1 Hamiltonians and EOM

1.1 Toy Problem

Consider simplest spin Hamiltonian $H = -\vec{B}\cdot\vec{s}$. It's clear that if we set up initial conditions $\vec{s}$ misaligned from $\vec{B}$, it will simply spin around $\vec{B}$, which is fixed. Thus, let $\hat{B}\cdot\hat{s} = \cos\theta$ the angle between the two, and let ϕ measure the azimuthal angle.

We claim that $\cos\theta, \phi$ are canonical variables. Since ϕ is ignorable, immediately $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\cos\theta}{\mathrm{d}t} = -\frac{\partial H}{\partial\phi} = 0$, while $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial H}{\partial(\cos\theta)} = Bs$ tells us the rate at which the spin precesses around $\vec{B}$.

1.2 Cassini State Hamilttonian

This Hamiltonian is Kassandras Eq. 13, in the co-rotating frame with the perturber's angular momentum:

$$\mathcal{H} = \frac{1}{2}\left(\hat{s}\cdot\hat{l}\right)^2 - \eta\left(\hat{s}\cdot\hat{l}_p\right). \tag{1}$$

In this frame, we can choose $\hat{l} \equiv \hat{z}$ fixed, and $\hat{l}_p = \cos I\hat{z} + \sin I\hat{x}$ fixed as well. Then

$$\hat{s} = \cos\theta\hat{z} - \sin\theta\left(\sin\phi\hat{y} + \cos\phi\hat{x}\right).$$

We can choose the convention for $\phi = \phi$ azimuthal angle requiring $\phi = 0, \pi$ mean coplanarity between $\hat{s}, \hat{l}, \hat{l}_p$ in the $\hat{x}, \hat{z}$ plane such that $\hat{l}_p, \hat{s}$ lie on the same side of $\hat{l}$. Then we can evaluate in coordinates

$$\begin{aligned}
\hat{s}\cdot\hat{l} &= \cos\theta,\\
\hat{s}\cdot\hat{l}_p &= \cos\theta\cos I - \sin I\sin\theta\cos\phi,\\
\mathcal{H} &= -\frac{1}{2}\cos^2\theta + \eta\left(\cos\theta\cos I - \sin I\sin\theta\cos\phi\right).
\end{aligned}$$

Note that if we take $\cos\theta$ to be our canonical variable, $\sin\theta = \sqrt{1-\cos^2\theta}$ can be used.

1.3 Equation of Motion

The correct EOM comes from Kassandra's Eq. 12:

$$\begin{aligned}
\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} &= \left(\hat{s}\cdot\hat{l}\right)\left(\hat{s}\times\hat{l}\right) - \eta\left(\hat{s}\times\hat{l}_p\right),\\
&= \left(s_y s_z - \eta s_y\cos(I)\right)\hat{x} - \left(s_x s_z + \eta(s_x\cos I - s_z\sin I)\right)\hat{y} + \eta s_y\sin(I)\hat{z}.
\end{aligned}$$

Alternatively, consider Hamilton's equations applied to the Hamiltonian:

$$\frac{\partial\phi}{\partial t} = \frac{\partial\mathcal{H}}{\partial(\cos\theta)} = -\cos\theta + \eta\left(\cos I + \sin I\cot\theta\cos\phi\right), \tag{2}$$

$$\frac{\partial(\cos\theta)}{\partial t} = -\frac{\partial\mathcal{H}}{\partial\phi} = -\eta\sin I\sin\theta\sin\phi. \tag{3}$$

This produces the same trajectories as the Cartesian EOM, so this is correct. However, since $\frac{\partial\phi}{\partial t} \propto 1/\sin\theta$, this is not a desirable system of equations to use, as they are very stiff near $\theta \approx 0$.

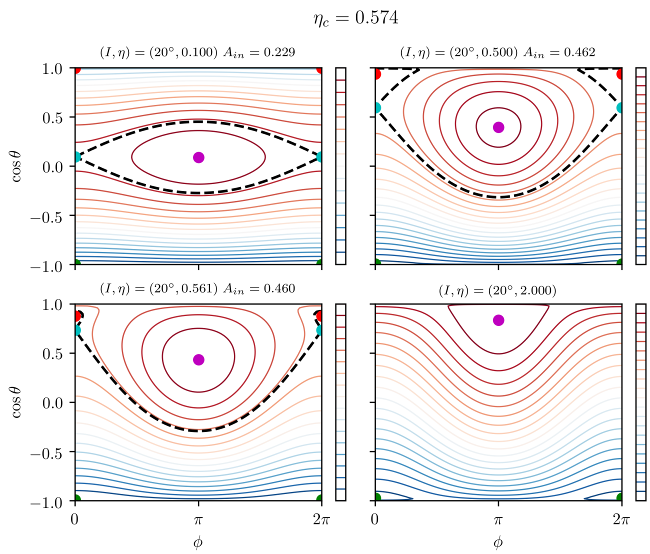

Figure 1: Separatrix for various values of η.

1.4 Cassini States

The zeros to Eq. 3 are the Cassini states; we will go to canonical variables $\mu = \cos\theta$. We can immediately see that $\sin\phi = 0$ is necessary, so $\cos\phi = \pm 1$ and we need only solve for $\frac{\partial\phi}{\partial t} = 0$. We can furthermore separate the problem into two regimes, $\eta \ll 1$ and $\eta \gg 1$.

For $\eta \ll 1$, it is clear that there will be two solutions near $\mu^2 = 1$ and two solutions near $\mu = 0$:

- For $\mu = 1 - \frac{\theta^2}{2}$, the dominant terms are $\frac{\partial\phi}{\partial t} \approx -1 + \eta \sin I \frac{1}{\theta} = 0$, where we've taken $\cos\phi = +1$ and $\phi = 0$. This forces $\theta = \eta \sin I$.
- Similarly, for $\mu = -1 + \frac{\epsilon^2}{2}$, $\phi = 0$ and $\epsilon = \eta\sin I$ again. This actually corresponds to $\theta = \pi - \eta\sin I$.
- For $\mu \approx 0$, we have instead $\frac{\partial\phi}{\partial t} = -\mu\left(1 - \eta\sin I\cos\phi\right) + \eta\cos I = 0$. This forces $\mu_\pm = \frac{\eta\cos I}{1 \pm \eta\sin I}$, where $\phi_\pm = \pi, 0$ respectively.

  Note that $\phi = 0, \mu \approx 0$ is conventionally CS4. The linearization locally has form $\frac{\partial\delta\phi}{\partial t} = -\delta\mu\left(1 - \eta\sin I\right)$ and $\frac{\partial\delta\mu}{\partial t} = -\eta\sin I\,\delta\phi$, so the eigenvalues are $\approx \mp\sqrt{\eta\sin I}$, and the two eigenvectors are $\left(1, \pm\sqrt{\eta\sin I}\right)$.

For $\eta \gg 1$, the solutions obviously just come from $\cos I \pm \sin I \cot\theta = 0$, which are just $\sin(I \pm \theta) = 0$

1.5 Separatrix Area

We can estimate the area enclosed by the separatrix, as shown in Fig. 1. Note that the separatrix joins Cassini State 4 to its $+2\pi$ image.

We notate $\mu = \cos\theta$; note that CS4 is $\mu_4 \approx \frac{\eta\cos I}{1 - \eta\sin I} \approx \eta\cos I$. Setting the Hamiltonian equal to its value at CS4 gives

$$\begin{aligned}
H_4 \equiv H\left(\mu_4, \phi_4\right) &\approx -\frac{\mu_4^2}{2} + \eta\mu_4 \cos I - \eta \sin I,\\
&= +\eta^2 \cos^2 I - \eta \sin I,\\
H(\mu_{sep}, \phi_{sep}) = H_4 &= -\eta \sin I \cos\phi_{sep} - \frac{\mu_{sep}^2}{2} + \eta\mu_{sep} \cos I + \mathcal{O}(\eta^3),\\
0 &\approx \frac{\mu_{sep}^2}{2} - \eta\mu_{sep}\cos I - \eta \sin I\left(1 - \cos\phi_{sep}\right) + \eta^2 \cos^2 I,\\
\mu_{sep}(\phi) &\approx \sqrt{2\eta \sin I\left(1-\cos\phi\right)} + \mathcal{O}(\eta).
\end{aligned}$$

We can then easily compute the area enclosed by the separatrix

$$\begin{aligned}
A_{sep} &= \int_0^{2\pi} 2\mu_{sep}\,\mathrm{d}\phi,\\
&\approx 16\sqrt{\eta \sin I}.
\end{aligned} \tag{4}$$

For $\eta = 0.1, I = 20^\circ$, this predicts $\frac{A_{sep}}{A_T} \approx 0.235$, which is pretty close to my numerically calculated $\frac{A_{sep}}{A_T} = 0.229$.

1.6 Tidal Dissipation

We can add a tidal dissipation term; we write it in form $\left(\frac{\mathrm{d}\hat{s}}{\mathrm{d}t}\right)_{tide} = \epsilon\hat{s} \times \left(\hat{l} \times \hat{s}\right) = \epsilon\left(\vec{l} - \left(\vec{s}\cdot\vec{l}\right)\vec{s}\right)$. Expanding,

$$\begin{aligned}
\left(\frac{\mathrm{d}\hat{s}}{\mathrm{d}t}\right)_{tide} &= \epsilon(\hat{z} - s_z\hat{s}),\\
&= \epsilon\left(-s_z s_x \hat{x} - s_z s_y \hat{y} + \left(1 - s_z^2\right)\hat{z}\right).
\end{aligned} \tag{5}$$

We run numerical simulations for weaker $\epsilon \ll \eta \ll 1$ and stronger $\epsilon \lesssim \eta \ll 1$.

We can seek equilibria of the the system including tides, which requires

$$\begin{aligned}
0 &= s_y s_z - \eta s_y \cos I - \epsilon s_z s_x,\\
0 &= -s_x s_z - \eta(s_x \cos I - s_z \sin I) - \epsilon s_z s_y,\\
0 &= \eta s_y \sin(I) + \epsilon\left(1 - s_z^2\right).
\end{aligned}$$

We expect at least two equilibria, based on the simulations: one near $s_z \approx 1$ and one $s_z \approx 0$.

For near alignment/near Cassini state 1, $1 - s_z \sim 1 - s_\perp^2$, so we can set $s_z = 1$ to first order: $s_y - \epsilon s_x - \eta s_y \cos I = -s_x - \eta(s_x \cos I - \sin I) - \epsilon s_y = \eta s_y \sin I = 0$. This can be satisfied if we set $s_x = \tan(I) \ll 1, s_y = \mathcal{O}(\epsilon s_x)$; this coarsely corresponds to Cassini state 1.

The other solution should be near Cassini state 2, where $s_x \approx 1$; dropping second order terms forces $\eta s_y + \epsilon s_z = -s_z - \eta(\cos I - s_z \sin I) = \eta s_y \sin(I) + \epsilon = 0$. This can thus be satisfied for $s_y \approx -\frac{\epsilon}{\eta \sin(I)}$. Thus, this explains why as ϵ is increased, we first start to get points that don't converge to Cassini state 2 in the absence of tides, before starting to see points that fail to converge to Cassini state 1.

2 Separatrix Hopping

Inspired by G&H, heteroclinic orbits are topologically unstable for any nonzero perturbation, but opened width ~ perturbation parameter.

2.1 Try 1: Qualitative

We zoom in on Cassini State 4, which has $\theta_4 = -\frac{\pi}{2} + \frac{\eta\cos I}{1-\eta\sin I}, \mu_4 = \frac{\eta\cos I}{1-\eta\sin I}, \phi_4 = 0$. Then, using equations of motion

$$\frac{\partial\phi}{\partial t} = \mu - \eta\left(\cos I + \sin I\frac{\mu}{\sqrt{1-\mu^2}}\cos\phi\right), \tag{6}$$

$$\frac{\partial\mu}{\partial t} = -\eta\sin I\sin\phi + \left[\epsilon\left(1-\mu^2\right)\right], \tag{7}$$

we can perturbatively require $\frac{\partial\theta}{\partial t} = 0$ for $\epsilon \neq 0$. This corresponds to $\eta\sin I\sin\left(\phi_4 + \delta\phi\right) \approx \epsilon$, or $\delta\phi_4 = +\frac{\epsilon}{\eta\sin I}$. This is in agreement with Dong's result. Note that $\delta\theta_2 = -\frac{\epsilon}{\eta\sin I}$, which I saw in my simulations.

This implies that the stable manifolds of the two saddle points, which once overlapped with each other's unstable manifolds (creating a heteroclinic orbit) now are offset from one another by distance $D \sim \frac{\epsilon}{\eta\sin I}$. The question is how likely it is to thread the needle.

Consider that, very near CS4, the angle of incidence on the desired gap is roughly $\tan\psi \approx \psi = \frac{\Delta\theta}{\Delta\phi}$. Over the course of one orbit, $\Delta\phi$ changes by 2π, while $\Delta\theta \sim \epsilon\sin\theta T$ where T is the period of an orbit. Examining the data, $T \sim 50$, and so $\frac{\Delta\theta}{\Delta\phi} \sim \frac{2\pi}{\epsilon(50)}$.

The effective probability of threading the opened gap between the stable/unstable manifolds is then just $P \propto D\sin\psi \sim \frac{2\pi}{T(\eta\sin I)}$. According to later analysis, this should really be $\frac{2\pi}{T}$. Plugging in some observational values $T \sim 50$ for $\eta = 0.1$gives $P \propto 0.13$. In reality, I find it asymptotes to ~ 0.08, so the constant of proportionality is of order unity. Not bad given the really crappy $\psi \sim \frac{\langle\dot\theta\rangle}{\dot\phi}$ argument.

2.2 Try 2: Melnikov Distance

We notice that the separatrix is a heteroclinic orbit, or a saddle connection, in the dissipation free problem. Introducing dissipation breaks the saddle connection by a distance that can be estimated with the Melnikov distance. This is G&H Equation 4.5.11 or something:

$$d(t_0) = \frac{\epsilon M(t_0)}{\left|f(q^0(0))\right|} + \mathcal{O}(\epsilon^2), \tag{8}$$

$$M(t_0) = \int\limits_{-\infty}^{\infty}\left[f \times g\right]_{hetero}\,\mathrm{d}t. \tag{9}$$

This is not a hard formula to understand; along the separatrix, motion is dominated by f, but the perpendicular component adds up to contribute to a total "perpendicular distance away from the original separatrix" necessary to hit the saddle point, at least intuitively.

We evaluate the Melnikov integral $M(t_0)$ on the heteroclinic orbit. Note that since in our problem our perturbation g is time-independent, so too is the Melnikov integral $M(t_0) = M$.

Let's apply this to the Cassini state Hamiltonian w/ dissipation. We first write down our EOM in Melnikov form (we use canonical variables μ, ϕ):

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} = \underbrace{\frac{\partial\mathcal{H}}{\partial\mu}\hat{\phi} - \frac{\partial\mathcal{H}}{\partial\phi}\hat{\mu}}_{f} + \epsilon\underbrace{\left(1-\mu^2\right)\hat{\mu}}_{g}. \tag{10}$$

Then $f \times g = f_\phi g_\mu = \frac{\partial \mathcal{H}}{\partial \mu}\left(1-\mu^2\right)$. We then want to integrate this along the heteroclinic orbit. We can make change of variables

$$M = \int_0^{2\pi} \frac{\partial \mathcal{H}}{\partial \mu}\left(1-\mu^2\right)\left(\frac{\partial \phi}{\partial t}\right)^{-1} \mathrm{d}\phi. \tag{11}$$

But thankfully, $\frac{\partial \mathcal{H}}{\partial \mu} = \frac{\partial \phi}{\partial t}$ in the absence of dissipation, and so $M = 2\pi\left(1-\mu^2\right) \approx 2\pi\left(1-2\eta\sin I\right)$. Thus, the Melnikov distance at point q^0, a point on the heteroclinic orbit of the unperturbed Hamiltonian, is just

$$d(q^0) = \frac{2\pi\epsilon\left(1-2\eta\sin I\right)}{\left|f(q^0)\right|}. \tag{12}$$

Note that the maximum value $\left|f(q^0)\right|$, which occurs at $\phi = \pi$, is just $f \approx \sqrt{4\eta\sin I}$.

It proves to be a bit difficult to make quantitative predictions though, since the phase diagram is very smushed where f is large, and d is rather inaccurate where f is small. Let's think about a Poincaré map instead.

2.3 Try 3: Poincaré Section

Let's consider the Poincaré section every time $\phi = \phi_4$ as the trajectory subject to tidal dissipation is moving $\theta < \theta_4 \to \theta_4$. To provide an estimate of $\Delta\theta(\theta) = \theta_{n-1} - \theta_n$, this is just ϵT where T is the time elapsed between θ_n, θ_{n+1}, the period of the orbit. T is dominated by when $\frac{\partial \phi}{\partial t} \ll 1$ though, or where the orbit is close to the saddle point.

Note that T is dominated by the time it spends near the saddle point. We showed earlier that near CS4, $\frac{\partial \phi}{\partial t} \approx \delta\mu$ where $\delta\mu = \mu - \mu_4$. Thus, we might surmise $\Delta\theta(\theta) \propto \theta^{-1}$ for sufficiently small $\theta - \theta_4$. Far away, T is roughly constant and $\Delta\theta(\theta)$ is roughly constant.

What is "far away"? Well, it probably depends on how affected our trajectory is by the separatrix; far away from the saddle point, we go along contours of roughly constant θ, while close by we follow the separatrix pretty well. We computed earlier that $\mu_{sep} \sim \sqrt{4\eta\sin I}$, so we might expect $\mu > \mu_{sep}, \Delta\mu \sim C$, while $\mu < \mu_{sep}, \Delta\mu \sim \delta\mu^{-1}$.

My $\mu > \mu_{sep}$ simulations don't seem to work very well, so I'll focus on the $\delta\mu^{-1}$ case. In this case, define $\delta\mu_c : \Delta\mu\left(\delta\mu_c\right) = -\delta\mu_c$, i.e. the point that jumps immediately to the saddle point. Furthermore, assume the inbound distribution is flat between $\delta\mu_c, f^{-1}\left(\delta\mu_c\right)$. TODO: empirically, $\mu_c \sim \epsilon T$ is *flat* with η, probably just because we're not getting sufficiently close to the saddle point for the $\propto \sqrt{\eta}$ to kick in.

Then, we can compare the empirical Poincaré section of the points that cross the separatrix versus the total predicted interval width $\delta\mu_c, f^{-1}\left(\delta\mu_c\right)$; this would predict $7.2\%, 18\%$. This does alright!

Abandoning the rest of this for the time being.

3 Weak Tidal Friction, changing η

Previously, we took the effect of tides to simply be $\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} = \epsilon\hat{s} \times \left(\hat{l} \times \hat{s}\right)$, but in reality, tides will spin down the body (in our case, planet) at the same rate as aligning $\hat{s}$ to $\hat{l}$. We must treat more carefully.

3.1 Equations of Motion

We first write out the full forms of the EOM without tidal friction. These are taken from Kassandra's Equations 1–3 except I replace subscript $\star$ with subscript s since we are interested in the case where the spin of planet 1 evolves with its coupling to its orbital angular momentum and perturber. We obtain (maybe?)

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} = \omega_{s1}(\hat{s}\cdot\hat{l}_1)(\hat{s}\times\hat{l}_1) - \omega_{1p}\cos(I)(\hat{s}\times\hat{l}_p), \tag{13}$$

$$\omega_{s1} = \frac{3k_q}{2k}\left(\frac{R_1}{a_1}\right)^3 s, \tag{14}$$

$$\omega_{1p} = \frac{3m_p}{4m_1}\left(\frac{a_1}{a_p(1-e_p^2)}\right)\Omega_1. \tag{15}$$

Note here that s is the spin frequency and $\Omega_1 = \sqrt{GM_1/a_1^3}$ is the Keplerian orbital frequency.

In the presence of tides, and further assuming $s \ll l_1$, we may write (Lai 2012, Equations 43–44, also Ward 1975 Equation 9 & 13)

$$\frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}t} = \frac{1}{t_s}\frac{L}{2S}\left[\cos\theta - \frac{s}{2\Omega_1}\left(1+\cos^2\theta\right)\right], \tag{16}$$

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\frac{1}{t_s}\frac{L}{2S}\sin\theta\left(1-\frac{s}{2\Omega_1}\cos\theta\right). \tag{17}$$

It is perhaps easiest to define $\frac{s}{s_c} = \frac{\omega_{s1}}{\omega_{1p}\cos I}$ and $\epsilon = \frac{L}{2St_s\omega_{1p}\cos I}$ while rescaling time $\tau = \omega_{1p}\cos(I)t$, so that we obtain equations of motion

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}\tau} = \frac{s}{s_c}(\hat{s}\cdot\hat{l}_1)(\hat{s}\times\hat{l}_1) - \hat{s}\times\hat{l}_p + \epsilon\left(1-\frac{s}{2\Omega_1}(\hat{l}_1\cdot\hat{s})\right)\hat{s}\times(\hat{l}_1\times\hat{s}), \tag{18}$$

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \epsilon s\left(\hat{s}\cdot\hat{l}_1 - \frac{s}{2\Omega_1}\left(1+(\hat{s}\cdot\hat{l}_1)^2\right)\right). \tag{19}$$

s_c has the interpretation of being the critical spin such that the $s1$ coupling is roughly equal strength to the $1p$ coupling. There then seem to be a few outcomes that we might expect:

- Fast evolution towards CS1, then tides will slowly change s without changing $\hat{s}$.
- Fast evolution towards CS2, then tides are strong while state lives inside separatrix maybe? Then will spin down rapidly near CS2 until spin-orbit coupling is weak.
- Slow evolution that trails behind separatrix, expect state to converge somewhere below separatrix? Would probably stay on level curve of high-η H from earlier? Includes anything that doesn't make it to separatrix, including almost fully anti-aligned.

3.2 Crude Analytic Estimate

To make the equations more anemable to analytic analysis (not simulation), let's write down the EOM in (μ, ϕ) coordinates again. The ϕ EOM does not change from the tide-free case, so we can reuse earlier equation:

$$\frac{\partial\phi}{\partial\tau} = \frac{s}{s_c}\mu - \left(\cos I + \sin I\frac{\mu}{\sqrt{1-\mu^2}}\cos\phi\right), \tag{20}$$

$$\frac{\partial\mu}{\partial\tau} = -\sin I\sin\phi + \epsilon(1-\mu^2)\left(1-\frac{s}{2\Omega_1}\mu\right), \tag{21}$$

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \epsilon s\left(\mu - \frac{s}{2\Omega_1}(1+\mu^2)\right). \tag{22}$$

Assuming $s \gg s_c$ the strong spin-orbit coupling regime, let's first try assuming μ is roughly constant over the course of a precession period, then we can average out the ϕ dependencies. Then ϕ drops out of the EOM, and we have approximate averaged equations

$$\begin{aligned}
\frac{\partial \mu}{\partial(\epsilon\tau)} &\approx \left(1-\mu^2\right)\left(1-\frac{s}{2\Omega_1}\mu\right),\\
&\approx \left(1-\mu^2\right)\left(-\frac{s}{2\Omega_1}\mu\right),
\end{aligned} \tag{23}$$

$$\begin{aligned}
\frac{1}{s}\frac{\mathrm{d}s}{\mathrm{d}(\epsilon\tau)} &\approx \mu - \frac{s}{2\Omega_1}\left(1+\mu^2\right),\\
&\approx -\frac{s}{2\Omega_1}.
\end{aligned} \tag{24}$$

In the last term, we note $s \gtrsim 2\Omega_1$ initially, while $\mu \le 1$, so we drop both the linear contribution from μ and approximate $\left(1+\mu^2\right) \approx 1$.

Making all these approximations, we identify approximate analytic solution $s(\epsilon\tau) \approx \frac{2\Omega_1}{\epsilon\tau + C}$ with constant of integration C set by initial conditions, or more precisely

$$s(\epsilon\tau) \approx \frac{2\Omega_1}{\epsilon\tau + \frac{2\Omega_1}{s(0)}}. \tag{25}$$

The critical timescale is clearly $\epsilon\tau \sim 1$ (since $2\Omega_1/s(0) \ll 1$) at which $\frac{s}{2\Omega_1} \sim 1$ and synchronization sets in.

The other timescale of interest is how long it takes $\hat{s}$ to reach a Cassini State. We may present crude estimates in the $s \gg s_c$ strong spin-orbit coupling limit; consider starting $\mu = -1 + \delta$, then to leading order

$$\begin{aligned}
\frac{\partial \delta}{\partial(\epsilon\tau)} &\approx 2\delta\frac{s}{2\Omega_1},\\
\delta(\epsilon\tau) &\approx \delta_0 \exp\left[\frac{\epsilon\tau}{s/\Omega_1}\right].
\end{aligned} \tag{26}$$

Setting $\delta \sim 1$ gives $(\epsilon\tau)_{CS} \sim -\frac{\Omega_1}{s}\ln\delta_0$ the timescale it takes to reach $\mu = 0$ which is of order the timescale to reach the CS. This tells us that at displacements of $\delta_0 \gtrsim e^{-s/\Omega_1}$ is where states will be able to reach Cassini States. So for any respectable s/Ω_1, the CS timescale will be short compared to the synchronization timescale, so we should expect the adiabatic evolution near CS1, CS2 as $\frac{s}{s_c}$ changes governs the long-term observed states. Specifically, states that end up near CS1 should (slowly!) hit the resonant obliquity excitation that Kassandra studied, while states near CS2 should stay on CS2 as η grows past η_{crit}, since it is far from bifurcation.

Under this assumption, it would appear that points that start inside the separatrix (rare) stay inside the separatrix, points that start above the separatrix stay above and undergo resonant obliquity excitation ($\sim 50\%$), and finally points that start outside the separatrix probably end up at either CS1 or CS2 probabilistically depending on $\eta(\Delta T(\mu_0))$ where $\Delta T(\mu_0)$ is the arrival time for a state starting at μ_0.