1 Equations

1.1 Boussinesq Equations

The physical equations in the Boussinesq approximation are:

$$\vec{\nabla}\cdot\vec{u} = 0, \tag{1a}$$

$$\frac{\partial \rho_1}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\rho_1 - \frac{\rho_0 u_z}{H} = 0, \tag{1b}$$

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} + \frac{\vec{\nabla}P_1}{\rho_0} + \frac{\rho_1\vec{g}}{\rho_0} = 0. \tag{1c}$$

In these equations, we take $\rho_0(z) = \rho_0$ a constant reference density.

Including numerical terms and driving terms, my full equations are

$$\vec{\nabla}\cdot\vec{u}_1 = 0, \tag{2a}$$

$$\frac{\partial \rho_1}{\partial t} - \frac{\rho_0 u_z}{H} - \nu\nabla^6\rho_1 = -\Gamma(z)\rho_1 - \left(\vec{u}\cdot\vec{\nabla}\right)\rho_1 + Fe^{-\frac{(z-z_0)^2}{2\sigma^2}}\cos(k_x x - \omega t), \tag{2b}$$

$$\frac{\partial u_x}{\partial t} + \frac{\partial_x P}{\rho_0} - \nu\nabla^6 u_x = -\Gamma(z)u_x - \left(\vec{u}\cdot\vec{\nabla}\right)u_x, \tag{2c}$$

$$\frac{\partial u_z}{\partial t} + \frac{\partial_z P}{\rho_0} + \frac{\rho_1 g}{\rho_0} - \nu\nabla^6 u_z = -\Gamma(z)u_z - \left(\vec{u}\cdot\vec{\nabla}\right)u_z, \tag{2d}$$

$$\Gamma(z) = 5\left[2 + \tanh\frac{z - z_T}{(z_{\max} - z_T)/2} + \tanh\frac{z_B - z}{z_B/2}\right], \tag{2e}$$

where $z_B = 0.07z_{\max}, z_T = 0.93z_{\max}$ are the boundaries of the damping zones, and my domain is $z \in [0, z_{\max}]$. The forcing term F is chosen to be weakly nonlinear, ω is chosen by inverting $\omega(k_x, k_z)$ dispersion relation for fixed $k_x = 2\pi/x_{\max}$ and some desired $k_z \ll H, z_{\max}$ (I've chosen $k_z \approx -\frac{2\pi}{z_{\max}/5}$ here), and $\sigma \lesssim \frac{1}{k_z}$ is used to excite a broad band of modes including the desired k_z mode.

1.2 Stratification

We simulate an incompressible, isothermal fluid, representatitive of degenerate matter in WD bulks. We assume a barotropic equation of state to simplify for the time being. The physical equations for an incompressible, barotropic fluid are:

$$\vec{\nabla}\cdot\vec{u} = 0, \tag{3a}$$

$$\frac{\partial \rho}{\partial t} + \vec{u}\cdot\vec{\nabla}\rho = 0, \tag{3b}$$

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} + \frac{\vec{\nabla}P}{\rho} + g\hat{z} = 0. \tag{3c}$$

We will introduce variables $T = P/\rho$. We then mandate ρ_0, T_0 backgrounds satisfy hydrostatic equilibrium $\vec{\nabla}T_0 + T_0\vec{\nabla}\rho_0 + \vec{g} = 0$. Taking isothermal stratification, we find $T_0 = gH$. Making then substitution of variables $\Upsilon = \ln\rho - \ln\rho_0$ and $T_1 = T - T_0$ deviations from the background state, the exact fluid equations in the new variables are:

$$\vec{\nabla}\cdot\vec{u} = 0, \tag{4a}$$

$$\frac{\partial\Upsilon}{\partial t} - \frac{u_z}{H} = 0, \tag{4b}$$

$$\frac{\partial u_x}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)u_x + \frac{\partial T}{\partial x} + gH\frac{\partial\Upsilon}{\partial x} + T_1\frac{\partial\Upsilon}{\partial x} = 0, \tag{4c}$$

$$\frac{\partial u_z}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)u_z + \frac{\partial T}{\partial z} + gH\frac{\partial\Upsilon}{\partial z} + T_1\frac{\partial\Upsilon}{\partial z} - \frac{T_1}{H} = 0. \tag{4d}$$

These are implemented as:

$$\vec{\nabla}\cdot\vec{u} = 0, \tag{5a}$$

$$\frac{\partial\Upsilon}{\partial t} - \frac{u_z}{H} - \nu\nabla^2\Upsilon = -\Gamma(z)\Upsilon - \left(\vec{u}\cdot\vec{\nabla}\right)\Upsilon + \frac{F}{\rho_0(z)}e^{-\frac{(z-z_0)^2}{2\sigma^2}}\cos(k_x x - \omega t), \tag{5b}$$

$$\frac{\partial u_x}{\partial t} + \frac{\partial T}{\partial x} + gH\frac{\partial\Upsilon}{\partial x} - \nu\nabla^2 u_x = -\Gamma(z)u_x - \left(\vec{u}\cdot\vec{\nabla}\right)u_x - T_1\frac{\partial\Upsilon}{\partial x}, \tag{5c}$$

$$\frac{\partial u_z}{\partial t} + \frac{\partial T}{\partial z} + gH\frac{\partial\Upsilon}{\partial z} - \frac{T_1}{H} - \nu\nabla^2 u_z = -\Gamma(z)u_z - \left(\vec{u}\cdot\vec{\nabla}\right)u_z - T_1\frac{\partial\Upsilon}{\partial z}. \tag{5d}$$

$\Gamma(z)$ is as before, but we use $k_z = -2\pi/H$ here, and ω in the forcing term accordingly.