# Research Notes

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# Chapter 1

# 2D Wave Breaking in Atmospheres

The goal of this will be to lay out a formalism that can reproduce Sutherland et al. 2011[1] and also investigate driven oscillations versus the breaking of a single wave packet. This represents wave breaking in the atmosphere.

## 1.1 Dynamical Setup

We adopt notation where $q_0$ is the background quantity and $q_1$ is the perturbed quantity from the propagating wave.

### 1.1.1 Nonturbulent Equations of Motion

The equations of motion are the usual fluid equations

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0, \tag{1.1a}$$

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\frac{\vec{\nabla} P}{\rho} - g\hat{z}, \tag{1.1b}$$

where $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla}$ denotes the Lagrangian derivative. We take $\vec{u}_0 = 0$ no background flow, $P_0(x,z) = P_0(z) \propto e^{-z/H}$ where $H$ is the scale height of the atmosphere. We take the fluid to be incompressible $\vec{\nabla} \cdot \vec{u} = 0$, which turns the continuity equation Eq. 1.1a into $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$. Finally, we take the atmosphere to be isothermal, implying $\rho_0 \propto P_0 \propto e^{-z/H}$, and more importantly $\frac{P_0}{\rho_0} = \frac{P_1}{\rho_1} = \frac{k_B T}{\mu m_p}$. We denote $P_0 = P_0(z=0), \rho_0 = \rho_0(z=0)$.

We write out the new fluid equations under these assumptions; note that we do not assume

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perturbed quantities are small and must keep all terms

$$\frac{\partial \rho_1}{\partial t} = \frac{u_{1z}}{H}\rho_0 e^{-z/H} - \vec{u}_1 \cdot \vec{\nabla}\rho_1, \tag{1.2a}$$

$$\frac{\partial \vec{u}_1}{\partial t} = -\left(\vec{u}_1 \cdot \vec{\nabla}\right)\vec{u}_1 - \frac{-\frac{\hat{z}}{H}P_0 e^{-z/H} + \frac{k_B T}{\mu m_p}\vec{\nabla}\rho_1}{\rho_0 e^{-z/H} + \rho_1} - g\hat{z},. \tag{1.2b}$$

This consists of four equations in four unknowns, $\rho_1$ and the three components of $\vec{u}_1$.

### 1.1.2 Turbulent Dissipation

In order to mimic the effect of turbulent dissipation, we introduce a viscosity term into the equations of motion. Per the incompressible Navier-Stokes equation, we quantify viscosity via $\nu\nabla^2\vec{u}_1$ where $\nu$ is the turbulent viscosity. For this problem, we simply use $\nu \neq 0.1\,\mathrm{cm}^2/\mathrm{s}$ for air.

We identify the kinetic energy density carried by the perturbation to be $\mathcal{U}_T = \frac{1}{2}\rho(\vec{u}_1 \cdot \vec{u}_1)$. We require $\frac{\mathrm{d}\mathcal{U}}{\mathrm{d}t} = 0$ under the dissipation-free fluid equations Eq. 1.1 where we denote $\mathcal{U} \equiv \mathcal{U}_T + \mathcal{U}_V$ the sum of the kinetic and potential energy densities; we assume incompressibility as per our present problem:

$$\begin{aligned}
\frac{\mathrm{d}(\mathcal{U}_T + \mathcal{U}_V)}{\mathrm{d}t} = 0 &= \frac{1}{2}u_1^2 \cancelto{0}{\frac{\mathrm{d}\rho}{\mathrm{d}t}} + \rho\vec{u}_1 \cdot \frac{\mathrm{d}\vec{u}_1}{\mathrm{d}t} + \frac{\mathrm{d}\mathcal{U}_V}{\mathrm{d}t}, \\
&= \rho\left(-\frac{\vec{u}_1 \cdot \vec{\nabla}P}{\rho} - u_{1z}g\right) + \frac{\mathrm{d}\mathcal{U}_V}{\mathrm{d}t}, \\
&= -\cancelto{0}{\frac{\mathrm{d}P}{\mathrm{d}t}} + \cancelto{0}{\frac{\partial P_0}{\partial t}} + \frac{\partial P_1}{\partial t} - \rho u_{1z}g + \frac{\mathrm{d}\mathcal{U}_V}{\mathrm{d}t}.
\end{aligned} \tag{1.3}$$

### 1.1.3 Boundary Conditions

We must bound this to a finite domain. We choose periodic boundary conditions in $x$ with total length $L_x$, $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$. We choose $z$ dimension to have total length $L_z$, $z \in [0, L_z]$.

To set up the boundary condition at $z = 0$, we recall that gravity waves in an atmosphere with $e^{-z/H}$ profile have form

$$u_{1z} \propto e^{\frac{z}{2H}} e^{i(k_x x + k_z z - \omega t)}, \tag{1.4}$$

where ($N^2 \equiv \frac{g}{H}$ is the Brunt-Väisälä frequency in an incompressible, stratified atmosphere)

$$\omega^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}. \tag{1.5}$$

Thus, at constant $z = 0$ we must have

$$\vec{u}_{1z}(z = 0) = e^{i(k_x x - \omega t)}. \tag{1.6}$$

The boundary condition at $z = L_z$ is much harder to determine. It is clear that $\vec{u}_1(z=\infty) = \rho_1(z=\infty) = 0$, and for $L_z \gg H$ this would be a reasonable approximation, if simply because we expect the majority of the wave to dissipate via turbulent dissipation as $z$ reaches many $H$. We will simply choose the BC to be many multiples of $H$. We can solve with both a Dirichlet and Neumann BC and compare the two solutions; if the solutions differ significantly then we must choose a larger $L_z$. These are the only two solutions that can be implemented where we do not need the phase of the linear wave, which we lose during the nonlinear breaking region, to relate the function and derivative at the boundary.

For the boundary conditions on $\rho$, we note that in the linear regime it should just have a phase offset from $u_{1z}$, thus we choose $\rho(z=0) \propto e^{\frac{z}{2H}} i e^{i(k_x x + k_z z - \omega t)}$ and a similar treatment at $z = L_z$, taking both a Dirichlet and Neumann BC. This should be sufficient information to constrain our problem, as satisfying incompressibility along the boundaries constrains $u_{1x}$.

**Old Stuff—**$u_{1z}(z = L_z)$

**Note:** The reason the below is difficult to make work is that it tries to restore a linear treatment to the wave. However, since we must capture nonlinear dynamics, we lose the concept of the “phase” of the wave in the wave breaking region, and it proves difficult to write down a BC that is phase-independent.

However, at any finite boundary there is nonzero reflection thanks to the continually varying impedance. *The correct BC to use when performing a finite truncation must match propagation characteristics with an untruncated medium.*

We assume the perturbations have once again become small at $L_z$, i.e. dissipation between $[0, L_z]$ is sufficient that only a small amplitude perturbation persists to $z = L_z$. This is a necessary criterion for choosing $L_z$ else we do not have a representative portrait of the energy dissipation profile of the propagating wave with our simulation. Then we recall that in the linear regime, the solution given by Eq. 1.4 is the propagating solution. We assume that $k_x, \omega$ are the same as the values at the $z = 0$ boundary, then

$$\frac{\rho_{,z}}{\rho} = \frac{u_{1z,z}}{u_{1z}} = \frac{1}{2H} + ik_z. \tag{1.7}$$

Recall that since our equations of motion are nonlinear, we cannot use complex notation except in very limited, time-averaged cases. Thus, how to implement this BC is tricky...

### 1.1.4 Simulation

We begin our simulation with $\rho_1 = 0, \vec{u}_1 = 0$ strictly within the domain of simulation. We will borrow some values from Sutherland's paper and use $k_z = 2\,\text{km}^{-1}$ then define $k_x = -0.4k_z, H = 10/k_z, A = 0.05/k_z, L_z = 300/k_z, L_x = 20/k_z$, We also use $\mu \approx 29, T = 273\,\text{K}, \rho_0 = 1\,\text{kg/m}^3, P_0 = \frac{\rho_0 k_B T}{\mu m_p}, g = 10\,\text{m/s}^2$.