Research Notes

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Contents

Chapter 1

Preliminary Problems

To get an intuition for how Dedalus and fluid mechanics works, we will solve some toy problems. Recall fluid equations in the presence of a uniform gravitational field $\vec{g} = -g\hat{z}$:

$$\begin{aligned}
\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{u}\right) &= 0,\\
\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \frac{\vec{\nabla}P}{\rho} - \vec{g} &= 0.
\end{aligned} \tag{1.1}$$

In the incompressible limit, $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$, which implies $\vec{\nabla}\cdot\vec{u} = 0$. We use subscripts to indicate perturbed quantities, Q_0 is background and Q_1 is perturbed. We will generally use $\vec{u}_0 = 0$ unless otherwise noted. We will also generally assume symmetry along all axes except z the vertical axis.

In the incompressible limit, the fluid equations become

$$\begin{aligned}
\vec{\nabla}\cdot\vec{u}_1 &= 0,\\
\frac{\partial\rho_1}{\partial t} + u_{1z}\frac{\partial\rho_0}{\partial z} &= 0,\\
\frac{\partial\vec{u}_1}{\partial t} + \frac{1}{\rho_0}\vec{\nabla}P_1 + \frac{\rho_1 g\hat{z}}{\rho_0} &= 0.
\end{aligned} \tag{1.2}$$

We have used $\vec{\nabla}P_0 = -\rho_0 g\hat{z}$ in the absence of perturbations.

1.1 Incompressible, No Gravity

We note that in the no gravity limit that ρ_1 does not have an effect on other dynamical variables, so the equations of motion we must solve are

$$\begin{aligned}
\vec{\nabla}\cdot\vec{u}_1 &= 0,\\
\frac{\partial\vec{u}_1}{\partial t} + \frac{\vec{\nabla}P_1}{\rho_0} &= 0.
\end{aligned} \tag{1.3}$$

We can take the divergence of the momentum equation and substitute the continuity equation to get $\nabla^2 P = 0$.

1.1.1 Dirichlet BCs

This is a Laplace equation, which we've solved countless times. Imposing periodic boundary conditions in the x direction and $P_1(z = L) = 0, P_1(z = 0) = \mathscr{P}(x,t)$, we obtain eigenfunctions

$$\begin{aligned}
P_{1,n}(x,z,t) &= \frac{\mathscr{P}_n(t)}{\sinh(k_n L)} e^{ik_n x} \sinh(k_n(L-z)),\\
u_{1x,n}(x,z,t) &= \int^t -\frac{1}{\rho_0}\frac{\partial P_{1,n}}{\partial x}\,\mathrm{d}t,\\
u_{1z,n}(x,z,t) &= \int^t -\frac{1}{\rho_0}\frac{\partial P_{1,n}}{\partial z}\,\mathrm{d}t.
\end{aligned} \tag{1.4}$$

We define $k_n = \frac{2\pi n}{L}, n \ge 0$ and $\mathscr{P}(x,t) = \sum\limits_n \mathscr{P}_n(t) e^{ik_n x}$.

Thus, if we impose BCs $\mathscr{P}(x,t) = \sin\frac{2\pi x}{L}$ and start with initial conditions such that all quantities are zero, we would expect after transients die out that

$$\begin{aligned}
P(x,z,t) &= \frac{\sin\frac{2\pi x}{L}}{\sinh 2\pi}\sinh\left(2\pi\frac{L-z}{L}\right),\\
u_{1x}(x,z,t) &= -\frac{2\pi t}{L\rho_0}\frac{\cos\frac{2\pi x}{L}}{\sinh 2\pi}\sinh\left(2\pi\frac{L-z}{L}\right),\\
u_{1z}(x,z,t) &= +\frac{2\pi t}{L\rho_0}\frac{\sin\frac{2\pi z}{L}}{\sinh 2\pi}\cosh\left(2\pi\frac{L-z}{L}\right).
\end{aligned} \tag{1.5}$$

This is in good agreement with the results, presented in Fig. 1.1. Note that P is constant while $\vec{u}$ increases linearly in time, and we observe the expected $\sim \sin x \sinh\frac{L-z}{z}$ dependence. In fact, u_{1x}, u_{1z} are exactly $\frac{2\pi}{10}$ at $t = 1$.

1.1.2 Things to Note

It is worth noting that, since our Eq. 1.3 reduced to a Laplace equation, we needed two z BCs and two x BCs (periodic BCs amount to equating the value and derivative of the function). This is in agreement with the observation that the original Eq. 1.3 had two derivatives in x, z apiece, so we needed two BCs each.

It is also worth seeing from our solution that P immediately goes to the equilibrium solution. This is not surprising since in the incompressible limit, sound speed goes to infinity which is the timescale on which the pressure field adjusts to net forces. Thus, the dynamics solely arise from the static pressure field pushing the velocity field to equilibrium.

We chose time-independent $\mathscr{P}(x,t)$, but it is clear that whatever $\mathscr{P}(x,t)$ we choose, the time dependence propagates to the velocities by way of an integral. If we had instead chosen to take Fourier

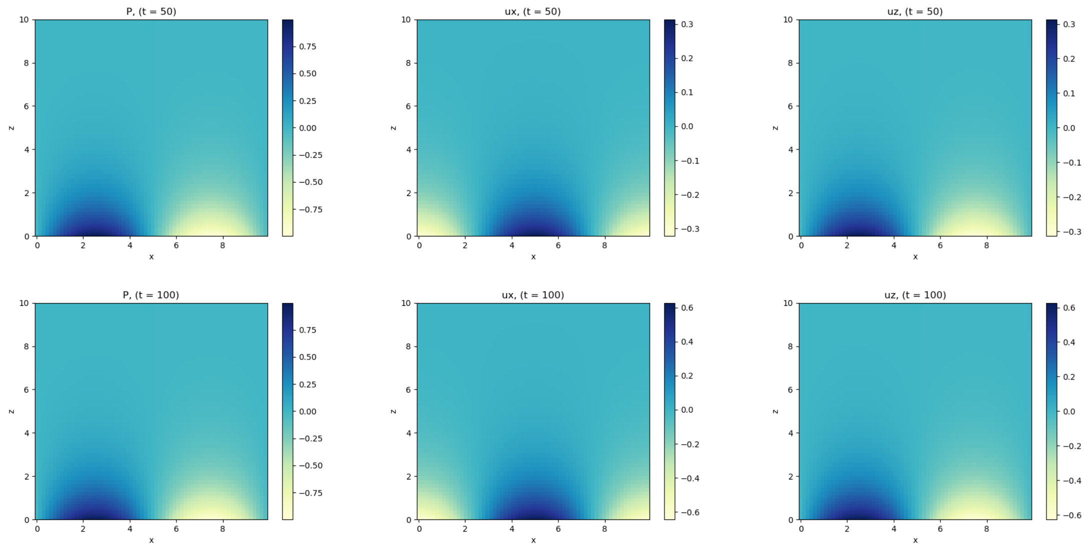

Figure 1.1: P, u_x, u_z at $t = 0.5$ and $t = 1$ for $\rho_0 = 1$. We choose a $L = 10$ square domain.

transform $t \to \omega$, we would have had to integrate the boundary condition against the eigenfunctions for each of the ω, which is still easily computable, to get the full $u_{1x}(x, z, t)$. We consider this in preparation for when we can only solve for a set of $\vec{k}, \omega$ in the next problem.

The next thing we would have wanted to do is solve a problem with radiative BCs, but we need to have wave solutions, which are missing in the absence of gravity. We thus move on to the next configuration.

1.2 Incompressible, Stratified w/ Gravity

1.2.1 Eigenfunctions

Let's restore the $\rho_1 g$ term now. For funsies, we begin by solving for arbitrary stratification $\rho_0(z)$ first. The fluid equations to first order reduce to

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \vec{u}_1 \cdot \left(\vec{\nabla} \rho_0\right) &= 0, \\ \vec{\nabla} \cdot \vec{u}_1 &= 0, \\ \frac{\partial \vec{u}_1}{\partial t} &= -\frac{\vec{\nabla} P_1}{\rho_0} - \frac{\rho_1 g}{\rho_0} \end{aligned} \tag{1.6}$$

We expect there to be some z dependence in the amplitude, so we substitute variables of form $e^{i(kx-\omega t)}$ and do not specify the z dependence. This gives us

$$\begin{aligned}
-i\omega\rho_1 - u_{1z}\frac{\partial\rho_0}{\partial z} &= 0,\\
iku_{1x} + \frac{\partial u_{1z}}{\partial z} &= 0,\\
-iwu_{1x} + \frac{ik_x P_1}{\rho_0} &= 0,\\
-iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1 g}{\rho_g} &= 0.
\end{aligned} \tag{1.7}$$

We substitute $N^2 = -\frac{g}{\rho_0}\frac{\partial\rho_0}{\partial z}$ to obtain

$$-i\omega\rho_1 - u_{1z}\frac{\rho_0 N^2}{g} = 0, \tag{1.8a}$$

$$iku_{1x} + \frac{\partial u_{1z}}{\partial z} = 0, \tag{1.8b}$$

$$-iwu_{1x} + \frac{ik_x P_1}{\rho_0} = 0, \tag{1.8c}$$

$$-iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1 g}{\rho_0} = 0. \tag{1.8d}$$

Eliminating u_{1x} by substituting (1.8b) into (1.8c) and ρ_1 by substituting (1.8a) into (1.8d) give

$$i\omega\frac{\partial u_{1z}}{\partial z} + \frac{k_x^2 P_1}{\rho_0} = 0, \tag{1.9a}$$

$$\left(\omega^2 - N^2\right)u_{1z} + \frac{i\omega}{\rho_0}\frac{\partial P_1}{\partial z} = 0. \tag{1.9b}$$

Finally, we multiply (1.9a) with ρ_0 and differentiate dz and combine with (1.9b) to give

$$\frac{\mathrm{d}^2 u_{1z}}{\mathrm{d}z^2} + \frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z}\frac{\partial u_{1z}}{\partial z} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right)u_{1z} = 0. \tag{1.10}$$

Let's now pick stratification $\rho \propto e^{-z/H}$ Eq. 1.10 clearly has exponential solutions $e^{\kappa z}$ for

$$\kappa^2 - \frac{\kappa}{H} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right) = 0. \tag{1.11}$$

We permit complex $\kappa = \frac{1}{2H} + ik_z$, and from the above clearly

$$\begin{aligned}
k_z^2 &= -\frac{1}{4H^2} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right),\\
\omega^2 &= \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}.
\end{aligned} \tag{1.12}$$

Thus the eigenfunctions are

$$\begin{aligned}
u_{1z} &= e^{z/2H} e^{i(k_z z + k_x x - \omega t)},\\
u_{1x} &= -\frac{k_z + i/2H}{k_x} u_{1z},\\
\rho_1 &= \frac{i\rho_0}{H\omega} u_{1z},\\
P_1 &= -\frac{\rho_0 \omega}{k_x^2}(k_z + i/2H) u_{1z}.
\end{aligned} \tag{1.13}$$

1.2.2 Analytically Solving an IVP, Dirichlet + Driving BCs

We will analyze everything in terms of u_{1z} since it has the simplest form; note that when actually choosing the BCs we will have to consider the gauge freedom of P and some considerations we defer to the computational section.

Currently, we have a set of eigenfunctions

$$u_{1z}\left(x,z,t\middle|\vec{k},\omega\right) = e^{z/2H} e^{i(k_z z + k_x x - \omega t)}. \tag{1.14}$$

Note that u_{1z} is really only a function of two parameters rather than the three (k_x, k_z, ω), since the three are related by dispersion relation Eq. 1.12.

Now, we implement BCs. Consider domain $x, z \in [0, L]$. We will use periodic BCs again in x, so then $k_{x,n} = \frac{2\pi n}{L}, n \ge 0$. Then, we will require $u_{1z,n}(x, L, t) = 0$, a Dirichlet condition at the top boundary, which restricts us to eigenfunctions of form

$$u_{1z,n}\left(x,z,t\middle|\vec{k},\omega\right) = e^{z/2H} e^{i(k_x x - \omega t)} \sin(k_z(L - z)). \tag{1.15}$$

Finally, we must choose a BC at $z = 0$. We will choose a general function $u_{1z}(x, 0, t) = F(x, t)$ where we can decompose

$$F(x,t) = \int \sum_n \mathscr{F}(k_{x,n}, \omega) e^{i(k_{x,n} x - \omega t)}\, \mathrm{d}\omega. \tag{1.16}$$

Matching BCs then gives us general solution for u_{1z} given an arbitrary driving function

$$u_{1z}\left(x,z,t\middle|\vec{k},\omega\right) = \int \sum_n \mathscr{F}\left(k_{x,n}, \omega\right) \frac{e^{z/2H} e^{i(k_{x,n} x - \omega t)} \sin(k_z(L - z))}{\sin k_z L}\, \mathrm{d}\omega. \tag{1.17}$$

For ease of computation, let's pick $F(x,t) = \cos\left(\frac{2\pi x}{L} - \omega_0 t\right)$, so our full expected solution is (note

$A + \epsilon = Ae^{\epsilon/A} + \mathcal{O}(\epsilon^2))$

$$\begin{aligned}
u_{1z}\left(x,z,t\middle|\vec{k},\omega\right) &= e^{z/2H}\frac{\cos\left(\frac{2\pi x}{L} - \omega_0 t\right)\sin(k_z(L-z))}{\sin k_z L},\\
u_{1x}\left(x,z,t\middle|\vec{k},\omega\right) &\approx \frac{k_z}{k_x}e^{z/2H}\frac{\cos\left(\frac{2\pi x}{L} - \omega_0 t + \frac{1}{2Hk_z}\right)\sin(k_z(L-z))}{\sin k_z L},\\
\rho_1\left(x,z,t\middle|\vec{k},\omega\right) &\approx \frac{\rho_0}{H\omega}e^{z/2H}\frac{-\sin\left(\frac{2\pi x}{L} - \omega_0 t\right)\sin(k_z(L-z))}{\sin k_z L},\\
P_1\left(x,z,t\middle|\vec{k},\omega\right) &\approx -\frac{\rho_0\omega k_z}{k_x^2}e^{z/2H}\frac{\cos\left(\frac{2\pi x}{L} - \omega_0 t + \frac{1}{2Hk_z}\right)\sin(k_z(L-z))}{\sin k_z L},
\end{aligned} \tag{1.18}$$

where $k_z : \omega(k_x, k_z) = \omega_0$ by the dispersion relation Eq. 1.12. Note that H contributes both to the overall exponential profile and to the phase lag of u_{1x}.

1.2.3 Computationally Solving an IVP, Dirichlet + Driving BCs

To solve this computationally with the aforementioned BCs, periodic in x, Dirichlet 0 at $z = L$ and $\cos\left(\frac{2\pi x}{L} - \omega_0 t\right)$ at $z = 0$, we must address the gauge freedom in P. This arises because for $k_x = 0$, the divergence-free condition $\vec{\nabla}\cdot\vec{u}_1 = \frac{\partial u_z}{\partial z} = 0$ specificies u_{1z} up to a constant already, so the bottom BC will fix the value of u_z when $k_x = 0$.

A different way of phrasing the same argument is as follows. Consider the discrete $N \times N$ (square for notational simplicity) grid. At the boundary, there is a list of values $f(\{x_i\}, z_0)$ that lives in an N dimensional space. We can thus pick a spanning set of N basis vectors, and for each of these N vectors, by enforcing $\vec{\nabla}\cdot\vec{u} = 0$ at the boundary we fix the allowed $f(\{x_i\}, z_{-1})$ boundary conditions we can implement. But there exists a choice of basis vectors for which one of the basis vectors is constant $f_i(\{x_i\}, z_0) = C$. For this basis vector, the boundary condition is fully determined, so we only have $N-1$ dimensions from which to choose the BCs $f(\{x_i\}, z_{-1})$. Since the dimensionality of a space cannot depend on the choice of basis vector, the divergence free condition actually yields an extra degree of freedom.

We can use this extra degreee of freedom to specify $P(z = L) = 0$ so the oscillations should have zero mean. Then we can just simulate away!

From here we can just tweak the parameters until we get something useful, can look at the videos. Note that we should simulate until at least $T > \frac{2L_z}{v_{p,z}}$ the z phase space velocity, so we can capture any reflections off the boundary. Turns out both Dirichlet and Neumann BCs give strong reflections that produce standing waves in the z and traveling waves in the x.

1.2.4 Phase/Group Velocity

Let's figure out the analytical forms for the phase, group velocity and the energy density/power flux.

Let's first consider the phase velocity. We traditionally think about the phase velocity in 1D $v_{ph} = \frac{\omega(k)}{k}$, but in general it is the function such that $\vec{k}\cdot\vec{r} - \omega t = \vec{k}\cdot\left(\vec{r} - \vec{v}_{ph}t\right)$ is constant (the phase of the wave). Thus, it must satisfy $\vec{k}\cdot\vec{v}_{ph} = \omega$, and so one sensible choice is

$$\vec{v}_{ph} = \frac{\omega\hat{k}}{|\vec{k}|} = \frac{\omega\vec{k}}{|\vec{k}|^2}. \tag{1.19}$$

For our stratified atmosphere problem, $\omega^2 = \frac{N^2 k_x^2}{k^2+\frac{1}{4H^2}}$. This corresponds to $\vec{v}_{ph} = \frac{N k_x}{(k_x^2+k_z^2+1/4H^2)^{3/2}}\vec{k}$.

On the other hand the group velocity is given $\vec{v}_g = \frac{\partial\omega}{\partial k_i}\hat{\imath}$. In our problem, $v_{g,z} = -\frac{N k_x k_z}{(k_x^2+k_z^2+1/4H^2)^{3/2}}$ while $v_{g,x} = \frac{N}{\sqrt{k_x^2+k_z^2+1/4H^2}} - \frac{N k_x^2}{(k_x^2+k_z^2+1/4H^2)^{3/2}} = \frac{N(k_z^2+1/4H^2)}{(k_x^2+k_z^2+1/4H^2)^{3/2}}$.

1.2.5 Energy/Power Flux

To compute the energy and power flux of the wave, we recall that for a general fluid the energy conservation equation reads

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2 + \rho\epsilon\right) = \vec{\nabla}\cdot\left(\rho\vec{v}\left(v^2+\epsilon+\frac{P}{\rho}\right)\right), \tag{1.20}$$

where ϵ is the internal energy. But since $P = (\gamma-1)\rho\epsilon$ and we take $\gamma\to\infty$ incompressible limit, $\epsilon = 0$, so the energy of the wave is just $\frac{1}{2}(\rho_0+\rho_1)u_1^2$ and the power flux is $(\rho_0+\rho_1)\vec{u}_1 u_1^2 + P_1\vec{u}_1$.

Another formula given by Sutherland 2011 is $\frac{\partial\langle E\rangle}{\partial t} = v_{g,z}\langle E\rangle$ where we average over x one wavelength. I wasn't able to show that these two agree in general; Sutherland's formula seems to be derived for a traveling wavepacket whereas we have no such thing. We do not use this expression since the Landau & Lifshitz expression works well.

1.3 Supressing Reflection

Sourced from `https://people.maths.ox.ac.uk/trefethen/6all.pdf`. There are largely two ways to do this. The first is to add a damping zone, a region in which $\dot{q} = -q/\tau_d(z)$ where $\tau_d(0) = 0$ and increases to some non-small number for some z_0 beyond which we want damping. One form, chosen in Ryan and Dong's paper, is to use multiplicative factor $f(z) = \max\left[0, 1-\frac{(z-z_b)^2}{(z_d-z_b)^2}\right]$ where z_b is where one begins supression and z_b is the boundary. Then a dynamical variable q can be supressed via $\dot{q} = (\dots) - f(z)\frac{q-q_0}{\tau}$ where τ is the dynamical timescale on which to damp and q_0 is the value to damp to. We will apply this only to the velocity variables with an eye to extending this approach to the nonlinear regime, where the velocity variables should still be damped to zero but ρ, P must capture the stratification and so cannot easily be artificially damped (maybe? Revisit if this does poorly compared to damping all linear variables, and damp the nonlinear variables to their equilibrium/stratified values).

1.3.1 First Order

In order to compute the reflecting boundary condition, we must find a sequence of differential operators that well approxmiates the system of PDEs. We can get this from the dispersion relation; since our radiating boundary is along z, we should try to solve for k_z in the dispersion relation to get a pseudodifferential operator form for $\frac{\partial}{\partial z}u_z$ (we take $k_xH \ll 1$ as well to simplify):

$$\begin{aligned} k_z^2 &= \frac{N^2}{\omega^2}k_x^2 - k_x^2 - \frac{1}{4H^2}, \\ &\approx k_x^2\left(\frac{N^2}{\omega^2} - 1\right), \end{aligned} \tag{1.21}$$

$$\frac{\partial u_z}{\partial z} \approx -\frac{\partial u_z}{\partial x}\sqrt{\frac{N^2}{\omega^2} - 1}. \tag{1.22}$$

We pick the negative sign in accordance with an outgoing wave, per the group velocity formulae for $k_z > k_x$.

From the numerical simulations, it is clear that reflections are well supressed initially, but the the simulation blows up. This is because our supression is imperfect (differential approximation to a pseudodifferential operator) and since our system is dissipation free, reflection grows. Moreover, since any reflected components seem to have different k_x, k_z, ω, the second time they're incident on the boundary, our above condition becomes wildly inaccurate.

1.4 The Difficulty of the Incompressible, Nonlinear Problem

We consider the problem where both advective terms are kept and $\rho_1 \sim \rho_0$. We write down thus nonlinear fluid equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{u}\cdot\left(\vec{\nabla}\rho\right) &= 0, \\ \vec{\nabla}\cdot\vec{u} &= 0, \\ \frac{\partial \vec{u}}{\partial t} &= -\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} - \frac{\vec{\nabla}P}{\rho} - g\hat{z}. \end{aligned} \tag{1.23}$$

Note that we cannot even subtract off hydrostatic equilibrium anymore since ρ can deviate greatly from the normal $\rho_0 e^{-z/H}$!

1.4.1 Rearrangement for Dedalus

In order to simulate this at all in a spectral program, recall the way that nonlinear PDEs are decomposed in a spectral code. Given a phase space Q, consider first a system of PDEs expressible in form

$$\dot{Q} + \mathbf{L}Q + f(Q) = g \tag{1.24}$$

where $\dot{Q}$ is the time derivative, $\mathbf{L}$ is some linear operator, $f(Q)$ are nonlinear terms and g is an arbitrary driving function. Spectral codes work by re-casting the operators to be purely algebraic in an appropriate spectral basis consisting of some φ_i trial and ξ_i test functions. The *tau spectral method* that Dedalus uses considers using the trial functions as the test functions, so the effective decomposition that is performed is

$$\langle\varphi_i\big|\dot{Q}+\mathbf{L}Q\big|\varphi_j\rangle = \dot{Q}_i + L_{ij}Q_q = \langle\varphi_i\big|-f(Q)+g\big|\varphi_i\rangle. \tag{1.25}$$

It is then clear that $-f(Q)+g$ can just be treated as inhomogeneous source terms, and over a short Δt it can be approximated via $-f(Q_0)+g$. This first-order inhomogeneous ODE then admits closed form exact solutions $e^{-L_{ij}t}(\ldots)$.

One can imagine, for this matter, that $\frac{\partial}{\partial t}$ can be wrapped in the L_{ij} operator and the timestepping computed implicitly; I believe this is what Dedalus does, implicitly timestepping the left hand side and explicitly the inhomogeneous terms.

Both of these procedures require $\mathbf{L}$ linear operator or $\frac{\partial}{\partial t}+\mathbf{L}$ to be full rank, or invertible. This is enforced analytically by requiring the same number of BCs as derivatives, and numerically this is little different. This is why it is possible to have components of Q that are not differentiated ∂_t but still have well-defined time evolutions; the only condition required is that the linear terms of the PDE make up a full-rank operator. Hence, in our linear incompressible equations, we permitted use of $\vec{\nabla}\cdot\vec{u}_1=0$ in lieu of a $\frac{\partial P}{\partial t}$ explicit equation of motion; so long as the gauge for P (which only appears in derivatives) is properly set, the operator is full rank and numerically well-defined.

In the case of the full nonlinear equations though, it bears noting that P only appears in nonlinear terms! Thus, the linear terms clearly cannot have full rank since they fail to reference P at all. We remedy this by re-casting the momentum equation as

$$\frac{\partial\vec{u}}{\partial t}+\frac{\vec{\nabla}P}{\rho_0} = -\vec{\nabla}P\left(\frac{1}{\rho}-\frac{1}{\rho_0}\right)+\vec{g}. \tag{1.26}$$

This ensures that, along with suitable gauge choice for P, the linear operators on the left hand side of the equation have full rank. The remaining equations

$$\frac{\partial\rho}{\partial t} = -\vec{u}\cdot\vec{\nabla}\rho, \tag{1.27a}$$

$$\vec{\nabla}\cdot\vec{u}=0, \tag{1.27b}$$

remain unchanged.

1.4.2 Pathology with u_z BCs

Our problem turns out to be ill-conditioned, the pathology arising from our driving term in conjunction with the incompressibility constraint. Since we want to model the propagation of waves from where they are excited, rather than exciting them directly in our problem, we must drive our region of interest via boundary conditions rather than forcing terms.

By driving via boundary conditions, we can only specify boundary conditions on the variables being differentiated ∂_z in our equations of motion, since we enforce fully periodic BCs in x via using a Fourier spectral decomposition. Thus, we are restricted to enforcing BCs on either $\vec{u}$ or P.

It turns out that by using BCs on $\vec{u}$, we run into a problem: the BCs we specify on u_z or its derivatives will in general affect grid points $\partial_z u_{z,1x}$ while those specified on u_x can only affect $\partial_x u_{x,0x}$. This means that $\partial_x u_x$ will pick up sharp jerks by being coupled to the BC, producing high frequency components in u_x as can be seen in Fig. 1.2. These come from enforcing the divergence-free condition too literally at the boundary. Not that the figure was produced with only a Dirichlet BC on u_z, no BCs on u_x, in a linear system with absorbing BCs.

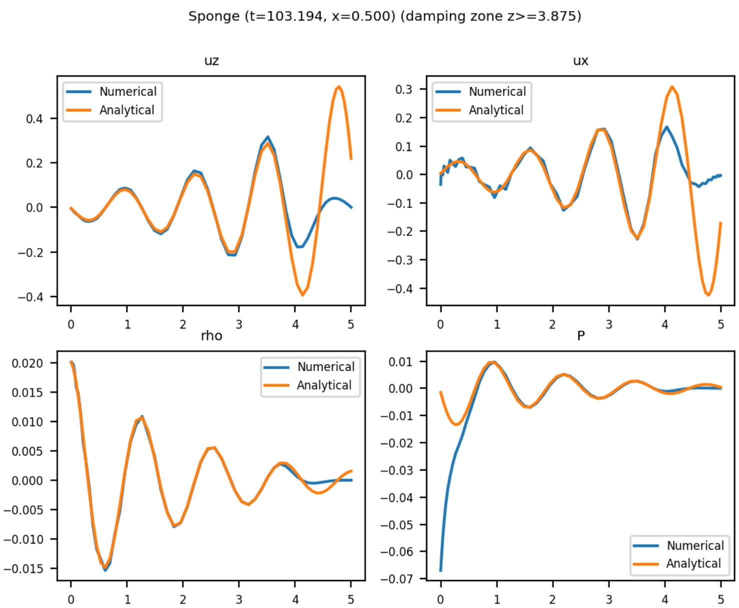

Figure 1.2: Note the jaggedness in u_x induced by a Dirichlet BC on u_z.

This problem seems to be remedied somewhat by instead enforcing Neumann BCs on u_z, or even when allowing dissipation enforcing Neumann BCs on $\partial_z u_z$ (the extra derivatives granted by $\nu\nabla^2\vec{u}$ let us specify higher order BCs). However, in the fully nonlinear system we still observe divergences near $z = 0$ regardless of how smooth of BCs we use on u_z. Moreover, since the discontinuities come from the nature of the BC and not its strength, we run into these instabilities regardless of how gradually we introduce the BC (e.g. we could consider multiplying by $1 - e^{-t/\tau}$ to gradually introduce the BC, but this turns out also to be unstable).

There are thus three possible solutions to this:

- Consider BCs only on P, the only other choice of dynamical variable. This seems to also exhibit instabilities near $z = 0$.
- Consider using a $\nabla^2 P$ equation instead of a $\vec{\nabla}\cdot\vec{u} = 0$ constraint. The correct choice of BCs in this case is discussed in Nordstrom et al, 2006. We choose not to investigate this.
- Change the system of equations to no longer consider an incompressible system; the anelastic system is an attractive alternative. This boasts the advantage of not having an algebraic constraint equation.

TODO Can taking $\rho_1 \ll \rho_0$, giving us a full rank operator w/o doing the janky double add thing, solve in the incompressible limit? Otherwise, go anelastic.

1.5 Limits of the Incompressibility and Anelasticity Assumptions

To get a better handle on the regimes of validity of the incompressible and anelastic approximations, we return to the full fluid equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{u}\right) &= 0, \\ \frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} + \frac{\vec{\nabla} P}{\rho} - \vec{g} &= 0. \end{aligned} \tag{1.28}$$

Furthermore, we assume small perturbations $\delta P = c_s^2 \delta\rho$ where c_s^2 is the sound speed, also the velocity at which an adiabatic perturbation is restored. We drop the $-\vec{g}$ from the analysis assuming that it is largely negated by an appropriate background P.

Given these considerations, we consider in what limit we may presume the fluid to be incompressible. Consider if $|\vec{u}| \ll c_s$, then for some characteristic frequency ω and wavenumber k the momentum equation scales as $\omega u + ku^2 + c_s^2 \delta\rho/\rho = 0$. Comparing scales it is thus clear that $\delta\rho \ll \rho$ in this limit. Then, considering the mass continuity equation, we recognize that the scalings are $\omega\delta\rho + \rho\vec{\nabla}\cdot\vec{u} + k\delta\rho u = 0$. Thus, $\vec{\nabla}\cdot\vec{u} = 0$ necessarily, since ρ is much larger than the other two terms, and we obtain the incompressible mass continuity equation from the other two terms. While this is obviously not a very careful treatment, it should suffice to convince us that $u \ll c, \delta\rho \ll \rho, \vec{\nabla}\cdot\vec{u} = 0$ are all roughly equivalent conditions.

On the other hand, when may we drop the total derivative terms? This seems to be a slightly different criterion, and indeed we may compare $\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} \sim \omega\vec{u} + ku^2$. Thus, the second term is dominated by the first term when $\vec{u} \ll \vec{v}_{ph}$ where $\vec{v}_{ph} \equiv \frac{\omega}{k}\hat{k}$ is the phase velocity. Recall that for our stratified waves that $\omega \le N = \sqrt{g/H}$ as a point of reference. So we require $u \ll v_{ph}$ to be able to drop the total derivative terms.

Since in general $v_{ph} \ll c_s$, we thus identify that there are three regimes. The first, $u \ll v_{ph}$, we solve the completely linear problem. Then, for $v_{ph} \lesssim u \ll c_s$, we must begin to consider the advective terms but are still allowed to assume $\rho_1 \ll \rho_0$, an assumption that is only violated when $u \sim c_s$.

If we look ahead to the anelastic system derivation, we can identify that the anelastic approximation corresponds to assuming $c_s^2 \gg uv_{ph}$, making no assertion of the scale between u, v_{ph}. Thus, we can also consider the anelastic approximation with advective terms just fine in teh regime $v_{ph} \lesssim u$, and if this is sufficient to induce wave breaking then we may never need to consider the $\rho_1 \sim \rho_0$ enormous perturbations.

Chapter 2

Simulation Setup

We describe now some concerns that arise in setting up the simulation.

2.1 Damping Zone

Since our code is a spectral code, instead of using the quadratic damping zone presented from before (whose second derivative is discontinuous over the entire domain), we prefer instead an arctan profile, such that

$$\Gamma(z) \propto 0.5 * \left(2 + \tanh C\left(\frac{z - z_t}{L_z - z_t}\right) + \tanh C\left(\frac{z - z_b}{z_b}\right)\right). \tag{2.1}$$

This is a profile that mostly vanishes $z \in [z_b, z_t]$ but is unity for $z \lesssim z_b, z \gtrsim z_t$, with steepness of transition governed by factor C. While this profile may have reflection issues if C is chosen to be too steep, it is smooth and so has exponentially convergent spectral expansion and empirically seems to do alright.

It is moreover prudent to note that, since implicit calculations are expensive, as long as the damping timescale Γ^{-1} is everywhere resolved it is cheaper to put $\Gamma\rho_1, \Gamma\vec{u}_1$ on the right hand sides of their equations in Dedalus, which is timestepped explicitly.

2.2 Volumetric Forcing

Due to the structure of `Dedalus`, we must use Chebyshev polynomials in the vertical direction (to be able to enforce BCs). But as discussed in §B.1, the grid spacing of the Chebyshev polynomials near boundaries goes as $1/N^2$, which means that interface forcing necessitates an extremely strict Courant condition to capture evolution near the interface; this is the explanation for the divergences we were seeing in earlier sections. In fact, in the literature, the spectral CFL condition meets a $\Delta t_{\max} \sim 1/N^2$ scaling thanks to poor bounds near the boundary.

Instead, we employ volumetric forcing. As a toy problem of seeing how this works, we will study the restricted 1D wave equation.

2.2.1 1D Wave Equation (Deprecated)

Consider the forced 1D wave equation (we set $c = 1$)

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t). \tag{2.2}$$

Note that the only allowed propagation modes here are $u \sim e^{iC(x \pm t)}$ for some constant C.

We wish to choose $f(x,t)$ such that $u(x \to -\infty) = 0$ while $u(x \to +\infty) \sim e^{i(x-t)}$. It is clear that if we consider the Fourier transform $\tilde{f}(k,t)$, the response in $\tilde{u}(k,t)$ will also contain components for each of the nonzero $\tilde{f}(k,t)$ components. Thus, if we want a pure $e^{i(x-t)}$ outgoing wave, we should have a pure $\tilde{f}(k,t) \propto \delta(k-1)$. However, it is clear that driving the oscillator exactly on resonance produces divergent oscillation amplitudes, and moreover it is impossible to drive with an $f(x,t)$ that doesn't vanish as $x \to \pm\infty$.

It is nonetheless clear that $\tilde{f}(k,t)$ should be strongly peaked at $k = 1$ and small elsewhere. We can thus consider a form $f(x,t) \propto e^{-x^2/2\sigma^2} e^{i(x-t)}$ such that $\sigma \gg 1$, i.e. the envelope contains many wavelengths. The Fourier transform $\tilde{f}(k,t)$ is then a Gaussian with width $1/\sigma \ll 1$ centered at $k = 1$ (convolution).

More precisely, consider $f(x,t) = f_0 e^{-x^2/2\sigma^2} e^{i(x-t)}$ and $u(x,t) = u_0(x) e^{i(x-t)}$ such that $u_0(x \to \infty) = 0$. Substituting this into the inhomogeneous PDE gives

$$i\frac{\partial u_0(x)}{\partial x} - \frac{\partial^2 u_0(x)}{\partial x^2} = f_0 e^{-x^2/2\sigma^2}. \tag{2.3}$$

We note that if $\sigma \gg 1$ then roughly $\frac{\partial}{\partial x} \ll 1$ or

$$\frac{\partial u_0(x)}{\partial x} \approx -i f_0 e^{-x^2/2\sigma^2}. \tag{2.4}$$

Enforcing $u_0(x \to -\infty) = 0$, then $|u_0(x \to +\infty)| = f_0 \sigma \sqrt{2\pi}$.

2.2.2 Narrow-band forcing

Instead of doing this on-frequency forcing, we choose instead to use a narrow forcing zone $f(x,t) \sim N_\sigma(z) e^{it}$ where σ is not large. Specifically, we need σ to be well-resolved spectrally, but we need $\tilde{f}(k_z, t)$ not to be small. Thus, since $1/k_z$ must be resolved by the simulation, choosing $\sigma = 1/k_z$ is a natural choice.

Note that this implies that waves will be excited going both upwards and downwards, as $\tilde{f}(k_z) = \tilde{f}(-k_z)$. We can however place the forcing zone near the bottom boundary so that downwards-propagating waves will be attenuated by damping zones.

Let us derive the resultant equations of motion. In the interest of maintaining a divergence-free velocity field, we place the driving term on the density equation. Since the forcing is purely to excite waves and not physical (for the time being), it should not worry us that driving the density equation is ill-justifiable physically.

Consider the linearized system of equations (to examine the excited perturbation in the linear legime)

$$\vec{\nabla}\cdot\vec{u} = 0, \tag{2.5a}$$

$$\frac{\partial \vec{u}_1}{\partial t} + \frac{\vec{\nabla}P}{\rho_0} + \frac{\rho_1\vec{g}}{\rho_0} = 0, \tag{2.5b}$$

$$\frac{\partial \rho_1}{\partial t} - \frac{u_{1z}\rho_0}{H} = F(x,z,t), \tag{2.5c}$$

$$F(x,z,t) = \mathrm{Re}\, F e^{-\frac{(z-z_0)^2}{2\sigma^2}} e^{i(k_x x - \omega t)}. \tag{2.5d}$$

For the linearized system, we can immediately substitute $\frac{\partial}{\partial t} \to -i\omega$, $\frac{\partial}{\partial x} \to ik_x$, giving

$$\begin{aligned}
\frac{\partial u_{1z}}{\partial z} + ik_x u_x &= 0,\\
-i\omega u_x + \frac{ik_x P}{\rho_0} &= 0,\\
-i\omega u_{1z} + \frac{1}{\rho_0}\frac{\partial P}{\partial z} + \frac{\rho_1 g}{\rho_0} &= 0,\\
-i\omega\rho_1 - \frac{u_{1z}\rho_0}{H} &= F.
\end{aligned}$$

This can be recast solely in terms of u_{1z} as

$$\begin{aligned}
-i\omega u_{1z} + \frac{\omega}{k_x}\left(-\frac{1}{ik_x}\frac{\partial^2 u_{1z}}{\partial z^2} + \frac{1}{ik_x H}\frac{\partial u_{1z}}{\partial z}\right) - \frac{u_{1z}g}{Hi\omega} - \frac{Fg}{i\omega\rho_0} &= 0,\\
\omega^2\left(k_x^2 u_{1z} + \frac{1}{H}\frac{\partial u_{1z}}{\partial z} - \frac{\partial^2 u_{1z}}{\partial z^2}\right) - u_{1z}N^2k_x^2 - \frac{Fgk_x^2}{\rho_0} &= 0.
\end{aligned}$$

Note that if $\frac{\partial}{\partial z} \to 1/2H + ik_z, F = 0$ then this simplifies to the expected dispersion relation $\omega^2 = \frac{N^2k_x^2}{k_x^2+k_z^2+\frac{1}{4H^2}}$.

If we approximate $F(x,z,t) \approx F_0\delta(z-z_0)e^{i(k_x x-\omega t)}$ for the time being, then clearly the only term that can be equally singular is the $\frac{\partial^2}{\partial z^2}$ term, leaving us with

$$-\omega^2\frac{\partial^2 u_{1z}}{\partial z^2} - \frac{Fgk_x^2}{\rho_0} = 0. \tag{2.6}$$

Since u_{1z} both above and below z_0 satisfy the homogeneous PDE, which has plane wave solutions with an $e^{z/2H}$ envelope where $k_zH \gg 1$, we can integrate about a small domain around z_0 and obtain

$$\left.\frac{\partial u_{1z}}{\partial z}\right|_{z>z_0} - \left.\frac{\partial u_{1z}}{\partial z}\right|_{z<z_0} = -F_0\frac{gk_x^2}{\rho_0\omega^2}e^{i(k_x x-\omega t)}. \tag{2.7}$$

It should then be apparent that the u_{1z} solution is

$$u_{1z} = \frac{F_0 g k_x^2}{\rho_0(z=z_0)\omega^2}\frac{1}{2ik_z} \times \begin{cases} e^{(z-z_0)/2H} e^{i(k_x x + k_z(z-z_0) - \omega t)} & z > z_0 \\ e^{(z-z_0)/2H} e^{i(k_x x - k_z(z-z_0) - \omega t)} & z < z_0 \end{cases}. \tag{2.8}$$

Given that we are actually using a Gaussian $F(x,z,t) \propto F_0 e^{-\frac{(z-z_0)^2}{2\sigma^2}}$, the integral under $F(z)$ is actually $F_0\sqrt{2\pi}\sigma$ rather than just F_0. Finally, since a delta function is flat in spatial frequency space while a Gaussian tapers off with width $1/\sigma$ (σ is the spatial width, so $1/\sigma$ is the spatial frequency width), we should expect an overall multiplicative factor of $e^{-k_z^2\sigma^2/2}$ in the k_z spatial component of $F(x,z,t)$ compared to the delta function case. Thus, we finally obtain

$$|u_{1z}(z=z_0)| = \frac{F_0\sqrt{2\pi}\sigma g k_x^2}{2\rho_0(z=z_0)\omega^2 k_z} e^{-k_z^2\sigma^2/2}. \tag{2.9}$$

2.3 Timestepping

2.3.1 Signatures of Insufficient Temporal Resolution

When running these simulations, it can occasionally be seen that the waves do not grow in amplitude like $e^{z/2H}$, such that the energy and flux are not constant (or the waves will grow too quickly; this is dependent on the exact timestepping used). This is caused by using an overly-long timestep, as can be seen with the below illustration.

We study the 1D advection equation $\psi_t = c\psi_x$ corresponding to rightwards propagating waves. Consider first order implicit Euler timestepping for wave mode $\psi_x = ik\psi$, then

$$\psi_{i+1} - \psi_i = ick\Delta t\psi_{i+1}. \tag{2.10}$$

Solving yields

$$\begin{aligned}\psi_{i+1} &= (1 - ick\Delta t)^{-1}\psi_i,\\ &= (1 - i\omega\Delta t)^{-1}\psi_i.\end{aligned} \tag{2.11}$$

If we then want to step forwards for some total time T, with $N\Delta t = T$ the number of timesteps taken, then we approximate

$$\psi(t_0 + T) \approx \left(1 - i\omega\frac{T}{N}\right)^{-N}\psi(t_0). \tag{2.12}$$

It is worth noting that in the $N \to \infty$ limit we recover $\psi(t_0+T) = e^{i\omega T}\psi(t_0)$.

However, consider if $\omega\Delta t \lesssim 1$, then we should see the wave amplitude decrease over time like

$$\left|\psi(t_0 + T)\right| \approx \left(1 + \omega^2\Delta t^2\right)^{-N/2}\psi(t_0). \tag{2.13}$$

Thus, since we have an implicit first order scheme, the wave amplitude *decreases* over time spuriously. An explicit method would produce an increase, but `Dedalus` uses implicit timestepping. The exact rate of decrease depends on the timestepper used, but it is clear that $\omega\Delta t \ll 1$ is necessary to avoid this spurious decrease (in numerics language, "the oscillation must be well-resolved").

2.4 Navier-Stokes Regularization

This was a major tripping point for quite a while. As the wave breaks to higher wavenumbers, it is expected that energy will accumulate in the higher k spectral modes, and so some numerical viscosity must be added. This is not a physical viscosity but a numerical one (since spectral codes have no numerical viscosity of their own), and so the viscous term is added to all dynamical equations $\frac{\mathrm{d}}{\mathrm{d}t} \rightarrow \frac{\partial}{\partial t} + \vec{u} \cdot \vec{\nabla} - \nu\nabla^2$. This makes the equations resemble the Navier-Stokes equations, but since ν is not grounded in any physical process, it is not helpful to think of it as a Navier-Stokes equation.

We choose the $\nu\nabla^2$ term, which scales $\sim k^2$, such that it becomes strong in the same regime that nonlinearity $\left(\vec{u} \cdot \vec{\nabla}\right)$ becomes important. Consider some k_{turb} turbulent wavenumber, then we will want $\nu k_{turb}^2 \sim u k_{turb}$. We can get an idea of what u will be when turbulence sets in using the wave breaking criterion, that breaking sets in when $u \sim \omega/k$ where k is the base wavelength of the wave. This substitution suggests we should use $\nu = \frac{\omega}{k_{turb}k}$. We can choose $k_{turb} = \frac{2\pi L}{N}$ the grid spacing, so that at this turbulent wavelength the regularization kicks in. If we then take k, ω to be the parent wave's properties, we have a prescription for ν.

This turns out to be insufficient for regularization, we need roughly $\nu \sim \frac{3\omega_0}{k_z k_{turb}}$.

This is because we performed the naive substitution k, ω from the parent wave. Instead, it is advisable to scale k, ω to the same k_{turb} length scale. We first make ansatz that the cascade is roughly Kolmogorov, in which case we do the hand-wavy argument that energy u_l^2 at a certain length scale l is dissipated on timescale $\omega_l = u_l/l$, and so the total energy dissipated from the original flow $\epsilon = u_l^3/l$. Thus, $\nu k_l^2 \sim u_l k_l$ gives $\frac{\omega_0}{\nu k_0^2}\left(\frac{k_0}{k_{turb}}\right)^{4/3} = 1$. This isn't quite right though, as per comparison.

My hypothesis is currently two fold: first, the turbulent cascade is not instantaneous, so we can solve a diffusion equation in k space to figure out how far the wave can propagate (and grow in amplitude) before the Kolmogorov spectrum is obtained. Second, the turbulence causes some energy to be reflected in higher k modes, which interact nonlinearly with the flow at lower amplitudes.

2.4.1 Gotcha: ν_x, ν_z and Grid Spacing

If the grid resolution is uneven in the x, z directions, it might be tempting to use different ν_x, ν_z per the above criteria, since their $k_{turb} = k_{\max}$ differ. This is dangerous however, since $\nu\nabla^2\vec{u}_1$ has to be divergenceless! Thus, it is more advisable to choose an equal grid spacing in the two directions.

Chapter 3

Stability Analysis

3.1 Modulational Instability of NLSE

We study the nonlinear Schrödinger Equation (NLSE)

$$i\psi_t + \psi_{xx} + \nu\left|\psi\right|^2\psi = 0. \tag{3.1}$$

Sutherland's paper demonstrates an interaction between the wave and the wave-induced mean flow ($\propto \left|\psi\right|^2$) that reduces the IGW system to the NLSE. Here we follow Whitham's textbook in rederiving the modulational instability criterion.

Consider for some fixed value k_0, then the linear SE (LSE) admits a plane wave solution $\psi = e^{i(k_0 x - \omega(k_0)t)}$ for some dispersion relation $\omega(k)$. If instead we permit ψ to be distributed in frequency space near k_0, then we may notate

$$\psi = \psi_0\varphi = e^{i(k_0 x - \omega(k_0)t)} \int \exp\left[i\left(k'x - k'\frac{\partial\omega}{\partial k}t - \frac{1}{2}k'^2\frac{\partial^2\omega}{\partial k^2}t\right)\right]\mathrm{d}k', \tag{3.2}$$

where $k' = k - k_0$ denotes the deviation from the central k_0 wavenumber. Defining $\omega' = \omega - \omega_0$, we may observe

$$\omega' = k'\frac{\partial\omega}{\partial k} + \frac{1}{2}k'^2\frac{\partial^2\omega}{\partial k^2}. \tag{3.3}$$

Making correspondence $-i\omega' \to \partial_t, ik' \to \partial_x$, we see that φ obeys dispersion relation

$$i\left(\varphi_t + \frac{\partial\omega}{\partial k}\varphi_x\right) + \frac{1}{2}\frac{\partial^2\omega}{\partial k^2}\varphi_{xx} = 0. \tag{3.4}$$

Thus, the modulation φ translates at the group velocity, and we find that φ obeys a modified LSE in a comoving frame.

If we reintroduce the nonlinearity, we observe that $\nu\left|\psi\right|^2\psi = \nu\left|\varphi\right|^2\psi_0\varphi$, and so the nonlinearity modifies the LSE as

$$i\left(\varphi_t + \frac{\partial \omega}{\partial k}\varphi_x\right) + \frac{1}{2}\frac{\partial^2 \omega}{\partial k^2}\varphi_{xx} + \nu\left|\varphi^2\right|\varphi = 0, \tag{3.5}$$

$$\omega' = k'\frac{\partial \omega}{\partial k} + \frac{1}{2}k'^2\frac{\partial^2 \omega}{\partial k^2} - \nu a^2. \tag{3.6}$$

Armed with this dispersion relation, we can apply the method of characteristics to the system of equations describing any wavetrain

$$\frac{\partial k}{\partial t} + \frac{\partial \omega}{\partial x} = 0, \tag{3.7a}$$

$$\frac{\partial A^2}{\partial t} + \frac{\partial}{\partial x}\left(c_g A^2\right) = 0. \tag{3.7b}$$

We denote A the wave amplitude and c_g the group velocity. In a nonlinear system, $\omega = \omega(A, k)$, and to leading order in A we expect $\omega = \omega_0 + \omega_2 A^2$, i.e. the dispersion should not depend on the sign of A. Thus we may rewrite Eq. 3.7 as

$$\frac{\partial k}{\partial t} + \left(\frac{\partial \omega_0}{\partial k} + \frac{\partial \omega_2}{\partial k}A^2\right)\frac{\partial k}{\partial x} + \omega_2\frac{\partial A^2}{\partial x} = 0, \tag{3.8a}$$

$$\frac{\partial A^2}{\partial t} + \frac{\partial A^2}{\partial x}\frac{\partial \omega_0}{\partial k} + A^2\frac{\partial^2 \omega_0}{\partial k^2}\frac{\partial k}{\partial x} = 0. \tag{3.8b}$$

We first note that, in $A \to 0$, the equations decouple and both equations describe propagation at the group velocity (they are already in characteristic form $Q_{,t} + cQ_{,x} = 0$). With some foresight, we anticipate that the $\frac{\partial A^2}{\partial x}$ term in the first equation will provide a $\mathcal{O}(A)$ correction to the group velocity, while the $\frac{\partial \omega_2}{\partial k}A^2\frac{\partial k}{\partial x}$ term provides an $\mathcal{O}(A^2)$ correction, so we drop it.

The method of characteristics then requires we seek linear combinations of the two equations such that all equations are in characteristic form. The Whitham prescription says that if the original system is of form $A_{ij}q_{j,t} + B_{ij}q_{j,x} = 0$, then the characteristic curves $X(T)$ satisfy $A_{ij}X + B_{ij}T = 0$. Here, this yields

$$X(T) = \left(\frac{\partial \omega_0}{\partial k} \pm A\sqrt{\omega_2\frac{\partial^2 \omega_0}{\partial k^2}}\right)T. \tag{3.9}$$

Comparing to the NLSE dispersion relation Eq. 3.6, we identify that $\sqrt{-\nu\frac{\partial^2 \omega_0}{\partial k^2}}A$ is the correction to the group velocity. Thus, if $\nu\frac{\partial^2 \omega_0}{\partial k^2} > 0$, we obtain a *complex* group velocity, which is identified with *modulational instability*. Naively, this might imply a steepening $k = c_{g,0}t - i\sqrt{\nu\omega_0''}At$ with timescale $\Gamma \sim \left(A^2\nu\omega_0''\right)^{-1/2}$.

Chapter 4

Stratified Flow

4.1 Linear Equations

We largely follow Booker & Bretherton from 1966 though do not adopt the Boussinesq approximation. Consider background shear flow $\vec{u}_0 = u_0(z)\hat{x}$ and vertical stratification $\rho_0 \propto e^{-z/H}$, then

$$\vec{\nabla}\cdot\vec{u}_1 = 0, \tag{4.1a}$$
$$\frac{\partial \rho_1}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\rho_1 - u_{1z}\frac{\rho_0}{H} = 0, \tag{4.1b}$$
$$\frac{\partial u_{1x}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)(u_{1x}+u_0) + \frac{1}{\rho_0}\frac{\partial P}{\partial x} = 0, \tag{4.1c}$$
$$\frac{\partial u_{1z}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)u_{1z} + \frac{1}{\rho_0}\frac{\partial P}{\partial z} + \frac{\rho_1 g}{\rho_0} = 0. \tag{4.1d}$$

Note that $\vec{u}\cdot\vec{\nabla} = (\vec{u}_0+\vec{u}_1)\cdot\vec{\nabla}$.

We again assume plane waves in x,t such that $u_{1z}(x,z,t) = u_{1z}(z)e^{i(k_x x-\omega t)}$. We bracket second-order terms below and set them to zero in the ensuing calculations

$$\frac{\partial u_{1z}}{\partial z} + ik_x u_{1x} = 0, \tag{4.2a}$$
$$-i\omega\rho_1 + u_0 ik_x\rho_1 - u_{1z}\frac{\rho_0}{H} + \left[\left(\vec{u}_1\cdot\vec{\nabla}\right)\rho_1\right] = 0, \tag{4.2b}$$
$$-i\omega u_{1x} + u_0 ik_x u_{1x} + \frac{1}{\rho_0}ik_x P + u_{1z}\frac{\partial u_0}{\partial z} + \left[\vec{u}_1\cdot\vec{\nabla}u_{1x}\right] = 0, \tag{4.2c}$$
$$-i\omega u_{1z} + u_0 ik_x u_{1z} + \frac{1}{\rho_0}\frac{\partial P}{\partial z} + \frac{\rho_1 g}{\rho_0} + \left[\vec{u}_1\cdot\vec{\nabla}u_{1z}\right] = 0. \tag{4.2d}$$
$$\tag{4.2e}$$

Denote $\tilde{\omega}(z) = \omega - u_0(z)k_x$ the corotating frequency. Using then that $u_{1x} = \frac{i}{k_x}\frac{\partial u_{1z}}{\partial z}$, $\rho_1 = \frac{i\rho_0}{H\tilde{\omega}}u_{1z}$ and

$P = \left[i\tilde{\omega}u_{1x} - u_{1z}\frac{\partial u_0}{\partial z}\right]\frac{\rho_0}{ik_x}$, we can finally arrive at linearized equation

$$\frac{\partial^2 u_{1z}}{\partial z^2} - \frac{1}{H}\frac{\partial u_{1z}}{\partial z} + \left[\frac{k_x^2 N^2}{\tilde{\omega}^2} - k_x^2 - \frac{k_x}{\tilde{\omega}H}\frac{\partial u_0}{\partial z} + \frac{k_x}{\tilde{\omega}}\frac{\partial^2 u_0}{\partial z^2}\right]u_{1z} = 0. \tag{4.3}$$

Note that if we make correspondences $ik_x \to \partial_x, i\tilde{\omega} \to \partial_t + u_0\partial_x$ and take $H \to \infty$ while holding N^2 constant, we recover Equation 2.1 of Booker & Bretherton. We can also note that if we take constant u_0 we recover the expected dispersion relation Eq. 1.12 for $\omega = \tilde{\omega}$.

As a brief exercise, we can take the $H \to \infty$ limit and compute the resultant shear-induced splitting; this is akin to rotational mode splitting, and is given without proof from $-k_z^2 - k_x^2 + \frac{k_x^2 N^2}{\omega^2} + \frac{k_x}{\omega}\frac{\partial^2 u_0}{\partial z^2} = 0$ as

$$\omega^2 = \frac{k_x^2 N^2}{k_x^2 + k_z^2}\left[1 + \frac{1}{\sqrt{k_x^2 + k_z^2}}\frac{\partial^2 u_0}{\partial z^2}\right]. \tag{4.4}$$

4.2 Reflection/Transmission in Linear Theory

One can solve Eq. 4.3 via *Frobenius Theory*, but if we consider only $\tilde{\omega} \to 0$, we obtain a very simple power series solution. In $\tilde{\omega} \to 0$, we have only leading order contributions

$$\frac{\partial^2 u_{1z}}{\partial z^2} + \frac{k_x^2 N^2}{\tilde{\omega}^2}u_{1z} = 0. \tag{4.5}$$

Consider a power series solution $u_{1z} \propto \tilde{\omega}^\alpha$, where $\tilde{\omega}(z) = \omega - k_x u_0(z)$. Note that very near the critical level, $\frac{\partial u_0}{\partial z}\delta z k_x = \tilde{\omega}$ where $\delta z = z - z_c, \tilde{\omega}(z_c) = 0$. We thus have

$$\left(\frac{\partial\tilde{\omega}}{\partial z}\right)^2 \alpha(\alpha - 1) + k_x^2 N^2 = 0, \tag{4.6}$$

$$\alpha = \frac{1}{2} \pm i\sqrt{\frac{N^2}{\left(\frac{\partial u_0}{\partial z}\right)^2} - \frac{1}{4}}. \tag{4.7}$$

The classic Booker & Bretherton paper notates this result $u_{1z} \propto (\tilde{\omega}(z))^{\frac{1}{2}\pm i\mu} \propto (\delta z)^{\frac{1}{2}\pm i\mu}$.

Now, consistent with a perturbation that vanishes as $t \to -\infty$ (alternatively, this is just the Landau prescription for deforming below a singularity on the real line), we must interpret $\tilde{\omega}$ as having a slight positive imaginary component. Given this, the expression for $u_{1z}(z)$ is multivalued, and we must specify a branch cut. For $\mathrm{Arg}(z - z_c) = 0$, or intuitively when we are above the critical layer, choose branch such that

$$(z - z_c)^{\frac{1}{2}\pm i\mu} \equiv |z - z_c|^{1/2 \pm i\mu}. \tag{4.8}$$

It is then clear that for $\mathrm{Arg}(z - z_c) = -\pi$, intuitively corresponding to being below the critical layer (note the arg is $-\pi$ since z_c is infinitesimally above the real axis), our branch cut enforces

$$(z - z_c)^{\frac{1}{2}\pm i\mu} = |z - z_c|^{\frac{1}{2}\pm i\mu} e^{-\frac{i\pi}{2}} e^{\mp\mu\pi}. \tag{4.9}$$

Lastly, we interpret $|z-z_c|^{\pm i\mu} = e^{\pm i\mu \ln|z-z_c|}$ to have the form of a positive/negative k_z value. Interestingly, this means that $e^{-i\mu\ln|z-z_c|}$ corresponds to the *positive group velocity*, which is $e^{\mu\pi}$ magnified compared to the $\mathrm{Arg}(z-z_c)=0$ case, and $e^{2\mu\pi}$ magnified over the negative group velocity component below z_c.

The interpretation we choose for this is that an incoming wavepacket moving with positive group velocity inbound on z_c is transmitted with amplitude $e^{-\mu\pi}$ and reflected with amplitude $e^{-2\mu\pi}$. The remainder is interpreted as being *absorbed* by the critical layer.

4.3 Competition between Viscous and Nonlinear Forces

As we zoom in on the critical layer, we expect to be able to take Boussinesq approximation $H\to\infty$ within the immediate vicinity of the critical layer.

In this limit, the group velocity is just $v_{g,z} = -\frac{Nk_xk_z}{(k_x^2+k_z^2)^{3/2}} = -\omega\frac{k_z}{k_x^2+k_z^2}$. As we approach corotation, $\omega\to 0, \omega\approx\frac{Nk_x}{k_z}, k_z\to\infty$, and so $v_{g,z}\to 0$. Then considering an energy flux incident on the critical layer clearly shows that energy must accumulate near the critical layer, and must be dissipated somehow. This must be either via viscous forces or nonlinear cascades.

The characteristic strength of the viscous force $\nu\nabla^2$ is νk_z^2, while the nonlinear advective term has characteristic strength $u_{1z}k_z$. Note that in the Boussinesq approximation $|u_{1z}|$ is constant during propagation. Thus, the ratio of forces $R=\frac{f_\nu}{f_{adv}}\sim\frac{\nu k_z}{u_{1z}}$. Note that this ratio varies spatially. Qualitatively, R is important since it dictates whether the wave is damped viscously before it needs to be treated in nonlinear fashion or whether a full nonlinear/hydrodynamic treatment is necessary.

We want to understand for what viscosity we expect linear behavior to dominate. Thus, at critical z_c where nonlinear effects become important, we want $R\gtrsim 1$. Shifting to coordinate system δz where $\delta z=0$ is the corotation resonance, we can write $|\tilde{\omega}| = k_x u_0'\delta z$ where $u_0' = \left.\frac{\partial u_0}{\partial z}\right|_{\delta z=0}$, and $k_z=\frac{N}{u_0'\delta z}$. Then the breaking criterion is $\frac{u_{1z}k_z(z)}{\tilde{\omega}(z)}\sim 1$, or $\delta z_c\sim\sqrt{\frac{u_{1z}N}{k_x u_0'^2}}$. We can then evaluate

$$R(z_c) = \frac{\nu k_z}{u_{1z}} = \nu\sqrt{\frac{Nk_x}{u_{1z}^3}}. \tag{4.10}$$

It is helpful also to write this down for any arbitrary hyperviscosity $f_{visc}=\nu^{(n)}\nabla^{(n)}$, which is cited without proof

$$R(z_c) = \nu^{(n)}\sqrt{\frac{N^{n-1}k_x^{n-1}}{u_{1z}^{n+1}}}. \tag{4.11}$$

4.4 Traveling Breaking Front

Motivated by simulations, we posit the existence of a hydraulic jump, similar to shocks in nature. We thus need to develop jump conditions, similar to the Rankine-Huginot jump conditions for shocks.

The R-H jump conditions take the conservation equations and integrate over a control volume.

Since in a hydrodynamic shock, all fluid quantities are advected, all conservation equations are necessary. A simpler treatment is merited for IGW breaking, since very few quantities are advected (notably, vertical momentum and density are not).

Let's consider first the advection of horizontal momentum $\rho_0 u_{1x}$ to leading order. This satisfies conservation equation

$$\frac{\partial P_x}{\partial t} + \vec{\nabla} \cdot \left(\rho_0 u_{1x} \vec{u}_1\right) = 0. \tag{4.12}$$

Integrating over control volume $V = [-\infty, \infty] \times [z < z_b, z > z_b]$ the full horizontal domain times the breaking zone, we require the total horizontal momentum of the zone be constant in time, if we permit z_b to move with velocity u_b the breaking front velocity. This equates to

$$\Delta P_x - \left[\rho_0 u_{1x} u_b\right]_{z > z_b} + \left[\rho_0 u_{1x}(u_{1z} - u_b)\right]_{z < z_b} = 0. \tag{4.13}$$

We've called ΔP_x the change in horizontal momentum in the breaking zone (so the boundary integral over the endcaps of the domain, effectively). Now, based on simulations, we identify that $u_{1x}(z > z_c) = B\cos\kappa z$ seems to form a standing wave pattern. Note that since this is a pure shear flow, it is consistent with an incompressible fluid and $u_{1z}(z > z_c) = 0$.

Permitting such a flow above the breaking zone, then we substitute our known linear solution $u_{1x} \approx \frac{k_z}{k_x} u_{1z} \propto A e^{z/2H} \cos\omega_0 t$ and $\rho_0 = \tilde{\rho} e^{-z/H}$ and evaluate harmonics of $\cos\omega_0 t$ to obtain

$$\Delta P_x(\omega = 0) = \tilde{\rho} \frac{A^2}{2} \frac{k_z}{k_x}, \tag{4.14}$$

$$\Delta P_x(\omega = \omega_0) = u_b A e^{-z/2H} \tilde{\rho} \cos\omega_0 t, \tag{4.15}$$

$$\Delta P_k(\omega = 2\omega_0) = -u_b \cos(\kappa u_b t) + \frac{k_z}{k_x} \frac{A^2}{2B} e^{z/H} \cos 2\omega_0 t. \tag{4.16}$$

We can easily interpret $\Delta P_x(0)$ to be the deposition of the horizontal momentum of the wave, and $\Delta P_x(\omega = \omega_0)$ to be just the wave flowing through the deposition zone; it time averages to zero anyways, so it's not super interesting.

Instead, if we dare make identification $u_b = \frac{2\omega_0}{\kappa}$ (the only way $u_b \neq 0$), then the deposition to

$$\Delta P_k(\omega = 2\omega_0) = \left(\frac{A^2}{2B} e^{z/H} - \frac{2\omega_0}{\kappa}\right) \cos 2\omega_0 t, \tag{4.17}$$

i.e. the standing wave pattern above can sap some $2\omega_0$ frequency P_x out of the mean flow via a breaking front.

4.4.1 Finding and Explaining the Breaking Front

It is a bit tricky to interpret this, so I'll just jot down some ideas lest I forget them in my sleep-deprived state. If we require that the breaking front only sap momentum from the wave (and not from the mean flow), then the breaking front shuts of when $\frac{A^2}{2B} e^{z/H} \sim \frac{2\omega_0}{\kappa}$. It's also unclear how κ is chosen as well. But based on preliminary simulations, this is actually pretty accurate for observed

κ, u_b!

Encouragingly, we can guess at where κ comes from. Since the waveform above the breaking zone has to lose its k_x component, we may guess that the incoming wave $\cos(k_x x + k_z z)$ must interact nonlinearly with another wave to knock out the $k_x x$ contribution. One possible candidate would have a $\cos(-k_x x + k_z z)$ such that their product $\cos 2k_z z$, in which case $\kappa = 2k_z$. This also seems to be reasonable from a few of our simulations, but this requires thus that $2k_z$ be much longer than the viscous length scale; this would explain why we don't see it on our low-resolution simulations at $z \in [0, 12H], N_Z = 512$ but only on $N_Z = 1024$. The desired $\cos(-k_x x + k_z z)$ is just a reflected wave in the x direction, so I guess it can come from the viscous subscales pushing against the horizontal momentum/mean flow? In any case it's not inconceivable it would exist, and it should be the dominant $-k_x x$ term compared e.g. to $-k_x x + 0z, -k_x x + 2k_z z$.

TODO We need simulations next of the front breaking down, to understand where it would stop, since we don't have a criterion for it yet.

Chapter 5

Cylindrical Stratified Flow

We fearlessly tackle the cylindrical coordinate problem, using Tsang & Lai 2008 non-barotropic flows as a guide but reproducing all algebra. We show that in an appropriate limit it recovers Eq. 4.3.

First, we take $\vec{\nabla} \cdot \vec{u}_0 = 0$, which we can define $u_0 = u_0(r)\hat{\phi} = r\Omega(r)\hat{\phi}$. Furthermore, under hydrostatic equilibrium if we assume a thin atmosphere (such that $\vec{\nabla}\Phi$ does not vary much) then $\frac{\vec{\nabla}P_0}{\rho_0} = -g\hat{r}$. We assume stratification $\rho \propto e^{-r/a_0}$, which gives buoyancy frequency $N^2 = g/a_0$ in the incompressible fluid limit. Finally, we make plane wave substitution $\propto e^{i(m\phi - \omega t)}$ and begin

$$0 = \frac{1}{r}\frac{\partial(u_{1r})}{\partial r} + \frac{im}{r}u_{1\phi}, \tag{5.1}$$

$$0 = -i\omega\rho_1 - u_{1r}\frac{\rho_0}{a_0} + im\Omega\rho_1\left\{+u_{1r}\frac{\partial\rho_1}{\partial r} + \frac{u_{1\phi}r}{r}\frac{\partial\rho_1}{\partial\phi}\right\}, \tag{5.2}$$

$$\begin{aligned} 0 = &-i\omega\vec{u}_1 + \left[\Omega imu_{1r} - 2\Omega u_{1\phi}\right]\hat{r} + \left[u_{1r}\frac{\mathrm{d}(r\Omega)}{\mathrm{d}r} + \Omega imu_{1\phi} + \Omega u_{1r}\right]\hat{\phi} + \frac{1}{\rho_0}\left[\frac{\partial P_1}{\partial r}\hat{r} + \frac{imP_1}{r}\hat{\phi}\right] + \frac{\rho_1 g}{\rho_0}\hat{r} \\ &+ \left\{\left[u_{1r}\frac{\partial u_{1r}}{\partial r} + \frac{u_{1\phi}}{r}\frac{\partial u_{1r}}{\partial\phi} - \frac{u_{1\phi}^2}{r}\right]\hat{r} + \left[u_{1r}\frac{\partial u_{1\phi}}{\partial r} + \frac{u_{1\phi}}{r}\frac{\partial u_{1\phi}}{\partial\phi} + \frac{u_{1\phi}u_{1r}}{r}\right]\hat{\phi}\right\}. \end{aligned} \tag{5.3}$$

We have bracketed nonlinear terms in curly braces. The natural definition $\tilde{\omega} = \omega - m\Omega$ and common definition $\frac{\kappa^2}{2\Omega} = \Omega + \frac{\mathrm{d}(r\Omega)}{\mathrm{d}r}$ may be used, which agrees with Tsang & Lai. Dropping bracketed nonlinear terms allows us to manipulate

$$u_{1\theta} = \frac{i}{m}\frac{\partial r u_{1r}}{\partial r}, \tag{5.4}$$

$$\rho_1 = u_{1r}\frac{\rho_0}{a_0}\frac{1}{-i\tilde{\omega}}, \tag{5.5}$$

$$P_1 = \frac{i\rho_0 r}{m}\left[-i\tilde{\omega}u_{1\theta} + u_{1r}\frac{\kappa^2}{2\Omega}\right], \tag{5.6}$$

$$0 = -i\tilde{\omega}u_{1r} - 2\Omega\left(\frac{i}{m}\frac{\partial(ru_{1r})}{\partial r}\right) + \frac{1}{\rho_0}\frac{\partial}{\partial r}\left[\frac{i\rho_0 r}{m}\left[-i\tilde{\omega}\frac{i}{m}\frac{\partial(ru_{1r})}{\partial r} + u_{1r}\frac{\kappa^2}{2\Omega}\right]\right] + \frac{iu_{1r}g}{a_0\tilde{\omega}}. \tag{5.7}$$

Expanding the rather ugly derivative gives

$$\begin{aligned}0 = &-i\tilde{\omega}u_{1r} - \frac{2i\Omega}{m}\left[r\frac{\partial u_{1r}}{\partial r} + u_{1r}\right] + \left(\frac{i}{m} - \frac{ir}{a_0 m}\right)\left[\frac{\tilde{\omega}}{m}\left(r\frac{\partial u_{1r}}{\partial r} + u_{1r}\right) + u_{1r}\frac{\kappa^2}{2\Omega}\right]\\ &+ \frac{ir}{m}\left[\frac{\tilde{\omega}}{m}\left(r\frac{\partial^2 u_{1r}}{\partial r^2} + 2\frac{\partial u_{1r}}{\partial r}\right) + \frac{\kappa^2}{2\Omega}\frac{\partial u_{1r}}{\partial r} + u_{1r}\frac{\partial}{\partial r}\left(\frac{\kappa^2}{2\Omega}\right)\right]\\ &+ \frac{r}{m}\frac{\partial\tilde{\omega}}{\partial r}\frac{i}{m}\left(r\frac{\partial u_{1r}}{\partial r} + u_{1r}\right) + \frac{iu_{1r}g}{a_0\tilde{\omega}}.\end{aligned} \tag{5.8}$$

Let's group terms to find

$$\begin{aligned}&\frac{\tilde{\omega}r^2}{m^2}\frac{\partial^2 u_{1r}}{\partial r^2} + \frac{\partial u_{1r}}{\partial r}\left[-\frac{2\Omega r}{m} + \frac{r\tilde{\omega}}{m}\left(\frac{1}{m} - \frac{r}{a_0 m}\right) + \frac{2r\tilde{\omega}}{m^2} + \frac{r}{m}\frac{\kappa^2}{2\Omega} + \frac{r^2}{m^2}\frac{\partial\tilde{\omega}}{\partial r}\right]\\ &\quad + u_{1r}\left[-\frac{2\Omega}{m} + \frac{\tilde{\omega}}{m}\left(\frac{1}{m} - \frac{r}{a_0 m}\right) + \left(\frac{1}{m} - \frac{r}{a_0 m}\right)\frac{\kappa^2}{2\Omega} + \frac{r}{m}\frac{\partial}{\partial r}\left(\frac{\kappa^2}{2\Omega}\right) + \frac{r}{m^2}\frac{\partial\tilde{\omega}}{\partial r} + \frac{N^2}{\tilde{\omega}} - \tilde{\omega}\right] = 0.\end{aligned} \tag{5.9}$$

One small simplification can be made: observe that $\frac{\kappa^2}{2\Omega} = \Omega + \frac{\partial(r\Omega)}{\partial r} = 2\Omega + r\frac{\partial\Omega}{\partial r}$, so

$$\frac{\tilde{\omega}r^2}{m^2}\frac{\partial^2 u_{1r}}{\partial r^2} + \frac{\partial u_{1r}}{\partial r}\frac{r\tilde{\omega}}{m^2}\left(3 - \frac{r}{a_0}\right) + u_{1r}\left[\frac{\tilde{\omega}}{m}\left(\frac{1}{m} - \frac{r}{a_0 m}\right) - \frac{r}{a_0 m}\frac{\kappa^2}{2\Omega} + \frac{r}{m}\frac{\partial}{\partial r}\left(\frac{\kappa^2}{2\Omega}\right) + \frac{N^2}{\tilde{\omega}} - \tilde{\omega}\right] = 0. \tag{5.10}$$

I think that the 3 actually makes sense, it's basically $\frac{\partial}{\partial r}r\frac{\partial(ru_r)}{\partial r} = \cdots + 2r\frac{\partial u_r}{\partial r}$ and $\frac{\partial(ru_r)}{\partial r} = r\frac{\partial u_r}{\partial r} + u_r$, which also explains the 1.

5.1 Comparison to Plane Parallel

5.1.1 Reduction in $r \to \infty$

First, let's consider the limit where $r \to \infty$, so we should have a plane parallel waveform. In this limit, $\frac{m}{r} \to k_x$ can be identified (and so m must also be taken to be infinitely large to have finite wavelengths), and of course $\Omega \to \frac{u_0}{r}$ in line with our plane parallel definitions. We need one last identification, that $\frac{\partial\tilde{\omega}}{\partial r} = -m\frac{\mathrm{d}\Omega}{\mathrm{d}r} \approx -k_x\frac{\mathrm{d}u_0}{\mathrm{d}r}$ also per the plane parallel result. These identifications allow us to write, approximating $\frac{\kappa^2}{2\Omega} \approx \frac{\mathrm{d}u_0}{\mathrm{d}r}$:

$$\frac{\partial^2 u_{1r}}{\partial r^2} - \frac{\partial u_{1r}}{\partial r}\frac{1}{a_0} + u_{1r}\left[\frac{N^2 k_x^2}{\tilde{\omega}^2} + \frac{k_x}{\omega}\frac{\mathrm{d}^2 u_0}{\mathrm{d}r^2} - \frac{k_x}{a_0\tilde{\omega}}\frac{\mathrm{d}u_0}{\mathrm{d}r} - k_x^2\right]u_{1r} = 0. \tag{5.11}$$

This can be seen to agree with Eq. 4.3, which is no small miracle for my usual algebra skills.

5.1.2 Boundary Matching at $\tilde{\omega} = 0$

Again, there is nothing more singular than a $\tilde{\omega}^{-2}$ coefficient on u_{1r}, after dividing out the leading coefficient, and so we may simply analyze

$$\frac{\partial^2 u_{1r}}{\partial r^2} + u_{1r}\frac{N^2 m^2}{\tilde{\omega}^2 r^2} = 0. \tag{5.12}$$

Using the usual $u_{1r} = (r - r_c)^\alpha = \delta r^\alpha$ and $\tilde{\omega} = -m \left.\frac{\partial \Omega}{\partial r}\right|_{r_c} \delta r \equiv -m\Omega_c' \delta r$ gives

$$\alpha(\alpha - 1) + \frac{N^2}{(\Omega_c')^2 r^2} = 0. \tag{5.13}$$

This solves to $\alpha = \frac{1}{2} \pm i\sqrt{\frac{N^2}{(\Omega_c')^2 r^2} - \frac{1}{4}}$. This agrees with the exponent derived in the Tsang & Lai paper. Thus, in the linear theory, transmission/reflection are attenuated by factor $\exp\left[-\pi\sqrt{\frac{N^2}{(\Omega_c')^2 r^2} - \frac{1}{4}}\right]$.

It can be observed in passing that if $N \sim \Omega_c' r$ then the scenario reduces to that considered in Tsang & Lai, where some variable such as ζ can induce superreflection. We don't compute it here, but it bears mentioning that the corresponding analysis for Tsang & Lai in plane parallel gives $\nu = -4\frac{\mathrm{d}\ln u_0'}{\mathrm{d}r}$, and so u_0' plays the role of ζ.

Note that the Høiland Criteria, such as given in Balbus & Hawley 1998, states that disks are hydrodynamically stable if

$$N^2 + \frac{1}{R^3}\frac{\partial R^4 \Omega^2}{\partial R} > 0. \tag{5.14}$$

We may compare this to our transmission/reflection criterion by computing at critical Ω_c, where

$$R^2 \left(\Omega_c'\right)^2 = \left(\frac{N^2 + 4\Omega^2}{2\Omega}\right)^2. \tag{5.15}$$

This was never going to agree with Eq. 5.13, since it doesn't depend simply on Ω_c'. This reinforces Booker & Bretherton's conclusion that the agreement of their calculation with convective stability was pure coincidence.

5.2 Wave Properties

Let's compute dispersion relation. I'm too lazy to sort out $u_{1r} \propto e^{r/2a_0}$ nonsense, so we will take Boussinesq approximation $a_0 \to \infty$, and so

$$\frac{\partial^2 u_{1r}}{\partial r^2} + \frac{\partial u_{1r}}{\partial r}\left(\frac{3}{r} - \frac{1}{a_0}\right) + u_{1r}\left[\left(\frac{1}{r^2} - \frac{1}{a_0 r}\right) + \frac{m}{r\tilde{\omega}}\frac{\partial}{\partial r}\left(\frac{\kappa^2}{2\Omega}\right) + \frac{N^2 m^2}{r^2 \tilde{\omega}^2} - \frac{m^2}{r^2}\right] = 0. \tag{5.16}$$

We've been a bit hand-wavy about what terms to drop, but we can identify now that some of the terms can come together to form (we have to drop the $\frac{\partial u_{1r}}{\partial r}$ coefficient, otherwise we'll have ik_r terms... ugh)

$$\tilde{\omega}^2 \simeq \frac{N^2 \left(m^2/r^2\right)}{k_r^2 + \frac{m^2}{r^2}}. \tag{5.17}$$

Thus, in a nutshell, we should expect the group velocity to largely vanish in cylindrical coordinates once the WKB regime is hit (where we're reasonably far away that we don't have to worry about the non-wave-like distortion of the geometry; is this the same as the plane wave limit though?).

Appendix A

Deriving Fluids Results

A.1 Equation of Energy Conservation

We follow Landau & Lifshitz's derivation for this expression. Consider that the total energy stored in a wave must be the sum of its kinetic and internal energy $\frac{1}{2}\rho v^2 + \rho\epsilon$ where ϵ is the internal energy density. To obtain an equation of energy conservation, we must take the time derivative of this expression. First, we consider

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) &= \frac{1}{2}v^2\frac{\partial\rho}{\partial t} + \rho\vec{v}\cdot\frac{\partial\vec{v}}{\partial t}, \\
&= -\frac{1}{2}v^2\left(\vec{\nabla}\cdot\rho\vec{v}\right) - \rho\vec{v}\cdot\left(\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} + \frac{\vec{\nabla}P}{\rho} - \vec{g}\right)_{T,V}, \\
&= -\frac{1}{2}v^2\left(\vec{\nabla}\cdot\rho\vec{v}\right) - \frac{1}{2}\rho\vec{v}\cdot\left(\vec{\nabla}v^2\right) - \vec{v}\cdot\left(\rho\vec{\nabla}w - \rho T\vec{\nabla}s\right) + \rho\vec{v}\cdot\vec{g}.
\end{aligned} \tag{A.1}$$

We have denoted s the specific internal entropy density of the fluid and $\mathrm{d}w = T\mathrm{d}s + \frac{\mathrm{d}P}{\rho} = T\mathrm{d}s + \epsilon$ the specific internal enthalpy density of the fluid. Recall enthalpy $\epsilon = w - Ts$ is the usual thermodynamic definition.

At the same time, consider

$$\begin{aligned}
\frac{\partial}{\partial t}(\rho\epsilon) &= \epsilon\frac{\partial\rho}{\partial t} + \rho\frac{\partial}{\partial t}\left(Ts - \frac{P}{\rho}\right)_{T,P}, \\
&= \epsilon\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t} + \frac{P}{\rho}\frac{\partial\rho}{\partial t}, \\
&= w\frac{\partial\rho}{\partial t} + \rho T\frac{\partial s}{\partial t}, \\
&= -w\vec{\nabla}(\rho\vec{v}) - \rho T\vec{v}\cdot\left(\vec{\nabla}s\right).
\end{aligned} \tag{A.2}$$

Summing the two, we find

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \rho\epsilon\right) = -\vec{\nabla}\cdot\left[\rho\vec{v}\left(v^2 + w\right)\right] = -\vec{\nabla}\cdot\left[\rho\vec{v}\left(v^2 + \epsilon + \frac{P}{\rho}\right)\right]. \tag{A.3}$$

A.2 Conservative Fluid Equations

This isn't particularly useful for our work since spectral methods don't benefit from having the dynamical equations in conservative form, but form a useful exercise for the ever-so-stupid writer.

The Reynolds transport theorem tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_V Q\,\mathrm{d}V = \int\limits_V \frac{\partial Q}{\partial t}\,\mathrm{d}V + \int\limits_{\partial V}(\vec{v}(\mathrm{d}A)\cdot\hat{n})Q\,\mathrm{d}A. \tag{A.4}$$

We have notated $\vec{v}(\mathrm{d}A)$ the velocity of the surface element. In fluid mechanics, this is just the velocity field. Thus, for conserved quantities where $\frac{\mathrm{d}}{\mathrm{d}t}\int\limits_V Q\,\mathrm{d}V = 0$, we can apply divergence theorem to obtain the simple result

$$\frac{\partial Q}{\partial t} + \vec{\nabla}\cdot(Q\vec{v}) = 0. \tag{A.5}$$

Of course, if there are any net forces etc. on the system, they are simply inserted as fileds on the right hand side of Eq. A.5.

The application of this to the three major conserved quantities: mass, momentum and energy, yield:

Conservation of Mass This produces simply $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{v}\right) = 0$ the continuity equation.

Conservation of Momentum This produces $\frac{\partial\rho v_i}{\partial t} + \vec{\nabla}\cdot\left(\rho v_i\vec{v}\right) = 0$. To obtain the traditional momentum equation, it is perhaps most clear in index notation where repeated indicies denote summation:

$$\begin{aligned}\frac{\partial\left(\rho v_i\right)}{\partial t} + \partial_j\rho v_i v_j &= 0,\\ v_i\frac{\partial\rho}{\partial t} + \rho\frac{\partial v_i}{\partial t} + v_I\partial_j\left(\rho v_j\right) + \rho v_j\partial_j v_i &= 0,\\ v_i\left(\frac{\partial\rho}{\partial t} + \partial_j\left(\rho v_j\right)\right) + \rho\left(\frac{\partial v_i}{\partial t} + v_j\partial_t v_i\right) &= 0.\end{aligned} \tag{A.6}$$

The first parenthetical term we recognize is just the continuity equation though. The remainder is the traditional source-free momentum equation.

Conservation of Energy This just reads $\frac{\partial e}{\partial t} + \vec{\nabla}\cdot(e\vec{v}) = 0$. It's a pretty uninteresting equation without pressure and thermodynamic effects, which would arise from a more careful treatment of the source terms.

In fact, pressure contributes a source term $P\vec{\nabla}\cdot\vec{v}$, so the full energy equation is often written

$$\frac{\mathrm{d}e}{\mathrm{d}t} + (e+P)\vec{\nabla}\cdot\vec{v} = 0. \tag{A.7}$$

A.3 The Anelastic/Boussinesq Approximations

A.3.1 Developing the Anelastic/Boussinesq Approximations

Let's relax the incompressibility constraint (we will expand the continuity equation to first order, but the momentum equation will merit a separate treatment):

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \vec{\nabla} \cdot \left(\rho_0 \vec{u}_1\right) = 0, \\ \frac{\partial \vec{u}_1}{\partial t} = -\frac{\vec{\nabla} P}{\rho} - \vec{g}. \end{aligned} \tag{A.8}$$

Suppose we are interested in phenomena with characteristic length scale L and time scale τ. Let's first examine the relative magnitudes of the terms in the continuity equation

$$\frac{\rho_1}{\tau} + \frac{\rho_0 |u_1|}{L} = 0.$$

Thus, if we are interested in time scales $\tau \gg \frac{\rho_1}{\rho_0} \frac{L}{|u_1|}$ then we neglect the first term, the time derivative. This corresponds to making the perturbation incompressible; note that $\frac{\partial \rho_1}{\partial t} \approx \frac{\mathrm{d}\rho_1}{\mathrm{d}t}$ to first order, so we drop the high frequency restoring forces in the perturbation.

For the momentum equation, we instead first manipulate to first order

$$\begin{aligned} -\frac{\vec{\nabla} P}{\rho} - \vec{g} &= -\frac{\vec{\nabla} P_0}{\rho} - \frac{\vec{\nabla} P_1}{\rho_0} - \vec{g}, \\ &= -\frac{\vec{\nabla} P_1}{\rho_0} + \left(\frac{\rho_0}{\rho} - 1\right)\vec{g}, \\ &= -\vec{\nabla}\left(\frac{P_1}{\rho_0}\right) - \frac{P_1}{\rho_0^2}\vec{\nabla}\rho_0 - \frac{\rho_1}{\rho_0}\vec{g}. \end{aligned} \tag{A.9}$$

We now have three equations for four variables, $\vec{u}_1, \rho_1, P_1$. We must introduce a fourth equation, a thermodynamic equation. For an adiabatic process $P\rho^{-\gamma} \propto P^{1-\gamma}T^{\gamma}$ is constant. We thus introduce the concept of the *potential temperature*

$$\theta = T\left(\frac{P_0}{P}\right)^{\kappa}. \tag{A.10}$$

For an adiabatic process, $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Motivated by this, we use

$$\begin{aligned} \frac{\partial 1}{\partial \rho_0} \frac{\partial \rho_0}{\partial z} = \frac{1}{\gamma P_0} \frac{\partial P_0}{\partial z} - \frac{1}{\theta_0} \frac{\partial \theta_0}{\partial z}, \\ \frac{\rho_1}{\rho_0} = \frac{1}{\gamma} \frac{P_1}{P_0} - \frac{\theta_1}{\theta_0}, \end{aligned} \tag{A.11}$$

to give the momentum equation form

$$\frac{\mathrm{d}\vec{u}_1}{\mathrm{d}t} = -\vec{\nabla}\left(\frac{P_1}{\rho_0}\right) + \frac{P_1}{\rho_0}\left(\frac{1}{\theta_0}\vec{\nabla}\theta_0\right) + \vec{g}\frac{\theta_1}{\theta_0}. \tag{A.12}$$

We also recognize $N^2 = \frac{g}{\theta_0}\frac{\partial\theta_0}{\partial z}$. We now do the same trick where we consider dynamics on length scale D and compare the first and second terms in Eq. A.12. Their ratio is $\frac{N^2 D}{g}$, and so as $N^2 \ll \frac{g}{D}$ the freefall time we neglect the second term.

The anelastic fluid equations thus read

$$\begin{aligned}
\vec{\nabla}\cdot\left(\rho_0\vec{u}\right) &= 0,\\
\frac{\partial\vec{u}_1}{\partial t} + \vec{\nabla}\left(\frac{P_1}{\rho_0}\right) - \vec{g}\frac{\theta_1}{\theta_0} &= 0,\\
\frac{\partial\theta_1}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\theta_0 &= 0.
\end{aligned} \tag{A.13}$$

The Boussinesq equations are obtained from these in the limit where $H \gg D$ the relevant length scale, thus we allow ρ_0 to be approximately constant.

A.3.2 Anelastic Solution to Stratified Atmosphere

We simply substitute $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ into Eq. A.13 with $\rho_0 \propto e^{-z/H}$ and obtain

$$\begin{bmatrix}
0 & 0 & ik_x\rho_0 & ik_z\rho_0 - \frac{\rho_0}{H}\\
0 & -i\omega & 0 & \frac{N^2\theta_0}{g}\\
\frac{ik_x}{\rho_0} & 0 & -i\omega & 0\\
\frac{ik_z}{\rho_0} + \frac{1}{\rho_0 H} & -\frac{g}{\theta_0} & 0 & -i\omega
\end{bmatrix}
\begin{bmatrix} P_1\\ \theta_1\\ u_{1x}\\ u_{1z}\end{bmatrix} = 0. \tag{A.14}$$

Taking the determinant of this matrix produces

$$\begin{aligned}
-k_x^2\left(-N^2+\omega^2\right) + \left(ik_z - \frac{1}{H}\right)\left(ik_z + \frac{1}{H}\right)\omega^2 &= 0,\\
\frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}} &= \omega^2.
\end{aligned} \tag{A.15}$$

Appendix B

Numerical Results

B.1 Chebyshev Polynomials

Note that, unlike a Fourier basis, a Chebyshev basis has N^2 grid spacing near its boundaries. By this, we mean that for the Chebyshev polynomials $T_n(x)$, the spacing between zeros of $T_n(x)$ near $x = \pm 1$ scales like n^{-2}. This contrasts with Fourier series for which $\psi_n(x) = \cos n\pi x, \sin n\pi x$ for which the zeros are spaced $\Delta x = \frac{1}{n}$. This is important because the CFL condition for PDEs requires that $u\frac{\Delta t}{\Delta x} < C$ for some constant C, otherwise the simulation will move fluid elements more than Δx in a single timestep, an illegal operation for the basis function.

If we consider instead $T_n(x)$ satisfying $T_n(\cos\theta) = \cos n\theta$, then within some spacing of the edge of the domain $\cos\theta = 1 - \Delta x$ then $\Delta x \approx \theta^2/2$ then the number of zeros seen is the number of solutions $\cos\phi = 0, \phi = n\theta = n\sqrt{2\Delta x}$ which requires $\sim k\pi \in [0, n\sqrt{2\Delta x}]$. Thus, for any interval Δx, the number of zeros contained scales with n^2!

Note: There exist papers on the Courant stability condition of Chebyshev spectral methods which cite exactly the N^{-2} minimum timestep scaling we can infer from the above. But if our field is small near the boundaries and is only interesting in a region isolated from the boundary (e.g. with our damping zones) then the bound used in the papers scales more like N^{-1}, which is much more desirable.

B.2 Algebraic Manipulations

B.2.1 Factoring out the exponential dependence

I've had to do this algebra too many times now, so let's just consider if we factor out the $e^{z/2H}$ exponential scaling on the dynamical variables and the $e^{-z/H}$ scaling on ρ_0. This leaves us with:

$$\vec{\nabla}\cdot\vec{u}_1 + \frac{u_{1z}}{2H} = 0, \tag{B.1a}$$

$$\frac{\partial \rho_1}{\partial t} - \frac{u_{1z}\rho_0}{H} = -e^{z/2H}\left[\left(\vec{u}_1\cdot\vec{\nabla}\rho_1\right) + \frac{u_{1z}\rho_1}{2H}\right], \tag{B.1b}$$

$$\frac{\partial \vec{u}_1}{\partial t} + \frac{\vec{\nabla}P}{\rho_0} + \frac{P}{2H\rho_0}\hat{z} + \frac{\rho_1\vec{g}}{\rho_0} = -e^{z/2H}\left[\left(\vec{u}\cdot\vec{\nabla}\vec{u}_1\right) + \frac{u_{1z}\vec{u}_1}{2H}\right]. \tag{B.1c}$$

I'm too lazy to re-notate my variables, so all of them are constant amplitude over z now.

B.3 Hydraulic Jumps in Surface Waves

We originally got started on this since it seems like a dimensionally-restricted version of what we're seeing in simulations, but presently unclear whether really the case. Also, Faber's book for fluids *Fluid Dynamics for Physicists* cites a critical breaking jump height of 1.3 which is markedly false; papers such as Chanson's 2009 hydraulic jumps review in Euro. J. Mech B/Fluids shows that undular jumps (instead of turbulent jumps) can be found easily up to 3.5/4.

B.3.1 1D Hydraulic Jump

This is one of the OG papers; Bélanger solved this as early as in 1838 but Rayleigh's work in 1916 is also pretty famous. Consider the 1D nonlinear Euler equations for a body of water whose surface can move $h(x)$, then give it velocity $u(x,z)$. The problem is really 2D, but we can integrate along the z direction, in which case $\frac{\partial P(x,z)}{\partial x} = \frac{\rho g h(x,z)}{x}$ simply integrates to $\frac{\rho g h(x)^2}{2} = \frac{g\Sigma(x)^2}{2\rho}$. Assuming also the velocity is uniform $u(x,z) = u(x)$ gives equations of motion

$$\frac{\partial \Sigma}{\partial t} + \frac{\partial(\Sigma u)}{\partial x} = 0, \tag{B.2}$$

$$\begin{aligned}\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} &= -\frac{1}{\Sigma}\frac{\partial}{\partial x}\left(\frac{g\Sigma^2}{2\rho}\right),\\ &= -\frac{g}{\rho}\frac{\partial \Sigma}{\partial x}.\end{aligned} \tag{B.3}$$

It proves easier to divide the density equation by ρ so that we can replace $\frac{\Sigma}{\rho} = h(x)$ in the above. A specific momentum conservation equation can be obtained

$$\frac{\partial(hu)}{\partial t} + \frac{\partial}{\partial x}\left(hu^2 + \frac{gh^2}{2}\right) = 0. \tag{B.4}$$

Now, consider a hydraulic jump $u_1, h_1 \to u_2, h_2$ such that the discontinuity is fixed (we've shifted to the jump front frame), so that $[uh], \left[hu^2 + gh^2/2\right]$ bracketed quantities are continuous across the jump. This results in

$$\begin{aligned}
h_1u_1^2 - h_2u_2^2 = h_1u_1^2\left(1 - \frac{u_2}{u_1}\right) &= h_1u_1^2\left(1 - \frac{h_1}{h_2}\right),\\
&= \frac{g\left(h_2^2 - h_1^2\right)}{2},\\
u_1^2 &= \frac{h_2}{h_1}\frac{g(h_1+h_2)}{2}.
\end{aligned} \tag{B.5}$$

Customarily one defines $a_1^2 = gh_1, \mathrm{Fr} = \frac{u_1}{a_1}$ and $\gamma = \frac{h_2}{h_1}$ which gives simplification

$$\begin{aligned}
2\mathrm{Fr}^2 &= \gamma(1+\gamma)\\
\gamma &= \frac{-1+\sqrt{1+8\mathrm{Fr}^2}}{2}.
\end{aligned} \tag{B.6}$$

Note now that $\mathrm{Fr}^2 > 1$ produces a $\gamma > 1$, so $h_2 > h_1$. Noting moreover that $\frac{u_1}{u_2} = \gamma$ and so $u_1 > u_2$. If we have a picture of a jet incident on a medium at rest, u_2 is just the velocity of the jump.

B.3.2 1D Viscous Theory

We follow Bohr's paper from J. Fluid Mech. in 1993 (also done by Tani in 1949 to some extent) but in 1D first so that the equations are a bit easier to intuit. It's completely unnecessary but illustrates some features of the approach.

Consider the same 2D equations as before (before integrating along z), but now permit a viscosity term. In the boundary layer approximation, the usual assumptions are that flow largely varies horizontally but variations are stronger vertically (i.e. we only consider the x velocity equation but only consider the z viscosity). We make one small modification, it seems that we permit an incompressibility assumption since $\frac{\mathrm{d}h(x,z,t)}{\mathrm{d}t} = 0$; this is just $\Delta P = \rho g h =$ the Lagrangian pressure boundary condition. Thus, the equations of motion are

$$\frac{\partial u_x}{\partial x} + \frac{\partial u_z}{\partial z} = 0, \tag{B.7}$$

$$u_x\frac{\partial u_x}{\partial x} + u_z\frac{\partial u_x}{\partial z} = -g\frac{\partial h}{\partial x} + \nu\frac{\partial^2 u_x}{\partial z^2}. \tag{B.8}$$

Integration along z of the momentum equation can be done with appropriate BCs: no-slip/penetration at the bottom $u_x(z=0) = u_z(z=0) = 0$, while no shear at the surface $\frac{\partial u_x}{\partial z}(z=h) = 0$. We handle the integral of $u_z u_{x,z}$ by integrating by parts, dropping the boundary term and continuity equation, giving

$$2u_x\int_0^h u_x\frac{\mathrm{d}u_x}{\mathrm{d}x}\,\mathrm{d}z = -g\frac{\partial h}{\partial x}h - \nu\left.\frac{\partial u_x}{\partial z}\right|_{z=0}. \tag{B.9}$$

We make a few handwavy substitutions for representative values $\partial_z u_x(z=0) \sim u_x/h$ and $\partial_x u_x^2 \sim \partial_x \bar{u}_x^2$ average value. This averaging procedure and dropping the factor of 2 (the two papers seem to disagree on this coefficient, I think Bohr's paper has an error) lets us examine

$$\bar{u}_x \frac{\partial \bar{u}_x}{\partial x} + g \frac{\partial h}{\partial x} \sim -\frac{\bar{u}_x}{h^2}. \tag{B.10}$$

Next, let's identify $\bar{u}_x h \equiv q$ volume flux constant in x, which leaves us with

$$\frac{\partial \bar{u}_x}{\partial x}\left(\bar{u}_x - \frac{gq}{\bar{u}_x^2}\right) \sim -\nu \frac{\bar{u}_x^3}{q^2}. \tag{B.11}$$

Let's study in terms of $\mathrm{Fr}^2 = \frac{\bar{u}_x^2}{gh} = \frac{\bar{u}_x^3}{gq}$ again, albeit a bit messier, which takes a bit of work but eventually lets us rearrange

$$\frac{\partial \mathrm{Fr}}{\partial x} = \frac{(gq)^{1/3}\nu}{q^2} \frac{\mathrm{Fr}^{5/3}}{1-\mathrm{Fr}^2}. \tag{B.12}$$

Note that the units work here, as $\frac{(gq)^{1/3}\nu}{q^2} \equiv L_\nu$ has units of length.

Let's interpret the equation we have now: we have exactly how the inbound $\mathrm{Fr}(x=-\infty)$ evolves over x, which should describe (extremely coarsely, owing to our approximations) the boundary layer of the jump.

Qualitatively, we can use a parameterization $\mathrm{Fr}(s), x(s)$ such that $\dot{\mathrm{Fr}}(s) = -\mathrm{Fr}^{5/3}, \dot{x}(s) = L_\nu\left(\mathrm{Fr}^2 - 1\right)$ where we define $L_\nu \equiv a_0 h^2/\nu$ and we've absorbed the negative sign into $\dot{\mathrm{Fr}}(s)$. I'm too lazy to plot this, but if we start at a point $\mathrm{Fr} > 1$, it's clear that Fr increases and x decreases until $\mathrm{Fr} = 1$. Since the sign convention on the parametric is arbitrary, we can flip the sign and see that the parameterized curve formally follows decreasing $\dot{\mathrm{Fr}}(s) \propto -1/s$ while $\dot{x}(s) \propto -L$. This is double-valued with the solution $\mathrm{Fr} > 1$ though, so with some physical intuition we discard this and consider what values of Fr can take on beyond its intersection with $\mathrm{Fr} = 1$.

It must discontinuously jump to another family of curves. But any nonzero value of Fr will live on another one of these such trajectories and increase without bound; indeed the only reasonable asymptotic solution is $\mathrm{Fr} = 0$. This completes the picture: for an inbound $\mathrm{Fr} \gg 1$, it follows roughly $\frac{\partial \mathrm{Fr}}{\partial x} \gtrsim -\mathrm{Fr}^{-1/3}$ until it reaches $\mathrm{Fr} = 1$, then it discontinously disappears. This is the upstream flow, and so by mass conservation the location of the discontinuity must travel.

This is exactly the picture we derived above: for an incoming $\mathrm{Fr} \gtrsim 1$, the upstream flow has zero velocity and the jump travels.

B.3.3 2D Hydraulic Jump

This is the case worked out in detail in Bohr's paper. First, perhaps the easiest thing to do is to enforce jump conditions $[rhu], [hu^2 + gh^2/2]$ subject to $q = 2\pi r h_i u_{ri}$ conserved total flow. This gives us

$$\begin{aligned} h_1 u_{r1}^2 - h_2 u_{r2}^2 = \frac{q}{2\pi r}(u_{r1} - u_{r2}) &= \frac{gh_1^2}{2} - \frac{gh_2^2}{2}, \qquad \text{(B.13)}\\ &= \frac{gq^2}{8\pi r^2 u_{r1}^2} - \frac{gq^2}{8\pi r^2 u_{r2}^2},\\ 2u_{r1}^2 u_{r2}^2 &= \frac{gq}{2\pi r}(u_{r2} + u_{r1}),\\ u_{r1} &= \frac{gq}{8\pi u_{r2}^2 r}\left(1 + \sqrt{1 + 16\frac{\pi u_{r2}^3 r}{gq}}\right), \qquad \text{(B.14)}\\ \frac{u_{r1}}{u_{r2}} &= \frac{1}{4\mathrm{Fr}^2}\left(1 + \sqrt{1 + 8\mathrm{Fr}^2}\right). \qquad \text{(B.15)} \end{aligned}$$

We've defined $\mathrm{Fr}^2 \equiv \frac{2\pi r u_r^3}{gq} = \frac{u_r^2}{gh}$. Note that this is in agreement with $\gamma = \frac{h_1}{h_2} = \frac{u_2}{u_1}$ from the plane parallel theory. The major difference is that now $\mathrm{Fr}^2 = \mathrm{Fr}^2(r)$, and so the location of the hydraulic jump can be pinned. One might expect that once $\mathrm{Fr}(r) > 1$ that the jump occurs, but this turns out not to be the case.

We now perform the boundary layer analysis. Following much the same approach but in cylindrical coordinates we may write down equations of motion

$$\begin{aligned} u_r\frac{\partial u_r}{\partial r} + u_z\frac{\partial u_r}{\partial z} &= -g\frac{\partial h}{\partial r} + \nu\frac{\partial^2 u_r}{\partial z^2},\\ \frac{1}{r}\frac{\partial(ru_r)}{\partial r} + \frac{\partial u_z}{\partial z} &= 0. \qquad \text{(B.16)} \end{aligned}$$

Note the slight difference here due to the $\vec{\nabla}\cdot$ in cylindrical coordinates. Interestingly, as a consequence of this $\frac{\partial P_r}{\partial t} + \frac{1}{r}\partial_r\left(hu_r^2 + gh^2/2\right) \neq 0$, since one term is a $\vec{\nabla}\cdot(hu_r)$ and the other is a $\vec{\nabla}gh^2/2$! Radial momentum is not conserved as the flow spreads out. Averaging using the same techniques as above gives

$$\begin{aligned} \bar{u}_r\frac{\partial \bar{u}_r}{\partial r} + g\frac{\partial h}{\partial r} &= -\nu\frac{\bar{u}_r}{h^2},\\ \bar{u}_r h r &= q. \qquad \text{(B.17)} \end{aligned}$$

Combining the two gives

$$\frac{\partial \bar{u}_r}{\partial r}\left(\bar{u}_r - \frac{gq}{\bar{u}_r^2 r}\right) = \frac{gq}{\bar{u}_r r^2} - \nu\frac{\bar{u}_r^3 r^2}{q^2}. \qquad \text{(B.18)}$$

It's not very worth converting into dimensionless Fr number since it would carry an r dependence (if we are to hold q constant) and we need to study the r dependence.

It may be observed that Eq. B.18 has only a single fixed point, $u_r^3 r = gq, u_r^4 r^4 = gq^3/\nu$. Call this point r_c the critical radius, then it turns out that the two parametric solutions (as we analyzed in the previous section) both spiral around this point. This has a few important properties:

- No asymptotic solutions: the solution for $r > r_c$ always diverges at some finite $r_s < \infty$. We can try an approximate solution in the form $u_r = y/r$; this is done in Bohr's paper.
- For a fixed $r_s \gg r_c, < \infty$, we can require the usual jump condition which takes the slightly modified form

Edit: Looks like Bhagat 2018 J. Fluid. Mech. does this more correctly, they find that surface tension is the proper term to invoke after performing this boundary layer analysis. I'm not going to finish reworking the rest of Bohr's paper here, since it's up through the boundary layer equations that is reusable.