1 Problem

To the end of simulating breaking of internal gravity waves (IGW) in white dwarfs (WDs), we begin with the toy problem of IGW breaking in a uniformly stratified atmosphere in 2D. To begin with the simplest treatment, we consider an incompressible fluid characterized by four dynamical variables ρ, P, u_x, u_z.

1.1 Physical Description

We consider perturbations, not necessarily small, in a medium that at equilibrium is at rest and has uniform density stratification

$$\rho_0(x,z) \propto e^{-z/H}. \tag{1}$$

For an incompressible fluid, the density of the fluid $\rho(x,z,t)$ decomposes into this background stratification $\rho_0(x,z)$ and a small perturbation $\rho_1(x,z,t) \ll \rho_0$ ($\rho_1 \ll \rho_0$ is a necessary condition for an incompressible treatment). Permitting a similar decomposition for $P(x,z,t) = P_0(x,z) + P_1(x,z,t)$ and using hydrostatic equilibrium $\frac{\mathrm{d}P_0(x,z)}{\mathrm{d}z} = -\rho g$ gives fluid equations

$$\vec{\nabla} \cdot \vec{u}_1 = 0, \tag{2a}$$

$$\frac{\partial \rho_1}{\partial t} - u_{1z}\frac{\rho_0}{H} = -\left(\vec{u}_1 \cdot \vec{\nabla}\right)\rho_1, \tag{2b}$$

$$\frac{\partial \vec{u}_1}{\partial t} + \frac{\vec{\nabla} P_1}{\rho_0} + \frac{\rho_1}{\rho_0}\vec{g} = -\left(\vec{u}_1 \cdot \vec{\nabla}\right)\vec{u}_1. \tag{2c}$$

I notate $\vec{u} \equiv \vec{u_1}$ to identify that it is a perturbation variable even though there is no distinction between the two without a background flow.

1.2 Linear Description

If the perturbation is sufficiently small $u_1 \ll v_{ph}$ the phase velocity, then we drop terms nonlinear in the perturbation quantities. The linearized equations admit wave-like solutions described by

$$u_{1z} = A e^{z/2H} e^{i\left(\vec{k}\cdot\vec{r} - \omega t\right)}, \tag{3a}$$

$$\frac{\partial u_{1x}}{\partial x} = -\frac{\partial u_{1z}}{\partial z}, \tag{3b}$$

$$\frac{\partial \rho_1}{\partial t} = u_{1z}\frac{\rho_0}{H}, \tag{3c}$$

$$\frac{\partial P}{\partial x} = -\rho_0 \frac{\partial u_{1x}}{\partial t}, \tag{3d}$$

$$\omega^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}. \tag{3e}$$

Note $N^2 = g/H$ the buoyancy frequency here.

2 Linear Numerical Simulation using `Dedalus`

In the WD problem, the IGW is generated deep within the WD and propagates outwards. Since we are only concerned with its breaking behavior, we initially seek to excite waves in a way agnostic of how the waves are generated. Put another way, we don't want to generate the wave, we just want to see how the wave evolves as it nears nonlinear amplitudes. We begin with the linear problem to see whether our mechanism produces expected results.

The code described in this section is attached as `strat.py`, a minimum working example that saves plots to `plots/` on the fly.

2.1 Numerical Setup

To this end, we simulate the linearized fluid equations

$$\vec{\nabla}\cdot\vec{u}_1 = 0, \tag{4a}$$

$$\frac{\partial \rho_1}{\partial t} - u_{1z}\frac{\rho_0}{H} = 0, \tag{4b}$$

$$\frac{\partial \vec{u}_1}{\partial t} + \frac{\vec{\nabla}P_1}{\rho_0} + \frac{\rho_1}{\rho_0}\vec{g} = 0. \tag{4c}$$

The domain of our simulation is $x \in [0,H], z \in [0,4H]$. We use a Fourier basis in the x and a Chebyshev in the z. We try with just 16 modes in the x and 64 modes in the z. Since we choose parameters of the problem such that $\frac{2\pi}{k_x} = H, \frac{2\pi}{k_z} = -\frac{4H}{\pi}$[1], this should be plenty of resolution to resolve the IGW[2]. The timescale is set by $N = 1$.

2.2 Reflection Suppression

To simulate a purely outgoing wave, we adopt a damping zone near the top boundary to suppress reflection. Specifically, we use

$$\vec{\nabla}\cdot\vec{u}_1 = 0, \tag{5a}$$

$$\frac{\partial \rho_1}{\partial t} + f(z)\rho_1 - u_{1z}\frac{\rho_0}{H} = 0, \tag{5b}$$

$$\frac{\partial \vec{u}_1}{\partial t} + f(z)\rho_1 + \frac{\vec{\nabla}P_1}{\rho_0} + \frac{\rho_1}{\rho_0}\vec{g} = 0, \tag{5c}$$

where

$$f(z) = f_0\frac{\max(z-z_0,0)^2}{(z_{\max}-z_0)^2}. \tag{6}$$

We use $f_0 = 1, z_0 = 0.7z_{\max}$, so the damping turns on quadratically starting at $z_0 = 0.7z_{\max}$.

[1] As seen later, k_z never explicitly enters into the problem description; instead, we specify $\omega(k_x,k_z)$ using the desired k_x, k_z values.

[2] I chose k_z as an irrational value to ensure any undesired reflections are far from any normal modes of the box.

2.3 Boundary Conditions

To excite the wave, we adopt bottom boundary condition (BC)

$$\left.\frac{\partial u_{1z}}{\partial z}\right|_{z=0} = -Ak_z \cos\left(k_x x - \omega t + \frac{1}{2H}\right). \tag{7}$$

We use $A = 0.05$. The reason our BC isn't just $u_{1z}(z=0)$ is because the initial condition $u_x(t=0) = u_{1z}(t=0) = 0$ is not divergence free and produces high-k modes in u_x. We just differentiate $\frac{\partial u_{1z}}{\partial z}$ in Eq. 3 and take real part.

Since we have the reflection suppression term above, it turns out we need fix the u_{1z} gauge, otherwise the $k_x = 0$ mode is numerically ill-conditioned and causes `Dedalus` to blow up. Thus, we enforce $u_{1z}(k_x = 0) = 0$.

The remaining boundary condition (there are two ∂_z terms in Eq. 4) need only fix the pressure gauge, so we choose $P_1(k_x = 0) = 0$ and a throwaway $u_{1x}(k_x \neq 0) = 0$.

2.4 Initial Conditions

We initialize everything $\rho_1 = P_1 = u_{1x} = u_{1z} = 0$ everywhere.

2.5 Results

A snapshot of the simulation produced by `strat.py` is shown below in Fig. 1. We see reasonable agreement between the analytical solution and the simulation data as the wave propagates. More plots can be found in the `plots/` folder.

3 Nonlinear Numerical Simulation

3.1 Low Amplitude

To move to the nonlinear problem, we first introduce the nonlinear terms as in Eq. 2 but use small A such that $u_{1z} \sim Ae^{z_{\max}/2H} \ll v_{ph}$; we use $A = 5 \times 10^{-5}$. All other problem setup is kept the same. The code is in `strat_nonlin_lowA.py`. We expect agreement with the analytical solution to the linear problem, and this is shown well in Fig. 2. Further plots are available in `plots_nonlin_lowA/`.

3.2 Non-negligible amplitude: Instability

We now increase the amplitude back to $A = 0.05$, and a numerical instability seems to develop. We explore this with `strat_nonlin.py`. An instability seems to grow near $z = 0$, and two plots of its evolution can be seen in Fig. 3. Further plots can be found in `plots_nonlin/`.

4 Attempted Solutions

It seems that the observed instability has to do with the interface driving. A few possible solutions have been considered:

- Doubling the temporal resolution doesn't seem to change when the instability sets in. Doubling the spatial resolution makes it set in earlier, and halving makes it set in later.

- I've considered other types of boundary conditions, such as $\frac{\partial^2 u_{1z}}{\partial z^2}$ or $\frac{\partial P}{\partial z}$, but all of these seem to produce similar behavior, instability near $z = 0$.

- Start the simulation with a small amplitude and use $z_{\max}$ very large, relying on the $u_{1z} \sim e^{-z/2H}$ to bring the wave to breaking amplitudes. This will require a much larger number of spatial modes to resolve the vertical wavelengths though and will be very slow.

- Bulk forcing has also been considered, where we use a driving term in the momentum equation $\frac{\partial \vec{u}_1}{\partial t}$ to generate IGW, then use reflection-suppressing damping zones on both sides of the z domain. This seems to draw focus away from the focus on the breaking portion.

  I was able to get bulk forcing to work, however, and the results seem reasonable. It would be a less desirable entry point though, since we're trying to study just the breaking irrespective of the forcing mechanism.

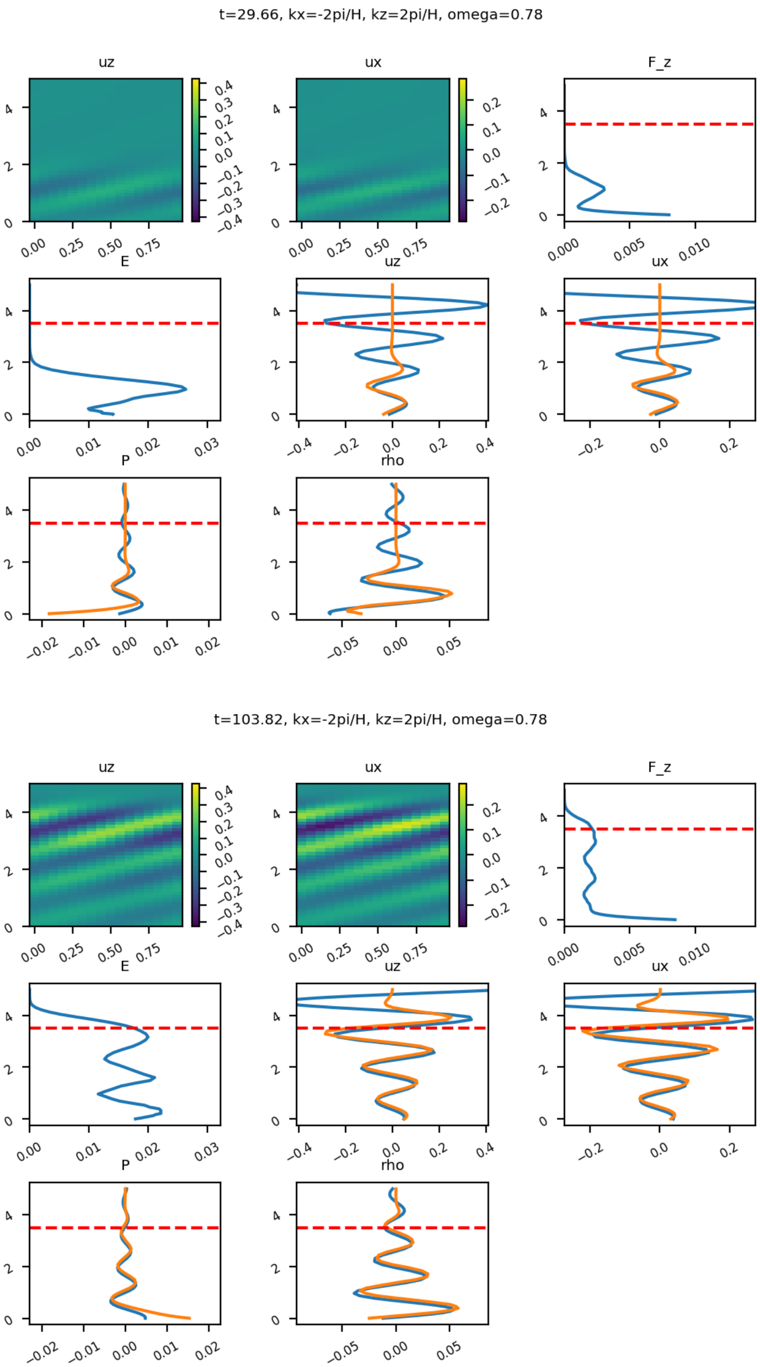

Figure 1: For line plots, red line indicates beginning of damping zone, blue the analytical solution and orange the simulation data. Variables are $u_{1z} = uz, u_{1x} = ux$, F_z the energy flux, E the energy, $P_1 = P, \rho_1 = rho$. Line plots of dynamical variables are slices through $x = 0$.

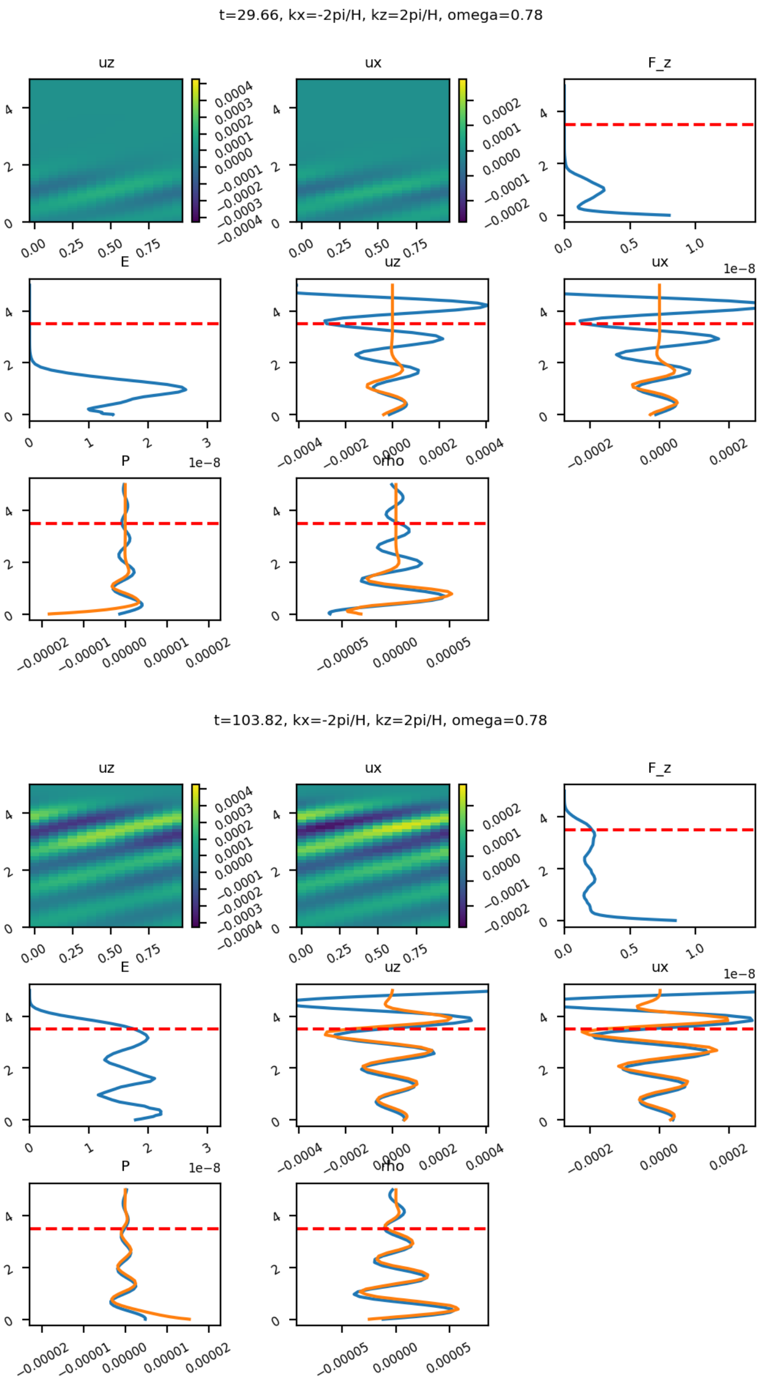

Figure 2: Same plots as Fig. 1 but with nonlinear terms added back to the fluid equations and very small $A = 5 \times 10^{-5}$.

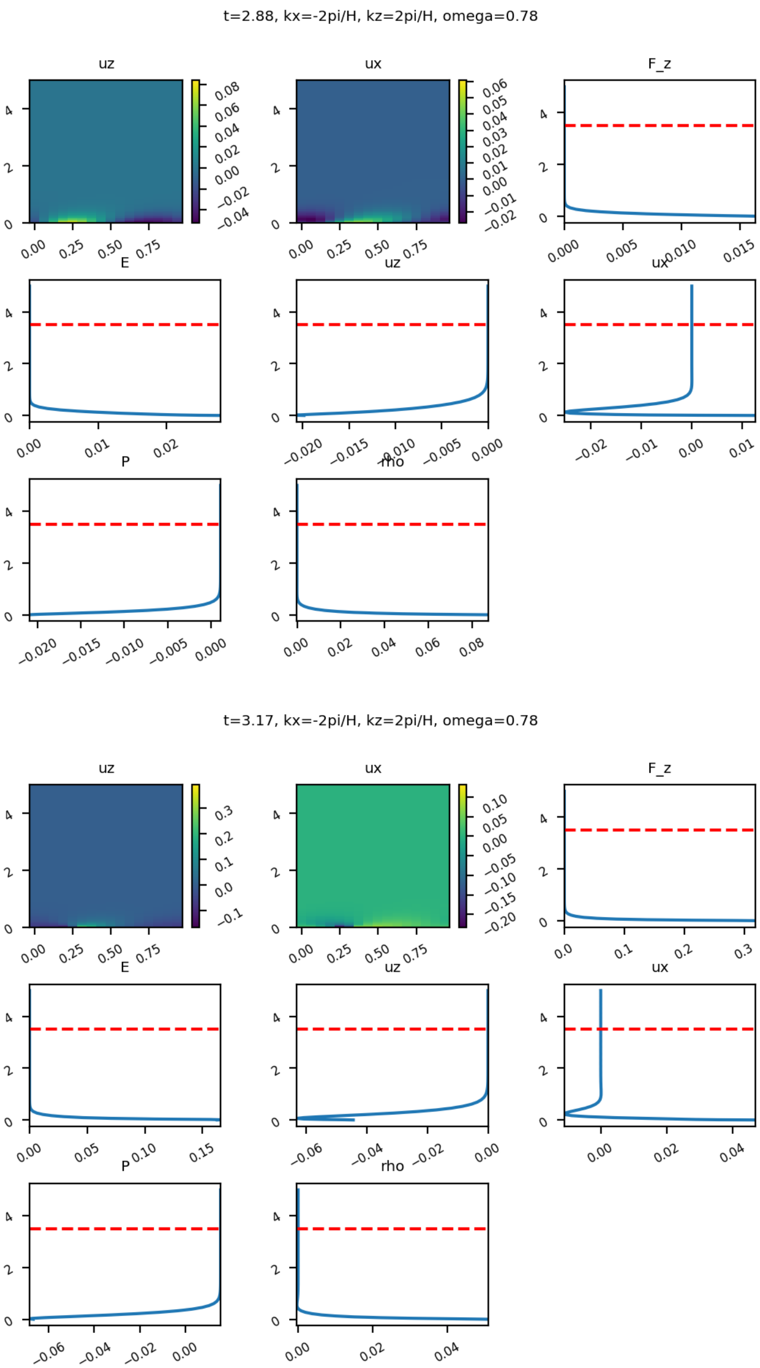

Figure 3: Nonlinear equations with $A = 0.05$ as it diverges near $z = 0$.