Chapter Seven

Divisible asset holdings

In this chapter, we considers an extension of the semi-centralized market model with divisible asset holdings: investors can choose the quantity q of assets they hold, and are no longer restricted to just two asset positions.

Section 7.1 characterizes the equilibrium: the asset demand, the equilibrium inter-dealer price, and the asset allocation. In the process, we establish economic and technical connections between this model and the one considered so far in previous chapter.

Next, in Section 7.2 we study special cases of the model that can be solved in closed form. These examples are workhorse tractable models that can be extended in a variety of directions. They also reveal the key novel prediction of the model with divisible asset holdings: search frictions lead investors to reduce the size of their trade. They demand less asset when they have a high valuation, and supply less when they have a low valuation. Because this shifts the supply and demand curve in opposite direction, this typically results in ambiguous price effects.

One of the main added value of considering divisible asset holding is to deliver a theory of trade size. This leads us, in Section 7.3, to study the corresponding empirical implications for trade size in OTC markets. First, we show that, with ex-ante homogeneous investors, proportional transaction costs are increasing with trade size, contrary to the evidence shown in Chapter 2. However, ex-ante heterogeneity in speed can reconcile the model with this evidence. Second, we show that the model has the counter-factual prediction that purchase and sales have the same average size.

7.1 EQUILIBRIUM

The model is the same as in Chapter 4 except for one difference: we assume that an investor can hold any quantity $q \geq q_0$ of asset with a flow utility $u(q, \delta)$. We assume that the utility flow $u(q, \delta)$ is twice continuously differentiable with respect to $(q, \delta) \in (q_0, \infty) \times \mathscr{D}$, strictly increasing in q near q_0, and strictly concave in $q \in (q_0, \infty)$. As before, we assume that the parameter δ creates positive shifts in asset demand: the cross derivative of the utility flow function is strictly positive, $u_{q\delta}(q, \delta) > 0$. Finally, for the asset demand to remain finite at all positive prices, and for equilibrium existence, we require that $\lim_{q\to\infty} u_q(q, \overline{\delta}) \leq 0$, $u_q(s, \underline{\delta}) > 0$, and $q_0 < s$.

7.1.1 Asset demand

Let $V(q, \delta)$ denote the value of an investor with holding q and utility type δ. The HJB equation for this value is:

$$rV(q, \delta) = u(q, \delta) + \gamma \mathbf{E}^F \left[V(q, x) - V(q, \delta) \right] + \lambda \left(V(q', \delta) - T' \right), \tag{7.1}$$

where q' the new asset holding of the investor and T' is the transfer that the investor makes to the dealer for this transaction. As in earlier chapters, we assume that (q', T') solves the generalized Nash bargaining problem:

$$\max \left(V(q', \delta) - V(q, \delta) - T' \right)^{1-\theta} \left(T' - P\left(q' - q\right) \right)^{\theta},$$

with respect to $q' \geq q_0$ and T'. The left-hand term of the Nash product is the net utility of the investor, while the right-hand term is the net profit of the dealer, the transfer made by the investor net of the cost of executing the desired transaction on the inter-dealer market. Changing variables from T' to $\Pi' = T' - P\left(q' - q\right)$, we obtain the equivalent problem:

$$\max \left(V(q', \delta) - V(q, \delta) - P(q' - q) - \Pi' \right)^{1-\theta} \left(\Pi' \right)^{\theta},$$

with respect to $q' \geq q_0$ and Π'. Since q' only appears on the left-hand term of the Nash product, it follows that it maximizes the trade surplus: $V(\delta, q') - V(\delta, q) - P(q' - q)$. Moreover, maximizing with respect to Π' reveals that

$$\Pi' = \theta \max\{V(\delta, q') - V(\delta, q) - P(q' - q)\},$$

with respect to $q' \geq q_0$. That is, the dealer appropriates a fraction θ of the maximum trade surplus. Plugging this back into the HJB equation (7.1), we obtain that the value of the investor solves:

$$\begin{aligned} rV(q,\delta) =& u(q,\delta) + \gamma \mathbf{E}^F \left[V(q,x) - V(q,\delta) \right] \\ &+ \lambda_\theta \max_{q' \geq q_0} \left\{ V(q',\delta) - V(q,\delta) - P\left(q' - q\right) \right\} \\ =& u(q,\delta) + \gamma \mathbf{E}^F \left[V(q,x) - V(q,\delta) \right] \\ &+ \lambda_\theta \left(Pq - V(q,\delta) \right) + \lambda_\theta \max_{q' \geq q_0} \left\{ V(q',\delta) - Pq' \right\}. \end{aligned} \tag{7.2}$$

In our earlier models with indivisible holdings, the optimal trading strategy could be entirely characterized by calculating the reservation value, $V(1,\delta) - V(0,\delta)$. The same is true here, but based instead on the marginal value $V_q(q,\delta)$. Namely, assuming that the value function is differentiable (see Exercise 24 for a proof) with respect to q, we have that

$$rV_q(q,\delta) = u_q(q,\delta) + \gamma \int \left[V_q(q,\delta') - V_q(q,\delta) \right] dF(\delta') + \lambda_\theta \left(P - V_q(q,\delta) \right),$$

keeping in mind that the term $\lambda_\theta \sup_{q' \geq q_0} \left\{ V(q',\delta) - Pq' \right\}$ has a zero derivative because it does not depend on the current holding, q. By inspection, this HJB equation has the same structure as our earlier equation for reservation value (4.6) except that δ is replaced by $u_q(q,\delta)$. Therefore, it has the same solution given by equation (4.11), namely:

$$V_q(q,\delta) = \frac{\lambda_\theta}{r + \lambda_\theta} P + \frac{r}{r + \lambda_\theta} U_q(q,\delta), \tag{7.3}$$

where

$$rU_q(q,\delta) \equiv \frac{\mathbf{E}_\delta \left[\int_0^{\tau_\theta} u(q,\delta_t) e^{-rt}\, dt \right]}{\mathbf{E} \left[\int_0^{\tau_\theta} e^{-rt}\, dt \right]},$$

and τ_θ is an exponentially distributed time with parameter λ_θ. As in Chapter (4), we interpret $U_q(q,\delta)$ is the certainty equivalent marginal utility that the investor derives until her next bargaining-adjusted contact time with dealers. This formula is exactly analogous to that of the reservation value in earlier chapters, with one difference: the certainty equivalent utility flow, $U_q(q,\delta)$ now depends on the quantity of asset held by the investor. Just as with the reservation value, we can use the marginal value determines to determine the optimal asset holding of an investor. Namely, the first-order condition of the investor is:

$$V_q(q,\delta) \leq P \Leftrightarrow U_q(q,\delta) \leq P, \tag{7.4}$$

with an equality if $q^\star(\delta,P) > q_0$, where the equivalence follows directly from (7.3). Given our maintained assumptions on the utility flow function $u(q,\delta)$, it follows that $U_q(q,\delta)$ is continuous, strictly increasing in δ, strictly decreasing in q, with $\lim_{q\to\infty} U_q(q,\delta) \leq 0$. This implies:

Lemma 4 (Optimal Asset Holding)**.** *For all $P > 0$, (7.4) has a unique solution $q^\star(\delta,P)$, which is continuous in (δ,P), increasing in $\delta \in \mathscr{D}$, decreasing in P, and strictly so if $q^\star(\delta,P) > q_0$. Moreover, for all $\delta \in (\underline{\delta},\bar{\delta})$, $\lim_{P\to 0} q^\star(\delta,P) > s$ and $\lim_{P\to\infty} q^\star(\delta,P) = q_0$.*

7.1.2 Market clearing and equilibrium inter-dealer price

To obtain the equilibrium inter-dealer price, we clear the market between investors in contact with dealers. As before, we equate the gross supply and the gross demand during a small interval of length h. The gross supply is equal to the total amount of assets held by those investors who contact the market during this small time interval. Given that contact times are independent across investors, with an intensity that is the same for all investor, the gross supply is equal to the measure of investors who contact the market, λh, multiplied by the average holding per investor's capita in the entire population, s. The gross demand originating from investors of type δ is equal to $\lambda h \, dF(\delta)$, the measure of investors of type δ who contact the market, multiplied by their optimal asset holding, $q^\star(\delta,P)$. Integrating across δ to obtain the aggregate gross demand, and equating with the gross supply, we obtain the market-clearing condition:

$$\lambda s = \lambda \int q^\star(\delta',P)\, dF(\delta'). \tag{7.5}$$

Lemma 4 imply that the right-hand side function is continuous, strictly decreasing as long as $q^\star(\delta,P) > q_0$ for some positive measure of investors, strictly larger (smaller) than s for all P large (small) enough. It follows that:

Lemma 5. *There is a unique P solving the market-clearing condition (7.5).*

7.1.3 Asset Allocation

We now characterize the resulting joint distribution of utility types and asset holdings. Let $\Phi(q,\delta)$ denote the cumulative measure of investors with holding less than q and utility type less than δ. We characterize $\Phi(q,\delta)$ in two steps.

First, we derive the measure of investors with holdings less than q,

$$\int \mathbb{I}_{\{q' \leq q\}}\, d\Phi(q',\delta').$$

As before we write the steady state equation in *gross* rather than *net* flow:

$$\lambda \int_{\delta} \mathbb{I}_{\{q^\star(\delta) \leq q\}}\, dF(\delta) = \lambda \int \mathbb{I}_{\{q' \leq q\}}\, d\Phi(q',\delta'),$$

where we suppress the explicit dependence of $q^\star(\delta)$ on P to simplify notations. The left-hand side is the gross inflow into the set of investors with holding less than q, generated by all investors with *optimal holding* less than q who contact dealers. The right-hand side is the gross outflow from this set, created by all investors with *current holding* less than q who contact dealers. Clearly, $q^\star(\delta) \leq q$ if and only if $\delta \leq \delta^\star(q)$ where $\delta^\star(q) \equiv \sup\{\delta \in \mathscr{D} : q^\star(\delta) \leq q\}$, with the convention that $\delta^\star(q) = \underline{\delta}$ if this set is empty. Hence, we obtain that

$$\int \mathbb{I}_{\{q' \leq q\}}\, d\Phi(q',\delta) = F(\delta^\star(q)).$$

This preliminary step facilitates the derivation of the entire distribution. Indeed, the inflow-outflow equation for $\Phi(q,\delta)$ can now be written:

$$\gamma F(\delta^\star(q))F(\delta) + \lambda \int \mathbb{I}_{\{\delta' \leq \delta \text{ and } q^\star(\delta) \leq q\}}\, dF(\delta') = (\gamma+\lambda)\Phi(q,\delta).$$

The first term on the left-hand side is the gross inflow created by investors with holding less than q who draw a new type less than δ. The second term is the gross outflow created by trade with dealers: investors with utility type less than δ and optimal holding less than $q^\star(\delta)$. The right-hand side is the gross outflow, created by all investors with type less than δ and holding less than q who either change type or contact dealers. Recalling the definition of $\delta^\star(q)$, this gives:

Lemma 6. *The cumulative measure of investors with holding less than q and utility* *type less than δ is:*

$$\Phi(q,\delta) = \frac{\gamma}{\gamma+\lambda}F(\delta^\star(q))F(\delta) + \frac{\lambda}{\gamma+\lambda}F(\min\{\delta,\delta^\star(q)\}).$$

Notice that, by letting $\delta \to \bar{\delta}$ we recover our earlier formula, $F(\delta^\star(q))$, for the measure of investors with holding less than q. The Lemma generalizes an earlier result in Chapter 4. The cumulative measure $\Phi(\delta,q)$ is a convex combination of two distribution. First, the random distribution of assets, $F(\delta^\star(q))F(\delta)$ where all investors with holdings less than q have a type δ drawn at random. Second, the perfect distribution of assets, where investors hold less than q if and only if their optimal holding is less than q.

7.1.4 Ex-ante Heterogeneity

Next, we propose a version of this model in which investors are ex-ante heterogenous: they can differ in their contact intensity, λ, their bargaining power θ, their preference shock intensity, γ, and in the distribution of their preference shocks, F. Such an extension is useful: as we will argue in Section 7.3.1 it helps reconcile the model with stylized facts about trade size and proportional transaction costs. In general, it offers a richer model to match many features of transaction data.

To see how to incorporate ex-ante heterogeneity in the model, let

$$\chi \equiv (\lambda,\theta,\gamma,F)$$

denote the type of an investor, and assume that χ is distributed according to the CDF $G(\chi)$. Then, we can derive the optimal asset holding of an investor of type χ, $q^\star(\delta,\chi,P)$ exactly as before. Also, just as before, equation (7.5) determines the total measure of asset per-investor of type χ

$$s(\chi,P) = \int q^\star(\delta,\chi,P)\,dF(\delta \mid \chi).$$

The difference is that $s(\chi,P)$ is endogenous: it is not in general equal to the exogenous supply per capita, since different types of investor may find it optimal to hold different amount of assets, on average. The market clearing condition is now that the amount of asset held by all types adds up to s, that is:

$$s = \int s(\chi, P)\, dG(\chi).$$

The same argument as before establishes the existence and uniqueness of an equilibrium inter-dealer price.

7.2 ANALYTICAL EXAMPLES

7.2.1 Quadratic utility flow

Consider first the quadratic specification:

$$u(q,\delta) = \mu q - \frac{\alpha\sigma^2}{2}q^2 + \alpha\sigma\delta q,$$

defined over $q \in \mathbb{R}$. This quadratic specification can be viewed as an approximation of a model with constant absolute risk aversion. Namely, in the utility flow function, μ and σ^2 are interpreted as the average and variance of the asset dividend. The parameter α corresponds to the investor's risk aversion. Finally, the parameter δ is related to the covariance between the asset dividend and the rest of the investor's portfolio. If the covariance is negative (positive), then the asset helps hedge these other risk, so δ is positive (negative).

In this case, elementary calculations similar to the one in Chapter 4 show that:

$$rU_q(q,\delta) = \mu - \alpha\sigma^2 q + \alpha\sigma r D(\delta)$$

where

$$rD(\delta) \equiv \frac{r+\lambda_\theta}{r+\lambda_\theta+\gamma}\delta + \frac{\gamma}{r+\lambda_\theta+\gamma}\mathbf{E}^F\left[\delta'\right] \tag{7.6}$$

is, as in Chapter 4, the certainty equivalent utility type of the investor until her next bargaining-adjusted contact time with dealers. Using (7.4), we obtain that the investor optimal asset holding is equal to

$$q^\star(\delta, P) = \max\left\{q_0, \frac{\mu + \alpha\sigma r D(\delta) - rP}{\alpha\sigma^2}\right\}$$

Consider first the case $q_0 \to -\infty$. Then, the constraint $q \geq q_0$ does not bind. Taking average with respect to F as in (7.5), noting that $\mathbf{E}^F\left[rD(\delta')\right] = \mathbf{E}^F\left[\delta'\right]$ for all λ_θ and solving the market clearing condition for the price, we obtain

$$rP = \mu - \alpha\sigma^2 s + \alpha\sigma\mathbf{E}^F\left[\delta'\right].$$

This formula has a standard interpretation: the price is equal to the present discounted value of dividend, appropriately risk-adjusted.

The most striking property of the above equilibrium price is that it does not depend on the level of frictions, λ_θ: in other words, when investors have quadratic utility flow and in the absence of short-selling constraints, the equilibrium price is equal to its frictionless counterpart for all λ_θ. This does not mean, of course, that friction do not matter for other equilibrium objects. Indeed, given that P does not depend on λ_θ in equilibrium, the equilibrium optimal asset holding of a type-δ investor satisfy:

$$\frac{\partial q^\star}{\partial \lambda_\theta} = \frac{1}{\alpha\sigma^2}\frac{\gamma}{(r+\lambda_\theta+\gamma)^2}\left(\delta - \mathbf{E}^F\left[\delta'\right]\right)$$

This mean that, when frictions decline and λ_θ increases, investors with higher-than-average δ demand more, while investor with lower-than-average δ supply more. This creates two opposite effects on the net asset demand, which cancel out exactly. In equilibrium, investors trade larger sizes but the price stay the same.

One sees however that this result is not robust to the introduction of a short-selling constraint, $q_0 = 0$. Indeed, when the short-selling constraint is binding, only one of the two effects operates. When λ_θ increases, investors with higher-than-average δ demand more, while investors with lower-than-average δ *cannot* supply more. The net effect is to increase demand and the equilibrium price.

This can be shown formally by noting that, when the short-selling constraint $q_0 = 0$ is binding, the optimal asset holding function becomes strictly convex in $D(\delta)$. Moreover, an increase in λ_θ creates a mean-preserving increase in the spread of $D(\delta)$. Correspondingly, aggregate demand increases at every price, and so does the equilibrium price.

7.2.2 Iso-elastic utility flow

Assume that the utility flow function is iso-elastic:

$$u(q,\delta) = \frac{q^{1-\alpha}-1}{1-\alpha}\,\delta$$

where $\alpha > 0$ and with the usual convention that $u(q,\delta) = \log(q)\delta$ if $\alpha = 1$. Indeed, the same calculations as above now show that

$$U_q(q,\delta) = q^{-\alpha} D(\delta),$$

where $D(\delta)$ is defined as (7.6). The optimal asset holding is always interior in this case and equal to

$$q^{\star}(\delta,P) = \left(\frac{D(\delta)}{P}\right)^{\frac{1}{\alpha}}.$$

Clearing the market as in equation (7.5), we obtain the equilibrium price:

$$P = s^{-\alpha}\mathbf{E}^F\left[D(\delta)^{\frac{1}{\alpha}}\right]^{\alpha}.$$

As before, we find that, when frictions are reduced, higher-than-average δ increase their demand, while lower-than-average δ increase their supply. Which of these two effects dominates depends on α. If $\alpha < 1$, then the optimal asset holding is a convex function of the effective type parameter $\hat{\delta}$. This means that the effect of λ_θ is stronger for higher- than for lower-than-average δ. Hence, the demand effect dominates. The opposite is true if $\alpha > 1$. When $\alpha = 1$, we obtain that the two effects exactly cancel out, so that frictions have no impact on price, similar to the quadratic utility flow case.

7.2.3 The indivisible holding limit

At first glance, the model with divisible holdings may appear fundamentally different from the model with indivisible holding, since it relaxes the $q \in \{0,1\}$ restrictions. It turns out, however, that the two are intimately related: the indivisible holding model can be viewed as a special case of the divisible holding case.

Perhaps the easiest way to make this point is to consider a Leontief utility flow function, $u(q, \delta) = \min\{q, 1\}\delta$: then, one easily sees that holding either zero or one is always optimal for an investor, and the analysis of equilibrium is exactly the same as in earlier chapters of this book.

One drawback of this argument is that the Leontief utility flow does not satisfies the regularity assumptions maintained in this chapter: it is not differentiable at $q = 1$ and it is not strictly concave. Arguably, the optimality of a $\{0, 1\}$ asset position relies on these departures from regularity. To address this criticism, one can go a step further and consider a utility flow function that is, in fact, differentiable and strictly concave $u(q, \delta) = m(q)\delta$ where

$$m(q) \equiv 1 - \frac{\log\left(1 + e^{1/\varepsilon\left[1 - q^{1-\varepsilon}/(1-\varepsilon)\right]}\right)}{\log\left(1 + e^{1/\varepsilon}\right)}, \tag{7.7}$$

and $\varepsilon \in (0, 1)$. One can show that the function $m(q)$ can be extended by continuity at $\varepsilon = 0$ where it is equal to $\delta \min\{q, 1\}$. For $\varepsilon > 0$, the function is strictly concave, twice continuously differentiable, and satisfies Inada conditions. Evidently, $\{0, 1\}$ asset holdings are no longer optimal for $\varepsilon > 0$, but they become approximately optimal as $\varepsilon \to 0$. Furthermore, one can show that the equilibrium price and allocation for $\varepsilon > 0$ converge to an equilibrium price and allocation for $\varepsilon = 0$ (see Exercise 25).

7.3 EMPIRICAL IMPLICATIONS

7.3.1 Trade size and proportional transaction costs

As shown in Chapter 2, a well-known observation about OTC markets is that large trades tend to be associated with low transaction costs. One of the advantage of this model with divisible asset asset holdings is to help address this evidence: the theory makes prediction about the relationship between trade size and transaction costs.

To do so, consider an investor seeks to purchase asset – the analysis of sale is symmetric. Specifically, assume that the investors has current utility flow δ and asset holding $q < q^\star(\delta)$. Hence, at his next contact time with dealers, this investor will make a purchase of size $q^\star(\delta) - q$, and will pay a fee equal to a

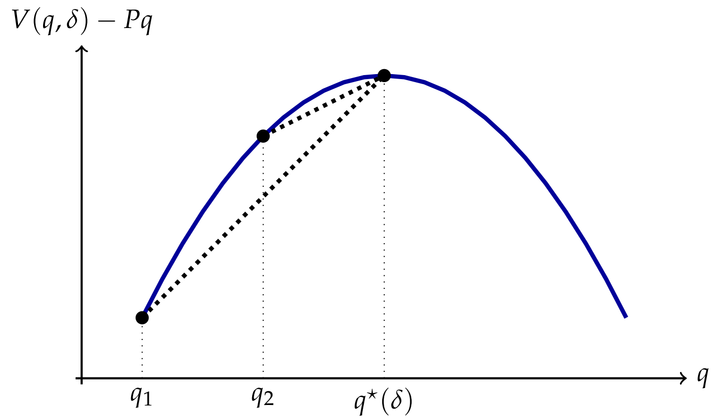

FIGURE 7.1: The relationship between transaction cost and trade size, holding the optimal demand constant.

fraction θ of the trade surplus. Therefore, the proportional transaction cost is

$$\theta \frac{V(q^\star(\delta),\delta) - V(q,\delta) - P(q^\star(\delta) - q)}{q^\star(\delta) - q}. \tag{7.8}$$

Figure 7.1 illustrates. The blue hump-shaped curve is the net value $V(q,\delta) - Pq$ of holding q units of the asset, which is maximized at the optimal holding, $q^\star(\delta)$. For any q, the proportional transaction cost (7.8) is proportional to the slope of the line segment between $(q, V(q,\delta) - Pq)$ and $(q^\star(\delta), V(q^\star(\delta),\delta) - Pq^\star(\delta))$. It is clear that the closer q is to $q^\star(\delta)$, the smaller the purchase, and the smaller the slope. Therefore, holding the optimal holding $q^\star(\delta)$ fixed, the proportional transaction cost increases with trade size $q^\star(\delta) - q$.

The economic intuition is simple and relies on two observations. First, investors have decreasing *marginal* value for holding assets. Second, the proportional transaction cost is equal to a fixed fraction θ of the *average* net value from trading. Taken together, this means that, if an investor's current holding, q, is further away from her optimal holding, $q^\star(\delta)$, then her trade surplus per quantity traded is larger.

This establishes that there is a increasing relationship between proportional transaction costs and trade size conditional on the optimal holding $q^\star(\delta)$. But this theoretical result does not apply directly to empirical observations, in regression analyses, it is difficult to control for an investor's optimal holding. The following lemma shows that the positive relationship holds unconditionally in an important case of interest: when investors have quadratic utility flow. In that case the proportional transaction costs (7.8) simplifies, because the value function $V(\delta, q)$ is quadratic. Indeed, a second-order Taylor expansion of $V(\delta, q)$ gives:

$$\begin{aligned} V(\delta, q) - &V(\delta, q^\star(\delta)) \\ &= V_q(q^\star(\delta), \delta)\,(q - q^\star(\delta)) + \frac{1}{2} V_{qq}(q^\star(\delta), \delta)\,(q - q^\star(\delta))^2 . \end{aligned}$$

The optimality of $q^\star(\delta)$ implies that the first derivative is $V_q(q^\star(\delta), \delta) = P$ when evaluated at $(\delta, q^\star(\delta))$. The calculations of Section 7.1 and 7.2.1 imply that the second derivative is:

$$V_{qq}(q, \delta) = \frac{rU_{qq}(q, \delta)}{r + \lambda_\theta} = -\frac{\alpha\sigma^2}{r + \lambda_\theta}.$$

It then follows from (7.8) and the Taylor approximation that:

Lemma 7. *With quadratic utility flow, the proportional transaction cost for an investor with current holding q and optimal holding $q^\star(\delta)$ is a linear and increasing function of trade size:*

$$\frac{\alpha\sigma^2\theta}{2(r + \lambda_\theta)} |q^\star(\delta) - q|. \tag{7.9}$$

Pinter, Wang, and Zou [50] use data from the U.K. bond market to shed new light about the empirical relationship between trade size and proportional transaction cost. They show that the negative unconditional relationship becomes positive after controlling for customers' fixed effect – such controls are not feasible with U.S. data because the customers' are not identified. This is lends support to this model since, conditional on investors characteristics captured by $(\lambda, \theta, \gamma, F)$ but also the preference type α, the relationship between proportional trading cost and size is indeed positive – though Pinter, Wang, and Zou argue empirically that the positive relationship after controlling for customer fixed effect is likely due to asymmetric information.

What type of customer heterogeneity can generate a negative relationship between trade size and proportional trading cost? The Lemma above suggests that heterogeneity in λ could work: indeed we already observed that a larger λ increases trade size. At the same time, in (7.9) an increase in λ reduces proportional transaction costs holding size constant. Exercise 26 studies an equilibrium with heterogeneous λ and derive condition for the cross-sectional relationship between trade size and proportional transaction cost to be negative.

7.3.2 Buy-sell symmetry

In Chapter 2 we have shown that, in the OTC market for U.S. Corporate Bonds, customer purchases have smaller size than sales. Correspondingly, since the total dollar value of purchase is approximately equal to that of sales, it follows that the *number* of customer purchase exceeds the number of sales. In this section we show that the model with divisible holding cannot account for this asymmetry: it implies that the number of customer purchases is equal to the number of customer sales.

To make this point let us denote by φ_p and φ_s the flow of purchases and sales. Assume to simplify the calculations that the distribution $F(\delta)$ is continuous. We have:

$$\varphi_p \equiv \lambda \int \mathbb{I}_{\{q^\star(\delta)>q\}}\, d\Phi(q,\delta) = \lambda \int \mathbb{I}_{\{\delta>\delta^\star(q)\}}\, d\Phi(q,\delta).$$

The first equality is definitional: it states that the flow of purchase is generated by those investors who contact dealers with an optimal holding that is strictly greater than their current holding – they buy $q^\star(\delta) - q > 0$. The second line follows from the definition of $\delta^\star(q)$. From Lemma 6, it follows that, in the domain of integration, $\delta > \delta^\star(q)$,

$$\Phi(q,\delta) = \frac{\gamma}{\gamma+\lambda} F(\delta^\star(q))F(\delta) + \frac{\lambda}{\gamma+\lambda} F(\delta).$$

Since the second term is constant in q for every (q,δ) in the domain of integration, it follows that

$$d\Phi(q,\delta) = \frac{\gamma}{\gamma+\lambda} d\left[F(\delta^\star(q))\right] dF(\delta).$$

Therefore, the flow of purchases is:

$$\varphi_p = \frac{\lambda\gamma}{\lambda+\gamma}\int \mathbb{I}_{\{\delta>\delta^\star(q)\}} F(\delta^\star(q))F(\delta) = \frac{\lambda\gamma}{\lambda+\gamma}\int \left[1-F(\delta^\star(q))\right]\, dF(\delta^\star(q)),$$

where the second equality follows by integrating over $\delta > \delta^\star(q)$ for each q. Now notice that, if the short-selling constraint is binding, $q^\star(\delta)$ is constant and equal to q_0 for all δ small enough and strictly increasing otherwise. Correspondingly, $\delta^\star(q)$ continuous at all q except at q_0, where it jumps from $\underline{\delta}$ to $\delta^\star(q_0)$. Figure 7.2 illustrates. Keeping in mind that $F(\delta^\star(q))$ inherits this discontinuity, we obtain:

$$\begin{aligned}\varphi_p &= \frac{\lambda\gamma}{\lambda+\gamma}\left(\left[1-F(\delta^\star(q_0))\right] F(\delta^\star(q_0)) + \frac{1}{2}\left[1-F(\delta^\star(q_0))\right]^2\right)\\ &= \frac{\lambda\gamma}{2\,(\lambda+\gamma)}\left[1-F(\delta^\star(q_0))^2\right].\end{aligned}$$

Proceeding similarly, we obtain the flow of sales:

$$\begin{aligned}\varphi_s &\equiv \lambda\int \mathbb{I}_{\{q^\star(\delta)<q\}}\, d\Phi(q,\delta) = \lambda\int \mathbb{I}_{\{\delta<\delta^\star(q-)\}}\, d\Phi(q,\delta)\\ &= \frac{\lambda\gamma}{\lambda+\gamma}\int F(\delta^\star(q-)))dF(\delta^\star(q)) = \frac{\lambda\gamma}{2(\lambda+\gamma)}\left[1-F(\delta^\star(q_0))^2\right].\end{aligned}$$

Taking stock, we obtain:

Lemma 8. *If the distribution of utility type is continuous, the flow of purchase and the flow of sales are both equal to*

$$\phi_s = \phi_p = \frac{\lambda\gamma}{2(\lambda+\gamma)}\left[1-F(\delta^\star(q_0))^2\right].$$

Correspondinly, the average trade size of a sale and of a purchase are also equal.

The formula when the short-selling constraint is slack and $\delta^\star(q_0) = \underline{\delta}$ has a natural intuition. Indeed, when the distribution is continuous, the fraction of investors whose seek to trade is equal to $\gamma/(\gamma+\lambda)$. This is because γ is the intensity with which an investor changes type and want to trade, while λ is the intensity with which an investor manages to trade. The perhaps less obvious result is that the measure of investors who want to sell is exactly equal to that who want to buy. When the short-selling constraint is binding, then the flow

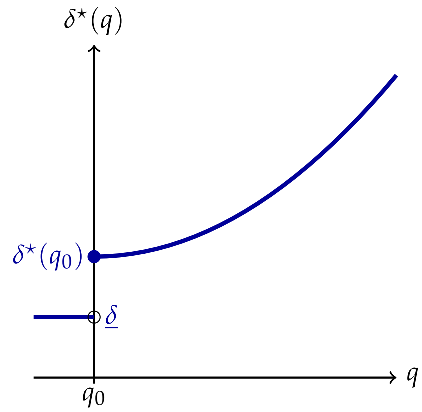

FIGURE 7.2: The function $\delta^\star(q)$.

of trade is reduced: for example an investor who already holds q_0 with utility $\delta < \delta^\star(q_0)$ may switch to an even lower utility type $\delta' < \delta$, but will not be able to sell because of the binding short-selling constraint. Symmetrically, if an investor with utility type δ' may switch to δ and not find it optimal to purchase.

All in all, we see that the model with divisible asset holding fails to explain a key feature of OTC market data: the apparent asymmetry between the number of sales and the number of purchase, and the corresponding asymmetry between the average size of a sale and that of a purchase.