(Q4) Let $\{\epsilon_t\} \sim WN(0, \sigma_\epsilon^2)$. Calculate the ACF of the following processes:

(i) $X_t = \epsilon_t + 0.5\epsilon_{t-1} + 0.4\epsilon_{t-2}$.

① Calculate $\gamma_X(k)$, $k \in \mathbb{Z}$ (ACVF)

② Calculate $\rho_X(k)$, from $\gamma_X(k)$ (ACF)

→ Since X_t is a finite-order MA process, it is stationary. ⇒ ACVF & ACF are even functions e.g. $\gamma_X(k) = \gamma_X(-k)$ $\forall k$. Can focus on positive lags.

→ X_t is MA(2) ⇒ ACVF & ACF $= 0$ for $|k| > 2$.

Lag 0 (Variance)
Note that for (i), $\mathbb{E}[X_t] = 0$ and so $Var[X_t] = \mathbb{E}[X_t^2] - 0^2$.

Calculate $X_t^2 = (\epsilon_t + 0.5\epsilon_{t-1} + 0.4\epsilon_{t-2})^2$

$$X_t^2 = \epsilon_t^2 + (0.5)^2\epsilon_{t-1}^2 + (0.4)^2\epsilon_{t-2}^2 + 2(0.5)\epsilon_t\epsilon_{t-1} + 2(0.4)\epsilon_t\epsilon_{t-2} + 2(0.5)(0.4)\epsilon_{t-1}\epsilon_{t-2}.$$

Then taking expectations of X_t^2,

$$\mathbb{E}[X_t^2] = \mathbb{E}[\epsilon_t^2] + (0.5)^2\mathbb{E}[\epsilon_{t-1}^2] + (0.4)^2\mathbb{E}[\epsilon_{t-2}^2] + 2(0.5)\mathbb{E}[\epsilon_t\epsilon_{t-1}] + 2(0.4)\mathbb{E}[\epsilon_t\epsilon_{t-2}] + 2(0.5)(0.4)\mathbb{E}[\epsilon_{t-1}\epsilon_{t-2}].$$

Use results from Q1.

$$\mathbb{E}[X_t^2] = \sigma_\epsilon^2 + (0.5)^2\sigma_\epsilon^2 + (0.4)^2\sigma_\epsilon^2$$
$$+ 2(0.5)(0) + 2(0.4)(0) + 2(0.5)(0.4)(0)$$
$$= (1 + 0.5^2 + 0.4^2)\sigma_\epsilon^2 = \frac{141}{100}\sigma_\epsilon^2$$

Hence $\gamma_X(0) = Var(X_t) = \boxed{\mathbb{E}[X_t^2] = \frac{141}{100}\sigma_\epsilon^2.}$

Lag 1

$$Cov(X_t, X_{t+1}) = Cov(\epsilon_t + 0.5\epsilon_{t-1} + 0.4\epsilon_{t-2},$$
$$\epsilon_{t+1} + 0.5\epsilon_t + 0.4\epsilon_{t-1})$$

$$= Cov(\epsilon_t, \epsilon_{t+1}) + 0.5\,Cov(\epsilon_t, \epsilon_t) + 0.4\,Cov(\epsilon_t, \epsilon_{t-1})$$
$$+ 0.5\,Cov(\epsilon_{t-1}, \epsilon_{t+1}) + (0.5)^2\,Cov(\epsilon_{t-1}, \epsilon_t)$$
$$+ (0.5)(0.4)\,Cov(\epsilon_{t-1}, \epsilon_{t-1}) + 0.4\,Cov(\epsilon_{t-2}, \epsilon_{t+1})$$
$$+ (0.4)(0.5)\,Cov(\epsilon_{t-2}, \epsilon_t) + (0.4)^2\,Cov(\epsilon_{t-2}, \epsilon_{t-1}).$$

Since $\{\epsilon_t\}$ are uncorrelated, for all $s \neq t$,
$Cov(\epsilon_s, \epsilon_t) = 0$

As $Cov(\epsilon_t, \epsilon_t) = Var(X_t)$,

$$Cov(X_t, X_{t+1}) = 0 + 0.5\,Var(\epsilon_t) + 0.4(0) + 0.5(0)$$
$$+ (0.5)^2(0) + (0.5)(0.4)\,Var(\epsilon_{t-1}) + 0.4(0)$$
$$+ (0.4)(0.5)$$

$$= 0.5\,Var(\epsilon_t) + (0.5)(0.4)\,Var(\epsilon_{t-1}) = 0.7\sigma_\epsilon^2.$$

Hence $\gamma_X(1) = Cov(X_t, X_{t+1}) = 0.7\sigma_\epsilon^2.$

To speed up the process, omit any ϵ term not on both sides of the comma in $Cov(X_t, X_{t+1})$, as correlation is zero.

Lag 2

$$\mathrm{Cov}(X_t, X_{t+2}) = \mathrm{Cov}(\epsilon_t + 0.5\,\epsilon_{t-1} + 0.4\,\epsilon_{t-2},\ \epsilon_{t+2} + 0.5\,\epsilon_{t+1} + 0.4\,\epsilon_t)$$
$$= \mathrm{Cov}(\epsilon_t, 0.4\,\epsilon_t)$$
$$= 0.4\,\mathrm{Var}(\epsilon_t) = 0.4\,\sigma_\epsilon^2.$$

Hence $\gamma_X(2) = \mathrm{Cov}(X_t, X_{t+2}) = 0.4\,\sigma_\epsilon^2$.

Using that the ACVF is an even function and $\rho_X(k) = \gamma_X(k)/\gamma_X(0)$,

$$\gamma_X(k) = \begin{cases} \frac{141}{100}\sigma_\epsilon^2, & \text{if } k = 0, \\ 0.7\,\sigma_\epsilon^2, & \text{if } |k| = 1, \\ 0.4\,\sigma_\epsilon^2, & \text{if } |k| = 2, \\ 0, & \text{if } |k| > 2. \end{cases}$$

$$\Downarrow$$

$$\rho_X(k) = \begin{cases} 1, & \text{if } k = 0, \\ \frac{70}{141}, & \text{if } |k| = 1, \\ \frac{40}{141}, & \text{if } |k| = 2, \\ 0, & \text{if } |k| > 2. \end{cases}$$

Omit (ii) and (iii).

Q5) Investigate whether or not each of the following
(a) process is stationary, where $\{\epsilon_t\} \sim WN(0, \sigma_\epsilon^2)$.

(i) $Y_t = Y_{t-1} + \epsilon_t$.

Since $\mathbb{E}(\epsilon_t) = 0$, $E(Y_t) = \mathbb{E}(Y_{t-1})$ for all t. Therefore, the mean of $\{Y_t\}$ is constant.

Note that Y_{t-1} depends on $\epsilon_{t-1}, \epsilon_{t-2}, \ldots$ and so it is uncorrelated with ϵ_t. Thus

$$\begin{aligned} Var(Y_t) = Var(Y_{t-1} + \epsilon_t) &= Var(Y_{t-1}) + 2Cov(Y_{t-1}, \epsilon_t) \\ &\quad + Var(\epsilon_t) \\ &= Var(Y_{t-1}) + Var(\epsilon_t). \end{aligned}$$

For stationarity, the variance should be the same constant value for all t. Since $Var(\epsilon_t) > 0$, the variance of $\{Y_t\}$ increases with time t. Hence $\{Y_t\}$ is not stationary. In fact, $\{Y_t\}$ is a random walk.

(ii) For a real-valued constant $\alpha \neq 0$, $Y_t = Y_{t-1} + \alpha + \epsilon_t$.

Since $\mathbb{E}(\epsilon_t) = 0$, $\mathbb{E}(Y_t) = \mathbb{E}(Y_{t-1}) + \alpha$ for all t. Therefore, the mean of $\{Y_t\}$ is not constant. Hence $\{Y_t\}$ is not stationary.

In fact, $\{Y_t\}$ is a random walk with drift α.

(b) Assume that, for a real-valued constant $\alpha \in (-1,1) \setminus \{0\}$, $Y_t = \alpha Y_{t-1} + \epsilon_t$ is stationary. Determine the mean and ACF of $\{Y_t\}$.

- Taking expectations and using the fact that $\mathbb{E}(\epsilon_t) = 0$, we get that $\mathbb{E}(Y_t) = \alpha \mathbb{E}(Y_{t-1})$.

  Since the process is stationary, then $\mathbb{E}(Y_t) = \mathbb{E}(Y_{t-1})$. This can hold only if either $\mathbb{E}(Y_t) = 0$ or $\alpha = 0$. But $\alpha = 0$ is specifically excluded as a possible value of α, so the only possibility is that $\mathbb{E}(Y_t) = 0$.

- For the variance,

$$\mathrm{Var}(Y_t) = \mathrm{Var}(\alpha Y_{t-1} + \epsilon_t) = \alpha^2 \mathrm{Var}(Y_{t-1}) + 2\alpha \mathrm{Cov}(Y_{t-1}, \epsilon_t) + \mathrm{Var}(\epsilon_t).$$

  Using that Y_{t-1} is a function of $\epsilon_{t-1}, \epsilon_{t-2}, \ldots$, all of which are uncorrelated with ϵ_t, and that $\mathrm{Var}(\epsilon_t) = \sigma_\epsilon^2$,

$$\mathrm{Var}(Y_t) = \alpha^2 \mathrm{Var}(Y_{t-1}) + \sigma_\epsilon^2.$$

  As stationarity is assumed, then set $\gamma_Y(0) = \mathrm{Var}(Y_t)$. Thus

$$\gamma_Y(0) = \alpha^2 \gamma_Y(0) + \sigma_\epsilon^2 \implies \gamma_Y(0) = \frac{\sigma_\epsilon^2}{1-\alpha^2}.$$

- For the autocovariance, let $\gamma_Y(k)$ be the autocovariance at lag k. For $k > 0$,

$$\begin{aligned} \mathrm{Cov}(Y_t, Y_{t+k}) &= \mathrm{Cov}(Y_t, \alpha Y_{t+k-1} + \epsilon_{t+k}) \\ &= \alpha \mathrm{Cov}(Y_t, Y_{t+k-1}) + \mathrm{Cov}(Y_t, \epsilon_{t+k}). \end{aligned}$$

As $k > 0$ then $\mathrm{Cov}(Y_t, \epsilon_{t+k}) = 0$. Substituting for the autocovariance $\{\gamma_Y(k)\}$ gives

$$\gamma_Y(k) = \alpha \gamma_Y(k-1) = \alpha^2 \gamma_Y(k-2) = \dots = \alpha^k \gamma_Y(0).$$

By stationarity, γ_Y is an even function and so $\gamma_Y(k) = \gamma_Y(-k)$ for all k. Hence for all k

$$\gamma_Y(k) = \alpha^{|k|} \gamma_Y(0).$$

Thus the autocorrelation function ρ_Y is calculated from $\rho_Y(k) = \gamma_Y(k)/\gamma_Y(0)$, as

$$\gamma_Y(k) = \begin{cases} \dfrac{\sigma_\epsilon^2}{1-\alpha^2}, & \text{if } k = 0, \\ \alpha^{|k|}\gamma_Y(0), & \text{if } |k| > 0 \end{cases}$$

$$\Downarrow$$

$$\rho_Y(k) = \begin{cases} 1, & \text{if } k = 0, \\ \alpha^{|k|}, & \text{if } |k| > 0. \end{cases}$$

(Q6) S'pose that $\{U_t\}$ and $\{V_t\}$ are two uncorrelated stationary processes, i.e. U_s is uncorrelated with V_t, for every s and t.

Show that $\{Y_t\}$ is a stationary process, where $Y_t := U_t + V_t$, with autocovariance function (ACVF) equal to the sum of the ACVFs of U_t and V_t.

- Let $\mu_U := \mathbb{E}(U_t)$ and $\mu_V := \mathbb{E}(V_t)$. Taking expectations,

$$\mathbb{E}(Y_t) = \mathbb{E}(U_t + V_t) = \mu_U + \mu_V,$$

for all t. Thus the mean of Y_t does not depend on t and so $\{Y_t\}$ is stationary in mean.

- For the second moment, fix lag $k \in \mathbb{Z}$. Let $\gamma_X(k)$ represent the autocovariance function (ACVF) at lag k for process $X \in \{Y, U, V\}$. Then using the zero correlation between $\{U_t\}$ and $\{V_t\}$,

$$\begin{aligned}
\operatorname{Cov}(Y_t, Y_{t+k}) &= \operatorname{Cov}(U_t + V_t,\ U_{t+k} + V_{t+k}) \\
&= \operatorname{Cov}(U_t, U_{t+k}) + \operatorname{Cov}(U_t, V_{t+k}) \\
&\quad + \operatorname{Cov}(V_t, U_{t+k}) + \operatorname{Cov}(V_t, V_{t+k}) \\
&= \operatorname{Cov}(U_t, U_{t+k}) + \operatorname{Cov}(V_t, V_{t+k}) \\
&= \gamma_U(k) + \gamma_V(k).
\end{aligned}$$

Thus $\{Y_t\}$ is stationary in second-order moments with ACVF equal to the sum of the ACVFs of $\{U_t\}$ and $\{V_t\}$.

Hence $\{Y_t\}$ is stationary.

(Q7) Suppose that the air temperature is measured at a site. The air temperature process $\{U_t\}$ represents the 'signal'. However, the observations of the air temperature are not exact, due to 'noise'. Noise is unwanted modifications that a signal may suffer during capture, storage, transmission, processing or conversion.

Mathematically, what is actually observed at each time are samples from the process $\{Y_t\}$, where $Y_t = U_t + \epsilon_t$. Thus $\{U_t\}$ represents what we are really interested in – the true air temperature – and $\{\epsilon_t\}$ represents the noise.

Assume that

- $\{\epsilon_t\} \sim WN(0, \sigma_\epsilon^2)$
- The signal $\{U_t\}$ is a stationary process with variance $\sigma_U^2 > 0$.
- The signal $\{U_t\}$ and the noise $\{\epsilon_t\}$ are uncorrelated processes.

Define the 'signal-to-noise ratio' $SNR = \sigma_U^2 / \sigma_\epsilon^2$ and let $\rho_X(k)$ represent the ACF at lag k of process $X \in \{U, Y\}$.

Show that $\{Y_t\}$ is a stationary process with ACF

$$\rho_Y(k) = \frac{\rho_U(k)}{1 + \frac{1}{SNR}} \quad \forall k.$$

Comment on the result.

As $\{Y_t\}$ is the sum of stationary processes; it is itself stationary with autocovariance function equal to the sum of the autocovariance functions of $\{\epsilon_t\}$ and $\{U_t\}$ as in Q6.

Thus for each lag k and with $\gamma_X(k) := Cov(X_t, X_{t+k})$ for $X \in \{\epsilon, U\}$.

$$\rho_Y(k) = \frac{\gamma_Y(k)}{\gamma_Y(0)} = \frac{\gamma_U(k) + \gamma_\epsilon(k)}{\gamma_U(0) + \gamma_\epsilon(0)}.$$

For $k \neq 0$, as $\{\epsilon_t\}$ is a sequence of uncorrelated r.v.s, $\gamma_\epsilon(k) = Cov(\epsilon_t, \epsilon_{t+k}) = 0$. Also $\gamma_X(0) = \sigma_X^2$, for $X \in \{\epsilon, U\}$.

Hence

$$\rho_Y(k) = \frac{\gamma_U(k)}{\sigma_U^2 + \sigma_\epsilon^2} = \frac{\frac{\gamma_U(k)}{\sigma_U^2}}{1 + \frac{\sigma_\epsilon^2}{\sigma_U^2}} = \frac{\rho_U(k)}{1 + \frac{1}{SNR}}.$$

Comment on the result.

- If the noise variance is small compared to that of the signal, the signal is not disturbed very much. In that case, the SNR is large and hence $\rho_Y(k) \approx \rho_U(k)$. This approximation means that the observed signal $\{Y_t\}$ has much the observed signal $\{Y_t\}$ has much the same autocorrelation structure as the pure signal $\{U_t\}$. A high SNR means that the signal is clear and easy to detect or interpret.

- If the noise variance is about the same size as that of the signal then $SNR \approx 1$ and $\rho_Y(k) \approx 0.5\,\rho_U(k)$.

- If the noise variance is large compared to that of the signal, the signal is seriously disturbed: the SNR is small and $\rho_Y(k)$ is much smaller than $\rho_U(k)$. This means that the observed signal $\{Y_t\}$ has much weaker autocorrelation structure than the pure signal $\{U_t\}$. A low SNR means that the signal is corrupted or obscured by noise and may be difficult to distinguish or recover.

In general, increasing the variance of the noise imposed on the pure signal weakens the autocorrelation structure of the observed signal.