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Dynamics of Colombo's Top: Tidal Dissipation and Resonance Capture

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ABSTRACT
Abstract here

Key words: planet-star interactions

1 INTRODUCTION

- Studying planetary obliquities (define) is important. Cassini States are key. More introduction.
- Resonance capture via separatrix crossing was first considered by (Henrard 1982) for non-dissipative perturbations (e.g. Su & Lai 2020). However, tidal friction is dissipative, so this formalism does not apply. We generalize this calculation and show that it reproduces results.

In Section XXX. . .

2 SPIN EVOLUTION EQUATIONS AND CASSINI STATES: REVIEW

In this section, we first briefly lay out the spin dynamics of the planet, introducing the Cassini State spin-orbit resonance (for more details, see Su & Lai 2020). We then introduce the weak friction theory of equilibrium tides used in this work (Lai 2012). While many different tidal effects may dominate in different planetary systems, our qualitative conclusions do not depend on the specific form of the tidal dissipation, so we use the classic weak friction theory for simplicity.

2.1 Spin Dynamics in the Absence of Tides

2.1.1 Equations of Motion

We consider a star of mass $M_\star$ hosting an inner oblate planet of mass m and radius R on a circular orbit with semi-major axis a and an outer perturber of mass $m_{\rm p}$ on a circular orbit with semi-major axis $a_{\rm p}$. We assume that the two orbits are mildly misaligned with mutual inclination I. Denote $\mathbf{S}$ the spin angular momentum and $\mathbf{L}$ the orbital angular momentum of the planet, and $\mathbf{L}_{\rm p}$ the angular momentum of the perturber. The corresponding unit vectors are $\hat{\mathbf{s}} \equiv \mathbf{S}/S$, $\hat{\mathbf{l}} \equiv \mathbf{L}/L$, and $\hat{\mathbf{l}}_{\rm p} \equiv \mathbf{L}_{\rm p}/L_{\rm p}$. The spin axis $\hat{\mathbf{s}}$ of the planet tends to precess around its orbital (angular momentum) axis $\hat{\mathbf{l}}$, driven by the gravitational torque from the host star acting on the planet's rotational bulge. On the other hand, $\hat{\mathbf{l}}$ and the disk axis $\hat{\mathbf{l}}_{\rm p}$ precess around each other due to gravitational interactions. We assume $S \ll L \ll L_{\rm p}$, so $\hat{\mathbf{l}}_{\rm p}$ and $\hat{\mathbf{l}}$ are nearly constant. The equations of motion for $\hat{\mathbf{s}}$ and $\hat{\mathbf{l}}$ in this limit are (Anderson & Lai 2018; Su & Lai 2020)

$$\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t} = \omega_{\rm sl}\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}\right) \equiv \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}\right), \tag{1}$$

$$\frac{\mathrm{d}\hat{\mathbf{l}}}{\mathrm{d}t} = \omega_{\rm lp}\left(\hat{\mathbf{l}}\cdot\hat{\mathbf{l}}_{\rm p}\right)\left(\hat{\mathbf{l}}\times\hat{\mathbf{l}}_{\rm p}\right) \equiv -g\left(\hat{\mathbf{l}}\times\hat{\mathbf{l}}_{\rm p}\right), \tag{2}$$

where

$$\omega_{\rm sl} \equiv \frac{3GJ_2 mR^2 M_\star}{2a^3 I\Omega_{\rm s}} = \frac{3k_q}{2k}\frac{M_\star}{m}\left(\frac{R}{a}\right)^3\Omega_{\rm s}, \tag{3}$$

$$\omega_{\rm lp} = \frac{3m_p}{4M_\star}\left(\frac{a}{a_{\rm p}}\right)^3 n. \tag{4}$$

In Eq. (3), $\Omega_{\rm s}$ is the spin frequency of the inner planet, $I = kmR^2$ (with k a constant) is its moment of inertia and $J_2 = k_{\rm q}\Omega_{\rm s}^2(R^3/Gm)$ (with k_q a constant) is its rotation-induced (dimensionless) quadrupole moment [for a body with uniform density, $k = 0.4, k_{\rm q} = 0.5$; for rocky planets, $k \simeq 0.2$ and $k_{\rm q} \simeq 0.17$ (e.g. Lainey 2016) ? not sure]. In other studies, $3k_{\rm q}/2k$ is often notated as $k_2/2C$ (e.g. Millholland & Batygin 2019). In Eq. (4), $n \equiv \sqrt{GM_\star/a^3}$ is the inner planet's orbital mean motion, and we have assumed $a_{\rm p} \gg a$ and included only the leading-order (quadrupole) interaction between the inner planet and perturber. Following standard notation, we define $\alpha = \omega_{\rm sl}$ and $g \equiv -\omega_{\rm lp}\cos I$ (e.g. Colombo 1966).

As in Su & Lai (2020), we combine Eqs. (1–2) into a single equation by transforming into a frame rotating about $\hat{\mathbf{l}}_{\rm p}$ with frequency g. In this frame, $\hat{\mathbf{l}}_{\rm p}$ and $\hat{\mathbf{l}}$ are both fixed, and $\hat{\mathbf{s}}$ evolves as:

$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\rm rot} = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}\right) + g\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}_{\rm p}\right). \tag{5}$$

In this reference frame, we choose the coordinate system such that $\hat{\mathbf{z}} = \hat{\mathbf{l}}$ and $\hat{\mathbf{l}}_{\rm p}$ lies in the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane. We describe $\hat{\mathbf{s}}$ in spherical coordinates using the polar angle θ, the planet's obliquity, and ϕ, the precessional phase of $\hat{\mathbf{s}}$ about $\hat{\mathbf{l}}$.

The equilibria of Eq. (5) are referred to as *Cassini States* (CSs; Colombo 1966; Peale 1969). We follow the notation of Su & Lai (2020), where the parameter

$$\eta \equiv -\frac{g}{\alpha}, \tag{6}$$

is used. For a given value of η, there can be either two or four CSs, all of which require $\hat{\mathbf{s}}$ lie in the plane of $\hat{\mathbf{l}}$ and $\hat{\mathbf{l}}_{\rm p}$. Following the standard

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Figure 1. Cassini State obliquities θ as a function of $\eta \equiv -g/\alpha$ (Eq. 6) for $I = 5°$. The vertical dashed line denotes η_c, where the number of Cassini States changes from four to just two (Eq. 8). The horizontal dashed line gives $\theta = I$, the asymptotic value of Cassini State 2's obliquity when $\eta \to \infty$.

Figure 2. Level curves of the Cassini State Hamiltonian (Eq. 9) for $I = 5°$, for which $\eta_c \approx 0.77$ (Eq. 8). For $\eta < \eta_c$, there are four Cassini States (labeled), while for $\eta > \eta_c$ there are only two. In the former case, the existence of a *separatrix* (solid black lines) separates phase space into three numbered zones (I/II/III, labeled). Finally, we denote the upper and lower legs of the separatrix by $C_\pm$ respectively, as shown in the upper two panels.

convention and nomenclature, CSs 1, 3, and 4 have $\phi = 0$ and $\theta < 0$, while CS2 has $\phi = \pi$ and $\theta > 0$. Figure 1 shows the CS obliquities as a function of η, each of which satisfies

$$\sin\theta\cos\theta - \eta\sin(\theta - I) = 0. \tag{7}$$

Note that there are four CSs when $\eta < \eta_c$ and only two when $\eta > \eta_c$, where

$$\eta_c \equiv \left(\sin^{2/3} I + \cos^{2/3} I\right)^{-3/2}. \tag{8}$$

The Hamiltonian corresponding to Eq. (5) is

$$\begin{aligned} H &= -\frac{\alpha}{2}\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)^2 - g\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}_{\rm d}\right) \\ &= -\frac{\alpha}{2}\cos^2\theta - g\left(\cos\theta\cos I - \sin I\sin\theta\cos\phi\right). \end{aligned} \tag{9}$$

Here, $\cos\theta$ and ϕ form a canonically conjugate pair of variables. Figure 2 shows the level curves of this Hamiltonian for $I = 20°$, for which $\eta_c \approx 0.77$ (Eq. 8). When $\eta < \eta_c$, CS4 exists and is a saddle point. The two trajectories originating and ending at CS4 are the only two infinite-period orbits in the phase space. Together, these two critical trajectories are referred to as the *separatrix* and divide phase space into three zones. Trajectories in zone II librate about CS2 while those in zones I and III circulate. On the other hand, when $\eta > \eta_c$, all trajectories circulate. When the separatrix exists, we divide it into two curves: C_+, the boundary between zones I and II, and C_-, the boundary between zones II and III.

3 SPIN EVOLUTION WITH ALIGNMENT TORQUE

In this section, we consider a simplified model of equilibrium tides that isolates the important new phenomenon presented in this paper. We assume that the spin magnitude of the planet is constant, so α and g are both fixed, while the spin orientation $\hat{\mathbf{s}}$ experiences a torque that drives it towards alignment with $\hat{\mathbf{l}}$ on the alignment timescale t_s:

$$\left(\frac{d\hat{\mathbf{s}}}{dt}\right)_{\rm tide} = \frac{1}{t_s}\hat{\mathbf{s}}\times\left(\hat{\mathbf{l}}\times\hat{\mathbf{s}}\right). \tag{10}$$

The full equations of motion for $\hat{\mathbf{s}}$ in the coordinates θ and ϕ can be written:

$$\frac{d\theta}{dt} = -g\sin I\sin\phi - \frac{1}{t_s}\sin\theta, \tag{11}$$

$$\frac{d\phi}{dt} = -\alpha\cos\theta - g\left(\cos I + \sin I\cot\theta\cos\phi\right). \tag{12}$$

3.1 Shifted Cassini States and Linear Stability Analysis

When t_s is finite, the CSs tilt out of the $\hat{\mathbf{l}}$-$\hat{\mathbf{l}}_p$ plane (as was first pointed out for CS2 by Fabrycky et al. 2007). A simple expression for the degree of this tilt can be obtained by solving for $\phi_{\rm cs}$, the azimuthal angle of the modified CS, when setting Eq. (11) equal to zero. This gives

$$\sin\phi_{\rm cs} = \frac{\sin^2\theta_{\rm cs}}{\sin I\,|g|\,t_s}. \tag{13}$$

Here, $\theta_{\rm cs}$ is the modified CS obliquity. Note that if t_s is shorter than some critical value $t_{s,c}$, which for a given CS is given by

$$t_{s,c} = \frac{\sin^2\theta_{\rm cs}}{|g|\sin I}, \tag{14}$$

then there are no solutions for $\phi_{\rm cs}$, and the CS is no longer a fixed point. This condition first breaks down for $\theta_{\rm cs} \approx 90°$, i.e. for CS2 (and CS4) in the limit $\eta \ll 1$. Thus, as long as $t_s \gtrsim (|g|\sin I)^{-1}$, then all CSs will exist.

We next seek to characterize the stability of small perturbations about each of the CSs in the presence of the alignment torque. We can linearize Eqs. (11–12) about a [modified] CS, yelding

$$\frac{d}{d\tau}\begin{bmatrix}\theta\\ \phi\end{bmatrix} = \begin{bmatrix} -\frac{\cos\theta}{t_s} & -g\sin I\cos\phi \\ \alpha\sin\theta + g\frac{\sin I\cos\phi}{\sin^2\theta} & 0\end{bmatrix}_{\rm cs}\begin{bmatrix}\Delta\theta\\ \Delta\phi\end{bmatrix}, \tag{15}$$

where the cs subscript indicates evaluating at a CS, $\Delta\theta = \theta - \theta_{\rm cs}$, and the same for $\Delta\phi$. The eigenvalues λ of Eq. (15) satisfy the equation

$$0 = \left(\lambda + \frac{\cos\theta_{\rm cs}}{t_s}\right)\lambda - \lambda_0^2, \tag{16}$$

$$\lambda_0^2 \equiv \left(\alpha\sin\theta_{\rm cs} + g\sin I\csc^2\theta_{\rm cs}\cos\phi_{\rm cs}\right)\left(-g\sin I\cos\phi_{\rm cs}\right), \tag{17}$$

$$\lambda \approx -\frac{\cos\theta_{\rm cs}}{t_s} \pm \sqrt{\lambda_0^2}. \tag{18}$$

The stability depends only on the real part of λ in Eq. (18). If we assume that the torque-induced modification to $\phi_{\rm cs}$ is small, then λ_0 reduces to the product of the eigenvalues in the torque-free limit. In this limit, $\lambda_0^2 < 0$ for each of CSs 1–3 and $\lambda_0^2 > 0$ for CS4 (see Fig. 17 of Su & Lai 2020). Thus, CS4 is always unstable, since there will always be a positive eigenvalue λ, while the stability of CSs 1–3 are solely determined by the sign of $\cos\theta_{\rm cs}$. From Fig. 1, we see that CSs 1 and 2 are stable while CS3 is unstable under the alignment torque. These calculations justify results long used in CS literature (e.g. Ward 1975).

3.2 Spin Obliquity Evolution Driven by Alignment Torque

With the above results, we are equipped to ask the key question of this section: for a given initial θ_0 and ϕ_0, which of CS1 and CS2 does the system evolve into? For initial conditions in Zone I (see Fig. 2), an unperturbed trajectory circulates and returns to $(\theta_0, \phi_0 + 2\pi)$ after a single precession period. During this period, the effect of the dissipation is a negative θ' everywhere during the cycle. Thus, for initial conditions in Zone I, θ decreases until the trajectory has converged to CS1. This is intuitively reasonable, as CS1 is stable. For initial conditions in Zone II, an equivalent calculation to that in Section A shows that all trajectories converge to CS2.

For illustrative purposes, we fix $|g|\,t_s = 10^{-3}$, and $\eta = -g/\alpha$ is varied among a few small values (small meaning much less than η_c). We specialize to the case where $\eta < \eta_c$ and there are four CSs.

However, initial conditions in Zone III pose a puzzle. There are no stable CSs in Zone III, so the system must evolve through the

Figure 3. TODO generate a better plot. Ignore bottom two panels, top two panels show which ICs end up in CS 1 (left) and CS2 (right). Probabilistic behavior in Zone III is evident.

Figure 4. TODO overplot separatrix, remove bad labels, better legend. Stable and unstable manifolds of CS4, showing how the separatrix splrits and gives a "path" from Zone III into Zones I/II.

separatrix to reach one of either CS1 or CS2. In Fig. 3, we find see that the outcome for initial conditions in Zone III is in fact *probabilistic*. Intuitively, this can be understood as probabilistic resonance capture: since $\eta \ll \eta_c$, $\alpha \gg -g$ (the spin-orbit precession rate, Eq. 1, and the orbit precession induced by the perturber, Eq. 2, respectively), but $\alpha\cos\theta$ can become commensurate with $-g$ if $\cos\theta \sim \eta$. This is achieved as θ evolves from an initially retrograde obliquity through 90° towards 0° under the influence of the dissipative term in Eq. (11).

While similar in behavior to previous studies of probabilistic resonance capture (Henrard 1982; Su & Lai 2020), the mechanism at play here is necessarily different: here, the Hamiltonian and underlying phase structure is kept constant while a non-Hamiltonian, dissipative perturbation is introduced. The origin of this resonance crossing process is easy to visualize. In the absence of dissipative effects, the separatrix is both the set of points that flow into CS4 and the set of points that flow away CS4, called the *stable* and *unstable* manifolds of the saddle point CS4 respectively (Guckenheimer & Holmes 1983). When the dissipative term is taken into account, these stable and unstable manifolds split and open a visible "path" from Zone III into both Zones I and II, as shown in Fig. **??**.

The amount that the stable and unstable manifolds split can be calculated perturbatively using *Melnikov's Method* (Guckenheimer & Holmes 1983). The essence of Melnikov's method lies at the heart of the perturbative approach used in Section A, where the evolution of H along the perturbed trajectories can be evaluated by integration along their original *unperturbed* trajectories. In particular: TODO the rest of this section is copy pasted from notes without even notation changes, needs to be written up correctly Now, let's reconsider the Melnikov integral when perturbing this finite (non-homoclinic orbit); this may no longer be an exact result but should give the correct scaling:

$$M_c = \int\limits_0^T \frac{\partial\phi}{\partial t}\epsilon\left(1-\mu^2\right)\,\mathrm{d}t. \tag{19}$$

Naively, we might claim that, since $\frac{\partial\phi}{\partial t}$ changes signs halfway through the interval of integration, that the only surviving component is $2\epsilon\bar{\mu}\mu'$, where $\bar{\mu} = \mu_4$ is the average value of μ and μ' is the fluctuation. This gives

$$M_c = 2\int\limits_0^{2\pi} 2\epsilon\frac{\eta\cos I}{1+\eta\sin I}\sqrt{2\eta\sin I\,(1-\cos\phi)}\,\mathrm{d}\phi. \tag{20}$$

Note that $M_c \propto \eta^{3/2}$, and since the gap opened $\Delta_c = \frac{M_c}{\frac{\partial\phi}{\partial t}} \propto \eta$, it seems like we're on the right track. Specifically:

$$M_c \approx \epsilon 2\eta\cos I A_{sep}, \tag{21}$$

$$\Delta_c \approx 2\epsilon\eta\cos I\left(16\sqrt{\eta\sin I}\right)\frac{1}{\sqrt{4\eta\sin I}}, \tag{22}$$

$$\approx 16\epsilon\eta\cos I. \tag{23}$$

Figure 5. Schematic illustrating the evolution of the planet's spin due to tidal friction. The green lines illustrate the directions of Eqs. (26–27), while the black and blue lines denote where the tidal Ω_s' and θ' change signs respectively. Overlaid are the obliquities of Cassini States 1 and 2, the two Cassini States that are stable under tidal dissipation (see Section 3.1). The points that are both Cassini States and satisfy $\Omega_s' = 0$ are the tidal Cassini Equilibria (tCE), circled in green. Finally, the vertical red lines illustrate the Ω_s below which tides causes tCE2 to become unstable; the dashed and dotted lines correspond to values of $\epsilon = 10^{-2}$ and 10^{-1} respectively.

This also agrees exceedingly well with our simulations This then gives us hopping probability

$$P_{hop} = \frac{\Delta_c}{\Delta_M} \approx \frac{16\eta^{3/2}\cos I\sqrt{\sin I}}{\pi}. \tag{24}$$

This agrees perfectly with the cases we've run.

Note: The full formula, by actually evaluating all the terms, comes out to be

$$P_{hop} = \frac{16\eta^{3/2}\cos I\sqrt{\sin I}}{\pi\left(1-2\eta\sin I-\eta^2\cos^2 I\right)+8\eta^{3/2}\cos I\sqrt{\sin I}}. \tag{25}$$

4 SPIN EVOLUTION WITH WEAK TIDAL FRICTION

4.1 Tidal Model: Equilibrium Tides

To model the dissipative effect of times, we use the weak friction theory of equilibrium tides (Lai 2012). In this model, tides cause both $\hat{\mathbf{s}}$ and Ω_s to evolve following:

$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}\tau}\right)_{\text{tide}} = \epsilon\left[\frac{2n}{\Omega_s} - \left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\right]\hat{\mathbf{s}}\times\left(\hat{\mathbf{l}}\times\hat{\mathbf{s}}\right), \tag{26}$$

$$\frac{1}{\Omega_s}\left(\frac{\mathrm{d}\Omega_s}{\mathrm{d}\tau}\right)_{\text{tide}} = \epsilon\left[\frac{2n}{\Omega_s}\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right) - \left(1+\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)^2\right)\right], \tag{27}$$

where ϵ is the dimensionless tidal evolution rate, given by

$$\epsilon \equiv -\frac{1}{gt_a}\frac{L}{2S}\frac{\Omega_s}{2n}, \tag{28}$$

$$\frac{1}{t_a} = \frac{3k_2}{Q}\left(\frac{m}{M}\right)\left(\frac{R}{a}\right)^5 n, \tag{29}$$

$$\epsilon \approx 0.003\frac{1}{\cos I}\left(\frac{2k_2/Q}{10^{-3}}\right)\left(\frac{m_p}{M_J}\right)^{-1}\left(\frac{a_p}{5\text{ AU}}\right)^3 \times\left(\frac{a}{0.4\text{ AU}}\right)^{-6}\left(\frac{\rho}{3\text{ g/cm}^3}\right)^{-1}\left(\frac{M_\star}{M_\odot}\right)^2. \tag{30}$$

Here, $L = ma^2n$ and $S = kmR^2 \approx mR^2/4$ are the orbital and spin angular momenta of the inner planet, respectively. In Eq. (26), it is clear that tides only acts on the polar component of $\hat{\mathbf{s}}$, i.e. θ, and not the azimuthal component ϕ. As such, the effect of tides on the dynamics of the system can be qualitatively understood in (Ω_s, θ) space. Figure 5 shows the behavior of Eqs. (26–27) in a few characteristic regions of (Ω_s, θ) space.

Component form

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = g\sin I\sin\phi - \epsilon\sin\theta, \tag{31}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = -\alpha\cos\theta - g\left(\cos I + \sin I\cot\theta\cos\phi\right), \tag{32}$$

$$\frac{\mathrm{d}\Omega_s}{\mathrm{d}t} = \epsilon\left[2n\cos\theta - \Omega_s\left(1+\cos^2\theta\right)\right]. \tag{33}$$

For the parameters considered here, the tidal evolution in Ω_s and

Figure 6. *Left:* Each dot indicates which tCE a given initial condition $(\theta_{\rm i}, \phi_{\rm i})$ evolve towards (labeled in legend), for $\Omega_{\rm c} = 0.06n$ and $I = 5^\circ$. The separatrix is shown as the black line. Note that points above the separatrix evolve towards tCE1, points inside the separatrix evolve towards tCE2, and points below the separatrix have a probabilistic outcome. *Right:* Histogram of which tCE a given initial obliquity $\theta_{\rm i}$ evolves towards.

Figure 7. Same as Fig. 6 but for $\Omega_{\rm c} = 0.2n$.

Figure 8. Same as Fig. 6 but for $\Omega_{\rm c} = 0.7n$. Note that even points above the separatrix can evolve towards tCE2 here.

Figure 9. Comparison of the right hand panel of Fig. 6 (red dots) with a semi-analytic calculation (blue line). The semi-analytic calculation is performed by numerically integrating Eqs.(31–33) on a grid of initial conditions uniform in $\cos\theta_{\rm i}$ and $\phi_{\rm i}$ until separatrix encounter, then using the $\Omega_{\rm s}$ at separatrix encounter to evaluate (TODO) to analytically compute the probability of evolution into tCE2.

θ occurs over characteristic time scale 2×10^8 yr. Note that while the semimajor axis a is not strictly constant under the influence of tidal dissipation, $\dot{a}/\dot{\theta} \sim S/L \ll 1$ (Lai 2012) and the evolution of a can be safely neglected in this paper.

Armed with the intuition provided by the toy model in Section 3, we now study the equations of motion including the full weak tidal friction theory, given by Eqs. (31–33). We imagine that the inner planet is born rapidly spinning, but for computational reasons, we restrict the initial spin to be $\Omega_{\rm s,i} = 10n$. The final results are not sensitive on the specific initial spin as long as $\Omega_{\rm s,i} \gg n$. The hierarchy of the system is determined by the parameter $\Omega_{\rm c}$, given by Eq. (35), for which we choose a few representative values. Then, for a given initial θ_0 and ϕ_0, the final outcome (between tCE1 and tCE2) of the system can be calculated. In Fig. 6, we show the final outcome for many randomly chosen θ_0 and ϕ_0 for $\Omega_{\rm c} = 0.06$ and $I = 5^\circ$. We see that tCE1 is generally reached for spins initially in Zone I, tCE2 is generally reached for spins initially in Zone II, and a probabilistic outcome is observed for spins initially in Zone III, very similar to the results found for the toy model in Section 3. Figures **??** show the same results but for $\Omega_{\rm c} = 0.2$ and $\Omega_{\rm c} = 0.7$. We see that more initial conditions are able to end up in tCE2, mostly because the initial Zone II is larger. In all three examples, probabilistic resonance capture for initial conditions in Zone III is responsible for a significant fraction of systems that end up in tCE2.

Similarly to Su & Lai (2020), these probabilities are the result of separatrix crossings. However, the calculation differs from Su & Lai (2020): since the perturbation is dissipative in phase space (i.e. it does not conserve phase space area), analysis following Henrard (1982) is not sufficient. A more general theory of separatrix crossing is still able to predict the outcome probabilities, extending the results in Section 3. Appendix B1 gives the appropriate generalization, which gives the resonance capture probability $P_{\rm III \to II}$ as a function of η upon separatrix encounter.

We can apply this probability to calculate the probabilities of the two outcomes when a numerically-integrated trajectory encounters the separatrix, and from this obtain a semi-analytical prediction of the distribution of outcomes. In Fig. 9, we show the same histogram as the vertical panel of Fig. 6 alongside the semi-analytical prediction. Good qualitative agreement is observed. Figs. **??** show the same for Figs. **??**.

Figure 10. Same as Fig. 9 but for $\Omega_{\rm c} = 0.2n$, shown in Fig. 7.

Figure 11. Same as Fig. 9 but for $\Omega_{\rm c} = 0.7n$, shown in Fig. 8.

Figure 12. *Top:* Total probability of ending up in tCE2 (red dots) for $I = 5^\circ$ for a range of $\Omega_{\rm c}$. *Bottom:* obliquities of the two possible tCEs.

4.2 Outcomes for Distribution of Initial Spin Orientations

Here, we define $\Omega_{\rm c}$ to be the critical spin where the precession frequencies are equal, i.e.

$$\Omega_{\rm c} \equiv -\frac{g}{\alpha}\Omega_{\rm s}, \tag{34}$$

$$\frac{\Omega_{\rm c}}{n} = 0.33 \left(\frac{k}{k_{\rm q}}\right)\left(\frac{m_{\rm p}}{M_{\rm J}}\right)\left(\frac{a_{\rm p}}{5\ {\rm AU}}\right)^{-3} \times \left(\frac{a}{0.4\ {\rm AU}}\right)^{6}\left(\frac{\rho}{3\ {\rm g/cm^3}}\right)\left(\frac{M_\star}{M_\odot}\right)^{-2}. \tag{35}$$

Here, $\rho = m/\left(4\pi R^3/3\right)$ is the average density of the inner planet, and $M_{\rm J}$ is the mass of Jupiter. Tides tend to drive $\Omega_{\rm s}$ to spin-orbit synchronization, where $\Omega_{\rm s} = n$ and thus $-g/\alpha = \Omega_{\rm c}/n$. As such, we see that the ratio $\Omega_{\rm c}/n$ quantifies the strength of the perturber relative to the spin-orbit coupling at the tidal equilibrium.

In the previous section, we considered the outcome as a function of the initial spin orientation, specified by θ_0 and ϕ_0. In this section, we consider the distribution of outcomes when averaging over a distribution of initial spin orientations. For simplicity, we just consider $\hat{\mathbf{s}}$ being isotropically distributed. Figure 12 shows this for $I = 5^\circ$ as a function of $\Omega_{\rm c}$.

5 SUMMARY AND DISCUSSION

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APPENDIX A: CONVERGENCE OF INITIAL CONDITIONS INSIDE THE SEPARATRIX TO TCE2

Based on the results from the previous section, we intuitively expect that all initial conditions in Zone II, i.e. inside the separatrix, will remain near CS2, and eventually converge to tCE2. However, this is not immediately justified, since not all points in Zone II are necessarily close enough to CS2 for the linear analysis in Section 3.1 to be valid. We present an alternative calculation illustrating that all points in Zone II remain near CS2. For simplicity, we neglect the evolution of $\Omega_{\rm s}$ here and take η to be constant; the full calculation presented in Appendix B includes the evolution of $\Omega_{\rm s}$ and reproduces this result.

The objective is to determine the change in H, the value of the unperturbed Hamiltonian, over a single libration cycle. If the system's has initial $H_0 \equiv H\left(\theta_0, \phi_0\right) = H_{\rm sep} + \Delta H$ where

$$H_{\rm sep} \equiv H\left(\cos\theta_4, \phi_4\right), \tag{A1}$$

$$\approx -\sin I + \frac{\eta}{2}\cos^2 I + O\left(\eta^2\right), \tag{A2}$$

is the value of H along the separatrix, and $\Delta H > 0$ for initial conditions inside the separatrix, then the unperturbed trajectory $\theta(\phi)$ can then be written explicitly via:

$$H_{\rm sep} + \Delta H = -\frac{\cos^2\theta}{2\eta} + \cos\theta\cos I - \sin I\cos\phi, \tag{A3}$$

$$\cos\theta_\pm \approx \eta\cos I \pm \sqrt{2\eta\left[\sin I\left(1 - \cos\phi\right) - \Delta H\right]}. \tag{A4}$$

We have taken $\sin\theta \approx 1$, a good approximation in Zone II since $\eta \ll 1$. When $\Delta H = 0$, this reproduces Eq. (B5) of Su & Lai (2020). When $\Delta H > 0$, there are some values of ϕ for which no solutions of θ exist, reflecting the fact that the libration does not extend over the full interval $\phi \in [0, 2\pi]$. During a libration cycle, θ_- $[\theta_+]$ is traversed while $\phi' > 0$ $[\phi' < 0]$, i.e. the trajectory librates counterclockwise in $(\cos\theta, \phi)$ phase space (see Fig. 2).

The above calculation assumes H is constant, yielding the unperturbed trajectory. The leading order change to H can be computed by integrating $\mathrm{d}H/\mathrm{d}\tau$ along this trajectory, where $\mathrm{d}H/\mathrm{d}\tau$ is given by

$$\frac{\mathrm{d}H}{\mathrm{d}\tau} = \frac{\partial H}{\partial(\cos\theta)}\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}\tau} + \frac{\partial H}{\partial\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau},$$

$$= -\left(\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}\tau}\right)_{\rm tide}\frac{\mathrm{d}\phi}{\mathrm{d}\tau}. \tag{A5}$$

We have used Hamilton's equations for the canonically conjugate variables $\cos\theta$ and ϕ to eliminate the non-tidal contributions. The total change in H over a single cycle is then:

$$\oint \frac{\mathrm{d}H}{\mathrm{d}\tau}\,\mathrm{d}\tau = \oint \frac{\mathrm{d}H}{\mathrm{d}\tau}\frac{\mathrm{d}\tau}{\mathrm{d}\phi}\,\mathrm{d}\phi,$$

$$= \oint -\left(\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}\tau}\right)_{\rm tide}\,\mathrm{d}\phi,$$

$$= \int_{\phi_{\rm min}}^{\phi_{\rm max}} \epsilon\left(\frac{2n}{\Omega_{\rm s}} - \cos\theta_-\right)\sin^2\theta_-\,\mathrm{d}\phi$$

$$+ \int_{\phi_{\rm min}}^{\phi_{\rm max}} \epsilon\left(\frac{2n}{\Omega_{\rm s}} - \cos\theta_+\right)\sin^2\theta_+\,\mathrm{d}\phi,$$

$$\approx \int_{\phi_{\rm min}}^{\phi_{\rm max}} \epsilon\left(\frac{2n}{\Omega_{\rm s}}\sin^2\theta_- - \cos\theta_-\right)\,\mathrm{d}\phi$$

$$+ \int_{\phi_{\rm min}}^{\phi_{\rm max}} \epsilon\left(\frac{2n}{\Omega_{\rm s}}\sin^2\theta_+ - \cos\theta_+\right)\,\mathrm{d}\phi. \tag{A6}$$

Finally, for every ϕ, $\sin\theta_- \geq \sin\theta_+$ and $\cos\theta_- \leq \cos\theta_+$ with equality only at the endpoints of the libration cycle where $\theta_- = \theta_+$. As such, the total change in H over a libration cycle is positive, and we conclude that every trajectory starting in Zone II remains near CS2 under the effect of tidal dissipation even when not infinitesimally close to CS2 initially.

APPENDIX B: SEPARATRIX CROSSING THEORY

Thus, the natural extension of the two above results should be

$$\Delta_\pm = \oint_{C_\pm} \frac{\mathrm{d}H}{\mathrm{d}t}\,\mathrm{d}t, \tag{B1}$$

$$= \oint_{C_\pm} \dot{\mu}^{(1)} + \frac{\dot{s}^{(1)}}{\dot{\phi}^{(0)}}\left(\frac{\partial H}{\partial s} - \frac{\partial H_4}{\partial s}\right)\,\mathrm{d}\phi. \tag{B2}$$

B1 Separatrix Crossing Probability: Tidal Friction

$$\begin{aligned}\frac{\mathrm{d}\,(\Delta H)}{\mathrm{d}\tau} &= \frac{\partial H}{\partial(\cos\theta)}\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}\tau} + \frac{\partial H}{\partial\phi}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \frac{\partial H}{\partial\Omega_{\rm s}}\frac{\mathrm{d}\Omega_{\rm s}}{\mathrm{d}\tau} - \frac{\partial H_{\rm sep}}{\partial\Omega_{\rm s}}\frac{\mathrm{d}\Omega_{\rm s}}{\mathrm{d}\tau},\\ &= -\left(\frac{\mathrm{d}(\cos\theta)}{\mathrm{d}\tau}\right)_{\rm tide}\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \left(\frac{\partial H}{\partial\Omega_{\rm s}} - \frac{\partial H_{\rm sep}}{\partial\Omega_{\rm s}}\right)\frac{\mathrm{d}\Omega_{\rm s}}{\mathrm{d}\tau},\end{aligned} \tag{B3}$$

$$\frac{1}{\epsilon}\frac{\mathrm{d}(\Delta H)}{\mathrm{d}\tau} \approx \sin^2\theta\left(\frac{2n}{\Omega_{\rm s}} - \cos\theta\right)\frac{\mathrm{d}\phi}{\mathrm{d}\tau} + \left[\frac{\cos^2\theta}{2\Omega_{\rm c}} - \frac{\Omega_{\rm c}}{2\Omega_{\rm s}^2}\cos^2 I\right]\frac{\mathrm{d}\Omega_{\rm s}}{\mathrm{d}\tau}. \tag{B4}$$

Application of the full formula presented in Section B. The key result is that one integrates

$$\frac{\mathrm{d}(\Delta H)}{\mathrm{d}(\epsilon t)} \approx (1-\mu^2)\left(\frac{2\Omega}{s} - \mu\right)\dot{\phi}^{(0)} + 2\Omega\left(1 + \frac{s}{2\Omega}(1+\mu^2)\right)\left[\frac{\mu^2}{2s_c} - \frac{s_c}{2s^2}\cos^2 I\right]. \tag{B5}$$

An analytical form that holds when $s \gg s_c$ is:

$$\begin{aligned}\frac{\Delta_\pm}{\epsilon} =& -2\cos I\left(\pm 2\pi\eta\cos I + 8\sqrt{\eta\sin I}\right) \pm 2\pi s\cos I + \eta\cos I\left(-8\sqrt{\sin I/\eta}\right) + \frac{s}{2}8\sqrt{\sin I/\eta}\\ &+ \frac{2\Omega}{s}\left(\mp 2\pi\left(1 - 2\eta\sin I\right) + 16\cos I\eta^{3/2}\sqrt{\sin I}\right) + 8\sqrt{\eta\sin I} \pm 2\pi\eta\cos I - \frac{64}{3}\left(\eta\sin I\right)^{3/2}.\end{aligned} \tag{B6}$$

The capture probability is then just

$$P_c = \frac{\Delta_+ + \Delta_-}{\Delta_-}. \tag{B7}$$