

# High Spin-Orbit Misalignment as an Attractor in Cassini State Systems with Weak Tidal Friction

Meeting XX/XX/XXXX

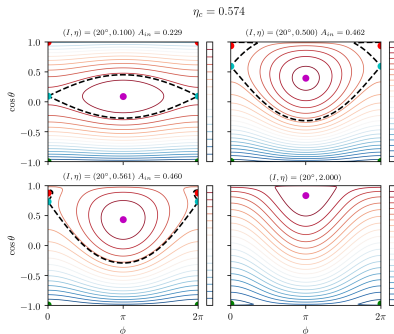
Yubo Su

Some day

# Problem 1

## Introduction

- Close-in planet to a star w/  $\vec{l}$  starts with random spin  $\hat{s}$  (e.g. collision). Evolves under tides + precession around perturber  $\hat{l}_p$ .
- **Toy Problem:** Assume constant tidal dissipation, fate?
- Cassini States:  $H^{(0)} = \frac{(\hat{s} \cdot \hat{l})^2}{2} + \eta \hat{s} \cdot \hat{l}_p$ . CS4 is saddle point, *separatrix*.



# Problem 1

## Constant Tides

- Constant tides  $\frac{d\theta}{dt} = \epsilon \sin \theta$ , EOM ( $\mu = \cos \theta$ ):

$$\frac{d\hat{s}}{d\tau} = (\hat{s} \cdot \hat{l})(\hat{s} \times \hat{l}) - \eta \hat{s} \times \hat{l}_p + \epsilon \hat{s} \times (\hat{l} \times \hat{s}),$$

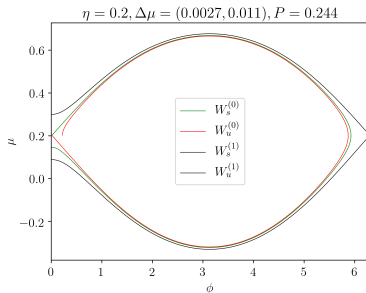
$$\frac{\partial \phi}{\partial t} = \mu - \eta \left( \cos I + \sin I \frac{\mu}{\sqrt{1 - \mu^2}} \cos \phi \right),$$

$$\frac{\partial \mu}{\partial t} = -\eta \sin I \sin \phi + \epsilon (1 - \mu^2),$$

- *Review:* Last meeting, found probability that initial state  $\cos \theta < 0$  jumps into separatrix  $\propto \eta^{3/2} \epsilon^0$ .

# Problem 1

## Flow Boundaries



- Notation: subscripts *stable*, *unstable* manifolds, superscripts are left/right copy of CS4.
  - Below black, continues orbit  $\mu < \mu_4$ .
  - Black/green, escapes.
  - Green/red, captures.
- Evaluate  $\Delta\mu$  at  $\phi = \pi$  reproduces probability (numerical was  $P_{hop}(\eta = 0.2) = 0.251$ ).

# Problem 1

## Melnikov's Method (coarsely)

- **Goal:** Under weak perturbation, how much does trajectory deviate from level curve of  $H$ ?

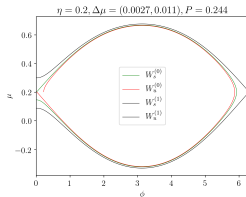
- Procedure:

- Compute  $\Delta H^{(0)}$  unperturbed over trajectory.
- Locate new level curve to find  $\Delta q$  along coordinate:

$$\Delta H^{(0)} = \frac{\partial H^{(0)}}{\partial q} \Delta q.$$

- So looks something like:

$$\frac{dH^{(0)}}{dt} = \underbrace{\frac{\partial H^{(0)}}{\partial \mu} \frac{d\mu^{(0)}}{dt} + \frac{\partial H^{(0)}}{\partial \phi} \frac{d\phi^{(0)}}{dt}}_{\dot{\phi}^{(0)} \dot{\mu}^{(0)} - \dot{\mu}^{(0)} \dot{\phi}^{(0)}} + \frac{\partial H^{(0)}}{\partial \mu} \frac{d\mu^{(1)}}{dt}.$$



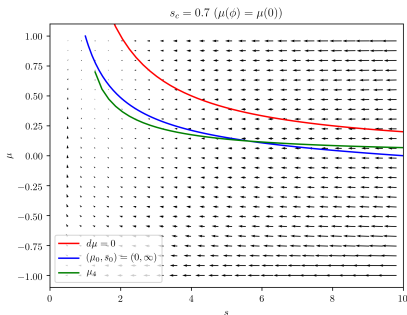
- Gives rise to Melnikov distance:

- $\Delta \mu(\phi) = \frac{1}{\dot{\phi}(\phi)} \oint \dot{\phi}^{(0)} \epsilon (1 - \mu^2) dt.$
- $\mu(\phi) \approx \eta \cos I \pm \mathcal{O}(\sqrt{\eta}).$
- $W_s^{(1)}, W_u^{(0)}$ : dominated by 1.
- $W_s^{(0)}, W_u^{(1)}$ : dominated by  $\eta^{3/2}$ !

# Problem 2

## Realistic Tides

- In realistic tides,  $\eta$  can evolve as  $s$  spins down.
- $\eta = \frac{s_c}{s}$ , full expression below.
- First, consider tides alone, phase portrait at right:
  - Like ignoring  $\frac{d\mu}{d\phi}$  over precession orbit.
  - Breaks down near  $\mu_4$ !



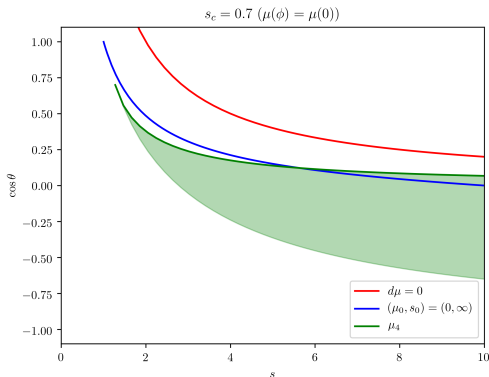
$$\frac{d\hat{s}}{d\tau} = \frac{s}{s_c} (\hat{s} \cdot \hat{l}) (\hat{s} \times \hat{l}) - \hat{s} \times \hat{l}_p + \frac{\epsilon 2\Omega}{s} \left( 1 - \frac{s}{2\Omega} (\hat{l} \cdot \hat{s}) \right) \hat{s} \times (\hat{l} \times \hat{s}), \quad (1)$$

$$\frac{ds}{d\tau} = \epsilon 2\Omega \left( \hat{s} \cdot \hat{l} - \frac{s}{2\Omega} \left( 1 + (\hat{s} \cdot \hat{l})^2 \right) \right). \quad (2)$$

# Problem 2

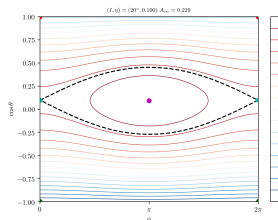
## Intuitive Guess

- Ignore perturber for now ( $\eta \ll 1$ ). Maybe: *points incident on CS4* have  $P_{hop}$  probability of entering separatrix?
- For instance, shaded green = capture region:



# Problem 2

Near CS4



- When  $\eta$  non-negligible, full phase space  $(\mu, \phi, s)$ .
- Study  $(\mu_0, s)$  where  $\mu_0 \equiv \mu(\phi = 0)$ .
  - $\mu_0 \approx \mu_4 = P_{hop}$  candidate.

- Update model:

- $|\mu_0 - \mu_4| \gtrsim \sqrt{\eta \sin I}$ , then  $\mu(\phi) \approx \mu_0$ .
- $|\mu_0 - \mu_4| \lesssim \sqrt{\eta \sin I}$ , turns out

$$\Delta\mu \sim \frac{\epsilon}{\mu_4 - \mu_0} \dot{\mu}^T(\mu_{eff}). \quad (3)$$

- $\dot{\mu}^T$  is just the tidal component.
- $\mu_{eff}$  such that  $\dot{\mu}^T(\mu_{eff}) \equiv \langle \dot{\mu}^T(\mu(t)) \rangle$ .

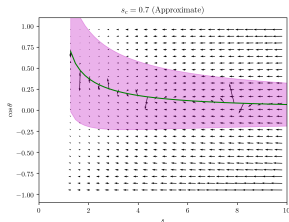
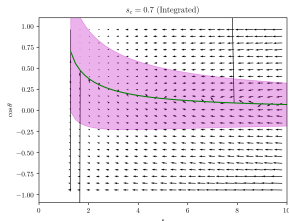


# Problem 2

## Effective Model

$$\Delta\mu \sim \frac{\epsilon}{\mu_4 - \mu_0} \dot{\mu}^T(\mu_{eff}).$$

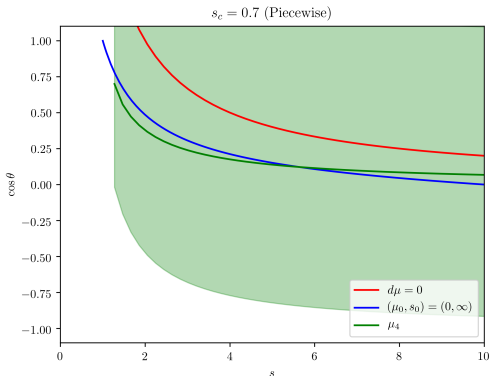
- Two key properties:
  - $\Delta\mu$ 's sign is set at  $\mu_{eff}$ , which changes steeply  $\mu_0 \approx \mu_4$ .
  - $\mu_0 \approx \mu_4$  means large  $\Delta\mu$ .
- Trajectories will cross CS4 *multiple times*!
- Effective “phase portrait”.
  - Shade  $|\mu_4 - \mu_0| < \sqrt{\eta \sin I}$ .  
*Attracting!*



# Problem 2

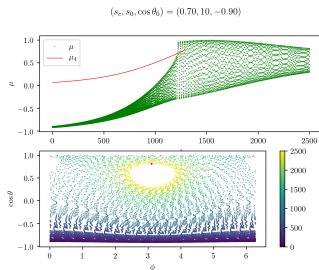
## Capture Region

- Effective CS4 cross section attracts onto CS4, trajectories cross CS4 multiple times,  $P_{hop} \approx 1$  to CS2.
- Capture region of cross section? Almost all!



# Problem 2

## Summary



- Slightly outside of capture region in previous slide, barely intersects  $\mu_4$  before  $\mu_4$  disappears. Near capture boundary.

- Capture region
  - Flow CS4 effective cross section backwards in time.
  - 100% goes to CS2.
- Above: stays near CS1, tides stay weak.
- Below: cannot catch up to CS4, go on level curves of large- $\eta$  Hamiltonian.