Analytical Predictions of Explanet Obliquities Generated by Planet-Disk Interactions

Yubo Su[1], Dong Lai[1,2]

[1] *Cornell Center for Astrophysics and Planetary Science, Department of Astronomy, Cornell University, Ithaca, NY 14853, USA*

[2] *Tsung-Dao Lee Institute, 800 Dongchuan Road, Shanghai, 200240, China*

Accepted XXX. Received YYY; in original form ZZZ

ABSTRACT

Abstract here

Key words: planet–star interactions

1 INTRODUCTION

Separatrix crossing was studied by (Henrard 1982).

2 THEORY AND EQUATIONS

We study an oblate planet orbiting around a star hosting a further out planet. The equations of motion in the absence of dissipation are:

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} = \omega_{s1}\left(\hat{s}\cdot\hat{l}_1\right)\left(\hat{s}\times\hat{l}_1\right) - \omega_{1p}\cos I\left(\hat{s}\times\hat{l}_p\right), \tag{1}$$

$$\omega_{s1} = \frac{3k_q}{2k}\frac{M_*}{m_1}\left(\frac{R_1}{a_1}\right)^3 s, \tag{2}$$

$$\omega_{1p} = \frac{3m_p}{4m_*}\left(\frac{a_1}{a_p\sqrt{1-e_p^2}}\right)^3 \Omega_1. \tag{3}$$

We define s_c to be the critical spin where the precession frequencies ω_{s1} and ω_{1p} are equal, or

$$\omega_{s1}\rvert_{s=s_c} = \omega_{1p}\cos I. \tag{4}$$

rhroughout this paper, we often set the orbital frequency of the inner planet $\Omega_1 = 1$.

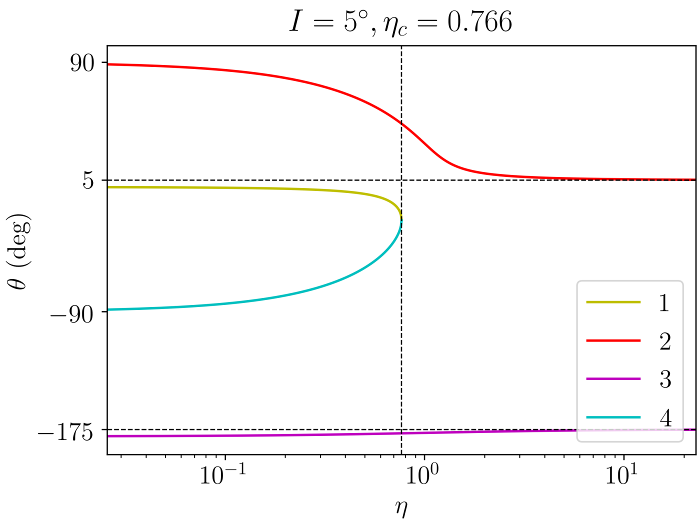

Figure 1. CS locations, TODO change axis labels etc.

2.1 Cassini States

Equilibria of spin dynamics are *Cassini States* (CSs). Refer to Su & Lai 2020 for more detailed discussion. In the notation of the previous paper,

$$\eta \equiv -\frac{g}{\alpha} = s_c/s. \tag{5}$$

Plots of CS locations as always in Figure 1.

Note that ϕ is conjugate to $\cos\theta$, and thus the spin dynamics exhibit Hamiltonian:

$$H(\mu, \phi; s) = -\frac{s}{s_c}\frac{\mu^2}{2} + \mu\cos I - \sin I\sqrt{1-\mu^2}\cos\phi. \tag{6}$$

Plot of level curves of Hamiltonian as before, including $C_\pm$ notation in Figure 2.

2.2 Weak Tidal Friction

We use the weak tidal friction model (Lai 2012). The EOM for θ, s become:

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}\tau} = \frac{s}{s_c}\left(\hat{s}\cdot\hat{l}_1\right)\left(\hat{s}\times\hat{l}_1\right) - \hat{s}\times\hat{l}_p + \frac{\epsilon 2\Omega_1}{s}\left(1 - \frac{s}{2\Omega_1}\left(\hat{l}_1\cdot\hat{s}\right)\right)\hat{s}\times\left(\hat{l}_1\times\hat{s}\right), \tag{7}$$

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \epsilon 2\Omega_1\left(\hat{s}\cdot\hat{l}_1 - \frac{s}{2\Omega_1}\left(1 + \left(\hat{s}\cdot\hat{l}_1\right)^2\right)\right). \tag{8}$$

The phase portrait for these EOM in (s, θ) space is shown in Fig. 3.

2.3 Stable Equilibria of Tidal Friction

Generally, these EOM have equilibria at the points that are both a CS and satisfy $\dot{s} = 0$ under weak tidal friction. These are stable as $\epsilon \to 0$, see Appendix A. On Fig. 3, these are the intersection of the locations of CS1 and CS2 with the line $\dot{s} = 0$. It is clear that the locations of these equilibria

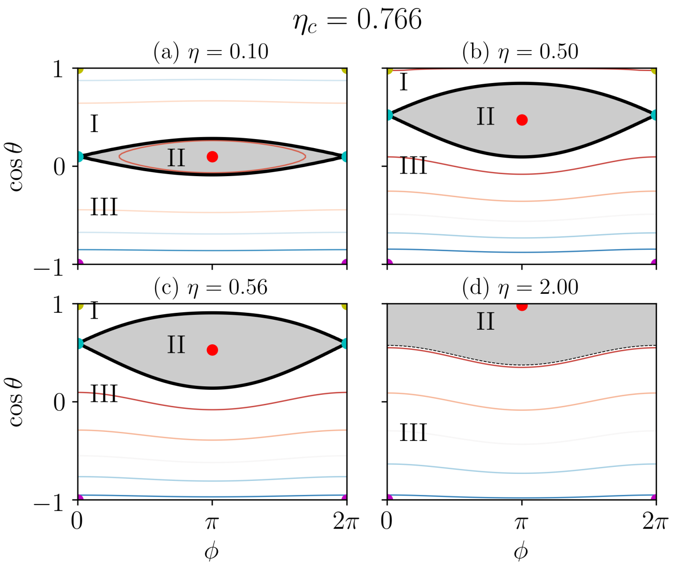

Figure 2. TODO label $C_{\pm}$.

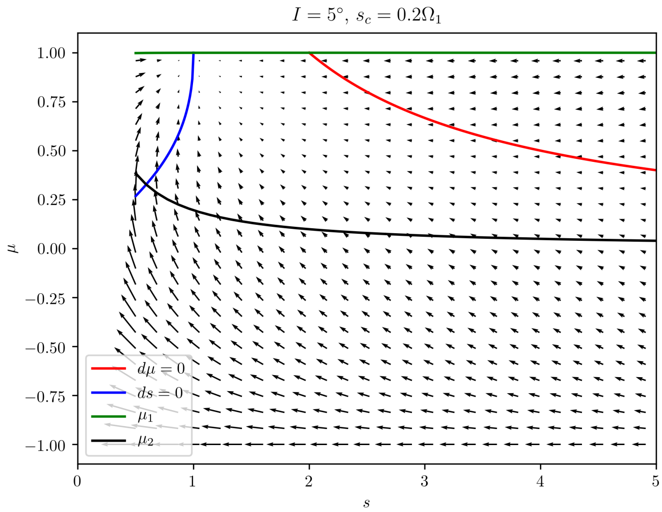

Figure 3. Phase portrait.

depend on the value of s_c. Call these generalized equilibria *tidal Cassini States* (tCS), and number them tCS1 and tCS2 depending on whether they are CS1 or CS2 states. Shown

Furthermore, if tides are too strong (large ϵ), tCS2 can become unstable (cite Fabrycky). The required ϵ for stability of tCS2 is given by

$$\epsilon \leq \frac{\eta_2 \sin I}{1 - \mu_2^2}, \tag{9}$$

where η_2 and μ_2 are evaluated at tCS2.

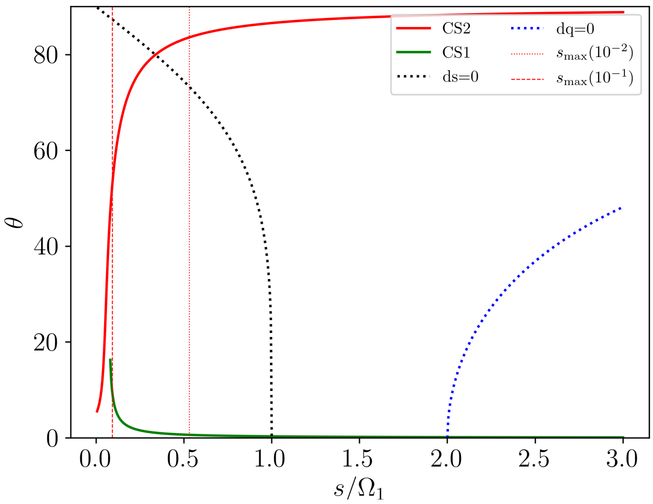

Figure 4. Location of tCS. Vertical line is spin rate below which tCS is destroyed.

3 PROBABILITY DISTRIBUTION OF OUTCOMES

The general question of the dynamics is then as follows: given problem parameters (including s_c) and an initial planet spin and obliquity (s, θ), what are the possible outcomes and their associated probabilities? We study this first at fixed s_c as a function of θ in Section 3.1, then as a function of s_c for an isotropic initial spin $\hat{s}$ in Section 3.2.

3.1 Distribution As a Function of θ

As a result of subsection 2.3, the tCS are the only possible final outcomes. We generate initial spin vectors $\hat{s}$ for isotropic initial distribution $(\cos\theta, \phi)$, and marginalize over ϕ (which generally cannot be fixed by any formation mechanism) in histograms. This gives histograms in Figs. **??**

The governing mechanism for these probabilities is probabilistic separatrix crossing, which is covered in Appendix B2. The calculation is difficult analytically, but can be performed semi-analytically (describe procedure here, calculate $\eta_\star$ numerically etc.). The agreement of this curve with the histograms is examined for the same three s_c values in Figs. **??**.

3.2 Distribution As a Function of s_c

Assuming that $\hat{s}$ is drawn from an isotropic distribution, we may then calculate the probabilities of going to either tCS as a function of s_c. This is shown below for $I = 5°$

Note that while only the results for an isotropic distribution of initial $\hat{s}$ are shown, in prin-

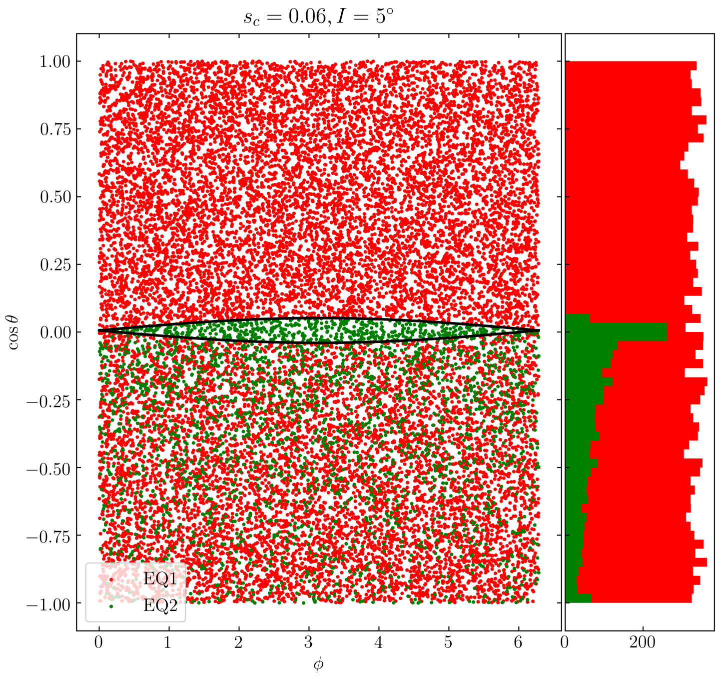

Figure 5. $s_c = 0.06$. Note that it is difficult to reach tCS2.

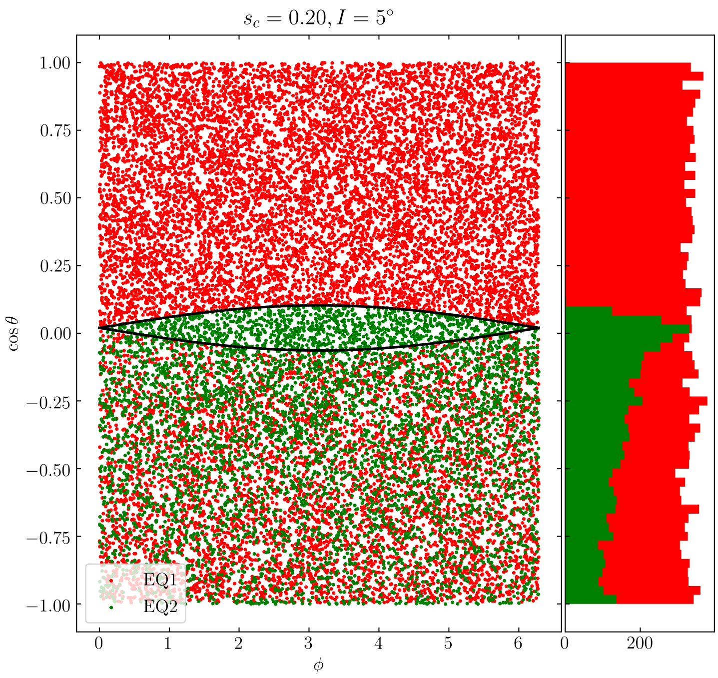

Figure 6. $s_c = 0.2$. Note that tCS2 is both reached with substantial probability and has substantial obliquity.

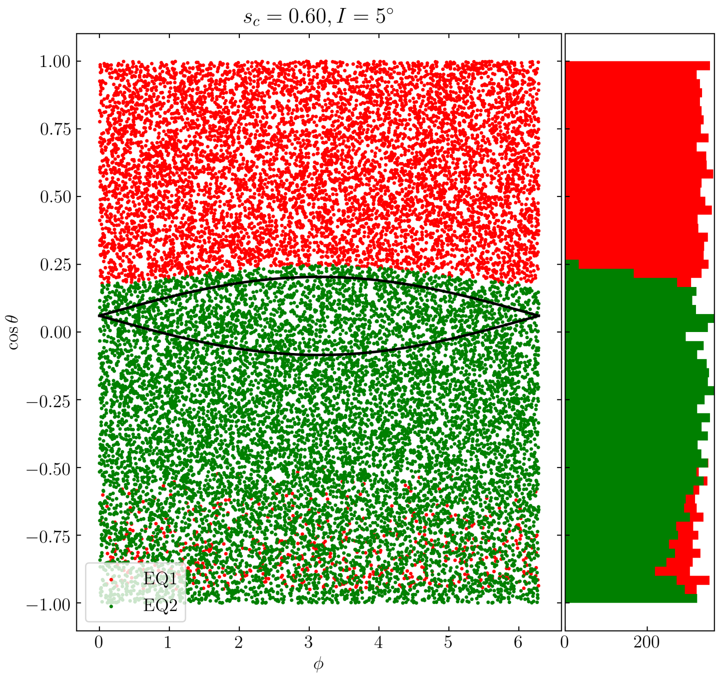

Figure 7. $s_c = 0.6$. Note that tCS2 becomes attracting over tCS1, but will have obliquity $\approx I$ and is uninteresting.

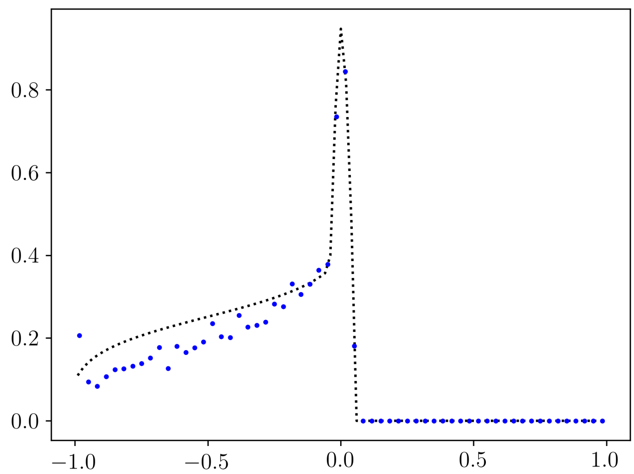

Figure 8. $s_c = 0.06$ prediction vs histogram.

ciple arbitrary distributions $P(\theta_i)$ can be convolved against the $P_{tCS2}(\theta_i)$ distributions shown in subsection 3.1.

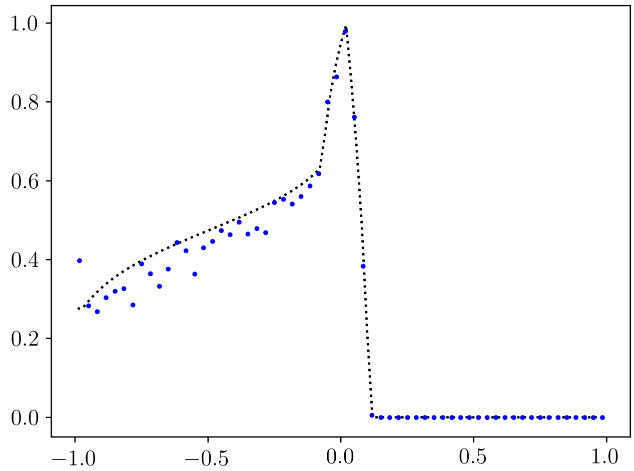

Figure 9. $s_c = 0.20$ prediction vs histogram.

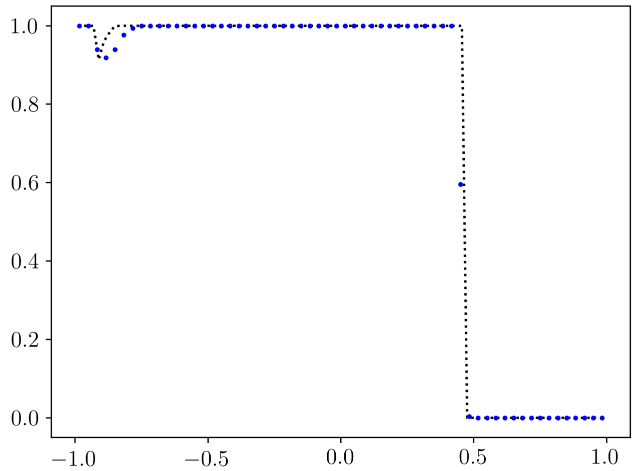

Figure 10. $s_c = 0.70$ prediction vs histogram.

4 SUMMARY AND DISCUSSION

REFERENCES

Henrard J., 1982, Celestial Mechanics and Dynamical Astronomy, 27, 3

Lai D., 2012, Monthly Notices of the Royal Astronomical Society, 423, 486

APPENDIX A: PROOF OF STABILITY OF CASSINI STATES 1 & 2

Insert here.

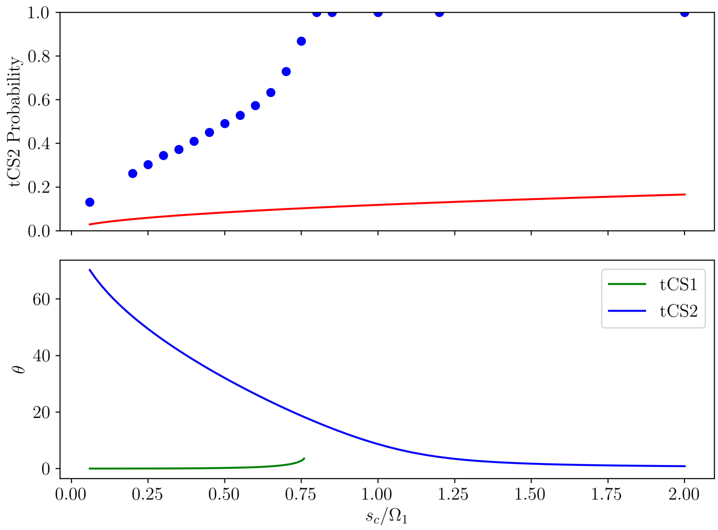

Figure 11. Top: total probability of ending up in tCS2 (blue dots) and the prediction ignoring separatrix capture (red line). Bottom: obliquities of the tCSs.

APPENDIX B: SEPARATRIX CROSSING DYNAMICS

B1 Theory

B1.1 Application of Adiabatic Invariance: Henrard Theory

Review Henrard result, is already very successful e.g. MMR capture.

B1.2 Melnikov Integral

The first substantial new result.

B1.3 Example: Constant s

"Toy problem 1", the nice $P_c \propto \eta^{3/2}$ result. Application of Section B1.2.

B1.4 Combined Result

Thus, the natural extension of the two above results should be

$$\Delta_{\pm} = \oint_{C_{\pm}} \frac{\mathrm{d}H}{\mathrm{d}t}\,\mathrm{d}t, \tag{B1}$$

$$= \oint_{C_{\pm}} \dot{\mu}^{(1)} + \frac{\dot{s}^{(1)}}{\dot{\phi}^{(0)}}\left(\frac{\partial H}{\partial s} - \frac{\partial H_4}{\partial s}\right)\,\mathrm{d}\phi. \tag{B2}$$

B2 Separatrix Crossing Probability: Tidal Friction

Application of the full formula presented in Section B1. The key result is that one integrates

$$\frac{\mathrm{d}(\Delta H)}{\mathrm{d}(\epsilon t)} \approx (1-\mu^2)\left(\frac{2\Omega}{s} - \mu\right)\dot{\phi}^{(0)} + 2\Omega\left(1 + \frac{s}{2\Omega}(1+\mu^2)\right)\left[\frac{\mu^2}{2s_c} - \frac{s_c}{2s^2}\cos^2 I\right]. \tag{B3}$$

An analytical form that holds when $s \gg s_c$ is:

$$\begin{aligned}\frac{\Delta_\pm}{\epsilon} = & -2\cos I\left(\pm 2\pi\eta\cos I + 8\sqrt{\eta\sin I}\right) \pm 2\pi s\cos I + \eta\cos I\left(-8\sqrt{\sin I/\eta}\right) + \frac{s}{2}8\sqrt{\sin I/\eta} \\ & + \frac{2\Omega}{s}\left(\mp 2\pi\left(1-2\eta\sin I\right) + 16\cos I\eta^{3/2}\sqrt{\sin I}\right) + 8\sqrt{\eta\sin I} \pm 2\pi\eta\cos I - \frac{64}{3}\left(\eta\sin I\right)^{3/2}.\end{aligned} \tag{B4}$$

The capture probability is then just

$$P_c = \frac{\Delta_+ + \Delta_-}{\Delta_-}. \tag{B5}$$

This paper has been typeset from a TEX/LATEX file prepared by the author.