Contents

1 06/02/21—Simple Parameterized Model

1.1 Orbital Dynamics

We consider the 3-planet case with an inner test mass, as Dong did, for simplicity. This is not quantitatively exact but gives us an intuitive parameterization (to leading order) of realistic dynamics.

Consider the mutual precession of an inner test particle and two outer planets. Define the complex inclination $\mathcal{I}_j = I_j \exp\left(i\Omega_j\right)$, then these evolve following

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathcal{I}_1\\ \mathcal{I}_2\\ \mathcal{I}_3\end{bmatrix} = \begin{bmatrix}-\omega_{21}-\omega_{31} & \omega_{21} & \omega_{31}\\ 0 & -\omega_{32} & \omega_{32}\\ 0 & \omega_{23} & -\omega_{23}\end{bmatrix}\begin{bmatrix}\mathcal{I}_1\\ \mathcal{I}_2\\ \mathcal{I}_3\end{bmatrix}, \tag{1}$$

$$\omega_{kj} = -\frac{3m_>}{4M_\star}\left(\frac{a_>}{a_<}\right)^3 n_< f\left(\frac{a_<}{a_>}\right) \times \min\left(\frac{L_k}{L_j}, 1\right), \tag{2}$$

$$\begin{aligned} f\left(\alpha\right) &= \frac{b_{3/2}^{(1)}}{3\alpha}\\ &= \frac{1}{3\alpha}\frac{1}{\pi}\int\limits_0^{2\pi}\frac{\cos\left(t\right)}{\left(\alpha^2 + 1 - 2\alpha\cos t\right)^{3/2}}\,\mathrm{d}t\\ &\approx 1 + \frac{15}{8}\alpha^2 + \mathcal{O}\left(\alpha^4\right). \end{aligned} \tag{3}$$

Note that this min term means e.g. that $\omega_{23} = \omega_{32}\frac{L_2}{L_3}$.

We seek the eigenmodes of Eq. (1). It is expected that there will only be two: a third is if $\mathcal{I}_i = 0$, i.e. if everything is aligned with the total angular momentum. Otherwise:

- The first obvious, non-trivial eigenmode is $\mathcal{I}_2 = \mathcal{I}_3 = 0$, then $\mathcal{I}_1$ executes free precession with the precession frequency $g_1 \equiv -\omega_{21} - \omega_{31}$.

- We expect the second eigenmode to describe precession of $\mathcal{I}_2$ and $\mathcal{I}_3$ about their total angular momentum axis. Upon inspection, we find that the second eigenvector must indeed satisfy:

$$\mathcal{I}_2 = -\frac{L_3}{L_2}\mathcal{I}_3. \tag{4}$$

The eigenvalue can also be directly read off in this case to be $\omega_{32}\left(-1 - L_2/L_3\right) \approx -\omega_{32}(J/L_3) \equiv g_2$, where $J \approx L_2 + L_3$. Then, we can find that the corresponding component of $\mathcal{I}_1$ must satisfy

$$g_1 \mathcal{I}_1 + \omega_{21} \mathcal{I}_2 + \omega_{31} \mathcal{I}_3 = g_2 \mathcal{I}_1, \tag{5}$$

$$\left(g_2 - g_1\right) \mathcal{I}_1 = \left(\omega_{21} \frac{L_3}{J} + \omega_{31} \frac{L_2}{J}\right) \mathcal{I}_{23}, \tag{6}$$

$$\mathcal{I}_1 = \frac{\left(\omega_{21} L_3 + \omega_{31} L_2\right) / J}{g_2 - g_1} \mathcal{I}_{23}. \tag{7}$$

Here, $\mathcal{I}_{23} = \mathcal{I}_3 - \mathcal{I}_2$ and corresponds to the complexified mutual inclination. Thus, the general solution to $\mathcal{I}_1(t)$ is given by:

$$\mathcal{I}_1(t) = i_1 e^{i g_1 t} + i_{1\mathrm{f}} e^{i(g_2 t + \phi_0)}, \tag{8}$$

where i_1, $i_{1\mathrm{f}}$, and ϕ_0 are real. Then, of course, we can construct the vector solution to $\hat{\mathbf{l}}_1$ via:

$$\begin{aligned} \hat{\mathbf{l}}_1 &= \mathrm{Re}\, \mathcal{I}_1 \hat{\mathbf{x}} + \mathrm{Im}\, \mathcal{I}_1 \hat{\mathbf{y}} + \sqrt{1 - \left|\mathcal{I}_1\right|^2} \hat{\mathbf{z}}, \\ &= \begin{bmatrix} i_1 \cos(g_1 t) + i_{1\mathrm{f}} \cos\left(g_2 t + \phi_0\right) \\ i_1 \sin(g_1 t) + i_{1\mathrm{f}} \sin\left(g_2 t + \phi_0\right) \\ \cos\left|\mathcal{I}_1\right|. \end{bmatrix} \end{aligned} \tag{9}$$

Note that $i_{1\mathrm{f}}$ is effectively set by the initial $\mathcal{I}_{23}$ (overlap with the eigenvector), and i_1 is set by whatever remaining IC is not described by the forced component.

1.2 Spin Dynamics

We now have the parameterized form for $\hat{\mathbf{l}}_1$ following Eq. (17) and the solution for $\mathcal{I}_1$, and the spin evolves following

$$\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t} = \alpha \left(\hat{\mathbf{s}} \cdot \hat{\mathbf{l}}_1\right) \left(\hat{\mathbf{s}} \times \hat{\mathbf{l}}_1\right). \tag{10}$$

1.2.1 Numerical Approach

We seek to understand the stability of CS2 in the regime $\alpha \gg g_1, g_2$. WLOG, we choose to perturb about the initial condition where $\hat{\mathbf{s}}$ points exactly along CS2, and $i_{1\mathrm{f}} = 0$ (as is in the case of aligned outer planets; note that if the inner planet is not a perfect test mass, aligned outer planets does not completely suppress this mode). There are then three perturbations we can make:

- We can increase $i_{1\mathrm{f}}$; the two relevant regimes are expected to be where $i_{1\mathrm{f}} \ll i_1$ and $i_{1\mathrm{f}} \gg i_1$.
- We can change g_2; the two relevant regimes are expected to be where $g_2 \gg g_1$ and $g_2 \ll g_1$, with possible resonance when they are close (since $i_{1\mathrm{f}}$ is prescribed explicitly and held constant, we can imagine that the mutual inclination is decreased in proportion to $g_2 - g_1$ such that $i_{1\mathrm{f}}$ is constant).
- We can perturb the initial system about the initial CS2; we do this by adding an angle offset $\Delta\theta_{\mathrm{i}}$ to the IC, so that there is a nonzero initial libration amplitude.

1.2.2 Analytical Approach

It seems like we might be seeing some resonance type behaviors. For convenience, we go to the co-rotating frame with g_1 where we then have

$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\mathrm{rot},1} = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}_1\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}_1\right) + g_1\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{J}}\right). \tag{11}$$

Here, now, the mode frequencies are $g_1 \Rightarrow 0$ and $g_2 \Rightarrow g_2 - g_1$. Decompose then $\hat{\mathbf{l}}$ (we drop the subscript) into a mean and fluctuating piece $\bar{\mathbf{l}}$ and $\mathbf{l}'$ (where the former is still a vector of unit length), then:

$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\mathrm{rot},1} = \alpha\left(\hat{\mathbf{s}}\cdot\bar{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\bar{\mathbf{l}}\right) + g_1\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{J}}\right) + \alpha\left[\left(\hat{\mathbf{s}}\cdot\bar{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\mathbf{l}'\right) + \left(\hat{\mathbf{s}}\cdot\mathbf{l}'\right)\left(\hat{\mathbf{s}}\times\bar{\mathbf{l}}\right)\right]. \tag{12}$$

This is already looking painful. Let's write down the Hamiltonian to have a faster path to EOM in coordinates:

$$\begin{aligned}
H_{\mathrm{rot},1} &= -\frac{\alpha}{2}\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)^2 - g\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{J}}\right), &(13)\\
&= -\frac{\alpha}{2}\left(\hat{\mathbf{s}}\cdot\bar{\mathbf{l}}\right)^2 - \alpha\left(\hat{\mathbf{s}}\cdot\bar{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\cdot\mathbf{l}'\right) - g\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{J}}\right), &(14)\\
&\approx -\frac{\alpha}{2}\cos^2\theta - g\left(\cos\theta\cos I - \sin I\sin\theta\cos\phi\right)\\
&\quad - \alpha\cos\theta\left(\cos\theta\cos i_{1\mathrm{f}} - \sin i_{1\mathrm{f}}\sin\theta\cos\left(\phi - \phi_0 - \left(g_2 - g_1\right)t\right)\right). &(15)
\end{aligned}$$

Is this right, or is $i_{1\mathrm{f}}$ also changing?

2 06/09/21

2.1 Analytical Time-Dependent Hamiltonian Perturbation Theory Setup (Abandoned)

We came up with at least a possible way to do this perturbatively, though it might not be useful, since we haven't included the effect of tidal dissipation yet. I'm going to just mostly copy this from my weekly update, since I don't want to retype the scratch work. We analyze Dong's equations, where:

$$\begin{aligned}
\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t} &= \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}_1\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}_1\right), &(16)\\
\hat{\mathbf{l}}_1 &= \mathrm{Re}\,\mathcal{I}_1\hat{\mathbf{x}} + \mathrm{Im}\,\mathcal{I}_1\hat{\mathbf{y}} + \sqrt{1 - |\mathcal{I}_1|^2}\hat{\mathbf{z}}, &(17)\\
&= \begin{bmatrix} i_1\cos(g_1 t) + i_{1\mathrm{f}}\cos\left(g_2 t + \phi_0\right)\\ i_1\sin(g_1 t) + i_{1\mathrm{f}}\sin\left(g_2 t + \phi_0\right)\\ \cos|\mathcal{I}_1| \end{bmatrix}. &(18)
\end{aligned}$$

I've taken $\phi_0 = 0$ for simplicity. We take $\alpha/g_1 = 10$ consistently. We permit $i_{1\mathrm{f}}$ and $g_{1,2}$ to vary independently for now, even though $i_{1\mathrm{f}} \propto \left(g_2 - g_1\right)^{-1}$.

Consider the Hamiltonian

$$H = -\frac{\alpha}{2}\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)^2. \tag{19}$$

Decompose $\hat{\mathbf{l}} = \bar{\mathbf{l}} + \mathbf{l}'$ where:

$$\bar{\mathbf{l}} = \left[i_1 \cos(g_1 t) i_1 \sin(g_1 t) \cos i_1\right], \tag{20}$$

$$\mathbf{l}' \equiv \hat{\mathbf{l}} - \bar{\mathbf{l}}, \tag{21}$$

$$\approx \begin{bmatrix} i_{1\mathrm{f}} \cos\left(g_2 t + \phi_0\right) \\ i_{1\mathrm{f}} \sin\left(g_2 t + \phi_0\right) \\ -\frac{i_{1\mathrm{f}} i_1}{\cos^2 i_1} \cos\left(g_2 t + \phi_0\right) \end{bmatrix}. \tag{22}$$

Note that $l_z' \approx 0$.

We next go to the corotating frame with g_1 about $\hat{\boldsymbol{\jmath}}$, so that

$$H_{\mathrm{rot}} = -\frac{\alpha}{2}\left(\hat{\mathbf{s}} \cdot \hat{\mathbf{l}}\right)^2 - g\left(\hat{\mathbf{s}} \cdot \hat{\boldsymbol{\jmath}}\right), \tag{23}$$

$$\equiv H_0 + H_1, \tag{24}$$

$$H_0 \equiv -\frac{\alpha}{2}\left(\hat{\mathbf{s}} \cdot \bar{\mathbf{l}}\right)^2 - g\left(\hat{\mathbf{s}} \cdot \hat{\boldsymbol{\jmath}}\right), \tag{25}$$

$$H_1 \approx -\alpha\left(\hat{\mathbf{s}} \cdot \bar{\mathbf{l}}\right)\left(\hat{\mathbf{s}} \cdot \mathbf{l}'\right) + \mathcal{O}\left((l')^2\right). \tag{26}$$

In the corotating frame, we find that

$$\left(\mathbf{l}'\right)_{\mathrm{rot}} \approx \begin{bmatrix} i_{1\mathrm{f}} \cos\left((g_2 - g_1)t + \phi_0\right) \\ i_{1\mathrm{f}} \sin\left((g_2 - g_1)t + \phi_0\right) \\ 0 \end{bmatrix}. \tag{27}$$

Thus, in the corotating frame, when $g_2 = g_1$, CS2 is simply *shifted*, since $\mathbf{l}'_{\mathrm{rot}}$ is fixed in space. This isn't entirely realistic, as $i_{1\mathrm{f}} \propto (g_2 - g_1)^{-1}$, but also gives us a chance at using time-dependent perturbation theory.

NB: Per Dong's recommendation, I'm going to mostly stay away from this, as the dynamics are already reasonably well understood in the absence of tidal dissipation due to resonance overlap. I suspect that there might be some other interesting behavior, like chaotic behavior near the separatrix due to another Melnikov's Method calculation, but it's a little bit difficult to justify doing too much work on this when there's a lot of canonical work on the subject already.

I do have a concern about resonance overlap: the resonances are only formally defined in their respective corotating frames (i.e. they don't share resonance angles $\phi_i = \phi_{\mathrm{inertial}} + g_i t$), so I'm not exactly sure the picture of "don't know which resonance a point belongs to" is entirely valid.