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Dynamics of Colombo's Top: Tidal Dissipation and Resonance Capture

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ABSTRACT

Abstract here

Keywords: planet—star interactions

1. INTRODUCTION

Separatrix crossing was studied by (Henrard 1982).

2. THEORY AND EQUATIONS

In this section, we first briefly lay out the spin dynamics of the planet, introducing the Cassini State spin-orbit resonance (for more details, see Su & Lai 2020). We then introduce the equilibrium tide model of weak tidal friction used in this work, though much of our analysis does not depend on the detailed tidal model used.

2.1. *Spin Dynamics Without Tidal Friction*

We study an oblate planet orbiting around a star hosting a further out planet. The equations of motion in the absence of dissipation are:

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} = \omega_{s1}\left(\hat{s}\cdot\hat{l}_1\right)\left(\hat{s}\times\hat{l}_1\right) - \omega_{1p}\cos I\left(\hat{s}\times\hat{l}_p\right), \tag{1}$$

$$\omega_{s1} = \frac{3k_q}{2k}\frac{M_*}{m_1}\left(\frac{R_1}{a_1}\right)^3 s, \tag{2}$$

$$\omega_{1p} = \frac{3m_p}{4m_*}\left(\frac{a_1}{a_p\sqrt{1-e_p^2}}\right)^3\Omega_1. \tag{3}$$

We define s_c to be the critical spin where the precession frequencies ω_{s1} and ω_{1p} are equal, or

$$\omega_{s1}|_{s=s_c} = \omega_{1p}\cos I. \tag{4}$$

rhroughout this paper, we often set the orbital frequency of the inner planet $\Omega_1 = 1$.

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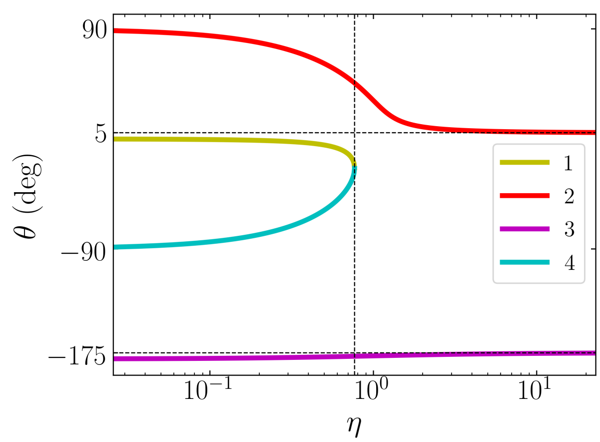

Figure 1. CS locations, TODO change axis labels etc.

2.2. *Cassini States*

Equilibria of spin dynamics are *Cassini States* (CSs). Refer to Su & Lai 2020 for more detailed discussion. In the notation of the previous paper,

$$\eta \equiv -\frac{g}{\alpha} = s_c/s. \tag{5}$$

Plots of CS locations as always in Figure 1.

Note that ϕ is conjugate to $\cos\theta$, and thus the spin dynamics exhibit Hamiltonian:

$$H(\mu,\phi;s) = -\frac{s}{s_c}\frac{\mu^2}{2} + \mu\cos I - \sin I\sqrt{1-\mu^2}\cos\phi. \tag{6}$$

Plot of level curves of Hamiltonian as before, including $C_\pm$ notation in Figure 2.

2.3. *Weak Tidal Friction*

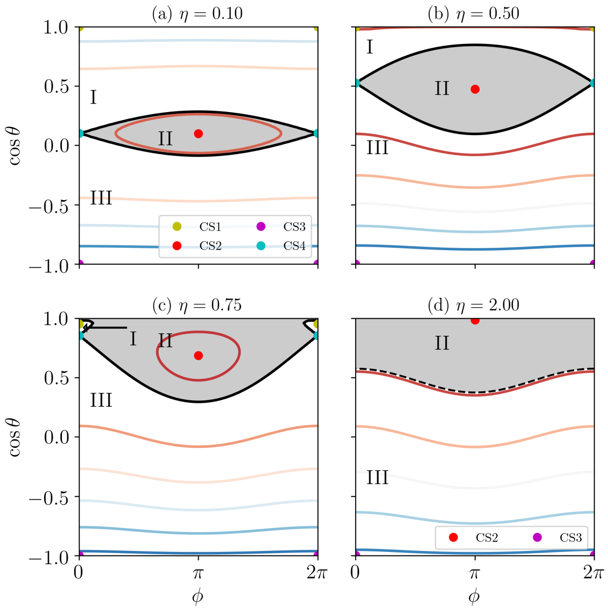

Figure 2. TODO label $C_{\pm}$.

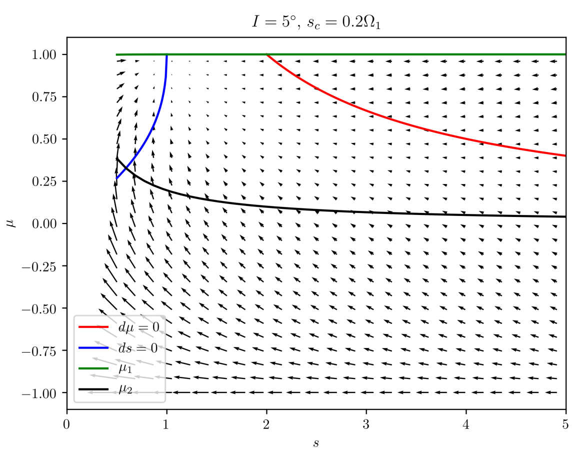

Figure 3. Phase portrait.

We use the weak tidal friction model (Lai 2012). The EOM for θ, s become:

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}\tau} = \frac{s}{s_c}\left(\hat{s}\cdot\hat{l}_1\right)\left(\hat{s}\times\hat{l}_1\right) - \hat{s}\times\hat{l}_p + \frac{\epsilon 2\Omega_1}{s}\left(1 - \frac{s}{2\Omega_1}\left(\hat{l}_1\cdot\hat{s}\right)\right)\hat{s}\times\left(\hat{l}_1\times\hat{s}\right), \tag{7}$$

$$\frac{\mathrm{d}s}{\mathrm{d}\tau} = \epsilon 2\Omega_1\left(\hat{s}\cdot\hat{l}_1 - \frac{s}{2\Omega_1}\left(1 + \left(\hat{s}\cdot\hat{l}_1\right)^2\right)\right). \tag{8}$$

The phase portrait for these EOM in (s, θ) space is shown in Fig. 3. Note that the semimajor axis a is not constant, but $\dot{a}/\dot{\theta} \sim S/L \ll 1$, and is negligible for our problem (but not always, Millholland tidal runaway cite).

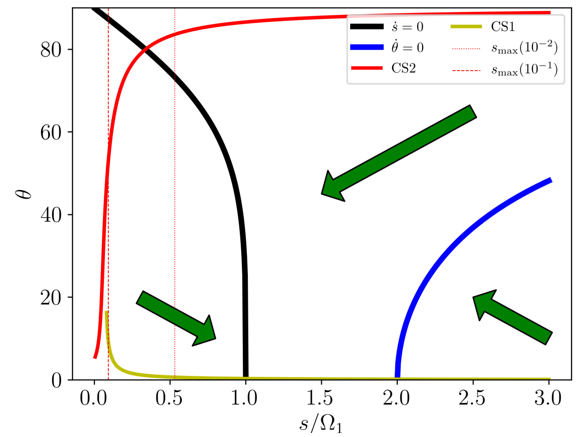

Figure 4. Location of tCE. Vertical line is spin rate below which tCE is destroyed.

2.4. *Stable Equilibria of Tidal Friction*

Generally, these EOM have equilibria at the points that are both a CS and satisfy $\dot{s} = 0$ under weak tidal friction. These are stable when ignoring tidal friction (Su & Lai 2020). On Fig. 3, these are the intersection of the locations of CS1 and CS2 with the line $\dot{s} = 0$. It is clear that the locations of these equilibria depend on the value of s_c. Call these generalized equilibria *tidal Cassini Equilibria* (tCE), and number them tCE1 and tCE2 depending on whether they are CS1 or CS2 states. Shown

Furthermore, if tides are too strong (large ϵ), tCE2 can become unstable (cite Fabrycky). The required ϵ for stability of tCE2 is given by

$$\epsilon \leq \frac{\eta_2 \sin I}{1 - \mu_2^2}, \tag{9}$$

where η_2 and μ_2 are evaluated at tCE2.

3. PROBABILITY DISTRIBUTION OF OUTCOMES

The general question of the dynamics is then as follows: given problem parameters (including s_c) and an initial planet spin and obliquity (s, θ), what are the possible outcomes and their associated probabilities? We study this first at fixed s_c as a function of θ in Section 3.1, then as a function of s_c for an isotropic initial spin $\hat{s}$ in Section 3.2.

3.1. *Distribution As a Function of θ*

As a result of subsection 2.4, the tCE are the only possible final outcomes. We generate initial spin vectors $\hat{s}$ for isotropic initial distribution $(\cos\theta, \phi)$, and average over ϕ in histograms. This gives histograms in Figs. **??**

Similarly to Su & Lai (2020), these probabilities are the result of separatrix crossings. However, the calculation differs from Su & Lai (2020): since the perturbation is dissipative in

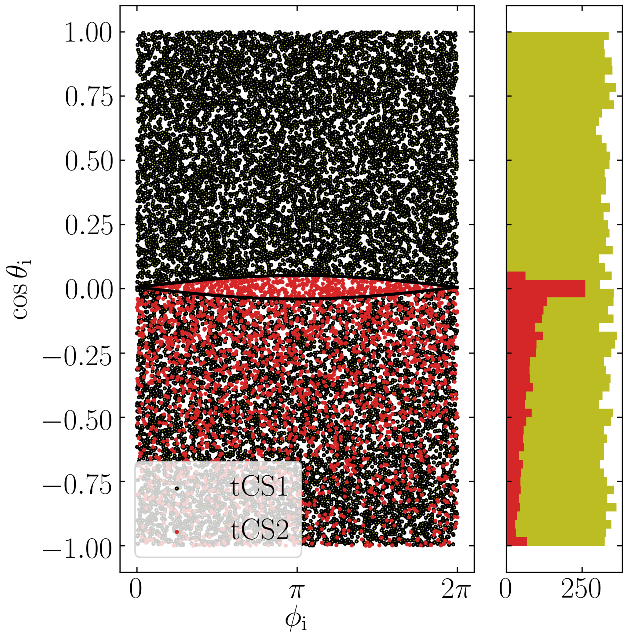

Figure 5. $s_c = 0.06$. Note that it is difficult to reach tCE2.

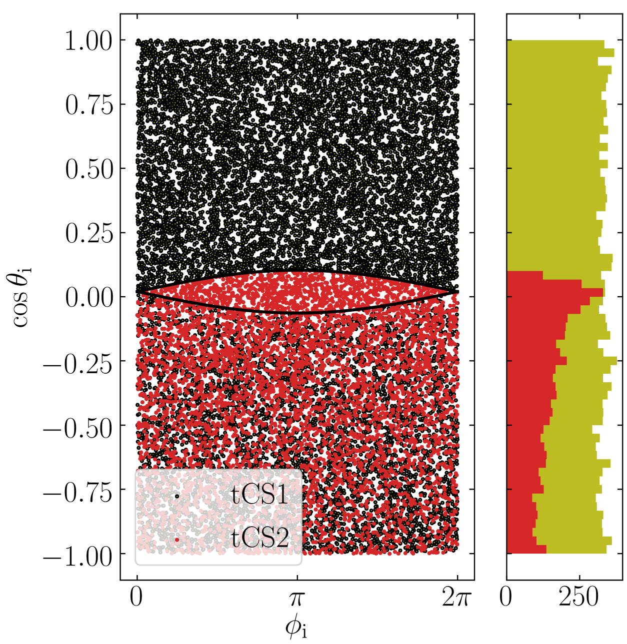

Figure 6. $s_c = 0.2$. Note that tCE2 is both reached with substantial probability and has substantial obliquity.

phase space (i.e. it does not conserve phase space area), analysis following Henrard (1982) is not sufficient. A more general theory of separatrix crossing indeed lets us semi-analytically calculate the outcome probabilities. Appendix A.2 gives a more thorough overview of separatrix crossing dynamics, but the key result is:

$$a^2 + b^2 = c^2. \tag{10}$$

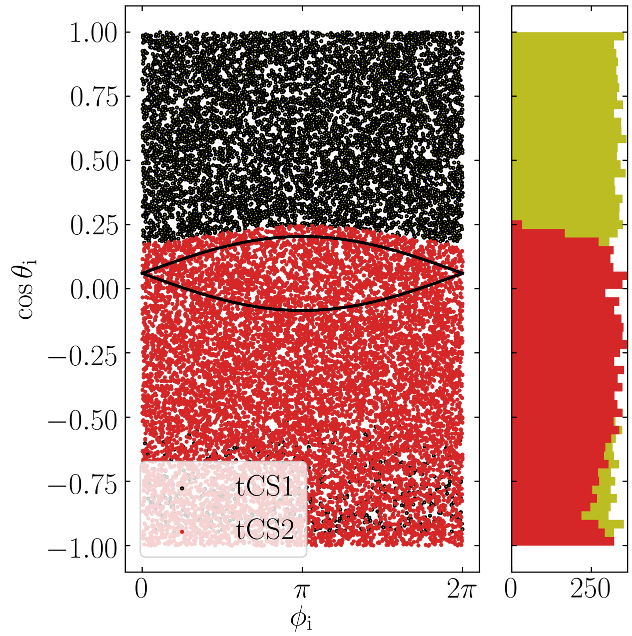

Figure 7. $s_c = 0.6$. Note that tCE2 becomes attracting over tCE1, but will have obliquity $\approx I$ and is uninteresting.

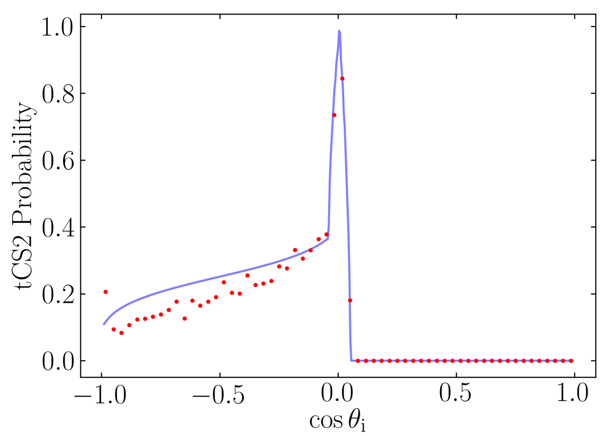

Figure 8. $s_c = 0.06$ prediction vs histogram.

We can apply this curve to calculate the likelihood of the tCEs when a particular trajectory encounters the separatrix, and from this obtain a semi-analytical prediction of the distribution of outcomes. The semi-analytical nature of this calculation arises because $\eta_\star$ cannot be calculated *a priori*. The agreement of this curve with the histograms is examined for the same three s_c values in Figs. **??**.

3.2. *Distribution As a Function of s_c*

Assuming that $\hat{s}$ is drawn from an isotropic distribution, we may then calculate the probabilities of going to either tCE as a function of s_c. This is shown below for $I = 5°$

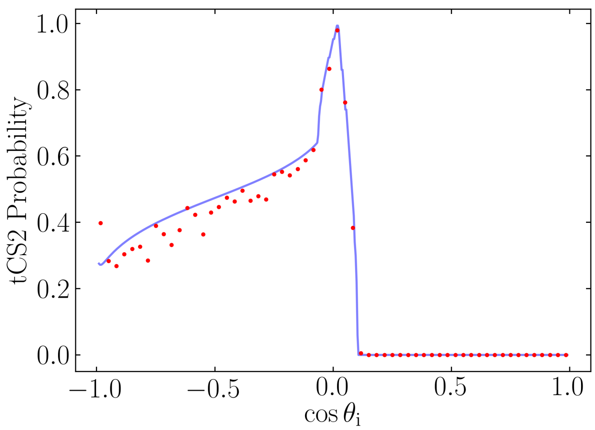

Figure 9. $s_c = 0.20$ prediction vs histogram.

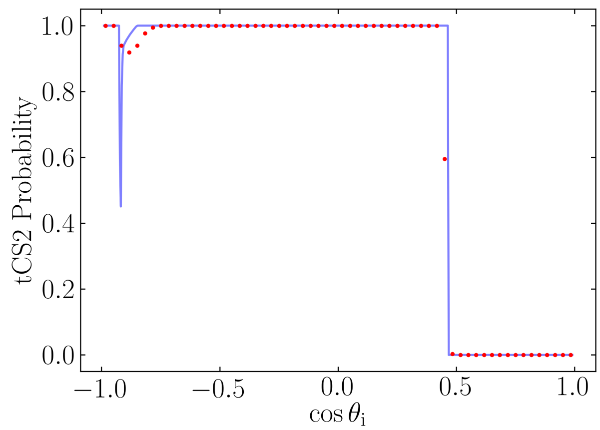

Figure 10. $s_c = 0.70$ prediction vs histogram.

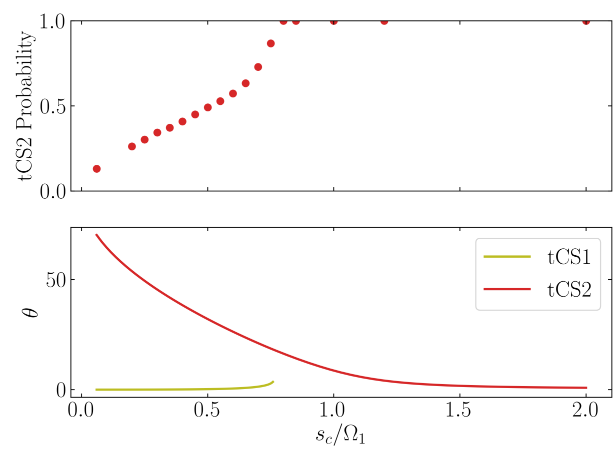

Figure 11. Top: total probability of ending up in tCE2 (blue dots) and the prediction ignoring separatrix capture (red line). Bottom: obliquities of the tCEs.

Note that while only the results for an isotropic distribution of initial $\hat{s}$ are shown, in principle arbitrary distributions $P(\theta_i)$ can be convolved against the $P_{tCE2}(\theta_i)$ distributions shown in subsection 3.1.

4. SUMMARY AND DISCUSSION

REFERENCES

Henrard, J. 1982, Celestial Mechanics and Dynamical Astronomy, 27, 3

Lai, D. 2012, Monthly Notices of the Royal Astronomical Society, 423, 486

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APPENDIX

A. SEPARATRIX CROSSING DYNAMICS

A.1. *Theory*

A.1.1. *Application of Adiabatic Invariance: Henrard Theory*

Review Henrard result, is already very successful e.g. MMR capture.

A.1.2. *Melnikov Integral*

The first substantial new result.

A.1.3. *Example: Constant s*

"Toy problem 1", the nice $P_c \propto \eta^{3/2}$ result. Application of Section A.1.2.

A.1.4. *Combined Result*

Thus, the natural extension of the two above results should be

$$\Delta_\pm = \oint_{C_\pm} \frac{\mathrm{d}H}{\mathrm{d}t}\;\mathrm{d}t, \tag{A1}$$

$$= \oint_{C_\pm} \dot{\mu}^{(1)} + \frac{\dot{s}^{(1)}}{\dot{\phi}^{(0)}}\left(\frac{\partial H}{\partial s} - \frac{\partial H_4}{\partial s}\right)\;\mathrm{d}\phi. \tag{A2}$$

A.2. *Separatrix Crossing Probability: Tidal Friction*

Application of the full formula presented in Section A.1. The key result is that one integrates

$$\frac{\mathrm{d}(\Delta H)}{\mathrm{d}(\epsilon t)} \approx (1-\mu^2)\left(\frac{2\Omega}{s} - \mu\right)\dot{\phi}^{(0)} + 2\Omega\left(1 + \frac{s}{2\Omega}(1+\mu^2)\right)\left[\frac{\mu^2}{2s_c} - \frac{s_c}{2s^2}\cos^2 I\right]. \tag{A3}$$

An analytical form that holds when $s \gg s_c$ is:

$$\begin{aligned}\frac{\Delta_\pm}{\epsilon} = &-2\cos I\left(\pm 2\pi\eta\cos I + 8\sqrt{\eta\sin I}\right) \pm 2\pi s\cos I + \eta\cos I\left(-8\sqrt{\sin I/\eta}\right) + \frac{s}{2}8\sqrt{\sin I/\eta}\\ &+ \frac{2\Omega}{s}\left(\mp 2\pi\left(1 - 2\eta\sin I\right) + 16\cos I\eta^{3/2}\sqrt{\sin I}\right) + 8\sqrt{\eta\sin I} \pm 2\pi\eta\cos I - \frac{64}{3}\left(\eta\sin I\right)^{3/2}.\end{aligned} \tag{A4}$$

The capture probability is then just

$$P_c = \frac{\Delta_+ + \Delta_-}{\Delta_-}. \tag{A5}$$