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Finite-time observer for a time-varying cascade of an ODE in a system of balance laws ⋆

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Abstract: In this paper, we design a finite-time convergent observer for linear time-varying ODE-hyperbolic balance law cascade systems. This work extends the study of output regulation for linear time-varying hyperbolic systems. In Bai et al. (2025), a state feedback regulator was developed to achieve finite-time output regulation under the assumption that all disturbances and system states are known. However, this assumption is unrealistic in practical applications. Therefore, we design a finite-time observer under the condition that only the to-be-tracked reference signal and boundary values of the system state are the available measurement. The key idea is to use two ODE observers to achieve finite-time observation of the ODE subsystem and to exploit the finite-time stability structure of the hyperbolic subsystem to estimate its state. In this work, we utilize results on the observability and estimability of linear time-varying ODE systems to establish the existence of the proposed observer.

Keywords: Lyapunov-based and backstepping techniques, observer design, hyperbolic systems, non-autonomous systems, output regulation

1. INTRODUCTION

The control problem of hyperbolic equations plays a pivotal role both in mathematics and in applied fields. Such systems appear in various aspects of nature and applications, such as the propagation of water in open channels, gas flow pipelines, the propagation of epidemics (see Bertaglia and Pareschi (2021)), or road traffic flow models (see Aw and Rascle (2000)). For more examples of hyperbolic systems in various applications, one can find numerous cases in (Bastin and Coron, 2016, Chap. 1) and the references therein.

This work is a continuation of Bai et al. (2025). In Bai et al. (2025), the finite-time output regulation problem for linear time-varying hyperbolic systems was addressed. Under the assumption that the system states and disturbances are known, a state feedback regulator was designed to solve the output regulation problem. However, such an assumption is unrealistic in practice. Therefore, this paper aims to design a finite-time convergent observer for this control problem under the condition that only the signal to be tracked and the boundary values of the system states can be measured. Based on this observer, an output feedback regulator is further developed.

Concerning the backstepping method for time-independent hyperbolic systems, one can refer to Coron et al. (2013); Hu et al. (2019). More recently, Coron et al. (2021) extended this approach to time-varying hyperbolic systems. For the design of observers, Vazquez et al. (2011) and Hu et al. (2016) both employed the backstepping method to design observers for 2×2 and $n\times n$ linear time-independent hyperbolic systems, respectively.

The main contribution of this paper is the design of an observer for linear time-varying ODE-hyperbolic systems. Inspired by Deutscher (2017a,b); Deutscher and Gabriel (2020); Coron et al. (2021); Hu et al. (2016); Ikeda et al. (1975), we employ a time-dependent backstepping approach to design a minimum-time convergent observer. For the observation of the reference signal system, based on the results of Ikeda et al. (1975), we design a finite-time convergent reference signal observer in a similar way as in Engel and Kreisselmeier (2002); Menold et al. (2003), under the assumption that the signal system is uniformly completely observable. This observer consists of two sub-observers and a time delay. For the disturbance and state observers, we use a three-step transformation to simplify the error system. The first transformation is a Volterra transformation of the second kind as in Coron et al. (2021). The second transformation is similar to that in Deutscher

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(2017b); Deutscher and Gabriel (2020), aiming to decouple the error system into a PDE-ODE cascaded system. The third transformation is a Fredholm transformation as in Coron et al. (2021), aiming to obtain the minimum convergent time. The remaining task is to determine the observer gain functions for the finite-dimensional disturbance system to achieve the design of a finite-time observer. Similar to the observer for the reference signal system, we utilize two sub-observers and a time delay to accomplish this goal. Contrary to the results in Deutscher (2017b); Deutscher and Gabriel (2020), all the transformations and observer gain functions in this paper need to be time-dependent.

The remaining part of this paper is organized as follows. In Section 2, we introduce the considered problems and review the results from Bai et al. (2025). Some preliminaries needed in the paper is given in Section 3. Then, Section 4 presents the main results of this paper, namely the design of the finite-time convergent observers and the design of the output feedback regulator.

Throughout the paper, we use the following notation. For $T > 0$ and $0 \leq t_0 < T$, define the domain $\mathcal{D}(t_0) = \{(t,x) | t_0 < t < T,\ 0 < x < 1\}$. For a vector ν and a matrix A, denote by $\|\nu\|$ the Euclidean norm and by $\|A\|$ the norm of A compatible with $\|\nu\|$. For symmetric matrices P and Q, $P > 0$ ($P \geq 0$) means that P is positive (nonnegative) definite, and $P > Q$ ($P \geq Q$) means $P - Q > 0$ ($P - Q \geq 0$). Denote by Id_n the $n \times n$ identity matrix and denote $\mathrm{B}_k = (\mathrm{Id}_k, \mathrm{Id}_k)^\top$. Denote by $\mathrm{diag}(A_1, \ldots, A_n)$ the block diagonal matrix with matrices $A_1, \ldots, A_n$ on the diagonal, where A_i are matrices of potentially different sizes.

2. PROBLEM STATEMENT

In this paper, combining the systems from Coron et al. (2021); Deutscher and Gabriel (2020), we consider the following linear time-varying $n \times n$ hyperbolic system as in Bai et al. (2025), for (t,x) in $\mathcal{D}(t_0)$,

$$\begin{aligned}
&\partial_t w(t,x) + \Lambda(t,x)\partial_x w(t,x) = A(t,x)w(t,x) + g_1(t,x)d(t),\\
&w_+(t,0) = Q_0(t)w_-(t,0) + g_2(t)d(t),\\
&w_-(t,1) = Q_1(t)w_+(t,1) + u(t) + g_3(t)d(t),\\
&w(t_0,x) = w^0(x), \quad y_m(t) = w_-(t,0), \quad y(t) = \mathcal{C}[w,d](t).
\end{aligned} \tag{1}$$

In (1), $w : \mathcal{D}(t_0) \to \mathbb{R}^n$ is the state, w^0 in $L^2(0,1)^n$ is the initial data at time t_0, $u(t)$ in $\mathbb{R}^m$ is the control input, $d(t)$ in $\mathbb{R}^h$ is the disturbance, $y(t)$ in $\mathbb{R}^q$ is the output to be controlled and $y_m(t)$ in $\mathbb{R}^m$ is the available measurement. The matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ couples the equations of the system inside the domain, the matrix Q_0 (resp. Q_1) couples the equations of the system on the boundary $x = 0$ (resp. $x = 1$) and the matrices g_i, $i = 1, \ldots, 4$ are disturbance input locations. Let us make the following assumptions on all coefficients involved in (1).

Assumption 1. The matrix Λ is diagonal, namely $\Lambda(t,x) = \mathrm{diag}(\lambda_1(t,x), \ldots, \lambda_n(t,x))$ for (t,x) in $[0,+\infty) \times [0,1]$.

Assumption 2. Assume that $n \geq 2$. Denote by m in $\{1, \ldots, n-1\}$ the number of equations with negative speeds and by $p = n - m$ in $\{1, \ldots, n-1\}$ the number of equations with positive speeds. We assume that there exists some $\varepsilon_0 > 0$ such that for (t,x) in $[0,+\infty) \times [0,1]$, we have

$$\begin{aligned}
\lambda_1(t,x) < \cdots < \lambda_m(t,x) < -\varepsilon_0 < 0&\\
< \varepsilon_0 < \lambda_{m+1}(t,x) < \cdots < \lambda_n(t,x),&
\end{aligned} \tag{2}$$

and, for every i in $\{1, \ldots, n-1\}$,

$$\lambda_{i+1}(t,x) - \lambda_i(t,x) > \varepsilon_0. \tag{3}$$

Assumption 2 is identical to the assumption in Coron et al. (2021). All along this paper, for a vector (or vector-valued function) ν in $\mathbb{R}^n$ and a matrix (or matrix-valued function) B in $\mathbb{R}^{n \times n}$, we use the notation $\nu = \begin{pmatrix} \nu_- \\ \nu_+ \end{pmatrix}$ and $B = \begin{pmatrix} B_{--} & B_{-+} \\ B_{+-} & B_{++} \end{pmatrix}$ with v_- in $\mathbb{R}^m$, v_+ in $\mathbb{R}^p$ and B_{--} in $\mathbb{R}^{m \times m}$, B_{-+} in $\mathbb{R}^{m \times p}$, B_{+-} in $\mathbb{R}^{p \times m}$, B_{++} in $\mathbb{R}^{p \times p}$. For the in-domain coupled term A, we make the following assumption.

Assumption 3. $a_{ii} = 0$ for $i = 1, \ldots, n$.

This assumption is made for the sake of expositional clarity. In fact, for any general A, we can use the transformation from Coron et al. (2021) (see (29) and (33) in Coron et al. (2021)) to eliminate the diagonal terms of A.

Assumption 4. Λ is of class C^1. Λ and $\partial_x \Lambda$ are bounded on $[0,+\infty) \times [0,1]$. A is continuous and bounded on $[0,+\infty) \times [0,1]$. Q_0 and Q_1 are continuous and bounded on $[0,+\infty)$.

Assumption 5. Matrix-valued functions g_i, $i = 1,2,3,4$, are known, with appropriate dimensions, and are continuous and bounded on their respective domains ($[0,+\infty) \times [0,1]$ or $[0,+\infty)$).

The disturbance $d(t)$ and the reference input $r(t)$ in $\mathbb{R}^q$ to be tracked by the output $y(t)$ are the solutions to the following finite-dimensional signal model, for $t > t_0$,

$$\dot v_d(t) = S_d(t)v_d(t),\ v_d(t_0) = v_d^0,\ d(t) = p_d(t)v_d(t), \tag{4a}$$

$$\dot v_r(t) = S_r(t)v_r(t),\ v_r(t_0) = v_r^0,\ r(t) = p_r(t)v_r(t), \tag{4b}$$

where v_d^0 and v_r^0 are in $\mathbb{R}^{n_d}$ and $\mathbb{R}^{n_r}$ respectively. The coefficients of (4) satisfy the following assumption.

Assumption 6. Matrix-valued functions S_d, S_r, p_d and p_r are known and have appropriate dimensions. Assume that S_d and S_r are bounded on $(0,+\infty)$, and p_d and p_r are continuous and bounded on $[0,+\infty)$.

By Assumption 6, there exists a unique continuous transition matrix $\Psi_d : [0,+\infty)^2 \to \mathbb{R}^{n_d \times n_d}$ of S_d (resp. $\Psi_r : [0,+\infty)^2 \to \mathbb{R}^{n_r \times n_r}$ of S_r) such that the solution of (4a) (resp. (4b)) is given by $v_d(t) = \Psi_d(t,t_0)v_d^0$ (resp. $v_r(t) = \Psi_r(t,t_0)v_r^0$). One can refer to (Coron, 2007, p. 5) for the properties of transition matrix Ψ_d and Ψ_r. For the design of the output feedback regulator, it is required that $r(t)$ is known. Denote by $e_y(t) = y(t) - r(t)$ the output tracking error and denote $v = (v_d^\top, v_r^\top)^\top$. Inspired by Coron et al. (2021); Deutscher and Gabriel (2020), let us give the notion of the uniform finite-time output regulation that we are interested in.

Definition 7. The output y of the system (1) achieves the uniform finite-time output regulation within settling time T_0 by state feedback regulator (resp. by output feedback regulator) if for any $T > T_0$, there exists a feedback regulator $u = \mathcal{K}_{s,T}[w,v]$ (resp. $u = \mathcal{K}_{o,T}[y_m, r]$), such that for all $0 \leq t_0 < T - T_0$, w^0 in $L^2(0,1)^n$ and v^0 in $\mathbb{R}^{n_v}$, the output tracking error e_y satisfies $e_y = 0$ a.e. in $(t_0 + T_0, T)$.

In this paper, we focus on the finite-time observers design for system (1) together with the signal model (4). Since the observer design is independent of the output $y(t)$, we omit its specific form here. The complete technical details are available in Bai et al. (2025). For the existence of the state feedback regulator, we make the following assumption, which is essentially the main result of Bai et al. (2025).

Assumption 8. There exists a settling time T_0 such that the output y of the system (1) achieves the uniform finite-time output regulation within settling time T_0 by a state feedback regulator $u = \mathcal{K}_{s,T}[w, v]$.

Remark 9. In Bai et al. (2025), it is proven that the settling time T_0 is

$$T_{\mathrm{unif}}(\Lambda) = \sup_{t\geq 0}[s_{m+1}^{\mathrm{out}}(s_m^{\mathrm{out}}(t,1),0) - t], \tag{5}$$

where s_{m+1}^{out} and s_m^{out} are defined in next section. It is the same settling time as in Coron et al. (2021).

3. PRELIMINARIES

In this section, we introduce some knowledges about the characteristics of linear time-varying hyperbolic systems and the observer design for finite-dimensional linear time-varying systems.

3.1 Preliminaries on characteristics

Let us introduce some known facts on the characteristics associated with system (1) and the entry and exit times for the interval $[0, 1]$, see Coron et al. (2021). To this end, we use the extension method introduced in Coron et al. (2021) to extend Λ to a function of $\mathbb{R}^2$ (still denoted by Λ) by keeping Assumptions 1 to 4. For every $j = 1, \ldots, n$, let χ_j be the flow associated with λ_j, namely for every (t, x) in $\mathbb{R}^2$, the function $s \mapsto \chi_j(s; t, x)$ is the solution to the ODE, for s in $\mathbb{R}$,

$$\partial_s\chi_j(s; t, x) = \lambda_j(s, \chi_j(s; t, x)), \quad \chi_j(t; t, x) = x. \tag{6}$$

The existence and uniqueness of the solutions to the ODE (6) follows the classical theory. Moreover, since λ_j is bounded, the solution χ_j is global and of class C^1 with respect to s, t and x.

Next we introduce the entry and exit times for the interval $[0, 1]$. For $j = 1, \ldots, n$, t in $\mathbb{R}$ and x in $[0, 1]$, let $s_j^{\mathrm{in}}(t, x)$ and $s_j^{\mathrm{out}}(t, x)$ be the entry and exit times of the flow $\chi_j(\cdot; t, x)$ inside the interval $[0, 1]$, namely the respective unique solutions to $\chi_j(s_j^{\mathrm{in}}(t, x); t, x) = 1$ and $\chi_j(s_j^{\mathrm{out}}(t, x); t, x) = 0$ for $j = 1, \ldots, m$, and $\chi_j(s_j^{\mathrm{in}}(t, x); t, x) = 0$ and $\chi_j(s_j^{\mathrm{out}}(t, x); t, x) = 1$ for $j = m+1, \ldots, n$. The existence and uniqueness of $s_j^{\mathrm{in}}(t, x)$ and $s_j^{\mathrm{out}}(t, x)$ are guaranteed by (2) in Assumption 2.

3.2 Preliminaries on linear time-varying systems

In this section, we introduce some known results on observability, estimatability and observer design for finite-dimensional linear time-varying systems, which are used in design of disturbance observer. Consider a finite-dimensional linear time-varying system

$$\dot{\nu}(t) = U(t)\nu(t), \quad \nu_y(t) = V(t)\nu(t), \tag{7}$$

where $\nu(t)$ in $\mathbb{R}^{\hat{n}}$ is the state vector, $\nu_y(t)$ in $\mathbb{R}^{\hat{m}}$ is the output vector, and $U(t)$ and $V(t)$ are matrix functions with appropriate dimension. In addition U and V are assumed to be measurable and bounded on every finite subinterval of time. The transition matrix associated with $\dot{\nu}(t) = U(t)\nu(t)$ is denoted by $\Psi(\cdot, \cdot)$. In view of he assumptions on $U(\cdot)$, a unique, continuous, nonsingular $\Psi(\cdot, \cdot)$ exists on $\mathbb{R}^2$. Since the system (7) is fully determined by $U(t)$ and $V(t)$, we also use $(V(t), U(t))$ to represent the system (7). In order to define observability, we introduce the following definitions. Let the observability Gramian and reconstructibility Gramian of (7) be denoted by $M_o(\cdot, \cdot)$ and $M_r(\cdot, \cdot)$ respectively, namely $M_o(t, s) = \int_t^s \Psi(\tau, t)^\top V(\tau)^\top V(\tau)\Psi(\tau, t)\mathrm{d}\tau$ for $t < s$, and $M_r(s, t) = \int_s^t \Psi(\tau, t)^\top V(\tau)^\top V(\tau)\Psi(\tau, t)\mathrm{d}\tau$ for $s < t$.

Definition 10. (Ikeda et al. (1975)). $(V(t), U(t))$ is said to be uniformly completely observable if there are positive numbers γ and $\kappa_i(\gamma)$, $i = 1, 2, 3, 4$, such that

$$0 < \kappa_1(\gamma)\mathrm{Id}_n \leq M_o(t, t+\gamma) \leq \kappa_2(\gamma)\mathrm{Id}_n, \tag{8}$$

$$0 < \kappa_3(\gamma)\mathrm{Id}_n \leq M_r(t, t+\gamma) \leq \kappa_4(\gamma)\mathrm{Id}_n \tag{9}$$

hold for all t in $\mathbb{R}$.

The next lemma reduces the condition of uniform completely observability for bounded system.

Lemma 11. (Ikeda et al. (1975)). Bounded system $(V(t), U(t))$ is uniformly completely observable if one of (8) and (9) holds for all t in $\mathbb{R}$.

For the estimatability, $(V(t), U(t))$ is estimated by the dynamical estimator $\dot{\hat{\nu}}(t) = U(t)\hat{\nu}(t) - K(t)(V(t)\hat{\nu}(t) - \nu_y(t))$. Then system of the error $e_\nu := \hat{\nu} - \nu$ reads

$$\dot{e_\nu}(t) = (U(t) - K(t)V(t))e_\nu(t). \tag{10}$$

Therefore, we can define the estimatability as follows.

Definition 12. (Ikeda et al. (1975)). $(V(t), U(t))$ is said to be uniformly completely estimatable if for any pair of real numbers m and M such that $m \leq M$, there are positive numbers $a = a(m, M)$, $b = b(m, M)$ and an estimator gain $K(\cdot)$ such that any solution to (10) satisfies $a\|e_\nu(t_1)\|\mathrm{e}^{m(t_2-t_1)} \leq \|e_\nu(t_2)\| \leq b\|e_\nu(t_1)\|\mathrm{e}^{M(t_2-t_1)}$ for all t_1 in $\mathbb{R}$ and $t_2 \geq t_1$.

The relationship between the uniform complete observability and the uniform complete estimatability is stated in the next theorem.

Theorem 13. (Ikeda et al. (1975)). (1) If $(V(t), U(t))$ is uniformly completely observable, then it is uniformly completely estimatable.
(2) A bounded system $(V(t), U(t))$ is uniformly completely estimatable by a bounded estimator $K(\cdot)$ if and only if it is uniformly completely observable.

4. MAIN RESULTS: OUTPUT FEEDBACK REGULATOR DESIGN

A reasonable extension of the result in Bai et al. (2025) is the design of a finite-time output feedback regulator. To this end, the states v_r, v_d and w have to be estimated in finite time. In this section, we are committed to addressing the observer design for system (1) and achieving the finite-time output regulation by an output feedback regulator.

4.1 Finite-time reference signal observer

As mentioned in Section 2, only the reference input $r(t)$ is known. Therefore, the state v_r of the reference model has to be estimated by an observer for $n_r > 1$. Based on the knowledges introduced in Section 3.2, i.e., observability and estimatability for finite-dimensional linear time-varying system, we design the finite-time convergent reference observer in a similar way as in Engel and Kreisselmeier (2002); Menold et al. (2003). Consider two identical reference observers, for $t > t_0$,

$$\begin{aligned}&\dot{\hat{\underline{v}}}_r(t) = \underline{S}_r(t)\hat{\underline{v}}_r(t) + \underline{l}_r(t)(\mathrm{B}_q r(t) - \underline{p}_r(t)\hat{\underline{v}}_r(t)),\\ &\hat{\underline{v}}_r(t_0) = \hat{\underline{v}}_r^0,\end{aligned} \tag{11}$$

where $\hat{\underline{v}}_r = (\hat{v}_r^{1\top}, \hat{v}_r^{2\top})^\top$ is the observer state, $\underline{S}_r = \mathrm{diag}(S_r, S_r)$, $\underline{p}_r = \mathrm{diag}(p_r, p_r)$, $\hat{\underline{v}}_r^0 = (\hat{v}_r^{0,1\top}, \hat{v}_r^{0,2\top})^\top$ is the initial data and $\underline{l}_r = \mathrm{diag}(l_r^1, l_r^2)$ is the observer gain function to be determined later. In order to utilize the results introduced in Section 3.2, let $S_r(t) = S_r(-t)$ and $p_r(t) = p_r(-t)$ for t in $(-\infty, 0)$. The next theorem provides the observer gain function $\underline{l}_r$ and the design of the finite-time reference observer.

Theorem 14. Assume that Assumption 6 holds and that $(p_r(t), S_r(t))$ is uniformly completely observable. Then there exist a bounded observer gain function $\underline{l}_r$ in $L^\infty(\mathbb{R})^{2n_r\times 2q}$ and a time delay $D_r > 0$ such that

$$\det(\mathrm{B}_{n_r}, \underline{\Theta}_r(t, t-D_r)\mathrm{B}_{n_r}) \neq 0, \quad \forall t \in \mathbb{R}, \tag{12}$$

where $\underline{\Theta}_r(\cdot,\cdot)$ is the transition matrix of $\underline{S}_r(t) - \underline{l}_r(t)\underline{p}_r(t)$. Moreover, we have the following reference estimate

$$\begin{aligned}\hat{v}_r^+(t) = (\mathrm{Id}_{n_r}, 0)(\mathrm{B}_{n_r}, \underline{\Theta}_r(t, t-D_r)\mathrm{B}_{n_r})^{-1}(\hat{\underline{v}}_r(t)\\ - \underline{\Theta}_r(t, t-D_r)\hat{\underline{v}}_r(t-D_r)), \quad t \geq t_0,\end{aligned} \tag{13}$$

for v_r in (4b) within time D_r. In (13), $\hat{\underline{v}}_r(t) = \hat{\underline{v}}_r^0$ for t in $[t_0 - D_r, t_0]$.

Proof. We only prove (12) here, since Menold et al. (2003) has already prove that $\hat{v}_r^+$ estimates v_r in D_r under condition (12). Notice that the matrix $\underline{S}_r(t) - \underline{l}_r(t)\underline{p}_r(t)$ is block diagonal. Therefore, the corresponding transition matrix $\underline{\Theta}_r(\cdot,\cdot)$ is block diagonal as well, namely $\underline{\Theta}_r(\cdot,\cdot) = \mathrm{diag}(\Theta_r^1(\cdot,\cdot), \Theta_r^2(\cdot,\cdot))$. Then, direct calculation shows that $\det(\mathrm{B}_{n_r}, \underline{\Theta}_r(t, t-D_r)\mathrm{B}_{n_r}) = (-1)^{n_r}\det(\Theta_r^1(t, t-D_r) - \Theta_r^2(t, t-D_r))$. Therefore, we only need to prove there exist an observer gain function $\underline{l}_r$ in $L^\infty(\mathbb{R})^{2n_r\times 2q}$ and a time delay $D_r > 0$ such that $\det(\Theta_r^1(t, t-D_r) - \Theta_r^2(t, t-D_r)) \neq 0$ for all t in $\mathbb{R}$. Since $(p_r(t), S_r(t))$ is uniformly completely observable, $(p_r(t), S_r(t))$ is uniformly completely estimatable by a bounded estimator due to Theorem 13. Therefore, for any $m_1 < M_1 < m_2 < M_2 < 0$, there exist positive numbers $a_1 = a_1(m_1, M_1)$, $b_1 = b_1(m_1, M_1)$, $a_2 = a_2(m_2, M_2)$, $b_2 = b_2(m_2, M_2)$ and estimator gains l_r^1, l_r^2 in $L^\infty(\mathbb{R})^{n_r\times q}$ such that for any ν_r in $\mathbb{R}^{n_r}$, $D > 0$, t in $\mathbb{R}$ and $i = 1, 2$, we have $a_i\|\nu_r\|\mathrm{e}^{m_i D} \leq \|\Theta_r^i(t, t-D)\nu_r\| \leq b_i\|\nu_r\|\mathrm{e}^{M_i D}$. Since $M_1 < m_2$, we can select $D_r > 0$ large enough such that $b_1(m_1, M_1)\mathrm{e}^{M_1 D_r} < a_2(m_2, M_2)\mathrm{e}^{m_2 D_r}$. Then, for any ν_r in $\mathbb{R}^{n_r}$ and t in $\mathbb{R}$, $\|\Theta_r^1(t, t-D_r)\nu_r\| < \|\Theta_r^2(t, t-D_r)\nu_r\|$, and thus, $\det(\Theta_r^1(t, t-D_r) - \Theta_r^2(t, t-D_r)) \neq 0$ for all t in $\mathbb{R}$, which completes the proof of (12).

4.2 Asymptotic disturbance observer

The first step for designing a finite-time disturbance observer is to design an asymptotic disturbance observer. To this end, let us consider the following disturbance observer, for (t, x) in $\mathcal{D}(t_0)$,

$$\begin{aligned}&\dot{\hat{v}}_d(t) = S_d(t)\hat{v}_d(t) + l_d(t)(y_m(t) - \hat{w}_-(t,0)),\\ &\partial_t\hat{w}(t,x) + \Lambda(t,x)\partial_x\hat{w}(t,x) = A(t,x)\hat{w}(t,x)\\ &\quad + G_1(t,x)\hat{v}_d(t) + l_w(t,x)(y_m(t) - \hat{w}_-(t,0)),\\ &\hat{w}_+(t,0) = Q_0(t)y_m(t) + G_2(t)\hat{v}_d(t),\\ &\hat{w}_-(t,1) = Q_1(t)\hat{w}_+(t,1) + u(t) + G_3(t)\hat{v}_d(t),\\ &\hat{v}_d(t_0) = \hat{v}_d^0, \quad \hat{w}(t_0,x) = \hat{w}^0(x),\end{aligned} \tag{14}$$

where $\hat{v}_d$ and $\hat{w}$ are the observer states, $\hat{v}_d^0$ and $\hat{w}^0$ are the initial data, y_m is the measurement defined in (1), $G_i = g_i p_d$, $i = 1, 2, 3$, and l_d and l_w are the observer gain functions to be determined later. By introducing the observer errors $e_d = v_d - \hat{v}_d$ and $e_w = w - \hat{w}$, we have the observer error system for (t, x) in $\mathcal{D}(t_0)$,

$$\begin{aligned}&\dot{e}_d(t) = S_d(t)e_d(t) - l_d(t)e_{w,-}(t,0),\\ &\partial_t e_w(t,x) + \Lambda(t,x)\partial_x e_w(t,x) = A(t,x)e_w(t,x)\\ &\quad + G_1(t,x)e_d(t) - l_w(t,x)e_{w,-}(t,0),\\ &e_{w,+}(t,0) = G_2(t)e_d(t),\\ &e_{w,-}(t,1) = Q_1(t)e_{w,+}(t,1) + G_3(t)e_d(t).\end{aligned} \tag{15}$$

We see that all the coefficients in (15) are defined in time domain $(0, +\infty)$. As explained in Bai et al. (2025), the reason we consider the control system in finite time domain (t_0, T) lies in the regularity of the control input u. Since the observer design is independent of the control input u, in the next part of this section, we consider system (15) for (t, x) in $(t_0, +\infty)\times(0,1)$ and we are going to determine gain functions $l_d(t)$ and $l_w(t, x)$ for (t, x) in $(0, +\infty)\times(0,1)$.

In order to determine the gain functions l_d and l_w, we use the time-varying backstepping approach introduced in Coron et al. (2021) to transform the error system (15) into an PDE-ODE cascade system. This method is inspired by the disturbance observer design in Deutscher (2017a); Deutscher and Gabriel (2020). We introduce the following transformations. Let

$$\begin{aligned}e_w(t,x) &= \tilde{e}_w(t,x) - \textstyle\int_0^x P_I(t,x,\zeta)\tilde{e}_w(t,\zeta)\mathrm{d}\zeta\\ &:= \mathcal{T}_0^{-1}(t)[\tilde{e}_w(t,\cdot)](x),\end{aligned} \tag{16}$$

$$\tilde{\varepsilon}_w(t,x) = \tilde{e}_w(t,x) - N(t,x)e_d(t), \tag{17}$$

with the integral kernel P_I defined on the infinite triangular prism $\Gamma = \{(t,x,\zeta) \in (0,+\infty)\times(0,1)\times(0,1),\ x > \zeta\}$ and N defined on $(0,+\infty)\times(0,1)$. By direct calculation, the system of e_d and $\tilde{\varepsilon}_w$ reads, for (t,x) in $(t_0,+\infty)\times(0,1)$,

$$\begin{aligned}&\dot{e}_d(t) = (S_d(t) - l_d(t)N_-(t,0))e_d(t) - l_d(t)\tilde{\varepsilon}_{w,-}(t,0),\\ &\partial_t\tilde{\varepsilon}_w(t,x) + \Lambda(t,x)\partial_x\tilde{\varepsilon}_w(t,x) = -(\tilde{l}_{w,-}^\top, 0)^\top(t,x)\tilde{\varepsilon}_{w,-}(t,0),\\ &\tilde{\varepsilon}_{w,+}(t,0) = 0,\\ &\tilde{\varepsilon}_{w,-}(t,1) = Q_1(t)\tilde{\varepsilon}_{w,+}(t,1) - \textstyle\int_0^1 F(t,\zeta)\tilde{\varepsilon}_w(t,\zeta)\mathrm{d}\zeta,\end{aligned} \tag{18}$$

if we have the following properties:

(1) P_I satisfies, for (t,x,ζ) in Γ,

$$\begin{aligned}&\partial_t P_I(t,x,\zeta) + \Lambda(t,x)\partial_x P_I(t,x,\zeta)\\ &\quad + \partial_\zeta(P_I(t,x,\zeta)\Lambda(t,\zeta)) = A(t,x)P_I(t,x,\zeta),\end{aligned} \tag{19}$$

$$P_I(t,x,x)\Lambda(t,x) - \Lambda(t,x)P_I(t,x,x) = A(t,x), \tag{20}$$

and

$$\begin{aligned}&Q_1(t)P_{I,+-}(t,1,\zeta) - P_{I,--}(t,1,\zeta)\\ &\qquad \text{is a strictly upper triangular matrix.}\end{aligned} \tag{21}$$

(2) $F := (F_-, F_+)$, where

$$\begin{aligned}F_-(t,\zeta) &= Q_1(t)P_{I,+-}(t,1,\zeta) - P_{I,--}(t,1,\zeta),\\ F_+(t,\zeta) &= Q_1(t)P_{I,++}(t,1,\zeta) - P_{I,-+}(t,1,\zeta).\end{aligned}$$

(3) N satisfies, for (t,x) in $(0,+\infty)\times(0,1)$,

$$\begin{aligned}&\partial_t N(t,x)+\Lambda(t,x)\partial_x N(t,x)+N(t,x)S_d(t)\\&=\tilde{G}_1(t,x)-(\tilde{l}_{w,-}^\top,0)^\top(t,x)N_-(t,0),\\&N_+(t,0)=G_2(t),\quad N_-(t,1)=Q_1(t)N_+(t,1)\\&-\textstyle\int_0^1 F(t,\zeta)N(t,\zeta)\mathrm{d}\zeta+G_3(t),\end{aligned}\tag{22}$$

where $\tilde{G}_1$ is defined by $\tilde{G}_1(t,x)=\mathcal{T}_0(t)[G_1(t,\cdot)](x)+\begin{pmatrix}P_{-+}(t,x,0)\\P_{++}(t,x,0)\end{pmatrix}\Lambda_{++}(t,0)G_2(t)$ and $\tilde{l}_{w,-}$ is a strictly upper triangular observer gain to be determined later. Therein, N_- (resp. N_+) is formed by the first m rows (resp. the last p rows) of N.

(4) l_w is given by, for (t,x) in $(0,+\infty)\times(0,1)$,

$$\begin{aligned}l_w(t,x)&=\mathcal{T}_0^{-1}(t)[(\tilde{l}_{w,-}^\top,0)^\top(t,\cdot)+N(t,\cdot)l_d(t)](x)\\&+\begin{pmatrix}P_{I,--}(t,x,0)\\P_{I,+-}(t,x,0)\end{pmatrix}\Lambda_{--}(t,0),\end{aligned}\tag{23}$$

with l_d and $\tilde{l}_{w,-}$ to be determined later.

The remaining thing is to prove the existence of the transformation kernel functions P_I and N. Inspired by Deutscher (2017b); Hu et al. (2016, 2019), applying the transformation $\xi=1-x$ and $y=1-\zeta$ of the independent variables to (19)-(21) and taking the transpose of (19)-(21), it can be verified that (19)-(21) can be transformed to the same form as in Coron et al. (2021). Then, there exists a bounded and piecewise continuous solution P_I to (19)-(21) such that its discontinuities can only occur on a C^1 surface within Γ (see (Coron et al., 2021, Theorem 2.6)). The inverse transformation $\tilde{e}_w(t,x)=e_w(t,x)+\int_0^x P(t,x,\zeta)e_w(t,\zeta)\mathrm{d}\zeta := \mathcal{T}_0(t)[e_w(t,\cdot)](x)$ of (16) exists and can be obtained from the reciprocity of kernels P_I and P (see Ch. 4.5 of Krstic and Smyshlyaev (2008)): $P(t,x,\zeta)=P_I(t,x,\zeta)+\int_\zeta^x P_I(t,x,\xi)P(t,\xi,\zeta)\mathrm{d}\xi$ for (t,x,ζ) in Γ.

The existence of the kernel function N is given by the following lemma. The main idea of the proof is to use the method of characteristics along with the specific structures of $\tilde{l}_{w,-}$ and F to solve N explicitly. We skip the proof for lack of space.

Lemma 15. Let $\tilde{l}_{w,-}$ in $L^\infty((0,+\infty)\times(0,1))^{m\times m}$ be a strictly upper triangular matrix function which is piecewise continuous. Then, there exists a solution N in $L^\infty((0,+\infty)\times(0,1))^{n\times n_d}$ to (22) such that N is piecewise continuous on $(0,+\infty)\times(0,1)$.

4.3 Minimum-time disturbance observer

In order to obtain a time optimal result, consider the Fredholm transformation

$$\begin{aligned}\bar{\varepsilon}_{w,-}(t,x)&=\tilde{\varepsilon}_{w,-}(t,x)-\textstyle\int_0^1 H(t,x,\zeta)\tilde{\varepsilon}_{w,-}(t,\zeta)\mathrm{d}\zeta\\&:=\mathcal{F}_0(t)[\tilde{\varepsilon}_{w,-}(t,\cdot)](x),\end{aligned}\tag{24}$$

with the strictly upper triangular integral kernel H defined on the infinite rectangular prism $(0,+\infty)\times(0,1)\times(0,1)$. Applying (24) to (18), the target system of e_d and $(\bar{\varepsilon}_{w,-}^\top,\tilde{\varepsilon}_{w,+}^\top)^\top$ reads, for (t,x) in $(t_0,+\infty)\times(0,1)$,

$$\begin{aligned}&\dot{e}_d(t)=(S_d(t)-l_d(t)N_-(t,0))e_d(t)\\&\quad-l_d(t)\mathcal{F}_0(t)[\bar{\varepsilon}_{w,-}(t,\cdot)](0),\\&\partial_t\bar{\varepsilon}_{w,-}(t,x)+\Lambda_{--}(t,x)\partial_x\bar{\varepsilon}_{w,-}(t,x)=0,\\&\partial_t\tilde{\varepsilon}_{w,+}(t,x)+\Lambda_{++}(t,x)\partial_x\tilde{\varepsilon}_{w,+}(t,x)=0,\\&\tilde{\varepsilon}_{w,+}(t,0)=0,\\&\bar{\varepsilon}_{w,-}(t,1)=Q_1(t)\tilde{\varepsilon}_{w,+}(t,1)-\textstyle\int_0^1 F_+(t,\zeta)\tilde{\varepsilon}_{w,+}(t,\zeta)\mathrm{d}\zeta,\end{aligned}\tag{25}$$

if H is the solution to the kernel equations, for (t,x,ζ) in $(0,+\infty)\times(0,1)\times(0,1)$,

$$\begin{aligned}&\partial_t H(t,x,\zeta)+\Lambda_{--}(t,x)\partial_x H(t,x,\zeta)\\&\quad+\partial_\zeta(H(t,x,\zeta)\Lambda_{--}(t,\zeta))=0,\\&H(t,1,\zeta)=-F_-(t,\zeta),\quad H(t,x,1)=0,\end{aligned}\tag{26}$$

and

$$\tilde{l}_{w,-}(t,x)=-\mathcal{F}_0^{-1}(t)[H(t,\cdot,0)\Lambda_{--}(t,0)](x).\tag{27}$$

Similar to proving the existence of the kernel function P_I, it can be verified that (26) can be transformed to the same form as in Coron et al. (2021). Therefore, there exists a bounded and piecewise continuous solution H to (26) such that its discontinuities can only occur on a C^1 surface within $(0,+\infty)\times(0,1)\times(0,1)$ (see (Coron et al., 2021, Theorem 2.13)). The strictly upper triangular structure in particular ensures that the Fredholm transformation (24) is invertible (see Coron et al. (2017); Deutscher and Gabriel (2020) for more details). Therefore, $\tilde{l}_{w,-}$ in (27) is well-defined, bounded and piecewise continuous.

The structure of the target system (25) directly implies that $(\bar{\varepsilon}_{w,-}^\top,\tilde{\varepsilon}_{w,+}^\top)^\top(t,\cdot)=0$ for $t\geq t_0+\tilde{T}_{\mathrm{unif}}(\Lambda)$, where

$$\tilde{T}_{\mathrm{unif}}(\Lambda):=\sup_{t\geq 0}[s_m^{\mathrm{out}}(s_{m+1}^{\mathrm{out}}(t,0),1)-t].\tag{28}$$

It follows from (23) and (27) that the observer gain function l_w is uniquely determined by l_d. In next step, we extend the observer (14) to achieve finite-time estimate for states v_d and w. To this end, let us provide the following claim, which is useful for designing the extended observer. For the lack of space, we skip the proof, which is based on a direct observation of the error system.

Claim 16. Let N be defined by (22). Then function

$$y_d(t):=y_m(t)-\hat{w}_-(t,0)+N_-(t,0)\hat{v}_d(t)\tag{29}$$

satisfies

$$y_d(t)=N_-(t,0)v_d(t),\quad\forall t\geq t_0+\tilde{T}_{\mathrm{unif}}(\Lambda).\tag{30}$$

By Claim 16, we consider the following finite-time disturbance observer, for (t,x) in $\mathcal{D}(t_0)$,

$$\begin{aligned}&\dot{\hat{\mu}}_d(t)=S_d(t)\hat{\mu}_d(t)+l_\mu(t)(y_d(t)-N_-(t,0)\hat{\mu}_d(t)),\\&\dot{\hat{v}}_d(t)=S_d(t)\hat{v}_d(t)+l_d(t)(y_m(t)-\hat{w}_-(t,0)),\\&\partial_t\hat{w}(t,x)+\Lambda(t,x)\partial_x\hat{w}(t,x)=A(t,x)\hat{w}(t,x)\\&\quad+G_1(t,x)\hat{v}_d(t)+l_w(t,x)(y_m(t)-\hat{w}_-(t,0)),\\&\hat{w}_+(t,0)=Q_0(t)y_m(t)+G_2(t)\hat{v}_d(t),\\&\hat{w}_-(t,1)=Q_1(t)\hat{w}_+(t,1)+u(t)+G_3(t)\hat{v}_d(t),\\&\hat{\mu}_d(t_0)=\hat{\mu}_d^0,\quad\hat{v}_d(t_0)=\hat{v}_d^0,\quad\hat{w}(t_0,x)=\hat{w}^0(x),\end{aligned}\tag{31}$$

with the additional initial condition $\hat{\mu}_d^0$. Therein, y_d is defined by (29) and l_μ is the observer gain function. By introducing the additional error $e_\mu=v_d-\hat{\mu}_d$ and noticing (25) and (30), we obtain that the error dynamics of observers $\hat{\mu}_d$ and $\hat{v}_d$ is given by, for $t\geq t_0+\tilde{T}_{\mathrm{unif}}(\Lambda)$,

$$\begin{aligned}&\dot{e}_\mu(t)=(S_d(t)-l_\mu(t)N_-(t,0))e_\mu(t),\\&\dot{e}_d(t)=(S_d(t)-l_d(t)N_-(t,0))e_d(t).\end{aligned}\tag{32}$$

Let $S_d(t) = S_d(-t)$ for $t < 0$. Similar to Theorem 14, we have the result for the finite-time disturbance observer.

Theorem 17. Assume that Assumptions 1 to 6 hold. Let $\tilde{T}_{\text{unif}}(\Lambda)$ be defined by (28). Let transformation kernels P_I, and H be given by (19)-(21), and (26) respectively. For $t \geq 0$, let $N(t, \cdot)$ be defined by (22) with $\tilde{l}_{w,-}$ given by (27) and for $t < 0$, let $N(t, \cdot) = N(-t, \cdot)$. Assume that $(N_-(t, 0), S_d(t))$ is uniformly completely observable. Then there exist observer gain functions l_μ and l_d in $L^\infty(\mathbb{R})^{n_d \times m}$ and a time delay $D_d > 0$ such that

$$\det \begin{pmatrix} \mathrm{Id}_{n_d} & \Theta_{d1}(t, t - D_d) \\ \mathrm{Id}_{n_d} & \Theta_{d2}(t, t - D_d) \end{pmatrix} \neq 0, \quad \forall t \in \mathbb{R}, \tag{33}$$

where $\Theta_{d1}(\cdot, \cdot)$ (resp. $\Theta_{d2}(\cdot, \cdot)$) is the transition matrix of $S_d(t) - l_\mu(t)N_-(t, 0)$ (resp. $S_d(t) - l_d(t)N_-(t, 0)$). Moreover, we have the disturbance estimate, for (t, x) in $\mathcal{D}(t_0)$,

$$\begin{aligned} \hat{v}_d^+(t) &= (\mathrm{Id}_{n_d}, 0) \begin{pmatrix} \mathrm{Id}_{n_d} & \Theta_{d1}(t, t - D_d) \\ \mathrm{Id}_{n_d} & \Theta_{d2}(t, t - D_d) \end{pmatrix}^{-1} \\ &\quad \cdot \begin{pmatrix} \hat{\mu}_d(t) - \Theta_{d1}(t, t - D_d)\hat{\mu}_d(t - D_d) \\ \hat{v}_d(t) - \Theta_{d2}(t, t - D_d)\hat{v}_d(t - D_d) \end{pmatrix}, \\ \hat{w}^+(t, x) &= \hat{w}(t, x) + \mathcal{T}_0^{-1}(t)[N(t, \cdot)](x)(\hat{v}_d^+(t) - \hat{v}_d(t)), \end{aligned} \tag{34}$$

for v_d in (4a) and w in (1) within time $\tilde{T}_{\text{unif}}(\Lambda) + D_d$. In (34), $\hat{\mu}_d(t) = \hat{\mu}_d^0$ and $\hat{v}_d(t) = \hat{v}_d^0$ for t in $[t_0 - D_d, t_0]$.

Proof. Due to Claim 16 and the error system (32), we obtain that there exist bounded observer gain functions l_μ and l_d and $D_d > 0$ such that (33) holds and $\hat{v}_d^+$ defined in (34) satisfies $\hat{v}_d^+(t) = v_d(t)$ for t in $[t_0 + \tilde{T}_{\text{unif}}(\Lambda) + D_d, T]$.

Then we prove $\hat{w}^+$ achieves finite-time estimate for w. The structure of target system (25) implies $(\tilde{\varepsilon}_{w,-}^\top, \tilde{\varepsilon}_{w,+}^\top)^\top(t, \cdot) = 0$ for $t \geq t_0 + \tilde{T}_{\text{unif}}(\Lambda)$. Since $\mathcal{F}_0$ is invertible, we have $\tilde{\varepsilon}_w(t, \cdot) = 0$ for $t \geq t_0 + \tilde{T}_{\text{unif}}(\Lambda)$. Then it follows from (16) and (17) that, for t in $[t_0 + \tilde{T}_{\text{unif}}(\Lambda), T]$ and x in $(0, 1)$,

$$w(t, x) - \hat{w}^+(t, x) = \mathcal{T}_0^{-1}(t)[N(t, \cdot)](x)(v_d(t) - \hat{v}_d^+(t)).$$

Consequently, for t in $[t_0 + \tilde{T}_{\text{unif}}(\Lambda) + D_d, T]$, it follows from $\hat{v}_d^+(t) = v_d(t)$ that $\hat{w}^+(t, \cdot) = w(t, \cdot)$.

4.4 Finite-time output feedback regulator

Combining the result of Bai et al. (2025) with Theorems 14 and 17, we obtain finite-time output feedback regulator.

Theorem 18. Assume that Assumption 8 and the assumptions of Theorems 14 and 17 hold. Let $T_{\text{unif}}(\Lambda)$ and $\tilde{T}_{\text{unif}}(\Lambda)$ be given by (5) and (28) respectively. Let the finite-time observers $\hat{v}_r^+$, $\hat{v}_d^+$ and $\hat{w}^+$, and time delays D_r and D_d be given by Theorems 14 and 17. Then the output y of the system (1) achieves the uniform finite-time output regulation within settling time $T_{\min} := T_{\text{unif}}(\Lambda) + \max\{D_r, \tilde{T}_{\text{unif}}(\Lambda) + D_d\}$ by output feedback regulator.

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