Moving Barriar option: Modeling Corporate Bond with Safety Covenant

Junming Yang, Yuhao Zheng

May 16, 2022

.

Name	Net ID	Tasks
Junming Yang	jy3295	Section 1, 2, 3, 6
Yuhao Zheng	yz3328	Section 4, 5, 6

1 Introduction

We are considering the credit modeling problem, the Black Cox model more specifically. In classic Merton credit model, equity S of a company is a call option on company's asset V with Strike = debt D, and bond default happens if such option is out-of-money at the expiration of Debt. The Black Cox model is a natural extension of Merton model. It allows for premature default by including a safety covenant which provides the company's bondholders with right to force the firm to bankruptcy or reorganization, if the company's asset value V falls below a time-dependent deterministic barrier $v_t = Ke^{-\gamma(T-t)}, \gamma > 0, \forall t \in [0, T)$.
In section 2, We are going to price the company's equity via a moving barrier call option, which naturally can be used to derive the value of the bond by subtraction from company's asset value. In section 3, we are going to derive the default probability of the debt under this moving covenant setting via a moving barrier digital call option.

2 Equity Pricing: Moving Barrier Call

To state the problem mathematically, following the notations above, equity S is a call option on asset value V with strike = debt D, with down-and-out moving barrier property. First, we are going to apply option pricing methodology to price the company's equity. Under GBM assumption for asset value V, we have a BS PDE for S:

$$\frac{\partial S}{\partial t} + rV\frac{\partial S}{\partial V} + \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 S}{\partial V^2} - rS = 0, \quad t \in [0, T), V > Ke^{-\gamma(T-t)} \tag{1}$$

coupled with boundary conditions:

$$S(V,T) = max(V-D,0), \quad S(Ke^{-\gamma(T-t)},t) = 0 \tag{2}$$

Obviously, for equity value S, its PDE has a time dependent boundary $v_t = Ke^{-\gamma(T-t)}$, which satisfies $0 < Ke^{-\gamma(T-t)} \leq De^{-r(T-t)}$, and thus $0 < K \leq D$. In order to make boundary time-independent, we change variable as

$$x = \ln\frac{V}{v_t}, \quad u(x,t) = S(V,t)$$

Then We have:

$$\frac{\partial S}{\partial t} = \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x}(-\gamma), \quad V\frac{\partial S}{\partial V} = \frac{\partial u}{\partial x}, \quad V^2\frac{\partial^2 S}{\partial V^2} = \frac{\partial^2 u}{\partial x^2} - \frac{\partial u}{\partial x} \tag{3}$$

Substituting them into PDE (1), we obtain

$$\frac{\partial u}{\partial t} + (r-\gamma-\frac{1}{2}\sigma_V^2)\frac{\partial u}{\partial x} + \frac{1}{2}\sigma_V^2\frac{\partial^2 u}{\partial x^2} - ru = 0 \tag{4}$$

with boundary conditions:

$$u(x,T) = max(Ke^x - D, 0), \quad u(0,t) = 0 \tag{5}$$

We introduce functions of time $a(t)$, $b(t)$ and transformation

$$u(x,t) = U(x,t)e^{a(t)x+b(t)}$$

Then we have

$$\frac{\partial u}{\partial t} = e^{a(t)x+b(t)}(\frac{\partial U}{\partial t} + U[a^{'}(t)x + b^{'}(t)]), \ \frac{\partial u}{\partial x} = e^{a(t)x+b(t)}(\frac{\partial U}{\partial x} + Ua(t)), \ \frac{\partial^2 u}{\partial x^2} = e^{a(t)x+b(t)}(\frac{\partial^2 U}{\partial x^2} + 2a(t)\frac{\partial U}{\partial x} + Ua^2(t))$$

Substituting them into PDE (4) and dividing both sides by $e^{a(t)x+b(t)}$, we obtain

$$\frac{\partial U}{\partial t} + U[a^{'}(t)x + b^{'}(t)] + (r-\gamma-\frac{1}{2}\sigma_V^2)(\frac{\partial U}{\partial x} + Ua(t)) + \frac{1}{2}\sigma_V^2(\frac{\partial^2 U}{\partial x^2} + 2a(t)\frac{\partial U}{\partial x} + Ua^2(t)) - rU = 0$$

We can eliminate terms with U and $\frac{\partial U}{\partial x}$ by imposing

$$a^{'}(t)x + b^{'}(t) + a(t)(r-\gamma-\frac{1}{2}\sigma_V^2) + \frac{1}{2}\sigma_V^2 a^2(t) - r = 0, \quad (r-\gamma-\frac{1}{2}\sigma_V^2) + a(t)\sigma_V^2 = 0 \tag{6}$$

The solution is

$$a(t) = a = -\frac{(r-\gamma-\frac{1}{2}\sigma_V^2)}{\sigma_V^2}, \quad b(t) = (\frac{(r-\gamma-\frac{1}{2}\sigma_V^2)^2}{2\sigma_V^2} + r)t \tag{7}$$

Meanwhile, we obtain the PDE satisfied by $U(x,t)$:

$$\frac{\partial U}{\partial t} + \frac{1}{2}\sigma_V^2\frac{\partial^2 U}{\partial x^2} = 0$$

with boundary conditions:

$$U(x,T) = max(Ke^x - D, 0)e^{-ax-b(T)}, \quad U(0,t) = 0$$

After a change of time variable $\tau = \frac{1}{2}\sigma_V^2(T-t)$, we finally reduce the problem to semi-infinite heat equation problem:

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}, \quad \tau \in (0,T], x > 0 \tag{8}$$

$$U(x,0) = e^{-ax-b(T)}max(Ke^x - D, 0), \quad U(0,\tau) = 0 \tag{9}$$

Since $K \leq D$, the reflected solution vanishes at $x \geq 0$, so we can apply the Method of Images without replicating payoff first. The value of a vanilla call option $C(V,t) = U_1(x,\tau)e^{ax+b(t)}$, then the down-and-out barrier call option value is $S(V,t) = (U_1(x,\tau) + U_2(x,\tau))e^{ax+b(t)}$, where $U_2(x,\tau) = -U_1(-x,\tau) = -C(\frac{v_t^2}{V}, t)e^{ax-b(t)}$. Note that here we take advantage of the fact that we already know the pricing formula for a standard vanilla European call. Therefore, we finally obtain

$$S(V,t) = C(V,t) - (\frac{V}{v_t})^{2a}C(\frac{v_t^2}{V}, t), \quad t \in [0,T], V \geq v_t \tag{10}$$

where

$$C(V,t) = V\mathcal{N}(d_+(V,t)) - De^{-r(T-t)}\mathcal{N}(d_-(V,t)), \quad d_\pm(V,t) = \frac{\ln\frac{V}{D} + (r \pm \frac{1}{2}\sigma_V^2)(T-t)}{\sigma_V\sqrt{T-t}}$$

We can verify that the solution (10) satisfies (1) and (2).
The delta of this down-and-out call option is

$$\Delta_{d/o} = \frac{\partial S}{\partial V} = \mathcal{N}(d_+(V,t)) - \frac{2a}{v_t}(\frac{V}{v_t})^{2a-1}C(\frac{v_t^2}{V}, t) + (\frac{V}{v_t})^{2a-2}\mathcal{N}(d_+(\frac{v_t^2}{V}, t)) \tag{11}$$

Notice that our solution is of the same form as the one with constant barrier, except that we have time-dependent barrier v_t instead of a constant B in the formula. Some might wonder, if we stand at a fixed time, e.g. $t = 0$, and let $B = v_0$, then how is this formula any different from a constant barrier one and imply a moving barrier setting? The difference lies on the value of a. In the constant barrier solution, recall that $a' = -\frac{r - \frac{1}{2}\sigma_V^2}{\sigma_V^2}$. Comparing it with our a defined in (7), we notice that a has extra γ term denoting barrier moving rate, and that $a > a'$ because of it. Therefore, the moving barrier option price given in (10) is smaller than a constant one, which makes sense since increasing barrier one is more likely to be knocked out. This higher knock-out probability is also implied in the same manner in the formula of Probability default given later.

3 Default Probability: Moving Barrier Digital Call

Next, we are going to calculate the probability of default of this company on the bond. To this end, we consider a European digital down-and-out call option with moving barrier, which pays at expiry one dollar if (1)the company's asset value V never falls below the safety covenant(moving barrier); (2)at expiry T, the company value $V \geq D$. Then, standing at time $t = 0$, obviously we have the following relation: $e^{-rT}\mathbb{P}(survival) = Dig_{d/o}$, where $Dig_{d/o}$ denotes the price of the digital down-and-out call mentioned above at $t = 0$. Therefore, we can derive probability of default by pricing the option.

We state the PDE for such option:

$$\frac{\partial Dig_{d/o}}{\partial t} + rV\frac{\partial Dig_{d/o}}{\partial V} + \frac{1}{2}\sigma_V^2 V^2 \frac{\partial^2 Dig_{d/o}}{\partial V^2} - rDig_{d/o} = 0, \quad t \in [0, T), V > Ke^{-\gamma(T-t)} \tag{12}$$

coupled with boundary conditions:

$$Dig_{d/o}(V, T) = \mathbb{1}_{\{V \geq D\}}, \quad Dig_{d/o}(Ke^{-\gamma(T-t)}, t) = 0 \tag{13}$$

Using the same changes of variables and function forms, we can finally get the counterpart of (8) and (9) for this digital option:

$$\frac{\partial U}{\partial \tau} = \frac{\partial^2 U}{\partial x^2}, \quad \tau \in (0, T], x > 0 \tag{14}$$

$$U(x, 0) = e^{-ax-b(T)}\mathbb{1}_{\{Ke^x \geq D\}}, \quad U(0, \tau) = 0 \tag{15}$$

To apply the Method of Image, we need to first consider the value of a standard European digital call $Dig(V, t)$, on underlying V and with strike D, without any barrier. We can easily derive its formula

$$Dig(V, t) = e^{-r(T-t)}\mathcal{N}(d_-(V, t)) \tag{16}$$

where d_- is defined above in (10), because $\mathcal{N}(d_-)$ is the probability of $V_T \geq D$.
Now, applying the same procedures of Method of Image as above, we can derive the pricing formula for our object option:

$$\begin{aligned} Dig_{d/o}(V, t) &= Dig(V, t) - (\frac{V}{v_t})^{2a} Dig(\frac{v_t^2}{V}, t) \\ &= e^{-r(T-t)}(\mathcal{N}(d_-(V, t)) - (\frac{V}{v_t})^{2a}\mathcal{N}(d_-(\frac{v_t^2}{V}, t))), \quad t \in [0, T], V \geq v_t \end{aligned} \tag{17}$$

Finally, the probability of default is given by

$$\mathbb{P}(default) = \mathcal{N}(-d_-(V, t)) + (\frac{V}{v_t})^{2a}\mathcal{N}(d_-(\frac{v_t^2}{V}, t)) \tag{18}$$

It has the extra second term compared to the default probability in Merton model, which is reasonable because it could default before debt expiration.

4 Test in Monte Carlo Simulation

In order to test our model, we calculate the value of the down-and-out call option and the digital call option with the moving barrier in Monte Carlo simulation. The steps of simulation for each iteration are shown as below:

- Simulate a path of company's value with the formula: $V_t = V_0 e^{(r-\frac{\sigma_V^2}{2})t+\sigma W_t}$, where $W_t \sim N(0,t)$
- Simulate a path of moving barrier with the formula: $v_t = Ke^{-\gamma(T-t)}$
- If $V_t < v_t$ for any $0 \leq t \leq T$, $Payoff_i = 0$, $Default_i = 1$
  else $Payoff_i = max(V_T - D, 0)$, $Default_i = 0$
- $S(V_0, 0) = e^{-rT}\frac{\sum_i^N Payoff_i}{N}$, $\mathbb{P}(default) = \frac{\sum_i^N Default_i}{N}$

We set $V = 60$, $K = 55$, $D = 55$, $T = 3$, $\sigma_V = \sqrt{360} \times 1.318\% = 25\%$, $r = 0.05$, and $\gamma = 0.1$, and for the Monte Carlo simulation, we set the number of iteration $N = 10000$ and the frequency of updating V_t is 10 times per day. Then, we have the result shown as below:

Table 1: Test in Monte Carlo Simulation

Item	Analytical Solution	Monte Carlo Simulation
Moving Barrier Call Option	14.6684	14.7966
Default Probability	0.5691	0.5602

It is obvious that there is a slight difference between the analytical price and the Monte Carlo simulated one within 1%. This slight difference would be magnified while pricing a company's equity, whose value is usually up to 10 billion dollars. However, we think the relative error is acceptable in this case.

Besides, we also test the model under some extreme conditions. When the strike price and the moving barrier are close to 0, we expect to have the value of the option S equal to the underlying price V, and when the parameter $\gamma = 0$, the barrier becomes static, which makes the option become a normal down-and-out call option. We have the results shown as below:

Table 2: Test in Extreme Condition

Condition	Analytical Solution	Theoretical Solution
$K = 10^{-8}$, $D = 10^{-8}$	60.00	60.00
$K = 50$, $\gamma = 0$	12.2603	12.2603

5 Example: *Dentsply Sirona* 1-Year Default Probability

DENTSPLY SIRONA Inc. ("Dentsply Sirona") (NASDAQ: XRAY) is a dental solutions company. On May 12$^{\text{th}}$, 2022, Moody's Investors Service ("Moody's") placed Dentsply Sirona's Baa2 issuer rating, Baa2 senior unsecured rating and Prime-2 commercial paper rating on review for downgrade[1]. In this section, we are going to estimate the default probability of Dentsply Sirona within one year and compare it with the Moody's Idealized Cumulative Expected Default Rates[2]. In our model for calculating the default probability, we need to know the debt, the value and the volatility of a company. However, in the real world, a company can have more than one debt with different maturity date, which means, besides the parameter γ, the moving barrier K and the maturity days T need to be well estimated.

5.1 Data

Since Dentsply Sirona completed the merger of DENTSPLY International Inc. and Sirona Dental Systems, Inc. on February 29$^{\text{th}}$, 2016[3], we only use the financial data after the first quarter of 2016. The total data period is from 03/31/2016 to 03/31/2022. Total Assets in the balance sheet represents the value of the company V, Total Equity Gross Minority Interest the equity S, Total Liabilities Net Minority Interest the debt D including Current Liabilities as short-term debt C and Total Non Current Liabilities Net Minority Interest as long-term debt $D - C$. For risk-free rate r_f, we use the mean value of the 10-year treasury yield (Yahoo Finance: T̂NX) in the past one year. All the data can be downloaded from Yahoo Finance[4].

5.2 Volatility of the Company

The next step is to estimate the volatility of the company's value. However, since the balance sheet is published quarterly, calculating the standard deviation of quarterly change of total assets V cannot provide us with the most recent volatility of the company. According to the paper of Jones in 1984[5], there is a following relationship between the volatility of the company's value V and the equity S:

$$\sigma_s = \sigma_v \frac{\partial S}{\partial V} \frac{V}{S} \tag{19}$$

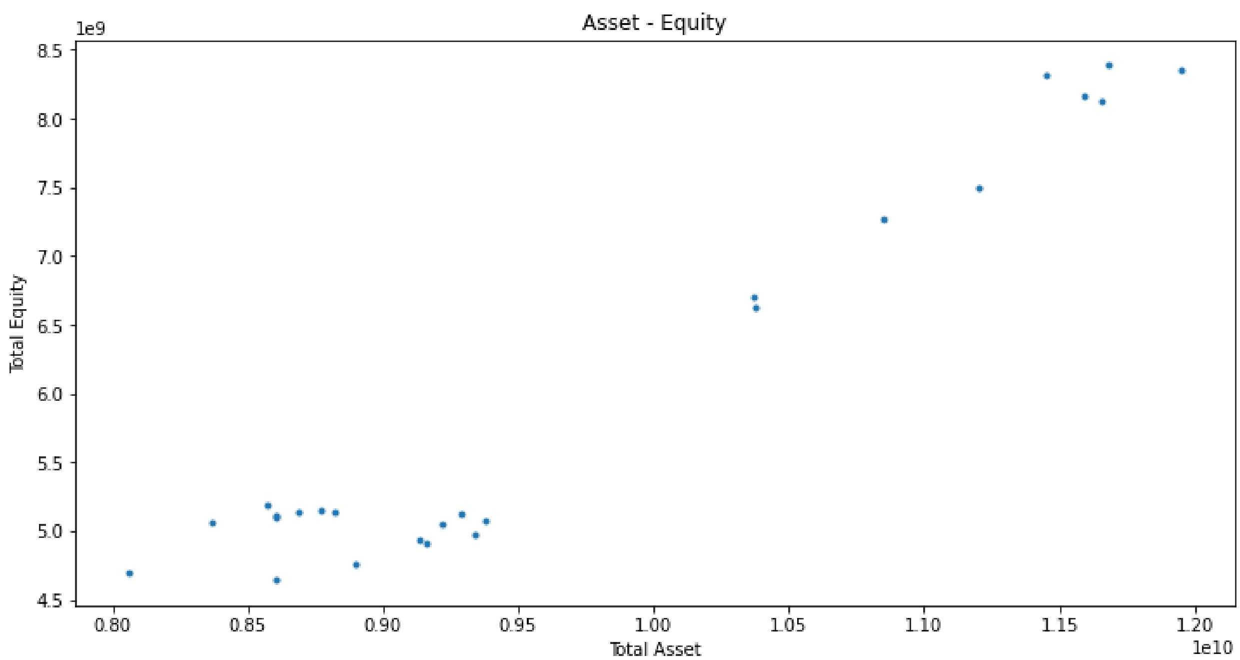
So, in this case, the first step is to calculate the volatility of the equity. Even though the Black-Scholes Model assumes that the volatility of the underlying asset is constant, we want to have an estimation that can well reflect the most recent volatility of the company. So, we figure out 4 different methods in calculating the volatility of the equity.

For Method 1, we simply use the standard deviation of quarterly change of total equity gross minority interest S since 03/31/2016 to represent the volatility of the equity. For Method 2, we use the daily return of the adjusted close price from 03/31/2016 to 03/31/2022 to calculate the standard deviation. For Method 3, in order to get the most recent volatility, we use the same method as Method 2 but with the data from 03/31/2022 to 05/12/2022. For Method 4, we use the implied volatility of options of the company, which provides an estimation for the volatility in the future. The implied volatility is calculated by using quote price data of all the out-of-the-money call and put options following the formula of Cboe VIX Index[6].

Table 3: Volatility of the Equity

Method	Formula	Value
1	$\sqrt{4} \times Std$(Quarterly Change of Equity)	0.11804993389484673
2	$\sqrt{252} \times Std$(Daily Return [03/31/16 - 03/31/22])	0.30656631888524577
3	$\sqrt{252} \times Std$(Daily Return [03/31/22 - 05/12/22])	0.5064703066353624
4	$\sqrt{(\sigma^2_{June} + \sigma^2_{July})/2}$	0.47827490264491196

The next step is to estimate the first-order derivative of equity to value. According to the Merton Model, the equity is an option of the company's value strikes at the debt. In this case, the derivative of equity to value is equal to the delta value of the option. Even though we have the model to calculate the option delta, the volatility of the company required by the model still need estimating. So, we decide to apply a linear regression on total equity gross minority interest S to total assets V from 03/31/2016 to 03/31/2022 and use the slope to represent the derivative of equity to value. It is clear that the value and the equity have a relationship close to linear shown as below:



Another method to estimate the derivative of equity to value is to use formula (11) to calculate the delta value. We use the standard deviation of quarterly change of total assets V from 03/31/16 to 03/31/22 as the initial volatility of the company's value.

Table 4: Linear Regression on Equity to Value

Variable	Equity S	Value V
Data	Total Equity Gross Minority Interest	Total Assets
Period	03/31/2016 - 03/31/2022	
Equation	$S = -4488004567 + 1.0784018144 \times V + \epsilon$	

Finally, with the volatility of equity calculated by Method 3 ($\sigma_s = 0.5064703066353624$), when we substitute the derivative of equity to value calculated by the former 2 methods into formula (19), we have the volatility of the company shown as below:

Table 5: Volatility of the Company

Method	Formula	Value
0	$\sqrt{4} \times Std$(Quarterly Change of Total Assets)	0.09051647732200913
1	$\sigma_s \frac{S}{V} / \beta$	0.25153906886125293
2	$\sigma_s \frac{S}{V} / \Delta_{d/o}$	0.2712601882481537

We decide to use Method 1 to calculate the volatility of the company, considering that the delta value in Method 2 is calculated with the standard deviation of quarterly change of total assets V as the initial volatility, who has a significant bias, and it also shares all the errors in our model, which would be mentioned in the following section.

5.3 1-Year Default Probability

According to the calculation in the former 2 subsections, we have the risk-free rate r_f, the value V and the volatility of the company σ_v, so the final step is to select the debt as the strike price K and the maturity date for the option. We know that the Current Liabilities C in the balance sheet includes all the short-term debts normally due within one year, so it is reasonable for us to set the maturity date right at one year. However, for the Total Non Current Liabilities Net Minority Interest $D - C$ whose maturity days are longer than one year, the question remains is how can we estimate the discount value of them at one year. A considerable method is to assume that they all have the same long-term maturity date at T, and we use the parameter γ in our moving barrier as the discount rate to calculate the value of the long-term debt at one year:

$$K = C + (D - C) \times e^{-\gamma(T-1)} \tag{20}$$

Theoretically speaking, since all the long-term debts are due at different maturity dates, there is no way to know the suitable one to be set and applied to all the debts. So, the first method is to set the long-term maturity date T as a parameter and use the historical data to optimize it in a back test, but together with discount rate γ, we have two parameters need estimating, which would be time consuming and make the model too sensitive to parameters.

Free Cash Flow FCF in the cash flow represents the cash available for the company to repay creditors and pay out dividends and interest to investors. For this item, interest to debt holders has already been paid, so together with another item Repayment of Debt Pay_{debt}, we can roughly estimate the max cash flow that can be used to repay the debt. In this case, we can calculate the minimum number of years that all the debt until now could be paid off:

$$T = \frac{D}{(FCF^{TTM} + ABS(Pay_{debt}^{TTM}))} \tag{21}$$

We use the trailing 12-month (TTM) data published by Yahoo Finance on 03/31/2022 to calculate the long-term maturity date T with the former formula, and we have the result shown as below:

Table 6: Long-Term Maturity Date

D	FCF	Pay_{debt}	T
4254000000	545000000	299000000	5.040284360189573

For the discount rate γ, we decide to leave it as a parameter that can be optimized in the further research. In this case, we simply set it to be the same as the risk-free rate r_f. Finally, with the formula (18), we can calculate the default probability of Dentsply Sirona within 1 year:

Table 7: 1-Year Default Probability of Dentsply Sirona

V	K	σ_v	$T-t$	rf
9160000000	4054658276.232226	0.25153906886125293	1	0.017310750988142286
1-Year Default Probability		Moody's Idealized Cumulative Expected Default Rates (Baa2)		
0.0014108485506072466		0.0017		

Our model is a little bit undervalue for the 1-year default probability of Dentsply Sirona compares with Moody's rating result. The key parameters might influence the result are the long-term maturity date T and the discount rate γ. For further research, we can use back-test technique to optimize these two parameters if historical data is large enough.

6 Conclusion

To model a company's equity value, bond value and corresponding default probability, we consider call option and digital option of European style with moving barrier. We define change of variables to incorporate time-variant barrier into new variables so as to fix the boundary of PDE, and then apply method of image to solve semi-infinite heat equations. We also compare our analytical formulas

with those of constant barrier options. Our Monte Carlo experiments prove the correctness of our analytical solutions.
In the end, we apply our model in calculating the 1-year default probability of the company DENTSPLY SIRONA Inc., whose Moody's rating is Baa2. Even though the result 0.14% is quite close to the Moody's Idealized Cumulative Expected Default Rates for Baa2, our model underestimates the default probability a little bit. The long-term maturity date T and the discount rate γ could be optimized in the further research for better the model.

References

[1] Kailash Chhaya, Ola Hannoun-Costa, *Rating Action: Moody's places Dentsply Sirona's ratings under review for downgrade*, Moody's Investors Service, 05/12/2022

[2] Kenneth Emery, *Rating Symbols and Definitions*, Moody's Investors Service, 03/22/2022, pp. 38

[3] Derek Leckow, Joshua Zable, *Dentsply Sirona Completes $14.5 Billion Merger of Equals*, Dentsply Sirona Global Headquarters, 02/29/2016

[4] Yahoo Finance: *https://finance.yahoo.com/quote/XRAY/financials?p=XRAY*

[5] E. Philip Jones, Scott P. Mason, Eric Rosenfeld, *Contingent Claims Analysis of Corporate Capital Structures: an Empirical Investigation*, The Journal of Finance, Jul., 1984, Vol. 39, No. 3, Papers and Proceedings, Forty-Second Annual Meeting, American Finance Association, San Francisco, CA, December 28-30, 1983 (Jul., 1984), pp. 617

[6] *The VIX Index: Basics*, Cboe Exchange, 10/08/2019