Exciting Internal Gravity Waves, Toy Problem

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- Eventual objective: simulate IGW breaking in WDs, compute energy/angular momentum dissipation profiles.
- Toy problem: 2D hydrodynamics with driving boundary condition.
- Begin with linear incompressible case, equations:

$$\frac{\partial \rho_1}{\partial t} = 0, \tag{1a}$$

$$\vec{\nabla} \cdot \vec{u}_1 = 0, \tag{1b}$$

$$\frac{\partial \vec{u}_1}{\partial t} = -\frac{\vec{\nabla} P_1}{\rho_0} + \vec{g}. \tag{1c}$$

- Consider stratified atmosphere $\rho_0(x,z,t) = \rho_0 e^{-z/H}$.

- Four variables $(\rho_1, P_1, u_{1x}, u_{1z})$, all first order in equations of motion, four boundary conditions.
- Periodic BCs in x (2).
- Wavelike solutions $u_{1z} \propto e^{z/2H} e^{i(k_x x + k_z z - \omega t)}$, u_{1x}, P_1, ρ_1 similar up to phase and terms $\sim \mathcal{O}\left(H^{-1}\right) \ll k_z$.
- Obeys dispersion relation

$$\omega^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}. \tag{2}$$

- Driven bottom BC: $u_{1z}(x, 0, t) = A\cos(k_x x + \omega t)$ (1).

- Top BC can be chosen Dirichlet $u_{1z}(x, L_z, t) = 0$, analytically tractable as linear combination of $\pm k_z$ solutions.
- Driving term will pump energy into the system, without dissipation becomes like driven undamped SHO. In full nonlinear problem not an issue. . .
- More realistically, two solutions: radiative boundary conditions and damping zone.
- We use damping zone, add term $\frac{\partial}{\partial t}\vec{q} = L\vec{q} - f(z)(\vec{q} - \vec{q}_0)$ where

$$f(z) = \begin{cases} \Gamma\left[1 - \frac{(z-L_z)^2}{(z_{damp}-L_z)^2}\right] & z > z_{damp} \\ 0 & z < z_{damp} \end{cases}. \tag{3}$$

- Linear incompressible 2D hydrodynamics with periodic BCs in x, driving term at $z = 0$ and either Dirichlet or reflection-free BCs at $z = L_z$.
- Spectral code Dedalus, exponential convergence for shock-free hydrodynamics.

Waveform Agreement

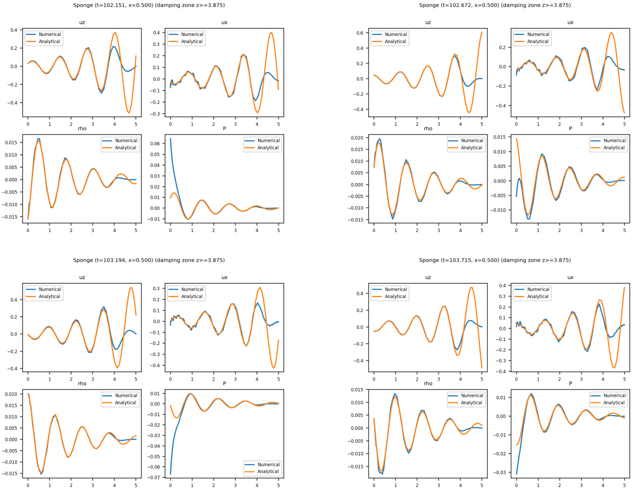

Figure: Numerical and analytical solutions agree. Numerics chooses $v_{gz} > 0$.

- $$RMS_i = \frac{\sqrt{\sum_{x,z,t}\left(u_{1z}^{(i)}(x,z,t)-u_{1z}^{(N)}(x,z,t)\right)^2}}{\sqrt{\sum_{x,z,t}\left(u_{1z}^{(N)}(x,z,t)\right)^2}}$$

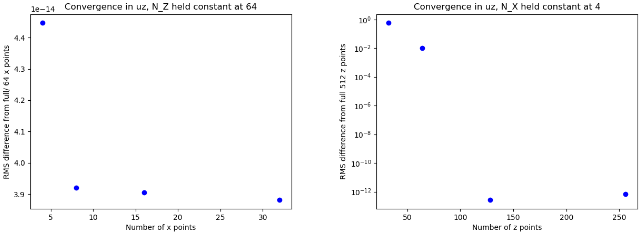

Figure: Convergences of RMS over grid resolution. Note the exponential convergence in z to machine precision (since x is always monochromatic, increasing resolution does little).

Future Work

- Nonlinear 2D incompressible hydro; does it reproduce the above results in the small-perturbation limit?
- Radiative boundary conditions?