# Research Notes

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# Contents

# Chapter 1

# Preliminary Problems

To get an intuition for how Dedalus and fluid mechanics works, we will solve some toy problems. Recall fluid equations in the presence of a uniform gravitational field $\vec{g} = -g\hat{z}$:

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{u}\right) &= 0, \\ \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \frac{\vec{\nabla} P}{\rho} - \vec{g} &= 0. \end{aligned} \tag{1.1}$$

In the incompressible limit, $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$, which implies $\vec{\nabla} \cdot \vec{u} = 0$. We use subscripts to indicate perturbed quantities, $Q_0$ is background and $Q_1$ is perturbed. We will generally use $\vec{u}_0 = 0$ unless otherwise noted. We will also generally assume symmetry along all axes except $z$ the vertical axis.

In the incompressible limit, the fluid equations become

$$\begin{aligned} \vec{\nabla} \cdot \vec{u}_1 &= 0, \\ \frac{\partial \rho_1}{\partial t} + u_{1z} \frac{\partial \rho_0}{\partial z} &= 0, \\ \frac{\partial \vec{u}_1}{\partial t} + \frac{1}{\rho_0} \vec{\nabla} P_1 + \frac{\rho_1 g \hat{z}}{\rho_0} &= 0. \end{aligned} \tag{1.2}$$

We have used $\vec{\nabla} P_0 = -\rho_0 g \hat{z}$ in the absence of perturbations.

## 1.1 Incompressible, No Gravity

We note that in the no gravity limit that $\rho_1$ does not have an effect on other dynamical variables, so the equations of motion we must solve are

$$\begin{aligned} \vec{\nabla} \cdot \vec{u}_1 &= 0, \\ \frac{\partial \vec{u}_1}{\partial t} + \frac{\vec{\nabla} P_1}{\rho_0} &= 0. \end{aligned} \tag{1.3}$$

We can take the divergence of the momentum equation and substitute the continuity equation to get $\nabla^2 P = 0$.

### 1.1.1 Dirichlet BCs

This is a Laplace equation, which we've solved countless times. Imposing periodic boundary conditions in the $x$ direction and $P_1(z=L)=0, P_1(z=0)=\mathscr{P}(x,t)$, we obtain eigenfunctions

$$\begin{aligned} P_{1,n}(x,z,t) &= \frac{\mathscr{P}_n(t)}{\sinh(k_n L)} e^{ik_n x}\sinh(k_n(L-z)), \\ u_{1x,n}(x,z,t) &= \int^t -\frac{1}{\rho_0}\frac{\partial P_{1,n}}{\partial x}\,\mathrm{d}t, \\ u_{1z,n}(x,z,t) &= \int^t -\frac{1}{\rho_0}\frac{\partial P_{1,n}}{\partial z}\,\mathrm{d}t. \end{aligned} \tag{1.4}$$

We define $k_n = \frac{2\pi n}{L}, n \geq 0$ and $\mathscr{P}(x,t) = \sum_n \mathscr{P}_n(t) e^{ik_n x}$.

Thus, if we impose BCs $\mathscr{P}(x,t) = \sin\frac{2\pi x}{L}$ and start with initial conditions such that all quantities are zero, we would expect after transients die out that

$$\begin{aligned} P(x,z,t) &= \frac{\sin\frac{2\pi x}{L}}{\sinh 2\pi}\sinh\left(2\pi\frac{L-z}{L}\right), \\ u_{1x}(x,z,t) &= -\frac{2\pi t}{L\rho_0}\frac{\cos\frac{2\pi x}{L}}{\sinh 2\pi}\sinh\left(2\pi\frac{L-z}{L}\right), \\ u_{1z}(x,z,t) &= +\frac{2\pi t}{L\rho_0}\frac{\sin\frac{2\pi z}{L}}{\sinh 2\pi}\cosh\left(2\pi\frac{L-z}{L}\right). \end{aligned} \tag{1.5}$$

This is in good agreement with the results, presented in Fig. 1.1. Note that $P$ is constant while $\vec{u}$ increases linearly in time, and we observe the expected $\sim \sin x \sinh\frac{L-z}{z}$ dependence. In fact, $u_{1x}, u_{1z}$ are exactly $\frac{2\pi}{10}$ at $t=1$.

### 1.1.2 Things to Note

It is worth noting that, since our Eq. 1.3 reduced to a Laplace equation, we needed two $z$ BCs and two $x$ BCs (periodic BCs amount to equating the value and derivative of the function). This is in agreement with the observation that the original Eq. 1.3 had two derivatives in $x,z$ apiece, so we needed two BCs each.

It is also worth seeing from our solution that $P$ immediately goes to the equilibrium solution. This is not surprising since in the incompressible limit, sound speed goes to infinity which is the timescale on which the pressure field adjusts to net forces. Thus, the dynamics solely arise from the static pressure field pushing the velocity field to equilibrium.

We chose time-independent $\mathscr{P}(x,t)$, but it is clear that whatever $\mathscr{P}(x,t)$ we choose, the time dependence propagates to the velocities by way of an integral. If we had instead chosen to take Fourier

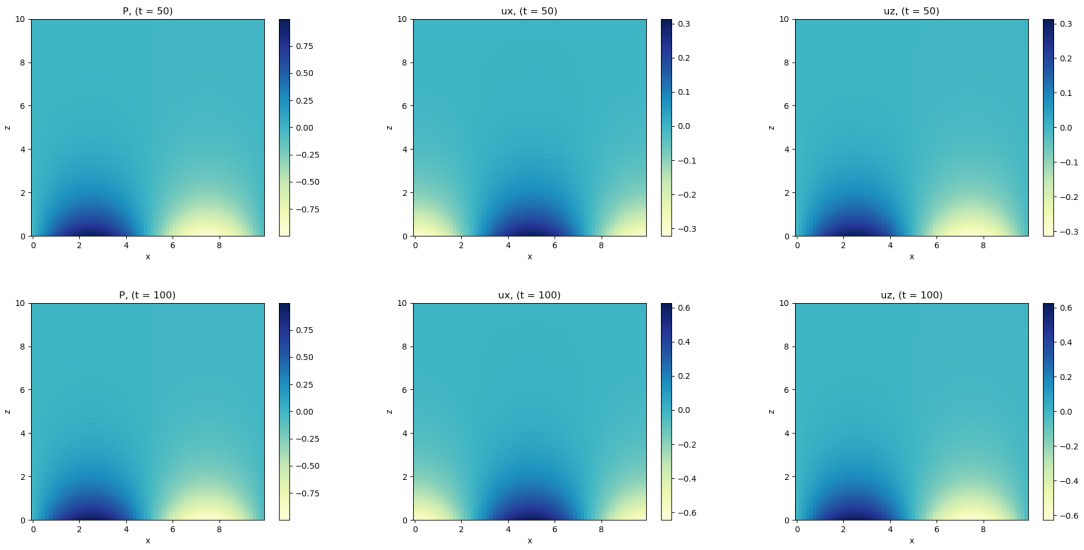


**Figure 1.1:** $P, u_x, u_z$ at $t = 0.5$ and $t = 1$ for $\rho_0 = 1$. We choose a $L = 10$ square domain.

transform $t \to \omega$, we would have had to integrate the boundary condition against the eigenfunctions for each of the $\omega$, which is still easily computable, to get the full $u_{1x}(x,z,t)$. We consider this in preparation for when we can only solve for a set of $\vec{k}, \omega$ in the next problem.

The next thing we would have wanted to do is solve a problem with radiative BCs, but we need to have wave solutions, which are missing in the absence of gravity. We thus move on to the next configuration.

## 1.2 Incompressible, Stratified w/ Gravity

### 1.2.1 Eigenfunctions

Let's restore the $\rho_1 g$ term now. For funsies, we begin by solving for arbitrary stratification $\rho_0(z)$ first. The fluid equations to first order reduce to

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \vec{u}_1 \cdot \left(\vec{\nabla} \rho_0\right) &= 0, \\ \vec{\nabla} \cdot \vec{u}_1 &= 0, \\ \frac{\partial \vec{u}_1}{\partial t} &= -\frac{\vec{\nabla} P_1}{\rho_0} - \frac{\rho_1 g}{\rho_0} \end{aligned} \tag{1.6}$$

We expect there to be some $z$ dependence in the amplitude, so we substitute variables of form $e^{i(kx-\omega t)}$ and do not specify the $z$ dependence. This gives us

$$\begin{aligned} -i\omega\rho_1 - u_{1z}\frac{\partial\rho_0}{\partial z} &= 0, \\ iku_{1x} + \frac{\partial u_{1z}}{\partial z} &= 0, \\ -iwu_{1x} + \frac{ik_xP_1}{\rho_0} &= 0, \\ -iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1 g}{\rho_g} &= 0. \end{aligned} \tag{1.7}$$

We substitute $N^2 = -\frac{g}{\rho_0}\frac{\partial\rho_0}{\partial z}$ to obtain

$$-i\omega\rho_1 - u_{1z}\frac{\rho_0 N^2}{g} = 0, \tag{1.8a}$$

$$iku_{1x} + \frac{\partial u_{1z}}{\partial z} = 0, \tag{1.8b}$$

$$-iwu_{1x} + \frac{ik_xP_1}{\rho_0} = 0, \tag{1.8c}$$

$$-iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1 g}{\rho_0} = 0. \tag{1.8d}$$

Eliminating $u_{1x}$ by substituting (1.8b) into (1.8c) and $\rho_1$ by substituting (1.8a) into (1.8d) give

$$i\omega\frac{\partial u_{1z}}{\partial z} + \frac{k_x^2P_1}{\rho_0} = 0, \tag{1.9a}$$

$$\left(\omega^2 - N^2\right)u_{1z} + \frac{i\omega}{\rho_0}\frac{\partial P_1}{\partial z} = 0. \tag{1.9b}$$

Finally, we multiply (1.9a) with $\rho_0$ and differentiate $\mathrm{d}z$ and combine with (1.9b) to give

$$\frac{\mathrm{d}^2u_{1z}}{\mathrm{d}z^2} + \frac{1}{\rho_0}\frac{\partial\rho_0}{\partial z}\frac{\partial u_{1z}}{\partial z} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right)u_{1z} = 0. \tag{1.10}$$

Let's now pick stratification $\rho \propto e^{-z/H}$ Eq. 1.10 clearly has exponential solutions $e^{\kappa z}$ for

$$\kappa^2 - \frac{\kappa}{H} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right) = 0. \tag{1.11}$$

We permit complex $\kappa = \frac{1}{2H} + ik_z$, and from the above clearly

$$\begin{aligned} k_z^2 &= -\frac{1}{4H^2} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right), \\ \omega^2 &= \frac{N^2k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}. \end{aligned} \tag{1.12}$$

Thus the eigenfunctions are

$$\begin{aligned} u_{1z} &= e^{z/2H} e^{i(k_z z + k_x x - \omega t)}, \\ u_{1x} &= -\frac{k_z + i/2H}{k_x} u_{1z}, \\ \rho_1 &= \frac{i\rho_0}{H\omega} u_{1z}, \\ P_1 &= -\frac{\rho_0 \omega}{k_x^2}(k_z + i/2H) u_{1z}. \end{aligned} \tag{1.13}$$

### 1.2.2 Analytically Solving an IVP, Dirichlet + Driving BCs

We will analyze everything in terms of $u_{1z}$ since it has the simplest form; note that when actually choosing the BCs we will have to consider the gauge freedom of $P$ and some considerations we defer to the computational section.

Currently, we have a set of eigenfunctions

$$u_{1z}\left(x,z,t\middle|\vec{k},\omega\right) = e^{z/2H} e^{i(k_z z + k_x x - \omega t)}. \tag{1.14}$$

Note that $u_{1z}$ is really only a function of two parameters rather than the three $(k_x, k_z, \omega)$, since the three are related by dispersion relation Eq. 1.12.

Now, we implement BCs. Consider domain $x, z \in [0, L]$. We will use periodic BCs again in $x$, so then $k_{x,n} = \frac{2\pi n}{L}, n \geq 0$. Then, we will require $u_{1z,n}(x, L, t) = 0$, a Dirichlet condition at the top boundary, which restricts us to eigenfunctions of form

$$u_{1z,n}\left(x,z,t\middle|\vec{k},\omega\right) = e^{z/2H} e^{i(k_x x - \omega t)} \sin(k_z(L - z)). \tag{1.15}$$

Finally, we must choose a BC at $z = 0$. We will choose a general function $u_{1z}(x, 0, t) = F(x, t)$ where we can decompose

$$F(x,t) = \int \sum_n \mathscr{F}(k_{x,n}, \omega) e^{i(k_{x,n} x - \omega t)} \, \mathrm{d}\omega. \tag{1.16}$$

Matching BCs then gives us general solution for $u_{1z}$ given an arbitrary driving function

$$u_{1z}\left(x,z,t\middle|\vec{k},\omega\right) = \int \sum_n \mathscr{F}\left(k_{x,n}, \omega\right) \frac{e^{z/2H} e^{i(k_{x,n} x - \omega t)} \sin(k_z(L - z))}{\sin k_z L} \, \mathrm{d}\omega. \tag{1.17}$$

For ease of computation, let's pick $F(x,t) = \cos\left(\frac{2\pi x}{L} - \omega_0 t\right)$, so our full expected solution is (note

$A+\epsilon = Ae^{\epsilon/A}+\mathcal{O}(\epsilon^2))$

$$\begin{aligned}
u_{1z}\left(x,z,t\middle|\vec{k},\omega\right) &= e^{z/2H}\frac{\cos\left(\frac{2\pi x}{L}-\omega_0 t\right)\sin(k_z(L-z))}{\sin k_z L},\\
u_{1x}\left(x,z,t\middle|\vec{k},\omega\right) &\approx \frac{k_z}{k_x}e^{z/2H}\frac{\cos\left(\frac{2\pi x}{L}-\omega_0 t+\frac{1}{2Hk_z}\right)\sin(k_z(L-z))}{\sin k_z L},\\
\rho_1\left(x,z,t\middle|\vec{k},\omega\right) &\approx \frac{\rho_0}{H\omega}e^{z/2H}\frac{-\sin\left(\frac{2\pi x}{L}-\omega_0 t\right)\sin(k_z(L-z))}{\sin k_z L},\\
P_1\left(x,z,t\middle|\vec{k},\omega\right) &\approx -\frac{\rho_0\omega k_z}{k_x^2}e^{z/2H}\frac{\cos\left(\frac{2\pi x}{L}-\omega_0 t+\frac{1}{2Hk_z}\right)\sin(k_z(L-z))}{\sin k_z L},
\end{aligned} \tag{1.18}$$

where $k_z : \omega(k_x,k_z)=\omega_0$ by the dispersion relation Eq. 1.12. Note that $H$ contributes both to the overall exponential profile and to the phase lag of $u_{1x}$.

### 1.2.3 Computationally Solving an IVP, Dirichlet + Driving BCs

To solve this computationally with the aforementioned BCs, periodic in $x$, Dirichlet 0 at $z=L$ and $\cos\left(\frac{2\pi x}{L}-\omega_0 t\right)$ at $z=0$, we must address the gauge freedom in $P$. This arises because for $k_x=0$, the divergence-free condition $\vec{\nabla}\cdot\vec{u}_1=\frac{\partial u_z}{\partial z}=0$ specificies $u_{1z}$ up to a constant already, so the bottom BC will fix the value of $u_z$ when $k_x=0$.

A different way of phrasing the same argument is as follows. Consider the discrete $N\times N$ (square for notational simplicity) grid. At the boundary, there is a list of values $f(\{x_i\},z_0)$ that lives in an $N$ dimensional space. We can thus pick a spanning set of $N$ basis vectors, and for each of these $N$ vectors, by enforcing $\vec{\nabla}\cdot\vec{u}=0$ at the boundary we fix the allowed $f(\{x_i\},z_{-1})$ boundary conditions we can implement. But there exists a choice of basis vectors for which one of the basis vectors is constant $f_i(\{x_i\},z_0)=C$. For this basis vector, the boundary condition is fully determined, so we only have $N-1$ dimensions from which to choose the BCs $f(\{x_i\},z_{-1})$. Since the dimensionality of a space cannot depend on the choice of basis vector, the divergence free condition actually yields an extra degree of freedom.

We can use this extra degreee of freedom to specify $P(z=L)=0$ so the oscillations should have zero mean. Then we can just simulate away!

From here we can just tweak the parameters until we get something useful, can look at the videos. Note that we should simulate until at least $T>\frac{2L_z}{v_{p,z}}$ the $z$ phase space velocity, so we can capture any reflections off the boundary. Turns out both Dirichlet and Neumann BCs give strong reflections that produce standing waves in the $z$ and traveling waves in the $x$.

### 1.2.4 Phase/Group Velocity

Let's figure out the analytical forms for the phase, group velocity and the energy density/power flux.

Let's first consider the phase velocity. We traditionally think about the phase velocity in 1D $v_{ph}=\frac{\omega(k)}{k}$, but in general it is the function such that $\vec{k}\cdot\vec{r}-\omega t=\vec{k}\cdot\left(\vec{r}-\vec{v}_{ph}t\right)$ is constant (the phase of

the wave). Thus, it must satisfy $\vec{k}\cdot\vec{v}_{ph}=\omega$, and so one sensible choice is

$$\vec{v}_{ph}=\frac{\omega\hat{k}}{\left|\vec{k}\right|}=\frac{\omega\vec{k}}{\left|\vec{k}\right|^2}. \tag{1.19}$$

For our stratified atmosphere problem, $\omega^2=\frac{N^2k_x^2}{k^2+\frac{1}{4H^2}}$. This corresponds to $\vec{v}_{ph}=\frac{Nk_x}{\sqrt{k_x^2+k_z^2+1/4H^2}}\vec{k}$.

On the other hand the group velocity is given $\vec{v}_g=\frac{\partial\omega}{\partial k_i}\hat{\imath}$. In our problem, $v_{g,z}=-\frac{Nk_xk_z}{\left(k_x^2+k_z^2+1/4H^2\right)^{3/2}}$ while $v_{g,x}=\frac{N}{\sqrt{k_x^2+k_z^2+1/4H^2}}-\frac{Nk_x^2}{\left(k_x^2+k_z^2+1/4H^2\right)^{3/2}}=\frac{N\left(k_z^2+1/4H^2\right)}{\left(k_x^2+k_z^2+1/4H^2\right)^{3/2}}$.

### 1.2.5 Energy/Power Flux

To compute the energy and power flux of the wave, we recall that for a general fluid the energy conservation equation reads

$$\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2+\rho\epsilon\right)=\vec{\nabla}\cdot\left(\rho\vec{v}\left(v^2+\epsilon+\frac{P}{\rho}\right)\right), \tag{1.20}$$

where $\epsilon$ is the internal energy. But since $P=(\gamma-1)\rho\epsilon$ and we take $\gamma\to\infty$ incompressible limit, $\epsilon=0$, so the energy of the wave is just $\frac{1}{2}\left(\rho_0+\rho_1\right)u_1^2$ and the power flux is $\left(\rho_0+\rho_1\right)\vec{u}_1u_1^2+P_1\vec{u}_1$.

Another formula given by Sutherland 2011 is $\frac{\partial\langle E\rangle}{\partial t}=v_{g,z}\langle E\rangle$ where we average over $x$ one wavelength. I wasn't able to show that these two agree in general; Sutherland's formula seems to be derived for a traveling wavepacket whereas we have no such thing. We do not use this expression since the Landau & Lifshitz expression works well.

## 1.3 Supressing Reflection

Sourced from `https://people.maths.ox.ac.uk/trefethen/6all.pdf`. There are largely two ways to do this. The first is to add a damping zone, a region in which $\dot{q}=-q/\tau_d(z)$ where $\tau_d(0)=0$ and increases to some non-small number for some $z_0$ beyond which we want damping. One form, chosen in Ryan and Dong's paper, is to use multiplicative factor $f(z)=\max\left[0,1-\frac{(z-z_b)^2}{(z_d-z_b)^2}\right]$ where $z_b$ is where one begins supression and $z_b$ is the boundary. Then a dynamical variable $q$ can be supressed via $\dot{q}=(\ldots)-f(z)\frac{q-q_0}{\tau}$ where $\tau$ is the dynamical timescale on which to damp and $q_0$ is the value to damp to. We will apply this only to the velocity variables with an eye to extending this approach to the nonlinear regime, where the velocity variables should still be damped to zero but $\rho,P$ must capture the stratification and so cannot easily be artificially damped (maybe? Revisit if this does poorly compared to damping all linear variables, and damp the nonlinear variables to their equilibrium/stratified values).

### 1.3.1 First Order

In order to compute the reflecting boundary condition, we must find a sequence of differential operators that well approxmiates the system of PDEs. We can get this from the dispersion relation; since our radiating boundary is along $z$, we should try to solve for $k_z$ in the dispersion relation to get a pseudodifferential operator form for $\frac{\partial}{\partial z}u_z$ (we take $k_x H \ll 1$ as well to simplify):

$$\begin{aligned} k_z^2 &= \frac{N^2}{\omega^2}k_x^2 - k_x^2 - \frac{1}{4H^2}, \\ &\approx k_x^2\left(\frac{N^2}{\omega^2} - 1\right), \end{aligned} \tag{1.21}$$

$$\frac{\partial u_z}{\partial z} \approx -\frac{\partial u_z}{\partial x}\sqrt{\frac{N^2}{\omega^2} - 1}. \tag{1.22}$$

We pick the negative sign in accordance with an outgoing wave, per the group velocity formulae for $k_z > k_x$.

From the numerical simulations, it is clear that reflections are well supressed initially, but the the simulation blows up. This is because our supression is imperfect (differential approximation to a pseudodifferential operator) and since our system is dissipation free, reflection grows. Moreover, since any reflected components seem to have different $k_x, k_z, \omega$, the second time they're incident on the boundary, our above condition becomes wildly inaccurate.

## 1.4 The Difficulty of the Incompressible, Nonlinear Problem

We consider the problem where both advective terms are kept and $\rho_1 \sim \rho_0$. We write down thus nonlinear fluid equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{u}\cdot\left(\vec{\nabla}\rho\right) &= 0, \\ \vec{\nabla}\cdot\vec{u} &= 0, \\ \frac{\partial \vec{u}}{\partial t} &= -\left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} - \frac{\vec{\nabla}P}{\rho} - g\hat{z}. \end{aligned} \tag{1.23}$$

Note that we cannot even subtract off hydrostatic equilibrium anymore since $\rho$ can deviate greatly from the normal $\rho_0 e^{-z/H}$!

### 1.4.1 Rearrangement for Dedalus

In order to simulate this at all in a spectral program, recall the way that nonlinear PDEs are decomposed in a spectral code. Given a phase space $Q$, consider first a system of PDEs expressible in form

$$\dot{Q} + \mathbf{L}Q + f(Q) = g \tag{1.24}$$

where $\dot{Q}$ is the time derivative, $\mathbf{L}$ is some linear operator, $f(Q)$ are nonlinear terms and $g$ is an arbitrary driving function. Spectral codes work by re-casting the operators to be purely algebraic in an appropriate spectral basis consisting of some $\varphi_i$ trial and $\xi_i$ test functions. The *tau spectral method* that Dedalus uses considers using the trial functions as the test functions, so the effective decomposition that is performed is

$$\langle * | \varphi_i \dot{Q} + \mathbf{L} Q | * \rangle \varphi_j = \dot{Q}_i + L_{ij} Q_q = \langle * | \varphi_i - f(Q) + g | * \rangle \varphi_i. \tag{1.25}$$

It is then clear that $-f(Q)+g$ can just be treated as inhomogeneous source terms, and over a short $\Delta t$ it can be approximated via $-f(Q_0)+g$. This first-order inhomegeneous ODE then admits closed form exact solutions $e^{-L_{ij}t}(\ldots)$.

One can imagine, for this matter, that $\frac{\mathrm{d}}{\mathrm{d}t}$ can be wrapped in the $L_{ij}$ operator and the timestepping computed implicitly; I believe this is what Dedalus does, implicitly timestepping the left hand side and explicitly the inhomogeneous terms.

Both of these procedures require $\mathbf{L}$ linear operator or $\frac{\mathrm{d}}{\mathrm{d}t} + \mathbf{L}$ to be full rank, or invertible. This is enforced analytically by requiring the same number of BCs as derivatives, and numerically this is little different. This is why it is possible to have components of $Q$ that are not differentiated $\partial_t$ but still have well-defined time evolutions; the only condition required is that the linear terms of the PDE make up a full-rank operator. Hence, in our linear incompressible equations, we permitted use of $\vec{\nabla} \cdot \vec{u}_1 = 0$ in lieu of a $\frac{\partial P}{\partial t}$ explicit equation of motion; so long as the gauge for $P$ (which only appears in derivatives) is properly set, the operator is full rank and numerically well-defined.

In the case of the full nonlinear equations though, it bears noting that $P$ only appears in nonlinear terms! Thus, the linear terms clearly cannot have full rank since they fail to reference $P$ at all. We remedy this by re-casting the momentum equation as

$$\frac{\partial \vec{u}}{\partial t} + \frac{\vec{\nabla} P}{\rho_0} = -\vec{\nabla} P \left( \frac{1}{\rho} - \frac{1}{\rho_0} \right) + \vec{g}. \tag{1.26}$$

This ensures that, along with suitable gauge choice for $P$, the linear operators on the left hand side of the equation have full rank. The remaining equations

$$\frac{\partial \rho}{\partial t} = -\vec{u} \cdot \vec{\nabla} \rho, \tag{1.27a}$$

$$\vec{\nabla} \cdot \vec{u} = 0, \tag{1.27b}$$

remain unchanged.

### 1.4.2 Pathology with $u_z$ BCs

Our problem turns out to be ill-conditioned, the pathology arising from our driving term in conjunction with the incompressibility constraint. Since we want to model the propagation of waves from where they are excited, rather than exciting them directly in our problem, we must drive our region

of interest via boundary conditions rather than forcing terms.

By driving via boundary conditions, we can only specify boundary conditions on the variables being differentiated $\partial_z$ in our equations of motion, since we enforce fully periodic BCs in $x$ via using a Fourier spectral decomposition. Thus, we are restricted to enforcing BCs on either $\vec{u}$ or $P$.

It turns out that by using BCs on $\vec{u}$, we run into a problem: the BCs we specify on $u_z$ or its derivatives will in general affect grid points $\partial_z u_{z,1x}$ while those specified on $u_x$ can only affect $\partial_x u_{x,0x}$. This means that $\partial_x u_x$ will pick up sharp jerks by being coupled to the BC, producing high frequency components in $u_x$ as can be seen in Fig. 1.2. These come from enforcing the divergence-free condition too literally at the boundary. Not that the figure was produced with only a Dirichlet BC on $u_z$, no BCs on $u_x$, in a linear system with absorbing BCs.

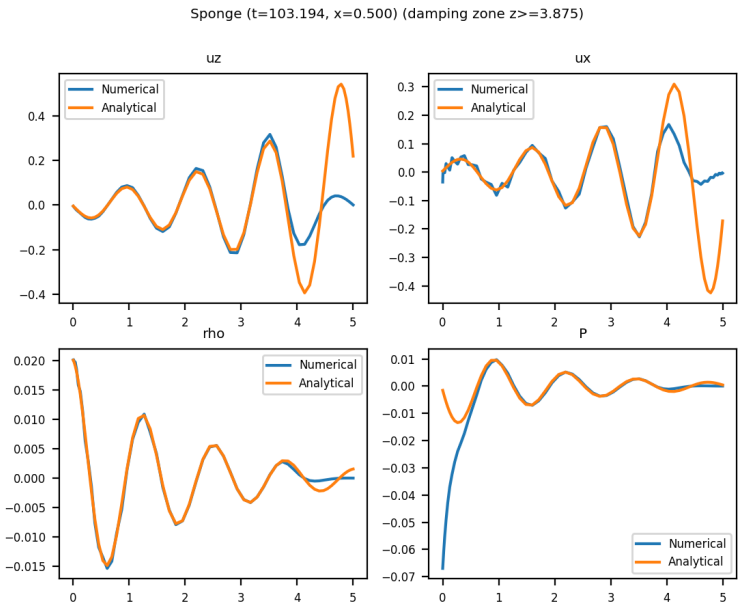


**Figure 1.2:** Note the jaggedness in $u_x$ induced by a Dirichlet BC on $u_z$.

This problem seems to be remedied somewhat by instead enforcing Neumann BCs on $u_z$, or even when allowing dissipation enforcing Neumann BCs on $\partial_z u_z$ (the extra derivatives granted by $\nu\nabla^2\vec{u}$ let us specify higher order BCs). However, in the fully nonlinear system we still observe divergences near $z = 0$ regardless of how smooth of BCs we use on $u_z$. Moreover, since the discontinuities come from the nature of the BC and not its strength, we run into these instabilities regardless of how gradually we introduce the BC (e.g. we could consider multiplying by $1 - e^{-t/\tau}$ to gradually introduce the BC, but this turns out also to be unstable).

There are thus three possible solutions to this:

- Consider BCs only on $P$, the only other choice of dynamical variable. This seems to also exhibit instabilities near $z = 0$.
- Consider using a $\nabla^2 P$ equation instead of a $\vec{\nabla}\cdot\vec{u} = 0$ constraint. The correct choice of BCs in

this case is discussed in Nordstrom et al, 2006. We choose not to investigate this.

- Change the system of equations to no longer consider an incompressible system; the anelastic system is an attractive alternative. This boasts the advantage of not having an algebraic constraint equation.

TODO Can taking $\rho_1 \ll \rho_0$, giving us a full rank operator w/o doing the janky double add thing, solve in the incompressible limit? Otherwise, go anelastic.

## 1.5 Limits of the Incompressibility and Anelasticity Assumptions

To get a better handle on the regimes of validity of the incompressible and anelastic approximations, we return to the full fluid equations

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \left(\rho \vec{u}\right) = 0, \\ \frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} + \frac{\vec{\nabla} P}{\rho} - \vec{g} = 0. \end{aligned} \tag{1.28}$$

Furthermore, we assume small perturbations $\delta P = c_s^2 \delta\rho$ where $c_s^2$ is the sound speed, also the velocity at which an adiabatic perturbation is restored. We drop the $-\vec{g}$ from the analysis assuming that it is largely negated by an appropriate background $P$.

Given these considerations, we consider in what limit we may presume the fluid to be incompressible. Consider if $|*|\vec{u} \ll c_s$, then for some characteristic frequency $\omega$ and wavenumber $k$ the momentum equation scales as $\omega u + k u^2 + c_s^2 \delta\rho/\rho = 0$. Comparing scales it is thus clear that $\delta\rho \ll \rho$ in this limit. Then, considering the mass continuity equation, we recognize that the scalings are $\omega\delta\rho + \rho\vec{\nabla}\cdot\vec{u} + k\delta\rho u = 0$. Thus, $\vec{\nabla}\cdot\vec{u} = 0$ necessarily, since $\rho$ is much larger than the other two terms, and we obtain the incompressible mass continuity equation from the other two terms. While this is obviously not a very careful treatment, it should suffice to convince us that $u \ll c, \delta\rho \ll \rho, \vec{\nabla}\cdot\vec{u} = 0$ are all roughly equivalent conditions.

On the other hand, when may we drop the total derivative terms? This seems to be a slightly different criterion, and indeed we may compare $\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} \sim \omega\vec{u} + k u^2$. Thus, the second term is dominated by the first term when $\vec{u} \ll \vec{v}_{ph}$ where $\vec{v}_{ph} \equiv \frac{\omega}{k}\hat{k}$ is the phase velocity. Recall that for our stratified waves that $\omega \leq N = \sqrt{g/H}$ as a point of reference. So we require $u \ll v_{ph}$ to be able to drop the total derivative terms.

Since in general $v_{ph} \ll c_s$, we thus identify that there are three regimes. The first, $u \ll v_{ph}$, we solve the completely linear problem. Then, for $v_{ph} \lesssim u \ll c_s$, we must begin to consider the advective terms but are still allowed to assume $\rho_1 \ll \rho_0$, an assumption that is only violated when $u \sim c_s$.

If we look ahead to the anelastic system derivation, we can identify that the anelastic approximation corresponds to assuming $c_s^2 \gg u v_{ph}$, making no assertion of the scale between $u, v_{ph}$. Thus, we can also consider the anelastic approximation with advective terms just fine in teh regime $v_{ph} \lesssim u$,

and if this is sufficient to induce wave breaking then we may never need to consider the $\rho_1 \sim \rho_0$ enormous perturbations.

# Chapter 2

# Simulation Setup

We describe now some concerns that arise in setting up the simulation.

## 2.1 Volumetric Forcing

Due to the structure of `Dedalus`, we must use Chebyshev polynomials in the vertical direction (to be able to enforce BCs). But as discussed in §B.1, the grid spacing of the Chebyshev polynomials near boundaries goes as $1/N^2$, which means that interface forcing necessitates an extremely strict Courant condition to capture evolution near the interface; this is the explanation for the divergences we were seeing in earlier sections. In fact, in the literature, the spectral CFL condition meets a $\Delta t_{\max} \sim 1/N^2$ scaling thanks to poor bounds near the boundary.

Instead, we employ volumetric forcing. As a toy problem of seeing how this works, we will study the restricted 1D wave equation.

### 2.1.1 1D Wave Equation (Deprecated)

Consider the forced 1D wave equation (we set $c = 1$)

$$\frac{\partial^2 u(x,t)}{\partial t^2} - \frac{\partial^2 u(x,t)}{\partial x^2} = f(x,t). \tag{2.1}$$

Note that the only allowed propagation modes here are $u \sim e^{iC(x\pm t)}$ for some constant $C$.

We wish to choose $f(x,t)$ such that $u(x \to -\infty) = 0$ while $u(x \to +\infty) \sim e^{i(x-t)}$. It is clear that if we consider the Fourier transform $\tilde{f}(k,t)$, the response in $\tilde{u}(k,t)$ will also contain components for each of the nonzero $\tilde{f}(k,t)$ components. Thus, if we want a pure $e^{i(x-t)}$ outgoing wave, we should have a pure $\tilde{f}(k,t) \propto \delta(k-1)$. However, it is clear that driving the oscillator exactly on resonance produces divergent oscillation amplitudes, and moreover it is impossible to drive with an $f(x,t)$ that doesn't vanish as $x \to \pm\infty$.

It is nonetheless clear that $\tilde{f}(k,t)$ should be strongly peaked at $k = 1$ and small elsewhere. We can thus consider a form $f(x,t) \propto e^{-x^2/2\sigma^2} e^{i(x-t)}$ such that $\sigma \gg 1$, i.e. the envelope contains many

wavelengths. The Fourier transform $\tilde{f}(k,t)$ is then a Gaussian with width $1/\sigma \ll 1$ centered at $k=1$ (convolution).

More precisely, consider $f(x,t) = f_0 e^{-x^2/2\sigma^2} e^{i(x-t)}$ and $u(x,t) = u_0(x)e^{i(x-t)}$ such that $u_0(x\to\infty)=0$. Substituting this into the inhomogeneous PDE gives

$$i\frac{\partial u_0(x)}{\partial x} - \frac{\partial^2 u_0(x)}{\partial x^2} = f_0 e^{-x^2/2\sigma^2}. \tag{2.2}$$

We note that if $\sigma \gg 1$ then roughly $\frac{\partial}{\partial x} \ll 1$ or

$$\frac{\partial u_0(x)}{\partial x} \approx -i f_0 e^{-x^2/2\sigma^2}. \tag{2.3}$$

Enforcing $u_0(x\to-\infty)=0$, then $|*|\, u_0(x\to+\infty) = f_0\sigma\sqrt{2\pi}$.

### 2.1.2 Narrow-band forcing

Instead of doing this on-frequency forcing, we choose instead to use a narrow forcing zone $f(x,t) \sim N_\sigma(z)e^{it}$ where $\sigma$ is not large. Specifically, we need $\sigma$ to be well-resolved spectrally, but we need $\tilde{f}(k_z,t)$ not to be small. Thus, since $1/k_z$ must be resolved by the simulation, choosing $\sigma = 1/k_z$ is a natural choice.

Note that this implies that waves will be excited going both upwards and downwards, as $\tilde{f}(k_z) = \tilde{f}(-k_z)$. We can however place the forcing zone near the bottom boundary so that downwards-propagating waves will be attenuated by damping zones.

## 2.2 Timestepping

### 2.2.1 Signatures of Insufficient Temporal Resolution

When running these simulations, it can occasionally be seen that the waves do not grow in amplitude like $e^{z/2H}$, such that the energy and flux are not constant (or the waves will grow too quickly; this is dependent on the exact timestepping used). This is caused by using an overly-long timestep, as can be seen with the below illustration.

We study the 1D advection equation $\psi_t = c\psi_x$ corresponding to rightwards propagating waves. Consider first order implicit Euler timestepping for wave mode $\psi_x = ik\psi$, then

$$\psi_{i+1} - \psi_i = ick\Delta t\psi_{i+1}. \tag{2.4}$$

Solving yields

$$\begin{aligned}\psi_{i+1} &= (1-ick\Delta t)^{-1}\psi_i,\\ &= (1-i\omega\Delta t)^{-1}\psi_i.\end{aligned} \tag{2.5}$$

If we then want to step forwards for some total time $T$, with $N\Delta t = T$ the number of timesteps taken, then we approximate

$$\psi(t_0 + T) \approx \left(1 - i\omega\frac{T}{N}\right)^{-N} \psi(t_0). \tag{2.6}$$

It is worth noting that in the $N \to \infty$ limit we recover $\psi(t_0 + T) = e^{i\omega T}\psi(t_0)$.

However, consider if $\omega\Delta t \lesssim 1$, then we should see the wave amplitude decrease over time like

$$|*|\,\psi(t_0 + T) \approx \left(1 + \omega^2\Delta t^2\right)^{-N/2}\psi(t_0). \tag{2.7}$$

Thus, since we have an implicit first order scheme, the wave amplitude *decreases* over time spuriously. An explicit method would produce an increase, but `Dedalus` uses implicit timestepping. The exact rate of decrease depends on the timestepper used, but it is clear that $\omega\Delta t \ll 1$ is necessary to avoid this spurious decrease (in numerics language, "the oscillation must be well-resolved").

# Appendix A

# Deriving Fluids Results

## A.1 Equation of Energy Conservation

We follow Landau & Lifshitz's derivation for this expression. Consider that the total energy stored in a wave must be the sum of its kinetic and internal energy $\frac{1}{2}\rho v^2 + \rho\epsilon$ where $\epsilon$ is the internal energy density. To obtain an equation of energy conservation, we must take the time derivative of this expression. First, we consider

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\frac{1}{2}\rho v^2\right) &= \frac{1}{2}v^2\frac{\partial \rho}{\partial t} + \rho\vec{v}\cdot\frac{\partial \vec{v}}{\partial t}, \\
&= -\frac{1}{2}v^2\left(\vec{\nabla}\cdot\rho\vec{v}\right) - \rho\vec{v}\cdot\left(\left(\vec{v}\cdot\vec{\nabla}\right)\vec{v} + \frac{\vec{\nabla}P}{\rho} - \vec{g}\right)_{T,V}, \\
&= -\frac{1}{2}v^2\left(\vec{\nabla}\cdot\rho\vec{v}\right) - \frac{1}{2}\rho\vec{v}\cdot\left(\vec{\nabla}v^2\right) - \vec{v}\cdot\left(\rho\vec{\nabla}w - \rho T\vec{\nabla}s\right) + \rho\vec{v}\cdot\vec{g}. \qquad \text{(A.1)}
\end{aligned}$$

We have denoted $s$ the specific internal entropy density of the fluid and $\mathrm{d}w = T\mathrm{d}s + \frac{\mathrm{d}P}{\rho} = T\mathrm{d}s + \epsilon$ the specific internal enthalpy density of the fluid. Recall enthalpy $\epsilon = w - Ts$ is the usual thermodynamic definition.

At the same time, consider

$$\begin{aligned}
\frac{\partial}{\partial t}\left(\rho\epsilon\right) &= \epsilon\frac{\partial \rho}{\partial t} + \rho\frac{\partial}{\partial t}\left(Ts - \frac{P}{\rho}\right)_{T,P}, \\
&= \epsilon\frac{\partial \rho}{\partial t} + \rho T\frac{\partial s}{\partial t} + \frac{P}{\rho}\frac{\partial \rho}{\partial t}, \\
&= w\frac{\partial \rho}{\partial t} + \rho T\frac{\partial s}{\partial t}, \\
&= -w\vec{\nabla}\left(\rho\vec{v}\right) - \rho T\vec{v}\cdot\left(\vec{\nabla}s\right). \qquad \text{(A.2)}
\end{aligned}$$

Summing the two, we find

$$\frac{\partial}{\partial t}\left(\frac{\rho v^2}{2} + \rho\epsilon\right) = -\vec{\nabla}\cdot\left[\rho\vec{v}\left(v^2 + w\right)\right] = -\vec{\nabla}\cdot\left[\rho\vec{v}\left(v^2 + \epsilon + \frac{P}{\rho}\right)\right]. \qquad \text{(A.3)}$$

## A.2 Conservative Fluid Equations

This isn't particularly useful for our work since spectral methods don't benefit from having the dynamical equations in conservative form, but form a useful exercise for the ever-so-stupid writer.

The Reynolds transport theorem tells us that

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_V Q\,\mathrm{d}V = \int_V \frac{\partial Q}{\partial t}\,\mathrm{d}V + \int_{\partial V}(\vec{v}(\mathrm{d}A)\cdot\hat{n})Q\,\mathrm{d}A. \tag{A.4}$$

We have notated $\vec{v}(\mathrm{d}A)$ the velocity of the surface element. In fluid mechanics, this is just the velocity field. Thus, for conserved quantities where $\frac{\mathrm{d}}{\mathrm{d}t}\int_V Q\,\mathrm{d}V = 0$, we can apply divergence theorem to obtain the simple result

$$\frac{\partial Q}{\partial t} + \vec{\nabla}\cdot(Q\vec{v}) = 0. \tag{A.5}$$

Of course, if there are any net forces etc. on the system, they are simply inserted as fileds on the right hand side of Eq. A.5.

The application of this to the three major conserved quantities: mass, momentum and energy, yield:

**Conservation of Mass** This produces simply $\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{v}\right) = 0$ the continuity equation.

**Conservation of Momentum** This produces $\frac{\partial\rho v_i}{\partial t} + \vec{\nabla}\cdot\left(\rho v_i\vec{v}\right) = 0$. To obtain the traditional momentum equation, it is perhaps most clear in index notation where repeated indicies denote summation:

$$\begin{aligned}
\frac{\partial\left(\rho v_i\right)}{\partial t} + \partial_j\rho v_i v_j &= 0,\\
v_i\frac{\partial\rho}{\partial t} + \rho\frac{\partial v_i}{\partial t} + v_I\partial_j\left(\rho v_j\right) + \rho v_j\partial_j v_i &= 0,\\
v_i\left(\frac{\partial\rho}{\partial t} + \partial_j\left(\rho v_j\right)\right) + \rho\left(\frac{\partial v_i}{\partial t} + v_j\partial_t v_i\right) &= 0.
\end{aligned} \tag{A.6}$$

The first parenthetical term we recognize is just the continuity equation though. The remainder is the traditional source-free momentum equation.

**Conservation of Energy** This just reads $\frac{\partial e}{\partial t} + \vec{\nabla}\cdot(e\vec{v}) = 0$. It's a pretty uninteresting equation without pressure and thermodynamic effects, which would arise from a more careful treatment of the source terms.

In fact, pressure contributes a source term $P\vec{\nabla}\cdot\vec{v}$, so the full energy equation is often written

$$\frac{\mathrm{d}e}{\mathrm{d}t} + (e+P)\vec{\nabla}\cdot\vec{v} = 0. \tag{A.7}$$

## A.3 The Anelastic/Boussinesq Approximations

### A.3.1 Developing the Anelastic/Boussinesq Approximations

Let's relax the incompressibility constraint (we will expand the continuity equation to first order, but the momentum equation will merit a separate treatment):

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} + \vec{\nabla}\cdot\left(\rho_0 \vec{u}_1\right) = 0,\\ \frac{\partial \vec{u}_1}{\partial t} = -\frac{\vec{\nabla} P}{\rho} - \vec{g}.\end{aligned} \tag{A.8}$$

Suppose we are interested in phenomena with characteristic length scale $L$ and time scale $\tau$. Let's first examine the relative magnitudes of the terms in the continuity equation

$$\frac{\rho_1}{\tau} + \frac{\rho_0 |u_1|}{L} = 0.$$

Thus, if we are interested in time scales $\tau \gg \frac{\rho_1}{\rho_0}\frac{L}{|u_1|}$ then we neglect the first term, the time derivative. This corresponds to making the perturbation incompressible; note that $\frac{\partial \rho_1}{\partial t} \approx \frac{\mathrm{d}\rho_1}{\mathrm{d}t}$ to first order, so we drop the high frequency restoring forces in the perturbation.

For the momentum equation, we instead first manipulate to first order

$$\begin{aligned}-\frac{\vec{\nabla} P}{\rho} - \vec{g} &= -\frac{\vec{\nabla} P_0}{\rho} - \frac{\vec{\nabla} P_1}{\rho_0} - \vec{g},\\ &= -\frac{\vec{\nabla} P_1}{\rho_0} + \left(\frac{\rho_0}{\rho} - 1\right)\vec{g},\\ &= -\vec{\nabla}\left(\frac{P_1}{\rho_0}\right) - \frac{P_1}{\rho_0^2}\vec{\nabla}\rho_0 - \frac{\rho_1}{\rho_0}\vec{g}.\end{aligned} \tag{A.9}$$

We now have three equations for four variables, $\vec{u}_1, \rho_1, P_1$. We must introduce a fourth equation, a thermodynamic equation. For an adiabatic process $P\rho^{-\gamma} \propto P^{1-\gamma}T^{\gamma}$ is constant. We thus introduce the concept of the *potential temperature*

$$\theta = T\left(\frac{P_0}{P}\right)^{\kappa}. \tag{A.10}$$

For an adiabatic process, $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Motivated by this, we use

$$\begin{aligned}\frac{\partial 1}{\partial \rho_0}\frac{\partial \rho_0}{\partial z} &= \frac{1}{\gamma P_0}\frac{\partial P_0}{\partial z} - \frac{1}{\theta_0}\frac{\partial \theta_0}{\partial z},\\ \frac{\rho_1}{\rho_0} &= \frac{1}{\gamma}\frac{P_1}{P_0} - \frac{\theta_1}{\theta_0},\end{aligned} \tag{A.11}$$

to give the momentum equation form

$$\frac{\mathrm{d}\vec{u}_1}{\mathrm{d}t} = -\vec{\nabla}\left(\frac{P_1}{\rho_0}\right) + \frac{P_1}{\rho_0}\left(\frac{1}{\theta_0}\vec{\nabla}\theta_0\right) + \vec{g}\frac{\theta_1}{\theta_0}. \tag{A.12}$$

We also recognize $N^2 = \frac{g}{\theta_0}\frac{\partial\theta_0}{\partial z}$. We now do the same trick where we consider dynamics on length scale $D$ and compare the first and second terms in Eq. A.12. Their ratio is $\frac{N^2 D}{g}$, and so as $N^2 \ll \frac{g}{D}$ the freefall time we neglect the second term.

The anelastic fluid equations thus read

$$\begin{aligned} \vec{\nabla}\cdot\left(\rho_0\vec{u}\right) &= 0, \\ \frac{\partial\vec{u}_1}{\partial t} + \vec{\nabla}\left(\frac{P_1}{\rho_0}\right) - \vec{g}\frac{\theta_1}{\theta_0} &= 0, \\ \frac{\partial\theta_1}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\theta_0 &= 0. \end{aligned} \tag{A.13}$$

The Boussinesq equations are obtained from these in the limit where $H \gg D$ the relevant length scale, thus we allow $\rho_0$ to be approximately constant.

### A.3.2 Anelastic Solution to Stratified Atmosphere

We simply substitute $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ into Eq. A.13 with $\rho_0 \propto e^{-z/H}$ and obtain

$$\begin{bmatrix} 0 & 0 & ik_x\rho_0 & ik_z\rho_0 - \frac{\rho_0}{H} \\ 0 & -i\omega & 0 & \frac{N^2\theta_0}{g} \\ \frac{ik_x}{\rho_0} & 0 & -i\omega & 0 \\ \frac{ik_z}{\rho_0} + \frac{1}{\rho_0 H} & -\frac{g}{\theta_0} & 0 & -i\omega \end{bmatrix} \begin{bmatrix} P_1 \\ \theta_1 \\ u_{1x} \\ u_{1z} \end{bmatrix} = 0. \tag{A.14}$$

Taking the determinant of this matrix produces

$$\begin{aligned} -k_x^2\left(-N^2+\omega^2\right) + \left(ik_z - \frac{1}{H}\right)\left(ik_z + \frac{1}{H}\right)\omega^2 &= 0, \\ \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}} &= \omega^2. \end{aligned} \tag{A.15}$$

# Appendix B

# Numerical Results

## B.1 Chebyshev Polynomials

Note that, unlike a Fourier basis, a Chebyshev basis has $N^2$ grid spacing near its boundaries. By this, we mean that for the Chebyshev polynomials $T_n(x)$, the spacing between zeros of $T_n(x)$ near $x = \pm 1$ scales like $n^{-2}$. This contrasts with Fourier series for which $\psi_n(x) = \cos n\pi x, \sin n\pi x$ for which the zeros are spaced $\Delta x = \frac{1}{n}$. This is important because the CFL condition for PDEs requires that $u\frac{\Delta t}{\Delta x} < C$ for some constant $C$, otherwise the simulation will move fluid elements more than $\Delta x$ in a single timestep, an illegal operation for the basis function.

If we consider instead $T_n(x)$ satisfying $T_n(\cos\theta) = \cos n\theta$, then within some spacing of the edge of the domain $\cos\theta = 1 - \Delta x$ then $\Delta x \approx \theta^2/2$ then the number of zeros seen is the number of solutions $\cos\phi = 0, \phi = n\theta = n\sqrt{2\Delta x}$ which requires $\sim k\pi \in [0, n\sqrt{2\Delta x}]$. Thus, for any interval $\Delta x$, the number of zeros contained scales with $n^2$!