# Research Notes

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# Chapter 1

# 2D Wave Breaking in Atmospheres

The goal of this will be to lay out a formalism that can reproduce Sutherland et al. 2011[1] and also investigate driven oscillations versus the breaking of a single wave packet. This represents wave breaking in the atmosphere.

## 1.1 Dynamical Setup

We adopt notation where $q_0$ is the background quantity and $q_1$ is the perturbed quantity from the propagating wave.

The fluid equations are

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \rho \vec{u} = 0, \tag{1.1a}$$

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} = -\vec{\nabla}\frac{P}{\rho} - g\hat{z}, \tag{1.1b}$$

where we will take gravity to be uniform throughout the domain of interest. We will study for no background flow $\vec{u}_0 = 0$ and in the presence of stratification $\rho_0 \propto e^{-z/H}$. In the absence of any perturbations it is then easy to show that $\frac{\mathrm{d}P_0}{\mathrm{d}z} = -\rho_0 g$.

[1]DOI:10.1175/JAS-D-11–097.1

### 1.1.1 Linear, Incompressible

#### Solution for Arbitrary Stratification

First, we solve the incompressible case $c_s^2 \to \infty, \vec{\nabla} \cdot \vec{u} = 0$ in the linear regime. For funsies, we solve for arbitrary stratification first. The fluid equations to first order reduce to

$$\begin{aligned} \frac{\partial \rho_1}{\partial t} + \left(\vec{u}_1 \cdot \vec{\nabla}\right)\rho_0 &= 0, \\ \vec{\nabla} \cdot \vec{u}_1 &= 0, \\ \frac{\partial \vec{u}_1}{\partial t} &= -\frac{\vec{\nabla} P_1}{\rho_0} - P_1 \vec{\nabla} \frac{1}{\rho_0} \end{aligned} \tag{1.2}$$

We expect there to be some $z$ dependence in the amplitude, so we substitute variables of form $e^{i(kx-\omega t)}$ and do not specify the $z$ dependence. This gives us

$$\begin{aligned} -i\omega\rho_1 - u_{1z}\frac{\mathrm{d}\rho_0}{\mathrm{d}z} &= 0, \\ iku_{1x} + \frac{\mathrm{d}u_{1z}}{\mathrm{d}z} &= 0, \\ -iwu_{1x} + \frac{ik_x P_1}{\rho_0} &= 0, \\ -iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1}{\rho_0^2}\frac{\mathrm{d}P_0}{\mathrm{d}z} &= 0. \end{aligned} \tag{1.3}$$

We substitute $N^2 = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}$ and $\frac{\mathrm{d}P_0}{\mathrm{d}z} = -\rho g$ to obtain

$$-i\omega\rho_1 - u_{1z}\frac{\rho_0 N^2}{g} = 0, \tag{1.4a}$$

$$iku_{1x} + \frac{\mathrm{d}u_{1z}}{\mathrm{d}z} = 0, \tag{1.4b}$$

$$-iwu_{1x} + \frac{ik_x P_1}{\rho_0} = 0, \tag{1.4c}$$

$$-iwu_{1z} + \frac{1}{\rho_0}\frac{\partial P_1}{\partial z} + \frac{\rho_1 g}{\rho_0} = 0. \tag{1.4d}$$

Eliminating $u_{1x}$ by substituting (1.4b) into (1.4c) and $\rho_1$ by substituting (1.4a) into (1.4d) give

$$i\omega\frac{\mathrm{d}u_{1z}}{\mathrm{d}z} + \frac{k_x^2 P_1}{\rho_0} = 0, \tag{1.5a}$$

$$\left(\omega^2 - N^2\right)u_{1z} + \frac{i\omega}{\rho_0}\frac{\mathrm{d}P_1}{\mathrm{d}z} = 0. \tag{1.5b}$$

Finally, we multiply (1.5a) with $\rho_0$ and differentiate d$z$ and combine with (1.5b) to give

$$\frac{\mathrm{d}^2 u_{1z}}{\mathrm{d}z^2} + \frac{1}{\rho_0}\frac{\mathrm{d}\rho_0}{\mathrm{d}z}\frac{\mathrm{d}u_{1z}}{\mathrm{d}z} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right)u_{1z} = 0. \tag{1.6}$$

**Introduce Stratification**

With stratification $\rho \propto e^{-z/H}$ Eq. 1.6 clearly has exponential solutions $e^{\kappa z}$ for

$$\kappa^2 - \frac{\kappa}{H} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right) = 0. \tag{1.7}$$

We permit complex $\kappa = -\frac{1}{2H} + ik_z$, and from the above clearly

$$k_z^2 = -\frac{1}{4H^2} + k_x^2\left(\frac{N^2}{\omega^2} - 1\right),$$
$$\omega^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}. \tag{1.8}$$

### 1.1.2 Linear Regime, Compressible

### 1.1.3 Boundary Conditions

We must bound this to a finite domain. We choose periodic boundary conditions in $x$ with total length $L_x$, $x \in \left[-\frac{L}{2}, \frac{L}{2}\right]$. We choose $z$ dimension to have total length $L_z$, $z \in [0, L_z]$.

To set up the boundary condition at $z = 0$, we recall that gravity waves in an atmosphere with $e^{-z/H}$ profile have form

$$u_{1z} \propto e^{\frac{z}{2H}} e^{i(k_x x + k_z z - \omega t)}, \tag{1.9}$$

where ($N^2 \equiv \frac{g}{H}$ is the Brunt-Väisälä frequency in an incompressible, stratified atmosphere)

$$\omega^2 = \frac{N^2 k_x^2}{k_x^2 + k_z^2 + \frac{1}{4H^2}}. \tag{1.10}$$

Thus, at constant $z = 0$ we must have

$$\vec{u}_{1z}(z=0) \propto e^{i(k_x x - \omega t)}. \tag{1.11}$$

The boundary condition at $z = L_z$ is much harder to determine. It is clear that $\vec{u}_1(z=\infty) = \rho_1(z=\infty) = 0$, and for $L_z \gg H$ this would be a reasonable approximation, if simply because we expect the majority of the wave to dissipate via turbulent dissipation as $z$ reaches many $H$. We will simply choose the BC to be many multiples of $H$. We can solve with both a Dirichlet and Neumann BC and compare the two solutions; if the solutions differ significantly then we must choose a larger $L_z$. These are the only two solutions that can be implemented where we do not need the phase of the linear wave, which we lose during the nonlinear breaking region, to relate the function and derivative at the boundary.

For the boundary conditions on $\rho$, we note that in the linear regime it should just have a phase offset from $u_{1z}$, thus we choose $\rho(z=0) \propto e^{\frac{z}{2H}} i e^{i(k_x x + k_z z - \omega t)}$ and a similar treatment at $z = L_z$, taking both a Dirichlet and Neumann BC.

### 1.1.4 Simulation

We begin our simulation with $\rho_1 = 0, \vec{u}_1 = 0$ strictly within the domain of simulation. We will borrow some values from Sutherland's paper and use $k_z = 2\,\mathrm{km}^{-1}$ then define $k_x = -0.4k_z, H = 10/k_z, A = 0.05/k_z, L_z = 300/k_z, L_x = 20/k_z$, We also use $\mu \approx 29, T = 273\,\mathrm{K}, \rho_0 = 1\,\mathrm{kg/m}^3, P_0 = \frac{\rho_0 k_B T}{\mu m_p}, g = 10\,\mathrm{m/s}^2$.