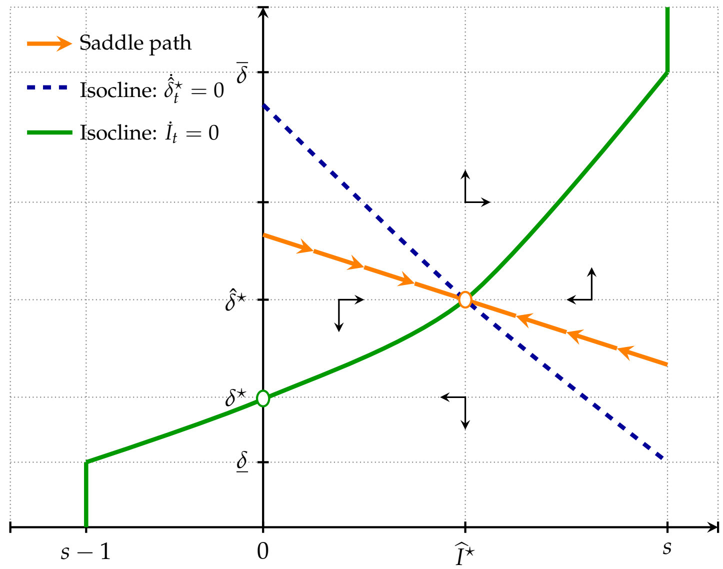

FIGURE 9.2: The phase plane diagram

The figure plots the phase plane diagram of the dynamic system formed by inventories and the utility flow of the marginal investor during the stress period.

where recovery never occurs. See Exercise 9.2 below for some results on the diagram associated to (9.17)–(9.18).

9.3.3 Taking stock

Figure 9.3 illustrates the finding of this section. It shows the realized path of inventories panel and the interdealer price during the stress period and the ensuing recovery. The upper panel shows that dealers accumulate inventories throughout the stress period $[0, \tau_\rho]$. The lower panel shows that the price initially jumps down and continues to fall until τ_ρ so as to induce dealers to hold more and more inventories. When the economy recovers, dealers start unwinding their positions. Correspondingly, the interdealer price jumps up at the recovery time and then grows at rate r until time $\tau_\rho + T$, at which point it has reached its steady state level.

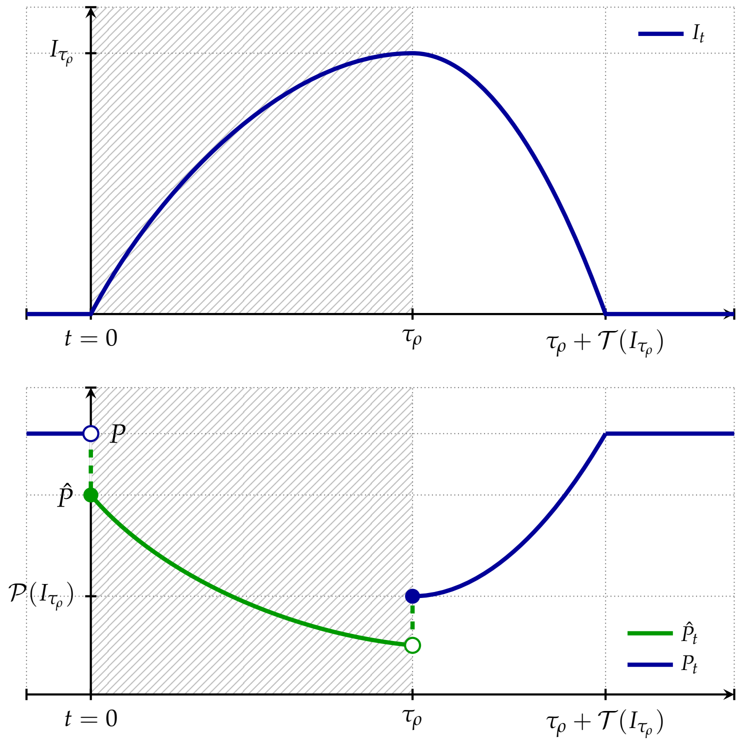

FIGURE 9.3: Inventory and price dynamics

The figure illustrates the time paths of a representative dealer's inventory (top panel) and the interdealer price (bottom panel) during the stress period (hatched region) and the ensuing recovery starting at time τ_ρ.

9.4 EXERCISES

Exercise 9.2. This exercise establishes some properties of the phase diagram corresponding to the dynamic system formed by (9.17) and (9.18).

1. Show that the two isoclines have a unique intersection with strictly positive inventories if and only if $\delta^\star > \mathbf{E}^F[x]$.
2. Letting $(\delta, I) \equiv x$, write the system as $\dot{x} = M(x)$. Show that the Jacobian of the vector valued function M has two real roots, one strictly positive and one strictly negative. Solve for the unique stable solution of the linear approximation to the ODE near the intersection point.

Exercise 9.3. Suppose that an asset in positive supply s which is initially held by infinitely-lived, risk neutral investors, called sellers, with subjective discount rate $r > 0$. Sellers can hold $q \in \{0, 1\}$ unit of the asset with flow utility $q\delta_L$. Starting at time zero, buyers slowly enter the economy. Just as sellers, they are risk-neutral with a discount rate $r > 0$ but when they hold $q \in \{0, 1\}$ unit of the asset their flow utility $q\delta_H$ is higher than that of sellers.

Assume that the mass of buyers in the market at time $t \geq 0$ is given by a continuous and strictly increasing function μ_{Ht} such that

$$\lim_{t\to\infty} (\mu_{Ht} - s) > \mu_{H0} = 0$$

and denote by T_s the unique solution to $\mu_{Ht} = s$.

1. Assume that the asset is traded in a *centralized* market.
   a) Solve for the demand of buyers and sellers.
   b) Argue that the marginal investor is a buyer if and only if $t > T_s$.
2. Assume next that the asset is traded in a semi-centralized market and that dealers are forbidden to hold any asset in inventory.
   a) Write the equations satisfied by the reservation values $R_t(\delta_H)$ and $R_t(\delta_L)$ of buyers and sellers.
   b) Show that $R_t(\delta_L) - R_t(\delta_H) = (\delta_H - \delta_L)/(r + \lambda_\theta)$.
   c) Argue that the marginal investor is a buyer if and only if $t > T_s$.
   d) Solve for the interdealer price over $[0, T_s)$ and (T_s, ∞). Compute the jump in the price at time T_s.
   e) Argue that dealers would have incentives to hold inventories if they were allowed to do so.
3. Assume finally that the asset is traded in a semi-centralized market in which dealers are allowed to have inventories. Guess that there exist $t_1 < T_s < t_2$ such that dealers hold inventories for all $t \in [t_1, t_2]$. Note that the lower endpoint t_1 may or may not be equal to zero.
   a) Argue that the interdealer price must be continuous.
   b) Show that

$$\dot{I}_t = \lambda (s - \mu_{Ht}), \qquad \forall t \in [t_1, t_2].$$

Use this ODE to show that

$$\int_{t_1}^{t_2} (s - \mu_{Ht})\, e^{\lambda t}\, dt = 0.$$

c) Write the equations for $R_t(\delta_L)$, $R_t(\delta_H)$ and P_t when $t \in [t_1, t_2]$.

d) Denote by

$$N_t = P_t - R_t(\delta_L)$$

the surplus in a match between a dealer and a seller at time t. Argue that $N_{t_2} = (\delta_H - \delta_L)\,/\,(r + \lambda_\theta)$ and that $N_{t_1} \geq 0$, with an equality if $t_1 \geq 0$. Solve for and verify that the latter condition can be written as

$$t_2 \leq t_1 + \frac{\log(\delta_H/\delta_L)}{r + \lambda_\theta} \text{ with `` = '' if } t_1 > 0.$$

e) Discuss comparative statics for the length of the inventory accumulation period $t_2 - t_1$.

9.5 NOTES AND REFERENCES

The representation of a stress event studied in this chapter is due to Grossman and Miller [1988]. The main substantive difference is that, in Grossman and Miller [1988], dealers have no advantage in timing the market over investors, so they do not provide liquidity services. Instead, they provide risk-sharing services until the recovery materializes. Weill [2004, 2007] first introduced a Grossman and Miller [1988] shock in a search model of an OTC market. Lagos, Rocheteau, and Weill [2011] extended his analysis in two ways: they assume a random recovery, as in this chapter, and they allow for unrestricted asset holdings. Di Maggio [2013] shows that dealers can have destabilizing equilibrium strategies in anticipation of a negative shock.

The model in this chapter is closely related to these papers, yet still novel in that it considers a specification with indivisible holdings and a continuum of types. Weill [2004, 2007] and Lagos, Rocheteau, and Weill [2011] go beyond the positive analysis of this chapter and derive normative implications. In particular, they show that, when the contact rate between investors and dealers is independent of the *ratio* of sellers to dealers, the equilibrium asset allocation maximizes welfare if and only if $\theta = 0$. Weill [2011] considers a market setting in which there are no search frictions but, instead, dealers face constraints on the flow of assets they can buy and sell from investors per unit of time. Duffie, Gârleanu, and Pedersen [2007], Feldhütter [2012], Trejos and Wright [2016], and Akın and Platt [2022] analyze the impact of shocks in OTC markets using models in which there are no dealers able to provide liquidity by holding inventories.

In this chapter, the stress event is modeled as a sudden increase in investors' demand for immediacy [Demsetz, 1968]. However, other types of negative shocks are also relevant. For example, during the 2008-2009 Global Financial Crisis, trading volume collapsed precisely in markets that were plagued by a surge in asymmetric information—namely, in asset-backed securities markets where investors had serious concerns about collateral quality. Camargo and Lester [2014] and Chiu and Koeppl [2016] have studied theoretically how the market recovers from such asymmetric information, and have characterized welfare-improving policy interventions, while Zou [2019] considers dynamic information acquisition decisions.

There is also a large empirical literature studying the impact of stress events in OTC markets. In addition to studying large (but rare) events, the empirical literature has also analyzed smaller events in which the shock that generates stress is better identified. In particular, researchers have studied supply shocks created by the exclusion of a bond from an index, when the bond crosses a maturity or credit threshold. The exclusion leads index funds to mechanically dispose of their holdings of the bond, creating a largely exogenous positive supply shock in the market; for example, see Bao, O'Hara, and Zhou [2018] and Dick-Nielsen and Rossi [2019]. The advantage of studying such a supply shock is that they are well identified. During a large stress event, such as the onset of the COVID-19 pandemic of March 2020, there are many shocks hitting the market at the same time—shocks that would impact the market even in the absence of liquidity frictions. In contrast, it is plausible that the impact of small supply shocks induced by index exclusion are only due to frictions.

Finally, recent research by Haddad, Moreira, and Muir [2021], Kargar et al. [2021a], O'Hara and Zhou [2021], and Boyarchenko, Kovner, and Shachar [2022] among others study the behavior of the corporate bond market during the acute selling pressure that emerged at the onset of the COVID-19 crisis. Duffie

[2020] and O'Hara and Zhou [2022] perform similar analyses of the Treasury and municipal bond markets, respectively. In general, they find that bond prices fell and transaction costs surged during the crisis period. However, several of these studies find that dealers did *not* accumulate inventories during these episodes. The theory above suggests conditions under which this can be the case; for example, if the stress event is expected to be long lived or if perceived frictions are small. Yet many have argued that dealers' unwillingness to to absorb assets could be caused by other forces not modeled directly in this chapter. For example, Duffie [2022] argues that post-GFC regulations increased dealers' cost of holding assets, and thus made them less willing to lean against the wind. However, formal theoretical studies of this claim have only just begun; see, e.g., Cohen, Kargar, Lester, and Weill [2023].