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We analyze Dong's equations, where:

$$\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t} = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}_1\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}_1\right), \tag{1}$$

$$\hat{\mathbf{l}}_1 = \mathrm{Re}\,\mathscr{I}_1\hat{\mathbf{x}} + \mathrm{Im}\,\mathscr{I}_1\hat{\mathbf{y}} + \sqrt{1-|\mathscr{I}_1|^2}\hat{\mathbf{z}}, \tag{2}$$

$$= \begin{bmatrix} i_1\cos(g_1 t) + i_{1\mathrm{f}}\cos\left(g_2 t + \phi_0\right) \\ i_1\sin(g_1 t) + i_{1\mathrm{f}}\sin\left(g_2 t + \phi_0\right) \\ \cos|\mathscr{I}_1| \end{bmatrix}. \tag{3}$$

I've taken $\phi_0 = 0$ for simplicity. We take $\alpha/g_1 = 10$ consistently. We permit $i_{1\mathrm{f}}$ and $g_{1,2}$ to vary independently for now, even though $i_{1\mathrm{f}} \propto (g_2 - g_1)^{-1}$.

1 Preliminary Numerical Work

We analyze the dynamics of ϕ_{rot}, the azimuthal angle in the co-rotating frame. This lets us both understand the locations of equilibria ($\phi_{\mathrm{rot}} \approx \phi_{\mathrm{rot},0}$) and the distortion of the separatrix (circulation-libration transition). We suppress the suffix. In particular, we consider the quantity:

$$\Delta\phi(i_{1\mathrm{f}}; g_2; \Delta\theta_{\mathrm{i}}) \equiv \phi_{\max} - \phi_{\min}. \tag{4}$$

Here, we fix the initial $\phi_{\mathrm{rot}} = 180^\circ$, and $\Delta\theta_{\mathrm{i}}$ is the angular distance from CS2. For instance, $\Delta\phi(0,0,0) = 0$. We show a few plots in Figs. 1–4.

It is clear that there are some resonances for $g_1 \simeq g_2$! In fact, there are some interesting behaviors. If we examine the phase space evolution (in $\cos\theta, \phi$ space as usual) of ICs near CS2 as we vary g_2/g_1, we obtain:

2 Analytical Approach

It is clear that there are some resonance behaviors, and in particular that $g_2/g_1 = 1$ is *not* a resonance; it is suspiciously quiet (see Fig. 5). Here is a simple explanation:

Consider the Hamiltonian

$$H = -\frac{\alpha}{2}\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)^2. \tag{5}$$

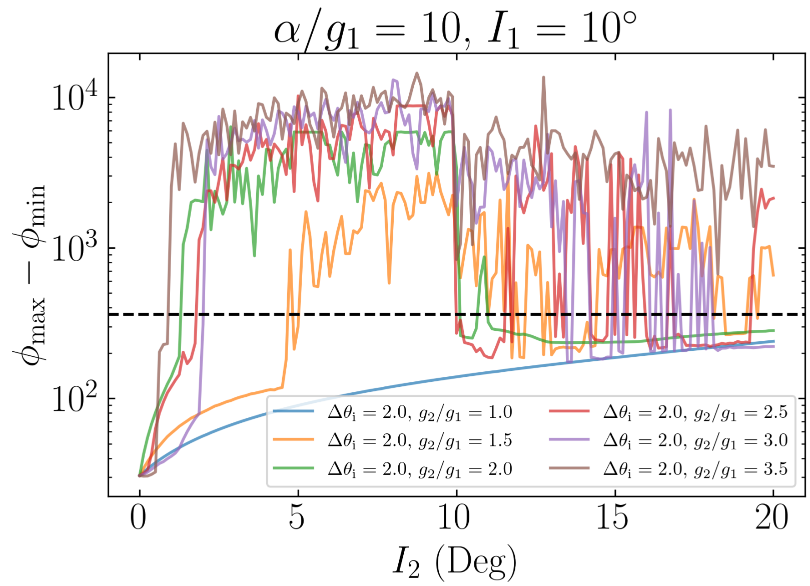

Figure 1: Scanning $\Delta\phi$ over $i_{1\mathrm{f}}$. Note that $i_1 = 10°$ for all of these.

Decompose $\hat{\mathbf{l}} = \bar{\mathbf{l}} + \mathbf{l}'$ where:

$$\bar{\mathbf{l}} = \left[i_1 \cos(g_1 t) i_1 \sin(g_1 t) \cos i_1 \right], \tag{6}$$

$$\mathbf{l}' \equiv \hat{\mathbf{l}} - \bar{\mathbf{l}}, \tag{7}$$

$$\approx \begin{bmatrix} i_{1\mathrm{f}} \cos\left(g_2 t + \phi_0\right) \\ i_{1\mathrm{f}} \sin\left(g_2 t + \phi_0\right) \\ -\frac{i_{1\mathrm{f}} i_1}{\cos^2 i_1} \cos\left(g_2 t + \phi_0\right) \end{bmatrix}. \tag{8}$$

Note that $l'_z \approx 0$.

We next go to the corotating frame with g_1 about $\hat{\boldsymbol{\jmath}}$, so that

$$H_{\rm rot} = -\frac{\alpha}{2}\left(\hat{\mathbf{s}} \cdot \hat{\mathbf{l}}\right)^2 - g\left(\hat{\mathbf{s}} \cdot \hat{\boldsymbol{\jmath}}\right), \tag{9}$$

$$\equiv H_0 + H_1, \tag{10}$$

$$H_0 \equiv -\frac{\alpha}{2}\left(\hat{\mathbf{s}} \cdot \bar{\mathbf{l}}\right)^2 - g\left(\hat{\mathbf{s}} \cdot \hat{\boldsymbol{\jmath}}\right), \tag{11}$$

$$H_1 \approx -\alpha\left(\hat{\mathbf{s}} \cdot \bar{\mathbf{l}}\right)\left(\hat{\mathbf{s}} \cdot \mathbf{l}'\right) + \mathcal{O}\left((l')^2\right). \tag{12}$$

In the corotating frame, we find that

$$\left(\mathbf{l}'\right)_{\rm rot} \approx \begin{bmatrix} i_{1\mathrm{f}} \cos\left((g_2 - g_1)t + \phi_0\right) \\ i_{1\mathrm{f}} \sin\left((g_2 - g_1)t + \phi_0\right) \\ 0 \end{bmatrix}. \tag{13}$$

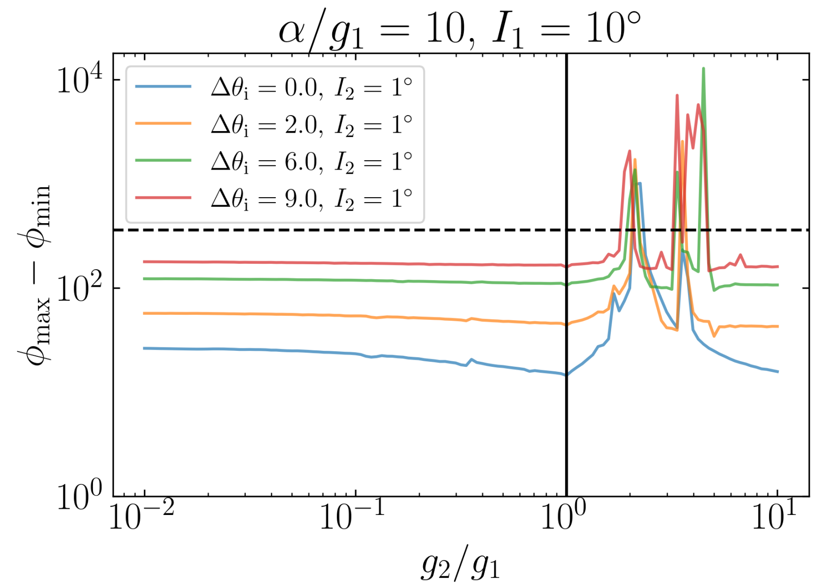

Figure 2: Scanning $\Delta\phi$ over g_2.

Thus, in the corotating frame, when $g_2 = g_1$, CS2 is simply *shifted*, since $\mathbf{l}'_{\rm rot}$ is fixed in space. This isn't entirely realistic, as $i_{1\rm f} \propto (g_2 - g_1)^{-1}$.

3 Numerical Searching for Resonant Behavior (WIP)

We look for resonances by fixing all frequencies of the system and varying $\Delta\theta_{\rm i}$, proximity to CS2.

What are the origin of any resonances? My hypothesis was that if $g_2 - g_1 \simeq \omega_{\rm lib}$, the libration frequency about CS2, then resonances can appear. Thus, we calculate $\omega_{\rm lib}$ as a function of $\Delta\theta_{\rm i}$ numerically.

Separately, is evolution chaotic? For every $\Delta\theta_{\rm i}$, we can compute the evolution for $\phi_{\rm i} = 0$ and $\phi_{\rm i} = 10^{-5}$, then record

$$\max \Delta\hat{\mathbf{s}} \equiv \max_t |\hat{\mathbf{s}}_1 - \hat{\mathbf{s}}_2|. \tag{14}$$

If $\max \Delta\hat{\mathbf{s}} \simeq 1$, then chaos is likely (I spot checked that these seemed to correspond to the case of exponential growth).

We can plot both of these criteria in Fig. 6. It seems like a plausible explanation that resonances occur when $g_2 - g_1 \simeq \omega_{\rm lib}$, and that these resonances give rise to chaotic behavior (likely when interacting with the separatrix).

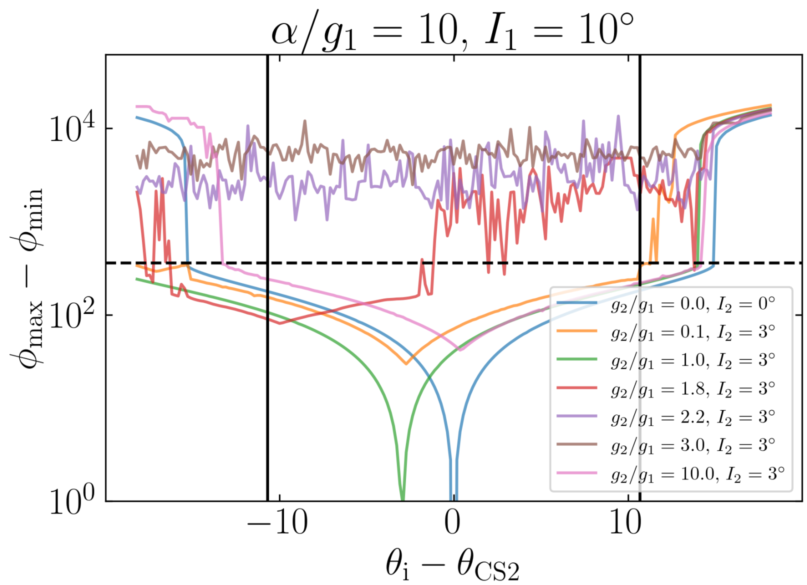

Figure 3: Scanning $\Delta\phi$ over $\Delta\theta_{\rm i}$.

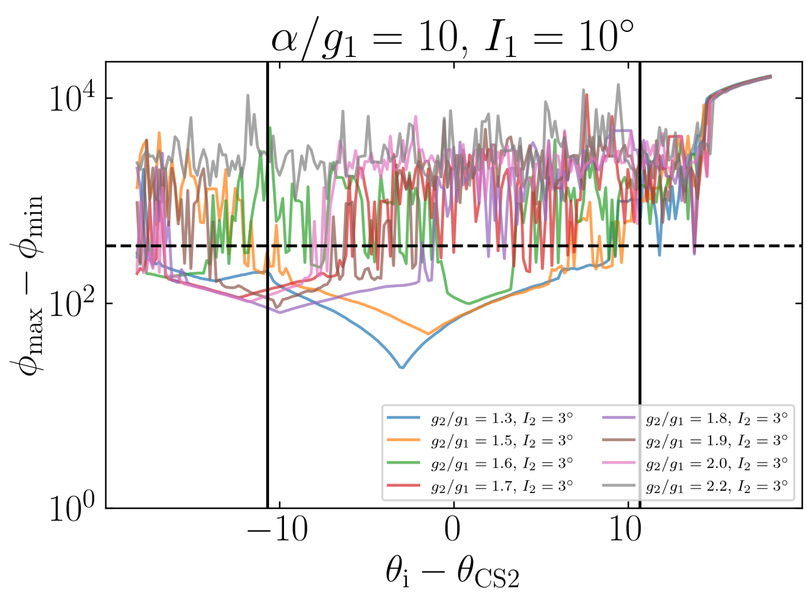

Figure 4: Scanning $\Delta\phi$ over $\Delta\theta_{\rm i}$ for many frequencies near the $g_2 = 2g_1$ resonance.

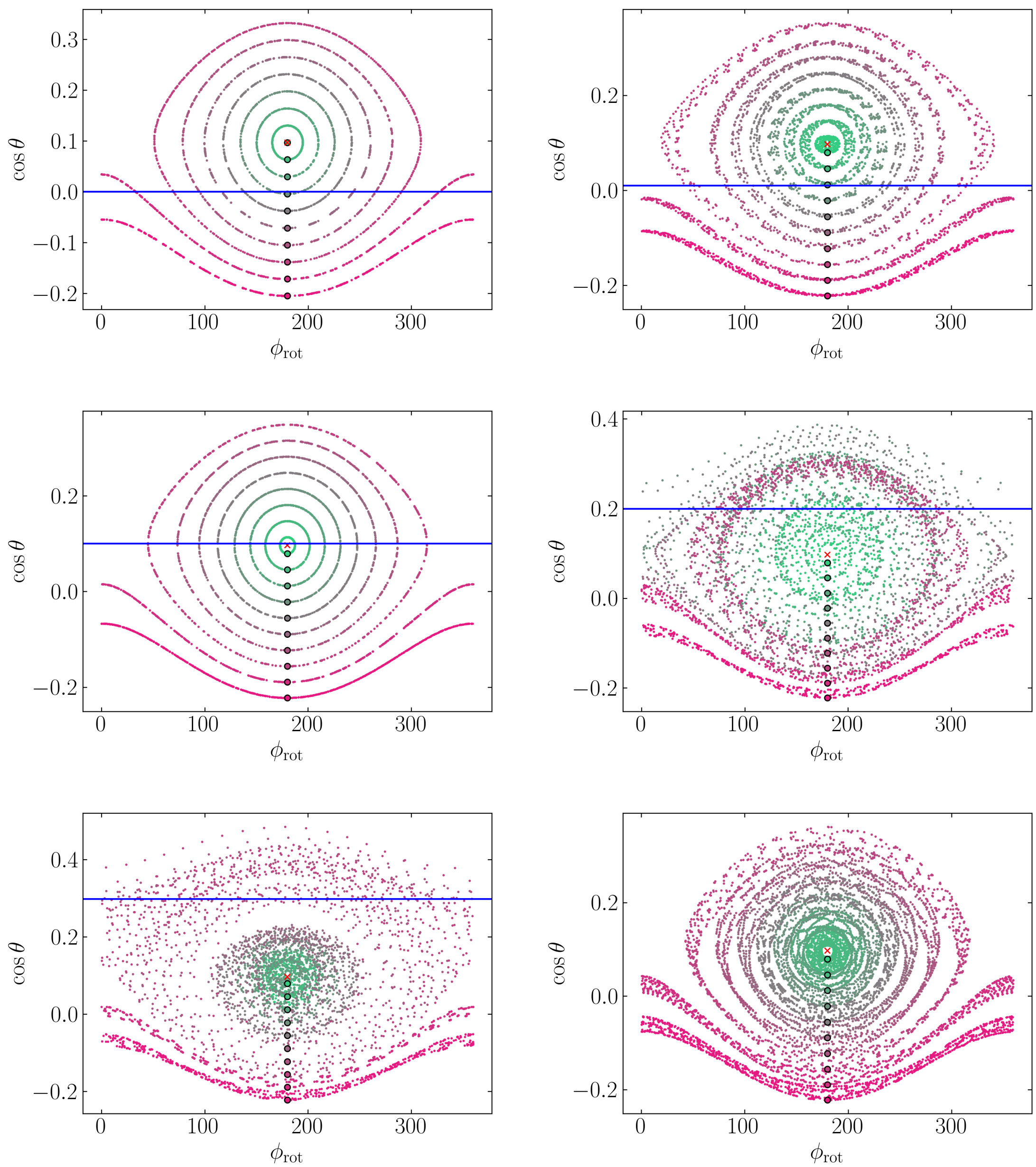

Figure 5: Evolution of some ICs for $g_2/g_1 = \{0, 0.1, 1, 2, 3, 10\}$. Note that for $g_2 = 0$, we've set $i_{1\rm f} = 0°$, else we set $i_{\rm f} = 1°$, and $i_1 = 10°$.

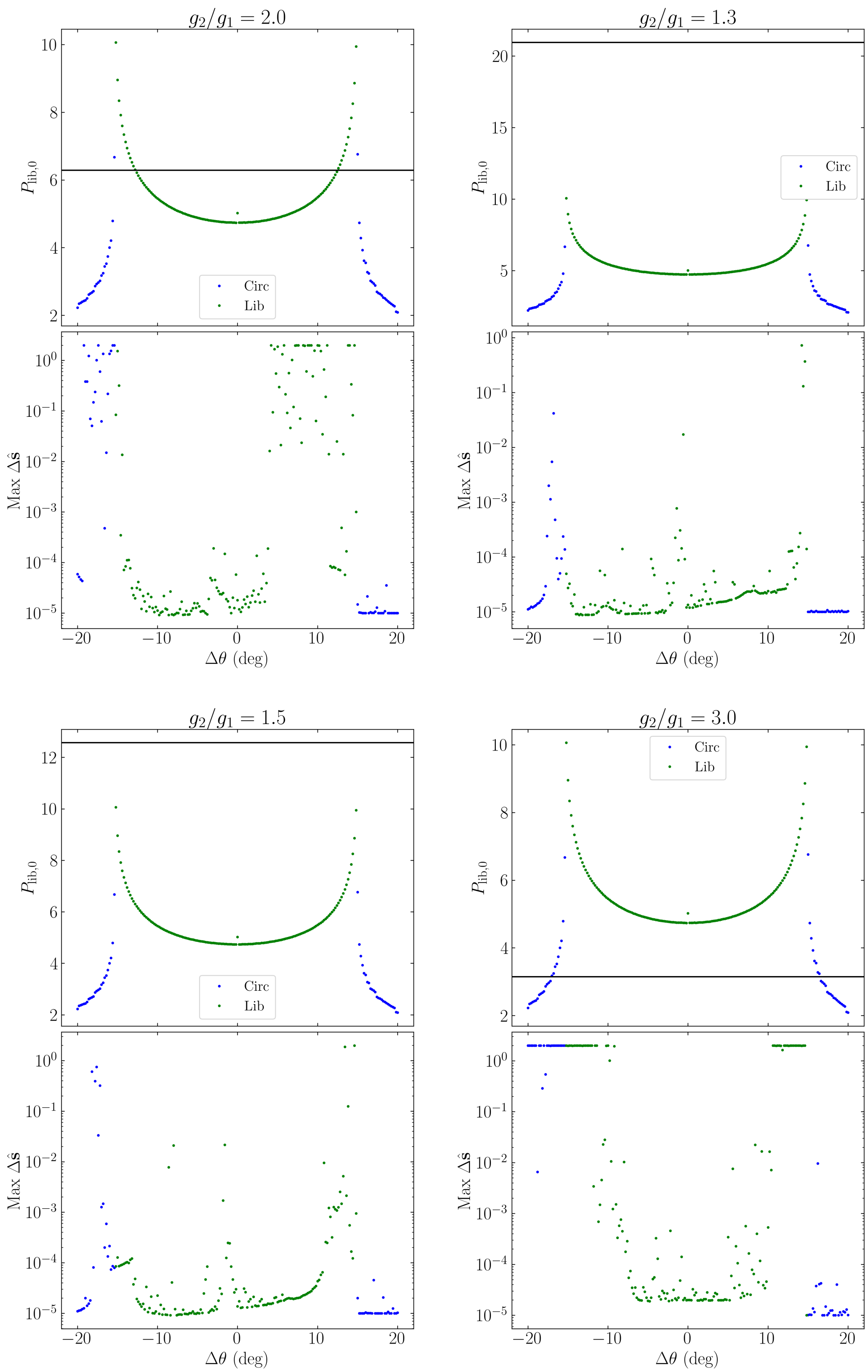

Figure 6: Top panels: $\omega_{\rm lib}$ (in the absence of any perturbation) as a function of distance to CS2. Horizontal line is the naive $2\pi/(g_2 - g_1)$. Bottom: $\max \Delta\hat{\mathbf{s}}$, where large values suggest chaos. $i_1 = 10^\circ$ and $i_{1\rm f} = 1^\circ$ as usual.