1 Equations

We begin with the fully compressible fluid equations and general equation of state

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} + \rho\left(\vec{\nabla}\cdot\vec{u}\right) = 0, \tag{1a}$$

$$\frac{\mathrm{d}S}{\mathrm{d}t} = 0, \tag{1b}$$

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \frac{\vec{\nabla}P}{\rho} + \vec{g} = 0, \tag{1c}$$

$$P = P(\rho, S). \tag{1d}$$

We notate $\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)$. Note this is 5 equations (in 2D) for 5 variables $(\rho, P, S, \vec{u})$ and so is properly closed.

1.1 Current Implementation

We assume $P(\rho, S) = P(\rho)$ a barotropic equation of state, such that S is entirely decoupled from the dynamical equations. We thus drop Eq. 1b of the full fluid equations Eq. 1. Then, we will decompose dynamical variables into background and fluctuation quantities $P = P_0 + P_1, \rho = \rho_0 + \rho_1$, where background quantities are time-independent. Assuming no background velocities, we write $\vec{u}$ to denote the fluctuation velocity with no ambiguity.

Furthermore, we will implement incompressibility by mandating $\left.\frac{\partial P(\rho)}{\partial \rho}\right|_{ad} \to \infty$ adiabatic derivative. This forces $\Delta P \gg \Delta\rho$ within a displaced fluid parcel, or $\frac{\mathrm{d}\rho}{\mathrm{d}t} = 0$ comoving derivative. This allows us to rewrite Eq. 1a, and so we arrive at the system of equations

$$\vec{\nabla}\cdot\vec{u} = 0, \tag{2a}$$

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \frac{\vec{\nabla}P}{\rho} + \vec{g} = 0, \tag{2b}$$

$$\frac{\mathrm{d}\rho}{\mathrm{d}t} = \frac{\mathrm{d}\rho_1}{\mathrm{d}t} + u_z\frac{\mathrm{d}\rho_0}{\mathrm{d}z} = 0. \tag{2c}$$

We now assume $P_0 = \rho_0 c_s^2 \propto e^{-z/H}$ isothermal exponential stratification. At hydrostatic equilibrium, Eq. 2b forces $\vec{\nabla}P_0 = -\rho_0\vec{g}$. We next claim that $\frac{\rho_1}{\rho_0} \sim \left(\frac{u}{c_s}\right)^2 = \mathrm{Ma}^2 \ll 1$[1]. This may be verified to be true by looking briefly at the momentum equation: $\omega u + k\left(\frac{P}{\rho} - \frac{P_0}{\rho_0}\right) = 0$, which then dividing through by $P_0/\rho_0 = c_s^2$ shows that $\frac{P/P_0}{\rho/\rho_0} - 1 \sim \mathrm{Ma}^2$.

Given this, we expand Eq. 2b by $\frac{\vec{\nabla}P}{\rho} \approx \frac{\vec{\nabla}P_0}{\rho_0} + \frac{\vec{\nabla}P_1}{\rho_0} - \frac{\rho_1\vec{\nabla}P_0}{\rho_0^2} + \ldots$ and obtain the full system of equations

[1] I guess this doesn't have to be true if $\frac{P_0}{\rho_0} = c_s^2 \neq \left.\frac{\partial P}{\partial \rho}\right|_{ad}$ which is the velocity we are sending to infinity.

I have been using

$$\vec{\nabla} \cdot \vec{u} = 0, \tag{3a}$$

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} + \frac{\vec{\nabla} P_1}{\rho_0} + \frac{\rho_1 \vec{g}}{\vec{0}} - \frac{\rho_1 \vec{\nabla} P_1}{\rho_0^2} = 0, \tag{3b}$$

$$\frac{\partial \rho_1}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\rho_1 - \frac{u_z \rho_0}{H} = 0. \tag{3c}$$

These are complete up to numerical and forcing terms.

1.2 Anelastic

We can argue for this in a very formal/nondimensionalized way, but we can also see this comparatively simply (albeit less rigorously). We will start from Eq. 1. Expand all variables in terms of $\epsilon \equiv \mathrm{Ma} = \frac{u}{c_s}$ where u is the characteristic velocity of the flows we want to study, then

$$P = P_0 + \epsilon P_1 + \ldots, \qquad \rho = \rho_0 + \epsilon \rho_1 + \ldots, \qquad S = S_0 + \epsilon S_1 + \ldots, \qquad u \sim \mathcal{O}(\epsilon^1),$$
$$\vec{\nabla} \sim k \sim \mathcal{O}(\epsilon^0), \qquad \partial_t \approx uk \sim \mathcal{O}\left(\epsilon^1\right).$$

Note that we pick a particular length scale with k and with typical flow velocity u, then $\partial_t = uk$ is the correct order to examine at[2].

With this dictionary, we may consider the incompressible equations at:

- $\mathcal{O}(\epsilon^0)$, which is just $\frac{\vec{\nabla} P_0}{\rho_0} + \vec{g} = 0$ hydrostatic equilibrium.

  At this point, we may make the same argument as in the preceeding section: $\frac{\mathrm{d}}{\mathrm{d}t} \sim \mathcal{O}(\epsilon^2)$, so $\frac{\vec{\nabla} P}{\rho} + \vec{g} = \frac{\vec{\nabla} P}{\rho} - \frac{\vec{\nabla} P_0}{\rho_0} \sim \mathcal{O}(\epsilon^2)$ as well. Thus, $P_1, \rho_1 = 0$ necessarily, otherwise $\frac{\vec{\nabla} P}{\rho} - \frac{\vec{\nabla} P_0}{\rho_0}$ must have $\mathcal{O}(\epsilon^1)$ terms that are not cancelled by $\frac{\mathrm{d}\vec{u}}{\mathrm{d}t}$ which is prohibited.

- $\mathcal{O}(\epsilon^1)$. Since $P_1 = \rho_1 = 0$, the only terms at this order are $\vec{\nabla} \cdot \left(\rho_0 \vec{u}\right) = 0$ and $\left(\vec{u}_1 \cdot \vec{\nabla}\right) S_0 = 0$.

- $\mathcal{O}(\epsilon^2)$. This is where the momentum equation gets most of its terms, it becomes the familiar

  $$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \frac{\vec{\nabla} P_2}{\rho_0} + \frac{\rho_2 \vec{g}}{\rho_0} = 0. \tag{4}$$

  Now we haven't argued for $S_1 \neq 0$, so we must permit it as well, so the entropy equation becomes

  $$\frac{\mathrm{d}S_1}{\mathrm{d}t} + \left(\vec{u} \cdot \vec{\nabla}\right) S_0 = 0. \tag{5}$$

- Higher order expansions are not hard to find, and relax $\vec{\nabla} \cdot \left(\rho_0 \vec{u}\right) = 0$.

[2] More formally, we can probably do $\partial_t = \partial_{t_0} + \partial_{t_1/\epsilon} + \ldots$

Thus, to order Ma^2, our equations are

$$\vec{\nabla} \cdot \left(\rho_0 \vec{u}\right) = 0, \tag{6a}$$

$$\frac{\mathrm{d}S_1}{\mathrm{d}t} + \vec{u} \cdot \vec{\nabla} S_0 = 0, \tag{6b}$$

$$\frac{\mathrm{d}\vec{u}}{\mathrm{d}t} + \frac{\vec{\nabla} P_2}{\rho_0} + \frac{\rho_2 \vec{g}}{\rho_0} = 0, \tag{6c}$$

$$P = P(\rho, S). \tag{6d}$$

The last step is just to remove ρ_2 from the equations using the equation of state so that the first three equations of Eq. 6 above can form a complete system. We simply follow Glatzmaier and rewrite

$$\vec{\nabla} \cdot \left(\rho_0 \vec{u}\right) = 0, \tag{7a}$$

$$\frac{\partial S_1}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right) S_1 + \vec{u} \cdot \vec{\nabla} S_0 = 0, \tag{7b}$$

$$\frac{\partial \vec{u}}{\partial t} + \left(\vec{u} \cdot \vec{\nabla}\right)\vec{u} + \vec{\nabla}\frac{P_2}{\rho_0} - \frac{S_1 \vec{g}}{S_0} = 0. \tag{7c}$$

Solving this system gives buoyancy waves with frequency $N^2 = \frac{g}{H_S} = \frac{g}{H} \frac{\partial \ln S_0}{\partial \ln \rho_0}\Big|_P \ll \frac{g}{H}$ as we obtain $\frac{\partial \ln S_0}{\partial z} = -\frac{1}{H_S}$ stratification using the EoS.