1 Hamiltonians and EOM

1.1 Toy Problem

Consider simplest spin Hamiltonian $H = -\vec{B}\cdot\vec{s}$. It's clear that if we set up initial conditions $\vec{s}$ misaligned from $\vec{B}$, it will simply spin around $\vec{B}$, which is fixed. Thus, let $\hat{B}\cdot\hat{s} = \cos\theta$ the angle between the two, and let ϕ measure the azimuthal angle.

We claim that $\cos\theta, \phi$ are canonical variables. Since ϕ is ignorable, immediately $\frac{\mathrm{d}\theta}{\mathrm{d}t} = \frac{\mathrm{d}\cos\theta}{\mathrm{d}t} = -\frac{\partial H}{\partial\phi} = 0$, while $\frac{\mathrm{d}\phi}{\mathrm{d}t} = \frac{\partial H}{\partial(\cos\theta)} = Bs$ tells us the rate at which the spin precesses around $\vec{B}$.

1.2 Cassini State Hamilttonian

This Hamiltonian is Kassandras Eq. 13, in the co-rotating frame with the perturber's angular momentum:

$$\mathscr{H} = \frac{1}{2}\left(\hat{s}\cdot\hat{l}\right)^2 - \eta\left(\hat{s}\cdot\hat{l}_p\right). \tag{1}$$

In this frame, we can choose $\hat{l} \equiv \hat{z}$ fixed, and $\hat{l}_p = \cos I\hat{z} + \sin I\hat{x}$ fixed as well. Then

$$\hat{s} = \cos\theta\hat{z} - \sin\theta\left(\sin\phi\hat{y} + \cos\phi\hat{x}\right).$$

We can choose the convention for $\phi = \phi$ azimuthal angle requiring $\phi = 0, \pi$ mean coplanarity between $\hat{s}, \hat{l}, \hat{l}_p$ in the $\hat{x}, \hat{z}$ plane such that $\hat{l}_p, \hat{s}$ lie on the same side of $\hat{l}$. Then we can evaluate in coordinates

$$\begin{aligned}
\hat{s}\cdot\hat{l} &= \cos\theta,\\
\hat{s}\cdot\hat{l}_p &= \cos\theta\cos I - \sin I\sin\theta\cos\phi,\\
\mathscr{H} &= \frac{1}{2}\cos^2\theta - \eta\left(\cos\theta\cos I - \sin I\sin\theta\cos\phi\right).
\end{aligned}$$

Note that if we take $\cos\theta$ to be our canonical variable, $\sin\theta = \sqrt{1-\cos^2\theta}$ can be used.

1.3 Equation of Motion

The correct EOM comes from Kassandra's Eq. 12:

$$\begin{aligned}
\frac{\mathrm{d}\hat{s}}{\mathrm{d}t} &= \left(\hat{s}\cdot\hat{l}\right)\left(\hat{s}\times\hat{l}\right) - \eta\left(\hat{s}\times\hat{l}_p\right),\\
&= \cos\theta\left[\hat{x}\left(-\sin\theta\sin\phi\right) - \hat{y}\left(-\sin\theta\cos\phi\right)\right]\\
&\quad - \eta\left[\hat{x}\left(-\sin\theta\sin\phi\cos I\right) - \hat{y}\left(-\sin\theta\cos\phi\cos I - \cos\theta\sin I\right) + \hat{z}\left(+\sin\theta\sin\phi\sin I\right)\right],\\
&= \left[-\sin\theta\sin\phi\cos\theta + \eta\sin\theta\sin\phi\cos I\right]\hat{x} + \left[\cos\theta\sin\theta\cos\phi - \eta\left(\sin\theta\cos\phi\cos I + \cos\theta\sin I\right)\right]\hat{y}\\
&\quad + \left[-\eta\sin\theta\sin\phi\sin I\right]\hat{z}.
\end{aligned}$$

Alternatively, consider Hamilton's equations applied to the Hamiltonian:

$$\frac{\partial\phi}{\partial t} = \frac{\partial\mathscr{H}}{\partial(\cos\theta)} = \cos\theta - \eta\left(\cos I + \sin I\cot\theta\cos\phi\right), \tag{2}$$

$$\frac{\partial(\cos\theta)}{\partial t} = -\frac{\partial\mathscr{H}}{\partial\phi} = +\eta\sin I\sin\theta\sin\phi. \tag{3}$$

This produces the same trajectories as the Cartesian EOM, so this is correct. However, since $\frac{\partial\phi}{\partial t} \propto 1/\sin\theta$, this is not a desirable system of equations to use, as they are very stiff near $\theta \approx 0$.

We can add a tidal dissipation term; this is just $\left(\frac{\mathrm{d}\hat{s}}{\mathrm{d}t}\right)_{tide} = \epsilon\hat{s}\times\left(\hat{l}\times\hat{s}\right)$.

We can check the stability of each of the equilibria of the EOM, there should be four corresponding to the four Cassini states: