

1 Hamiltonians and EOM

1.1 Toy Problem

Consider simplest spin Hamiltonian $H = -\vec{B} \cdot \vec{s}$. It's clear that if we set up initial conditions \vec{s} misaligned from \vec{B} , it will simply spin around \vec{B} , which is fixed. Thus, let $\hat{B} \cdot \hat{s} = \cos \theta$ the angle between the two, and let ϕ measure the azimuthal angle.

We claim that $\cos \theta, \phi$ are canonical variables. Since ϕ is ignorable, immediately $\frac{d\theta}{dt} = \frac{d \cos \theta}{dt} = -\frac{\partial H}{\partial \phi} = 0$, while $\frac{d\phi}{dt} = \frac{\partial H}{\partial(\cos \theta)} = Bs$ tells us the rate at which the spin precesses around \vec{B} .

1.2 Cassini State Hamiltonian

This Hamiltonian is Kassandra's Eq. 13, in the co-rotating frame with the perturber's angular momentum:

$$\mathcal{H} = \frac{1}{2}(\hat{s} \cdot \hat{l})^2 - \eta(\hat{s} \cdot \hat{l}_p). \quad (1)$$

In this frame, we can choose $\hat{l} \equiv \hat{z}$ fixed, and $\hat{l}_p = \cos I \hat{z} + \sin I \hat{x}$ fixed as well. Then

$$\hat{s} = \cos \theta \hat{z} - \sin \theta (\sin \phi \hat{y} + \cos \phi \hat{x}).$$

We can choose the convention for $\phi = \phi$ azimuthal angle requiring $\phi = 0, \pi$ mean coplanarity between $\hat{s}, \hat{l}, \hat{l}_p$ in the \hat{x}, \hat{z} plane such that \hat{l}_p, \hat{s} lie on the same side of \hat{l} . Then we can evaluate in coordinates

$$\begin{aligned} \hat{s} \cdot \hat{l} &= \cos \theta, \\ \hat{s} \cdot \hat{l}_p &= \cos \theta \cos I - \sin I \sin \theta \cos \phi, \\ \mathcal{H} &= \frac{1}{2} \cos^2 \theta - \eta(\cos \theta \cos I - \sin I \sin \theta \cos \phi). \end{aligned}$$

Note that if we take $\cos \theta$ to be our canonical variable, $\sin \theta = \sqrt{1 - \cos^2 \theta}$ can be used.

1.3 Equation of Motion

Consider Hamilton's equations applied to the Hamiltonian

$$\frac{\partial \phi}{\partial t} = -\frac{\partial \mathcal{H}}{\partial(\cos \theta)} = -\cos \theta + \eta(\cos I + \sin I \cot \theta \cos \phi), \quad (2)$$

$$\frac{\partial \theta}{\partial t} = +\frac{\partial \mathcal{H}}{\partial \phi} = -\eta \sin I \sin \theta \sin \phi. \quad (3)$$

This can be compared to the original EOM given in Kassandra's Eq. 12,

$$\frac{d\hat{s}}{dt} = (\hat{s} \cdot \hat{l})(\hat{s} \times \hat{l}) - \eta(\hat{s} \times \hat{l}_p). \quad (4)$$

Recalling that $|\dot{\phi}| = \sin \theta$ (the differential geometry result), we can verify the expressions are equivalent for $\phi = \pi$, alignment with \hat{s}, \hat{l}_p on the same side of \hat{l} as in her Fig. 2, checks our signs.