Research Notes

Yubo Su

February 5, 2018

Contents

Chapter 1

2D Wave Breaking in Atmospheres

The goal of this will be to lay out a formalism that can reproduce Sutherland et al. 2011[1] and also investigate driven oscillations versus the breaking of a single wave packet. This represents wave breaking in the atmosphere.

1.1 Dynamical Setup

We adopt notation where q_0 is the background quantity and q_1 is the perturbed quantity from the propagating wave.

The fluid equations are

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0, \tag{1.1a}$$

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\vec{\nabla}\frac{P}{\rho} - g\hat{z}, \tag{1.1b}$$

where we will take gravity to be uniform throughout the domain of interest. We will study for no background flow $\vec{u}_0 = 0$ and in the presence of stratification $\rho_0 \propto e^{-z/H}$. In the absence of any perturbations it is then easy to show that $\frac{\mathrm{d}P_0}{\mathrm{d}z} = -\rho_0 g$.

[1]DOI:10.1175/JAS-D-11–097.1

1.1.1 Linear, Incompressible

Solution for Arbitrary Stratification

First, we solve the incompressible case $c_s^2 \to \infty, \vec{\nabla} \cdot \vec{u} = 0$ in the linear regime. For funsies, we solve for arbitrary stratification first. The fluid equations to first order reduce to

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} + \left(\vec{u}_1 \cdot \vec{\nabla}\right)\rho_0 &= 0,\\ \vec{\nabla} \cdot \vec{u}_1 &= 0,\\ \frac{\partial \vec{u}_1}{\partial t} &= -\frac{\vec{\nabla} P_1}{\rho_0} - P_1 \vec{\nabla} \frac{1}{\rho_0}\end{aligned} \tag{1.2}$$

We expect there to be some z dependence in the amplitude, so we substitute variables of form $e^{i(kx-\omega t)}$ and do not specify the z dependence. This gives us

$$\begin{aligned}-i\omega\rho_1 - u_{1z}\frac{\partial \rho_0}{\partial z} &= 0,\\ iku_{1x} + \frac{\partial u_{1z}}{\partial z} &= 0,\\ -iwu_{1x} + \frac{ik_x P_1}{\rho_0} &= 0,\\ -iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1}{\rho_0^2}\frac{\partial P_0}{\partial z} &= 0.\end{aligned} \tag{1.3}$$

We substitute $N^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}$ and $\frac{\partial P_0}{\partial z} = -\rho g$ to obtain

$$-i\omega\rho_1 - u_{1z}\frac{\rho_0 N^2}{g} = 0, \tag{1.4a}$$

$$iku_{1x} + \frac{\partial u_{1z}}{\partial z} = 0, \tag{1.4b}$$

$$-iwu_{1x} + \frac{ik_x P_1}{\rho_0} = 0, \tag{1.4c}$$

$$-iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1 g}{\rho_0} = 0. \tag{1.4d}$$

Eliminating u_{1x} by substituting (1.4b) into (1.4c) and ρ_1 by substituting (1.4a) into (1.4d) give

$$i\omega\frac{\partial u_{1z}}{\partial z} + \frac{k_x^2 P_1}{\rho_0} = 0, \tag{1.5a}$$

$$\left(\omega^2 - N^2\right)u_{1z} + \frac{i\omega}{\rho_0}\frac{\partial P_1}{\partial z} = 0. \tag{1.5b}$$

Finally, we multiply (1.5a) with ρ_0 and differentiate dz and combine with (1.5b) to give

$$\frac{\mathrm{d}^2 u_{1z}}{\mathrm{d}z^2} + \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z}\frac{\partial u_{1z}}{\partial z} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right)u_{1z} = 0. \tag{1.6}$$

Introduce Stratification

With stratification $\rho \propto e^{-z/H}$ Eq. 1.6 clearly has exponential solutions $e^{\kappa z}$ for

$$\kappa^2 - \frac{\kappa}{H} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right) = 0. \tag{1.7}$$

We permit complex $\kappa = -\frac{1}{2H} + ik_z$, and from the above clearly

$$\begin{aligned} k_z^2 &= -\frac{1}{4H^2} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right), \\ \omega^2 &= \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}. \end{aligned} \tag{1.8}$$

1.1.2 Developing the Anelastic/Boussinesq Approximations

Let's relax the incompressibility constraint (we will expand the continuity equation to first order, but the momentum equation will merit a separate treatment):

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \vec{\nabla}\cdot\left(\rho_0 \vec{u}_1\right) = 0, \\ \frac{\partial \vec{u}_1}{\partial t} = -\frac{\vec{\nabla} P}{\rho} - \vec{g}. \end{aligned} \tag{1.9}$$

Suppose we are interested in phenomena with characteristic length scale L and time scale τ. Let's first examine the relative magnitudes of the terms in the continuity equation

$$\frac{\rho_1}{\tau} + \frac{\rho_0 |u_1|}{L} = 0.$$

Thus, if we are interested in time scales $\tau \gg \frac{\rho_1}{\rho_0}\frac{L}{|u_1|}$ then we neglect the first term, the time derivative. This corresponds to making the perturbation incompressible; note that $\frac{\partial \rho_1}{\partial t} \approx \frac{\mathrm{d}\rho_1}{\mathrm{d}t}$ to first order, so we drop the high frequency restoring forces in the perturbation.

For the momentum equation, we instead first manipulate to first order

$$\begin{aligned} -\frac{\vec{\nabla} P}{\rho} - \vec{g} &= -\frac{\vec{\nabla} P_0}{\rho} - \frac{\vec{\nabla} P_1}{\rho_0} - \vec{g}, \\ &= -\frac{\vec{\nabla} P_1}{\rho_0} + \left(\frac{\rho_0}{\rho} - 1\right)\vec{g}, \\ &= -\vec{\nabla}\left(\frac{P_1}{\rho_0}\right) - \frac{P_1}{\rho_0^2}\vec{\nabla}\rho_0 - \frac{\rho_1}{\rho_0}\vec{g}. \end{aligned} \tag{1.10}$$

We now have three equations for four variables, $\vec{u}_1, \rho_1, P_1$. We must introduce a fourth equation, a thermodynamic equation. For an adiabatic process $P\rho^{-\gamma} \propto P^{1-\gamma}T^{\gamma}$ is constant. We thus introduce the concept of the *potential temperature*

$$\theta = T\left(\frac{P_0}{P}\right)^{\kappa}. \tag{1.11}$$

For an adiabatic process, $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Motivated by this, we use

$$\begin{aligned}\frac{\partial 1}{\partial \rho_0}\frac{\partial \rho_0}{\partial z} &= \frac{1}{\gamma P_0}\frac{\partial P_0}{\partial z} - \frac{1}{\theta_0}\frac{\partial \theta_0}{\partial z},\\ \frac{\rho_1}{\rho_0} &= \frac{1}{\gamma}\frac{P_1}{P_0} - \frac{\theta_1}{\theta_0},\end{aligned} \tag{1.12}$$

to give the momentum equation form

$$\frac{\mathrm{d}\vec{u}_1}{\mathrm{d}t} = -\vec{\nabla}\left(\frac{P_1}{\rho_0}\right) + \frac{P_1}{\rho_0}\left(\frac{1}{\theta_0}\vec{\nabla}\theta_0\right) + \vec{g}\frac{\theta_1}{\theta_0}. \tag{1.13}$$

We also recognize $N^2 = \frac{g}{\theta_0}\frac{\partial \theta_0}{\partial z}$. We now do the same trick where we consider dynamics on length scale D and compare the first and second terms in Eq. 1.13. Their ratio is $\frac{N^2 D}{g}$, and so as $N^2 \ll \frac{g}{D}$ the freefall time we neglect the second term.

The anelastic fluid equations thus read

$$\begin{aligned}\vec{\nabla}\cdot\left(\rho_0\vec{u}\right) &= 0,\\ \frac{\partial \vec{u}_1}{\partial t} + \vec{\nabla}\left(\frac{P_1}{\rho_0}\right) - \vec{g}\frac{\theta_1}{\theta_0} &= 0,\\ \frac{\partial \theta_1}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\theta_0 &= 0.\end{aligned} \tag{1.14}$$

The Boussinesq equations are obtained from these in the limit where $H \gg D$ the relevant length scale, thus we allow ρ_0 to be approximately constant.

1.1.3 Anelastic Solution

We simply substitute $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ into Eq. 1.14 with $\rho_0 \propto e^{-z/H}$ and obtain

$$\begin{bmatrix} 0 & 0 & ik_x\rho_0 & ik_z\rho_0 - \frac{\rho_0}{H}\\ 0 & -i\omega & 0 & \frac{N^2\theta_0}{g}\\ \frac{ik_x}{\rho_0} & 0 & -i\omega & 0\\ \frac{ik_z}{\rho_0} + \frac{1}{\rho_0 H} & -\frac{g}{\theta_0} & 0 & -i\omega \end{bmatrix}\begin{bmatrix} P_1\\ \theta_1\\ u_{1x}\\ u_{1z}\end{bmatrix} = 0. \tag{1.15}$$

Taking the determinant of this matrix produces

$$\begin{aligned}-k_x^2\left(-N^2+\omega^2\right) + \left(ik_z - \frac{1}{H}\right)\left(ik_z + \frac{1}{H}\right)\omega^2 &= 0,\\ \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}} &= \omega^2.\end{aligned} \tag{1.16}$$

1.2 Boundary Conditions

1.2.1 Incompressible

We must bound this to a finite domain. We choose periodic boundary conditions in x with total length L_x, $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$. We choose z dimension to have total length L_z, $z \in [0, L_z]$.

To set up the boundary condition at $z = 0$, we recall that gravity waves in an atmosphere with $e^{-z/H}$ profile have form

$$u_{1z} \propto e^{\frac{z}{2H}} e^{i(k_x x + k_z z - \omega t)}. \tag{1.17}$$

We will take Boussinesq/WKB approximation $H \to 0$, reasonable in the linear regime, to simplify the BCs. Then $\rho_1 = 0$ (reasonable in the incompressible limit), $u_{1x} = -\frac{k_z}{k_x} u_{1z}$ and $P_1 = -\frac{\omega\rho_0}{k_z} u_{1z}$, from the linearized incompressible equations.

The boundary condition at $z = L_z$ is much harder to determine. It is clear that $\vec{u}_1(z = \infty) = \rho_1(z = \infty) = 0$, and for $L_z \gg H$ this would be a reasonable approximation, if simply because we expect the majority of the wave to dissipate via turbulent dissipation as z reaches many H. We will simply choose the BC to be many multiples of H. We can solve with both a Dirichlet and Neumann BC and compare the two solutions; if the solutions differ significantly then we must choose a larger L_z. These are the only two solutions that can be implemented where we do not need the phase of the linear wave, which we lose during the nonlinear breaking region, to relate the function and derivative at the boundary.

1.3 Simulation

We begin our simulation with $\rho_1 = 0, \vec{u}_1 = 0$ strictly within the domain of simulation. We will borrow some values from Sutherland's paper and use $k_z = 2\,\mathrm{km}^{-1}$ then define $k_x = -0.4k_z, H = 10/k_z, A = 0.05/k_z, L_z = 300/k_z, L_x = 20/k_z$, We also use $\mu \approx 29, T = 273\,\mathrm{K}, \rho_0 = 1\,\mathrm{kg/m^3}, P_0 = \frac{\rho_0 k_B T}{\mu m_p}, g = 10\,\mathrm{m/s^2}$.