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Non-linear internal gravity waves in a slightly stratified atmosphere

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Internal gravity waves in a stratified atmosphere of unbounded inviscid incompressible fluid are considered. A class of non-linear waves is found in the Boussinesq approximation, whereby the inertial variation of density is neglected but the buoyancy is not, by reduction of the equations of motion to ordinary differential equations. To find similar non-linear waves when the atmosphere is slightly stratified, i.e. when the inertial variation of density is small but not entirely negligible, the equations of motion are first expressed in Lagrangian variables and derived from a variational principle. The Lagrangian variables are transformed to systematize the Boussinesq approximation. Finally, the properties of the non-linear waves in a slightly stratified atmosphere are found by Whitham's method of averaging. In terms of the transformed Lagrangian variables, strong non-linearity affects the linearized solution only by adding to the pressure a term proportional to the square of the wave amplitude. It follows that the amplitude of the waves is inversely proportional to the density of the atmosphere, even where the amplitude is not small and the linear approximation becomes invalid.

1. Introduction

Observations and theory of the upper atmosphere (cf. Hines 1963; Hines & Reddy 1967; Booker & Bretherton 1967) suggest that internal gravity waves are generated in the troposphere but that most are strongly reflected or absorbed before penetrating very high. However, some wave components penetrate to the mesosphere or higher, where the density is much less than the density down at the source of the wave. Now the variation of the inertial density (as opposed to the weight density) leads to the amplitude of velocity in a wave increasing with height z like the inverse square-root of the mean density $\bar{\rho}(z)$ in the linear theory of small wave amplitudes. In the earth's atmosphere $\bar{\rho} = \rho_0 \exp(-z/H)$ approximately, where ρ_0 is the density of air at ground level and the scale height H is 8 km or so. So the wave amplitude increases with height like $\exp(z/2H)$, which is four orders of magnitude at a height of 100 km. Thus those waves that do penetrate high up are heavily amplified, and their non-linear behaviour will be important unless they are damped down by the effects of viscosity, turbulence, thermal conductivity, or ohmic dissipation first. These diffusive effects also increase with height. They have been studied by Pitteway & Hines (1963) and

Yanowitch (1967), the former authors having found significant damping above heights of the order of magnitude of 100 km which vary according to the lengths and frequencies of the waves. It seems appropriate now to study the competing effect of non-linearity to find which effect modifies the linear wave first.

We shall take a model atmosphere of unbounded inviscid incompressible fluid of variable density under the influence of gravity. We shall also assume that its motion is two-dimensional as a convenience, though in fact adaptation of the work for three-dimensional flow is straightforward. To study the non-linearity we assume that all waves except those within a narrow range of wavelengths and frequencies are reflected or absorbed by linear mechanisms discussed in the papers to which we have referred, and that the mean velocity of the basic flow is uniform where non-linearity is significant. The inertial variation of density is assumed to be small, i.e. the scale height of the atmosphere to be much greater than the wavelength. These assumptions make the mathematics tractable while retaining essential features of the non-linear waves.

First, the classical linear theory of internal gravity waves is recapitulated. This linearization is not uniformly valid where the density tends to zero, as occurs high up in the atmosphere, because wave amplitudes grow like the inverse square-root of the mean density.

Whitham's (1965*a*, *b*) recent application of the method of averaging to non-linear dispersive waves was thought to be a likely method of resolving this non-uniformity of the linear theory. Accordingly, we seek a special kind of non-linear gravity wave in § 3. A non-linear wave is found in terms of Eulerian variables but seems unsuitable for generalization by Whitham's method.

This leads to the adoption of Lagrangian variables in § 4. They have the advantages of simplicity and of enabling the equations of motion to be derived from a variational principle. As a final preparation for averaging the non-linear wave and finding how it varies slowly where the density varies little in a wave-length, the Lagrangian dependent variables are modified. This clarifies the Boussinesq approximation whereby the buoyancy is of order one but the inertial variation of density is small.

In § 5 the method of averaging gives all waves that behave locally like the special non-linear wave found in § 3. It is shown that, to the first approximation in slow variation of density with height, the energy of any special non-linear wave is propagated as if the wave were linear. This description of the change of the amplitude holds for the modified Lagrangian variables, but not the Lagrangian variables or Eulerian variables.

2. Linear theory

The classical linear theory of internal gravity waves (Rayleigh 1883) may be summarized as follows. The Euler equations of motion of inviscid fluid are

$$\rho\{\partial\mathbf{u}/\partial t+(\mathbf{u}.\mathrm{grad})\,\mathbf{u}\} = -\mathrm{grad}\,p-g\rho\mathbf{k}, \tag{2.1}$$

where $\mathbf{u}$ is the Eulerian velocity, ρ the density and p the pressure of the fluid at position $\mathbf{r}$ at time t and where the gravitational acceleration is g. The equation of incompressibility is

$$\partial\rho/\partial t + (\mathbf{u}.\mathrm{grad})\,\rho = 0, \tag{2.2}$$

so the equation of continuity gives

$$\mathrm{div}\,\mathbf{u} = 0. \tag{2.3}$$

Consider small perturbations of fluid at rest in hydrostatic equilibrium with density $\bar{\rho}(z)$ and pressure $\bar{p}(z)$. Then $\bar{p}_z + g\bar{\rho} = 0$, where the subscript denotes differentiation with respect to height. The linearized equations of motion now give

$$\bar{\rho}\,\partial\mathbf{u}/\partial t = -\mathrm{grad}\,(p-\bar{p}) - g(\rho-\bar{\rho})\,\mathbf{k},$$

$$\partial(\rho-\bar{\rho})/\partial t + w\,d\bar{\rho}/dz = 0,$$

$$\mathrm{div}\;\mathbf{u} = 0.$$

The perturbations may be resolved into independent wave components. Further, by rotation of the axes about the vertical, it is sufficient to consider each wave as a motion in the (x,z)-plane. Thus we put

$$u, w, p-\bar{p}, \rho-\bar{\rho} = \mathrm{Re}\,\{\hat{u}(z),\ \hat{w}(z),\ \hat{p}(z),\ \hat{\rho}(z)\exp\,[i(kx-\omega t)]\}$$

respectively, where k is a real wave-number and ω a real frequency. We assume $\bar{\rho}_z < 0$ so that the mean state is stable. Now elimination of $\hat{p}$, $\hat{\rho}$, $\hat{u}$ gives

$$\hat{w}_{zz} + \beta\hat{w}_z - k^2\hat{w} + k^2N^2\omega^{-2}\hat{w} = 0, \tag{2.4}$$

where the inverse scale height $\beta(z) = -\bar{\rho}_z/\bar{\rho}$ and the Brunt–Väisälä frequency $N(z) \equiv +\sqrt{(g\beta)}$.

When, for example, $\bar{\rho} = \rho_0\exp\,(-\beta z)$ with constant β, the solution

$$\hat{w} = \epsilon\exp\{\tfrac{1}{2}\beta z + i(kx+mz-\omega t)\}, \tag{2.5}$$

where ϵ is any small complex constant and the vertical wave-number m satisfies the frequency relation

$$\omega^2 = g\beta k^2/(k^2+m^2+\tfrac{1}{4}\beta^2). \tag{2.6}$$

It can be seen how the amplitude of the wave grows like $\bar{\rho}^{-\frac{1}{2}}$. So the linearization cannot be uniformly valid for small ϵ as $z \to +\infty$. In fact iteration of this solution gives

$$w = \mathrm{Re}\left\{\epsilon\exp\{\tfrac{1}{2}\beta z + i(kx+mz-\omega t)\} - \frac{ik\beta}{2\omega}\,\frac{(k^2+m^2-\beta^2)-i\beta m}{(3k^2+3m^2-\beta^2)-2i\beta m}\right.$$
$$\left.\times\,\epsilon^2\exp\{\beta z + 2i(kx+mz-\omega t)\} + \sum_{n=3}^{\infty} O(\epsilon\exp\,[\tfrac{1}{2}\beta z])^n\right\}. \tag{2.7}$$

on a suitable choice of complementary function at each stage of the iteration (or, equivalently, on a suitable redefinition of ϵ). This solution seems to break down where $\epsilon\exp\,(\frac{1}{2}\beta z) \gtrsim 1$, i.e. at and above heights $z \approx -2(\log\epsilon)/\beta$.

In the Boussinesq approximation β is neglected although $g\beta$ is not. More formally, one may take the limit as $\beta/m \to 0$ for fixed N^2/ω^2, m/k, ϵ. Thus, in the above example, the solution becomes

$$\hat{w} = \epsilon\exp\{i(kx+mz-\omega t)\} \tag{2.8}$$

and the frequency
$$\omega = \{g\beta k^2/(k^2+m^2)\}^{\frac{1}{2}}. \tag{2.9}$$
The non-uniformity of the linearization is removed (though it occurs for any given $\beta > 0$, however small) and there are linear waves with phase velocity
$$\mathbf{v} = \omega(k^2+m^2)^{-1}(k,m) \tag{2.10}$$
and group velocity
$$\mathbf{V} = \frac{\partial\omega}{\partial k}\mathbf{i}+\frac{\partial\omega}{\partial m}\mathbf{k}$$
$$= \omega mk^{-1}(k^2+m^2)^{-1}(m,-k). \tag{2.11}$$
This gives the vertical component of the group velocity
$$W = -\omega^2(N^2-\omega^2)^{\frac{1}{2}}/kN^2\,\mathrm{sgn}\,m. \tag{2.12}$$
These results can be extended by the JWKB approximation when N varies slowly with height (Bretherton 1966). The energy density of the waves varies like $N^2(N^2-\omega^2)^{-\frac{1}{2}}$, and reflexion occurs at a level where $\omega = N(z)$.

3. A non-linear wave

Whitham (1965*a*, *b*) has recently developed a method of finding the properties of non-linear dispersive waves by generalizing the method of averaging of ordinary differential equations. The breakdown of the linearization of internal gravity waves where $\bar{\rho}$ is small occurs very slowly in the Boussinesq approximation and so seems suitable for treatment by Whitham's method. Accordingly let us follow it, first looking for a single non-linear wave of the assumed form
$$u = u(\theta), \quad w = w(\theta), \quad \rho = \bar{\rho}(z)\{1+R(\theta)\}, \quad p = \bar{p}(z)+\bar{\rho}(z)P(\theta),$$
for some periodic functions u, w, R, P of $\theta \equiv kx+mz-\omega t$. Then the equations of motion give
$$ku_\theta+mw_\theta = 0$$
and therefore
$$ku+mw = \text{const} = \Omega, \quad \text{say}; \tag{3.1}$$
$$(\Omega-\omega)R_\theta = \beta(1+R)w, \tag{3.2}$$
$$(\Omega-\omega)(1+R)u_\theta = -kP_\theta, \tag{3.3}$$
$$(\Omega-\omega)(1+R)w_\theta = -mP_\theta+\beta P-gR. \tag{3.4}$$
These equations can have no solution except in regions where $\beta(z)$ is constant. There we may deduce
$$u = \Omega/k-mw/k,$$
$$R = C\exp\left\{\frac{\beta}{\Omega-\omega}\int_0^\theta w\,d\theta\right\}-1,$$
$$P = B+Cmk^{-2}(\Omega-\omega)\int_0^\theta w_\theta\exp\left\{\frac{\beta}{\Omega-\omega}\int_0^\theta w\,d\theta\right\}d\theta,$$
for any constants B, C where
$$w_{\theta\theta}+g\beta k^2(k^2+m^2)^{-1}(\Omega-\omega)^{-2}w = \beta w_\theta\{m(k^2+m^2)^{-1}-w/(\Omega-\omega)\} \tag{3.5}$$
and
$$\beta Bk^2 = gk^2(C-1)+B(\Omega-\omega)(k^2+m^2)\,w_\theta\,|_{\theta=0}.$$

It can be seen that the complete solution follows easily once w is determined from (3.5). To solve this equation, first note that we may choose a new normalization of θ without loss of generality so that

$$g\beta k^2 = (k^2+m^2)(\Omega-\omega)^2.$$

Then
$$w_{\theta\theta}+w = \beta w_\theta\{m(k^2+m^2)^{-1}-w/(\Omega-\omega)\}. \tag{3.6}$$

Analysis of this equation in the phase plane of (w, w_θ) shows that there is only one singularity, an unstable spiral point at the origin, and that each trajectory comes from the origin and goes to infinity. Therefore there are no periodic solutions in general. However, when $\beta = 0$,

$$w = A\cos(\theta+\eta)$$

for arbitrary constant amplitude A and phase η. When $0<\beta \ll 1$ we anticipate that A, η change slowly with θ but that the solution retains the above form. This process whereby A, η may change by order one when θ increases by order β^{-1} is well known in the theory of non-linear oscillations (cf. Bogoliubov & Mitropolsky 1961). An approximation to the solution for small β can be found by the method of averaging, which may be applied as follows. Equation (3.6) can be written as

$$(d/d\theta)\,\tfrac{1}{2}(w_\theta^2+w^2) = \beta w_\theta^2\{m/(k^2+m^2)-w/(\Omega-\omega)\}$$

exactly. To first order in β we average this equation over the 'period' 2π of the slowly varying solution $A\cos(\theta+\eta)$ and thereby pick up the cumulative effects of the small right-hand side. This gives

$$\frac{d}{d\theta}\frac{1}{2\pi}\int_0^{2\pi}\tfrac{1}{2}A^2\sin^2(\theta+\eta)+\tfrac{1}{2}A^2\cos^2(\theta+\eta)\,d\theta$$
$$= \frac{\beta}{2\pi}\int_0^{2\pi}A^2\sin^2(\theta+\eta)\{m/(k^2+m^2)-A(\Omega-\omega)^{-1}\cos(\theta+\eta)\}\,d\theta,$$

i.e.
$$dA^2/d\theta = \beta m(k^2+m^2)^{-1}A^2.$$

Therefore
$$A = A_0\exp\{\tfrac{1}{2}\beta m(k^2+m^2)^{-1}\theta\}.$$

A similar argument shows that η is constant to this order of approximation. Therefore
$$w = A_0\exp\{\tfrac{1}{2}\beta m(k^2+m^2)^{-1}\theta\}\cos(\theta+\eta_0).$$

As θ increases A will finally become so large that this approximation breaks down.

At this point one could put $\beta = 0$ to get the periodic solution $w = A\cos(\theta+\eta)$ and hence get periodic u, P, R. This complete periodic solution could be generalized for small β (not necessarily constant) by averaging conservation relations that follow exactly from the partial differential equations of motion. In this way one might find partial differential equations to show how the 'constants' A, B, C, Ω vary slowly when β is small. However, I felt that it would be technically easier to reformulate the whole problem in Lagrangian variables. Unfamiliarity with Lagrangian variables seems to be more than offset by the reduction of the dependent variables from four to three, by the ease of getting a variational principle (which is important in Whitham's (1965*b*) later work), and by the simplicity of handling the Boussinesq approximation. When this work was in progress, Seliger & Whitham (1968) found a variational principle for Euler variables in the Clebsch representation. The representation shares some of the advantages and disadvantages of Lagrangian co-ordinates, and the two sets of co-ordinates offer an interesting comparison. Accordingly, let us start again in Lagrangian variables.

4. Lagrangian formulation

Lagrangian co-ordinates may be chosen so that the Cartesian co-ordinates of a fluid particle are (x, z) at time t and $(x, z) = (a, c)$ at an initial time, $t = t_0$ say. Then continuity and incompressibility gives the Jacobian

$$\frac{d(x,z)}{d(a,c)} = 1.$$

This classical theory is conveniently described in the notation of Eckart (1960), whereby $t^i = a, c, t$; $x^m = x, z, p$; $x_i^m = dx^m/dt^i$ $(i, m = 1, 2, 3)$, the summation convention of repeated indices is used,

$$\frac{dF(x^m, x_i^m, t^i)}{da} = \frac{\partial F}{\partial x^m} x_a^m + \frac{\partial F}{\partial x_i^m} x_{ia}^m + \frac{\partial F}{\partial a}$$

denotes partial differentiation of any function F for constant c, t, etc., and $\partial F/\partial x^m$, $\partial F/\partial x_i^m$ denote formal partial derivatives. Thus we identify (x_t, z_t) as the Eulerian velocity, and we can rewrite the equation of incompressibility as

$$x_a z_c - x_c z_a = 1. \tag{4.1}$$

The equations of motion of inviscid fluid, when gravity acts in the negative c-direction, become

$$x_{tt} x_a + (z_{tt} + g) z_a + p_a/\rho = 0, \tag{4.2}$$

$$x_{tt} x_c + (z_{tt} + g) z_c + p_c/\rho = 0. \tag{4.3}$$

For incompressible fluid, $\rho = \rho(a, c)$ is independent of t. Let us suppose that the fluid is stratified with $\rho = \rho(c)$ at time t_0 and therefore $\rho = \rho(c)$ for all time.

Following Eckart (1960), consider the Lagrangian function

$$L \equiv T - V, \tag{4.4}$$

where the kinetic energy density

$$T \equiv \tfrac{1}{2}\rho(x_t^2 + z_t^2) \tag{4.5}$$

and the potential energy density

$$V \equiv g\rho z - p\left\{\frac{d(x,z)}{d(a,c)} - 1\right\}. \tag{4.6}$$

Now define the integral

$$I \equiv \int_{a_1}^{a_2} \int_{c_1}^{c_2} \int_{t_1}^{t_2} L \, dt \, dc \, da \tag{4.7}$$

and take $\delta I = 0$ to give I an extremum over the class of well-behaved functions x, z, p which have prescribed values at the end points of integration. This variational principle gives the Euler–Lagrange equations for p, x, z which are, respectively (4.1),

$$\rho x_{tt} + \frac{d(p,z)}{d(a,c)} = 0 \tag{4.8}$$

and

$$\rho(z_{tt}+g) + \frac{d(x,p)}{d(a,c)} = 0. \tag{4.9}$$

Equations (4.8), (4.9) are in fact the x- and z-equations of momentum in Lagrangian co-ordinates, and with (4.1) they are equivalent to equations (4.1)–(4.3). Thus the variational principle implies the equations of motion.

Conservation laws of the form $\operatorname{div}\mathbf{F} \equiv dF^i/dt^i = 0$ play an important part in Whitham's (1965*a*) method of averaging. Sometimes they can be derived by inspection. Alternatively the property (cf. Eckart 1960) that

$$\frac{dL_i^j}{dt^j} + \frac{\partial L}{\partial t^i} = 0,$$

where the energy-momentum tensor

$$L_i^j \equiv x_i^m \frac{\partial L}{\partial x_j^m} - L\delta_j^i$$

gives a conservation relation for each co-ordinate t^i that L does not explicitly depend upon. Here L depends explicitly on neither t nor a. A more systematic way to seek conservation relations is to find Lie groups under which I is invariant and then to use Noether's theorem (cf. Gelfand & Fomin 1963, pp. 81–2, 176–8). In our case I is invariant under the three groups of translations $t^* = t+\alpha$, $a^* = a+\alpha$, $x^* = x^*+\alpha$ for arbitrary parameter α, and the corresponding conservation relations turn out to be the conservation of energy (equivalent to $dL_t^j/dt^j = 0$), $dL_a^j/dt^j = 0$ and the conservation of x-momentum respectively. Also the translation $p^* = p+\alpha$ changes L by only the divergence of a vector and hence does not affect δI; this gives the conservation of mass.

Reverting to our problem of internal gravity waves, we may put

$$x = a+X, \quad z = c+Z, \quad p = p_0 - g\int_0^c \rho(c')\,dc' + \rho P$$

into equations (4.1)–(4.3) and linearize for small perturbations X, Z, P. This gives the wave solution of § 2, on identifying X_t, Z_t with the Eulerian variables u, w and a, c with x, z, respectively, in this linear approximation.

Similarly, the non-linear solution of § 3 is recovered if we suppose $X = X(\theta)$, $Z = Z(\theta)$, $P = P(\theta)$, where $\theta \equiv ka+mc-\omega t$, and substitute into equations (4.1)–(4.3) without approximation. The next step might seem to be Whitham's method of averaging. However, trial of this method has shown that informal use of the Boussinesq approximation, crudely treating $\beta(c)$ as small when not multiplied by g, is liable to lead to errors. So first we shall transform the Lagrangian co-ordinates in order to unambiguously sort out the relative magnitudes of terms when β is small.

This transformation, namely

$$\xi \equiv \rho^{\frac{1}{2}}(x-a), \quad \zeta \equiv \rho^{\frac{1}{2}}(z-c), \quad \varpi \equiv \rho^{-\frac{1}{2}}\left\{p - p_0 + g\int_0^c \rho(c')\,dc'\right\} + g\rho^{\frac{1}{2}}(z-c), \tag{4.10}$$

serves at once to represent the displacement of a particle from its basic position (a, c), to give waves whose average kinetic energy varies slowly with height, and to systematize the Boussinesq approximation. It follows that

$$x = a + \rho^{-\frac{1}{2}}\xi, \quad z = c + \rho^{-\frac{1}{2}}\zeta, \quad p = p_0 - g\int_0^c \rho(c')\,dc' - g\rho^{\frac{1}{2}}\xi + \rho^{\frac{1}{2}}\varpi, \tag{4.11}$$

and by routine calculus and algebra that

$$\begin{aligned} L = T^* - V^* + \frac{d}{da}\bigg\{\rho^{-\frac{1}{2}}\left(p_0 - g\int_0^c \rho(c')\,dc'\right)(\xi - \rho^{-\frac{1}{2}}\xi_c\zeta - \tfrac{1}{2}\beta\rho^{-\frac{1}{2}}\xi\zeta) \\ - g\rho^{-\frac{1}{2}}\xi\zeta\zeta_c\bigg\} + \frac{d}{dc}\bigg\{\rho^{-\frac{1}{2}}\left(p_0 - g\int_0^c \rho(c')\,dc'\right)(1 + \rho^{-\frac{1}{2}}\xi_a)\,\zeta \\ - g\int_0^c c'\rho(c')\,dc' - g\zeta^2 + g\rho^{-\frac{1}{2}}\xi\zeta\zeta_a\bigg\}, \end{aligned}$$

where
$$T^* \equiv \tfrac{1}{2}(\xi_t^2 + \zeta_t^2) \tag{4.12}$$

and
$$\begin{aligned} V^* \equiv \tfrac{1}{2}g\beta(1 + \rho^{-\frac{1}{2}}\xi_a)\,\zeta^2 - \varpi\{\xi_a + \zeta_c + \rho^{-\frac{1}{2}}(\xi_a\zeta_c - \xi_c\zeta_a) \\ + \tfrac{1}{2}\beta\zeta + \tfrac{1}{2}\beta\rho^{-\frac{1}{2}}(\xi_a\zeta - \xi\zeta_a)\}. \end{aligned} \tag{4.13}$$

The divergence terms in L make no contribution to δI, so the equations of motion in ξ, ζ, ϖ are given by the new variational principle $\delta I^* = 0$, with

$$I^* \equiv \iiint L^*\,da\,dc\,dt \quad \text{and} \quad L^* \equiv T^* - V^*.$$

With this new variational principle the Euler–Lagrange equations for ϖ, ξ, ζ are respectively

$$\xi_a + \zeta_c + \rho^{-\frac{1}{2}}(\xi_a\zeta_c - \xi_c\zeta_a) = -\tfrac{1}{2}\beta\zeta + \tfrac{1}{2}\beta\rho^{-\frac{1}{2}}(\xi\zeta_a - \xi_a\zeta), \tag{4.14}$$

$$\xi_{tt} + \varpi_a - g\beta\rho^{-\frac{1}{2}}\zeta\zeta_a + \rho^{-\frac{1}{2}}(\varpi_a\zeta_c - \varpi_c\zeta_a) = -\tfrac{1}{2}\beta\rho^{-\frac{1}{2}}(\varpi_a\zeta + \varpi\zeta_a), \tag{4.15}$$

$$\zeta_{tt} + \varpi_c + g\beta\zeta(1 + \rho^{-\frac{1}{2}}\xi_a) + \rho^{-\frac{1}{2}}(\xi_a\varpi_c - \xi_c\varpi_a) = \tfrac{1}{2}\beta\varpi + \tfrac{1}{2}\beta\rho^{-\frac{1}{2}}(\xi\varpi_a + \xi_a\varpi). \tag{4.16}$$

These exact equations of motion can alternatively be derived directly from (4.1), (4.8), (4.9) with (4.11).

Equations (4.14)–(4.16) make explicit the Boussinesq approximation. The orders of magnitude of the terms are apparent as $\beta \to 0$ for fixed $g\beta \neq 0$. This is in contrast to the derivation of (2.4) and (3.5) for which one can only put $\beta = 0$, $g\beta \neq 0$ *after* elimination of all the dependent variables but w. Here the Boussinesq approximation comes simply from neglecting the right-hand sides of equations (4.14)–(4.16). Alternatively the Lagrangian L^* may be replaced by $L_0 \equiv T^* - V_0$ in the variational principle, where

$$V_0 \equiv \tfrac{1}{2}g\beta(1 + \rho^{-\frac{1}{2}}\xi_a)\,\zeta^2 - \varpi\{\xi_a + \zeta_c + \rho^{-\frac{1}{2}}(\xi_a\zeta_c - \xi_c\zeta_a)\}. \tag{4.17}$$

In preparation for averaging we shall now get some conservation relations from equations (4.14)–(4.16) without approximation. The integral I^* is invariant under translations of t, a and ξ. Therefore Noether's theorem gives the exact conservation relations

$$\frac{d}{dt}(T^*+V^*)+\frac{d}{da}(-\tfrac{1}{2}g\beta\rho^{-\frac{1}{2}}\xi_t\zeta^2+\varpi\{\xi_t+\rho^{-\frac{1}{2}}(\xi_t\zeta_c-\xi_c\zeta_t)$$
$$+\tfrac{1}{2}\beta\rho^{-\frac{1}{2}}(\xi_t\zeta-\xi\zeta_t)\})+\frac{d}{dc}(\varpi\{\zeta_t+\rho^{-\frac{1}{2}}(\xi_a\zeta_t-\xi_t\zeta_a)\})=0, \qquad (4.18)$$

$$\frac{d}{dt}(\xi_a\xi_t+\zeta_a\zeta_t)+\frac{d}{da}\{-T^*+\tfrac{1}{2}g\beta\zeta^2-\varpi(\zeta_c+\tfrac{1}{2}\beta\zeta)\}+\frac{d}{dc}(\varpi\zeta_a)=0, \qquad (4.19)$$

$$\frac{d}{dt}(\rho^{\frac{1}{2}}\xi_t)+\frac{d}{da}(-\tfrac{1}{2}g\beta\zeta^2+\varpi\{\rho^{\frac{1}{2}}+\zeta_c+\tfrac{1}{2}\beta\zeta\})-\frac{d}{dc}(\varpi\zeta_a)=0 \qquad (4.20)$$

respectively. These equations are equivalent to (4.1)–(4.3). Also L^* is altered by only a divergence under a translation of ϖ, and so δI^* is invariant under translation of ϖ. Thus Noether's theorem gives conservation of volume of incompressible fluid,

$$\frac{d}{da}\{\rho^{-\frac{1}{2}}\xi-\rho^{-1}(\xi_c+\tfrac{1}{2}\beta\xi)\,\zeta\}+\frac{d}{dc}\{\rho^{-\frac{1}{2}}\zeta(1+\rho^{-\frac{1}{2}}\xi_a)\}=0. \qquad (4.21)$$

Also $dL_c^{*j}/dt^j+\partial L^*/\partial c=0$ gives

$$\frac{d}{dt}(\xi_c\xi_t+\zeta_c\zeta_t)+\frac{d}{da}\{\xi_c\varpi-\tfrac{1}{2}g\beta\rho^{-\frac{1}{2}}\xi_c\zeta^2+\tfrac{1}{2}\beta\rho^{-\frac{1}{2}}\varpi(\xi_c\zeta-\xi\zeta_c)\}$$
$$+\frac{d}{dc}(-T^*+\tfrac{1}{2}g\beta(1+\rho^{-\frac{1}{2}}\xi_a)\,\zeta^2-\varpi\{\xi_a+\tfrac{1}{2}\beta\zeta+\tfrac{1}{2}\beta\rho^{-\frac{1}{2}}(\xi_a\zeta-\xi\zeta_a)\})=\frac{\partial V^*}{\partial c}. \qquad (4.22)$$

There are an infinity of conservation relations that can be found by trial. Of these, we shall need only the three following:

$$(d/dt_i)\,(\xi\xi_{it}+\zeta\zeta_{it}+g\beta\zeta^2+g\beta\rho^{-\frac{1}{2}}\zeta(\xi_a\zeta-\xi\zeta_a)+\varpi(\xi_a+\zeta_c+\tfrac{1}{2}\beta\zeta)$$
$$+\{\varpi[\xi+\rho^{-\frac{1}{2}}(\xi\zeta_c-\xi_c\zeta)]\}_a+\{\varpi[\zeta+\rho^{-\frac{1}{2}}(\xi_a\zeta-\xi\zeta_a)]\}_c)=0. \qquad (4.23)$$

5. Slowly varying waves

The non-linear wave solution $w = A\cos\theta$ of the Euler equations (3.1)–(3.4) can be easily found in terms of the present variables. Put $\beta = 0$ for fixed $g\beta \neq 0$, i.e. neglect the right-hand sides of equations (4.14)–(4.16), and look for a solution for which the physical variables ξ, ζ, ϖ are periodic functions of θ alone, where θ is now any linear function of a, c and t. This gives, as before,

$$\zeta = A\cos\theta, \qquad (5.1)$$

$$\xi = \rho^{\frac{1}{2}}D-\theta_c\theta_a^{-1}A\cos\theta, \qquad (5.2)$$

$$\varpi = \rho^{-\frac{1}{2}}B-\theta_c\theta_t^2\theta_a^{-2}A\sin\theta+\tfrac{1}{4}g\beta\rho^{-\frac{1}{2}}A^2\cos 2\theta, \qquad (5.3)$$

for new arbitrary constants A, B, D, provided that

$$G \equiv \tfrac{1}{2}\{(\theta_a^2+\theta_c^2)\,\theta_t^2\theta_a^{-2}-g\beta\}=0. \qquad (5.4)$$

Note that $\rho^{\frac{1}{2}}D$, $\rho^{-\frac{1}{2}}B$ have been chosen as the average values of ξ, ϖ respectively over their period 2π in θ.

When $\beta = 0$ the quantities $A^m = A, B, D\ (m = 1, 2, 3)$ and θ_i are constants. When β is small and positive they are assumed to vary slowly and so can be found by Whitham's method of averaging. This method is too complicated for there to be any rigorous theory yet, so it may be helpful to bear in mind that the basic idea here is to generalize for partial differential equations the method of averaging used on the ordinary differential equation (3.6).

We have already derived exactly from the equations of motion relations of the form

$$\frac{dF^i(\xi^m, \xi^m_j, c)}{dt^i} = \beta H(\xi^m, \xi^m_j, c),$$

where F^i, H are functions of order one as $\beta \to 0$ for fixed constant $g\beta \neq 0$. One can eliminate ξ^m in favour of A^m, θ to put these relations in the form

$$\frac{dF^i(A^m, A^m_j, \theta_j, \theta_{jk}, c)}{dt^i} = \beta H(A^m, A^m_j, \theta_j, \theta_{jk}, c).$$

This is without any approximation, because it may be regarded as a change of dependent variables. To find how the 'constants' A^m, θ_i vary, the relations above are averaged over a 'period' 2π of the wave (5.1)–(5.4). The justification of this approximation and of averaging a relation of the above form rather than any equation of motion is discussed further by Whitham (1965*a*) and Luke (1966). This averaging gives approximate relations of the form

$$\frac{d\bar{F}^i(A^m, \theta_j, c)}{dt^i} = \beta \bar{H}(A^m, \theta_j, c), \tag{5.5}$$

where the average of any function $f(\theta)$ is defined so that

$$\bar{f} \equiv \frac{1}{2\pi}\int_0^{2\pi} f\, d\theta$$

for fixed A^m, θ_j. In all Whitham's examples, he found that increasing indefinitely the number of conservation relations to average led to redundant information, there being a basic independent set of averaged equations (5.5) for each problem. Later Whitham (1965*b*) found that these could be equivalently derived by finding an exact variational principle for the equations of motion, by averaging the Lagrangian as a function of θ and the 'constants', and then by using the Euler–Lagrange equations for the averaged Lagrangian.

Putting the local solution (5.1)–(5.3) into the conservation relation (4.18) and averaging over a 'period' 2π in θ, one finds

$$\frac{d}{dt}\{E(G+g\beta)\} - \frac{d}{da}\left\{\frac{\theta_c^2\theta_t^3}{\theta_a^3}E\right\} + \frac{d}{dc}\left\{\frac{\theta_c\theta_t^3}{\theta_a^2}E\right\}$$
$$= -\frac{d}{dt}\left\{\frac{\frac{1}{2}\beta\theta_c\theta_t^2}{\theta_a}DE\right\} + \frac{d}{da}\left\{\frac{\frac{1}{2}\beta\theta_c\theta_t^3}{\theta_a^2}DE\right\},$$

where $E \equiv \frac{1}{2}A^2$. The right-hand side is of order of magnitude β times slowly varying quantities, i.e. of order β^2, and therefore negligible to the present order of approximation. Therefore

$$\frac{d}{dt}\{E(G+g\beta)\}-\frac{d}{da}\left\{\frac{\theta_c^2\theta_t^3}{\theta_a^3}E\right\}+\frac{d}{dc}\left\{\frac{\theta_c\theta_t^3}{\theta_a^2}E\right\}=0. \tag{5.6}$$

Similarly, the average of (4.19) gives

$$\frac{d}{dt}\left\{\frac{\theta_t(\theta_a^2+\theta_c^2)}{\theta_a}E\right\}-\frac{d}{da}\{E(G+\theta_c^2\theta_t^2/\theta_a^2)\}+\frac{d}{dc}\left\{\frac{\theta_c\theta_t^2}{\theta_a}E\right\}=0; \tag{5.7}$$

of (4.22) gives

$$\begin{aligned}\frac{d}{dt}\left\{\frac{\theta_c(2G+g\beta)}{\theta_t}E\right\}-\frac{d}{da}\left\{\frac{\theta_c^3\theta_t^2}{\theta_a^3}E\right\}+\frac{d}{dc}\{E(\theta_c^2\theta_t^2/\theta_a^2-G)\}&\\ =\tfrac{1}{2}g\beta_cE+\tfrac{1}{2}(\beta_c+\tfrac{1}{2}\beta^2)\frac{\theta_c\theta_t^2}{\theta_a}DE+\frac{d}{da}\left\{\frac{\frac{1}{2}\beta\theta_c^2\theta_t^2}{\theta_a^2}DE\right\}-\frac{d}{dc}\left\{\frac{\frac{1}{2}\beta\theta_c\theta_t^2}{\theta_a}DE\right\}&\\ =\tfrac{1}{2}g\beta_cE,&\end{aligned} \tag{5.8}$$

on neglect of β^2, with the assumption that $\beta_c = O(\beta^2)$; and of (4.23) gives

$$\frac{d(EG)}{dt^i}=0. \tag{5.9}$$

The average of (4.20) gives

$$\frac{d}{da}\left\{B-\tfrac{1}{2}g\beta E+\frac{\theta_c^2\theta_t^2}{\theta_a^2}E\right\}-\frac{d}{dc}\left\{\frac{\theta_c\theta_t^2}{\theta_a}E\right\}=0; \tag{5.10}$$

and of (4.21) gives

$$\begin{aligned}\frac{dD}{da}&=-\frac{d}{da}\left\{\frac{\frac{1}{2}\beta\rho^{-1}\theta_c}{\theta_a}E\right\}\\ &=0,\end{aligned} \tag{5.11}$$

on neglect of $O(\beta^2)$.

Equations (5.9) imply that EG is constant, and therefore that $G=0$ globally as well as locally for the wave (5.1)–(5.4). Now

$$\frac{d\theta_i}{dt}+V_a\frac{d\theta_i}{da}+V_c\frac{d\theta_i}{dc}\equiv\left(\frac{dG}{dt^i}-\frac{\partial G}{\partial t^i}\right)\Big/\frac{\partial G}{\partial\theta_t} \tag{5.12}$$

identically, where the group velocity

$$\begin{aligned}\mathbf{V}=(V_a,V_c)&=-\frac{\partial\theta_t}{\partial\theta_i}=\frac{\partial G}{\partial\theta_i}\Big/\frac{\partial G}{\partial\theta_t}\quad(i=1,2)\\ &=\frac{\theta_c\theta_t}{\theta_a(\theta_a^2+\theta_c^2)}(-\theta_c,\theta_a).\end{aligned} \tag{5.13}$$

But G is zero, and depends explicitly upon c but not a or t. Therefore

$$\frac{d\theta_i}{dt}+V_a\frac{d\theta_i}{da}+V_c\frac{d\theta_i}{dc}=-\frac{\partial G}{\partial c}\delta_i^3\Big/\frac{\partial G}{\partial\theta_t}\quad(i=1,2,3). \tag{5.14}$$

Then equation (5.6) gives

$$\frac{d\mathscr{E}}{dt}+\frac{d(V_a\mathscr{E})}{da}+\frac{d(V_c\mathscr{E})}{dc}=0, \tag{5.15}$$

where
$$\mathscr{E} \equiv \frac{\partial G}{\partial \theta_t} E = \frac{\theta_t(\theta_a^2+\theta_c^2)}{\theta_a^2} E. \tag{5.16}$$

Similarly equations (5.7), (5.8) give (5.15). Thus the six equations (5.6)–(5.9) are compatible, being equivalent to $G = 0$ and (5.15).

To apply Whitham's (1965*b*) alternative method, one notes that

$$\overline{T^*} = \tfrac{1}{2}\theta_t^2(\theta_a^2+\theta_c^2)\,\theta_a^{-2}E, \quad \overline{V^*} = \tfrac{1}{2}(g\beta+\tfrac{1}{2}\beta\theta_c\theta_t^2\theta_a^{-1}D)\,E,$$

so the averaged Lagrangian

$$\overline{L^*} = (G-\tfrac{1}{2}\beta\theta_c\theta_t^2\theta_a^{-1}D)\,E. \tag{5.17}$$

The Euler–Lagrange equation for A gives $\partial\overline{L^*}/\partial A = 0$, i.e.

$$G = \tfrac{1}{2}\beta\theta_c\theta_t^2\theta_a^{-1}D. \tag{5.18}$$

The Euler–Lagrange equation for θ gives

$$\frac{d\mathscr{E}}{dt}+\frac{d(V_a\mathscr{E})}{da}+\frac{d(V_c\mathscr{E})}{dc} = \frac{d}{dt}\left\{\frac{\beta\theta_c\theta_t}{\theta_a}DE\right\}-\frac{d}{da}\left\{\frac{\frac{1}{2}\beta\theta_c\theta_t^2}{\theta_a^2}DE\right\}+\frac{d}{dc}\left\{\frac{\frac{1}{2}\beta\theta_t^2}{\theta_a}DE\right\}, \tag{5.19}$$

in agreement with (5.15) on neglect of β^2. $\overline{L^*}$ is a linear function of D and independent of D_t, so there is no extremum of $\overline{L^*}$ due to variations of D. This seems to imply that D is zero to the present order of approximation. Indeed, this can be imposed at the outset by a translation of the origin of the horizontal co-ordinate a. Also the averaged Lagrangian is inherently unable to give any variation of B because the pressure-like variable ϖ appears as a Lagrangian multiplier in $\overline{L^*}$. This is associated with invariance of the action integral I^* under translation of ϖ. These aspects of invariance of I^* in the formulation with Lagrangian co-ordinates seem to be new in applications of Whitham's method of averaging and to deserve deeper analysis.

In any event both approaches give the variation of A according to equation (5.15), which does not involve B or D. To the present order of approximation for small β, non-linearity gives only the term in A^2 to (5.3). Equation (5.15) is the same as if the wave were linear (Whitham 1965*b*, § 10). Its characteristics coincide, and the characteristic velocity is the group velocity $\mathbf{V}$. In the simplest case, the amplitude, wavelengths and frequency of the two-dimensional wave vary slowly with height only, i.e.

$$A = A(c), \quad \theta = k(c)\,a+m(c)\,c-\omega(c)\,t,$$

where k_c, m_c, ω_c are small. Then $\theta_c = m+(k_ca+m_cc-\omega_ct) \doteqdot m, \theta_{ac} = k_c$, $\theta_{tc} = -\omega_c$. Therefore equation (5.15) gives

$$-\frac{2\theta_c\theta_t^2}{\theta_a^2}E\theta_{ac}+\frac{d}{dc}\left(\frac{\theta_c\theta_t^2}{\theta_a^2}E\right) = 0$$

and thence

$$m\omega^2k^{-4}A^2 = \text{const.} \tag{5.20}$$

This is the adiabatic invariant as k, m, ω change slowly with height such that $G = 0$ everywhere. It can be seen that

$$A^2 = \text{const} \times \omega k^3 (N^2 - \omega^2)^{-\frac{1}{2}} \tag{5.21}$$

and thus that A becomes large at any level where $\omega = N(z)$. Therefore the present non-linear approximation breaks down locally, but reflexions may be expected as in linear theory (Bretherton 1966).

6. Discussion

In order to make the problem of non-linear internal gravity waves tractable, we have been somewhat restrictive. In addition to the limitations of the model atmosphere already mentioned, inviscid incompressible fluid, no mean flow, no magnetic field, it should be remembered that only slow variation of a single wave component was considered. Non-linear interactions of waves of significantly different lengths or frequencies were ignored.

However, the strongly non-linear behaviour of a single wave in a slightly stratified atmosphere has been revealed. The linear theory is valid only where the amplitude $A \ll \rho^{\frac{1}{2}}/\theta_c$, but if β is small the non-linear theory is valid elsewhere, even high up, where the density is small. If the transformed Lagrangian variables ξ, ζ, ϖ are used, non-linearity affects the solution only in ϖ, the pressure-like variable, for which a term in the square of A is added. Otherwise the non-linear waves behave like groups of linear waves. This means that $z = c + \rho^{-\frac{1}{2}} A \cos\theta$, etc. even when $\rho^{-\frac{1}{2}} A$ is not small. Therefore waves are reflected near levels where $\omega = N$ and the increase of amplitude of the displacement of a fluid particle increases with height like $\rho^{-\frac{1}{2}}$, even where the linear theory breaks down.

This non-linearity cannot be described so simply in the Eulerian variables, because the velocity components $u = x_t = \rho^{-\frac{1}{2}}\xi_t$ and $w = z_t = \rho^{-\frac{1}{2}}\zeta_t$ inextricably involve the Lagrangian co-ordinate c in $\rho(c)$ where $z - c$ is not small. This seems to justify working with the less familiar Lagrangian variables for this particular non-linear wave.

Finally, it should be emphasized that this theory gives the behaviour of internal gravity waves even where the density is not exponentially stratified. If the density does become exponentially small, as in the earth's atmosphere, then the amplitude of the waves becomes large and the solution will be as described above in the non-linear régime. Eventually a sinusoidal wave solution of the ordinary differential equation (3.6) will be no longer approximately a solution where $A \approx 1/\beta$, and the whole waves solution will be invalid. However, the theory is equally valid in an ocean in which the density tends to a non-zero constant. Then the frequency relation gives waves with vanishing frequency θ_t or vertical wavelength $2\pi/\theta_c$ in the region of uniform density.

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