

Dynamical Tides Reexpressed

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ABSTRACT

Zahn (1975) first put forward and calculated in detail the torque experienced by stars in a close binary systems due to dynamical tides. His widely used formula for stars with radiative envelopes and convective cores is expressed in terms of the stellar radius, even though the torque is actually being applied to the convective core at the core radius. This results in a large prefactor, which is very sensitive to the global properties of the star, that multiplies the torque. This large factor is compensated by a very small multiplicative factor, E_2 . Although this is mathematically accurate, depending on the application this can lead to significant errors. The problem is even more severe, since the calculation of E_2 itself is non-trivial, and different authors have obtained inconsistent values of E_2 . Moreover, many codes (e.g. BSE, StarTrack, MESA) interpolate (and sometimes extrapolate) a fit of E_2 values to the stellar mass, often in regimes where this is not sound practice. We express the torque in an alternate form, cast in terms of parameters at the envelope-core boundary and a dimensionless coefficient, β_2 . Previous attempts to express the torque in such a form are either missing an important factor, which depends on the density profile of the star, or are not easy to implement. We show that β_2 is almost independent of the properties of the star and its value is approximately unity. Our formula for the torque is simple to implement and avoids the difficulties associated with the classic expression.

Key words: binaries: close – stars: Wolf-Rayet

1 INTRODUCTION

The rate at which a binary system evolves to a state of a minimum kinetic energy (circular orbit, all spins aligned and synchronous with the orbital motion) depends on the dissipation of the tidal kinetic energy (see Zahn 2008, for a review). When a radiative zone exists, radiative damping operates on the dynamical tide (Zahn 1975; Goodman & Dickson 1998). It results from the excitation of internal gravity waves near the boundary of the convective zone and the radiative zone that then travel outwards (inwards) in the radiative envelope (core).

In what follows, we concentrate on the synchronization of stars with radiative envelopes, as the derivations for circulation and for radiative cores is analogous. The torque applied to the star in this case is given by (Zahn 1975)

$$\tau = \frac{3}{2} \frac{G(qM)^2}{R} \left(\frac{R}{d} \right)^6 E_2 s^{8/3}, \quad (1)$$

where M is the mass of the star, qM is the mass of the companion, R is the radius of the star, d is the distance between the stars, E_2 is a parameter to be discussed below, $s = 2|\omega - \Omega|/\sqrt{R^3/GM}$ is the

normalized apparent frequency of the tide, ω is the orbital angular velocity and Ω is the rotational angular velocity of the star. We consider the case $s^{-1} \gg 1$, in which the forced oscillations in the envelope behave like a purely travelling wave, and the function $p(s)$ multiplying τ in Zahn (1975) is ≈ 1 .

The torque is being applied to the convective core at radius r_c , but in order to cast the torque in terms of the global stellar properties (radius, R and mass qM), Zahn (1975) introduced the factor E_2 that depends strongly on r_c/R and can thus compensate for this. The numeric value of E_2 varies by many orders of magnitude for different stars owing to the strong dependence on the radius of the star, $\propto R^9$.

The physics of the internal gravity waves and how they lead to torques is now well understood. A simple order of magnitude of the effect can be found, for example, in Goldreich & Nicholson (1989) (see also Savonije & Papaloizou 1984):

$$\tau \sim \rho \lambda (\omega - \Omega)^2 r^4 \left(\frac{\Phi_{\text{ext}}}{gr} \right)^2, \quad (2)$$

where $\Phi_{\text{ext}} = G(qM)r^2/d^3$ is the perturbing tidal potential, ρ is the density, g is the gravitational acceleration, ω is the orbital angular velocity and r is the radius. All quantities are evaluated at the boundary of the convective core and λ is a length scale constructed

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using the derivative of the Brunt-Väisälä frequency with radius at that location and determines the scale of variation of the waves in the radial direction. We show below that this estimate (as well as Savonije & Papaloizou 1984) lacks a factor $(1 - \rho/\bar{\rho})^2$, where $\bar{\rho}$ is the average density of the core. This factor tends to decrease the torque for cores with density distribution which is close to uniform, and cannot be neglected. For example, for $n = 3$ polytrope with $r_c/R=0.1$ this factor is $\approx 7.9 \cdot 10^{-3}$, leading to an error of more than two orders of magnitude.

Goodman & Dickson (1998) repeated a simplified version of the Zahn (1975) analysis for low mass stars where the core is radiative while the envelope is convective. These cases only differ by whether the waves propagate towards the surface of the star or towards the centre. As long as the waves get damped before they can get reflected back to the radiative-convective boundary there is no practical difference between these cases. However, it is not apparent that the torque derived by Goodman & Dickson (1998) is indeed identical to the expression derived by Zahn (1975). In Appendix A we summarise the physics of the internal gravity waves and directly compare the formulas in Zahn (1975), Goldreich & Nicholson (1989) and Goodman & Dickson (1998).

Although it is well understood that the torque does not depend on the radius of the star but on the location and properties at the boundary of the convective core, Equation (1) is being routinely used in the literature. To reiterate the problem with doing so, a large factor of $(R/r_c)^9 (M/M_c)^{-4/3} \sim 10^5 - 10^{10}$, which is very sensitive to the properties of the star, multiplies the torque by using the values at the stellar radius instead of at the core radius, and then a small factor of $10^{-5} - 10^{-10}$ from E_2 is again multiplied to compensate for this large factor. While technically correct, in a given application this can lead to significant errors as both the large and the small factors are very sensitive to the properties of the stars and are not known exactly for many astrophysical systems of interest. The problem is worse in practice, and different authors have obtained apparently inconsistent values of E_2 over time (Zahn 1975; Claret & Cunha 1997; Siess et al. 2013). Moreover, Hurley et al. (2002) fit the E_2 values of Zahn (1975) as a function of M , and this fit is adopted in many codes [e.g., BSE (Hurley et al. 2002), StarTrack (Belczynski et al. 2008), MESA (Paxton et al. 2015)] to interpolate (and sometimes extrapolate) E_2 values, in a regime where this is wholly inappropriate. Siess et al. (2013) recognised this problem and pointed out that this procedure leads to at least an order of magnitude error for the value of the torque. We emphasise here that the basic problem is the mere use of Equation (1) and E_2 , and we suggest an alternate expression for the torque, which is normalised to the core boundary with a dimensionless coefficient of order one. Previous attempts to express the torque in such a form are either missing the $(1 - \rho/\bar{\rho})^2$ factor (Savonije & Papaloizou 1984; Goldreich & Nicholson 1989) or not easy to implement (Goodman & Dickson 1998).

In Section 2, we review the calculation of the parameter E_2 , and we resolve the apparent discrepancy between the inconsistent E_2 values in the literature, by showing that the value of E_2 can be accurately derived from a polytrope model. We further supplement E_2 values for main sequence (MS) models and Wolf-Rayet (WR) models calculated by the MESA code (Paxton et al. 2011, see Appendix B for details). We use these WR models in a companion paper that analyses the consequences of the torque applied by a black hole to a WR star to the gravitational-wave emission from the subsequent merger of two black holes (Kushnir et al. 2016). We demonstrate the sensitivity of E_2 on the properties of the star. In Section 3, we express the torque with an alternate form, Equa-

tion (8), which is the main result of this paper. Readers not interested in the form of the torque that contains E_2 , may skip directly to Section 3.

2 THE VALUE OF E_2

The parameter E_2 introduced in Zahn (1975) can be written as

$$\begin{aligned} E_2 &= \frac{3}{4\pi} \left(\frac{3}{2}\right)^{4/3} \frac{\Gamma^2(4/3)}{5} \left[\frac{\rho}{\bar{\rho}} \left(\frac{r}{g} \frac{dN^2}{d\ln r} \right)^{-1/3} \right]_c \times \\ &\quad \left(\frac{M_c}{M} \right)^{2/3} \left(\frac{r_c}{R} \right)^{-3} H_2^2 \\ &\equiv \bar{E}_2 \left(\frac{r}{g} \frac{dN^2}{d\ln r} \right)_c^{-1/3} \equiv \bar{E}_2 \bar{n}_c, \end{aligned} \quad (3)$$

where Γ is the gamma function, the subscript c indicates values at the convective core boundary, $\bar{\rho}$ is the mean density inside the sphere of radius r , N is the Brunt-Väisälä frequency, and we assumed that $N_c^2 = 0$. The parameter H_2 is given by

$$H_2 = \frac{5}{[Y'(1) - Y(1)]X(x_c)} \int_0^{x_c} \left(Y'' - \frac{6Y}{x^2} \right) X dx, \quad (4)$$

where Y is the solution of

$$Y'' - 6 \left(1 - \frac{\rho}{\bar{\rho}} \right) \frac{Y'}{x} - 6 \left[1 - 2 \left(1 - \frac{\rho}{\bar{\rho}} \right) \right] \frac{Y}{x^2} = 0, \quad (5)$$

regular at the centre, x is the normalised radius r/R , prime indicates a derivative with respect to x , and X is the solution of

$$X'' - \frac{\rho'}{\rho} X' - 6 \frac{X}{x^2} = 0, \quad (6)$$

regular at the centre. Note that the factor $5/[Y'(1) - Y(1)]$ in Equation (4) is sometimes replaced with $1/Y(1)$ (Claret & Cunha 1997; Siess et al. 2013), which is justified when the apsidal motion constant,

$$k_2 = \frac{6Y(1) - Y'(1)}{2[Y'(1) - Y(1)]}, \quad (7)$$

satisfies $k_2 \ll 1$.

The value of E_2 is completely determined by the density profile, $\rho(r)$, the location of the convective core boundary and \bar{n}_c . Assuming that $\rho(r)$ can be reasonably approximated by a polytrope and that \bar{n}_c is of the order of unity (both assumptions hold for our MS and WR models), then it is useful to inspect E_2 as a function of x_c for a given stellar model in comparison with the results from the relevant polytrope. In Figure 1 we show E_2 as a function of x_c for the following:

- (i) The results of Zahn (1975) for the unpublished, simplified $1.6 - 15 M_\odot$ MS stellar models of Aizenman.
- (ii) The results of Claret & Cunha (1997) for the MS stellar models of Claret (1995); Claret & Gimenez (1995) over the mass range of $1.6 - 32 M_\odot$. Since the results of Claret & Cunha (1997) are given as a function of M_c/M , we assumed $n = 3$ polytrope to estimate the x_c values.
- (iii) The results of Siess et al. (2013) for the MS stellar models of Siess (2006) over the mass range $2 - 20 M_\odot$.
- (iv) Our results for the MS and WR models (see Appendix B). Some details regarding the numerical calculation of E_2 are given in Appendix C.
- (v) Polytrope models, for which only \bar{E}_2 can be calculated (E_2 up to a factor \bar{n}_c).

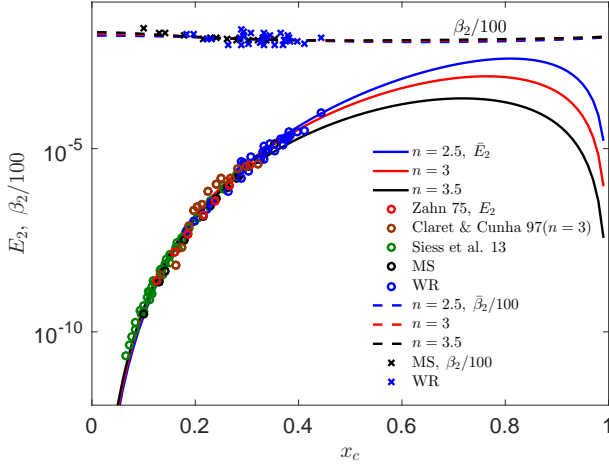


Figure 1. The values of E_2 and $\beta_2/100$ as function of the normalised location of the convective core boundary $x_c = r_c/R$. Lines: Polytrope models, for which only \bar{E}_2 (solid) and $\bar{\beta}_2$ (dashed) can be calculated (E_2 and β_2 up to a factor $\bar{n}_c = (r/g \cdot dN^2/d \ln r)^{-1/3}$, respectively); with blue, red and black for $n = 2.5, 3$ and 3.5 , respectively. Circles: E_2 values for different models; red: Zahn (1975) for the unpublished, simplified $1.6 - 15 M_\odot$ MS stellar models of Aizenman, brown: Claret & Cunha (1997) for the MS stellar models of Claret (1995); Claret & Gimenez (1995) over the mass range of $1.6 - 32 M_\odot$, where the x_c values are estimated by assuming $n = 3$ polytrope, green: Siess et al. (2013) for the MS stellar models of Siess (2006) over the mass range of $2 - 20 M_\odot$, black: our MS models, blue: our WR models. x's: $\beta_2/100$ for our models; black: MS, blue: WR. Our WR results are an extension of the previous MS results to larger x_c values. The variation of E_2 over many orders of magnitude is a direct consequence of the choice of parameters in Equation (1). β_2 is the dimensionless coefficient in our alternate form for the torque, expressed by using the values at the core boundary (Equation (8)). β_2 is almost independent of the properties of the star and its value is approximately unity.

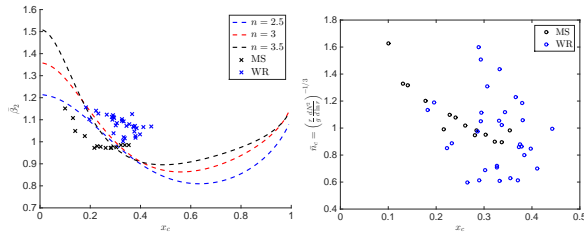


Figure 2. Left-hand panel: $\bar{\beta}_2$ as function of the normalised location of the convective core boundary $x_c = r_c/R$. Lines: Polytrope models, with blue, red, and black for $n = 2.5, 3$ and 3.5 , respectively. x's: our models; black: MS, blue: WR. The deviation between our MS and WR models and between the polytrope models are less than 20%, which is due to the small deviation of the stellar density profiles from polytropes. Right-hand panel: $\bar{n}_c = (r/g \cdot dN^2/d \ln r)^{-1/3}$ for our MS and WR models. The values of \bar{n}_c deviate from unity by less than a factor of 2.

We see that our MS results are consistent with the results of Siess et al. (2013), of Claret & Cunha (1997), and of Zahn (1975). For our MS and WR results, we can directly verify that the profiles are well described by polytropes with $n \approx 2.5 - 3.5$, \bar{n}_c is in the range $\approx 0.6 - 1.6$ (see right-hand panel of Figure 2) and that the E_2 estimate from the relevant polytrope is accurate to better than a factor 2. Our WR results are an extension of the previous MS results to larger x_c values.

Most crucially, the variation of E_2 over many orders of magnitude is a direct consequence of the choice of parameters in Equa-

tion (1). In the next section, we rewrite Equation (1) with physical parameters that minimise this large variance.

3 ALTERNATE FORM FOR THE TORQUE

In this Section, we rewrite Equation (1) to a form somewhat more natural from a theoretical point of view. The torque applied to the star is being applied to the convective core, so one would expect that the relevant quantity is the core radius $r_c = x_c R$ rather than the stellar radius R . As described in Appendix A, Equation (1) can be written as:

$$\begin{aligned} \tau &= \frac{G(qM)^2}{r_c} \left(\frac{r_c}{d} \right)^6 s_c^{8/3} \left[\frac{r_c}{g_c} \left(\frac{dN^2}{d \ln r} \right)_{r_c} \right]^{-1/3} \times \\ &\quad \frac{\rho_c}{\bar{\rho}_c} \left(1 - \frac{\rho_c}{\bar{\rho}_c} \right)^2 \left[\frac{3}{2} \frac{3^{2/3} \Gamma^2(1/3)}{5 \cdot 6^{4/3}} \frac{3}{4\pi} \alpha_2^2 \right], \\ &\equiv \beta_2 \frac{G(qM)^2}{r_c} \left(\frac{r_c}{d} \right)^6 s_c^{8/3} \frac{\rho_c}{\bar{\rho}_c} \left(1 - \frac{\rho_c}{\bar{\rho}_c} \right)^2 \end{aligned} \quad (8)$$

where $s_c \equiv 2|\omega - \Omega| \sqrt{r_c^3/GM_c}$, and α_2 is an order-unity constant of proportionality that relates the displacement at the convective-radiative boundary in the dynamical tide relative to the equilibrium tide times the deviation from constant density. Note that all quantities in this expression are evaluated at the core boundary. Direct inspection of Equation (1) shows that

$$\alpha_2 = \left(\frac{r_c}{R} \right)^{-5} \left(\frac{M_c}{M} \right) \left(1 - \frac{\rho}{\bar{\rho}} \right)^{-1} H_2. \quad (9)$$

In Figure 1, we plot β_2 for our MS and WR models and for the polytrope models (for which one can calculate only $\bar{\beta}_2 = \beta_2/\bar{n}_c$ with $\bar{n}_c = (r/g \cdot dN^2/d \ln r)^{-1/3}$). The value of β_2 is almost independent of x_c and is approximately unity. Since the torque depends on high powers of the core radius and of the distance between the stars, one can take $\beta_2 = 1$ for most applications of this theory. We nevertheless plot, for completeness, the values of $\bar{\beta}_2$ in the left-hand panel of Figure 2 using a linear scale. The deviation between our MS and WR models and between the polytrope models are less than 20%, which is due to the small deviation of the stellar density profiles from polytropes. The values of \bar{n}_c for our MS and WR models are shown in the right-hand panel of Figure 2, and they deviate from unity by less than a factor of 2.

In summary, we provide a simple alternative version of the Zahn (1975) expression for the dynamical tidal torque experienced by a star in a close binary system. Our alternate formulation references the core radius rather than the stellar radius and thus eliminates the need to compute E_2 , and the associated pitfalls in doing so, for the majority of modern applications. Previous attempts to express the torque in such a form are either missing an important factor, which depends on the density profile of the star, or not easy to implement.

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APPENDIX A: INTERNAL GRAVITY WAVES

Here we aim to clarify the various formulae in the literature for the energy carried by the internal gravity waves excited at the boundary between the convective core and the radiative envelope. This will motivate using the location and other parameters of this boundary to express the torque.

As a simple easily understood example, it is useful to consider the case of a plane parallel atmosphere where z is the vertical direction and g the acceleration of gravity. We are interested in internal gravity waves as those can have frequencies well below the dynamical frequency of the star that characterises the frequency of the sound waves. The restoring force for these waves is buoyancy and is characterised by the Brunt-Väisälä frequency N :

$$N^2 = g \frac{d}{dz} \left(\frac{\ln p}{\Gamma_1} - \ln \rho \right), \quad (\text{A1})$$

where p is the pressure and $\Gamma_1 = (d \ln p / d \ln \rho)_s$. When a fluid element moves a distance δz the resulting acceleration is given by $-N^2 \delta z \hat{z}$.

When considering internal gravity waves, one takes the density of the fluid as near constant with a value ρ_c . The acceleration of a fluid element is given by

$$\frac{D\vec{v}}{Dt} = -\frac{1}{\rho_c} \nabla p + b \hat{z}, \quad (\text{A2})$$

where b is the buoyancy and it satisfies:

$$\frac{Db}{Dt} = -N^2 \hat{z} \cdot \vec{v}. \quad (\text{A3})$$

Finally, motions do not lead to compressions in the fluid as sound waves result in much higher frequency. Thus we also have

$$\nabla \cdot \vec{v} = 0. \quad (\text{A4})$$

These equations are enough to recover the dynamics of internal

gravity waves. One can combine them to obtain an equation for the \hat{z} component of the velocity $w = \hat{z} \cdot \vec{v}$, accurate to first order in the perturbation:

$$\frac{\partial^2}{\partial t^2} \nabla^2 w = -N^2 \nabla_h^2 w, \quad (\text{A5})$$

where $\nabla_h^2 = \nabla^2 - \partial^2 / \partial z^2$ is the Laplacian in the horizontal direction. In the case where N is constant one can easily derive the dispersion relation:

$$\omega^2 = N^2 \frac{k_\perp^2}{k^2}, \quad (\text{A6})$$

where ω is the angular frequency, $k_\perp^2 = k_x^2 + k_y^2$ and $k^2 = k_x^2 + k_y^2 + k_z^2$. Note that the frequency decreases as k_z increases, such that the group velocity is opposite in sign to the phase velocity. To have excitation of frequencies well below N^2 one needs $k_\perp^2 \ll k^2$; the modes need to have a very short wavelength in the vertical direction. For our application the \hat{z} direction will be the radial direction of the star. The high spatial frequency of the oscillations will mean that tides will not easily excite these waves. As described in [Goldreich & Nicholson \(1989\)](#), the situation changes as one approaches the boundary of the radiative and convective regions as N^2 approaches zero there. There is a turning point for the waves at that location and the waves become evanescent in the convective zone. The waves vary more slowly in this part of the star. It is in there that tides can excite the waves.

Because the crucial dynamics for the excitation of the waves happens near the radiative-convective boundary, which is near the turning point where $\omega^2 = N^2$, we cannot treat N^2 as constant. To get an analytic handle one can approximate N^2 as varying linearly with z . This was the approach adopted by [Zahn \(1975\)](#) for massive stars and by [Goodman & Dickson \(1998\)](#) for low mass stars. At the level of discussion here, the main difference between the cases is whether the waves propagate towards the surface of the star or towards the centre. However, as long as the waves get damped before they can get reflected back to the radiative-convective boundary there is no practical difference between these cases for the present discussion.

To analyse this case we approximate N^2 as:

$$N^2 = N_0^2 + \frac{dN_0^2}{dz} z. \quad (\text{A7})$$

We will keep the z dependence of all quantities explicitly and use a Fourier decomposition in the $x - y$ plane. We write for example:

$$w = W(z) e^{i(k_x x + k_y y - \omega t)}. \quad (\text{A8})$$

The equation for w becomes:

$$W'' + k_\perp^2 \left(\frac{N^2}{\omega^2} - 1 \right) W = 0. \quad (\text{A9})$$

The turning point is located where $N^2 = \omega^2$. In the case of a linear dependence of N^2 the equation becomes:

$$W'' + \tilde{k}^2 (z + z_0) W = 0, \quad (\text{A10})$$

where we introduced:

$$\begin{aligned} \tilde{k}^2 &= \frac{k_\perp^2}{\omega^2} \left(\frac{dN_0^2}{dz} \right)_c \\ z_0 &= (N_0^2 - \omega^2) \left(\frac{dN_0^2}{dz} \right)^{-1}. \end{aligned} \quad (\text{A11})$$

The solutions of equation (A10) can be written in terms of Airy functions:

$$\begin{aligned} W(z) &= a\text{Ai}\left(-\frac{z+z_0}{\lambda}\right) + b\text{Bi}\left(-\frac{z+z_0}{\lambda}\right) \\ \lambda &= \tilde{k}^{-2/3} = \left(\frac{k_{\perp}^2}{\omega^2} \frac{dN_0^2}{dz}\right)^{-1/3}. \end{aligned} \quad (\text{A12})$$

Assuming that waves excited at the radiative convective-boundary get damped before being reflected back, then W describe outgoing waves, which are given by

$$W(z) = \tilde{a} \left[-i\text{Ai}\left(-\frac{z+z_0}{\lambda}\right) + \text{Bi}\left(-\frac{z+z_0}{\lambda}\right) \right], \quad (\text{A13})$$

since $\text{Bi}(x)$ differs in phase from $\text{Ai}(x)$ by $\pi/2$ as $x \rightarrow \infty$ and the group velocity is opposite in sign to the phase velocity for these waves.

To obtain the normalisation \tilde{a} one needs to compute how the equilibrium tide excites the waves at the radiative-convective boundary. The detailed calculation can be found in Zahn (1975) for massive stars and in Goodman & Dickson (1998) or low mass stars, but the order of magnitude that will allow us to understand the dependence on the properties of the star can be found in Goldreich & Nicholson (1989).

To make this connection more explicit we can express the constant \tilde{a} in terms of the displacement it implies at the turning point, $\delta z(-z_0) = \frac{i}{\omega} W'(-z_0)$. It is convenient to write the expression for the derivative of δz

$$\left. \frac{d\delta z}{dz} \right|_{-z_0} = \frac{i}{\omega} W'(-z_0) = -\frac{i\tilde{a}}{\lambda\omega} \frac{2}{3^{1/3}\Gamma(1/3)} \frac{(3^{1/2}+i)}{2}, \quad (\text{A14})$$

or equivalently:

$$\tilde{a} = i\lambda\omega \frac{3^{1/3}\Gamma(1/3)}{2} \frac{(3^{1/2}-i)}{2} \left. \frac{d\delta z}{dz} \right|_{-z_0}. \quad (\text{A15})$$

We have used the fact that $\text{Ai}'(0) = -1/3^{1/3}\Gamma(1/3)$ and $\text{Bi}'(0) = 3^{1/6}/\Gamma(1/3)$.

Once we have the form of the outgoing waves we need to calculate the energy flux. The energy density is the sum of kinetic energy and potential energy that describes the buoyancy force. The energy conservation equation to first order in the perturbation is:

$$\begin{aligned} \frac{\partial u}{\partial t} + \nabla \cdot \vec{F} &= 0 \\ u &= \frac{1}{2}\rho_0(\vec{v}^2 + \frac{b^2}{N^2}) \\ \vec{F} &= \vec{v}p, \end{aligned} \quad (\text{A16})$$

where u is the energy density and \vec{F} the energy flux. It is straightforward to use the linearized equations for the waves to show that this conservation law is satisfied. Thus to compute the energy flux carried by the waves away from the boundary (in the \hat{z} direction) we need to calculate the time average of wp .

For the pressure we write:

$$p = P(z)e^{i(k_x x + k_y y - \omega t)}. \quad (\text{A17})$$

Substituting this form into the equations of motion we find:

$$P = \frac{i\omega\rho_c}{k_{\perp}^2} W', \quad (\text{A18})$$

thus the energy flux per unit area is given by:

$$\frac{dL}{dA} = \frac{1}{2}\text{Re}(PW^*) = -\frac{\omega\rho_c}{2k_{\perp}^2}\text{Im}(W'W^*). \quad (\text{A19})$$

Using the identity $\text{Ai}(x)\text{Bi}'(x) - \text{Ai}'(x)\text{Bi}(x) = \pi^{-1}$ (Abramowitz & Stegun 1972), we get:

$$\frac{dL}{dA} = \frac{\omega\rho_c}{2k_{\perp}^2} \frac{|\tilde{a}|^2}{\pi\lambda} = \frac{3^{2/3}\Gamma^2(1/3)}{8\pi} \frac{\lambda\omega^3\rho_c}{k_{\perp}^2} \left(\left. \frac{d\delta z}{dz} \right|_{-z_0} \right)^2. \quad (\text{A20})$$

We transform now to the spherical case, assuming no rotation of the star. We identify the z direction with the radial direction and use $k_{\perp}^2 = l(l+1)/r^2$. With this replacement the scale λ is given by:

$$\lambda = \left(\frac{l(l+1)}{\omega^2} \frac{dN^2}{d\ln r} \right)^{-1/3} r. \quad (\text{A21})$$

The amplitude of the wave was determined at the turning point, which now we identify with the radiative-core boundary, r_c . This is justified, since $dN^2/d\ln r$ is on the order of the dynamical frequency of the star, and we have $\lambda/r \ll 1$. To get the energy flux we need to integrate over the area of the core, which introduces another factor $\int d\Omega r_c^2 |Y_{lm}|^2 = r_c^2$. After this translation our expression for the luminosity agrees with Equation (13) of Goodman & Dickson (1998), which reads:

$$\begin{aligned} L^G &= \frac{3^{2/3}\Gamma^2(1/3)}{8\pi} [l(l+1)]^{-4/3} \omega^{11/3} \\ &\times \left[\rho r^5 \left(\frac{dN^2}{d\ln r} \right)^{-1/3} \left(\frac{d\delta r}{dr} \right)^2 \right]_{r=r_c}. \end{aligned} \quad (\text{A22})$$

The order of magnitude of the displacement at the radiative convective boundary is given by the value for the equilibrium tide, the ratio of the external tidal potential $\Phi_{\text{ext}} \propto -G(qM)r^2/d^3$ to the local gravitational acceleration g , times the deviation from constant density (see exact derivation below):

$$\delta r \propto -\frac{\Phi_{\text{ext}}}{g} \left(1 - \frac{\rho}{\bar{\rho}} \right), \quad (\text{A23})$$

where $\bar{\rho}$ is the mean density inside the sphere of radius r , or equivalently

$$\frac{d\delta r}{dr} = \alpha \frac{\Phi_{\text{ext}}}{gr} \left(1 - \frac{\rho}{\bar{\rho}} \right), \quad (\text{A24})$$

where α is a proportionality constant. To get this proportionality constant, which is of order unity, requires solving the forced equations for the modes (Zahn 1975; Goodman & Dickson 1998). However it is clear from the expression for L that it only depends on the properties of the star near the radiative-convective boundary. Thus if we use the location, density and enclosed mass of this region to make dimensionless the expression of the energy flux, all we are doing when we solve the details of the modes across the star is calibrating a numerical constant of order one.

Equation (A22) is also equivalent to equation (19) in Goldreich & Nicholson (1989):

$$L^G \sim \rho\lambda\omega^3 r^2 \left(\frac{\Phi_{\text{ext}}}{g} \right), \quad (\text{A25})$$

although without the factor $(1 - \rho/\bar{\rho})^2$. Similar equation was derived by Savonije & Papaloizou (1984), once again without the $(1 - \rho/\bar{\rho})^2$ factor. This factor tends to decrease the torque for cores with density distribution which is close to uniform, and cannot be neglected. For example, for $n = 3$ polytrope with $r_c/R = 0.1$ this factor is $\approx 7.9 \cdot 10^{-3}$, leading to an error of more than two orders of magnitude.

Next we make the connection with the work of Zahn (1975), where the radiated energy is given there in section 2.e. The flux

of energy far away from the radiative convective boundary is computed using the asymptotic form of the modes far from the boundary, but as explained in [Zahn \(1975\)](#) the waves are excited interior to this region. Solutions for the excited waves are found by matching solutions in the various regions of the star. In our notation equation (2.50) of [Zahn \(1975\)](#) reads

$$L^Z = \frac{1}{2} \frac{\omega^2 R^3 K_0^2}{[l(l+1)]^{1/2}}, \quad (\text{A26})$$

where R is the stellar radius and because we want to make the connection with the notation of [Goodman & Dickson \(1998\)](#) we have used spherical harmonics rather than Legendre polynomials to define the modes and thus we have removed a factor of $4\pi(l+m)!(2l+1)(l-m)!$. We have also assumed that in equation (2.50) of [Zahn \(1975\)](#) $\gamma = 1$, valid in the regime where waves do not reflect back to the radiative-convective boundary. The constant K_0 is given by equation (2.40) of [Zahn \(1975\)](#):

$$K_0 = \frac{\sqrt{3}\Gamma(4/3)}{2\sqrt{\pi}} \left(\frac{\nu}{3}\right)^{-5/6} \rho_c^{1/2} \left[\left(\frac{-gA}{x^2} \right)' \right]_c^{1/4} \times \int_0^{x_c} \left[\left(\frac{x^2\Psi}{g} \right)'' - \frac{l(l+1)}{x^2} \left(\frac{x^2\Psi}{g} \right) \right] \frac{X}{X_c} dx, \quad (\text{A27})$$

where quantities with a subscript c (rather than f) are evaluated in the convective-radiative boundary, $x = r/R$ with R the radius of the star, primes are derivatives with respect to x , Ψ is the perturbation in the potential and X is the solution of

$$X'' - \frac{\rho'}{\rho} X' - l(l+1) \frac{X}{x^2} = 0 \quad (\text{A28})$$

regular at the centre. In terms of our notation, the parameter in K_0 are:

$$\begin{aligned} -gA &= N^2 \\ \nu^2 &= \frac{l(l+1)}{r_c^3 \omega^2} \left(\frac{dN^2}{d \ln r} \right)_{r=r_c} R^3 = \lambda^3 R^3 \\ \left(\frac{-gA}{x^2} \right)'_c &= \frac{\omega^2}{l(l+1)} \nu^2. \end{aligned} \quad (\text{A29})$$

In order to estimate the integral in equation (A27) we use the proportionality of $x^2\Psi/g$ to the function $Y(x)$ defined by [Zahn \(1975\)](#), which is the solution of

$$Y'' - 6 \left(1 - \frac{\rho}{\bar{\rho}} \right) \frac{Y'}{x} - \left[l(l+1) - 12 \left(1 - \frac{\rho}{\bar{\rho}} \right) \right] \frac{Y}{x^2} = 0 \quad (\text{A30})$$

regular at the centre. The integral is therefore

$$\int_0^{x_c} 6 \left(1 - \frac{\rho}{\bar{\rho}} \right) \left[\frac{1}{x} \left(\frac{x^2\Psi}{g} \right)' - \frac{2}{x^2} \left(\frac{x^2\Psi}{g} \right) \right] \frac{X}{X_c} dx. \quad (\text{A31})$$

Since the integral is dominated by $x \approx x_c$ and the perturbation in the potential is dominated by the external tide Φ_{ext} , it is useful to define α :

$$\begin{aligned} \alpha &= \left(\frac{g}{\Phi_{\text{ext}} x_c} \right)_{x_c} \left(1 - \frac{\rho}{\bar{\rho}} \right)_{x_c}^{-1} \\ &\times \int_0^{x_c} \left[\left(\frac{x^2\Psi}{g} \right)'' - \frac{l(l+1)}{x^2} \left(\frac{x^2\Psi}{g} \right) \right] \frac{X}{X_c} dx \end{aligned} \quad (\text{A32})$$

where α is a dimensionless parameter of order one. Replacing this

definition into L^Z we get:

$$\begin{aligned} L^Z &= \frac{3^{2/3}\Gamma^2(1/3)}{8\pi} [l(l+1)]^{-4/3} \omega^{11/3} \alpha^2 \\ &\times \left[\rho r^5 \left(\frac{dN^2}{d \ln r} \right)^{-1/3} \left(\frac{\Phi_{\text{ext}}}{gr} \right)^2 \left(1 - \frac{\rho}{\bar{\rho}} \right)^2 \right]_{r=r_c} \end{aligned} \quad (\text{A33})$$

Thus this is identical to the formula in [Goodman & Dickson \(1998\)](#) where α is the constant of proportionality that relates the displacement at the convective-radiative boundary in the dynamical tide relative to the equilibrium tide times the deviation from constant density.

Finally in the main text we are interested in the torque carried away by the waves τ , which is simply given by $\tau = 2L/\omega$. By specifying to $l = 2$, such that $\Phi_{\text{ext}} = -(6\pi/5)^{1/2} G(qM)r^2/d^3$, and replacing the angular frequency ω with the apparent angular frequency of the tide $2(\omega - \Omega)$, we get

$$\begin{aligned} \tau &= \frac{G(qM)^2}{r_c} \left(\frac{r_c}{d} \right)^6 s_c^{8/3} \left[\frac{r_c}{g_c} \left(\frac{dN^2}{d \ln r} \right)_{r_c} \right]^{-1/3} \frac{\rho_c}{\bar{\rho}_c} \left(1 - \frac{\rho_c}{\bar{\rho}_c} \right)^2 \\ &\times \left[\frac{3}{2} \frac{3^{2/3}\Gamma^2(1/3)}{5 \cdot 6^{4/3}} \frac{3}{4\pi} \alpha^2 \right], \end{aligned} \quad (\text{A34})$$

where $s_c \equiv 2|\omega - \Omega|/\sqrt{r_c^3/GM_c}$. The important point is that all quantities in this expression are evaluated at the core boundary.

We will use this expression in the main text and to re-express the torque in the formulas of [Zahn](#).

APPENDIX B: STELLAR MODELS

We follow the evolution of the stellar models using the publicly available package MESA version 6596 ([Paxton et al. 2011, 2013, 2015](#)). For the MS models, we vary the zero-age main sequence (ZAMS) mass between $[1.6, 40]M_\odot$, thus covering the mass range of main-sequence stars with the convective core and the radiative envelope. All MS models had solar metallicity ($Z = 0.02$) and no rotation.

For the WR models, we aim at covering a wide range of masses during the WR phase $\approx [5, 30]M_\odot$, and for that we select from a set of models having ZAMS masses in the range of $[40, 100]M_\odot$, metallicities between $[0.01, 1]Z_\odot$ and initial rotation between $[0.4, 0.6]$ of breakup rate. We use the WR model profiles during the epoch where time to core-collapse is greater than 10^4 yr and the stellar radius is smaller than $2R_{\text{SM}}$, where R_{SM} is the WR radius according to [Schaerer & Maeder \(1992\)](#).

In all models, mass-loss was determined according to the ‘Dutch’ recipe in MESA, combining the rates from [Glebbeek et al. \(2009\)](#); [Nieuwenhuijzen & de Jager \(1990\)](#); [Nugis & Lamers \(2000\)](#); [Vink et al. \(2001\)](#), with a coefficient $\eta = 1$, convection according to the Ledoux criterion, with mixing length parameter $\alpha_{\text{mlt}} = 2$, semi-convection efficiency parameter $\alpha_{\text{sc}} = 0.1$ ([Paxton et al. 2013](#), eq. 12) and exponential overshoot parameter $f = 0.008$ ([Paxton et al. 2011](#), eq. 2). For the atmosphere boundary conditions we use the ‘simple’ option of MESA ([Paxton et al. 2011](#), eq. 3).

APPENDIX C: NUMERICAL INTEGRATION OF THE FUNCTIONS Y AND X

In the limit $x \rightarrow 0$, we can write Equation (5) as

$$Y'' - 6\frac{Y}{x^2} = 0, \quad (\text{C1})$$

such that we can choose $Y(x) = x^3$ for the regular solution at the centre. Defining $Y \equiv f(x)x^3$, we get an equation for f

$$f'' + 6\frac{\rho}{\bar{\rho}}\frac{f'}{x} - 6\left(1 - \frac{\rho}{\bar{\rho}}\right)\frac{f}{x^2} = 0, \quad (\text{C2})$$

with $f(0) = 1$. Expanding near $x = 0$ as $f \approx 1 + a_1x + a_2x^2$ and $\rho \approx \rho_0 + \rho'_0x + \rho''_0x^2/2$ we get

$$\begin{aligned} \frac{\rho}{\bar{\rho}} &\approx 1 + \frac{1}{4}\frac{\rho'_0}{\rho_0}x + \frac{1}{5}\frac{\rho''_0}{\rho_0}x^2 - \frac{3}{4}\left(\frac{\rho'_0}{\rho_0}\right)^2x^2, \\ \Rightarrow 2a_2 + 6\left(1 + \frac{1}{4}\frac{\rho'_0}{\rho_0}x\right)\left(\frac{a_1}{x} + 2a_2\right) \\ &+ \left[\frac{3}{2}\frac{\rho'_0}{\rho_0} + \frac{6}{5}\frac{\rho''_0}{\rho_0}x - \frac{9}{2}\left(\frac{\rho'_0}{\rho_0}\right)^2x\right]\left(\frac{1}{x} + a_1\right) = 0. \end{aligned} \quad (\text{C3})$$

Equating order by order we get

$$\begin{aligned} f'(0) &= -\frac{1}{4}\frac{\rho'_0}{\rho_0}, \\ f''(0) &= \frac{3}{4}\left(\frac{\rho'_0}{\rho_0}\right)^2 - \frac{6}{35}\frac{\rho''_0}{\rho_0}. \end{aligned} \quad (\text{C4})$$

For numerical integration we can begin at some small x_{\min} and estimate the boundary conditions using Equation (C4).

Similarly, in the limit $x \rightarrow 0$, we can write Equation (6) as

$$X'' - 6\frac{X}{x^2} = 0, \quad (\text{C5})$$

such that we can choose $X(x) = x^3$ for the regular solution at the centre. Defining $X \equiv g(x)x^3$, we get an equation for g

$$g'' + \left(\frac{6}{x} - \frac{\rho'}{\rho}\right)g' - 3\frac{\rho'}{\rho}\frac{g}{x} = 0, \quad (\text{C6})$$

with $g(0) = 1$. Expanding near $x = 0$ as $g \approx 1 + b_1x + b_2x^2$ we get (assuming $\rho'_0 \neq 0$)

$$\begin{aligned} \frac{\rho'}{\rho} &\approx \frac{\rho'_0}{\rho_0}\left(1 - \frac{\rho'_0}{\rho_0}x + \frac{\rho''_0}{\rho'_0}x\right), \\ \Rightarrow 2b_2 + (b_1 + 2b_2x)\left(\frac{6}{x} - \frac{\rho'_0}{\rho_0}\right) \\ &- 3\frac{\rho'_0}{\rho_0}\left(1 - \frac{\rho'_0}{\rho_0}x + \frac{\rho''_0}{\rho'_0}x\right)\left(\frac{1}{x} + b_1\right) = 0. \end{aligned} \quad (\text{C7})$$

Equating order by order we get

$$\begin{aligned} g'(0) &= \frac{1}{2}\frac{\rho'_0}{\rho_0}, \\ g''(0) &= -\frac{1}{7}\left(\frac{\rho'_0}{\rho_0}\right)^2 + \frac{3}{7}\frac{\rho''_0}{\rho_0}. \end{aligned} \quad (\text{C8})$$

One can verify that Equation (C8) is also correct for the case $\rho'_0 = 0$. For numerical integration we can begin at some small x_{\min} and estimate the boundary conditions using Equation (C8).

The derivation of the boundary conditions by Siess et al. (2013) is erroneous, since they did not expand ρ near $x = 0$, but rather used the value at $x = 0$, which is inconsistent with the order

of expansion. That leads to wrong values of $f''(0)$ and $g''(0)$, but has a negligible effect on the E_2 values.

For a polytrope with index n , the dimensionless density of the star, θ , satisfies the Lane-Emden equation of index n :

$$\frac{1}{\xi^2}\frac{d}{d\xi}\left(\xi^2\frac{d\theta}{d\xi}\right) = -\theta^n, \quad (\text{C9})$$

where ξ is the radius and ξ_1 is the stellar radius ($\theta(\xi_1) = 0$). In the limit $\xi \rightarrow 0$ we get $\theta \approx 1 - \xi^2/6$, leading to $\rho'_0 = 0$ and $\rho''_0/\rho_0 = -n\xi_1^2/3$.

C1 Analytical solutions for a special case

For the case $\rho(x) = 1 - x^2$ we can find an analytical solutions for f and g , which are useful for testing the numerical integration scheme. In this case we need to solve

$$f'' + \frac{30(1-x^2)}{x(5-3x^2)}f' - \frac{12}{5-3x^2}f = 0, \quad (\text{C10})$$

with $f(0) = 1$ and $f'(0) = 0$. The solution is

$$f = {}_2F_1\left(\frac{9-\sqrt{65}}{4}, \frac{9+\sqrt{65}}{4}; \frac{7}{2}; \frac{3x^2}{5}\right), \quad (\text{C11})$$

where ${}_2F_1$ is the hypergeometric function. The equation for g is

$$g'' + \frac{6-4x^2}{x(1-x^2)}g' + \frac{6}{1-x^2}g = 0, \quad (\text{C12})$$

with $g(0) = 1$ and $g'(0) = 0$. The solution is

$$g = {}_2F_1\left(\frac{3-\sqrt{33}}{4}, \frac{3+\sqrt{33}}{4}; \frac{7}{2}; x^2\right). \quad (\text{C13})$$

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