1 Misc Insights

- To understand the DOF counting for the shooting method, let's first imagine shooting only from one direction. We have four first-order ODEs and one pair of BCs on either side. Thus, when we shoot from one direction, we apply two BCs and have two remaining constraints. Thus, there should be a 2D space of solutions, so there should be two linearly independent solutions that we can find. For them to be linearly independent, they cannot differ only by a normalization; we thus choose different y_3/y_1 for them.

  For a given pair of linearly independent solutions, we then evaluate the outer BCs (upon arrival). In general, the values of these BCs will be nonzero for both of our trial solutions. However, if a linear combination of our linearly independent solutions can satisfy both BCs, then we will have found a solution satisfying all four BCs. Of course, for a general ω, this should not be expected; it is not easy to satisfy both BCs at once. As such, it's not surprising that only isolated values of ω give us a chance at solving at the outer boundary. To find whether a linear combination works, we evaluate the Wronskian at the outer boundary and require it to be zero.

  Now, what about two-sided shooting? This happens if there are singular points at both boundaries. Here, we do the same procedure as above, and we just require the two solutions to be stitchable at some interior fitting point. We can think of this as two extra DOF (y_3/y_1 and normalization at outer point), but we have two matching conditions at the interior fitting point, so we again have the same argument for isolated eigenvalues as above.

We follow Appendicies B-C of Le Bihan & Burrows 2013. Note that we can seek two independent solutions to the ODEs because it is not a simple Sturm-Liouville form problem: the structure is quite complex, so we aren't guaranteed simple roots. Perhaps it will be fun to try and do the properties of Sturm-Liouville proofs again someday...

Update: we are able to get seemingly correct oscillation modes with dummy stellar profiles, but we don't have the correct non-singular solution near the core. Due to this, the matrix determinants in the core-surface solution matching are large, and so we suffer from some loss of numerical precision.

2 Polytrope Notes

Let's review how to obtain the structure of a polytrope. The equations of hydrostatic equilibrum are:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r}{r^2}\rho, \tag{1}$$

$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2\rho, \tag{2}$$

$$P = K\rho^{\Gamma} = K\rho^{1+1/n}. \tag{3}$$

Here, K is constant. This can easily be rearranged to obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = \frac{\mathrm{d}}{\mathrm{d}r}(-GM_r) = -G4\pi r^2\rho. \tag{4}$$

Then, since $\partial P/\partial r = K\rho^{\Gamma-1}\Gamma\mathrm{d}\rho/\mathrm{d}r$, we have that

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2K\rho^{\Gamma-2}\Gamma\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -G4\pi\rho. \tag{5}$$

Now, let's guess that $\rho(r) = \rho_c\theta^p(r)$, where p must be chosen such that $\rho^{\Gamma-2}\mathrm{d}\rho/\mathrm{d}r \propto \mathrm{d}\theta/\mathrm{d}r$. This can be seen to yield that $p = 1/(\Gamma-1) = n$, so then

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Gamma r^2Kn\rho_c^{\Gamma-1}\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -G4\pi\rho_c\theta^n, \tag{6}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\theta^n\times\left(\frac{G4\pi\rho_c^{1-1/n}}{K(n+1)}\right) \equiv -\theta^n\frac{1}{\alpha^2}. \tag{7}$$

Let's define $\xi \equiv r/\alpha$, then the Lane Emden-Equation reduces to (let primes denote $\mathrm{d}/\mathrm{d}\xi$)

$$\theta'' + \frac{2\theta'}{\xi} + \theta^n = 0. \tag{8}$$

Note that the ICs for θ are

$$\theta(0) = 1, \qquad \theta'(0) = 0. \tag{9}$$

It's clear that $\theta'(0) = 0$ is required for the equation to be non-singular. Let's guess that $\theta'(\xi \ll 1) \propto \xi$, then upon inspection we find that

$$\theta'(\xi \ll 1) = -\xi/3 \tag{10}$$

is correct, which is useful later. In addition, $\theta(\xi_1) = 0$ exists if $n < 5$, so ξ_1 is the outer boundary.

From here, we seek to solve for all of the quantities in closed form; first the stellar properties, then the relevant dimensionless quantities for the oscillation equations. First, it is clear that our natural parameterization of the stellar structure above relies on K and ρ_c, while we want to express these in terms of M and R.

$$
\begin{aligned}
\alpha &= \left(\frac{K(n+1)}{4\pi G\rho_c^{1-1/n}}\right)^{1/2}, && (11)\\
C &\equiv \left(\frac{K(n+1)}{4\pi G}\right)^{1/2} = \alpha\rho^{\frac{n-1}{2n}}, && (12)\\
R &= \alpha\xi_1\\
&\equiv C^{1/2}\rho_c^{\frac{1-n}{2n}}\xi_1, && (13)\\
M &= \int^R 4\pi r^2\rho(r)\,\mathrm{d}r\\
&= 4\pi\rho_c\alpha^3\int^{\xi_1}\xi^2\theta^n(\xi)\,\mathrm{d}\xi\\
&= 4\pi\rho_c\alpha^3\int^{\xi_1} -\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\mathrm{d}\xi\\
&= 4\pi\rho_c^{\frac{3-n}{2n}}C^{3/2}\left[-\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\right]_{\xi_1}, && (14)\\
M &= 4\pi\rho_c^{\frac{3-n}{2n}}\left(\frac{R}{\rho_c^{(1-n)/(2n)}\xi_1}\right)^3\left[-\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\right]_{\xi_1}\\
&= 4\pi\rho_c\left(\frac{R}{\xi_1}\right)^3\left[-\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\right]_{\xi_1}\\
\rho_c &= \left(\frac{M}{4\pi R^3/3}\right)\left(-\frac{\xi_1}{3\theta'(\xi_1)}\right), && (15)\\
R &= \left(\frac{K(n+1)}{4\pi G}\right)^{1/2}\rho_c^{\frac{1-n}{2n}}\xi_1\\
K &= \frac{R^2}{\xi_1^2}\rho_c^{\frac{n-1}{n}}\frac{4\pi G}{n+1}. && (16)
\end{aligned}
$$

As such, we have $(K,\rho_c) \leftrightarrow (M,R)$, so we use K,ρ_c for simplicity. The rest follow:

$$
\begin{aligned}
\rho(\xi) &= \rho_c\theta^n(\xi), && (17)\\
P(\xi) &= K\rho_c^\Gamma\theta^{n+1}(\xi) \propto \theta^{n+1}(\xi), && (18)\\
M_r(\xi) &= \int^{\xi\alpha} 4\pi r^2\rho\,\mathrm{d}r\\
&= 4\pi\alpha^3\rho_c\int^{\xi}\xi^2\theta^n(\xi)\,\mathrm{d}\xi\\
&= -4\pi\alpha^3\rho_c\xi^2\theta'(\xi), && (19)\\
g(\xi) &= \frac{GM_r}{(\xi\alpha)^2}\\
&= -4\pi G\alpha\rho_c\theta'(\xi), && (20)\\
U(\xi) &= \frac{4\pi\rho(\xi\alpha)^3}{M_r}\\
&= -\frac{\theta^n\xi}{\theta'(\xi)} && (21)\\
c_1 &= \frac{r/R}{M_r/M}\\
&= \frac{1}{(\xi/\xi_1)(\theta'(\xi)/\theta'(\xi_1))}, && (22)\\
c_1 &= \frac{(r/R)^3}{M_r/M} = \frac{\xi/\xi_1}{\theta'(\xi)/\theta'(\xi_1)}. && (23)
\end{aligned}
$$

Note that $U(\xi\to 1) = 3$ and $c_1(\xi\to 0) \propto -1/\xi^2$. Correct c_1 is shown for easier reference.

There's a small point of contention about c_s and N, since they depend on the adiabatic index Γ_1 and not just the polytropic index Γ. It takes a bit of digging, but Mullan & Ulrich 1988 point out that $\Gamma_1 = 5/3$ is used in all of their models. Thus, we have the remaining quantities in the Le Bihan formulation:

$$\begin{aligned}
c_s^2(\xi) &= \Gamma_1 \frac{P}{\rho} = \Gamma_1 K \rho_c^{1/n} \theta(\xi), && (24)\\
N^2(\xi) &= g^2 \left(\frac{\mathrm{d}\rho}{\mathrm{d}P} - \frac{1}{c_s^2}\right)\\
&= \left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right) g^2 \frac{\rho}{P}\\
A^* &= \frac{N^2 r}{g} = \frac{\rho g r}{P}\left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right)\\
&= \left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right)\left(-4\pi G \alpha \rho_c \theta'(\xi)\right)(\alpha\xi)\frac{1}{K\rho_c^{1/n}\theta(\xi)}\\
&= \left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right)\left(-4\pi G \rho_c^{1-1/n}\right)\alpha^2 \frac{\xi\theta'(\xi)}{K\theta(\xi)}\\
&= -\left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right)\frac{(n+1)\xi\theta'(\xi)}{\theta(\xi)}\\
V_g &= \frac{gr}{c_s^2}\\
&= A^* \frac{1/\Gamma_1}{1/\Gamma - 1/\Gamma_1}\\
&= -\frac{1}{\Gamma_1}\frac{(n+1)\xi\theta'(\xi)}{\theta(\xi)} && (25)
\end{aligned}$$

For convenience, we also provide the inversion of these fields back into physical quantities. We choose (P_0, ρ_0, c_s^2, g) as our four physical quantities (any four will do):

$$\begin{aligned}
g &= \frac{(GM/R^3)r}{c_1}, && (26)\\
c_s^2 &= \frac{gr}{V_g} = \frac{(GM/R^3)r^2}{c_1 V_g}, && (27)\\
\rho_0 &= \frac{gU}{4\pi G r} = U\frac{M}{4\pi c_1 R^3}, && (28)\\
A^* &= gr\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}P_0} - \frac{1}{c_s^2}\right),\\
P_0 &= \int \mathrm{d}r \left(\frac{A^*}{gr} + \frac{1}{c_s^2}\right)^{-1}\frac{\mathrm{d}\rho_0}{\mathrm{d}r}. && (29)
\end{aligned}$$

In the case of a polytrope with $\Gamma = 1 + 1/n$, the P_0 inversion is a bit cleaner:

$$P_0 = \frac{\rho_0 g r}{\Gamma(A^* + V_g)}. \tag{30}$$

Also possibly useful is

$$M_r = \frac{(r/R)^3}{c_1} M. \tag{31}$$

It's clear that we cannot calculate M_r until we have integrated the entire polytrope, nor can we calculate the end point of the first simulation ξ_c since we don't know R yet. Thus, if we want a specific core mass / radius, we will have to proceed iteratively.

2.1 Piecewise Polytrope

Simplest model for a core + envelope structure. Let's imagine that there is a interior and envelope with polytropic indicies n_i and n_e. How do we stitch these together? Well, qualitatively, looking at Le Bihan, we note that the pressure should be continuous, but the density can change (drop), resulting in a change to the sound speed. Thus, in both regions, we are solving the usual hydrostatic equilibrium equations:

$$\begin{aligned}
\frac{\mathrm{d}P}{\mathrm{d}r} &= -\frac{GM_r}{r^2}\rho, && (32)\\
\frac{\mathrm{d}M_r}{\mathrm{d}r} &= 4\pi r^2 \rho, && (33)\\
P &= K\rho^\Gamma = K_{i,e}\rho^{1+1/n_{i,e}}, && (34)\\
\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 K \rho^{\Gamma_{i,e}-2}\Gamma_{i,e}\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) &= -G4\pi\rho. && (35)
\end{aligned}$$

The solution within the core is exactly what we've done before, except that we terminate the solution based on a particular M_r instead of when $\theta = 0$.

However, how do we solve in the envelope? Well, the inner BC at $r = R_c$ (initial conditions) must satisfy

$$\begin{aligned}
\rho_i(R_c) &= C_\rho \rho_e(R_c), && (36)\\
K_i \rho_i^{\Gamma_i}(R_c) &= K_e \rho_e^{\Gamma_e}(R_c). && (37)
\end{aligned}$$

This C_ρ can be specified by hand. Note though that the structure for a polytrope is scale-free: we translate the dimensionless θ, θ' to physical quantities via M, R. So with two polytropic zones and two BCs, we have four quantities $M_{i,e}, R_{i,e}$ that are connected via the BCs to have only two DOF which reduce the problem to the standard form. To relate the two M, R, we can plug in our known forms for K and ρ_c (note that ξ_1 depends only on n):

$$\rho_c = \left(\frac{M}{4\pi R^3/3}\right)\left(-\frac{\xi_1}{3\theta'(\xi_1)}\right)$$
$$\equiv \frac{M}{R^3}\eta_\rho(n), \tag{38}$$
$$\eta_\rho(n) \equiv -\frac{3}{4\pi}\frac{\xi_1}{3\theta'(\xi_1)}, \tag{39}$$
$$K = \frac{R^2}{\xi_1^2}\rho_c^{\frac{n-1}{n}}\frac{4\pi G}{n+1}$$
$$= GM^{\frac{n-1}{n}}R^{\frac{3}{n}-1}\eta_K(n), \tag{40}$$
$$\eta_K(n) = \frac{4\pi}{(n+1)\xi_1^2}\eta_\rho^{\frac{n-1}{n}}(n). \tag{41}$$

The units on K should check out: the ρ factor is just $\rho^2\rho^{-\Gamma}$, so $K\rho^\Gamma \sim R^2\rho^2 G = GM^2/R^4$ which has units of pressure indeed. The BCs then become

$$\frac{M_i}{R_i^3}\eta_\rho(n_i)\theta_i^{n_i}\left(\xi_{1,i}\frac{R_c}{R_i}\right) = C_\rho\frac{M_e}{R_e^3}\eta_\rho(n_e)\theta_e^{n_e}\left(\xi_{1,e}\frac{R_c}{R_e}\right), \tag{42}$$
$$M_i^{\frac{n_i-1}{n_i}}R_i^{\frac{3}{n_i}-1}\eta_K(n_i)\left[\frac{M_i}{R_i^3}\eta_\rho(n_i)\theta_i^{n_i}\right]^{\Gamma_i} = M_e^{\frac{n_e-1}{n_e}}R_e^{\frac{3}{n_e}-1}\eta_K(n_e)\left[\frac{M_e}{R_e^3}\eta_\rho(n_e)\theta_e^{n_e}\right]^{\Gamma_e}$$
$$\frac{M_i^2}{R_i^4}\eta_K(n_i)\eta_\rho^{\Gamma_i}(n_i)\theta_i^{n_i+1}\left(\xi_{1,i}\frac{R_c}{R_i}\right) = \frac{M_e^2}{R_e^4}\eta_K(n_e)\eta_\rho^{\Gamma_e}(n_e)\theta_e^{n_e+1}\left(\xi_{1,e}\frac{R_c}{R_e}\right)$$
$$R_i^2\eta_K\eta_\rho^{\frac{1-n_i}{n_i}}\theta_i^{1-n_i}\left(\xi_{1,i}\frac{R_c}{R_i}\right) = \frac{R_e^2}{C_\rho^2}\eta_K\eta_\rho^{\frac{1-n_e}{n_e}}\theta_e^{1-n_e}\left(\xi_{1,e}\frac{R_c}{R_e}\right). \tag{43}$$

Of course the pressure condition was going to be equal to this, since GM^2/R^4 was the only term that had units of pressure. These equations are, of course, implicit, since the argument to $\theta_{i,e}$ depends on $R_{i,e}$ (the subscript to θ is because different $n_{i,e}$ will give different solutions to the L-E equation).

If we assume that $n_i = n_e = n$, these conditions simplify quite a bit, since the η factors cancel:

$$R_i^2\theta^{1-n}\left(\xi_1\frac{R_c}{R_i}\right) = \frac{R_e^2}{C_\rho^2}\theta^{1-n}\left(\xi_1\frac{R_c}{R_e}\right), \tag{44}$$
$$M_e = M_i\frac{R_e^3}{C_\rho R_i^3}\theta^n\left(\xi_1\frac{R_c}{R_i}\right)\theta^{-n}\left(\xi_1\frac{R_c}{R_e}\right). \tag{45}$$

In the code, we will take $M_i = R_i = 1$, so we effectively solve for M_e/M_i and R_e/R_i. Then, the total radius of the star is R_e, and:

- For $r < R_c$, we need to evaluate the interior polytrope at $\xi_i = r\frac{\xi_{1i}}{R_i}$.
- For $r > R_c$, we need to evaluate the exterior polytrope at $\xi_e = r\frac{\xi_{1e}}{R_e}$.

We want however to express the stellar structure exclusively in terms of the overall radius variable $\xi \equiv \xi_e$. This means that for all interior quantities $Q_i(\xi_i)$, we should just be evaluating them at

$$Q_i(\xi_i) = Q_i\left(\xi\frac{\xi_{1i}/\xi_{1e}}{R_i/R_e}\right). \tag{46}$$

Note: This is surprisingly difficult to get in closed form, and I don't know if I'm doing all the transformation of variables correctly. Probably best at this point to just move on to a better structure model.

3 Oscillation Equations

3.1 Physical Form

Let's rederive the oscillation equations too, all the way through the nondimensionalized form (we did this in Dong's class). We start with the Euler Equations and Poisson equation

$$\frac{\partial\rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{v}\right) = 0, \tag{47}$$
$$\frac{\partial\vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} + \frac{\vec{\nabla}P}{\rho} = -\vec{\nabla}\Phi, \tag{48}$$
$$\nabla^2\Phi = 4\pi G\rho. \tag{49}$$

We expand $x = x_0 + \delta x$ for each of $x = P, \rho, \Phi$. Then we assume an adiabatic EOS

$$\frac{\Delta P}{\Delta\rho} = c_s^2, \tag{50}$$

where Δ is the Lagrangian perturbation, and we have the useful relation

$$\delta\vec{v} = \frac{\mathrm{d}\vec{\xi}}{\mathrm{d}t}, \qquad \Delta x = \delta x + \left(\vec{\xi}\cdot\vec{\nabla}\right)x. \tag{51}$$

equilibrium $\mathrm{d}P_0/\mathrm{d}r = -\rho_0 g$. The goal is to take FTs and cast everything into ξ_r, δP, and $\delta\Phi$ with only three equations of motion.

Then we first tackle the density equation. There is no leading order term, so we directly analyze the perturbative component. There are two forms that I think might be useful:

$$\frac{\cancel{\partial\rho_0}}{\cancel{\partial t}} + \frac{\partial(\delta\rho)}{\partial t} + \vec{\nabla}\cdot\left(\rho_0\delta\vec{u}\right) = 0,$$

$$\delta\rho = -\vec{\nabla}\cdot\left(\rho_0\vec{\xi}\right), \tag{52}$$

$$\frac{\mathrm{d}(\delta\rho)}{\mathrm{d}t} = -\rho_0\left(\vec{\nabla}\cdot(\delta\vec{u})\right),$$

$$\Delta\rho = -\rho_0\left(\vec{\nabla}\cdot\vec{\xi}\right). \tag{53}$$

Next, we tackle the momentum equation. The zeroth order term is just hydrostatic equilibrium, corresponding to $\vec{\nabla}P_0/\rho_0 = -\vec{\nabla}\Phi_0$, which gives $g = \vec{\nabla}\Phi_0$. The first order term is

$$\frac{\partial(\delta\vec{u})}{\partial t} + \frac{\vec{\nabla}(\delta P)}{\rho_0} - \frac{(\delta\rho)\vec{\nabla}P_0}{\rho_0^2} = -\vec{\nabla}(\delta\Phi),$$

$$\frac{\partial^2\vec{\xi}}{\partial t^2} + \frac{\vec{\nabla}(\delta P)}{\rho_0} - \frac{\delta\rho\vec{\nabla}P_0}{\rho_0^2} = \vec{\nabla}(\delta\Phi). \tag{54}$$

We go ahead and substitute $\partial_t \to i\omega$. The perpendicular and parallel components of Eq. (54) are:

$$\hat{r}: -\omega^2\xi_r + \frac{1}{\rho_0}\frac{\partial}{\partial r}(\delta P) + \frac{\delta r}{\rho_0}g = -\frac{\partial}{\partial r}\delta\Phi, \tag{55}$$

$$\perp: -\omega^2\vec{\xi}_\perp + \frac{\vec{\nabla}_\perp(\delta P)}{\rho_0} = -\vec{\nabla}_\perp(\delta\Phi)$$

$$-\omega^2\vec{\xi}_\perp = -\vec{\nabla}_\perp\left(\frac{\delta P}{\rho_0} + \delta\Phi\right). \tag{56}$$

Combining the perpendicular component with Eq. (53), we can rewrite (and substitute in Y_{lm}'s)

$$\Delta\rho = -\rho_0\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2\xi_r\right) + \nabla_\perp^2\left(\frac{\delta P}{\rho_0\omega^2} + \frac{\delta\Phi}{\omega^2}\right)\right),$$

$$-\frac{\Delta\rho}{\rho_0} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\xi_r\right) - \frac{l(l+1)}{r^2\omega^2}\left(\frac{\delta P}{\rho_0} + \delta\Phi\right),$$

$$= -\frac{\Delta P}{c_s^2\rho_0} = -\frac{\delta P}{c_s^2\rho_0} + \frac{\xi_r g}{c_s^2}. \tag{57}$$

To rewrite the radial component of the momentum equation, we first define N^2 and rewrite:

$$N^2 \equiv g^2\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}P_0} - \frac{1}{c_s^2}\right), \tag{58}$$

$$\frac{N^2}{g^2} + \frac{1}{c_s^2} = \frac{\mathrm{d}\rho_0/\mathrm{d}r}{\mathrm{d}P_0/\mathrm{d}r},$$

$$\delta\rho = \Delta\rho - \xi_r\frac{\partial\rho_0}{\partial r}$$

$$= \frac{\Delta P}{c_s^2} - \xi_r\left(\frac{N^2}{g^2} + \frac{1}{c_s^2}\right)(-\rho_0 g)$$

$$= \frac{\delta P}{c_s^2} + \xi_r\frac{N^2}{g^2}. \tag{59}$$

Then the radial component is straightforwardly

$$\frac{1}{\rho_0}\frac{\partial}{\partial r}\delta P + \frac{\partial}{\partial r}(\delta\Phi) - \omega^2\xi_r + \frac{g}{\rho_0}\left(\frac{\delta P}{c_s^2} + \xi_r\frac{N^2\rho_0}{g}\right) = 0. \tag{60}$$

Finally, the Poisson equation is simplified straightforwardly (with spherical harmonic dependence) to

$$\nabla^2(\delta\Phi) = 4\pi G\delta\rho,$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(\delta\Phi)\right) - \frac{l(l+1)}{r^2}(\delta\Phi) = 4\pi G\rho_0\left(\frac{\delta P}{\rho_0 c_s^2} + \xi_r\frac{N^2}{g}\right). \tag{61}$$

Thus, the three equations for radial oscillations are a combination of: the perpendicular momentum equation in combination with the density equation, the radial momentum equation, and the Poisson equation. For posterity, they are (with slight rearrangements):

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\xi_r\right) - \frac{g}{c_s^2}\xi_r + \left(1 - \frac{l(l+1)c_s^2}{r^2\omega^2}\right)\frac{\delta P}{c_s^2\rho_0} - \frac{l(l+1)}{r^2\omega^2}\delta\Phi = 0, \tag{62}$$

$$\frac{1}{\rho_0}\frac{\partial}{\partial r}\delta P + \frac{\partial}{\partial r}\delta\Phi + \frac{g}{\rho_0 c_s^2}\delta P + \left(N^2 - \omega^2\right)\xi_r = 0, \tag{63}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(\delta\Phi)\right) - \frac{l(l+1)}{r^2}(\delta\Phi) - 4\pi G\rho_0\left(\frac{\delta P}{\rho_0 c_s^2} + \xi_r\frac{N^2}{g}\right) = 0. \tag{64}$$

This constitutes 3 ODEs for 3 quantities, but one is second order. To make a reduction to first order, $(\delta\Phi)'$ constitutes a fourth variable.

3.2 Nondimensional Form

To nondimensionalize, we use the variables recommended in Unno's 1979 book (and by Le Bihan). They are:

$$y_1 = \frac{\xi_r}{r}, \tag{65}$$

$$y_2 = \frac{1}{gr}\left(\frac{\delta P}{\rho_0} + \delta\Phi\right), \tag{66}$$

$$y_3 = \frac{1}{gr}\delta\Phi, \tag{67}$$

$$y_4 = \frac{1}{g}\frac{\mathrm{d}}{\mathrm{d}r}\delta\Phi, \tag{68}$$

$$x = \frac{r}{R}. \tag{69}$$

To make this substitution, it helps not to use the fully-simplified forms initially. For instance, to simplify Eq. (62), it helps to start with the earlier form (primes denote ∂_r; we will explicitly denote x derivatives)

$$\begin{aligned}
-\frac{\delta P}{c_s^2\rho_0} + \frac{\xi_r g}{c_s^2} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\xi_r) - \frac{l(l+1)}{r^2\omega^2}\left(\frac{\delta P}{\rho_0} + \delta\Phi\right),\\
-\frac{1}{c_s^2}(gry_2 - \delta\Phi) + \frac{ry_1 g}{c_s^2} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^3 y_1) - \frac{l(l+1)}{r^2\omega^2}gry_2,\\
-\frac{gr}{c_s^2}(y_2 - y_3) + \frac{ry_1 g}{c_s^2} &= 3y_1 + ry_1' - \frac{l(l+1)}{r^2\omega^2}gry_2,\\
ry_1' &= -\frac{gr}{c_s^2}(y_2 - y_3) + \frac{y_1 gr}{c_s^2} + \frac{l(l+1)}{r^2\omega^2}gry_2 - 3y_1\\
xy_1'(x) &= \left(\frac{gr}{c_s^2} - 3\right)y_1 + \left[\frac{l(l+1)g}{r\omega^2} - \frac{gr}{c_s^2}\right]y_2 + \frac{gr}{c_s^2}y_3.
\end{aligned} \tag{70}$$

To simplify Eq. (63), we do

$$\delta P = (y_2 - y_3)gr\rho_0, \tag{71}$$

$$\begin{aligned}
0 &= \frac{1}{\rho_0}\frac{\partial((y_2 - y_3)gr\rho_0)}{\partial r} + \frac{\partial(gry_2)}{\partial r} + \frac{g(y_2 - y_3)gr\rho_0}{\rho_0 c_s^2} + \left(N^2 - \omega^2\right)ry_1,\\
&= \frac{\rho_0'(y_2 - y_3)gr}{\rho_0} + \frac{\partial(gry_2)}{\partial r} + \frac{g(y_2 - y_3)gr\rho_0}{\rho_0 c_s^2} + \left(N^2 - \omega^2\right)ry_1,\\
ry_2' &= -\frac{\rho_0'(y_2 - y_3)r}{\rho_0} - y_2\frac{(gr)'}{g} - \frac{(y_2 - y_3)gr}{c_s^2} - \frac{\left(N^2 - \omega^2\right)}{g}ry_1,\\
xy_2'(x) &= \frac{\left(\omega^2 - N^2\right)}{g}ry_1 - \left(\frac{\rho_0' r}{\rho_0} + \frac{(gr)'}{g} + \frac{gr}{c_s^2}\right)y_2 + \left(\frac{\rho_0' r}{\rho_0} + \frac{gr}{c_s^2}\right)y_3,\\
\frac{\rho_0' r}{\rho_0} &= \left(\frac{N^2}{g^2} + \frac{1}{c_s^2}\right)(-gr),
\end{aligned} \tag{72}$$

$$\begin{aligned}
xy_2'(x) &= \frac{\left(\omega^2 - N^2\right)}{g}ry_1 + \left(\frac{N^2 r}{g} - \frac{(gr)'}{g}\right)y_2 - \frac{N^2 r}{g}y_3,\\
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\Phi_0}{\partial r}\right) &= 4\pi G\rho_0,\\
\frac{\partial}{\partial r}\left(r^2 g\right) &= 4\pi G\rho_0 r^2,\\
r(gr)' &= 4\pi G\rho_0 r^2 - gr,\\
\frac{(gr)'}{g} &= \frac{4\pi\rho_0 r^3}{M_r} - 1,
\end{aligned} \tag{73}$$

$$xy_2'(x) = \left[\frac{\omega^2 r}{g} - \frac{N^2 r}{g}\right]y_1 + \left(\frac{N^2 r}{g} + 1 - \frac{4\pi\rho_0 r^3}{M_r}\right)y_2 - \frac{N^2 r}{g}y_3. \tag{74}$$

Eq. (64) also follows:

$$\begin{aligned}
0 &= \frac{1}{r^2}\left(r^2 g y_4\right)' - \frac{l(l+1)}{r^2}gry_3 - 4\pi G\rho_0\left(\frac{(y_2 - y_3)gr\rho_0}{\rho_0 c_s^2} + ry_1\frac{N^2}{g}\right),\\
0 &= (gr^2)'\frac{y_4}{r} + gry_4' - (l(l+1))gy_3 - 4\pi G\rho_0 r\left(\frac{(y_2 - y_3)gr}{c_s^2} + y_1\frac{N^2 r}{g}\right),\\
ry_4' &= -(4\pi G\rho_0 r^2)\frac{y_4}{gr} + (l(l+1))y_3 + \frac{4\pi G\rho_0 r}{g}\left(\frac{(y_2 - y_3)gr}{c_s^2} + y_1\frac{N^2 r}{g}\right),\\
&= \frac{4\pi\rho_0 r^3}{M_r}\left[-y_4 + \frac{(y_2 - y_3)gr}{c_s^2} + \frac{N^2 r}{g}y_1\right] + l(l+1)y_3.
\end{aligned} \tag{75}$$

Finally, the implicit equation is

$$\begin{aligned}
(gry_3)' &= gy_4, \\
ry_3' &= y_4 - \frac{(gr)'}{g} y_3 \\
&= y_4 - \left[\frac{4\pi\rho_0 r^3}{M_r} - 1\right] y_3.
\end{aligned} \tag{76}$$

Defining the constants

$$V_g = \frac{gr}{c_s^2}, \tag{77}$$

$$U = \frac{4\pi\rho_0 r^3}{M_r}, \tag{78}$$

$$c_1 = \frac{(r/R)^3}{M_r/M}, \tag{79}$$

$$\bar{\omega}^2 = \frac{\omega^2 R^3}{GM}, \tag{80}$$

$$A^* = \frac{N^2 r}{g}, \tag{81}$$

then we obtain

$$\frac{g}{r\omega^2} = \frac{GM_r/r^2}{rGM\bar{\omega}^2/R^3} = \frac{M_r/M}{x^3}\frac{1}{\bar{\omega}^2} = \frac{1}{c_1\bar{\omega}^2},$$

$$xy_1' = \left(V_g - 3\right)y_1 + \left[\frac{l(l+1)}{c_1\bar{\omega}^2} - V_g\right]y_2 + V_g y_3, \tag{82}$$

$$xy_2' = \left[c_1\bar{\omega}^2 - A^*\right]y_1 + \left(A^* + 1 - U\right)y_2 - A^* y_3, \tag{83}$$

$$xy_3' = (1-U)y_3 + y_4, \tag{84}$$

$$xy_4' = UA^* y_1 + UV_g y_2 + \left[l(l+1) - UV_g\right]y_3 - Uy_4. \tag{85}$$

The red parts differ from the Le Bihan paper. Notably, c_1 differs from the Le Bihan paper, and we should obtain

$$c_1 = \frac{(r/R)^3}{M_r/M} = \frac{\xi/\xi_1}{\theta'(\xi)/\theta'(\xi_1)}. \tag{86}$$

Note that in this case, $c_1(\xi\to 0) = -3\theta'(\xi_1)/\xi_1$ is finite. As such, we have fully constrained the eigenvalue problem for $\{y_i(x), \omega\}$ in terms of the fields V_g, c_1, A^*, and U.

3.3 Boundary Conditions / Behavior

Near $x = 0$, $V_g, A^* \propto x^2$ while $U \approx 3$ and $c_1 \approx -3\mathrm{d}\theta(1)/\mathrm{d}x$. The inner BCs are standard (we read from Le Bihan, but Fuller 2017 also seems to have similar):

$$0 = \xi_r - \frac{l}{\omega^2 r}\left(\frac{\delta P}{\rho_0} + \delta\Phi\right), \tag{87}$$

$$0 = \frac{\mathrm{d}(\delta\Phi)}{\mathrm{d}r} - \frac{l}{r}\delta\Phi. \tag{88}$$

These nondimensionalize to

$$\begin{aligned}
0 &= ry_1 - \frac{l}{\omega^2 r} gry_2 \\
&= y_1 - l\frac{GM_r/r^2}{rGM\bar{\omega}^2/R^3} y_2 \\
&= y_1 - \frac{l}{c_1\bar{\omega}^2} y_2,
\end{aligned} \tag{89}$$

$$\begin{aligned}
0 &= gy_4 - \frac{l}{r} gry_3 \\
&= y_4 - ly_3.
\end{aligned} \tag{90}$$

In order then for $xy_3' \to 0$ as $x \to 0$, we require $xy_3' \propto (l-2)y_3$ or that

$$y_3(x \ll 1) \sim x^{l-2}. \tag{91}$$

In addition, for $xy_1' \to 0$, we obtain $-3y_1 + (l+1)y_1 = xy_1'$, so we also find

$$y_1(x \ll 1) \sim x^{l-2}. \tag{92}$$

What about the other two y_i' equations? Well, $xy_2' = ly_2 - 2y_2$, so $y_2 \sim x^{l-2}$ as well, and $xy_4' = (l+1)y_4 - 3y_4$, so $y_4 \sim x^{l-2}$ at last. These two BCs constrain two DOF; the overall normalization is a third DOF, and the ratio y_3/y_1 is a fourth. So we choose two different y_3/y_1 in constructing our solutions.

As for the outer BCs, they are (small correction to Le Bihan; consistent with Fuller)

$$0 = \frac{\mathrm{d}(\delta\Phi)}{\mathrm{d}r} + \frac{(l+1)}{r}\delta\Phi, \tag{93}$$

$$0 = \Delta P = \delta P - \xi_r g\rho_0. \tag{94}$$

Nondimensionalizing, we obtain

$$0 = gy_4 + \frac{(l+1)}{r} gry_3$$
$$= y_4 + (l+1)y_3, \tag{95}$$
$$0 = gr\rho_0(y_2 - y_3) - gr\rho_0 y_1$$
$$= y_1 - y_2 + y_3. \tag{96}$$

Near the surface, V_g and A^* diverge, $U \to 0$, and $c_1 \to 1$. First, we note that $xy_3' = y_3 + y_4 = y_3 - (l+1)y_3 = -ly_3$, and so

$$y_3(x \to 1) \sim x^{-l}. \tag{97}$$

For y_1, we look at the singular terms in xy_1' and xy_2', but $xy_1' = V_g(y_1 - y_2 + y_3)$ and $xy_2' = -A^*(y_1 - y_2 + y_3)$ both identically cancel (which is good, since the LHS doesn't have a divergence to cancel these out!). So, doing a bit more work, the other equations are

$$xy_1' = -3y_1 + \frac{l(l+1)}{\bar{\omega}^2} y_2, \tag{98}$$
$$xy_2' = \bar{\omega}^2 y_1 + y_2, \tag{99}$$
$$xy_4' = UA^* y_1 + UV_g(y_2 + y_4/(l+1)) - ly_4. \tag{100}$$

Note that

$$UV_g \propto UA^* = \frac{4\pi\rho_0 R^3}{M_r} \frac{GM_r/R}{c_s^2} \propto \frac{4\pi G\rho_c \theta^n(1) R^2}{\theta(1)} \propto \theta^{n-1}(1). \tag{101}$$

This is zero if $n > 1$. Thus, the last equation shows $y_4 \propto x^{-l}$ as well. If we also want $y_1, y_2 \propto x^p$, then we need ($z \equiv y_1\bar{\omega}^2$)

$$pz = -3z + l(l+1)y_2, \tag{102}$$
$$py_2 = z + y_2, \tag{103}$$
$$(p+3)(p-1) = l(l+1). \tag{104}$$

This doesn't work! Thus, y_1 and y_2 have different power law behaviors at the edge, so we should instead write

$$p_1 z = -3z + l(l+1)y_2, \tag{105}$$
$$p_2 y_2 = z + y_2, \tag{106}$$
$$(p_1+3)(p_2-1) = l(l+1). \tag{107}$$

Then either $p_2 = l+1, p_1 = l-2$ or $p_2 = l+2$ and $p_1 = l-3$. Le Bihan says that the correct one to use is

$$y_1 \sim x^{l-2}, \quad y_2 \sim x^{l+1}. \tag{108}$$

Note: After correcting all the expressions and frequencies and checking the BCs, it seems that I was passing in x to an interpolation routine that accepted ξ instead... welp.

3.4 Checking Signs with Previous Papers

I changed the signs back to those in Le Bihan, and I get frequencies that are closer to theirs. So could it be possible that they didn't just make a typo, and have a different version of the Equations? Let's check whether the Christiansen-Dalsgaard paper also has the same signs as Le Bihan. Their nondimensional quantities are

$$z_1 = \frac{\xi_r}{R} = xy_1, \tag{109}$$
$$z_3 = x\frac{\delta\Phi}{gr} = xy_3, \tag{110}$$
$$z_4 = x^2 \frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{\delta\Phi}{gr}\right)$$
$$= x^2\left(\frac{1}{gr}\frac{\mathrm{d}}{\mathrm{d}x}\delta\Phi + \delta\Phi\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{1}{gr}\right)\right)$$
$$= x^2\left(\frac{R}{gr}\frac{\mathrm{d}}{\mathrm{d}r}\delta\Phi + R\delta\Phi\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{1}{gr}\right)\right)$$
$$= x^2\left(\frac{y_4}{x} - Ry_3\frac{(gr)'}{gr}\right)$$
$$= xy_4 - xy_3(U-1), \tag{111}$$
$$z_2 = x\left(\frac{\delta P}{\rho} - \delta\Phi\right)\frac{l(l+1)}{\omega^2 r^2} = \frac{l(l+1)}{R}\xi_\perp$$
$$= \frac{1}{gr}\left(\frac{\delta P}{\rho} + \delta\Phi - 2\delta\Phi\right) grx\frac{l(l+1)}{\omega^2 r^2}$$
$$= (y_2 - 2y_3)\frac{xl(l+1)}{\bar{\omega}^2 c_1} = (y_2 - 2y_3)x\eta. \tag{112}$$

And the equations are:

$$x\frac{\mathrm{d}z_1}{\mathrm{d}x} = \left(V_g-2\right)z_1 + \left(1-\frac{V_g}{\eta}\right)z_2 - V_g z_3, \tag{113}$$

$$x\frac{\mathrm{d}z_2}{\mathrm{d}x} = \left[l(l+1)-\eta A\right]z_1 + (A-1)z_2 + \eta A z_3, \tag{114}$$

$$x\frac{\mathrm{d}z_3}{\mathrm{d}x} = z_3 + z_4, \tag{115}$$

$$x\frac{\mathrm{d}z_4}{\mathrm{d}x} = -AUz_1 - U\frac{V_g}{\eta}z_2 + [l(l+1)+U(A-2)]z_3 + 2(1-U)z_4. \tag{116}$$

These are expressed in terms of

$$V_g = -\frac{1}{\Gamma_1}\frac{\mathrm{d}\ln P_0}{\mathrm{d}\ln r}, \tag{117}$$

$$= -\frac{1}{\Gamma_1}\frac{r}{P_0}\left(-\rho_0 g\right)$$

$$= \frac{gr}{c_s^2}, \tag{118}$$

$$A = \frac{1}{\Gamma_1}\frac{\mathrm{d}\ln P_0}{\mathrm{d}\ln r} - \frac{\mathrm{d}\ln\rho_0}{\mathrm{d}\ln r}, \tag{119}$$

$$= -\frac{gr}{c_s^2} - \frac{r}{\rho_0}\frac{\mathrm{d}P_0}{\mathrm{d}r}\frac{\mathrm{d}\rho_0}{\mathrm{d}P_0}$$

$$= -\frac{gr}{c_s^2} + gr\frac{\mathrm{d}\rho_0}{\mathrm{d}P_0} \equiv \frac{N^2 r}{g}, \tag{120}$$

$$U = \frac{4\pi\rho_0 r^3}{M_r}, \tag{121}$$

$$\eta = \frac{l(l+1)g}{\omega^2 r}, \tag{122}$$

$$= \frac{l(l+1)GM_r}{\bar{\omega}^2 GMx^3}$$

$$= \frac{l(l+1)}{\bar{\omega}^2 c_1}. \tag{123}$$

Note that V_g, A, U are all the same as in Le Bihan, but η is new. We replace the z's with the Le Bihan ys:

$$x\frac{\mathrm{d}}{\mathrm{d}x}(xy_1) = x^2\frac{\mathrm{d}y_1}{\mathrm{d}x} + xy_1$$

$$= \left(V_g-2\right)xy_1 + \left(1-\frac{V_g\bar{\omega}^2c_1}{l(l+1)}\right)(y_2-2y_3)\frac{xl(l+1)}{\bar{\omega}^2c_1} - V_g xy_3$$

$$x\frac{\mathrm{d}y_1}{\mathrm{d}x} = \left(V_g-3\right)y_1 + \left(\frac{l(l+1)}{\bar{\omega}^2c_1} - V_g\right)y_2 + \left(V_g - 2\frac{l(l+1)}{\bar{\omega}^2c_1}\right)y_3, \tag{124}$$

$$x\frac{\mathrm{d}}{\mathrm{d}x}(xy_3) = x^2\frac{\mathrm{d}y_3}{\mathrm{d}x} + xy_3$$

$$= xy_3 + xy_4 + xy_3(1-U)$$

$$x\frac{\mathrm{d}y_3}{\mathrm{d}x} = (1-U)y_3 + y_4, \tag{125}$$

$$x\frac{\mathrm{d}z_4}{\mathrm{d}x} = x^2\frac{\mathrm{d}y_4}{\mathrm{d}x} + x^2(1-U)\frac{\mathrm{d}y_3}{\mathrm{d}x} + xy_4 + xy_3(1-U) - x^2y_3\frac{\mathrm{d}U}{\mathrm{d}x}$$

$$= -AUxy_1 - U\frac{V_g}{\eta}z_2 + [l(l+1)+U(A-2)]xy_3 + 2(1-U)z_4$$

$$\frac{\mathrm{d}U}{\mathrm{d}r} = \frac{1}{M_r}\frac{\mathrm{d}(4\pi\rho_0r^3)}{\mathrm{d}r} - \frac{4\pi\rho_0r^3}{M_r^2}(4\pi\rho_0r^2)$$

$$= -\frac{U^2}{r} + \frac{3U}{r} + U\frac{\mathrm{d}\ln\rho_0}{\mathrm{d}r}$$

$$x\frac{\mathrm{d}U}{\mathrm{d}x} = -U^2 + 3U - U\left(A+V_g\right). \tag{126}$$

This gets messy, but we chug along:

$$\begin{aligned} x\frac{\mathrm{d}y_4}{\mathrm{d}x} = & -AUy_1 - UV_g(y_2-2y_3) + [l(l+1)+U(A-2)]y_3 \\ & + 2(1-U)[y_4+(1-U)y_3] - (1-U)[(1-U)y_3+y_4] - y_4 \\ & - y_3(1-U) + y_3\left(-U^2+3U-U(A+V_g)\right) \\ = & -UAy_1 - UV_gy_2 \\ & + \Big(2UV_g + l(l+1) + U(A-2) + 2(1-U)^2 - (1-U)^2 \\ & \quad -(1-U) - U^2 + 3U - U(A+V_g)\Big)y_3 \\ & + (2(1-U)-(1-U)-1)y_4 \\ = & -UAy_1 - UV_gy_2 + \left(UV_g + l(l+1)\right)y_3 + Uy_4. \end{aligned} \tag{127}$$

This has so many discrepancies in the signs, basically everything but the $l(l+1)$ is flipped. And as for the very last one:

$$
\begin{aligned}
x\frac{\mathrm{d}z_2}{\mathrm{d}x} &= x\eta(y_2-2y_3) + x^2\eta\frac{\mathrm{d}}{\mathrm{d}x}(y_2-y_3) + x^2(y_2-y_3)\frac{\mathrm{d}\eta}{\mathrm{d}x}\\
&= \left[l(l+1)-\eta A\right]xy_1 + (A-1)(y_2-2y_3)x\eta + \eta A x y_3,\\
\frac{\mathrm{d}\eta}{\mathrm{d}x} &= \frac{l(l+1)}{\bar{\omega}^2 M/R^3}\frac{\mathrm{d}}{\mathrm{d}x}\left(\frac{M_r}{r^3}\right)\\
&= \frac{l(l+1)R^4}{\bar{\omega}^2 M}\left(\frac{4\pi\rho_0 r^2}{r^3} - 3\frac{M_r}{r^4}\right)\\
&= \frac{l(l+1)R^4}{\bar{\omega}^2 M}\left(U\frac{M_r}{r^4} - 3\frac{M_r}{r^4}\right)\\
&= \frac{l(l+1)}{\bar{\omega}^2 c_1 x}(U-3) = \frac{\eta}{x}(U-3),\\
x\frac{\mathrm{d}y_2}{\mathrm{d}x} &= \left[\frac{l(l+1)}{\eta} - A\right]y_1 + (A-1)(y_2-2y_3) + Ay_3 - (y_2-2y_3)\\
&\quad + (1-U)y_3 + y_4 - (y_2-y_3)(U-3)\\
&= \left[c_1\bar{\omega}^2 - A\right]y_1 + (A-1-1-(U-3))y_2\\
&\quad + (2(1-A)+A+2+(1-U)+(U-3))y_3 + y_4\\
&= \left[c_1\bar{\omega}^2 - A\right]y_1 + (A+1-U)y_2 + 2y_3 + y_4. \qquad (128)
\end{aligned}
$$

At this point, it's clear we're dealing with something wrong: there's no y_4 coefficient in our expressions for $y_2'(x)$!

3.4.1 Why is Everything Wrong? Here's a Guess:

Maybe the Christiansen-Dalsgaard paper uses the opposite sign convention for Φ. In this case, the expressions would be amended to:

$$
\begin{aligned}
z_1 &= xy_1, && (129)\\
z_2 &= x\eta y_2, && (130)\\
z_3 &= -xy_3, && (131)\\
z_4 &= -xy_4 + xy_3(U-1), && (132)\\
x\frac{\mathrm{d}y_1}{\mathrm{d}x} &= \left(V_g-3\right)y_1 + \left(\frac{l(l+1)}{\bar{\omega}^2 c_1} - V_g\right)y_2 + V_g y_3, && (133)\\
x\frac{\mathrm{d}y_2}{\mathrm{d}x} &= \left[\frac{l(l+1)}{\eta} - A\right]y_1 + (A-1)y_2 - Ay_3 - y_2\\
&\quad + -y_2(U-3)\\
&= \left[\frac{l(l+1)}{\eta} - A\right]y_1 + (A+1-U)y_2 - Ay_3, && (134)\\
x\frac{\mathrm{d}y_3}{\mathrm{d}x} &= (1-U)y_3 + y_4, && (135)\\
x\frac{\mathrm{d}y_4}{\mathrm{d}x} &= AUy_1 + UV_g y_2 + [l(l+1) + U(A-2)]y_3\\
&\quad + 2(1-U)[y_4 + (1-U)y_3] - (1-U)[(1-U)y_3 + y_4] - y_4\\
&\quad - y_3(1-U) + y_3\left(-U^2 + 3U - U(A+V_g)\right)\\
&= AUy_1 + UV_g y_2 + \left[l(l+1) - U(V_g)\right]y_3 - Uy_4. && (136)
\end{aligned}
$$

Remarkably, this is in agreement with what we found. Thus, **we find that our derived equations are in agreement with Christiansen-Dalsgaard, but have small sign discrepancies with Le Bihan.**

Note: In hindsight, this had to just be a simple typo in the Le Bihan paper, otherwise the outer BC doesn't make sense; the coefficient of A^* in y_2' would be $y_1 - y_2 - y_3$, which doesn't vanish by the BCs.

Also, I finally found the last bug: I decided to drop the outermost point in my polytrope mesh since we have some divergent quantities there, but this is insufficient precision. Instead, we should just evaluate the polytropic model for $x \in [\epsilon, 1-\epsilon]$ and take $\epsilon = 10^{-4}$ or so.

3.5 Trick to pass in dense interpolant to Pickle!

We want better precision for our polytrope model, and the return of `solve_ivp` has a very good dense solution for θ. However, since V_g etc. are derived functions of θ, we need to find a way to pass this accurate interpolant to the actual stellar oscillation calculations. This is not easy, because we cannot just define lambdas that wrap the θ interpolant, since pickle won't let us multiprocess using them.

Instead, while computing the solution for the polytrope, let's also solve for V_g and the others at the same time! Thus, the call to `solve_ivp` will also return a dense interpolant for V_g and others, which we can pass into the `dydt` for the oscillation code.

To do this, we will need the expressions for

$$\frac{\mathrm{d}V_g}{\mathrm{d}\xi} = -\frac{n+1}{\Gamma_1}\left[\frac{\theta'}{\theta} + \frac{\xi}{\theta}\theta'' - \frac{\xi\left(\theta'\right)^2}{\theta^2}\right], \tag{137}$$

$$\frac{\mathrm{d}U}{\mathrm{d}\xi} = -\left[n\xi\theta^{n-1} + \frac{\theta^n}{\theta'} - \frac{\xi\theta^n\theta''}{(\theta')^2}\right], \tag{138}$$

$$\frac{\mathrm{d}c_1}{\mathrm{d}\xi} = \frac{\theta'(\xi_1)}{\xi_1}\left(\frac{1}{\theta'} - \frac{\xi\theta''}{(\theta')^2}\right), \tag{139}$$

$$\frac{\mathrm{d}A^*}{\mathrm{d}\xi} = -(n+1)\left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right)\left[\frac{\theta'}{\theta} + \frac{\xi}{\theta}\theta'' - \frac{\xi\left(\theta'\right)^2}{\theta^2}\right]. \tag{140}$$

Note that c_1 depends on the surface values for θ, so we can just normalize c_1 by its surface value, i.e.

$$\frac{\mathrm{d}\bar{c}_1}{\mathrm{d}\xi} = \frac{1}{\theta'} - \frac{\xi\theta''}{(\theta')^2}, \tag{141}$$

$$c_1 = \bar{c}_1/\bar{c}_1(\xi_1). \tag{142}$$

Recall that the central values for these are

$$V_g(\epsilon) = \frac{(n+1)\epsilon^2}{3\Gamma_1}, \tag{143}$$

$$U(\epsilon) \approx 3, \tag{144}$$

$$\bar{c}_1(\epsilon) \approx -3, \tag{145}$$

$$A^*(0) = \frac{(n+1)\epsilon^2}{3}\left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right). \tag{146}$$

This procedure seems to be giving the correct eigenfrequencies now!

3.6 Mode Order: Another sign error?

In Le Bihan, the mode order is given by

$$n \equiv \sum_{x_{z1}>0} {\color{red}+}\mathrm{sgn}\left(y_2\frac{\mathrm{d}y_1}{\mathrm{d}x}\right) + n_0, \tag{147}$$

where $y_1\left(x_{z_1}\right) = 0$ are the nodes of the radial displacement $\xi_r \equiv r y_1$. However, this seems to give the wrong sign for us, so we again check the citation for Christiensen-Dalsgaard 1994. Here, they define the sign of the mode to be whether the mode moves clockwise (g; negative) or counterclockwise (p; positive) in $\sqrt{\rho r}\{\xi_r, \xi_h\}$ coordinate space. In other words, if ξ_r has a zero, ξ_r' (the derivative) is negative, and $\xi_h > 0$, then motion is counterclockwise in this plane. We require this contribution to be positive to n, so there should be a missing negative sign.

Let's do a few checks. Do we agree that $z_2 \propto \xi_h$? Well, referring back to our derivation of the oscillation equations, we had Eq. (56) where

$$-\omega^2\vec{\xi}_\perp = -\vec{\nabla}_\perp\left(\frac{\delta P}{\rho_0} + \delta\Phi\right). \tag{148}$$

Well, actually, I noticed in my ASTRO6531 notes that

$$\vec{\xi}_\perp = \xi_\perp(r) r \vec{\nabla}_\perp Y_{lm}. \tag{149}$$

This directly pops out that $\xi_\perp(r) r = \left(\delta P/\rho_0 + \delta\Phi\right)$. This seems clear enough in hindsight:

$$-\frac{\Delta\rho}{\rho_0} \propto Y_{lm} = \vec{\nabla}\cdot\vec{\xi} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\xi_r\right) + \vec{\nabla}_\perp\cdot\xi_\perp. \tag{150}$$

In order for the last quantity to be $\propto Y_{lm}$, we of course need $\xi_\perp \propto \vec{\nabla}_\perp Y_{lm}$.

3.7 Brief Aside: p vs g modes

Let's also think about why we identify these signs with p and g modes respectively. Following my notes from Dong's ASTRO6531, we begin with Eqs. (62–64) and take the Cowling approximation + WKB limit. Then $\delta\Phi \to 0$ and $\partial_r \to ik_r$, and we obtain ($\partial_r(r^2\xi_r) = ik_r r^2\xi_r$ is the best prescription)

$$i\xi_r k_r - \frac{g}{c_s^2}\xi_r + \left(1 - \frac{l(l+1)c_s^2}{r^2\omega^2}\right)\frac{\delta P}{c_s^2\rho_0} = 0, \tag{151}$$

$$i\frac{1}{\rho_0}\delta P k_r + \frac{g}{\rho_0 c_s^2}\delta P + \left(N^2 - \omega^2\right)\xi_r = 0, \tag{152}$$

$$\begin{aligned} ik_r - \frac{g}{c_s^2} - \left(1 - \frac{l(l+1)c_s^2}{r^2\omega^2}\right)\frac{N^2 - \omega^2}{c_s^2\rho_0\left(\frac{1}{\rho_0}ik_r + \frac{g}{\rho_0 c_s^2}\right)} &= 0 \\ ik_r - \frac{g}{c_s^2} - \frac{\left(\omega^2 - L^2\right)\left(N^2 - \omega^2\right)}{\omega^2(ik_r c_s^2 + g)} &= \\ \frac{\left(\omega^2 - L^2\right)\left(\omega^2 - N^2\right)}{\omega^2} &= k_r^2 c_s^2 + \frac{g^2}{c_s^2}. \end{aligned} \tag{153}$$

Note that $L^2 = l(l+1)c_s^2/r^2$. I'm not sure how my ASTRO6531 notes can ignore the g^2/c_s^2 term, but maybe it's small? Additionally, the WKB limit should only be valid for large k_r (short wavelength), so this also checks out. Then, in this case, we find the usual result that $\omega > N, \omega > L$ for p-modes and $\omega < N, \omega < L$ for g-modes.

What does this mean in terms of the wavefunctions? Well, we found that

$$\begin{aligned}\delta P &= \frac{\omega^2 - N^2}{\frac{1}{\rho_0} i k_r} \xi_r \\ &= -i\rho_0 \frac{\omega^2 - N^2}{k_r} \xi_r. \end{aligned} \tag{154}$$

Thus, if $\omega > N$, then δP *trails* ξ_r in phase, and if $\omega < N$, δP *leads* ξ_r in phase. Thus, for p-modes, δP should trail ξ_r in phase, and so when ξ_r crosses zero (from positive to negative), δP should still be positive. Thus, indeed, our sign convention is correct and Le Bihan is missing a negative sign!

Finally, note that

$$n_0 = \begin{cases} 1 & y_1(x_{\text{in}})/y_2(x_{\text{in}}) < 0 \\ 0 & y_1(x_{\text{in}})/y_2(x_{\text{in}}) > 0, \end{cases} \tag{155}$$

is only relevant when we start solving for an envelope rather than the full body, since if $x_{\text{in}} = 0$ then the inner BC always forces $y_1/y_2 > 0$ and thus $n_0 = 0$.

4 Structure Models

In general, to get a structure, we solve the Poisson equation and that of hydrostatic equilibrium

$$\vec{\nabla} P = -\rho \vec{\nabla}\Phi, \tag{156}$$

$$\nabla^2 \Phi = 4\pi G\rho. \tag{157}$$

For a spherically symmetric structure, this becomes

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\rho \frac{\mathrm{d}\Phi}{\mathrm{d}r}, \tag{158}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2 \frac{\mathrm{d}\Phi}{\mathrm{d}r}\right) = 4\pi G\rho. \tag{159}$$

This constitutes two equations for the three fields $\left(P(r), \rho(r), \Phi(r)\right)$. Thus, we need one more equation to close the system.

4.1 Polytrope

Again, let's do this as a sanity check. A polytropic EoS gives

$$P = K\rho^\Gamma, \tag{160}$$

$$\Gamma K \rho^{\Gamma-2}\frac{\mathrm{d}\rho}{\mathrm{d}r} = -\Phi', \tag{161}$$

$$\frac{\mathrm{d}\Phi}{\mathrm{d}r} \equiv \Phi', \tag{162}$$

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\Phi'\right) = 4\pi G\rho. \tag{163}$$

Thus, we now have three first-order ODEs for the three fields $\rho(r)$, $\Phi(r)$, and $\Phi'(r)$. For a given $\Gamma > 6/5$ (corresponding to $n < 5$), we integrate out until $\rho = 0$.

4.2 General EoS

In general, we are given a more complex EOS. Suppose for now that we have a single species (easily generalizable to uniform composition probably) and a uniform entropy S_0. The EOS for this single species is a 4-parameter table mapping any two of $\left(P, \rho, T, S\right)$ to the other two. Thus, we should be able to just read off $P\left(\rho, S_0\right)$ for a given uniform entropy S_0. Does this work?

Note that if this is the case, then $\mathrm{d}P/\mathrm{d}\rho = (\partial P/\partial \rho)_S$, and $N^2 = 0$. This is in accordance with our expectations.