1 Hamiltonians and EOM

1.1 Toy Problem

Consider simplest spin Hamiltonian $H = -\vec{B}\cdot\vec{s}$. It's clear that if we set up initial conditions $\vec{s}$ misaligned from $\vec{B}$, it will simply spin around $\vec{B}$, which is fixed. Thus, let $\hat{B}\cdot\hat{s} = \cos\theta$ the angle between the two, and let ϕ measure the azimuthal angle.

We claim that $\cos\theta, \phi$ are canonical variables. Since ϕ is ignorable, immediately $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\cos\theta}{\mathrm{d}t} = -\frac{\partial H}{\partial\phi} = 0$, while $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial H}{\partial(\cos\theta)} = Bs$ tells us the rate at which the spin precesses around $\vec{B}$.

1.2 Cassini State Hamilttonian

This Hamiltonian is Kassandras Eq. 13, in the co-rotating frame with the perturber's angular momentum:

$$\mathcal{H} = \frac{1}{2}\left(\hat{s}\cdot\hat{l}\right)^2 - \eta\left(\hat{s}\cdot\hat{l}_p\right). \tag{1}$$

In this frame, we can choose $\hat{l} \equiv \hat{z}$ fixed, and $\hat{l}_p = \cos I\hat{z} + \sin I\hat{x}$ fixed as well. Then

$$\hat{s} = \cos\theta\hat{z} - \sin\theta\left(\sin\phi\hat{y} + \cos\phi\hat{x}\right).$$

We can choose the convention for $\phi = \phi$ azimuthal angle requiring $\phi = 0, \pi$ mean coplanarity between $\hat{s}, \hat{l}, \hat{l}_p$ in the $\hat{x}, \hat{z}$ plane such that $\hat{l}_p, \hat{s}$ lie on the same side of $\hat{l}$. Then we can evaluate in coordinates

$$\begin{aligned}
\hat{s}\cdot\hat{l} &= \cos\theta,\\
\hat{s}\cdot\hat{l}_p &= \cos\theta\cos I - \sin I\sin\theta\cos\phi,\\
\mathcal{H} &= -\frac{1}{2}\cos^2\theta + \eta\left(\cos\theta\cos I - \sin I\sin\theta\cos\phi\right).
\end{aligned}$$

Note that if we take $\cos\theta$ to be our canonical variable, $\sin\theta = \sqrt{1-\cos^2\theta}$ can be used.

1.3 Equation of Motion

The correct EOM comes from Kassandra's Eq. 12:

$$\begin{aligned}
\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} &= \left(\hat{s}\cdot\hat{l}\right)\left(\hat{s}\times\hat{l}\right) - \eta\left(\hat{s}\times\hat{l}_p\right),\\
&= \left(s_y s_z - \eta s_y\cos(I)\right)\hat{x} - \left(s_x s_z + \eta(s_x\cos I - s_z\sin I)\right)\hat{y} + \eta s_y\sin(I)\hat{z}.
\end{aligned}$$

Alternatively, consider Hamilton's equations applied to the Hamiltonian:

$$\frac{\partial\phi}{\partial t} = \frac{\partial\mathcal{H}}{\partial(\cos\theta)} = -\cos\theta + \eta\left(\cos I + \sin I\cot\theta\cos\phi\right), \tag{2}$$

$$\frac{\partial(\cos\theta)}{\partial t} = -\frac{\partial\mathcal{H}}{\partial\phi} = -\eta\sin I\sin\theta\sin\phi. \tag{3}$$

This produces the same trajectories as the Cartesian EOM, so this is correct. However, since $\frac{\partial\phi}{\partial t} \propto 1/\sin\theta$, this is not a desirable system of equations to use, as they are very stiff near $\theta \approx 0$.

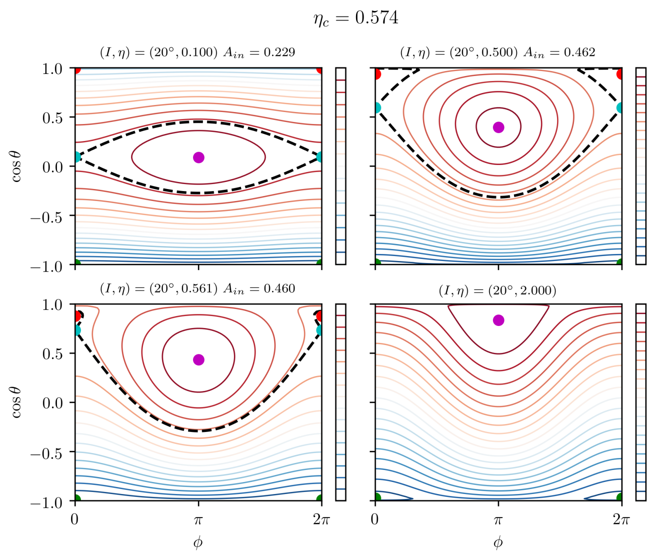

Figure 1: Separatrix for various values of η.

1.4 Cassini States

The zeros to Eq. 3 are the Cassini states; we will go to canonical variables $\mu = \cos\theta$. We can immediately see that $\sin\phi = 0$ is necessary, so $\cos\phi = \pm 1$ and we need only solve for $\frac{\partial\phi}{\partial t} = 0$. We can furthermore separate the problem into two regimes, $\eta \ll 1$ and $\eta \gg 1$.

For $\eta \ll 1$, it is clear that there will be two solutions near $\mu^2 = 1$ and two solutions near $\mu = 0$:

- For $\mu = 1 - \frac{\theta^2}{2}$, the dominant terms are $\frac{\partial\phi}{\partial t} \approx -1 + \eta \sin I \frac{1}{\theta} = 0$, where we've taken $\cos\phi = +1$ and $\phi = 0$. This forces $\theta = \eta \sin I$.
- Similarly, for $\mu = -1 + \frac{\epsilon^2}{2}$, $\phi = 0$ and $\epsilon = \eta \sin I$ again. This actually corresponds to $\theta = \pi - \eta \sin I$.
- For $\mu \approx 0$, we have instead $\frac{\partial\phi}{\partial t} = -\mu(1 - \eta \sin I \cos\phi) + \eta \cos I = 0$. This forces $\mu_\pm = \frac{\eta \cos I}{1 \pm \eta \sin I}$, where $\phi_\pm = \pi, 0$ respectively.

  Note that $\phi = 0, \mu \approx 0$ is conventionally CS4. The linearization locally has form $\frac{\partial\delta\phi}{\partial t} = -\delta\mu(1 - \eta \sin I)$ and $\frac{\partial\delta\mu}{\partial t} = -\eta \sin I \delta\phi$, so the eigenvalues are $\approx \mp\sqrt{\eta \sin I}$, and the two eigenvectors are $\left(1, \pm\sqrt{\eta \sin I}\right)$.

For $\eta \gg 1$, the solutions obviously just come from $\cos I \pm \sin I \cot\theta = 0$, which are just $\sin(I \pm \theta) = 0$

1.5 Separatrix Area

We can estimate the area enclosed by the separatrix, as shown in Fig. 1. Note that the separatrix joins Cassini State 4 to its $+2\pi$ image.

We notate $\mu = \cos\theta$; note that CS4 is $\mu_4 \approx \frac{\eta \cos I}{1 - \eta \sin I} \approx \eta \cos I$. Setting the Hamiltonian equal to its value at CS4 gives

$$\begin{aligned}
H_4 \equiv H(\mu_4, \phi_4) &\approx -\frac{\mu_4^2}{2} + \eta\mu_4 \cos I - \eta \sin I,\\
&= +\eta^2 \cos^2 I - \eta \sin I,\\
H(\mu_{sep}, \phi_{sep}) = H_4 &= -\eta \sin I \cos\phi_{sep} - \frac{\mu_{sep}^2}{2} + \eta\mu_{sep}\cos I + \mathcal{O}(\eta^3),\\
0 &\approx \frac{\mu_{sep}^2}{2} - \eta\mu_{sep}\cos I - \eta \sin I\left(1 - \cos\phi_{sep}\right) + \eta^2\cos^2 I,\\
\mu_{sep}(\phi) &\approx \sqrt{2\eta \sin I\left(1 - \cos\phi\right)} + \mathcal{O}(\eta).
\end{aligned}$$

We can then easily compute the area enclosed by the separatrix

$$\begin{aligned}
A_{sep} &= \int_0^{2\pi} 2\mu_{sep}\,\mathrm{d}\phi,\\
&\approx 16\sqrt{\eta \sin I}.
\end{aligned} \tag{4}$$

For $\eta = 0.1, I = 20^\circ$, this predicts $\frac{A_{sep}}{A_T} \approx 0.235$, which is pretty close to my numerically calculated $\frac{A_{sep}}{A_T} = 0.229$.

1.6 Tidal Dissipation

We can add a tidal dissipation term; we write it in form $\left(\frac{\mathrm{d}\hat{s}}{\mathrm{d}t}\right)_{tide} = \epsilon\hat{s} \times \left(\hat{l} \times \hat{s}\right)$. Expanding,

$$\begin{aligned}
\left(\frac{\mathrm{d}\hat{s}}{\mathrm{d}t}\right)_{tide} &= \epsilon(\hat{z} - s_z\hat{s}),\\
&= \epsilon\left(-s_z s_x \hat{x} - s_z s_y \hat{y} + \left(1 - s_z^2\right)\hat{z}\right).
\end{aligned} \tag{5}$$

We run numerical simulations for weaker $\epsilon \ll \eta \ll 1$ and stronger $\epsilon \lesssim \eta \ll 1$.

We can seek equilibria of the the system including tides, which requires

$$\begin{aligned}
0 &= s_y s_z - \eta s_y \cos I - \epsilon s_z s_x,\\
0 &= -s_x s_z - \eta(s_x \cos I - s_z \sin I) - \epsilon s_z s_y,\\
0 &= \eta s_y \sin(I) + \epsilon\left(1 - s_z^2\right).
\end{aligned}$$

We expect at least two equilibria, based on the simulations: one near $s_z \approx 1$ and one $s_z \approx 0$.

For near alignment/near Cassini state 1, $1 - s_z \sim 1 - s_\perp^2$, so we can set $s_z = 1$ to first order: $s_y - \epsilon s_x - \eta s_y \cos I = -s_x - \eta(s_x \cos I - \sin I) - \epsilon s_y = \eta s_y \sin I = 0$. This can be satisfied if we set $s_x = \tan(I) \ll 1, s_y = \mathcal{O}(\epsilon s_x)$; this coarsely corresponds to Cassini state 1.

The other solution should be near Cassini state 2, where $s_x \approx 1$; dropping second order terms forces $\eta s_y + \epsilon s_z = -s_z - \eta(\cos I - s_z \sin I) = \eta s_y \sin(I) + \epsilon = 0$. This can thus be satisfied for $s_y \approx -\frac{\epsilon}{\eta \sin(I)}$. Thus, this explains why as ϵ is increased, we first start to get points that don't converge to Cassini state 2 in the absence of tides, before starting to see points that fail to converge to Cassini state 1.

2 Separatrix Hopping

Inspired by G&H, heteroclinic orbits are topologically unstable for any nonzero perturbation, but opened width $\sim$ perturbation parameter.

2.1 Try 1: Qualitative

We zoom in on Cassini State 4, which has $\theta_4 = -\frac{\pi}{2} + \frac{\eta\cos I}{1-\eta\sin I}, \mu_4 = \frac{\eta\cos I}{1-\eta\sin I}, \phi_4 = 0$. Then, using equations of motion

$$\frac{\partial\phi}{\partial t} = \mu - \eta\left(\cos I + \sin I \frac{\mu}{\sqrt{1-\mu^2}}\cos\phi\right), \tag{6}$$

$$\frac{\partial\mu}{\partial t} = -\eta\sin I\sin\phi + \left[\epsilon\left(1-\mu^2\right)\right], \tag{7}$$

we can perturbatively require $\frac{\partial\theta}{\partial t} = 0$ for $\epsilon \neq 0$. This corresponds to $\eta\sin I\sin\left(\phi_4 + \delta\phi\right) \approx \epsilon$, or $\delta\phi_4 = +\frac{\epsilon}{\eta\sin I}$. This is in agreement with Dong's result. Note that $\delta\theta_2 = -\frac{\epsilon}{\eta\sin I}$, which I saw in my simulations.

This implies that the stable manifolds of the two saddle points, which once overlapped with each other's unstable manifolds (creating a heteroclinic orbit) now are offset from one another by distance $D \sim \frac{\epsilon}{\eta\sin I}$. The question is how likely it is to thread the needle.

Consider that, very near CS4, the angle of incidence on the desired gap is roughly $\tan\psi \approx \psi = \frac{\Delta\theta}{\Delta\phi}$. Over the course of one orbit, $\Delta\phi$ changes by 2π, while $\Delta\theta \sim \epsilon\sin\theta T$ where T is the period of an orbit. Examining the data, $T \sim 50$, and so $\frac{\Delta\theta}{\Delta\phi} \sim \frac{2\pi}{\epsilon(50)}$.

The effective probability of threading the opened gap between the stable/unstable manifolds is then just $P \propto D\sin\psi \sim \frac{2\pi}{T(\eta\sin I)}$. According to later analysis, this should really be $\frac{2\pi}{T}$. Plugging in some observational values $T \sim 50$ for $\eta = 0.1$gives $P \propto 0.13$. In reality, I find it asymptotes to ~ 0.08, so the constant of proportionality is of order unity. Not bad given the really crappy $\psi \sim \frac{\langle\dot\theta\rangle}{\dot\phi}$ argument.

2.2 Try 2: Melnikov Distance

We notice that the separatrix is a heteroclinic orbit, or a saddle connection, in the dissipation free problem. Introducing dissipation breaks the saddle connection by a distance that can be estimated with the Melnikov distance. This is G&H Equation 4.5.11 or something:

$$d(t_0) = \frac{\epsilon M(t_0)}{\left|f(q^0(0))\right|} + \mathcal{O}(\epsilon^2), \tag{8}$$

$$M(t_0) = \int\limits_{-\infty}^{\infty} [f \times g]_{hetero}\,\mathrm{d}t. \tag{9}$$

This is not a hard formula to understand; along the separatrix, motion is dominated by f, but the perpendicular component adds up to contribute to a total "perpendicular distance away from the original separatrix" necessary to hit the saddle point, at least intuitively.

We evaluate the Melnikov integral $M(t_0)$ on the heteroclinic orbit. Note that since in our problem our perturbation g is time-independent, so too is the Melnikov integral $M(t_0) = M$.

Let's apply this to the Cassini state Hamiltonian w/ dissipation. We first write down our EOM in Melnikov form (we use canonical variables μ, ϕ):

$$\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} = \underbrace{\frac{\partial\mathcal{H}}{\partial\mu}\hat{\phi} - \frac{\partial\mathcal{H}}{\partial\phi}\hat{\mu}}_{f} + \epsilon\underbrace{\left(1-\mu^2\right)\hat{\mu}}_{g}. \tag{10}$$

Then $f \times g = f_\phi g_\mu = \frac{\partial \mathcal{H}}{\partial \mu}\left(1-\mu^2\right)$. We then want to integrate this along the heteroclinic orbit. We can make change of variables

$$M = \int\limits_0^{2\pi} \frac{\partial \mathcal{H}}{\partial \mu}\left(1-\mu^2\right)\left(\frac{\partial \phi}{\partial t}\right)^{-1} \mathrm{d}\phi. \tag{11}$$

But thankfully, $\frac{\partial \mathcal{H}}{\partial \mu} = \frac{\partial \phi}{\partial t}$ in the absence of dissipation, and so $M = 2\pi\left(1-\mu^2\right) \approx 2\pi\left(1-2\eta\sin I\right)$. Thus, the Melnikov distance at point q^0, a point on the heteroclinic orbit of the unperturbed Hamiltonian, is just

$$d(q^0) = \frac{2\pi\epsilon\left(1-2\eta\sin I\right)}{\left|f(q^0)\right|}. \tag{12}$$

Note that the maximum value $\left|f(q^0)\right|$, which occurs at $\phi = \pi$, is just $f \approx \sqrt{4\eta\sin I}$.

It proves to be a bit difficult to make quantitative predictions though, since the phase diagram is very smushed where f is large, and d is rather inaccurate where f is small. Let's think about a Poincaré map instead.

2.3 Try 3: Poincaré Section

Let's consider the Poincaré section every time $\phi = \phi_4$ as the trajectory subject to tidal dissipation is moving $\theta < \theta_4 \to \theta_4$. To provide an estimate of $\Delta\theta(\theta) = \theta_{n-1} - \theta_n$, this is just ϵT where T is the time elapsed between θ_n, θ_{n+1}, the period of the orbit. T is dominated by when $\frac{\partial \phi}{\partial t} \ll 1$ though, or where the orbit is close to the saddle point.

Note that T is dominated by the time it spends near the saddle point. We showed earlier that near CS4, $\frac{\partial \phi}{\partial t} \approx \delta\mu$ where $\delta\mu = \mu - \mu_4$. Thus, we might surmise $\Delta\theta(\theta) \propto \theta^{-1}$ for sufficiently small $\theta - \theta_4$. Far away, T is roughly constant and $\Delta\theta(\theta)$ is roughly constant.

What is "far away"? Well, it probably depends on how affected our trajectory is by the separatrix; far away from the saddle point, we go along contours of roughly constant θ, while close by we follow the separatrix pretty well. We computed earlier that $\mu_{sep} \sim \sqrt{4\eta\sin I}$, so we might expect $\mu > \mu_{sep}, \Delta\mu \sim C$, while $\mu < \mu_{sep}, \Delta\mu \sim \delta\mu^{-1}$.

My $\mu > \mu_{sep}$ simulations don't seem to work very well, so I'll focus on the $\delta\mu^{-1}$ case. In this case, define $\delta\mu_c : \Delta\mu\left(\delta\mu_c\right) = -\delta\mu_c$, i.e. the point that jumps immediately to the saddle point. Furthermore, assume the inbound distribution is flat between $\delta\mu_c, f^{-1}\left(\delta\mu_c\right)$. TODO: empirically, $\mu_c \sim \epsilon T$ is *flat* with η, probably just because we're not getting sufficiently close to the saddle point for the $\propto \sqrt{\eta}$ to kick in.

Then, we can compare the empirical Poincaré section of the points that cross the separatrix versus the total predicted interval width $\delta\mu_c, f^{-1}\left(\delta\mu_c\right)$; this would predict 7.2%, 18%. This does alright!