

# 1 Hamiltonians and EOM

## 1.1 Toy Problem

Consider simplest spin Hamiltonian  $H = -\vec{B} \cdot \vec{s}$ . It's clear that if we set up initial conditions  $\vec{s}$  misaligned from  $\vec{B}$ , it will simply spin around  $\vec{B}$ , which is fixed. Thus, let  $\hat{B} \cdot \hat{s} = \cos\theta$  the angle between the two, and let  $\phi$  measure the azimuthal angle.

We claim that  $\cos\theta, \phi$  are canonical variables. Since  $\phi$  is ignorable, immediately  $\frac{d\theta}{dt} = \frac{d\cos\theta}{dt} = -\frac{\partial H}{\partial \phi} = 0$ , while  $\frac{d\phi}{dt} = \frac{\partial H}{\partial(\cos\theta)} = Bs$  tells us the rate at which the spin precesses around  $\vec{B}$ .

## 1.2 Cassini State Hamiltonian

This Hamiltonian is Kassandra's Eq. 13, in the co-rotating frame with the perturber's angular momentum:

$$\mathcal{H} = \frac{1}{2}(\hat{s} \cdot \hat{l})^2 - \eta(\hat{s} \cdot \hat{l}_p). \quad (1)$$

In this frame, we can choose  $\hat{l} \equiv \hat{z}$  fixed, and  $\hat{l}_p = \cos I \hat{z} + \sin I \hat{x}$  fixed as well. Then

$$\hat{s} = \cos\theta \hat{z} - \sin\theta(\sin\phi \hat{y} + \cos\phi \hat{x}).$$

We can choose the convention for  $\phi = \phi$  azimuthal angle requiring  $\phi = 0, \pi$  mean coplanarity between  $\hat{s}, \hat{l}, \hat{l}_p$  in the  $\hat{x}, \hat{z}$  plane such that  $\hat{l}_p, \hat{s}$  lie on the same side of  $\hat{l}$ . Then we can evaluate in coordinates

$$\begin{aligned} \hat{s} \cdot \hat{l} &= \cos\theta, \\ \hat{s} \cdot \hat{l}_p &= \cos\theta \cos I - \sin I \sin\theta \cos\phi, \\ \mathcal{H} &= -\frac{1}{2}\cos^2\theta + \eta(\cos\theta \cos I - \sin I \sin\theta \cos\phi). \end{aligned}$$

Note that if we take  $\cos\theta$  to be our canonical variable,  $\sin\theta = \sqrt{1 - \cos^2\theta}$  can be used.

## 1.3 Equation of Motion

The correct EOM comes from Kassandra's Eq. 12:

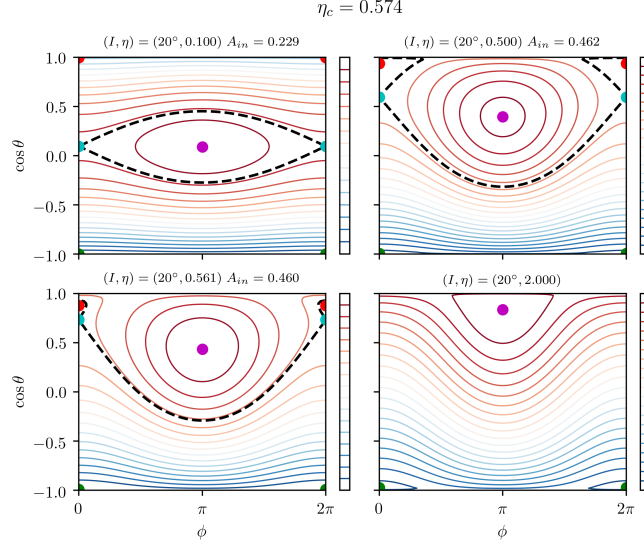
$$\begin{aligned} \frac{d\hat{s}}{dt} &= (\hat{s} \cdot \hat{l})(\hat{s} \times \hat{l}) - \eta(\hat{s} \times \hat{l}_p), \\ &= (s_y s_z - \eta s_y \cos I)\hat{x} - (s_x s_z + \eta(s_x \cos I - s_z \sin I))\hat{y} + \eta s_y \sin I \hat{z}. \end{aligned}$$

Alternatively, consider Hamilton's equations applied to the Hamiltonian:

$$\frac{\partial \phi}{\partial t} = \frac{\partial \mathcal{H}}{\partial(\cos\theta)} = -\cos\theta + \eta(\cos I + \sin I \cot\theta \cos\phi), \quad (2)$$

$$\frac{\partial(\cos\theta)}{\partial t} = -\frac{\partial \mathcal{H}}{\partial \phi} = -\eta \sin I \sin\theta \sin\phi. \quad (3)$$

This produces the same trajectories as the Cartesian EOM, so this is correct. However, since  $\frac{\partial \phi}{\partial t} \propto 1/\sin\theta$ , this is not a desirable system of equations to use, as they are very stiff near  $\theta \approx 0$ .



**Figure 1:** Separatrix for various values of  $\eta$ .

## 1.4 Cassini States

The zeros to Eq. 3 are the Cassini states; we will go to canonical variables  $\mu = \cos \theta$ . We can immediately see that  $\sin \phi = 0$  is necessary, so  $\cos \phi = \pm 1$  and we need only solve for  $\frac{\partial \phi}{\partial t} = 0$ . We can furthermore separate the problem into two regimes,  $\eta \ll 1$  and  $\eta \gg 1$ .

For  $\eta \ll 1$ , it is clear that there will be two solutions near  $\mu^2 = 1$  and two solutions near  $\mu = 0$ :

- For  $\mu = 1 - \frac{\theta^2}{2}$ , the dominant terms are  $\frac{\partial \phi}{\partial t} \approx -1 + \eta \sin I \frac{1}{\theta} = 0$ , where we've taken  $\cos \phi = +1$  and  $\phi = 0$ . This forces  $\theta = \eta \sin I$ .
- Similarly, for  $\mu = -1 + \frac{\epsilon^2}{2}$ ,  $\phi = 0$  and  $\epsilon = \eta \sin I$  again. This actually corresponds to  $\theta = \pi - \eta \sin I$ .
- For  $\mu \approx 0$ , we have instead  $\frac{\partial \phi}{\partial t} = -\mu(1 - \eta \sin I \cos \phi) + \eta \cos I = 0$ . This forces  $\mu_{\pm} = \frac{\eta \cos I}{1 \pm \eta \sin I}$ , where  $\phi_{\pm} = \pi, 0$  respectively.

Note that  $\phi = 0, \mu \approx 0$  is conventionally CS4. The linearization locally has form  $\frac{\partial \delta \phi}{\partial t} = -\delta \mu(1 - \eta \sin I)$  and  $\frac{\partial \delta \mu}{\partial t} = -\eta \sin I \delta \phi$ , so the eigenvalues are  $\approx \mp \sqrt{\eta \sin I}$ , and the two eigenvectors are  $(1, \pm \sqrt{\eta \sin I})$ .

For  $\eta \gg 1$ , the solutions obviously just come from  $\cos I \pm \sin I \cot \theta = 0$ , which are just  $\sin(I \pm \theta) = 0$

## 1.5 Separatrix Area

We can estimate the area enclosed by the separatrix, as shown in Fig. 1. Note that the separatrix joins Cassini State 4 to its  $+2\pi$  image.

We notate  $\mu = \cos \theta$ ; note that CS4 is  $\mu_4 \approx \frac{\eta \cos I}{1 - \eta \sin I} \approx \eta \cos I$ . Setting the Hamiltonian equal to its

value at CS4 gives

$$\begin{aligned}
H_4 &\equiv H(\mu_4, \phi_4) \approx -\frac{\mu_4^2}{2} + \eta\mu_4 \cos I - \eta \sin I, \\
&= +\eta^2 \cos^2 I - \eta \sin I, \\
H(\mu_{sep}, \phi_{sep}) &= H_4 = -\eta \sin I \cos \phi_{sep} - \frac{\mu_{sep}^2}{2} + \eta\mu_{sep} \cos I + \mathcal{O}(\eta^3), \\
0 &\approx \frac{\mu_{sep}^2}{2} - \eta\mu_{sep} \cos I - \eta \sin I (1 - \cos \phi_{sep}) + \eta^2 \cos^2 I, \\
\mu_{sep}(\phi) &\approx \sqrt{2\eta \sin I (1 - \cos \phi)} + \mathcal{O}(\eta).
\end{aligned}$$

We can then easily compute the area enclosed by the separatrix

$$\begin{aligned}
A_{sep} &= \int_0^{2\pi} 2\mu_{sep} d\phi, \\
&\approx 16\sqrt{\eta \sin I}.
\end{aligned} \tag{4}$$

For  $\eta = 0.1, I = 20^\circ$ , this predicts  $\frac{A_{sep}}{A_T} \approx 0.235$ , which is pretty close to my numerically calculated  $\frac{A_{sep}}{A_T} = 0.229$ .

## 1.6 Tidal Dissipation

We can add a tidal dissipation term; we write it in form  $\left(\frac{d\hat{s}}{dt}\right)_{tide} = \epsilon \hat{s} \times (\hat{l} \times \hat{s})$ . Expanding,

$$\begin{aligned}
\left(\frac{d\hat{s}}{dt}\right)_{tide} &= \epsilon(\hat{z} - s_z \hat{s}), \\
&= \epsilon(-s_z s_x \hat{x} - s_z s_y \hat{y} + (1 - s_z^2) \hat{z}).
\end{aligned} \tag{5}$$

We run numerical simulations for weaker  $\epsilon \ll \eta \ll 1$  and stronger  $\epsilon \lesssim \eta \ll 1$ .

We can seek equilibria of the the system including tides, which requires

$$\begin{aligned}
0 &= s_y s_z - \eta s_y \cos I - \epsilon s_z s_x, \\
0 &= -s_x s_z - \eta(s_x \cos I - s_z \sin I) - \epsilon s_z s_y, \\
0 &= \eta s_y \sin I + \epsilon(1 - s_z^2).
\end{aligned}$$

We expect at least two equilibria, based on the simulations: one near  $s_z \approx 1$  and one  $s_z \approx 0$ .

For near alignment/near Cassini state 1,  $1 - s_z \sim 1 - s_z^2$ , so we can set  $s_z = 1$  to first order:  $s_y - \epsilon s_x - \eta s_y \cos I = -s_x - \eta(s_x \cos I - \sin I) - \epsilon s_y = \eta s_y \sin I = 0$ . This can be satisfied if we set  $s_x = \tan(I) \ll 1, s_y = \mathcal{O}(\epsilon s_x)$ ; this coarsely corresponds to Cassini state 1.

The other solution should be near Cassini state 2, where  $s_x \approx 1$ ; dropping second order terms forces  $\eta s_y + \epsilon s_z = -s_z - \eta(\cos I - s_z \sin I) = \eta s_y \sin I + \epsilon = 0$ . This can thus be satisfied for  $s_y \approx -\frac{\epsilon}{\eta \sin I}$ . Thus, this explains why as  $\epsilon$  is increased, we first start to get points that don't converge to Cassini state 2 in the absence of tides, before starting to see points that fail to converge to Cassini state 1.

## 2 Separatrix Hopping

Inspired by G&H, heteroclinic orbits are topologically unstable for any nonzero perturbation, but opened width  $\sim$  perturbation parameter.

## 2.1 Try 1: Qualitative

We zoom in on Cassini State 4, which has  $\theta_4 = -\frac{\pi}{2} + \frac{\eta \cos I}{1 - \eta \sin I}$ ,  $\mu_4 = \frac{\eta \cos I}{1 - \eta \sin I}$ ,  $\phi_4 = 0$ . Then, using equations of motion

$$\frac{\partial \phi}{\partial t} = \mu - \eta \left( \cos I + \sin I \frac{\mu}{\sqrt{1 - \mu^2}} \cos \phi \right), \quad (6)$$

$$\frac{\partial \mu}{\partial t} = -\eta \sin I \sin \phi + [\epsilon(1 - \mu^2)], \quad (7)$$

we can perturbatively require  $\frac{\partial \theta}{\partial t} = 0$  for  $\epsilon \neq 0$ . This corresponds to  $\eta \sin I \sin(\phi_4 + \delta \phi) \approx \epsilon$ , or  $\delta \phi_4 = +\frac{\epsilon}{\eta \sin I}$ . This is in agreement with Dong's result. Note that  $\delta \theta_2 = -\frac{\epsilon}{\eta \sin I}$ , which I saw in my simulations.

This implies that the stable manifolds of the two saddle points, which once overlapped with each other's unstable manifolds (creating a heteroclinic orbit) now are offset from one another by distance  $D \sim \frac{\epsilon}{\eta \sin I}$ . The question is how likely it is to thread the needle.

Consider that, very near CS4, the angle of incidence on the desired gap is roughly  $\tan \psi \approx \psi = \frac{\Delta \theta}{\Delta \phi}$ . Over the course of one orbit,  $\Delta \phi$  changes by  $2\pi$ , while  $\Delta \theta \sim \epsilon \sin \theta T$  where  $T$  is the period of an orbit. Examining the data,  $T \sim 50$ , and so  $\frac{\Delta \theta}{\Delta \phi} \sim \frac{2\pi}{\epsilon(50)}$ .

The effective probability of threading the opened gap between the stable/unstable manifolds is then just  $P \propto D \sin \psi \sim \frac{2\pi}{T(\eta \sin I)}$ . According to later analysis, this should really be  $\frac{2\pi}{T}$ . Plugging in some observational values  $T \sim 50$  for  $\eta = 0.1$  gives  $P \propto 0.13$ . In reality, I find it asymptotes to  $\sim 0.08$ , so the constant of proportionality is of order unity. Not bad given the really crappy  $\psi \sim \frac{\langle \dot{\theta} \rangle}{\dot{\phi}}$  argument.

## 2.2 Try 2: Melnikov Distance

We notice that the separatrix is a heteroclinic orbit, or a saddle connection, in the dissipation free problem. Introducing dissipation breaks the saddle connection by a distance that can be estimated with the Melnikov distance. This is G&H Equation 4.5.11 or something:

$$d(t_0) = \frac{\epsilon M(t_0)}{|f(q^0(0))|} + \mathcal{O}(\epsilon^2), \quad (8)$$

$$M(t_0) = \int_{-\infty}^{\infty} [f \times g]_{hetero} dt. \quad (9)$$

This is not a hard formula to understand; along the separatrix, motion is dominated by  $f$ , but the perpendicular component adds up to contribute to a total “perpendicular distance away from the original separatrix” necessary to hit the saddle point, at least intuitively.

We evaluate the Melnikov integral  $M(t_0)$  on the heteroclinic orbit. Note that since in our problem our perturbation  $g$  is time-independent, so too is the Melnikov integral  $M(t_0) = M$ .

Let's apply this to the Cassini state Hamiltonian w/ dissipation. We first write down our EOM in Melnikov form (we use canonical variables  $\mu, \phi$ ):

$$\frac{d\hat{s}}{dt} = \underbrace{\frac{\partial \mathcal{H}}{\partial \mu} \hat{\phi} - \frac{\partial \mathcal{H}}{\partial \phi} \hat{\mu}}_f + \underbrace{\epsilon(1 - \mu^2) \hat{\mu}}_g. \quad (10)$$

Then  $f \times g = f_\phi g_\mu = \frac{\partial \mathcal{H}}{\partial \mu} (1 - \mu^2)$ . We then want to integrate this along the heteroclinic orbit. We can make change of variables

$$M = \int_0^{2\pi} \frac{\partial \mathcal{H}}{\partial \mu} (1 - \mu^2) \left( \frac{\partial \phi}{\partial t} \right)^{-1} d\phi. \quad (11)$$

But thankfully,  $\frac{\partial \mathcal{H}}{\partial \mu} = \frac{\partial \phi}{\partial t}$  in the absence of dissipation, and so  $M = 2\pi(1 - \mu^2) \approx 2\pi(1 - 2\eta \sin I)$ . Thus, the Melnikov distance at point  $q^0$ , a point on the heteroclinic orbit of the unperturbed Hamiltonian, is just

$$d(q^0) = \frac{2\pi\epsilon(1 - 2\eta \sin I)}{|f(q^0)|}. \quad (12)$$

Note that the maximum value  $|f(q^0)|$ , which occurs at  $\phi = \pi$ , is just  $f \approx \sqrt{4\eta \sin I}$ .

It proves to be a bit difficult to make quantitative predictions though, since the phase diagram is very smushed where  $f$  is large, and  $d$  is rather inaccurate where  $f$  is small. Let's think about a Poincaré map instead.

### 2.3 Try 3: Poincaré Section

Let's consider the Poincaré section every time  $\phi = \phi_4$  as the trajectory subject to tidal dissipation is moving  $\theta < \theta_4 \rightarrow \theta_4$ . To provide an estimate of  $\Delta\theta(\theta) = \theta_{n-1} - \theta_n$ , this is just  $\epsilon T$  where  $T$  is the time elapsed between  $\theta_n, \theta_{n+1}$ , the period of the orbit.  $T$  is dominated by when  $\frac{\partial \phi}{\partial t} \ll 1$  though, or where the orbit is close to the saddle point.

Note that  $T$  is dominated by the time it spends near the saddle point. We showed earlier that near CS4,  $\frac{\partial \phi}{\partial t} \approx \delta\mu$  where  $\delta\mu = \mu - \mu_4$ . Thus, we might surmise  $\Delta\theta(\theta) \propto \theta^{-1}$  for sufficiently small  $\theta - \theta_4$ . Far away,  $T$  is roughly constant and  $\Delta\theta(\theta)$  is roughly constant.

What is "far away"? Well, it probably depends on how affected our trajectory is by the separatrix; far away from the saddle point, we go along contours of roughly constant  $\theta$ , while close by we follow the separatrix pretty well. We computed earlier that  $\mu_{sep} \sim \sqrt{4\eta \sin I}$ , so we might expect  $\mu > \mu_{sep}, \Delta\mu \sim C$ , while  $\mu < \mu_{sep}, \Delta\mu \sim \delta\mu^{-1}$ .

My  $\mu > \mu_{sep}$  simulations don't seem to work very well, so I'll focus on the  $\delta\mu^{-1}$  case. In this case, define  $\delta\mu_c : \Delta\mu(\delta\mu_c) = -\delta\mu_c$ , i.e. the point that jumps immediately to the saddle point. Furthermore, assume the inbound distribution is flat between  $\delta\mu_c, f^{-1}(\delta\mu_c)$ . TODO: empirically,  $\mu_c \sim \epsilon T$  is flat with  $\eta$ , probably just because we're not getting sufficiently close to the saddle point for the  $\propto \sqrt{\eta}$  to kick in.

Then, we can compare the empirical Poincaré section of the points that cross the separatrix versus the total predicted interval width  $\delta\mu_c, f^{-1}(\delta\mu_c)$ ; this would predict 7.2%, 18%. This does alright!

Now, one final way to think about this, and I think this is the important one: consider if we are some  $\delta\mu_0$  at  $\phi = 0$  such that we barely hop the separatrix; this corresponds to  $P_{hop} = \frac{\delta\mu_0}{\mu_c}$ . In order for  $\delta\mu_0$  to be hopping,  $\delta\mu_\phi$  must be within  $d_\phi \approx \frac{2\pi\epsilon}{\mu}$  of the separatrix. We can compute the ratio between  $\delta\mu_0, \delta\mu_\phi$  by setting their Hamiltonian variations equal:

$$\begin{aligned} \delta H_0 &\approx -\mu_4 \delta\mu_0 + \eta \cos I \delta\mu_0 \approx \eta^2 \cos I \sin I \delta\mu_0, \\ \delta H_\phi &\approx -\sqrt{4\eta \sin I} \delta\mu_\phi, \\ \frac{\delta\mu_\phi}{\delta\mu_0} &\propto \eta^{3/2} \cos I \sqrt{\sin I}. \end{aligned}$$

We can assemble