Pricing Defaultable Debt in the Black & Cox Model

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Define S as the value of a company, $C(S,t)$ the equity value, D the nominal value of debt at maturity, $Ke^{-r(T-t)}$ the moving barrier where $K \leq D$ and r is the risk-free rate. So, we have the equity value as a down-and-out call option $C(S,t)$ strikes at D with a moving barrier $Ke^{-r(T-t)}$ to price.

1. Transform BSM Pricing PDE to a Heat Equation

$$[1] \quad \frac{\partial C}{\partial t} + rS\frac{\partial C}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 C}{\partial S^2} - rC = 0$$

Set $\tau = \frac{1}{2}\sigma^2(T-t), \quad S = Ke^{-r\frac{2\tau}{\sigma^2}}e^x, \quad C(S,t) = Ke^{-r\frac{2\tau}{\sigma^2}}v(x,\tau),$ so we have:

$$[2] \begin{cases} \frac{\partial C}{\partial t} = \frac{\partial C}{\partial \tau}\frac{\partial \tau}{\partial t} + \frac{\partial C}{\partial x}\frac{\partial x}{\partial \tau}\frac{\partial \tau}{\partial t} = Ke^{-r\frac{2\tau}{\sigma^2}}(rv - \frac{1}{2}\sigma^2 v_\tau - rv_x) \\ \frac{\partial C}{\partial S} = \frac{\partial C}{\partial x}\frac{\partial x}{\partial S} = Ke^{-r\frac{2\tau}{\sigma^2}}\frac{v_x}{S} \\ \frac{\partial^2 C}{\partial S^2} = Ke^{-r\frac{2\tau}{\sigma^2}}\frac{v_{xx}-v_x}{S^2} \end{cases}$$

Substitute [2] into [1] and remove $Ke^{-r\frac{2\tau}{\sigma^2}}$, we have:

$$[3] \quad rv - \frac{1}{2}\sigma^2 v_\tau - rv_x + rv_x + \frac{1}{2}\sigma^2(v_x - v_x x) - rv = \frac{1}{2}\sigma^2(v_x x - v_x - v_\tau)$$

Set $v(x,\tau) = e^{a(\tau)x+b(\tau)}u(x,\tau)$, so we have:

$$[4] \begin{cases} \frac{\partial v}{\partial \tau} = e^{a(\tau)x+b(\tau)}((a_\tau x + b_\tau)u + u_\tau) \\ \frac{\partial v}{\partial x} = e^{a(\tau)x+b(\tau)}(a(\tau)u(x,\tau) + u_x) \\ \frac{\partial^2 v}{\partial x^2} = e^{a(\tau)x+b(\tau)}(a^2(\tau)u(x,\tau) + 2a(\tau)u_x + u_{xx}) \end{cases}$$

Substitute [4] into [3] and remove $e^{a(\tau)x+b(\tau)}$, we have:

$$[5] \quad \frac{1}{2}\sigma^2(a^2u + 2au_x + u_{xx} - au - u_x - (a_\tau x + b_\tau)u - u_\tau)$$
$$= \frac{1}{2}\sigma^2(u_{xx} - u_\tau) + \frac{1}{2}\sigma^2(a^2 - a - (a_\tau x + b_\tau))u + \frac{1}{2}\sigma^2(2a-1)u_x$$

In order to have the heat equation, we need to have:

$$[6] \begin{cases} a^2 - a - (a_\tau x + b_\tau) = 0 \\ 2a - 1 = 0 \end{cases} \Rightarrow \begin{cases} b = -\frac{1}{4}(1-k)^2\tau \\ a = \frac{1}{2}(1-k) \end{cases} \quad k = 0 \text{ (to have a similar form as BSM PDE)}$$

So until now, we have $C(S,t) = Ke^{-r\frac{2\tau}{\sigma^2}}v(x,\tau) = Ke^{-r\frac{2\tau}{\sigma^2}}e^{\frac{1}{2}(1-k)x-\frac{1}{4}(1-k)^2\tau}u(x,\tau)$ and boundaries:

$$\begin{cases} C(S,T) = max(S_T - D, 0) \\ C(Ke^{-r(T-t)}, t) = 0 \end{cases}$$, so we have the heat equation:

$$\begin{cases} u_{xx} = u_\tau \\ u(x,0) = max(e^{\frac{1}{2}(k+1)x} - \frac{D}{K}e^{\frac{1}{2}(k-1)x}, 0) \\ u(0,\tau) = 0 \end{cases}$$

2. Solving Heat Equation on Semi-Infinite Domain

A fundamental solution of the heat equation on infinite domain is shown as below:

[7] $u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{+\infty} \phi(s) e^{-\frac{(x-s)^2}{4\tau}} ds$, where $\phi(s) = u(s,0)$

For the boundary condition $u(0,\tau) = 0$, we solve the heat equation on semi-infinite domain.

Set $\Phi(s) = \begin{cases} \phi(s) & s > 0 \\ -\phi(-s) & s < 0 \end{cases}$, so we have:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{-\infty}^{+\infty} \Phi(s) e^{-\frac{(x-s)^2}{4\tau}} ds = \frac{1}{2\sqrt{\pi\tau}} \int_{0}^{+\infty} \phi(s)(e^{-\frac{(x-s)^2}{4\tau}} - e^{-\frac{(x+s)^2}{4\tau}}) ds$$

For the boundary condition $\phi(s) = u(s,0) = max(e^{\frac{1}{2}(k+1)s} - \frac{D}{K} e^{\frac{1}{2}(k-1)s}, 0)$, we have:

$e^{\frac{1}{2}(k+1)s} - \frac{D}{K} e^{\frac{1}{2}(k-1)s} \geq 0 \quad \Rightarrow \quad s \geq \ln\frac{D}{K}$, so we have:

$$u(x,\tau) = \frac{1}{2\sqrt{\pi\tau}} \int_{\ln\frac{D}{K}}^{+\infty} (e^{\frac{1}{2}(k+1)s} - \frac{D}{K} e^{\frac{1}{2}(k-1)s})(e^{-\frac{(x-s)^2}{4\tau}} - e^{-\frac{(x+s)^2}{4\tau}}) ds$$

Set $y = \frac{s-x}{\sqrt{2\tau}}$, so we have:

$$l_1(x,\tau) = \frac{1}{\sqrt{2\pi}} \int_{\frac{\ln\frac{D}{K}+x}{2\tau}}^{+\infty} e^{\frac{1}{2}(k+1)(\sqrt{2\tau}y+x)} e^{-\frac{y^2}{2}} dy = e^{\frac{(k+1)}{2}x + \frac{(k+1)^2}{4}\tau} N(d_1). \text{ where } d_1 = \frac{\ln\frac{K}{D}+x}{2\tau} + \frac{\sqrt{2\tau}(k+1)}{2}$$

With the same method, we can have the rest three integrals as below: $(d_2 = \frac{\ln\frac{K}{D}+x}{2\tau} + \frac{\sqrt{2\tau}(k-1)}{2})$

$$u(x,\tau) = (e^{\frac{(k+1)x}{2} + \frac{(k+1)^2\tau}{4}} N(d_1) - \frac{D}{K} e^{\frac{(k-1)x}{2} + \frac{(k-1)^2\tau}{4}} N(d_2)) - (e^{-\frac{(k+1)x}{2} + \frac{(k+1)^2\tau}{4}} N(d_1) - \frac{D}{K} e^{-\frac{(k-1)x}{2} + \frac{(k-1)^2\tau}{4}} N(d_2))$$

3. Substitute $u(x,\tau)$ into $C(S,t)$

So until now, we have $C(S,t) = Ke^{-r\frac{2\tau}{\sigma^2}} v(x,\tau) = Ke^{-r\frac{2\tau}{\sigma^2}} e^{\frac{1}{2}(1-k)x - \frac{1}{4}(1-k)^2\tau} u(x,\tau)$.

$$v(x,\tau) = e^{\frac{1}{2}(1-k)x - \frac{1}{4}(1-k)^2\tau} u(x,\tau) = (e^{x+k\tau} N(d_1) - \frac{D}{K} N(d_2)) - (e^{-kx+k\tau} N(d_1) - e^{(1-k)x} N(d_2))$$

Since $k = 0$ and $C(S,t) = Ke^{-r\frac{2\tau}{\sigma^2}} v(x,\tau)$, we have:

$$C(S,t) = Ke^{-r\frac{2\tau}{\sigma^2}} e^x N(d_1) - De^{-r\frac{2\tau}{\sigma^2}} N(d_2) - (Ke^{-r\frac{2\tau}{\sigma^2}} N(d_1) - De^{-r\frac{2\tau}{\sigma^2}} e^x N(d_2))$$

Since $\tau = \frac{1}{2}\sigma^2(T-t)$, $\quad S = Ke^{-r\frac{2\tau}{\sigma^2}}$, we have:

$$C(S,t) = SN(d_1) - De^{-r(T-t)} N(d_2) - \frac{S}{Ke^{-r(T-t)}} (\frac{(Ke^{-r(T-t)})^2}{S} N(d_1) - De^{-r(T-t)} N(d_2))$$

d_1 and $d_2$have the same definitions as those in the BSM Pricing formula, so we finally have:

[8] $C(S,t) = C_{eu}(S,t) - \frac{S}{Ke^{-r(T-t)}} C_{eu}(\frac{(Ke^{-r(T-t)})^2}{S}, t)$,

where $C_{eu}(S,t)$ is the value of an European call option strikes at D.

4. Calculate Default Probability

For the down-and-out call option with moving barrier, we have:

$$C(S,t) = C_{eu}(S,t) - \frac{S}{Ke^{-r(T-t)}}C_{eu}(\frac{(Ke^{-r(T-t)})^2}{S}, t)$$

It is easy to deduce the down-and-out digital call option with moving barrier:

$$C_d(S,t) = C_{d/eu}(S,t) - \frac{S}{Ke^{-r(T-t)}}C_{d/eu}(\frac{(Ke^{-r(T-t)})^2}{S}, t),$$

where $C_{d/eu}(S,t) = N(d_2)$ is the value of an European digital call option strikes at D.

For an European call option, $N(d_2)$ is the probability of the option to be in-the-money at maturity.

So, it is reasonable to have $N(d_2)$ as the value of the digital option.

In order to calculate the default probability of a company, we need to have the digital option 0 value.

[9] $$\mathbb{P}(default) = 1 - C_d(S,t) = N(-d_2) + \frac{S}{Ke^{-r(T-t)}}N(d_2(\frac{(Ke^{-r(T-t)})^2}{S}))$$

5. Example: ***Dentsply Sirona*** 1-Year Default Probability

DENTSPLY SIRONA Inc. ("Dentsply Sirona") (NASDAQ: XRAY) is a dental solutions company. On May 12th, 2022, Moody's Investors Service ("Moody's") placed Dentsply Sirona's Baa2 issuer rating, Baa2 senior unsecured rating and Prime-2 commercial paper rating on review for downgrade[1]. In this section, we are going to estimate the default probability of Dentsply Sirona within one year and compare it with the Moody's Idealized Cumulative Expected Default Rates[2]. In our model for calculating the default probability, we need to know the debt, the value and the volatility of a company. However, in the real world, a company can have more than one debt with different maturity date, which means the moving barrier K and the maturity days T need to be well estimated.

5.1 Data

Since Dentsply Sirona completed the merger of DENTSPLY International Inc. and Sirona Dental Systems, Inc. on February 29th, 2016[3], we only use the financial data after the first quarter of 2016. The total data period is from 03/31/2016 to 03/31/2022. Total Assets in the balance sheet represents the value of the company V, Total Equity Gross Minority Interest the equity S, Total Liabilities Net Minority Interest the debt D including Current Liabilities as short-term debt C and Total Non Current Liabilities Net Minority Interest as long-term debt $D - C$. For risk-free rate r_f, we use the mean value of the 10-year treasury yield (Yahoo Finance: $\hat{\text{T}}$NX) in the past one year. All the data can be downloaded from Yahoo Finance[4].

5.2 Volatility of the Company

The next step is to estimate the volatility of the company's value. However, since the balance sheet is published quarterly, calculating the standard deviation of quarterly change of total assets V cannot provide us with the most recent volatility of the company. According to the paper of Jones in 1984[5], there is a following relationship between the volatility of the company's value V and the equity S:

$$\sigma_s = \sigma_v \frac{\partial S}{\partial V} \frac{V}{S} \tag{1}$$

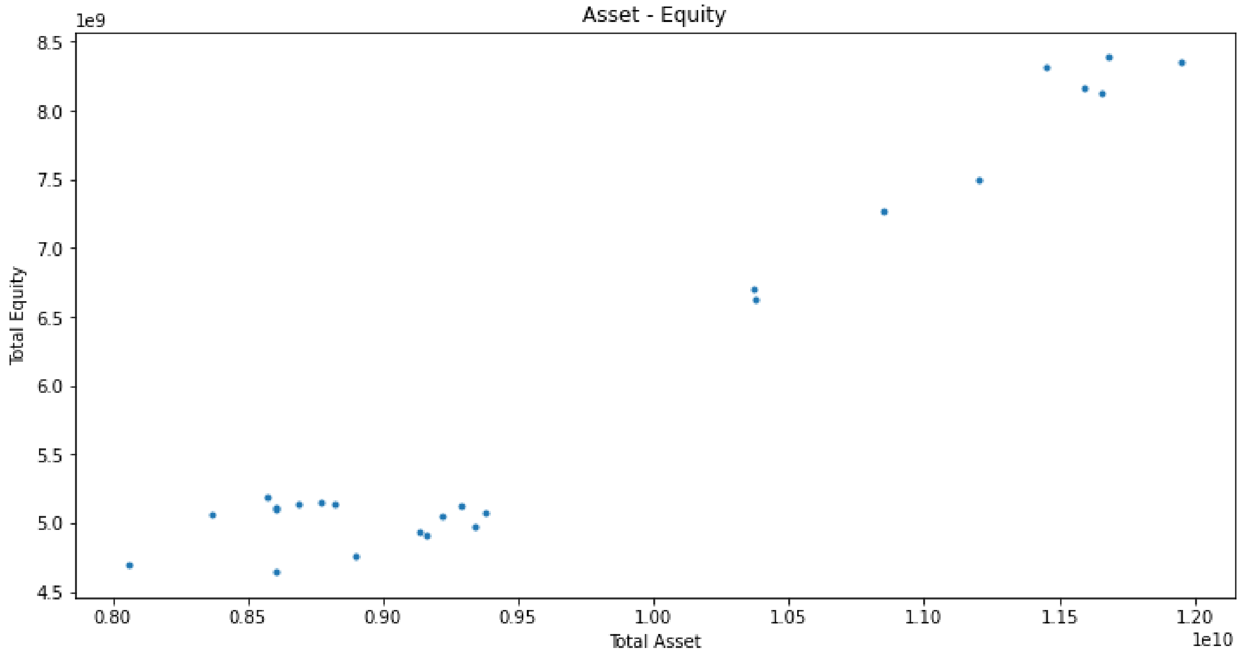
So, in this case, the first step is to calculate the volatility of the equity. Even though the Black-Scholes Model assumes that the volatility of the underlying asset is constant, we want to have an estimation that can well reflect the most recent volatility of the company. So, we figure out 4 different methods in calculating the volatility of the equity.

For Method 1, we simply use the standard deviation of quarterly change of total equity gross minority interest S since 03/31/2016 to represent the volatility of the equity. For Method 2, we use the daily return of the adjusted close price from 03/31/2016 to 03/31/2022 to calculate the standard deviation. For Method 3, in order to get the most recent volatility, we use the same method as Method 2 but with the data from 03/31/2022 to 05/12/2022. For Method 4, we use the implied volatility of options of the company, which provides an estimation for the volatility in the future. The implied volatility is calculated by using quote price data of all the out-of-the-money call and put options following the formula of Cboe VIX Index[6].

Table 1: Volatility of the Equity

Method	Formula	Value
1	$\sqrt{4} \times Std$(Quarterly Change of Equity)	0.11804993389484673
2	$\sqrt{252} \times Std$(Daily Return [03/31/16 - 03/31/22])	0.30656631888524577
3	$\sqrt{252} \times Std$(Daily Return [03/31/22 - 05/12/22])	0.5064703066353624
4	$\sqrt{(\sigma^2_{June} + \sigma^2_{July})/2}$	0.47827490264491196

The next step is to estimate the first-order derivative of equity to value. According to the Merton Model, the equity is an option of the company's value strikes at the debt. In this case, the derivative of equity to value is equal to the delta value of the option. Even though we have the model to calculate the option delta, the volatility of the company required by the model still need estimating. So, we decide to apply a linear regression on total equity gross minority interest S to total assets V from 03/31/2016 to 03/31/2022 and use the slope to represent the derivative of equity to value. It is clear that the value and the equity have a relationship close to linear shown as below:



Another method to estimate the derivative of equity to value is to use formula [8] to calculate the

Table 2: Linear Regression on Equity to Value

Variable	Equity S	Value V
Data	Total Equity Gross Minority Interest	Total Assets
Period	03/31/2016 - 03/31/2022	
Equation	$S = -4488004567 + 1.0784018144 \times V + \epsilon$	

delta value. We use the standard deviation of quarterly change of total assets V from 03/31/16 to 03/31/22 as the initial volatility of the company's value.

Finally, with the volatility of equity calculated by Method 3 ($\sigma_s = 0.5064703066353624$), when we substitute the derivative of equity to value calculated by the former 2 methods into formula (1), we have the volatility of the company shown as below: We decide to use Method 1 to calculate

Table 3: Volatility of the Company

Method	Formula	Value
0	$\sqrt{4} \times Std$(Quarterly Change of Total Assets)	0.09051647732200913
1	$\sigma_s \frac{S}{V} / \beta$	0.25153906886125293
2	$\sigma_s \frac{S}{V} / \Delta_{d/o}$	0.2712601882481537

the volatility of the company, considering that the delta value in Method 2 is calculated with the standard deviation of quarterly change of total assets V as the initial volatility, who has a significant bias, and it also shares all the errors in our model, which would be mentioned in the following section.

5.3 1-Year Default Probability

According to the calculation in the former 2 subsections, we have the risk-free rate r_f, the value V and the volatility of the company σ_v, so the final step is to select the debt as the strike price K and the maturity date for the option. We know that the Current Liabilities C in the balance sheet includes all the short-term debts normally due within one year, so it is reasonable for us to set the maturity date right at one year. However, for the Total Non Current Liabilities Net Minority Interest $D - C$ whose maturity days are longer than one year, the question remains is how can we estimate the discount value of them at one year. A considerable method is to assume that they all have the same long-term maturity date at T, and we use the risk-free rate as the discount rate to calculate the value of the long-term debt at one year:

$$K = C + (D - C) \times e^{-rf(T-1)} \tag{2}$$

Theoretically speaking, since all the long-term debts are due at different maturity dates, there is no way to know the suitable one to be set and applied to all the debts. So, the first method is to set the long-term maturity date T as a parameter and use the historical data to optimize it in a back test, but together with discount rate, we have two parameters need estimating, which would be time consuming and make the model too sensitive to parameters.

Free Cash Flow FCF in the cash flow represents the cash available for the company to repay creditors and pay out dividends and interest to investors. For this item, interest to debt holders has already been paid, so together with another item Repayment of Debt Pay_{debt}, we can roughly estimate the max cash flow that can be used to repay the debt. In this case, we can calculate the minimum number of years that all the debt until now could be paid off:

$$T = \frac{D}{(FCF^{TTM} + ABS(Pay_{debt}^{TTM}))} \tag{3}$$

We use the trailing 12-month (TTM) data published by Yahoo Finance on 03/31/2022 to calculate the long-term maturity date T with the former formula, and we have the result shown as below: For

Table 4: Long-Term Maturity Date

D	FCF	Pay_{debt}	T
4254000000	545000000	299000000	5.040284360189573

the discount rate, we decide to leave it as a parameter that can be optimized in the further research. In this case, we simply set it to be the same as the risk-free rate r_f. Finally, with the formula [9], we can calculate the default probability of Dentsply Sirona within 1 year:

Table 5: 1-Year Default Probability of Dentsply Sirona

V	K	σ_v	$T-t$	rf
9160000000	4054658276.232226	0.25153906886125293	1	0.017310750988142286
1-Year Default Probability		Moody's Idealized Cumulative Expected Default Rates (Baa2)		
0.0014108485506072466		0.0017		

Our model is a little bit undervalue for the 1-year default probability of Dentsply Sirona compares with Moody's rating result. The key parameters might influence the result are the long-term maturity date T and the discount rate. For further research, we can use back-test technique to optimize these two parameters if historical data is large enough.

References

[1] Kailash Chhaya, Ola Hannoun-Costa, *Rating Action: Moody's places Dentsply Sirona's ratings under review for downgrade*, Moody's Investors Service, 05/12/2022

[2] Kenneth Emery, *Rating Symbols and Definitions*, Moody's Investors Service, 03/22/2022, pp. 38

[3] Derek Leckow, Joshua Zable, *Dentsply Sirona Completes $14.5 Billion Merger of Equals*, Dentsply Sirona Global Headquarters, 02/29/2016

[4] Yahoo Finance: *https://finance.yahoo.com/quote/XRAY/financials?p=XRAY*

[5] E. Philip Jones, Scott P. Mason, Eric Rosenfeld, *Contingent Claims Analysis of Corporate Capital Structures: an Empirical Investigation*, The Journal of Finance, Jul., 1984, Vol. 39, No. 3, Papers and Proceedings, Forty-Second Annual Meeting, American Finance Association, San Francisco, CA, December 28-30, 1983 (Jul., 1984), pp. 617

[6] *The VIX Index: Basics*, Cboe Exchange, 10/08/2019