Tutorial 1 - Q3

Monday, September 15, 2025 4:31 PM

Question 3 Let $\{\epsilon_t\} \sim \mathrm{WN}(0, \sigma_\epsilon^2)$. Define a process $X = \{X_t; t \in \mathbb{Z}\}$ by

$$X_t = \beta\epsilon_{t-1} + \epsilon_t \quad \text{for } \beta \in \mathbb{R} \setminus \{0\}.$$

Show that

(i) X is stationary. ←

(ii) The autocorrelation function (ACF) of X has the form

$$\rho_X(k) = \begin{cases} 1, & \text{if } k = 0, \\ \rho, & \text{if } |k| = 1, \\ 0, & \text{if } |k| \geq 2, \end{cases}$$

for a constant ρ, with $\rho = \frac{\beta}{1+\beta^2}$.

(iii) Show that $|\rho| \leq 0.5$.

(iv) Show that the ACF of X is not unique by finding two values of β such that ρ has the same value. (For this reason, the restriction $|\beta| < 1$ is often made, over which range there is a unique value of ρ corresponding to each $\beta \in (-1, 1)$).

Defⁿ 1.3 (Weak Stationarity)

(i) $E[X_t^2] < \infty$ ✓ $\forall t \in \mathbb{R}$

(ii) $E[X_t] = \mu$ ✓

(iii) $\gamma(r,s) = \gamma(r+t, s+t)$ ✓

(ii) $E[X_t] = \beta E[\epsilon_{t-1}] + E[\epsilon_t] = 0$

(i) $E[X_t^2] = E[\beta^2\epsilon_{t-1}^2 + \cancelto{0}{2\beta\epsilon_{t-1}\epsilon_t} + \epsilon_t^2]$

$= E[\beta^2\epsilon_{t-1}^2 + \epsilon_t^2]$ $\quad Var(\epsilon_t) = \sigma_\epsilon^2$, $\quad E(\epsilon_t^2) = \sigma_\epsilon^2$

$= \beta^2 \underbrace{E[\epsilon_{t-1}^2]}_{\sigma_\epsilon^2} + \underbrace{E[\epsilon_t^2]}_{\sigma_\epsilon^2}$

$= \sigma_\epsilon^2(\beta^2 + 1) < \infty$

(iii) ACVF $\gamma_X(k)$

→ $Var(X_t) = E[X_t^2]$

$\gamma_K(0) = (\beta^2 + 1)\sigma_\epsilon^2$

→ $\gamma_K(1)$:

$Cov(X_t, X_{t+1}) = E((X_t)(X_{t+1}))$

$= E((\beta\epsilon_{t-1} + \epsilon_t)(\beta\epsilon_t + \epsilon_{t+1}))$

$= \beta^2 \cancelto{0}{E(\epsilon_{t-1}\epsilon_t)} + \beta \cancelto{0}{E(\epsilon_{t-1}\epsilon_{t+1})}$

$+ \beta E(\epsilon_t^2) + \cancelto{0}{E(\epsilon_t\epsilon_{t+1})}$

$\gamma_X(1) = \beta\sigma_\epsilon^2$

→ $\gamma_K(2)$:

$\Rightarrow$ 0 k l 2 l

$$Cov(X_t, X_{t+2}) = E((X_t)(X_{t+2}))$$

$$= E((\beta\epsilon_{t-1} + \epsilon_t)(\beta\epsilon_{t+1} + \epsilon_{t+2}))$$

$$= 0$$ b/c only $\epsilon_t^2 = 1$

$$\gamma_X(k) = 0 \quad \text{for} \quad |k| \geq 2$$

$$ACVF = \gamma_X(k) = \begin{cases} (\beta^2+1)\sigma_\epsilon^2 & k=0 \\ \beta\sigma_\epsilon^2 & |k|=1 \\ 0 & |k| \geq 2 \end{cases}$$

(ii)

$$\rho_X(k) = \frac{\gamma(k)}{\gamma(0)} = \begin{cases} 1 & k=0 \\ \dfrac{\beta}{\beta^2+1} = \rho & |k|=1 \\ 0 & |k| \geq 2 \end{cases}$$