$Z \sim N(0,1)$

Question 2 Let $\{Z_t\}$ be a sequence of iid random variables which are normally distributed with mean zero and variance one. Define

Cor = 0

$$X_t = \begin{cases} Z_t, & \text{if } t \text{ is even,} \\ \frac{Z_{t-1}^2 - 1}{\sqrt{2}}, & \text{if } t \text{ is odd.} \end{cases}$$

$X_t = Z_t$ t even

Show that $\{X_t\} \sim \mathrm{WN}(0,1)$ but $\{X_t\}$ is not iid noise with mean zero and variance one.

✓ not indp ✓ ✓

Mean

t even : $\mathbb{E}[X_t] = \mathbb{E}[Z_t] = 0$

$X_t = \alpha X_{t-1}$

t odd : $\mathbb{E}[X_t] = \mathbb{E}\left[\frac{Z_{t-1}^2 - 1}{\sqrt{2}}\right]$

$$= \frac{\overbrace{\mathbb{E}[Z_{t-1}^2]}^{1} - 1}{\sqrt{2}} = 0$$

ACVF of X_t := $\gamma_X(k)$ at lag k

$\gamma_X(0)$ = Variance

$$\mathrm{Var}(X_t) = \mathbb{E}[X_t^2] - \underbrace{\mathbb{E}[X_t]^2}_{0}$$

$$X_t^2 = \begin{cases} Z_t^2 & t \text{ is even} \\ \frac{Z_{t-1}^4 - 2Z_{t-1}^2 + 1}{2} & t \text{ is odd} \end{cases}$$

$$\frac{3 - 2 + 1}{2} = 1$$

t even : $\mathbb{E}[X_t^2] = \mathbb{E}[Z_t^2] = \mathrm{Var}(Z_t)$

$= 1$ ✓

t odd : $\mathbb{E}[Z_{t-1}^4] = 3$

$N(0,1)$

Kurtosis = 3

$$\mathbb{E}[X_t^2] = \mathbb{E}\left[\frac{\cancel{Z_{t-1}^4}^{\,3} - 2\cancel{Z_{t-1}^2}^{\,1} + 1}{2}\right]$$

$$= \frac{3 - 2 + 1}{2} = 1$$ ✓

X_t has $\mu = 0$ $\sigma^2 = 1$

Lag 1 : X_t, X_{t+1}

Lag -1 : X_{t-1}, X_t

$\mathrm{Cov}(X_t, X_{t+1})$

$\mathrm{Cov}(X_{t-1}, X_t) = 0$

t even : $\mathbb{E}[Z_t^3] = 0$

$$\mathrm{Cov}(X_t, X_{t+1}) = \mathrm{Cov}\left(Z_t, \frac{Z_t^2 - 1}{\sqrt{2}}\right) = 0$$

$$= \mathbb{E}\left(Z_t \cdot \frac{Z_t^2 - 1}{\sqrt{2}}\right)$$

$$= \mathbb{E}\left(\cancel{Z_t^3}^{\,0} - \cancel{Z_t}^{\,0}\right) = 0$$

$$= \mathbb{E}\left(\frac{\cancel{Z_t^3}^{\to 0} - \cancel{Z_t}^{\to 0}}{\sqrt{2}}\right) = 0$$

$Z_t^3 = Z_t^2 Z_t$

$$= \frac{\cancel{\mathbb{E}[Z_t]}^{\to 0}(\mathbb{E}[Z_t]^2 - 1)}{\sqrt{2}}$$

t odd: Lag $\underline{1}$ X_t, X_{t+1}

$$\mathrm{Cov}\left(\frac{Z_{t-1}^2 - 1}{\sqrt{2}}, Z_{t+1}\right) = 0$$

$X_t = Z_t$

$X_t = \frac{Z_{t-1}^2 -}{\sqrt{2}}$

Hence, $\gamma_X(1) = \gamma_X(-1) = 0$

Since X_t is indp of $X_{t+2}, X_{t+3}, \ldots$ an

$X_{t-2}, X_{t-3}, \ldots,$ $\gamma_X(k) = 0$ for $|k|$

Hence $\gamma_X(k) = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } |k| > \end{cases}$

WN(0,1) WN = no correlation

$\gamma_X(k) = 0$ for $|k| > 0 \Rightarrow$ WN

<u>Not iid:</u> $X_t = Z_t$ $X_{t+1} = \frac{Z}{\quad}$

$\_\;1$

d

$= 2, 3, \ldots$

)

0

$$\frac{\;^2 - 1}{\sqrt{2}}$$

if $z_t = 5$ $\quad$ $X_t = 5$ $\quad$ $X_{t+1} = \frac{4}{\sqrt{2}}$