High Spin-Orbit Misalignment as an Attractor in Cassini State Systems with Weak Tidal Friction

Meeting XX/XX/XXXX

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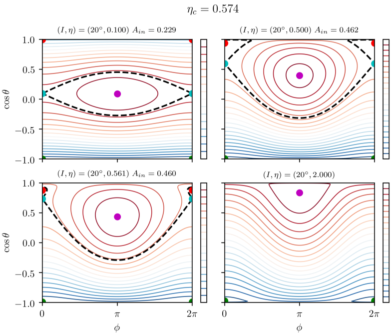
Some day

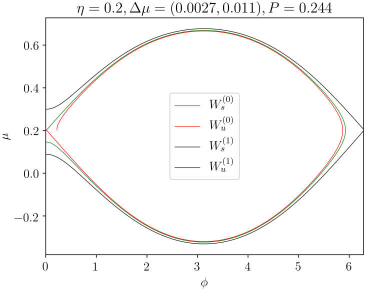
- Close-in planet to a star w/ $\vec{l}$ gstarts with random spin $\hat{s}$ (e.g. collision). Evolves under tides + precession around perturber $\hat{l}_p$.
- **Toy Problem:** Assume constant tidal dissipation, fate?
- Cassini States: $H^{(0)} = \frac{\left(\hat{s}\cdot\hat{l}\right)^2}{2} + \eta \hat{s}\cdot\hat{l}_p$. CS4 is saddle point, *separatrix*.



- Constant tides $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \epsilon \sin\theta$, EOM ($\mu = \cos\theta$):

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}\tau} = \left(\hat{s}\cdot\hat{l}\right)\left(\hat{s}\times\hat{l}\right) - \eta\hat{s}\times\hat{l}_p + \epsilon\hat{s}\times\left(\hat{l}\times\hat{s}\right),$$
$$\frac{\partial\phi}{\partial t} = \mu - \eta\left(\cos I + \sin I\frac{\mu}{\sqrt{1-\mu^2}}\cos\phi\right),$$
$$\frac{\partial\mu}{\partial t} = -\eta\sin I\sin\phi + \epsilon\left(1-\mu^2\right),$$

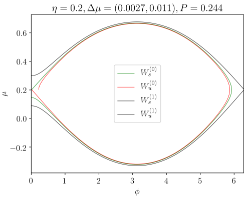
- *Review:* Last meeting, found probability that initial state $\cos\theta < 0$ jumps into separatrix $\propto \eta^{3/2}\epsilon^0$.



- Notation: subscripts *stable*, *unstable* manifolds, superscripts are left/right copy of CS4.
    - Below black, continues orbit $\mu < \mu_4$.
    - Black/green, escapes.
    - Green/red, captures.
- Evaluate $\Delta\mu$ at $\phi = \pi$ reproduces probability (numerical was $P_{hop}(\eta = 0.2) = 0.251$).

- **Goal:** Under weak perturbation, how much does trajectory deviate from level curve of H?
- Procedure:
  - Compute $\Delta H^{(0)}$ unperturbed over trajectory.
  - Locate new level curve to find Δq along coordinate: $\Delta H^{(0)} = \frac{\partial H^{(0)}}{\partial q}\Delta q$.
- So looks something like:

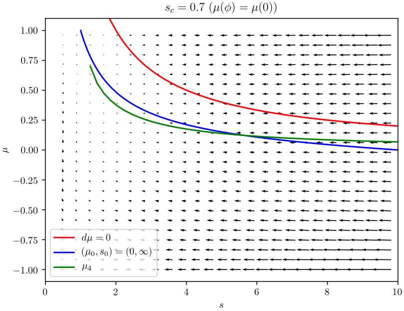
$$\frac{\mathrm{d}H^{(0)}}{\mathrm{d}t} = \underbrace{\cancel{\frac{\partial H^{(0)}}{\partial \mu}\frac{\mathrm{d}\mu^{(0)}}{\mathrm{d}t} + \frac{\partial H^{(0)}}{\partial \phi}\frac{\mathrm{d}\phi^{(0)}}{\mathrm{d}t}}}_{\dot{\phi}^{(0)}\dot{\mu}^{(0)} - \dot{\mu}^{(0)}\dot{\phi}^{(0)}} + \frac{\partial H^{(0)}}{\partial \mu}\frac{\mathrm{d}\mu^{(1)}}{\mathrm{d}t}.$$



- Gives rise to Melnikov distance:
  - $\Delta\mu(\phi) = \frac{1}{\dot{\phi}(\phi)}\oint \dot{\phi}^{(0)}\epsilon\left(1-\mu^2\right)\mathrm{d}t.$
  - $\mu(\phi) \approx \eta\cos I \pm \mathcal{O}\left(\sqrt{\eta}\right).$
  - $W_s^{(1)}, W_u^{(0)}$: dominated by 1.
  - $W_s^{(0)}, W_u^{(0)}$: dominated by $\eta^{3/2}$!

Problem 2
Realistic Tides

- In realistic tides, η can evolve as s spins down.
- $\eta = \frac{s_c}{s}$, full expression below.
- First, consider tides alone, phase portrait at right:
  - Like ignoring $\frac{\mathrm{d}\mu}{\mathrm{d}\phi}$ over precession orbit.
  - Breaks down near μ_4!

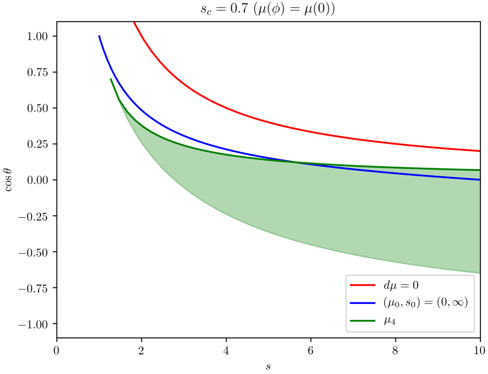


$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}\tau} = \frac{s}{s_c}(\hat{s}\cdot\hat{l})(\hat{s}\times\hat{l}) - \hat{s}\times\hat{l}_p + \frac{\epsilon 2\Omega}{s}\left(1 - \frac{s}{2\Omega}(\hat{l}\cdot\hat{s})\right)\hat{s}\times(\hat{l}\times\hat{s}), \tag{1}$$

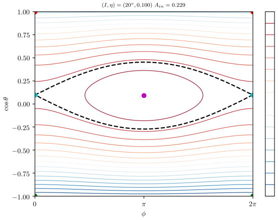
$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \epsilon 2\Omega\left(\hat{s}\cdot\hat{l} - \frac{s}{2\Omega}\left(1 + (\hat{s}\cdot\hat{l})^2\right)\right). \tag{2}$$

Problem 2
Intuitive Guess

- Ignore perturber for now ($\eta \ll 1$). Maybe: *points incident on CS4* have P_{hop} probability of entering separatrix?
- For instance, shaded green = capture region:



Problem 2
Near CS4



- When η non-negligible, full phase space (μ, ϕ, s).
- Study (μ_0, s) where $\mu_0 \equiv \mu(\phi = 0)$.
  - $\mu_0 \approx \mu_4 = P_{hop}$ candidate.

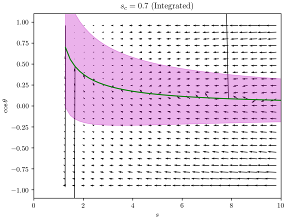
- Update model:
  - $\left|\mu_0 - \mu_4\right| \gtrsim \sqrt{\eta \sin I}$, then $\mu(\phi) \approx \mu_0$.
  - $\left|\mu_0 - \mu_4\right| \lesssim \sqrt{\eta \sin I}$, turns out

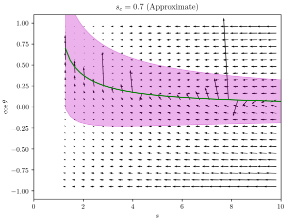
$$\Delta\mu \sim \frac{\epsilon}{\mu_4 - \mu_0} \dot{\mu}^T(\mu_{eff}). \quad (3)$$

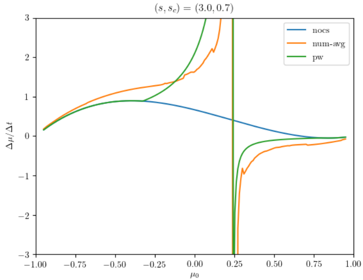
  - $\dot{\mu}^T$ is just the tidal component.
  - μ_{eff} such that $\dot{\mu}^T(\mu_{eff}) \equiv \langle \dot{\mu}^T(\mu(t)) \rangle$, $\approx \mu_4 \pm 2\sqrt{\eta \sin I}$.

$$\Delta\mu \sim \frac{\epsilon}{\mu_4 - \mu_0}\dot{\mu}^T(\mu_{eff}).$$

- Two key properties:
  - $\Delta\mu$'s sign is set at μ_{eff}, which can be far from μ_0.
  - $\mu_0 \approx \mu_4$ means large $\Delta\mu$.
- Trajectories might cross CS4 *multiple times* if $\Delta\mu$ is different signs on two sides!
- Effective "phase portrait".
  - Shade $|\mu_4 - \mu_0| < \sqrt{\eta \sin I}$. *Can be attracting*!

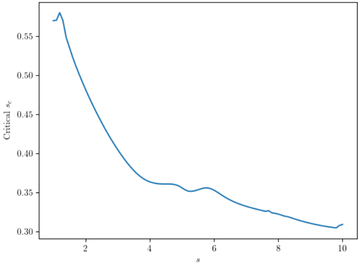




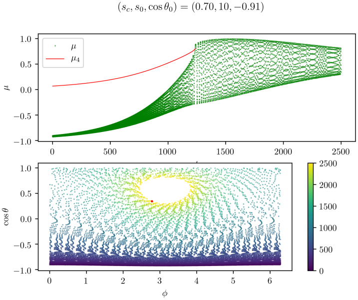


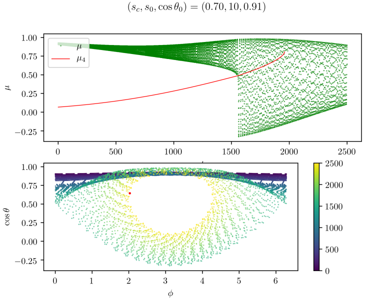
- $\Delta\mu/\Delta t$ in attracting case.
- Depends on sign of $\frac{\mathrm{d}\mu^T}{\mathrm{d}t}(\mu_{eff,+})$.

- Critical s_c: smaller means $\Delta\mu/\Delta t(\mu > \mu_4) > 0$.



- CS4 $\pm\sqrt{2\eta \sin I}$ attracts onto CS4, trajectories cross CS4 multiple times, $P_{hop} \approx 1$, goes to CS2.
- Capture region of attraction basin? Almost all!

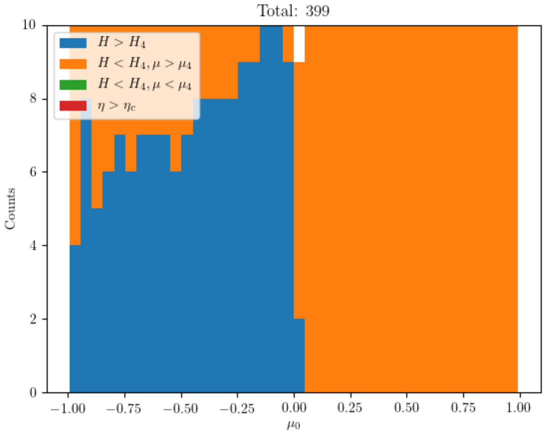




Problem 2

Capture Region, Non-attracting Regime

- Only points below hit CS4, probabilistically hop $P_{hop} \propto \sqrt{s}$. *Closer points have higher hop probability.*



- Two regimes based on s_c, attracting/non-attracting μ_4.
- Attracting regime, almost all ICs separatrix hop and go to CS2.
- Non-attracting regime, only below ICs probabilistically separatrix hop, rest go straight to CS1.