# Notes

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Date

I'm trying a new format for research notes. I'm just going to go chronologically, and never delete anything. So there will probably be a lot of unclear/wrong stuff at the beginning, but that's the case with all my research notes anyway.

## Contents

# 1 09/1/20—The Kuramoto Model

This is mostly review of the Kuramoto model as we learned in class in MATH 6270, but where I actually try to do the algebra.

## 1.1 Basic, Nonlinear Theory

In the Kuramoto model, we consider $N$ phase-coupled oscillators

$$\begin{aligned}\dot{\theta}_i &= \omega_i + \sum_{j\neq i}^{N} \frac{K}{N}\sin\left(\theta_j - \theta_i\right), &(1)\\ &= \omega_i + \mathrm{Re}\left[\frac{K}{N}\sum_{j\neq i}^{N}\frac{1}{i}e^{i(\theta_j - \theta_i)}\right]. &(2)\end{aligned}$$

For a bit of an interlude, we define the complex variable $x_i = e^{i\theta_i}$ and obtain

$$\begin{aligned}\dot{x}_i &= ie^{i\theta_i}\frac{\mathrm{d}\theta_i}{\mathrm{d}t}, &(3)\\ &= ix_i\,\mathrm{Re}\left[\omega_i - \frac{iK}{N}\sum_{j\neq i}^{N} x_j x_i^*\right], &(4)\\ &= ix_i\,\mathrm{Re}\left[x_i^*\left(\omega_i x_i - \frac{iK}{N}\sum_{j\neq i}^{N} x_j\right)\right]. &(5)\end{aligned}$$

We have intentionally rearranged the terms a little bit for a bit more insight. If the Re is dropped above, the EOM becomes completely linear, of form $\dot{x} = Ax$. However, as it stands, $\dot{x}_i$ is always $\pi/2$ out of phase with $x_i$, so the magnitude of the $x_i$ do not change. So the Kuramoto model can be thought of as $N$ linearly coupled complex variables whose magnitudes are then constrained to be fixed; that's the origin of the nonlinearity.

For now, let's study the basic $\theta_i$ variable. Consider mean field variables $re^{i\psi} = \langle e^{i\theta}\rangle$, then

$$\dot{\theta}_i = \omega_i + \mathrm{Im}\left[Ke^{-i\theta_i}re^{i\psi}\right]. \tag{6}$$

We seek a steady state solution. Assume then $r$ is constant and $\psi = \Omega t$, then we can always choose a corotating frame of reference that $\psi = 0$ (define $\theta_i + \Omega t \equiv \theta_i'$ and $\omega_i + \Omega \equiv \omega_i'$ and drop the primes), which gives

$$\dot{\theta}_i = \omega_i - Kr\sin\theta_i. \tag{7}$$

This is the EOM Kuramoto analyzed.

Now, for a given $\theta_i$, if $|\omega_i| > Kr$ then it will have no equilibria, while if $|\omega_i| < Kr$ then there will be fixed points where $\omega_i - Kr\sin\theta_i = 0$; we call these *drifters* and *locked* oscillators respectively. Can we find a self-consistent solution for $r$ now? We assume the $\omega_i$ are drawn from some symmetric distribution $g(\omega) = g(-\omega)$. Then the problem is symmetric for $\theta \leftrightarrow -\theta$, so $\mathrm{Im}\left[\langle e^{i\theta_i}\rangle\right] = 0$ (this is expected/necessary, since we are in the corotating frame where $\psi = 0$). For the real parts:

- For the drifting oscillators, they spend more time in some parts of the unit circle than others, so we think to measure their contribution in a time-averaged sense. This isn't strictly valid (time averages and ensemble averages are only equal in the ergodic limit), so an alternative way of stating this is that the drifting oscillators form a stationary distribution over the unit circle.

  In this approximation, the distribution $\rho(\theta, \omega) \propto 1/\left|\dot\theta_i\right|$, and is given w/ normalization (my notes)
$$\rho\left(\theta, \omega\right) = \frac{1}{2\pi}\frac{\sqrt{\omega^2 - (Kr)^2}}{\left|\omega - Kr\sin\theta\right|}. \tag{8}$$
  Then, $\int\limits_0^{2\pi} \cos\theta\rho(\theta,\omega)\,\mathrm{d}\theta = 0$ because every contribution at $\theta$ gets cancelled by the contribution at $\theta + \pi$.

- For the locked oscillators, they are locked where $\sin\theta_i = \omega_i/Kr$, so the distribution
$$f(\theta) = g(\omega)\frac{\mathrm{d}\omega}{\mathrm{d}\theta} = g\left(Kr\sin\theta\right)Kr\cos\theta. \tag{9}$$

The value of $r$ is then just the integral over the real part of the locked oscillators
$$r = \int\limits_{-Kr}^{Kr} g(\omega)\cos(\theta)\,\mathrm{d}\omega, \tag{10}$$
$$= \int\limits_{-\pi/2}^{\pi/2} Kr\cos^2\theta g\left(Kr\sin\theta\right)\,\mathrm{d}\theta. \tag{11}$$

Of course $r = 0$ is a trivial solution; we want to know under what conditions a solution appears for $r > 0$, so we assume $r$ is small. Then the simplest thing to do is $g\left(Kr\sin\theta\right) \approx g(0)$, and we obtain
$$1 \approx \int\limits_{-\pi/2}^{\pi/2} K\cos^2\theta g(0)\,\mathrm{d}\theta, \tag{12}$$
$$= \frac{\pi K g(0)}{2}. \tag{13}$$

Thus, when $K = 2/(\pi g(0))$, we have a small, positive solution for $r$.

In fact, $K_c \equiv 2/(\pi g(0))$ is the onset of collective synchronization, i.e. for any $K > K_c$, there are solutions $r > 0$. We can't really see this in our calculation above though, so we expand Eq. (11) to quadratic order in $r$ (the linear term vanishes since $g$ is even)

$$1 = \int_{-\pi/2}^{\pi/2} K \cos^2\theta \left(g(0) + \frac{g''(0)}{2}\left(Kr\sin\theta\right)^2\right)\,\mathrm{d}\theta, \tag{14}$$

$$= \frac{\pi K g(0)}{2} + \frac{g''(0)}{2}K^3 r^2 \int_{-\pi/2}^{\pi/2} \cos^2\theta \sin^2\theta\,\mathrm{d}\theta, \tag{15}$$

$$= \frac{\pi K g(0)}{2} + \frac{g''(0)}{8}K^3 r^2 \int_{-\pi/2}^{\pi/2} \sin^2\left(2\theta\right)\,\mathrm{d}\theta, \tag{16}$$

$$= \frac{\pi K g(0)}{2} + \frac{g''(0)}{8}K^3 r^2 \int_{-\pi/2}^{\pi/2} \frac{1 + \cos\left(2\theta\right)}{2}\,\mathrm{d}\theta, \tag{17}$$

$$= \frac{\pi K g(0)}{2} + \pi\frac{g''(0)}{16}K^3 r^2, \tag{18}$$

$$r^2 = \frac{16 - 8\pi K g(0)}{\pi K^3 g''(0)}, \tag{19}$$

$$= -\frac{8 g(0)}{K^2 g''(0)}\frac{K - K_c}{K}. \tag{20}$$

The dimensions check out, as $g'' K^2$ is dimensionless. Then if $g'' < 0$, the case for most distributions, we see that solutions for $r$ only exist for $K > K_c$, as expected. This is a supercritical bifurcation.

In conclusion, for the standard $g(\omega)$, a synchronized solution $r > 0$ spontaneously appears for $K > K_c$. The stability analysis is not too hard for $r = 0$ and is very hard for $r > 0$, so I'll push off on it for the time being, but the result is that $r = 0$ is linearly neutrally stable for $K < K_c$, unstable for $K > K_c$, while $r > 0$ is locally stable.

# 2 09/21/20—Laplace Lagrange Secular Perturbation Theory

In this section, we learn the derivation of the Laplace-Lagrange results that couple the complex eccentricity and inclination vectors (see e.g. Pu & Lai *Eccentricities and inclinations of multiplanet systems with external perturbers*, though we will follow Murray & Dermott).

## 2.1 Leading Order Laplace-Lagrange Solution

The idea behind Lagrange's planetary equations (M&D §6.8) is to rewrite the potential in terms of the orbital elements (M&D use $R = -\Phi$ the *disturbing function*), then the EOM are related to derivatives of $\Phi$. If $\Phi$ does not explicitly depend on $\lambda$ the mean longitude, then $a$ is constant as well, and the EOM for the remaining for orbital elements are

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial\Phi}{\partial\varpi} \approx \frac{1}{na^2 e}\frac{\partial\Phi}{\partial\varpi}, \tag{21a}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = -\frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial\Phi}{\partial I} \approx -\frac{1}{na^2 I}\frac{\partial\Phi}{\partial I}, \tag{21b}$$

$$\frac{\mathrm{d}\varpi}{\mathrm{d}t} = -\frac{\sqrt{1-e^2}}{na^2 e}\frac{\partial\Phi}{\partial e} - \frac{\tan(I/2)}{na^2\sqrt{1-e^2}}\frac{\partial\Phi}{\partial I}, \approx -\frac{1}{na^2 e}\frac{\partial\Phi}{\partial e} \tag{21c}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{\tan(I/2)}{na^2\sqrt{1-e^2}}\frac{\partial\Phi}{\partial\varpi} + \frac{1}{na^2\sqrt{1-e^2}\sin I}\frac{\partial\Phi}{\partial\Omega} \approx \frac{1}{na^2 I}\frac{\partial\Phi}{\partial\Omega}. \tag{21d}$$

We include the leading order approximations for small $e, I$ as well. Recall $n = \sqrt{G(M_\star + m)/a^3}$ is the mean motion.

We then grab the disturbing function between two planets from M&D §7.2. We assume $GM_\star \approx n_i^2 a_i^3$ for the $i$th planet, and expand to second order in eccentricities and inclinations and to leading order in the masses, then apply Lagrange's planetary equations. It's convenient to define the components of the eccentricity and inclination vectors

$$h_j = e_j \sin\varpi_j \qquad k_j = e_j\cos\varpi_j, \tag{22}$$

$$p_j = I_j \sin\Omega_j \qquad q_j = I_j\cos\Omega_j. \tag{23}$$

The equations for the evolution of these components is

$$\dot{h}_i = \sum_j A_{ij}k_j \qquad \dot{k}_i = \sum_j -A_{ij}h_j, \tag{24}$$

$$\dot{p}_i = \sum_j B_{ij}q_j \qquad \dot{q}_i = \sum_j -B_{ij}p_j, \tag{25}$$

where

$$A_{ii} = +n_i\sum_{j\neq i}\frac{m_j}{M_\star + m_i}\frac{a_>}{a_<}b_{3/2}^{(1)}\left(\alpha_{ij}\right), \qquad A_{ij} = -n_i\frac{m_j}{M_\star + m_i}\frac{a_>}{a_<}b_{3/2}^{(2)}\left(\alpha_{ij}\right), \tag{26}$$

$$B_{ii} = -n_i\sum_{j\neq i}\frac{m_j}{M_\star + m_i}\frac{a_>}{a_<}b_{3/2}^{(1)}\left(\alpha_{ij}\right), \qquad B_{ij} = +n_i\frac{m_j}{M_\star + m_i}\frac{a_>}{a_<}b_{3/2}^{(1)}\left(\alpha_{ij}\right), \tag{27}$$

where $\alpha_{ij} = a_</a_>$, and we follow PL18's convention for the Laplace coefficients. A useful approximation for $\alpha \ll 1$ is

$$b_{3/2}^{(n)}(\alpha) \equiv \frac{1}{2\pi}\int\limits_0^\pi \frac{\cos(nt)}{(\alpha^2+1-2\alpha\cos t)^{3/2}}\,\mathrm{d}t, \tag{28}$$

$$b_{3/2}^{(1)}(\alpha) \approx \frac{3\alpha}{4} + \frac{43\alpha^3}{32} + \frac{525\alpha^5}{256} + \dots, \tag{29}$$

$$b_{3/2}^{(2)}(\alpha) \approx \frac{15\alpha^2}{16} + \frac{105\alpha^4}{64}, \tag{30}$$

while when $\alpha \to 1$ the Laplace coefficients diverge such that $b_{3/2}^{(2)}/b_{3/2}^{(1)} \to 1$. To be more precise, we note from M&D that (using our convention for the factor of 4)

$$b_s^{(j)} = 2\binom{s+j-1}{j}\alpha^j F\left(s, s+j, j+1; \alpha^2\right), \tag{31}$$

where $F$ is the standard hypergeometric function. According to Wikipedia (and Abramowitz & Stegun apparently), the singular solution as $\alpha^2 \to 1$ goes like $\left(1-\alpha^2\right)^{(j+1)-s-(s+j)} = \left(1-\alpha^2\right)^{1-2s}$, which is borne out by my numerical check, so all of the $b_{3/2}^{(n)}$ grow like $(1-\alpha)^{-2}$.

A very easy choice is to define the complex eccentricies and inclinations such that $\mathcal{E}_i = e\exp(i\varpi)$ and $\mathcal{I} = I\exp(i\Omega)$, so that

$$\dot{\mathcal{E}} = i\mathbf{A}\mathcal{E}, \qquad\qquad \dot{\mathcal{I}} = i\mathbf{B}\mathcal{I}. \tag{32}$$

This is the general form of PL18, §2.3 Eq. (23), except they choose slightly more symmetric form for the matrix elements

$$A_{ii} = \sum_{j\neq i}\frac{Gm_im_ja_<}{a_>^2L_i}b_{3/2}^{(1)}(\alpha_{ij}), \qquad A_{ij} = -\frac{Gm_im_ja_<}{a_>^2L_i}b_{3/2}^{(2)}(\alpha_{ij}), \tag{33}$$

$$B_{ii} = -\sum_{j\neq i}\frac{Gm_im_ja_<}{a_>^2L_i}b_{3/2}^{(1)}(\alpha_{ij}), \qquad B_{ij} = +\frac{Gm_im_ja_<}{a_>^2L_i}b_{3/2}^{(1)}(\alpha_{ij}), \tag{34}$$

where $L_i \approx m_i\sqrt{GM_\star a_i}$ is the angular momentum. It bears noting that $A_{ij}L_i = A_{ji}L_j$ and similarly for $B_{ij}$.

We can write down the general solution in terms of the eigenvectors $v_n$ and eigenvalues $\lambda_n$ of $\mathbf{A}$, given as follows (unfortunately, $\mathbf{A}$ is not symmetric, so we don't have many guarantees on $v_n$ and $\lambda_n$):

$$\mathcal{E}(t) = \sum_n A_n v_n \exp\left(i\lambda_n t\right), \tag{35}$$

where the coefficients $A_n$ are obtained by matching to initial conditions. Of course, we can do similarly for $\mathcal{I}$. Note that the eigenvectors $v_n$ are real, so the $A_n$ are real as well. This solution corresponds to the free eccentricity that is sometimes talked about in M&D.

What are we interested in? We wonder whether there are are any situations in which the eccentricity vectors spontaneously align, which means clustering in $\varpi$. To linear order, it is obvious this is not possible: the only way the phases of all the individual components of $\mathcal{E}$ vary in a synchronized fashion is when the ICs are tuned such that the only $A_n$ that are nonzero are those that have identical $\lambda_n$.

### 2.2 Disk Expansion?

What if we try naively to take the limit for a swarm of particles in a disk that are initially coplanar? Assume a disk over some small extent $a \in [1 - \Delta a/2, 1 + \Delta a/2]$ where $\Delta a \ll 1$, with total mass $M_d$. Partition it into rings of width $\delta a$ such that, for simplicity, each ring has mass $\sim M_d \frac{\Delta a}{\Delta a}$. Furthermore, the smallest $\alpha_{ij}$ has $1 - \alpha_{ij} \approx \delta a$, so $1 - \alpha_{ij}^2 \approx 2\delta a$. Then, $b_{3/2}^{(n)}(\alpha_{ij}) \propto (\delta a)^{-2}$, and the matrix elements $A_{ij}, B_{ij}$ still diverge like $(\delta a)^{-1}$ due to the $L_i^{-1}$ term.

### 2.3 Connection to Kuramoto

What sorts of nonlinear restrictions would give a possibility of a Kuramoto-like spontaneous synchronization? Recall our complexified Kuramoto model

$$\dot{x}_i = i x_i \operatorname{Re}\left[x_i^*\left(\omega_i x_i - \frac{iK}{N}\sum_{j\neq i}^{N} x_j\right)\right]. \tag{36}$$

Note that if we ignore taking the real part above, we obtain something that looks like $\dot{x}_i = i\mathbf{M}x_i$, though $\mathbf{M}$ is complex and symmetric. The effect of taking the real part is to project $\dot{x}_i$ to be out of phase with $x_i$, such that the magnitude of $x_i$ never changes. We can imagine that such a similar restriction to the Laplace-Lagrange solution can cause synchronization, perhaps effected via the higher-order terms neglected in this solution.

## 3 09/22/20—More Test Applications of LL

### 3.1 Secular Disk

The problem with our disk expansion above is obvious in hindsight: the secular approximation breaks down if the rates of change $\gg n$. In practice, the secular approximation will only work if $n$ grows in a way that the secular timescale is constant, i.e. the matricies $\mathbf{A}, \mathbf{B}$ are everywhere finite. This is obviously most strict on the diagonal terms, where there are $N = \Delta a/\delta a$ terms in the summand while each term diverges like $(\delta a)^{-1}$ as well. To keep the diagonal terms finite then, we need $M_\star \propto N^4$. More precisely, choose all of the $m_i \approx M/N$, write $a_< \approx a_> \approx a_0$ for simplicity, $L_i \approx M/N\sqrt{G\left(M_\star N^4\right)a_0}$, then we obtain (we really should have used the original forms instead of the PL18 ones, they work better)

$$A_{ij} \approx -B_{ij} \propto -\frac{\sqrt{G}\left(M/N\right)\left(1 - \frac{|j-i|}{N}\frac{\Delta a}{a_0}\right)}{N^2\sqrt{M_\star a_0^{3/2}}}\frac{1}{N^2}, \tag{37}$$

$$\propto -\frac{nM/M_\star}{N}\left(1 - \frac{|j-i|}{N}\frac{\Delta a}{a_0}\right), \tag{38}$$

$$A_{ii} \approx -B_{ii} \propto \frac{\sqrt{G}\left(M/N\right)}{N^2\sqrt{M_\star a_0^{3/2}}}\frac{1}{N^2}\sum_{j\neq i}\left(1 - \frac{|j-i|}{N}\frac{\Delta a}{a_0}\right), \tag{39}$$

$$\propto n\left(M/M_\star\right)\left(1 - \frac{i^2 + (N-i)^2}{N^2}\frac{\Delta a}{a_0}\right). \tag{40}$$

The basic structure is a matrix whose diagonal elements have a slight dip in the middle, and whose off-diagonal elements decrease going away from the diagonal but depend only on $|j-i|$. However, they don't fall off very quickly, only linearly. Nevertheless, it is now possible, in theory, to get a solution for all of the rings.

We note however a substantial discrepancy between this coupling and the Kuramoto coupling: the off diagonal terms are *real*, while in the Kuramoto model they are imaginary. This comes because the oscillators couple via a sine in the Kuramoto model, and via a cosine in the LL secular solution, as $\dot{\varpi} \propto \partial\Phi/\partial e$ where for two planets [M&D Eq. (7.6)]

$$\Phi_1 = -\frac{n_1^2 a_1^2 m_2}{M_\star + m_1}\left[\frac{\alpha_{12}^2 b_{3/2}^{(1)} e_1^2}{8} - \frac{\alpha_{12}^2 b_{3/2}^{(1)} I_1^2}{8} - \frac{\alpha_{12}^2 b_{3/2}^{(2)} e_1 e_2}{4}\cos\left(\varpi_1 - \varpi_2\right) + \frac{\alpha_{12}^2 b_{3/2}^{(2)} I_1 I_2}{4}\cos\left(\Omega_1 - \Omega_2\right)\right]. \tag{41}$$

It seems like this may be a dead end then.