Research Notes

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Contents

Chapter 1

Preliminary Problems

To get an intuition for how Dedalus and fluid mechanics works, we will solve some toy problems. Recall fluid equations in the presence of a uniform gravitational field $\vec{g} = -g\hat{z}$:

$$\begin{aligned}\frac{\mathrm{d}\rho}{\mathrm{d}t} + \vec{\nabla}\cdot\vec{u} &= 0,\\ \frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \frac{\vec{\nabla}P}{\rho} - \vec{g} &= 0.\end{aligned} \tag{1.1}$$

In the incompressible limit, $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$, which implies $\vec{\nabla}\cdot\vec{u} = 0$. We use subscripts to indicate perturbed quantities, Q_0 is background and Q_1 is perturbed. We will generally use $\vec{u}_0 = 0$ unless otherwise noted. We will also generally assume symmetry along all axes except z the vertical axis.

In the incompressible limit, the fluid equations become

$$\begin{aligned}\vec{\nabla}\cdot\vec{u}_1 &= 0,\\ \frac{\partial\rho_1}{\partial t} + u_{1z}\frac{\partial\rho_0}{\partial z} &= 0,\\ \frac{\partial\vec{u}_1}{\partial t} + \frac{1}{\rho_0}\vec{\nabla}P_1 + \frac{\rho_1 g\hat{z}}{\rho_0} &= 0.\end{aligned} \tag{1.2}$$

We have used $\vec{\nabla}P_0 = -\rho_0 g\hat{z}$ in the absence of perturbations.

1.1 Incompressible, No Gravity

We note that in the no gravity limit that ρ_1 does not have an effect on other dynamical variables, so the equations of motion we must solve are

$$\begin{aligned}\vec{\nabla}\cdot\vec{u}_1 &= 0,\\ \frac{\partial\vec{u}_1}{\partial t} + \frac{\vec{\nabla}P_1}{\rho_0} &= 0.\end{aligned} \tag{1.3}$$

We can take the divergence of the momentum equation and substitute the continuity equation to get $\nabla^2 P = 0$.

1.1.1 Dirichlet BCs

This is a Laplace equation, which we've solved countless times. Imposing periodic boundary conditions in the x direction and $P_1(z=L)=0, P_1(z=0)=\mathscr{P}(x,t)$, we obtain eigenfunctions

$$\begin{aligned}
P_{1,n}(x,z,t) &= \frac{\mathscr{P}_n(t)}{\sinh(k_n L)} e^{ik_n x}\sinh(k_n(L-z)),\\
u_{1x,n}(x,z,t) &= \int^t -\frac{1}{\rho_0}\frac{\partial P_{1,n}}{\partial x}\,\mathrm{d}t,\\
u_{1z,n}(x,z,t) &= \int^t -\frac{1}{\rho_0}\frac{\partial P_{1,n}}{\partial z}\,\mathrm{d}t.
\end{aligned} \tag{1.4}$$

We define $k_n = \frac{2\pi n}{L}, n \geq 0$ and $\mathscr{P}(x,t) = \sum\limits_n \mathscr{P}_n(t) e^{ik_n x}$.

Thus, if we impose BCs $\mathscr{P}(x,t) = \sin\frac{2\pi x}{L}$ and start with initial conditions such that all quantities are zero, we would expect after transients die out that

$$\begin{aligned}
P(x,z,t) &= \frac{\sin\frac{2\pi x}{L}}{\sinh 2\pi}\sinh\left(2\pi\frac{L-z}{L}\right),\\
u_{1x}(x,z,t) &= -\frac{2\pi t}{L\rho_0}\frac{\cos\frac{2\pi x}{L}}{\sinh 2\pi}\sinh\left(2\pi\frac{L-z}{L}\right),\\
u_{1z}(x,z,t) &= +\frac{2\pi t}{L\rho_0}\frac{\sin\frac{2\pi z}{L}}{\sinh 2\pi}\cosh\left(2\pi\frac{L-z}{L}\right).
\end{aligned} \tag{1.5}$$

This is in good agreement with the results, presented in Fig. 1.1. Note that P is constant while $\vec{u}$ increases linearly in time, and we observe the expected $\sim \sin x \sinh\frac{L-z}{z}$ dependence. In fact, u_{1x}, u_{1z} are exactly $\frac{2\pi}{10}$ at $t=1$.

It is worth noting that, since our Eq. 1.3 reduced to a Laplace equation, we needed two z BCs and two x BCs (periodic BCs amount to equating the value and derivative of the function). This is in agreement with the observation that the original Eq. 1.3 had two derivatives in x, z apiece, so we needed two BCs each.

1.1.2 Sommerfield/Radiative + Driving BCs

This is the more interesting case. Let's go back to the Laplace equation $\nabla^2 P = 0$ but instead implement a driving term at $z=0$ and radiative BCs at $z=L$. This

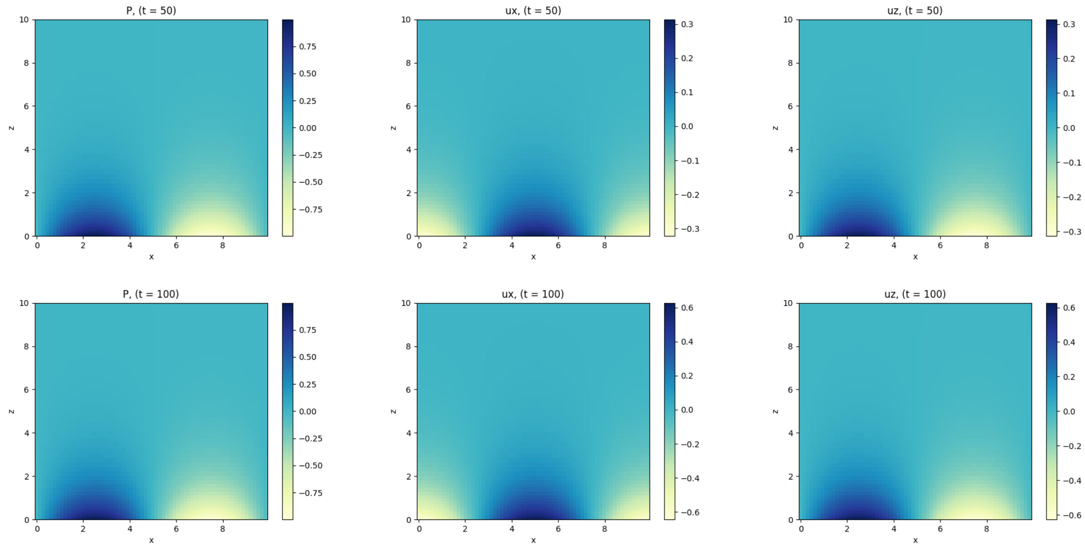

Figure 1.1: P, u_x, u_z at $t = 0.5$ and $t = 1$ for $\rho_0 = 1$. We choose a $L = 10$ square domain.

1.2 Incompressible, Stratified w/ Gravity

1.2.1 Eigenfunctions

Let's restore the $\rho_1 g$ term now. For funsies, we begin by solving for arbitrary stratification $\rho_0(z)$ first. The fluid equations to first order reduce to

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} + \left(\vec{u}_1 \cdot \vec{\nabla}\right)\rho_0 &= 0,\\ \vec{\nabla} \cdot \vec{u}_1 &= 0,\\ \frac{\partial \vec{u}_1}{\partial t} &= -\frac{\vec{\nabla} P_1}{\rho_0} - \frac{\rho_1 g}{\rho_0}\end{aligned} \tag{1.6}$$

We expect there to be some z dependence in the amplitude, so we substitute variables of form $e^{i(kx-\omega t)}$ and do not specify the z dependence. This gives us

$$\begin{aligned}-i\omega\rho_1 - u_{1z}\frac{\partial \rho_0}{\partial z} &= 0,\\ iku_{1x} + \frac{\partial u_{1z}}{\partial z} &= 0,\\ -iwu_{1x} + \frac{ik_x P_1}{\rho_0} &= 0,\\ -iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1 g}{\rho_g} &= 0.\end{aligned} \tag{1.7}$$

We substitute $N^2 = -\frac{g}{\rho_0}\frac{\partial \rho_0}{\partial z}$ to obtain

$$-i\omega\rho_1 - u_{1z}\frac{\rho_0 N^2}{g} = 0, \tag{1.8a}$$

$$iku_{1x} + \frac{\partial u_{1z}}{\partial z} = 0, \tag{1.8b}$$

$$-iwu_{1x} + \frac{ik_x P_1}{\rho_0} = 0, \tag{1.8c}$$

$$-iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1 g}{\rho_0} = 0. \tag{1.8d}$$

Eliminating u_{1x} by substituting (1.8b) into (1.8c) and ρ_1 by substituting (1.8a) into (1.8d) give

$$i\omega\frac{\partial u_{1z}}{\partial z} + \frac{k_x^2 P_1}{\rho_0} = 0, \tag{1.9a}$$

$$\left(\omega^2 - N^2\right)u_{1z} + \frac{i\omega}{\rho_0}\frac{\partial P_1}{\partial z} = 0. \tag{1.9b}$$

Finally, we multiply (1.9a) with ρ_0 and differentiate $\mathrm{d}z$ and combine with (1.9b) to give

$$\frac{\mathrm{d}^2 u_{1z}}{\mathrm{d}z^2} + \frac{1}{\rho_0}\frac{\partial \rho_0}{\partial z}\frac{\partial u_{1z}}{\partial z} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right)u_{1z} = 0. \tag{1.10}$$

Let's now pick stratification $\rho \propto e^{-z/H}$ Eq. 1.10 clearly has exponential solutions $e^{\kappa z}$ for

$$\kappa^2 - \frac{\kappa}{H} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right) = 0. \tag{1.11}$$

We permit complex $\kappa = \frac{1}{2H} + ik_z$, and from the above clearly

$$\begin{aligned} k_z^2 &= -\frac{1}{4H^2} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right), \\ \omega^2 &= \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}. \end{aligned} \tag{1.12}$$

Thus the eigenfunctions are

$$\begin{aligned} u_{1z} &= e^{z/2H} e^{i(k_z z + k_x x - \omega t)}, \\ u_{1x} &= -\frac{k_z + i/2H}{k_x} u_{1z}, \\ \rho_1 &= \frac{i\rho_0}{H\omega} u_{1z}, \\ P_1 &= -\frac{\rho_0 \omega}{k_x^2}(k_z + i/2H) u_{1z}. \end{aligned} \tag{1.13}$$

1.2.2 Solving an IVP

1.3 (Algebra) The Anelastic/Boussinesq Approximations

1.3.1 Developing the Anelastic/Boussinesq Approximations

Let's relax the incompressibility constraint (we will expand the continuity equation to first order, but the momentum equation will merit a separate treatment):

$$\begin{aligned}\frac{\partial \rho_1}{\partial t} + \vec{\nabla}\cdot\left(\rho_0 \vec{u}_1\right) &= 0,\\ \frac{\partial \vec{u}_1}{\partial t} &= -\frac{\vec{\nabla}P}{\rho} - \vec{g}.\end{aligned} \tag{1.14}$$

Suppose we are interested in phenomena with characteristic length scale L and time scale τ. Let's first examine the relative magnitudes of the terms in the continuity equation

$$\frac{\rho_1}{\tau} + \frac{\rho_0 |u_1|}{L} = 0.$$

Thus, if we are interested in time scales $\tau \gg \frac{\rho_1}{\rho_0}\frac{L}{|u_1|}$ then we neglect the first term, the time derivative. This corresponds to making the perturbation incompressible; note that $\frac{\partial \rho_1}{\partial t} \approx \frac{\mathrm{d}\rho_1}{\mathrm{d}t}$ to first order, so we drop the high frequency restoring forces in the perturbation.

For the momentum equation, we instead first manipulate to first order

$$\begin{aligned}-\frac{\vec{\nabla}P}{\rho} - \vec{g} &= -\frac{\vec{\nabla}P_0}{\rho} - \frac{\vec{\nabla}P_1}{\rho_0} - \vec{g},\\ &= -\frac{\vec{\nabla}P_1}{\rho_0} + \left(\frac{\rho_0}{\rho} - 1\right)\vec{g},\\ &= -\vec{\nabla}\left(\frac{P_1}{\rho_0}\right) - \frac{P_1}{\rho_0^2}\vec{\nabla}\rho_0 - \frac{\rho_1}{\rho_0}\vec{g}.\end{aligned} \tag{1.15}$$

We now have three equations for four variables, $\vec{u}_1, \rho_1, P_1$. We must introduce a fourth equation, a thermodynamic equation. For an adiabatic process $P\rho^{-\gamma} \propto P^{1-\gamma}T^{\gamma}$ is constant. We thus introduce the concept of the *potential temperature*

$$\theta = T\left(\frac{P_0}{P}\right)^{\kappa}. \tag{1.16}$$

For an adiabatic process, $\frac{\mathrm{d}\theta}{\mathrm{d}t} = 0$. Motivated by this, we use

$$\begin{aligned}\frac{\partial 1}{\partial \rho_0}\frac{\partial \rho_0}{\partial z} &= \frac{1}{\gamma P_0}\frac{\partial P_0}{\partial z} - \frac{1}{\theta_0}\frac{\partial \theta_0}{\partial z},\\ \frac{\rho_1}{\rho_0} &= \frac{1}{\gamma}\frac{P_1}{P_0} - \frac{\theta_1}{\theta_0},\end{aligned} \tag{1.17}$$

to give the momentum equation form

$$\frac{\mathrm{d}\vec{u}_1}{\mathrm{d}t} = -\vec{\nabla}\left(\frac{P_1}{\rho_0}\right) + \frac{P_1}{\rho_0}\left(\frac{1}{\theta_0}\vec{\nabla}\theta_0\right) + \vec{g}\frac{\theta_1}{\theta_0}. \tag{1.18}$$

We also recognize $N^2 = \frac{g}{\theta_0}\frac{\partial\theta_0}{\partial z}$. We now do the same trick where we consider dynamics on length scale D and compare the first and second terms in Eq. 1.18. Their ratio is $\frac{N^2 D}{g}$, and so as $N^2 \ll \frac{g}{D}$ the freefall time we neglect the second term.

The anelastic fluid equations thus read

$$\begin{aligned} \vec{\nabla}\cdot\left(\rho_0\vec{u}\right) &= 0,\\ \frac{\partial\vec{u}_1}{\partial t} + \vec{\nabla}\left(\frac{P_1}{\rho_0}\right) - \vec{g}\frac{\theta_1}{\theta_0} &= 0,\\ \frac{\partial\theta_1}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\theta_0 &= 0. \end{aligned} \tag{1.19}$$

The Boussinesq equations are obtained from these in the limit where $H \gg D$ the relevant length scale, thus we allow ρ_0 to be approximately constant.

1.3.2 Anelastic Solution to Stratified Atmosphere

We simply substitute $e^{i(\vec{k}\cdot\vec{r}-\omega t)}$ into Eq. 1.19 with $\rho_0 \propto e^{-z/H}$ and obtain

$$\begin{bmatrix} 0 & 0 & ik_x\rho_0 & ik_z\rho_0 - \frac{\rho_0}{H}\\ 0 & -i\omega & 0 & \frac{N^2\theta_0}{g}\\ \frac{ik_x}{\rho_0} & 0 & -i\omega & 0\\ \frac{ik_z}{\rho_0} + \frac{1}{\rho_0 H} & -\frac{g}{\theta_0} & 0 & -i\omega \end{bmatrix} \begin{bmatrix} P_1\\ \theta_1\\ u_{1x}\\ u_{1z} \end{bmatrix} = 0. \tag{1.20}$$

Taking the determinant of this matrix produces

$$\begin{aligned} -k_x^2\left(-N^2+\omega^2\right) + \left(ik_z - \frac{1}{H}\right)\left(ik_z + \frac{1}{H}\right)\omega^2 &= 0,\\ \frac{N^2k_x^2}{k_x^2+k_z^2+\frac{1}{4H^2}} &= \omega^2. \end{aligned} \tag{1.21}$$

Chapter 2

2D Wave Breaking in Atmospheres

2.1 Dynamical Setup

TODO (fluid equations, driven on bottom, parameters)

2.2 Boundary Conditions

TODO (periodic in x, show that right number of BCs in z, discuss whether gauge choice).

2.3 Simulation

TODO (CFL conditions etc.)