Research Notes

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Chapter 1

Preliminary Problems

To get an intuition for how Dedalus and fluid mechanics works, we will solve some toy problems. Recall fluid equations in the presence of a uniform gravitational field $\vec{g} = -g\hat{z}$:

$$\begin{aligned} \frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{u}\right) &= 0, \\ \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \frac{\vec{\nabla}P}{\rho} - \vec{g} &= 0. \end{aligned} \tag{1.1}$$

In the incompressible limit, $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$, which implies $\vec{\nabla}\cdot\vec{u} = 0$. We use subscripts to indicate perturbed quantities, Q_0 is background and Q_1 is perturbed. We will generally use $\vec{u}_0 = 0$ unless otherwise noted. We will also generally assume symmetry along all axes except z the vertical axis.

In the incompressible limit, the fluid equations become

$$\begin{aligned} \vec{\nabla}\cdot\vec{u}_1 &= 0, \\ \frac{\partial\rho_1}{\partial t} + u_{1z}\frac{\partial\rho_0}{\partial z} &= 0, \\ \frac{\partial\vec{u}_1}{\partial t} + \frac{1}{\rho_0}\vec{\nabla}P_1 + \frac{\rho_1 g\hat{z}}{\rho_0} &= 0. \end{aligned} \tag{1.2}$$

We have used $\vec{\nabla}P_0 = -\rho_0 g\hat{z}$ in the absence of perturbations.

1.1 Incompressible, No Gravity

We note that in the no gravity limit that ρ_1 does not have an effect on other dynamical variables, so the equations of motion we must solve are

$$\begin{aligned} \vec{\nabla}\cdot\vec{u}_1 &= 0, \\ \frac{\partial\vec{u}_1}{\partial t} + \frac{\vec{\nabla}P_1}{\rho_0} &= 0. \end{aligned} \tag{1.3}$$

We can take the divergence of the momentum equation and substitute the continuity equation to get $\nabla^2 P = 0$.

1.1.1 Dirichlet BCs

This is a Laplace equation, which we've solved countless times. Imposing periodic boundary conditions in the x direction and $P_1(z=L)=0, P_1(z=0)=\mathscr{P}(x,t)$, we obtain eigenfunctions

$$\begin{aligned} P_{1,n}(x,z,t) &= \frac{\mathscr{P}_n(t)}{\sinh(k_n L)} e^{ik_n x} \sinh(k_n(L-z)), \\ u_{1x,n}(x,z,t) &= \int^t -\frac{1}{\rho_0}\frac{\partial P_{1,n}}{\partial x}\,\mathrm{d}t, \\ u_{1z,n}(x,z,t) &= \int^t -\frac{1}{\rho_0}\frac{\partial P_{1,n}}{\partial z}\,\mathrm{d}t. \end{aligned} \tag{1.4}$$

We define $k_n = \frac{2\pi n}{L}, n \geq 0$ and $\mathscr{P}(x,t) = \sum\limits_n \mathscr{P}_n(t) e^{ik_n x}$.

Thus, if we impose BCs $\mathscr{P}(x,t) = \sin\frac{2\pi x}{L}$ and start with initial conditions such that all quantities are zero, we would expect after transients die out that

$$\begin{aligned} P(x,z,t) &= \frac{\sin\frac{2\pi x}{L}}{\sinh 2\pi}\sinh\left(2\pi\frac{L-z}{L}\right), \\ u_{1x}(x,z,t) &= -\frac{2\pi t}{L\rho_0}\frac{\cos\frac{2\pi x}{L}}{\sinh 2\pi}\sinh\left(2\pi\frac{L-z}{L}\right), \\ u_{1z}(x,z,t) &= +\frac{2\pi t}{L\rho_0}\frac{\sin\frac{2\pi z}{L}}{\sinh 2\pi}\cosh\left(2\pi\frac{L-z}{L}\right). \end{aligned} \tag{1.5}$$

This is in good agreement with the results, presented in Fig. 1.1. Note that P is constant while $\vec{u}$ increases linearly in time, and we observe the expected $\sim \sin x \sinh\frac{L-z}{z}$ dependence. In fact, u_{1x}, u_{1z} are exactly $\frac{2\pi}{10}$ at $t=1$.

1.1.2 Things to Note

It is worth noting that, since our Eq. 1.3 reduced to a Laplace equation, we needed two z BCs and two x BCs (periodic BCs amount to equating the value and derivative of the function). This is in agreement with the observation that the original Eq. 1.3 had two derivatives in x,z apiece, so we needed two BCs each.

It is also worth seeing from our solution that P immediately goes to the equilibrium solution. This is not surprising since in the incompressible limit, sound speed goes to infinity which is the timescale on which the pressure field adjusts to net forces. Thus, the dynamics solely arise from the static pressure field pushing the velocity field to equilibrium.

We chose time-independent $\mathscr{P}(x,t)$, but it is clear that whatever $\mathscr{P}(x,t)$ we choose, the time dependence propagates to the velocities by way of an integral. If we had instead chosen to take Fourier

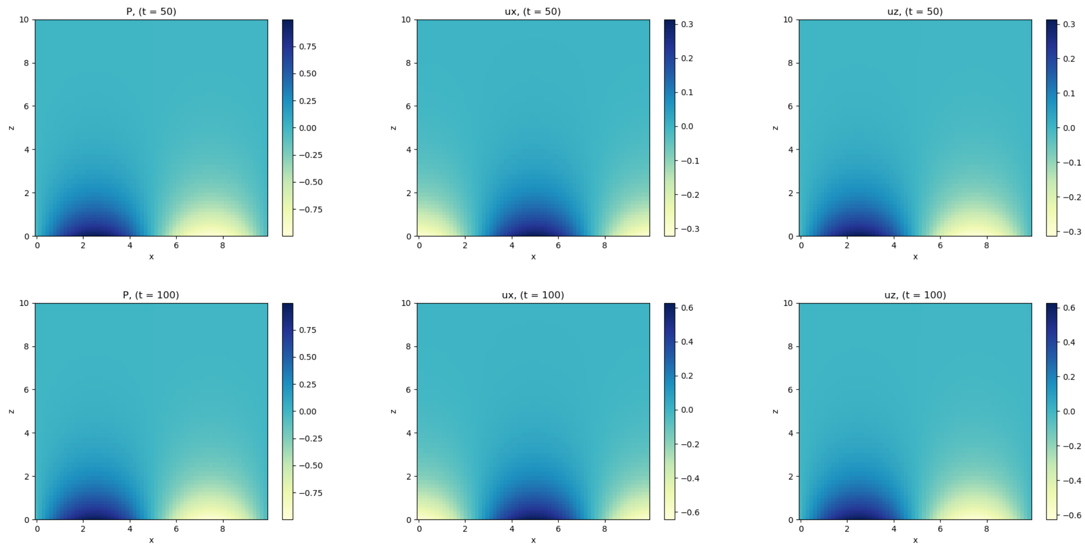

Figure 1.1: P, u_x, u_z at $t = 0.5$ and $t = 1$ for $\rho_0 = 1$. We choose a $L = 10$ square domain.

transform $t \to \omega$, we would have had to integrate the boundary condition against the eigenfunctions for each of the ω, which is still easily computable, to get the full $u_{1x}(x,z,t)$. We consider this in preparation for when we can only solve for a set of $\vec{k}, \omega$ in the next problem.

The next thing we would have wanted to do is solve a problem with radiative BCs, but we need to have wave solutions, which are missing in the absence of gravity. We thus move on to the next configuration.

1.2 Incompressible, Stratified w/ Gravity

1.2.1 Eigenfunctions

Let's restore the $\rho_1 g$ term now. For funsies, we begin by solving for arbitrary stratification $\rho_0(z)$ first. The fluid equations to first order reduce to

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \vec{u}_1 \cdot \left(\vec{\nabla}\rho_0\right) &= 0, \\ \vec{\nabla} \cdot \vec{u}_1 &= 0, \\ \frac{\partial \vec{u}_1}{\partial t} &= -\frac{\vec{\nabla} P_1}{\rho_0} - \frac{\rho_1 g}{\rho_0} \end{aligned} \tag{1.6}$$

We expect there to be some z dependence in the amplitude, so we substitute variables of form $e^{i(kx-\omega t)}$ and do not specify the z dependence. This gives us

$$\begin{aligned}
-i\omega\rho_1 - u_{1z}\frac{\partial\rho_0}{\partial z} &= 0,\\
iku_{1x} + \frac{\partial u_{1z}}{\partial z} &= 0,\\
-iwu_{1x} + \frac{ik_x P_1}{\rho_0} &= 0,\\
-iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1 g}{\rho_g} &= 0.
\end{aligned} \tag{1.7}$$

We substitute $N^2 = -\frac{g}{\rho_0}\frac{\partial\rho_0}{\partial z}$ to obtain

$$-i\omega\rho_1 - u_{1z}\frac{\rho_0 N^2}{g} = 0, \tag{1.8a}$$

$$iku_{1x} + \frac{\partial u_{1z}}{\partial z} = 0, \tag{1.8b}$$

$$-iwu_{1x} + \frac{ik_x P_1}{\rho_0} = 0, \tag{1.8c}$$

$$-iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1 g}{\rho_0} = 0. \tag{1.8d}$$

Eliminating u_{1x} by substituting (1.8b) into (1.8c) and ρ_1 by substituting (1.8a) into (1.8d) give

$$i\omega\frac{\partial u_{1z}}{\partial z} + \frac{k_x^2 P_1}{\rho_0} = 0, \tag{1.9a}$$

$$\left(\omega^2 - N^2\right)u_{1z} + \frac{i\omega}{\rho_0}\frac{\partial P_1}{\partial z} = 0. \tag{1.9b}$$

Finally, we multiply (1.9a) with ρ_0 and differentiate dz and combine with (1.9b) to give

$$\frac{\mathrm{d}^2 u_{1z}}{\mathrm{d}z^2} + \frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z}\frac{\partial u_{1z}}{\partial z} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right)u_{1z} = 0. \tag{1.10}$$

Let's now pick stratification $\rho \propto e^{-z/H}$ Eq. 1.10 clearly has exponential solutions $e^{\kappa z}$ for

$$\kappa^2 - \frac{\kappa}{H} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right) = 0. \tag{1.11}$$

We permit complex $\kappa = \frac{1}{2H} + ik_z$, and from the above clearly

$$\begin{aligned}
k_z^2 &= -\frac{1}{4H^2} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right),\\
\omega^2 &= \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}.
\end{aligned} \tag{1.12}$$

Thus the eigenfunctions are

$$\begin{aligned}
u_{1z} &= e^{z/2H} e^{i(k_z z + k_x x - \omega t)},\\
u_{1x} &= -\frac{k_z + i/2H}{k_x} u_{1z},\\
\rho_1 &= \frac{i\rho_0}{H\omega} u_{1z},\\
P_1 &= -\frac{\rho_0 \omega}{k_x^2}(k_z + i/2H) u_{1z}.
\end{aligned} \tag{1.13}$$

1.2.2 Analytically Solving an IVP, Dirichlet + Driving BCs

We will analyze everything in terms of u_{1z} since it has the simplest form; note that when actually choosing the BCs we will have to consider the gauge freedom of P and some considerations we defer to the computational section.

Currently, we have a set of eigenfunctions

$$u_{1z}\left(x,z,t\middle|\vec{k},\omega\right) = e^{z/2H} e^{i(k_z z + k_x x - \omega t)}. \tag{1.14}$$

Note that u_{1z} is really only a function of two parameters rather than the three (k_x, k_z, ω), since the three are related by dispersion relation Eq. 1.12.

Now, we implement BCs. Consider domain $x, z \in [0, L]$. We will use periodic BCs again in x, so then $k_{x,n} = \frac{2\pi n}{L}, n \ge 0$. Then, we will require $u_{1z,n}(x, L, t) = 0$, a Dirichlet condition at the top boundary, which restricts us to eigenfunctions of form

$$u_{1z,n}\left(x,z,t\middle|\vec{k},\omega\right) = e^{z/2H} e^{i(k_x x - \omega t)} \sin(k_z(L - z)). \tag{1.15}$$

Finally, we must choose a BC at $z = 0$. We will choose a general function $u_{1z}(x, 0, t) = F(x, t)$ where we can decompose

$$F(x,t) = \int \sum_n \mathscr{F}(k_{x,n}, \omega) e^{i(k_{x,n} x - \omega t)}\,\mathrm{d}\omega. \tag{1.16}$$

Matching BCs then gives us general solution for u_{1z} given an arbitrary driving function

$$u_{1z}\left(x,z,t\middle|\vec{k},\omega\right) = \int \sum_n \mathscr{F}\left(k_{x,n}, \omega\right) \frac{e^{z/2H} e^{i(k_{x,n} x - \omega t)} \sin(k_z(L - z))}{\sin k_z L}\,\mathrm{d}\omega. \tag{1.17}$$

For ease of computation, let's pick $F(x,t) = \cos\left(\frac{2\pi x}{L} - \omega_0 t\right)$, so our full expected solution is (note

$A + \epsilon = Ae^{\epsilon/A} + \mathcal{O}(\epsilon^2))$

$$\begin{aligned}
u_{1z}\left(x,z,t\middle|\vec{k},\omega\right) &= e^{z/2H}\frac{\cos\left(\frac{2\pi x}{L}-\omega_0 t\right)\sin(k_z(L-z))}{\sin k_z L},\\
u_{1x}\left(x,z,t\middle|\vec{k},\omega\right) &\approx \frac{k_z}{k_x}e^{z/2H}\frac{\cos\left(\frac{2\pi x}{L}-\omega_0 t+\frac{1}{2Hk_z}\right)\sin(k_z(L-z))}{\sin k_z L},\\
\rho_1\left(x,z,t\middle|\vec{k},\omega\right) &\approx \frac{\rho_0}{H\omega}e^{z/2H}\frac{-\sin\left(\frac{2\pi x}{L}-\omega_0 t\right)\sin(k_z(L-z))}{\sin k_z L},\\
P_1\left(x,z,t\middle|\vec{k},\omega\right) &\approx -\frac{\rho_0\omega k_z}{k_x^2}e^{z/2H}\frac{\cos\left(\frac{2\pi x}{L}-\omega_0 t+\frac{1}{2Hk_z}\right)\sin(k_z(L-z))}{\sin k_z L},
\end{aligned} \tag{1.18}$$

where $k_z : \omega(k_x, k_z) = \omega_0$ by the dispersion relation Eq. 1.12. Note that H contributes both to the overall exponential profile and to the phase lag of u_{1x}.

1.2.3 Computationally Solving an IVP, Dirichlet + Driving BCs

To solve this computationally with the aforementioned BCs, periodic in x, Dirichlet 0 at $z = L$ and $\cos\left(\frac{2\pi x}{L} - \omega_0 t\right)$ at $z = 0$, we must address the gauge freedom in P. This arises because for $k_x = 0$, the divergence-free condition $\vec{\nabla}\cdot\vec{u}_1 = \frac{\partial u_z}{\partial z} = 0$ specificies u_{1z} up to a constant already, so the bottom BC will fix the value of u_z when $k_x = 0$.

A different way of phrasing the same argument is as follows. Consider the discrete $N \times N$ (square for notational simplicity) grid. At the boundary, there is a list of values $f(\{x_i\}, z_0)$ that lives in an N dimensional space. We can thus pick a spanning set of N basis vectors, and for each of these N vectors, by enforcing $\vec{\nabla}\cdot\vec{u} = 0$ at the boundary we fix the allowed $f(\{x_i\}, z_{-1})$ boundary conditions we can implement. But there exists a choice of basis vectors for which one of the basis vectors is constant $f_i(\{x_i\}, z_0) = C$. For this basis vector, the boundary condition is fully determined, so we only have $N-1$ dimensions from which to choose the BCs $f(\{x_i\}, z_{-1})$. Since the dimensionality of a space cannot depend on the choice of basis vector, the divergence free condition actually yields an extra degree of freedom.

We can use this extra degreee of freedom to specify $P(z = L) = 0$ so the oscillations should have zero mean. Then we can just simulate away!

From here we can just tweak the parameters until we get something useful, can look at the videos. Note that we should simulate until at least $T > \frac{2L_z}{v_{p,z}}$ the z phase space velocity, so we can capture any reflections off the boundary. Turns out both Dirichlet and Neumann BCs give strong reflections that produce standing waves in the z and traveling waves in the x.

1.2.4 Phase/Group Velocity

Let's figure out the analytical forms for the phase, group velocity and the energy density/power flux.

Let's first consider the phase velocity. We traditionally think about the phase velocity in 1D $v_{ph} = \frac{\omega(k)}{k}$, but in general it is the function such that $\vec{k}\cdot\vec{r} - \omega t = \vec{k}\cdot\left(\vec{r} - \vec{v}_{ph} t\right)$ is constant (the phase of the wave). Thus, it must satisfy $\vec{k}\cdot\vec{v}_{ph}=\omega$, and so one sensible choice is

$$\vec{v}_{ph}=\frac{\omega\hat{k}}{\left|\vec{k}\right|}=\frac{\omega\vec{k}}{\left|\vec{k}\right|^2}. \tag{1.19}$$

For our stratified atmosphere problem, $\omega^2=\frac{N^2k_x^2}{k^2+\frac{1}{4H^2}}$. This corresponds to $\vec{v}_{ph}=\frac{Nk_x}{\sqrt{k_x^2+k_z^2+1/4H^2}}\vec{k}$.

On the other hand the group velocity is given $\vec{v}_g=\frac{\partial\omega}{\partial k_i}\hat{\imath}$. In our problem, $v_{g,z}=-\frac{Nk_xk_z}{\left(k_x^2+k_z^2+1/4H^2\right)^{3/2}}$ while $v_{g,x}=\frac{N}{\sqrt{k_x^2+k_z^2+1/4H^2}}-\frac{Nk_x^2}{\left(k_x^2+k_z^2+1/4H^2\right)^{3/2}}=\frac{N(k_z^2+1/4H^2)}{\left(k_x^2+k_z^2+1/4H^2\right)^{3/2}}$.

1.2.5 Energy/Power Flux

To compute the energy and power flux of the wave, we recall that for a general fluid the energy conservation equation reads

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho\epsilon\right)=\vec{\nabla}\cdot\left(\rho\vec{v}\left(v^2+\epsilon+\frac{P}{\rho}\right)\right), \tag{1.20}$$

where ϵ is the internal energy. But since $P=(\gamma-1)\rho\epsilon$ and we take $\gamma\to\infty$ incompressible limit, $\epsilon=0$, so the energy of the wave is just $\frac{1}{2}\left(\rho_0+\rho_1\right)u_1^2$ and the power flux is $\left(\rho_0+\rho_1\right)\vec{u}_1u_1^2+P_1\vec{u}_1$.

Another formula given by Sutherland 2011 is $\frac{\partial\langle E\rangle}{\partial t}=v_{g,z}\langle E\rangle$ where we average over x one wavelength. I wasn't able to show that these two agree in general; Sutherland's formula seems to be derived for a traveling wavepacket whereas we have no such thing. We do not use this expression since the Landau & Lifshitz expression works well.

1.2.6 Supressing Reflection

Sourced from `https://people.maths.ox.ac.uk/trefethen/6all.pdf`. There are largely two ways to do this. The first is to add a damping zone, a region in which $\dot{q}=-q/\tau_d(z)$ where $\tau_d(0)=0$ and increases to some non-small number for some z_0 beyond which we want damping. One form, chosen in Ryan and Dong's paper, is to use multiplicative factor $f(z)=\max\left[0,1-\frac{(z-z_b)^2}{(z_d-z_b)^2}\right]$ where z_b is where one begins supression and z_b is the boundary. Then a dynamical variable q can be supressed via $\dot{q}=(\dots)-f(z)\frac{q-q_0}{\tau}$ where τ is the dynamical timescale on which to damp and q_0 is the value to damp to. We will apply this only to the velocity variables with an eye to extending this approach to the nonlinear regime, where the velocity variables should still be damped to zero but ρ,P must capture the stratification and so cannot easily be artificially damped (maybe? Revisit if this does poorly compared to damping all linear variables, and damp the nonlinear variables to their equilibrium/stratified values).

In order to compute the reflecting boundary condition, we must find a sequence of differential operators that well approxmiates the system of PDEs. We can get this from the dispersion relation; since our radiating boundary is along z, we should try to solve for k_z in the dispersion relation to get a pseudodifferential operator form for $\frac{\partial}{\partial z} u_z$ (we take $k_x H \ll 1$ as well to simplify):

$$\begin{aligned} k_z^2 &= \frac{N^2}{\omega^2} k_x^2 - k_x^2 - \frac{1}{4H^2}, \\ &\approx k_x^2 \left(\frac{N^2}{\omega^2} - 1 \right), \end{aligned} \tag{1.21}$$

$$\frac{\partial u_z}{\partial z} \approx -\frac{\partial u_z}{\partial x} \sqrt{\frac{N^2}{\omega^2} - 1}. \tag{1.22}$$

We pick the negative sign in accordance with an outgoing wave, per the group velocity formulae for $k_z > k_x$.

From the numerical simulations, it is clear that reflections are well supressed initially, but the the simulation blows up. This is because our supression is imperfect (differential approximation to a pseudodifferential operator) and since our system is dissipation free, reflection grows. Moreover, since any reflected components seem to have different k_x, k_z, ω, the second time they're incident on the boundary, our above condition becomes wildly inaccurate.

1.3 Incompressible, Stratified w/ Gravity, Nonlinear

We consider the problem where $\rho_1 \sim \rho_0$, where we cannot make such a perturbative expansion. We write down thus nonlinear fluid equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{u} \cdot \left(\vec{\nabla} \rho \right) &= 0, \\ \vec{\nabla} \cdot \vec{u} &= 0, \\ \frac{\partial \vec{u}}{\partial t} &= -\frac{\vec{\nabla} P}{\rho} - g\hat{z}. \end{aligned} \tag{1.23}$$

Note that we cannot even subtract off hydrostatic equilibrium anymore since ρ can deviate greatly from the normal $\rho_0 e^{-z/H}$!

Since Dedalus cannot handle a field quantity in the denominator, we must get rid of ρ somehow. We choose to do this by recasting the equations in terms of ρ^{-1}, which is everywhere nonzero and finite since ρ is everywhere nonzero and finite. But then $\frac{\partial \rho}{\partial t} = -\rho^2 \frac{\partial \rho^{-1}}{\partial t}$ while $\vec{\nabla} \rho = -\rho^2 \vec{\nabla} \rho^{-1}$, and so factoring out the ρ^2 we can simply rewrite

$$\begin{aligned} \frac{\partial \rho^{-1}}{\partial t} + \vec{u} \cdot \left(\vec{\nabla} \rho^{-1} \right) &= 0, \\ \vec{\nabla} \cdot \vec{u} &= 0, \\ \frac{\partial \vec{u}}{\partial t} &= -\rho^{-1} \vec{\nabla} P - g\hat{z}. \end{aligned} \tag{1.24}$$

We must also consider how we implement the outgoing boundary condition in this regime. If we use the radiative boundary condition formula, thankfully we have little to change, though it also doesn't do a great job of supressing reflections. If we instead use a sponge zone, we should be careful about how we damp ρ^{-1}, since we obviously should not try damping it to zero! This is why we initially chose to only apply our sponge zone to the velocity variables, even in the linear problem.

Appendix A

Deriving Fluids Results

A.1 Equation of Energy Conservation

We follow Landau & Lifshitz's derivation for this expression. Consider that the total energy stored in a wave must be the sum of its kinetic and internal energy $\frac{1}{2}\rho v^2 + \rho\epsilon$ where ϵ is the internal energy density. To obtain an equation of energy conservation, we must take the time derivative of this expression. First, we consider

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) &= \frac{1}{2}v^2\frac{\partial\rho}{\partial t} + \rho\vec{v}\cdot\frac{\partial\vec{v}}{\partial t}, \\
&= -\frac{1}{2}v^2\left(\vec{\nabla}\cdot\rho\vec{v}\right) - \rho\vec{v}\cdot\left(\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} + \frac{\vec{\nabla}P}{\rho} - \vec{g}\right)_{T,V}, \\
&= -\frac{1}{2}v^2\left(\vec{\nabla}\cdot\rho\vec{v}\right) - \frac{1}{2}\rho\vec{v}\cdot\left(\vec{\nabla}v^2\right) - \vec{v}\cdot\left(\rho\vec{\nabla}w - \rho T\vec{\nabla}s\right) + \rho\vec{v}\cdot\vec{g}.
\end{aligned} \tag{A.1}$$

We have denoted s the specific internal entropy density of the fluid and $\mathrm{d}w = T\mathrm{d}s + \frac{\mathrm{d}P}{\rho} = T\mathrm{d}s + \epsilon$ the specific internal enthalpy density of the fluid. Recall enthalpy $\epsilon = w - Ts$ is the usual thermodynamic definition.

At the same time, consider

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho\epsilon) &= \epsilon\frac{\partial\rho}{\partial t} + \rho\frac{\partial}{\partial t}\left(Ts - \frac{P}{\rho}\right)_{T,P}, \\
&= \epsilon\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t} + \frac{P}{\rho}\frac{\partial\rho}{\partial t}, \\
&= w\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t}, \\
&= -w\vec{\nabla}(\rho\vec{v}) - \rho T\vec{v}\cdot\left(\vec{\nabla}s\right).
\end{aligned} \tag{A.2}$$

Summing the two, we find

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \rho\epsilon\right) = -\vec{\nabla}\cdot\left[\rho\vec{v}\left(v^2 + w\right)\right] = -\vec{\nabla}\cdot\left[\rho\vec{v}\left(v^2 + \epsilon + \frac{P}{\rho}\right)\right]. \tag{A.3}$$

A.2 The Anelastic/Boussinesq Approximations

A.2.1 Developing the Anelastic/Boussinesq Approximations

Let's relax the incompressibility constraint (we will expand the continuity equation to first order, but the momentum equation will merit a separate treatment):

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} + \vec{\nabla}\cdot\left(\rho_0 \vec{u}_1\right) = 0,\\ \frac{\partial \vec{u}_1}{\partial t} = -\frac{\vec{\nabla}P}{\rho} - \vec{g}.\end{aligned} \tag{A.4}$$

Suppose we are interested in phenomena with characteristic length scale L and time scale τ. Let's first examine the relative magnitudes of the terms in the continuity equation

$$\frac{\rho_1}{\tau} + \frac{\rho_0 \left|u_1\right|}{L} = 0.$$

Thus, if we are interested in time scales $\tau \gg \frac{\rho_1}{\rho_0}\frac{L}{|u_1|}$ then we neglect the first term, the time derivative. This corresponds to making the perturbation incompressible; note that $\frac{\partial \rho_1}{\partial t} \approx \frac{\mathrm{d}\rho_1}{\mathrm{d}t}$ to first order, so we drop the high frequency restoring forces in the perturbation.

For the momentum equation, we instead first manipulate to first order

$$\begin{aligned}-\frac{\vec{\nabla}P}{\rho} - \vec{g} &= -\frac{\vec{\nabla}P_0}{\rho} - \frac{\vec{\nabla}P_1}{\rho_0} - \vec{g},\\ &= -\frac{\vec{\nabla}P_1}{\rho_0} + \left(\frac{\rho_0}{\rho} - 1\right)\vec{g},\\ &= -\vec{\nabla}\left(\frac{P_1}{\rho_0}\right) - \frac{P_1}{\rho_0^2}\vec{\nabla}\rho_0 - \frac{\rho_1}{\rho_0}\vec{g}.\end{aligned} \tag{A.5}$$

We now have three equations for four variables, $\vec{u}_1, \rho_1, P_1$. We must introduce a fourth equation, a thermodynamic equation. For an adiabatic process $P\rho^{-\gamma} \propto P^{1-\gamma}T^{\gamma}$ is constant. We thus introduce the concept of the *potential temperature*

$$\theta = T\left(\frac{P_0}{P}\right)^{\kappa}. \tag{A.6}$$

For an adiabatic process, $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Motivated by this, we use

$$\begin{aligned}\frac{\partial 1}{\partial \rho_0}\frac{\partial \rho_0}{\partial z} = \frac{1}{\gamma P_0}\frac{\partial P_0}{\partial z} - \frac{1}{\theta_0}\frac{\partial \theta_0}{\partial z},\\ \frac{\rho_1}{\rho_0} = \frac{1}{\gamma}\frac{P_1}{P_0} - \frac{\theta_1}{\theta_0},\end{aligned} \tag{A.7}$$

to give the momentum equation form

$$\frac{\mathrm{d}\vec{u}_1}{\mathrm{d}t} = -\vec{\nabla}\left(\frac{P_1}{\rho_0}\right) + \frac{P_1}{\rho_0}\left(\frac{1}{\theta_0}\vec{\nabla}\theta_0\right) + \vec{g}\frac{\theta_1}{\theta_0}. \tag{A.8}$$

We also recognize $N^2 = \frac{g}{\theta_0}\frac{\partial\theta_0}{\partial z}$. We now do the same trick where we consider dynamics on length scale D and compare the first and second terms in Eq. A.8. Their ratio is $\frac{N^2 D}{g}$, and so as $N^2 \ll \frac{g}{D}$ the freefall time we neglect the second term.

The anelastic fluid equations thus read

$$\begin{aligned}
\vec{\nabla}\cdot\left(\rho_0\vec{u}\right) &= 0,\\
\frac{\partial\vec{u}_1}{\partial t} + \vec{\nabla}\left(\frac{P_1}{\rho_0}\right) - \vec{g}\frac{\theta_1}{\theta_0} &= 0,\\
\frac{\partial\theta_1}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\theta_0 &= 0.
\end{aligned} \tag{A.9}$$

The Boussinesq equations are obtained from these in the limit where $H \gg D$ the relevant length scale, thus we allow ρ_0 to be approximately constant.

A.2.2 Anelastic Solution to Stratified Atmosphere

We simply substitute $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ into Eq. A.9 with $\rho_0 \propto e^{-z/H}$ and obtain

$$\begin{bmatrix}
0 & 0 & ik_x\rho_0 & ik_z\rho_0 - \frac{\rho_0}{H}\\
0 & -i\omega & 0 & \frac{N^2\theta_0}{g}\\
\frac{ik_x}{\rho_0} & 0 & -i\omega & 0\\
\frac{ik_z}{\rho_0} + \frac{1}{\rho_0 H} & -\frac{g}{\theta_0} & 0 & -i\omega
\end{bmatrix}
\begin{bmatrix}
P_1\\ \theta_1\\ u_{1x}\\ u_{1z}
\end{bmatrix} = 0. \tag{A.10}$$

Taking the determinant of this matrix produces

$$\begin{aligned}
-k_x^2\left(-N^2+\omega^2\right) + \left(ik_z - \frac{1}{H}\right)\left(ik_z + \frac{1}{H}\right)\omega^2 &= 0,\\
\frac{N^2k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}} &= \omega^2.
\end{aligned} \tag{A.11}$$