1 11/29/22

We follow Appendicies B-C of Le Bihan & Burrows 2013. Note that we can seek two independent solutions to the ODEs because it is not a simple Sturm-Liouville form problem: the structure is quite complex, so we aren't guaranteed simple roots. Perhaps it will be fun to try and do the properties of Sturm-Liouville proofs again someday...

Update: we are able to get seemingly correct oscillation modes with dummy stellar profiles, but we don't have the correct non-singular solution near the core. Due to this, the matrix determinants in the core-surface solution matching are large, and so we suffer from some loss of numerical precision.

2 12/07/22

2.1 Polytrope Solution

Let's review how to obtain the structure of a polytrope. The equations of hydrostatic equilibrum are:

$$\frac{\mathrm{d}P}{\mathrm{d}r} = -\frac{GM_r}{r^2}\rho, \tag{1}$$
$$\frac{\mathrm{d}M_r}{\mathrm{d}r} = 4\pi r^2\rho, \tag{2}$$
$$P = K\rho^\Gamma = K\rho^{1+1/n}. \tag{3}$$

Here, K is constant. This can easily be rearranged to obtain

$$\frac{\mathrm{d}}{\mathrm{d}r}\left(\frac{r^2}{\rho}\frac{\mathrm{d}P}{\mathrm{d}r}\right) = \frac{\mathrm{d}}{\mathrm{d}r}(-GM_r) = -G4\pi r^2\rho. \tag{4}$$

Then, since $\partial P/\partial r = K\rho^{\Gamma-1}\Gamma\mathrm{d}p/\mathrm{d}r$, we have that

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2K\rho^{\Gamma-2}\Gamma\frac{\mathrm{d}\rho}{\mathrm{d}r}\right) = -G4\pi\rho. \tag{5}$$

Now, let's guess that $\rho(r) = \rho_c\theta^p(r)$, where p must be chosen such that $\rho^{\Gamma-2}\mathrm{d}\rho/\mathrm{d}r \propto \mathrm{d}\theta/\mathrm{d}r$. This can be seen to yield that $p = 1/(\Gamma-1) = n$, so then

$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(\Gamma r^2Kn\rho_c^{\Gamma-1}\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -G4\pi\rho_c\theta^n, \tag{6}$$
$$\frac{1}{r^2}\frac{\mathrm{d}}{\mathrm{d}r}\left(r^2\frac{\mathrm{d}\theta}{\mathrm{d}r}\right) = -\theta^n\times\left(\frac{G4\pi\rho_c^{1-1/n}}{K(n+1)}\right) \equiv -\theta^n\frac{1}{\alpha^2}. \tag{7}$$

Let's define $\xi \equiv r/\alpha$, then the Lane Emden-Equation reduces to (let primes denote $\mathrm{d}/\mathrm{d}\xi$)

$$\theta'' + \frac{2\theta'}{\xi} + \theta^n = 0. \tag{8}$$

Note that the ICs for θ are

$$\theta(0) = 1, \qquad \theta'(0) = 0. \tag{9}$$

It's clear that $\theta'(0) = 0$ is required for the equation to be non-singular. Let's guess that $\theta'(\xi \ll 1) \propto \xi$, then upon inspection we find that

$$\theta'(\xi \ll 1) = -\xi/3 \tag{10}$$

is correct, which is useful later. In addition, $\theta(\xi_1) = 0$ exists if $n < 5$, so ξ_1 is the outer boundary.

From here, we seek to solve for all of the quantities in closed form; first the stellar properties, then the relevant dimensionless quantities for the oscillation equations. First, it is clear that our natural parameterization of the stellar structure above relies on K and ρ_c, while we want to express these in terms of M and R.

$$\alpha = \left(\frac{K(n+1)}{4\pi G\rho_c^{1-1/n}}\right)^{1/2}, \tag{11}$$

$$C \equiv \left(\frac{K(n+1)}{4\pi G}\right)^{1/2} = \alpha\rho^{\frac{n-1}{2n}}, \tag{12}$$

$$\begin{aligned} R &= \alpha\xi_1 \\ &\equiv C^{1/2}\rho_c^{\frac{1-n}{2n}}\xi_1, \end{aligned} \tag{13}$$

$$\begin{aligned} M &= \int^{R} 4\pi r^2\rho(r)\,\mathrm{d}r \\ &= 4\pi\rho_c\alpha^3\int^{\xi_1}\xi^2\theta^n(\xi)\,\mathrm{d}\xi \\ &= 4\pi\rho_c\alpha^3\int^{\xi_1} -\frac{\mathrm{d}}{\mathrm{d}\xi}\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\mathrm{d}\xi \\ &= 4\pi\rho_c^{\frac{3-n}{2n}}C^{3/2}\left[-\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\right]_{\xi_1}, \end{aligned} \tag{14}$$

$$\begin{aligned} M &= 4\pi\rho_c^{\frac{3-n}{2n}}\left(\frac{R}{\rho_c^{(1-n)/(2n)}\xi_1}\right)^3\left[-\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\right]_{\xi_1} \\ &= 4\pi\rho_c\left(\frac{R}{\xi_1}\right)^3\left[-\left(\xi^2\frac{\mathrm{d}\theta}{\mathrm{d}\xi}\right)\right]_{\xi_1} \end{aligned}$$

$$\rho_c = \left(\frac{M}{4\pi R^3/3}\right)\left(-\frac{\xi_1}{3\theta'(\xi_1)}\right), \tag{15}$$

$$\begin{aligned} R &= \left(\frac{K(n+1)}{4\pi G}\right)^{1/2}\rho_c^{\frac{1-n}{2n}}\xi_1 \\ K &= \frac{R^2}{\xi_1^2}\rho_c^{\frac{n-1}{n}}\frac{4\pi G}{n+1}. \end{aligned} \tag{16}$$

As such, we have $(K,\rho_c) \leftrightarrow (M,R)$, so we use K,ρ_c for simplicity. The rest follow:

$$\rho(\xi) = \rho_c\theta^n(\xi), \tag{17}$$

$$P(\xi) = K\rho_c^\Gamma\theta^{n+1}(\xi) \propto \theta^{n+1}(\xi), \tag{18}$$

$$\begin{aligned} M_r(\xi) &= \int^{\xi\alpha} 4\pi r^2\rho\,\mathrm{d}r \\ &= 4\pi\alpha^3\rho_c\int^{\xi}\xi^2\theta^n(\xi)\,\mathrm{d}\xi \\ &= -4\pi\alpha^3\rho_c\xi^2\theta'(\xi), \end{aligned} \tag{19}$$

$$\begin{aligned} g(\xi) &= \frac{GM_r}{(\xi\alpha)^2} \\ &= -4\pi G\alpha\rho_c\theta'(\xi), \end{aligned} \tag{20}$$

$$\begin{aligned} U(\xi) &= \frac{4\pi\rho(\xi\alpha)^3}{M_r} \\ &= -\frac{\theta^n\xi}{\theta'(\xi)} \end{aligned} \tag{21}$$

$$\begin{aligned} c_1 &= \frac{r/R}{M_r/M} \\ &= \frac{1}{(\xi/\xi_1)(\theta'(\xi)/\theta'(\xi_1))}. \end{aligned} \tag{22}$$

Note that $U(\xi \to 1) = 3$ and $c_1(\xi \to 0) \propto -1/\xi^2$.

There's a small point of contention about c_s and N, since they depend on the adiabatic index Γ_1 and not just the polytropic index Γ. It takes a bit of digging, but Mullan & Ulrich 1988 point out that $\Gamma_1 = 5/3$ is used in all of their models. Thus, we have the remaining quantities in the Le Bihan formulation:

$$\begin{aligned}
c_s^2(\xi) &= \Gamma_1 \frac{P}{\rho} = \Gamma_1 K \rho_c^{1/n} \theta(\xi), && (23)\\
N^2(\xi) &= g^2 \left(\frac{\mathrm{d}\rho}{\mathrm{d}P} - \frac{1}{c_s^2}\right)\\
&= \left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right) g^2 \frac{\rho}{P}\\
A^* &= \frac{N^2 r}{g} = \frac{\rho g r}{P}\left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right)\\
&= \left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right)\left(-4\pi G \alpha \rho_c \theta'(\xi)\right)(\alpha\xi)\frac{1}{K\rho_c^{1/n}\theta(\xi)}\\
&= \left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right)\left(-4\pi G \rho_c^{1-1/n}\right)\alpha^2 \frac{\xi\theta'(\xi)}{K\theta(\xi)}\\
&= -\left(\frac{1}{\Gamma} - \frac{1}{\Gamma_1}\right)\frac{(n+1)\xi\theta'(\xi)}{\theta(\xi)}\\
V_g &= \frac{gr}{c_s^2}\\
&= A^* \frac{1/\Gamma_1}{1/\Gamma - 1/\Gamma_1}\\
&= -\frac{1}{\Gamma_1}\frac{(n+1)\xi\theta'(\xi)}{\theta(\xi)} && (24)
\end{aligned}$$

2.2 Oscillation Equations

2.2.1 Physical Form

Let's rederive the oscillation equations too, all the way through the nondimensionalized form (we did this in Dong's class). We start with the Euler Equations and Poisson equation

$$\begin{aligned}
\frac{\partial \rho}{\partial t} + \vec{\nabla}\cdot\left(\rho\vec{v}\right) &= 0, && (25)\\
\frac{\partial \vec{u}}{\partial t} + \left(\vec{u}\cdot\vec{\nabla}\right)\vec{u} + \frac{\vec{\nabla}P}{\rho} &= -\vec{\nabla}\Phi, && (26)\\
\nabla^2\Phi &= 4\pi G\rho. && (27)
\end{aligned}$$

We expand $x = x_0 + \delta x$ for each of $x = P, \rho, \Phi$. Then we assume an adiabatic EOS

$$\frac{\Delta P}{\Delta \rho} = c_s^2, \tag{28}$$

where Δ is the Lagrangian perturbation, and we have the useful relation

$$\delta\vec{v} = \frac{\mathrm{d}\vec{\xi}}{\mathrm{d}t}, \qquad \Delta x = \delta x + \left(\vec{\xi}\cdot\vec{\nabla}\right)x. \tag{29}$$

equilibrium $\mathrm{d}P_0/\mathrm{d}r = -\rho_0 g$. The goal is to take FTs and cast everything into ξ_r, δP, and $\delta\Phi$ with only three equations of motion.

Then we first tackle the density equation. There is no leading order term, so we directly analyze the perturbative component. There are two forms that I think might be useful:

$$\begin{aligned}
\cancel{\frac{\partial \rho_0}{\partial t}} + \frac{\partial(\delta\rho)}{\partial t} + \vec{\nabla}\cdot\left(\rho_0\delta\vec{u}\right) &= 0,\\
\delta\rho &= -\vec{\nabla}\cdot\left(\rho_0\vec{\xi}\right), && (30)\\
\frac{\mathrm{d}(\delta\rho)}{\mathrm{d}t} &= -\rho_0\left(\vec{\nabla}\cdot(\delta\vec{u})\right),\\
\Delta\rho &= -\rho_0\left(\vec{\nabla}\cdot\vec{\xi}\right). && (31)
\end{aligned}$$

Next, we tackle the momentum equation. The zeroth order term is just hydrostatic equilibrium, corresponding to $\vec{\nabla}P_0/\rho_0 = -\vec{\nabla}\Phi_0$, which gives $g = \vec{\nabla}\Phi_0$. The first order term is

$$\begin{aligned}
\frac{\partial(\delta\vec{u})}{\partial t} + \frac{\vec{\nabla}(\delta P)}{\rho_0} - \frac{(\delta\rho)\vec{\nabla}P_0}{\rho_0^2} &= -\vec{\nabla}(\delta\Phi),\\
\frac{\partial^2\vec{\xi}}{\partial t^2} + \frac{\vec{\nabla}(\delta P)}{\rho_0} - \frac{\delta\rho\vec{\nabla}P_0}{\rho_0^2} &= \vec{\nabla}(\delta\Phi). && (32)
\end{aligned}$$

We go ahead and substitute $\partial_t \to i\omega$. The perpendicular and parallel components of Eq. (32) are:

$$\begin{aligned}
\hat{r} : -\omega^2\xi_r + \frac{1}{\rho_0}\frac{\partial}{\partial r}(\delta P) + \frac{\delta r}{\rho_0}g &= -\frac{\partial}{\partial r}\delta\Phi, && (33)\\
\perp : -\omega^2\vec{\xi}_\perp + \frac{\vec{\nabla}_\perp(\delta P)}{\rho_0} &= -\vec{\nabla}(\delta\Phi)\\
-\omega^2\vec{\xi}_\perp &= -\vec{\nabla}\left(\frac{\delta P}{\rho_0} + \delta\Phi\right). && (34)
\end{aligned}$$

Combining the perpendicular component with Eq. (31), we can rewrite (and substitute in Y_{lm}'s)

$$
\begin{aligned}
\Delta\rho &= -\rho_0\left(\frac{1}{r^2}\frac{\partial^2}{\partial r^2}\left(r^2\xi_r\right)+\nabla_\perp^2\left(\frac{\delta P}{\rho_0\omega^2}+\frac{\delta\Phi}{\omega^2}\right)\right),\\
-\frac{\Delta\rho}{\rho_0} &= \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\xi_r\right)-\frac{l(l+1)}{r^2\omega^2}\left(\frac{\delta P}{\rho_0}+\delta\Phi\right),\\
&= -\frac{\Delta P}{c_s^2\rho_0} = -\frac{\delta P}{c_s^2\rho_0}+\frac{\xi_r g}{c_s^2}.
\end{aligned}
\tag{35}
$$

To rewrite the radial component of the momentum equation, we first define N^2 and rewrite:

$$
N^2 \equiv g^2\left(\frac{\mathrm{d}\rho_0}{\mathrm{d}P_0}-\frac{1}{c_s^2}\right),\tag{36}
$$

$$
\begin{aligned}
\frac{N^2}{g^2}+\frac{1}{c_s^2} &= \frac{\mathrm{d}\rho_0/\mathrm{d}r}{\mathrm{d}P_0/\mathrm{d}r},\\
\delta\rho &= \Delta\rho-\xi_r\frac{\partial\rho_0}{\partial r}\\
&= \frac{\Delta P}{c_s^2}-\xi_r\left(\frac{N^2}{g^2}+\frac{1}{c_s^2}\right)(-\rho_0 g)\\
&= \frac{\delta P}{c_s^2}+\xi_r\frac{N^2}{g^2}.
\end{aligned}
\tag{37}
$$

Then the radial component is straightforwardly

$$
\frac{1}{\rho_0}\frac{\partial}{\partial r}\delta P+\frac{\partial}{\partial r}(\delta\Phi)-\omega^2\xi_r+\frac{g}{\rho_0}\left(\frac{\delta P}{c_s^2}+\xi_r\frac{N^2\rho_0}{g}\right)=0.\tag{38}
$$

Finally, the Poisson equation is simplified straightforwardly (with spherical harmonic dependence) to

$$
\begin{aligned}
\nabla^2(\delta\Phi) &= 4\pi G\delta\rho,\\
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(\delta\Phi)\right)-\frac{l(l+1)}{r^2}(\delta\Phi) &= 4\pi G\rho_0\left(\frac{\delta P}{\rho_0 c_s^2}+\xi_r\frac{N^2}{g}\right).
\end{aligned}
\tag{39}
$$

Thus, the three equations for radial oscillations are a combination of: the perpendicular momentum equation in combination with the density equation, the radial momentum equation, and the Poisson equation. For posterity, they are (with slight rearrangements):

$$
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\xi_r\right)-\frac{g}{c_s^2}\xi_r+\left(1-\frac{l(l+1)c_s^2}{r^2\omega^2}\right)\frac{\delta P}{c_s^2\rho_0}-\frac{l(l+1)}{r^2\omega^2}\delta\Phi=0,\tag{40}
$$

$$
\frac{1}{\rho_0}\frac{\partial}{\partial r}\delta P+\frac{\partial}{\partial r}\delta\Phi+\frac{g}{\rho_0 c_s^2}\delta P+\left(N^2-\omega^2\right)\xi_r=0,\tag{41}
$$

$$
\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial}{\partial r}(\delta\Phi)\right)-\frac{l(l+1)}{r^2}(\delta\Phi)-4\pi G\rho_0\left(\frac{\delta P}{\rho_0 c_s^2}+\xi_r\frac{N^2}{g}\right)=0.\tag{42}
$$

This constitutes 3 ODEs for 3 quantities, but one is second order. To make a reduction to first order, $(\delta\Phi)'$ constitutes a fourth variable.

2.3 Nondimensional Form

To nondimensionalize, we use the variables recommended in Unno's 1979 book (and by Le Bihan). They are:

$$
y_1 = \frac{\xi_r}{r},\tag{43}
$$

$$
y_2 = \frac{1}{gr}\left(\frac{\delta P}{\rho_0}+\delta\Phi\right),\tag{44}
$$

$$
y_3 = \frac{1}{gr}\delta\Phi,\tag{45}
$$

$$
y_4 = \frac{1}{g}\frac{\mathrm{d}}{\mathrm{d}r}\delta\Phi,\tag{46}
$$

$$
x = \frac{r}{R}.\tag{47}
$$

To make this substitution, it helps not to use the fully-simplified forms initially. For instance, to simplify Eq. (40), it helps to start with the earlier form (primes denote ∂_r; we will explicitly denote x derivatives)

$$
\begin{aligned}
-\frac{\delta P}{c_s^2\rho_0}+\frac{\xi_r g}{c_s^2} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\xi_r)-\frac{l(l+1)}{r^2\omega^2}\left(\frac{\delta P}{\rho_0}+\delta\Phi\right),\\
-\frac{1}{c_s^2}(gry_2-\delta\Phi)+\frac{ry_1g}{c_s^2} &= \frac{1}{r^2}\frac{\partial}{\partial r}(r^3y_1)-\frac{l(l+1)}{r^2\omega^2}gry_2,\\
-\frac{gr}{c_s^2}(y_2-y_3)+\frac{ry_1g}{c_s^2} &= 3y_1+ry_1'-\frac{l(l+1)}{r^2\omega^2}gry_2,\\
ry_1' &= -\frac{gr}{c_s^2}(y_2-y_3)+\frac{y_1gr}{c_s^2}+\frac{l(l+1)}{r^2\omega^2}gry_2-3y_1\\
xy_1'(x) &= \left(\frac{gr}{c_s^2}-3\right)y_1+\left[\frac{l(l+1)g}{r\omega^2}-\frac{gr}{c_s^2}\right]y_2+\frac{gr}{c_s^2}y_3.
\end{aligned}
\tag{48}
$$

To simplify Eq. (41), we do

$$\delta P = (y_2 - y_3) g r \rho_0, \tag{49}$$

$$0 = \frac{1}{\rho_0}\frac{\partial((y_2-y_3)gr\rho_0)}{\partial r} + \frac{\partial(gry_2)}{\partial r} + \frac{g(y_2-y_3)gr\rho_0}{\rho_0 c_s^2} + \left(N^2-\omega^2\right) r y_1,$$

$$= \frac{\rho_0'(y_2-y_3)gr}{\rho_0} + \frac{\partial(gry_2)}{\partial r} + \frac{g(y_2-y_3)gr\rho_0}{\rho_0 c_s^2} + \left(N^2-\omega^2\right) r y_1,$$

$$r y_2' = -\frac{\rho_0'(y_2-y_3)r}{\rho_0} - y_2\frac{(gr)'}{g} - \frac{(y_2-y_3)gr}{c_s^2} - \frac{\left(N^2-\omega^2\right)}{g} r y_1,$$

$$x y_2'(x) = \frac{\left(\omega^2-N^2\right)}{g} r y_1 - \left(\frac{\rho_0' r}{\rho_0} + \frac{(gr)'}{g} + \frac{gr}{c_s^2}\right) y_2 + \left(\frac{\rho_0' r}{\rho_0} + \frac{gr}{c_s^2}\right) y_3,$$

$$\frac{\rho_0' r}{\rho_0} = \left(\frac{N^2}{g^2} + \frac{1}{c_s^2}\right)(-gr), \tag{50}$$

$$x y_2'(x) = \frac{\left(\omega^2-N^2\right)}{g} r y_1 + \left(\frac{N^2 r}{g} - \frac{(gr)'}{g}\right) y_2 - \frac{N^2 r}{g} y_3,$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial \Phi_0}{\partial r}\right) = 4\pi G\rho_0,$$

$$\frac{\partial}{\partial r}\left(r^2 g\right) = 4\pi G\rho_0 r^2,$$

$$r(gr)' = 4\pi G\rho_0 r^2 - gr,$$

$$\frac{(gr)'}{g} = \frac{4\pi\rho_0 r^3}{M_r} - 1, \tag{51}$$

$$x y_2'(x) = \left[\frac{\omega^2 r}{g} - \frac{N^2 r}{g}\right] y_1 + \left(\frac{N^2 r}{g} + 1 - \frac{4\pi\rho_0 r^3}{M_r}\right) y_2 - \frac{N^2 r}{g} y_3. \tag{52}$$

Eq. (42) also follows:

$$0 = \frac{1}{r^2}\left(r^2 g y_4\right)' - \frac{l(l+1)}{r^2} g r y_3 - 4\pi G\rho_0\left(\frac{(y_2-y_3)gr\rho_0}{\rho_0 c_s^2} + r y_1\frac{N^2}{g}\right),$$

$$0 = (gr^2)'\frac{y_4}{r} + g r y_4' - (l(l+1)) g y_3 - 4\pi G\rho_0 r\left(\frac{(y_2-y_3)gr}{c_s^2} + y_1\frac{N^2 r}{g}\right),$$

$$r y_4' = -(4\pi G\rho_0 r^2)\frac{y_4}{gr} + (l(l+1)) y_3 + \frac{4\pi G\rho_0 r}{g}\left(\frac{(y_2-y_3)gr}{c_s^2} + y_1\frac{N^2 r}{g}\right),$$

$$= \frac{4\pi\rho_0 r^3}{M_r}\left[-y_4 + \frac{(y_2-y_3)gr}{c_s^2} + \frac{N^2 r}{g} y_1\right] + l(l+1) y_3. \tag{53}$$

Finally, the implicit equation is

$$(g r y_3)' = g y_4,$$

$$r y_3' = y_4 - \frac{(gr)'}{g} y_3$$

$$= y_4 - \left[\frac{4\pi\rho_0 r^3}{M_r} - 1\right] y_3. \tag{54}$$

Defining the constants

$$V_g = \frac{gr}{c_s^2}, \tag{55}$$

$$U = \frac{4\pi\rho_0 r^3}{M_r}, \tag{56}$$

$$c_1 = \frac{(r/R)^3}{M_r/M}, \tag{57}$$

$$\bar{\omega}^2 = \frac{\omega^2 R^3}{GM}, \tag{58}$$

$$A^* = \frac{N^2 r}{g}, \tag{59}$$

then we obtain

$$\frac{g}{r\omega^2} = \frac{GM_r/r^2}{rGM\bar{\omega}^2/R^3} = \frac{M_r/M}{x^3}\frac{1}{\bar{\omega}^2} = \frac{1}{c_1\bar{\omega}^2},$$

$$x y_1' = \left(V_g - 3\right) y_1 + \left[\frac{l(l+1)}{c_1\bar{\omega}^2} - V_g\right] y_2 + V_g y_3, \tag{60}$$

$$x y_2' = \left[c_1\bar{\omega}^2 - A^*\right] y_1 + \left(A^* + 1 - U\right) y_2 - A^* y_3, \tag{61}$$

$$x y_3' = (1-U) y_3 + y_4, \tag{62}$$

$$x y_4' = U A^* y_1 + U V_g y_2 + \left[l(l+1) - U V_g\right] y_3 - U y_4. \tag{63}$$

The red parts differ from the Le Bihan paper. Notably, c_1 differs from the Le Bihan paper, and we should obtain

$$c_1 = \frac{(r/R)^3}{M_r/M} = \frac{\xi/\xi_1}{\theta'(\xi)/\theta'(\xi_1)}. \tag{64}$$

Note that in this case, $c_1(\xi \to 0) = -3\theta'(\xi_1)/\xi_1$ is finite. As such, we have fully constrained the eigenvalue problem for $\{y_i(x), \omega\}$ in terms of the fields V_g, c_1, A^*, and U.

2.4 Boundary Conditions / Behavior

Near $x = 0$, $V_g, A^* \propto x^2$ while $U \approx 3$ and $c_1 \approx -3\mathrm{d}\theta(1)/\mathrm{d}x$. The inner BCs are standard (we read from Le Bihan, but Fuller 2017 also seems to have similar):

$$0 = \xi_r - \frac{l}{\omega^2 r}\left(\frac{\delta P}{\rho_0} + \delta\Phi\right), \tag{65}$$

$$0 = \frac{\mathrm{d}(\delta\Phi)}{\mathrm{d}r} - \frac{l}{r}\delta\Phi. \tag{66}$$

These nondimensionalize to

$$\begin{aligned} 0 &= r y_1 - \frac{l}{\omega^2 r} g r y_2 \\ &= y_1 - l\frac{GM_r/r^2}{rGM\bar{\omega}^2/R^3} y_2 \\ &= y_1 - \frac{l}{c_1\bar{\omega}^2} y_2, \end{aligned} \tag{67}$$

$$\begin{aligned} 0 &= g y_4 - \frac{l}{r} g r y_3 \\ &= y_4 - l y_3. \end{aligned} \tag{68}$$

In order then for $xy_3' \to 0$ as $x \to 0$, we require $xy_3' \propto (l-2)y_3$ or that

$$y_3(x \ll 1) \sim x^{l-2}. \tag{69}$$

In addition, for $xy_1' \to 0$, we obtain $-3y_1 + (l+1)y_1 = xy_1'$, so we also find

$$y_1(x \ll 1) \sim x^{l-2}. \tag{70}$$

What about the other two y_i' equations? Well, $xy_2' = l y_2 - 2y_2$, so $y_2 \sim x^{l-2}$ as well, and $xy_4' = (l+1)y_4 - 3y_4$, so $y_4 \sim x^{l-2}$ at last. These two BCs constrain two DOF; the overall normalization is a third DOF, and the ratio y_3/y_1 is a fourth. So we choose two different y_3/y_1 in constructing our solutions.

As for the outer BCs, they are (small correction to Le Bihan; consistent with Fuller)

$$0 = \frac{\mathrm{d}(\delta\Phi)}{\mathrm{d}r} + \frac{(l+1)}{r}\delta\Phi, \tag{71}$$

$$0 = \Delta P = \delta P - \xi_r g \rho_0. \tag{72}$$

Nondimensionalizing, we obtain

$$\begin{aligned} 0 &= g y_4 + \frac{(l+1)}{r} g r y_3 \\ &= y_4 + (l+1)y_3, \end{aligned} \tag{73}$$

$$\begin{aligned} 0 &= g r \rho_0 (y_2 - y_3) - g r \rho_0 y_1 \\ &= y_1 - y_2 + y_3. \end{aligned} \tag{74}$$

Near the surface, V_g and A^* diverge, $U \to 0$, and $c_1 \to 1$. First, we note that $xy_3' = y_3 + y_4 = y_3 - (l+1)y_3 = -l y_3$, and so

$$y_3(x \to 1) \sim x^{-l}. \tag{75}$$

For y_1, we look at the singular terms in xy_1' and xy_2', but $xy_1' = V_g(y_1 - y_2 + y_3)$ and $xy_2' = -A^*(y_1 - y_2 + y_3)$ both identically cancel (which is good, since the LHS doesn't have a divergence to cancel these out!). So, doing a bit more work, the other equations are

$$xy_1' = -3y_1 + \frac{l(l+1)}{\bar{\omega}^2} y_2, \tag{76}$$

$$xy_2' = \bar{\omega}^2 y_1 + y_2, \tag{77}$$

$$xy_4' = UA^* y_1 + UV_g(y_2 + y_4/(l+1)) - l y_4. \tag{78}$$

Note that

$$UV_g \propto UA^* = \frac{4\pi\rho_0 R^3}{M_r}\frac{GM_r/R}{c_s^2} \propto \frac{4\pi G\rho_c\theta^n(1)R^2}{\theta(1)} \propto \theta^{n-1}(1). \tag{79}$$

This is zero if $n > 1$. Thus, the last equation shows $y_4 \propto x^{-l}$ as well. If we also want $y_1, y_2 \propto x^p$, then we need ($z \equiv y_1\bar{\omega}^2$)

$$pz = -3z + l(l+1)y_2, \tag{80}$$

$$py_2 = z + y_2, \tag{81}$$

$$(p+3)(p-1) = l(l+1). \tag{82}$$

This doesn't work! Thus, y_1 and y_2 have different power law behaviors at the edge, so we should instead write

$$p_1 z = -3z + l(l+1)y_2, \tag{83}$$

$$p_2 y_2 = z + y_2, \tag{84}$$

$$(p_1+3)(p_2-1) = l(l+1). \tag{85}$$

Then either $p_2 = l+1, p_1 = l-2$ or $p_2 = l+2$ and $p_1 = l-3$. Le Bihan says that the correct one to use is

$$y_1 \sim x^{l-2}, \quad y_2 \sim x^{l+1}. \tag{86}$$