# Mean Flow Steepening in Internal Gravity Wave Breaking

## Group Meeting Presentation

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Nov 30, 2018

- 2D plane parallel atmosphere, continuous train IGW from below.
- Equations to filter sound waves (actually surprisingly contentious):

$$\vec{\nabla} \cdot \vec{u} = 0, \tag{1}$$

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} - \frac{u_{1z}\rho_0}{H} + \vec{u} \cdot \vec{\nabla}\rho_1 \\ = [-\Gamma(z) + \mathfrak{D}]\rho_1 + \left\{ F e^{-\frac{(z-z_0)^2}{2\sigma^2}} \cos(k_x x - \omega t) \right\},\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} + \frac{\vec{\nabla}P_1}{\rho_0} + \frac{\rho_1 \vec{g}}{\rho_0} - \frac{\vec{\nabla}P_1}{\rho_0^2}\rho_1 \\ = [-\Gamma(z) + \mathfrak{D}]\vec{u}\end{aligned} \tag{3}$$

- $\Gamma(z) \propto 2 + \tanh\left(\frac{z - z_H}{w}\right) + \tanh\left(\frac{z_b - z}{w}\right), \mathfrak{D} \sim \nabla^k$. ([] numerical, {} forcing).
- If $\frac{\mathrm{d}\ln\rho_0}{\mathrm{d}z} = H$ scale height $\gg$ domain of simulation, can use $\rho_0(z) = \rho_0$, *Boussinesq approximation*. Else, ~ *anelastic*.

- The above system is a strange mix of the incompressible/anelastic approximations!
- In the stars, $N^2 = g^2\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}P_0} - \frac{1}{c_s^2}\right)$ where $c_s^2 \to \infty$ is the sound speed.
- In atmospheric sciences, $\frac{\mathrm{d}\rho_0}{\mathrm{d}P_0} = \frac{1}{gH} = \frac{1}{(282\,\mathrm{m/s})^2} \sim \frac{1}{c_s^2}$!
- Introduce entropy: $S = c_v \ln P - c_p \ln \rho$. If $c_p \gg c_v$, equivalent to $c_s^2 \to \infty, \Delta\rho = 0$.
  If $c_p \sim c_v \gg 1$ however, $\Delta S = 0$ instead for atmospheric *anelastic* approximation.
- In our system:
  - $gH \ll c_s^2$ and so indeed we may use the equations on the previous slide rather than the atmospheric anelastic equations.
  - Our equations do not conserve horizontal momentum $\rho u_x$ exactly, but do so up to second order in perturbation quantities, and mean flow interactions are a second order effect.

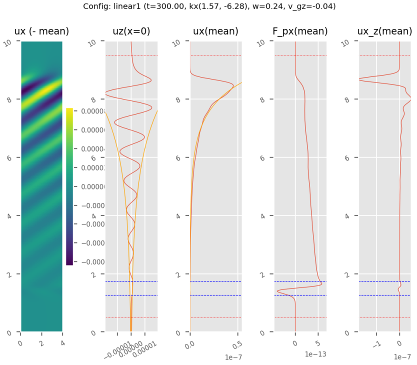
- Perturbation $(\vec{u}, \rho_1, P_1)$ carries average horizontal momentum flux $\langle F_{p,x}\rangle_x = \langle \rho u_x u_z \rangle$.
- Induces mean flow
$$\langle u_x \rangle_x \equiv \bar{U}_x(z) \neq 0 = \frac{\langle u_x u_z \rangle_x}{c_{g,z}}. \tag{4}$$
- Critical layer (equivalent to corotation resonance in other systems): where Doppler-shifted frequency (in fluid rest frame) $\omega \Rightarrow \omega - k_x \bar{U}_x = 0$.
- Since $\vec{u} \propto e^{z/2H}$, so $\bar{U}_x \propto e^{z/H}$, $\exists z_c : \omega - k_x \bar{U}_x(z_c) = 0$.

- Since critical layers almost always induce full absorption (recall, $\propto \exp\left[-\pi\sqrt{\mathrm{Ri}^2 - \frac{1}{4}}\right], \mathrm{Ri} = \frac{N}{\bar{U}_x'}$), hypothesis:
  - $\vec{u}_1$ is excited, induces $\bar{U}_x$ mean flow.
  - Once $\bar{U}_x$ satisfies critical layer criterion, $F_{p,x}$ is fully absorbed.
  - Horizontal momentum goes into spinning up more fluid up to $\bar{U}_{x,crit} = \frac{\omega}{k_x}$.
  - Thus, critical layer should propagate down.
- Exactly *quasi-biennial oscillation* theory (Lindzen 1980, 1982), assume critical layer breaks down eventually.
- Already different from naive "goes nonlinear and deposits locally" theory!

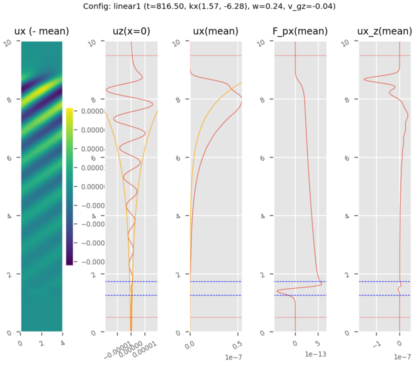
# Simulations

## Large Anelastic, Low-Amplitude

- Permit $z \in [0H, 10H]$, full $\rho_0 \propto e^{-z/H}$, allow $\vec{u} \propto e^{z/2H}$ to source growing mean flow.
- Low-Amplitude ($k_z \xi_z \ll 1$ everywhere). Orange = analytical solution.




**(a)** Early Low-A




**(b)** Later Low-A

# Simulations

## Large Anelastic, Low-Amplitude

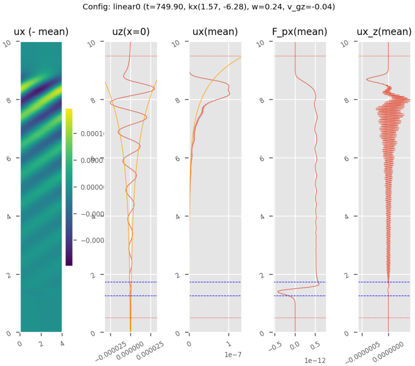
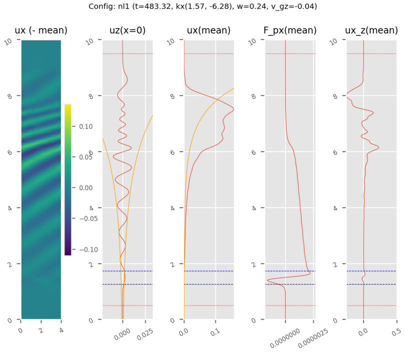
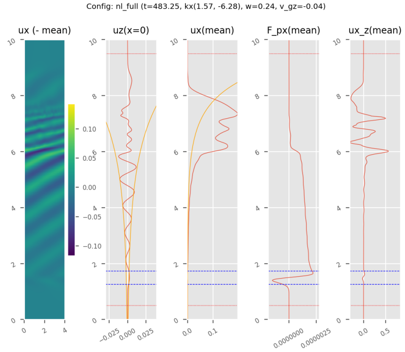



**Figure:** Low-Amplitude, nearly zero viscosity.

- Note steepening region ($N = 1$, so $\frac{\partial \bar{U}_x}{\partial z} = \mathrm{Ri}^{-1}$).




**(a)** Lower-res.

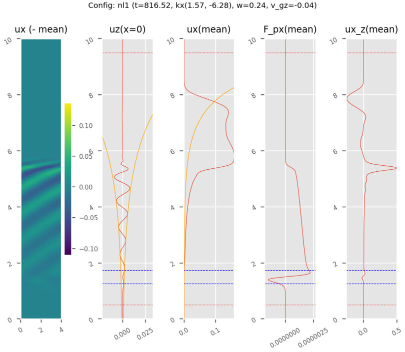



**(b)** Double $N_z$.

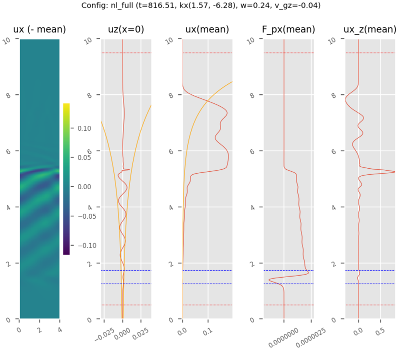
# Simulations

## Large Anelastic, High-Amplitude

- Later




**(a)** Lower-res.




**(b)** Double $N_z$.

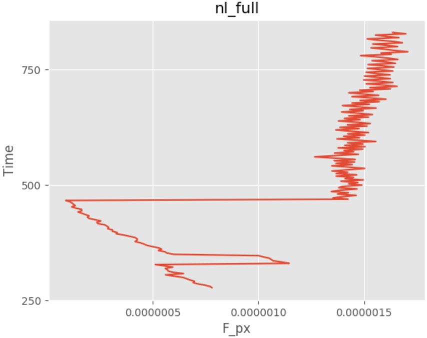
- While damping layers pump up $\bar{U}_x$, ran on higher amplitude, damping layer is not chief cause of $\bar{U}_x$ critical layer behavior.
- Would expect reflections due to $\mathrm{Ri} \gtrsim 1/2$, different from hypothesis!
  - No reflections seen, could be *viscously limited*?
  - Indeed, $\mathrm{Ri}^{-1} \sim \left|\vec{k}\right| d$ where $d$ is the width of the spinup layer, then $\nu\rho_0 \frac{\bar{U}_{x,c}}{d^2} \sim \frac{F_{p,x}}{d}$.
- Separately, matching $F_{p,x} = \langle \rho_0 u_x u_z \rangle$ with spinning up mass $F_{p,x} = u_{front}\rho_0 \bar{U}_{x,crit}$ lets us predict $u_{front}$, next page.

# Simulations

## Large Anelastic, Predicting $u_{front}$

- Front position is $z_f = \text{argmax}_z \frac{\mathrm{d}\bar{U}_x}{\mathrm{d}z}$, while $F_{p,x} = 2F_{p,x}(z_f)$ ($\sim$ halfway in front).
- Predicts $u_{front} = \frac{F_{p,x}}{\rho_0(z)\bar{U}_{x,c}} \approx 2.2 \times 10^{-3} NH$ (using $z \approx 5.5H$), or $2H/3$ in $300N$. Pretty close, seems to imply perfect absorption.


**(a)** Front Position. Slight exponential.

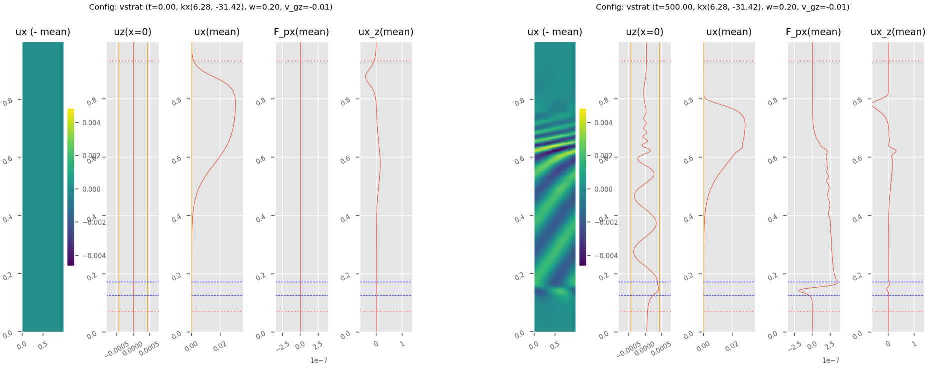



**(b)** $F_{p,x} = \langle \rho_0 u_x u_z \rangle_x$.

# Simulations

## Local Boussinesq

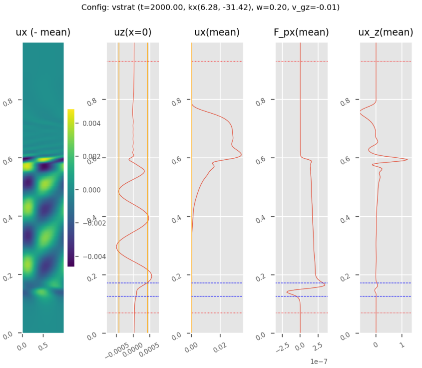
- Go to Boussinesq, "zoom in." Use $\mathfrak{D} = \nabla^6$ regularization. Set up initial mean flow $\bar{U}_x(z)$ such that $\max_z \bar{U}_x(z) = \bar{U}_{x,c}$.
- Reflection indeed develops!

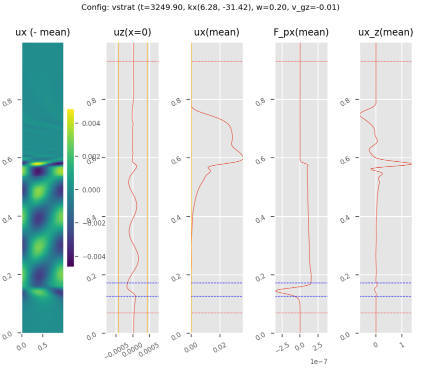


# Simulations

## Local Boussinesq

- Expect Kelvin-Helmholtz instability to develop; resolution-limit? Tried with random noise, no qualitative difference.
- Not viscosity-limited: $\nu^{(6)}\rho_0 \frac{\bar{U}_{x,c}}{d^6} \ll \frac{F_{p,x}}{d}$!

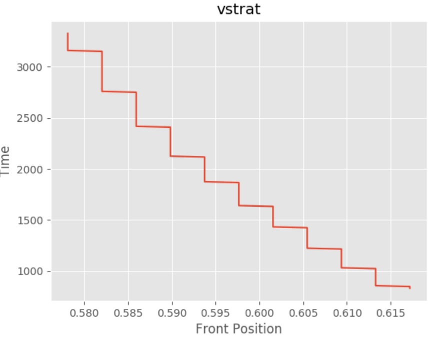


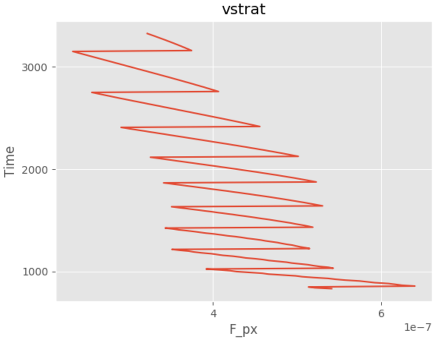
# Simulations

## Comparing $u_{front}$

- Same as for anelastic, but predicts $u_{front} \approx \frac{4.5\times 10^{-7}}{\rho_0 \bar{U}_{x,c}} \approx 1.4\times 10^{-5} HN$ or $0.014H$ in $1000N$. $t \in [1000, 2000]$ accurate, $t \in [2000, 3000]$ less so.
- No front slowdown is expected since $\rho_0$ is constant in space, but is observed; could be explained by increasing reflectivity.




**(a)** Front Position. Slowing down.




**(b)** $F_{p,x}$.