1 Two Planet Mutual Precession

Consider two planets mutually precessing. We calculate this in two ways below, which are equivalent: the vector formulation, and as the eigenvalue of Laplace-Lagrange theory. We compare these to the output of the RINGS code as a third calculation.

1.1 Vector Formulation

We have

$$\begin{aligned}\frac{\mathrm{d}\hat{\mathbf{l}}_1}{\mathrm{d}t} &= \omega_{21}\left(\hat{\mathbf{l}}_1\cdot\hat{\mathbf{l}}_2\right)\left(\hat{\mathbf{l}}_1\times\hat{\mathbf{l}}_2\right)\\ &= \omega_{21}\cos I_{12}\left(\hat{\mathbf{l}}_1\times\hat{\mathbf{l}}_2\right), \end{aligned}\tag{1}$$

$$\frac{\mathrm{d}\hat{\mathbf{l}}_2}{\mathrm{d}t} = \frac{L_1}{L_2}\omega_{21}\cos I_{12}\left(\hat{\mathbf{l}}_2\times\hat{\mathbf{l}}_1\right),\tag{2}$$

$$\omega_{21} = \frac{3m_2}{4M_\star}\left(\frac{a_1}{a_2}\right)^3 n_1 f\left(\frac{a_1}{a_2}\right),\tag{3}$$

$$\begin{aligned}f(\alpha) &= \frac{b_{3/2}^{(1)}}{3\alpha}\\ &= \frac{1}{3\alpha}\frac{1}{\pi}\int_0^{2\pi}\frac{\cos(t)}{\left(\alpha^2+1-2\alpha\cos t\right)^{3/2}}\,\mathrm{d}t\\ &\approx 1+\frac{15}{8}\alpha^2+\mathcal{O}\left(\alpha^4\right).\end{aligned}\tag{4}$$

In all numerical work, we calculate $f(\alpha)$ numerically, via direct integration.

To get the precession rate of $\hat{\mathbf{l}}_1$, we have shown many times that it precesses around $\hat{\mathbf{j}}$ with $\mathbf{J} = J\hat{\mathbf{j}} = \mathbf{L}_1+\mathbf{L}_2$ such that

$$\begin{aligned}\frac{\mathrm{d}\hat{\mathbf{l}}_1}{\mathrm{d}t} &= \omega_{21}\cos I_{21}\left(\hat{\mathbf{l}}_1\times\frac{\mathbf{L}_1+\mathbf{L}_2}{L_2}\right)\\ &= \frac{J}{L_2}\omega_{21}\left(\hat{\mathbf{l}}_1\cdot\hat{\mathbf{l}}_2\right)\left(\hat{\mathbf{l}}_1\times\hat{\mathbf{j}}\right),\end{aligned}\tag{5}$$

where again ω_{21} is given by Eq. (3).

1.2 Laplace-Lagrange Formulation

Defining $\mathscr{I}_i = |I_i| e^{i\Omega_i}$, the Laplace-Lagrange secular theory says that:

$$\frac{\mathrm{d}}{\mathrm{d}t}\begin{bmatrix}\mathscr{I}_1\\ \dots\\ \mathscr{I}_N\end{bmatrix} = \tilde{\mathbf{B}}\begin{bmatrix}\mathscr{I}_1\\ \dots\\ \mathscr{I}_N\end{bmatrix}, \tag{6}$$

$$B_{jk} = -\frac{3m_k}{4M_\star}\left(\frac{a_i}{a_j}\right)^2 \min\left(\frac{a_i}{a_j}, 1\right) n_j f\left(\frac{a_i}{a_j}\right), \tag{7}$$

$$B_{jj} = \sum_{k\neq j} -B_{jk}. \tag{8}$$

In the two-planet case, $\tilde{\mathbf{M}}$ is just a 2×2 matrix, and the only nonzero eigenvalue is the precession frequency (the other has eigenvalue zero and corresponds to the total angular momentum). It is believable that this is in agreement with Eq. (5) if we construct the corresponding eigenvector and calculate its eigenvalue, and this appears to be the case.

1.3 RINGS calculation

For a given set of two-planet parameters, we can use RINGS to calculate their dynamical evolution. The fiducial parameters we choose are:

$$a_1 = 0.035\,\mathrm{AU} \qquad m_1 = M_\oplus \qquad I_1 = 1^\circ \tag{9}$$

$$m_2 = 10M_\oplus \qquad I_2 = 5^\circ. \tag{10}$$

We choose $\Omega_i = \omega_i = 0$ for simplicity. We can then run RINGS and try to extract the precession frequencies. To do this, I took the Fourier Transform of $I_1(t)$ and found the frequency with the largest amplitude.

The results of the comparison among these three methods where a_2 is varied is shown in Fig. 1.

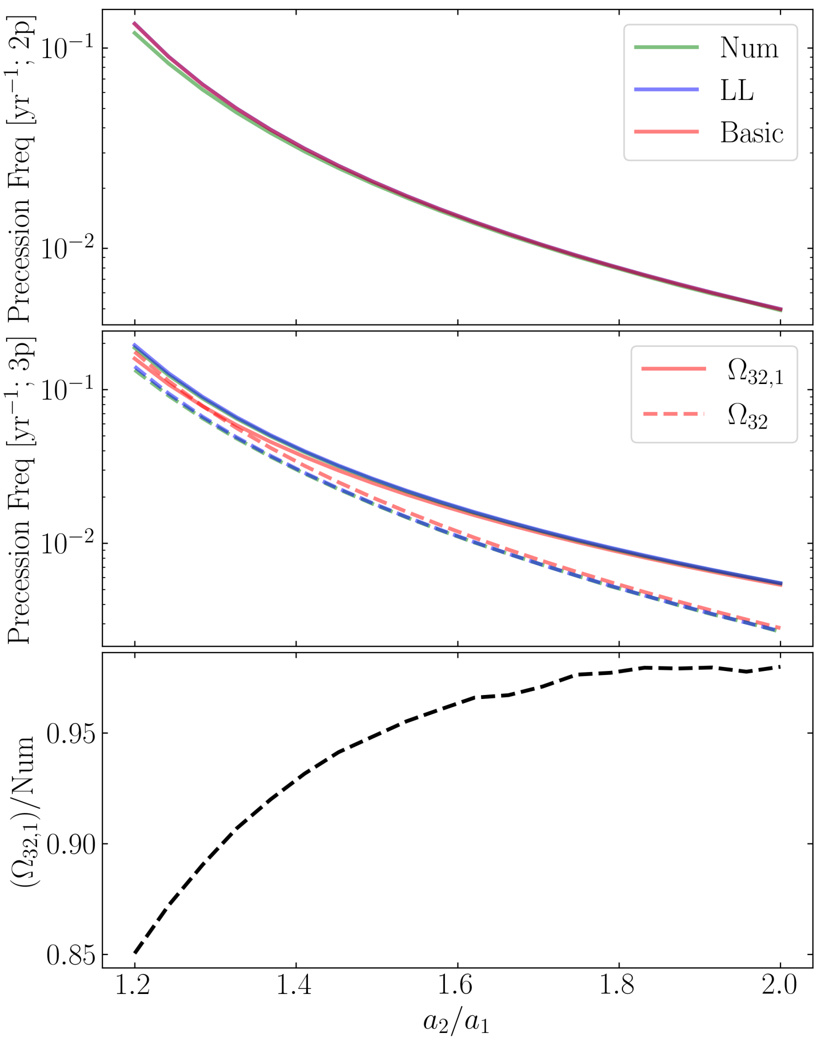

Figure 1: (Top & middle) plot of precession frequencies obtained using three methods as a function of a_2 for two and three-planet systems. Good agreement is observed in two-planet cases, while the three-planet systems have some interesting behavior. The bottom panel shows the quotient of the largest vector-calculated frequency and the numerical value for the 3-planet case; convincing agreement is observed. Frequencies are

2 Three-Planet Case

2.1 Vector Calculation

We now add a third planet. For simplicity, we will neglect the precession induced on $\mathbf{L}_2$ and $\mathbf{L}_3$ by $\mathbf{L}_1$, then:

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\mathbf{l}}_1}{\mathrm{d}t} &= \omega_{21}\cos I_{12}\left(\hat{\mathbf{l}}_1\times\hat{\mathbf{l}}_2\right)+\omega_{31}\cos I_{13}\left(\hat{\mathbf{l}}_1\times\hat{\mathbf{l}}_3\right)\\
\frac{\mathrm{d}\hat{\mathbf{l}}_2}{\mathrm{d}t} &= \omega_{32}\cos I_{23}\left(\hat{\mathbf{l}}_2\times\hat{\mathbf{l}}_3\right), && (11)\\
\frac{\mathrm{d}\hat{\mathbf{l}}_3}{\mathrm{d}t} &= \frac{L_2}{L_3}\omega_{32}\cos I_{23}\left(\hat{\mathbf{l}}_3\times\hat{\mathbf{l}}_2\right), && (12)\\
\omega_{jk} &= \frac{3m_k}{4M_\star}\left(\frac{a_j}{a_k}\right)^3 n_j f\left(\frac{a_j}{a_k}\right). && (13)
\end{aligned}
$$

We expect two eigenvectors, one where $\hat{\mathbf{l}}_1$ is evolving and one where the other two are evolving. The latter has the precession frequency

$$\Omega_{32} = \frac{J_{23}}{L_3}\omega_{32}. \tag{14}$$

The former has the precession frequency:

$$
\begin{aligned}
\frac{\mathrm{d}\hat{\mathbf{l}}_1}{\mathrm{d}t} &= -\left[\omega_{21}\cos I_{12}\hat{\mathbf{l}}_2+\omega_{31}\cos I_{13}\hat{\mathbf{l}}_3\right]\times\hat{\mathbf{l}}_1\\
&\equiv \boldsymbol{\Omega}_{32,1}\times\hat{\mathbf{l}}_1, && (15)\\
\Omega_{32,1} &\approx \omega_{21}\cos I_{12}+\omega_{31}\cos I_{13}+\mathcal{O}\left(\cos I_{23}\right). && (16)
\end{aligned}
$$

Here, we have simply assmued that $\hat{\mathbf{l}}_2$ is approximately aligned with $\hat{\mathbf{l}}_3$; by the law of cosines, the deviation scales with $\cos I_{23} \sim 2^\circ$ for compact architectures excepting the innermost planet.

We want to know which of Ω_{32} and $\Omega_{32,1}$ are larger. To calculate scalings, let's approximate $I_{12} \approx I_{13} \approx I_{23} \approx 0$. If we assume the semimajor axis ratios are constant, $a_3/a_2 = a_2/a_1 = \alpha$, then

$$
\begin{aligned}
\frac{\Omega_{32,1}}{\Omega_{32}} &= \frac{m_2\alpha^3 n_1 f(\alpha)+m_3\alpha^6 n_1 f(\alpha^2)}{m_3\alpha^3 n_2 f(\alpha)}\frac{L_3}{J_{23}}\\
&= \frac{L_3}{J_{23}}\left(\frac{m_2}{m_3}\alpha^{-3/2}+\alpha^{3/2}\frac{f\left(\alpha^2\right)}{f(\alpha)}\right). && (17)
\end{aligned}
$$

Note that $J_{23}/L_3 \approx (m_2/m_3)\alpha^{1/2}+1$. There are thus two ways to obtain $\Omega_{32,1} < \Omega_{32}$:

- If we have $\alpha \approx m_2/m_3 \approx f\left(\alpha^2\right)/f(\alpha) \approx 1$, then $J_{23}/L_3 \approx 2$ and $\Omega_{32,1}/\Omega_{32} \approx 0.5$, so not a very large ratio.

- A much larger ratio can be obtained if $m_2/m_3 \ll \alpha^{3/2} \ll 1$.

Thus, in general, we find that the $\Omega_{32,1}$ precession frequency should generally be larger or comparable except for very small a_{j+1}/a_j.

2.2 Comparison with Other Results

The LL results follow straightforwardly from Section 1.2. The RINGS simulations can yield two precession frequencies if we seek the first two non-commensurate, non-adjacent frequencies in the FT of $I_1(t)$; note that choosing $\Omega_i = \omega_i = 0$ means that $\hat{\mathbf{l}}_2 = \hat{\mathbf{l}}_3$ initially, but due to different backreaction torques from $\hat{\mathbf{l}}_1$ they eventually misalign and give rise to the Ω_{32} mode. This is shown in the middle panel of Fig. 1. Finally, in the bottom panel, we show the quotient of $\Omega_{32,1}$ and the numerically-determined maximum precession frequency. We see fractional deviations, showing that $\Omega_{32,1}$ is a good estimate of $g_{\rm max}$.

2.3 Inner Mass Dependence

In the above, we have considered the case where $m_1 = M_\odot$ and $m_2 = m_3 = 10M_\odot$, satisfying our "no backreaction" assumption. However, we can also consider the case closer to the Millholland paper, where $m_i = 5M_\odot$. These results are shown in Fig. 2. For some reason, the agreement *improves*?

3 Resulting Impact on USP Formation Story

In the 3p systems, we see that $|g_{\rm max}| \lesssim 0.2\,{\rm yr}^{-1}$. At the same time, we can calculate the spin-orbit precession frequency:

$$\alpha = \frac{3k_q}{2k}\frac{M_\star}{m}\left(\frac{R}{a}\right)^3\Omega_{\rm s}, \tag{18}$$

$$\approx 0.86\,{\rm yr}^{-1}\frac{k_q}{k}\left(\frac{M_\star}{M_\odot}\right)^{3/2}\left(\frac{m}{M_\oplus}\right)^{-1}\left(\frac{R}{R_\oplus}\right)^3\left(\frac{a}{0.035\,{\rm AU}}\right)^{-9/2}\left(\frac{\Omega_{\rm s}}{n}\right). \tag{19}$$

This shows that $\eta_{\rm sync} \equiv |g_{\rm max}|/\alpha \lesssim 0.15$ very optimistically.

Using Millholland's formula, we have yet another small correction. They use $3k_q = 0.4$ and $k = 0.35$, so their α is depressed by a factor of 8/21, leaving $\eta_{\rm sync} \sim 0.4$ for extremal values (note that $a_2/a_1 = 1.2$ corresponds to a period ratio of 1.3). So indeed, we should expect that $\eta_{\rm sync}$ is only $\gtrsim 1$ if $a_2/a_1 \lesssim 1.2$.

3.1 Double Checking $\eta_{\rm sync}$ for 2p

Recall that we had:

$$\eta_{\rm sync} = \frac{m_p m}{2M_\star^2}\left(\frac{a}{a_{\rm p}}\right)^3\left(\frac{a}{R}\right)^3, \tag{20}$$

$$= 4.67\times10^{-4}\cos I\frac{k}{k_{\rm q}}f\left(\frac{1}{2}\right)\frac{m_p m}{8(M_\odot^2)}\left(\frac{M_\star}{M_\odot}\right)^{-2}\left(\frac{a/a_{\rm p}}{1/2}\right)^3\left(\frac{a}{0.04\,{\rm AU}}\right)^3\left(\frac{R}{2R_\oplus}\right)^{-3}, \tag{21}$$

$$= 0.0145\cos I\frac{k}{k_{\rm q}}f\left(\frac{1}{1.2}\right)\left(\frac{m_p}{10M_\odot}\right)\left(\frac{\rho}{\rho_\oplus}\right)\left(\frac{M_\star}{M_\odot}\right)^{-2}\left(\frac{a/a_{\rm p}}{1/1.2}\right)^3\left(\frac{a}{0.035\,{\rm AU}}\right)^3. \tag{22}$$

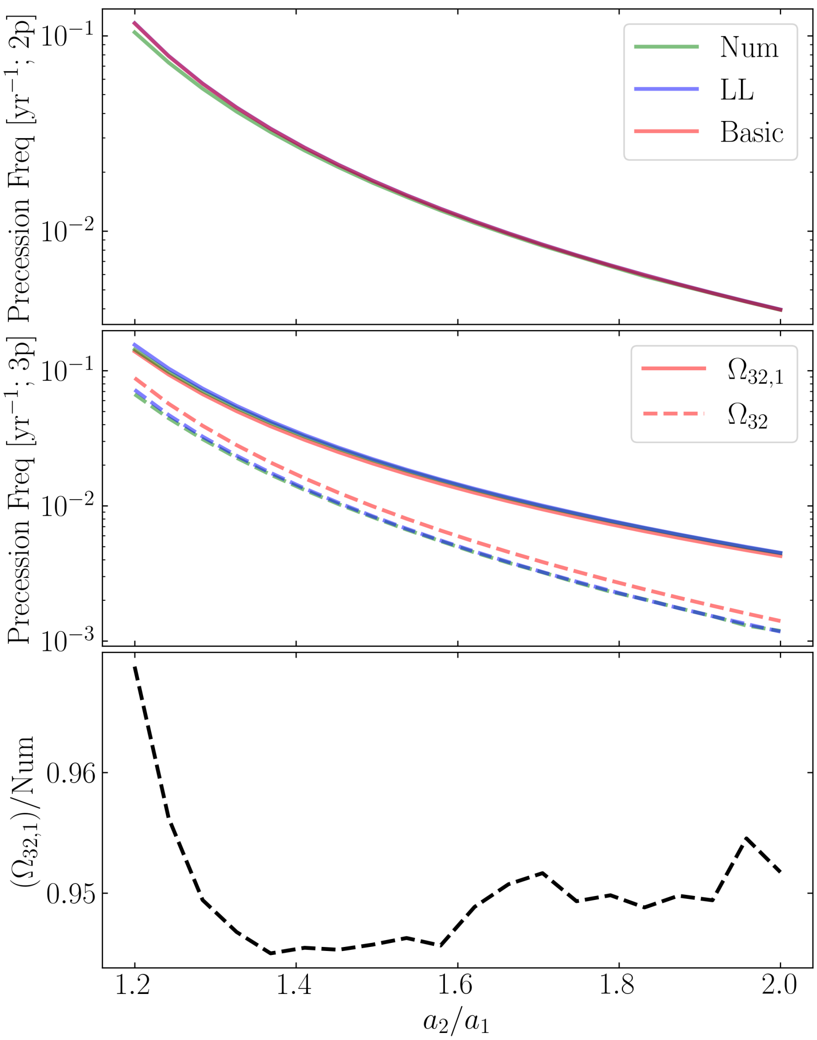

Figure 2: Same as Fig. 1 but for equal masses $m_i = 5M_\odot$.

Note that $f(0.5) = 1.72$ while $f(1/1.2) = 9.69$ (NB: the power series for α converges way too slowly for the latter result, have to evaluate numerically), so this is already a bit more promising.

3.2 Towards $\eta_{\rm sync}$ for 3p? No

We showed above that

$$\begin{aligned}|g_{\max}| &\approx \Omega_{32,1}\\ &\approx \omega_{21}\cos I_{12}\left(1+\frac{m_3}{m_2}\alpha^3\frac{f\left(\alpha^2\right)}{f(\alpha)}\frac{\cos I_{13}}{\cos I_{12}}\right).\end{aligned} \tag{23}$$

Trying to express this in any sort of scaling way is difficult since $f(\alpha)$ cannot be expressed analytically for sufficiently large α. Nevertheless, for the fiducial $\alpha = 1/1.2$ and $m_3 = m_2$, we find that this enhancement is only 1.2×, and if we account for the fact that our naive prediction is small by about 15%, the third planet only contributes 40% of the precession rate of the inner. This is unsurprising, and is not enough to lift $\eta_{\rm sync}$ to values of order unity; it raises it only to $\sim 0.2k/k_{\rm q}$, which is in line with our calculation above. So everything seems consistent!