
Exciting Internal Gravity Waves, Toy Problem

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Yubo Su

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Problem Setup

Problem Statement

- Eventual objective: simulate IGW breaking in WDs, compute energy/angular momentum dissipation profiles.
- Toy problem: 2D hydrodynamics with driving boundary condition.
- Begin with linear incompressible case, equations:

$$\frac{\partial \rho_1}{\partial t} = 0, \quad (1a)$$

$$\vec{\nabla} \cdot \vec{u}_1 = 0, \quad (1b)$$

$$\frac{\partial \vec{u}_1}{\partial t} = -\frac{\vec{\nabla} P_1}{\rho_0} + \vec{g}. \quad (1c)$$

- Consider stratified atmosphere $\rho_0(x, z, t) = \rho_0 e^{-z/H}$.

Problem Setup

Boundary Conditions

- Four variables ($\rho_1, P_1, u_{1x}, u_{1z}$), all first order in equations of motion, four boundary conditions.
- Periodic BCs in x (2).
- Wavelike solutions $u_{1z} \propto e^{z/2H} e^{i(k_x x + k_z z - \omega t)}$, u_{1x}, P_1, ρ_1 similar up to phase and terms $\sim \mathcal{O}(H^{-1}) \ll k_z$.
- Obeys dispersion relation

$$\omega^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}. \quad (2)$$

- Driven bottom BC: $u_{1z}(x, 0, t) = A \cos(k_x x + \omega t)$ (1).

Problem Setup

Top Boundary Condition

- Top BC can be chosen Dirichlet $u_{1z}(x, L_z, t) = 0$, analytically tractable as linear combination of $\pm k_z$ solutions.
- Driving term will pump energy into the system, without dissipation becomes like driven undamped SHO. In full nonlinear problem not an issue. . .
- More realistically, two solutions: radiative boundary conditions and damping zone.
- We use damping zone, add term $\frac{\partial}{\partial t} \vec{q} = L \vec{q} - f(z)(\vec{q} - \vec{q}_0)$ where

$$f(z) = \begin{cases} \Gamma \left[1 - \frac{(z - L_z)^2}{(z_{damp} - L_z)^2} \right] & z > z_{damp} \\ 0 & z < z_{damp} \end{cases} . \quad (3)$$

Results

Problem Statement Summary

- Linear incompressible 2D hydrodynamics with periodic BCs in x , driving term at $z = 0$ and either Dirichlet or reflection-free BCs at $z = L_z$.
- Spectral code Dedalus, exponential convergence for shock-free hydrodynamics.

Results

Waveform Agreement

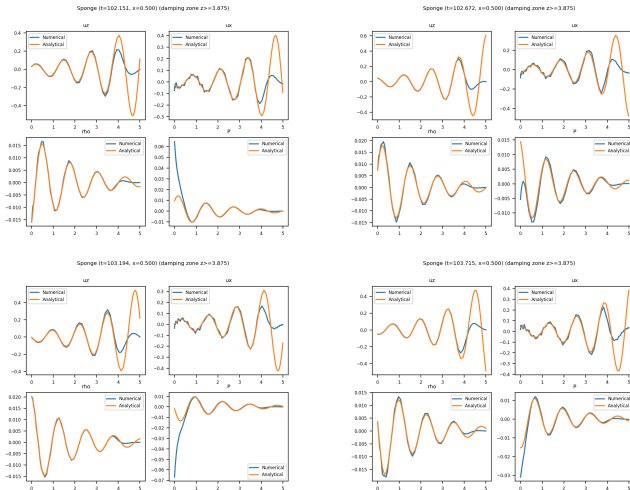


Figure: Numerical and analytical solutions agree. Numerics chooses $v_{gz} > 0$.

Results

Convergence Tests

$$\bullet \text{ } RMS_i = \frac{\sqrt{\sum_{x,z,t} \left(u_{1z}^{(i)}(x,z,t) - u_{1z}^{(N)}(x,z,t) \right)^2}}{\sqrt{\sum_{x,z,t} \left(u_{1z}^{(N)}(x,z,t) \right)^2}}$$

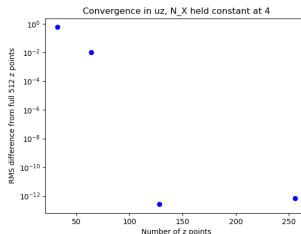
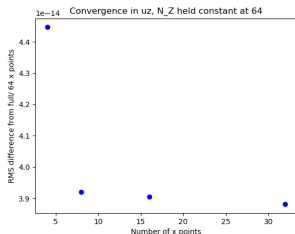


Figure: Convergences of RMS over grid resolution. Note the exponential convergence in z to machine precision (since x is always monochromatic, increasing resolution does little).