Eccentric Tides

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1 Kushnir et. al., 2016

We coarsely follow the derivation of Kushnir et. al., 2016 (KZ) to express the traveling wave regime of dynamical tides in high-mass stars (convective core, radiative envelope) in analytical form.

1.1 Plane Parallel Case

We will consider IGW in a plane parallel atmosphere in the Boussinesq approximation. Consider buoyancy frequency

$$N^2 = -g\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{g}{c_s^2}\right), \tag{1}$$

where $c_s \to \infty$ is the sound speed in the fluid. Then the Boussinesq equations can be written in terms of some *buoyancy* variable b

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \frac{\vec{\nabla}P}{\rho_0} + b\hat{z}, \tag{2a}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} = -N^2 u_z, \tag{2b}$$

$$\vec{\nabla}\cdot\vec{u} = 0. \tag{2c}$$

Note that $b \equiv -\frac{\rho'}{\rho_0}g$, as can be verified via direct substitution into the Euler equations:

$$0 = \frac{\mathrm{D}\rho'}{\mathrm{D}t} + \vec{u}\cdot\vec{\nabla}\rho_0 = \frac{\mathrm{D}\rho'}{\mathrm{D}t} - u_z\frac{N^2}{g}\rho', \tag{3a}$$

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \frac{\vec{\nabla}P'}{\rho_0} - \frac{\rho'}{\rho_0^2}\vec{\nabla}P_0 = \frac{\vec{\nabla}P'}{\rho_0} - \frac{\rho'}{\rho_0}g\hat{z}. \tag{3b}$$

These equations can be solved for u_z, or we can just recall the IGW dispersion relation $\omega^2 k^2 = N^2 k_\perp^2$ and write down PDE

$$\frac{\partial^2}{\partial t^2}\nabla^2 u_z = -N^2\nabla_\perp^2 u_z. \tag{4}$$

Now, we might recall that in tidally-forced stars, ω the tidal forcing frequency obeys $\omega \ll N$, or $k \gg k_\perp$. But the tidal potential, the quadrupolar expansion of the gravitational perturbation from the companion, has no quickly-varying directions, or can only excite $k \simeq k_\perp$ modes. Thus, we intuit that waves must be excited where N is much smaller than its typical value, or near the *radiative-convective boundary* (RCB). At the RCB, $N^2 = 0$, and we are concerned with the turning point where $\omega^2 = N^2$. We perform linear expansion about this turning point z_c, and for convenience we set $z_c = 0$, then

$$N^2 \approx \omega^2 + \frac{\mathrm{d}N^2}{\mathrm{d}z}z, \tag{5}$$

where compared to KZ I've taken $N_0^2 = \omega^2$, there seems to be little harm here. Making then general ansatz $u_z(z,\vec{r}_\perp,t) = \tilde{u}_z(z)e^{i\left(\vec{k}_\perp\cdot\vec{r}_\perp - \omega t\right)}$ we obtain

$$-\omega^2\left(-k_\perp^2\tilde{u}_z + \tilde{u}_z''\right) = N^2k_\perp^2u_z, \tag{6}$$

$$\tilde{u}_z'' + k_\perp^2\left(\frac{N^2}{\omega^2} - 1\right)\tilde{u}_z = 0, \tag{7}$$

$$\tilde{u}_z'' + k_\perp^2\frac{\mathrm{d}N^2}{\mathrm{d}z}\frac{z}{\omega^2}\tilde{u}_z = 0. \tag{8}$$

It's easiest now to rescale $\tilde{k}_\perp^2 \equiv k_\perp^2 3\frac{\mathrm{d}N^2}{\mathrm{d}z}\frac{1}{\omega^2}$ so that

$$\tilde{u}_z'' + \tilde{k}_\perp z\tilde{u}_z = 0. \tag{9}$$

The general solution to this ODE is written in terms of Airy functions for arbitrary constants a, b

$$\tilde{u}_z(z) = a\mathrm{Ai}\left(-\frac{z}{\lambda}\right) + b\mathrm{Bi}\left(-\frac{z}{\lambda}\right), \tag{10}$$

where $\lambda = \tilde{k}_\perp^{-2/3}$. For large $-z$, it turns out that

$$\mathrm{Ai}(-z) \sim \frac{\sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)}{z^{1/4}} + \mathcal{O}\left(z^{-7/4}\right), \tag{11}$$

$$\mathrm{Bi}(-z) \sim \frac{\cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)}{z^{1/4}} + \mathcal{O}\left(z^{-7/4}\right). \tag{12}$$

In order for us to get traveling waves with *group velocity* going outwards (towards $z > 0$), we need $\tilde{u}_z(z) \sim e^{-ik_zz}$ such that $u(z,t) \propto e^{i(-k_zz-\omega t)}$ (phase velocity goes inwards, group velocity goes outwards for IGW). Thus, $a = -ib$ in Eq. 13, and we obtain

$$\tilde{u}_z(z) = b\left(-i\mathrm{Ai}\left(-\frac{z}{\lambda}\right) + \mathrm{Bi}\left(-\frac{z}{\lambda}\right)\right), \tag{13}$$

Now we just need to fix b. This is traditionally accomplished by mandating a particular $\frac{\mathrm{d}\delta z}{\mathrm{d}z}$ the displacement of the mode at the turning point $z = 0$. That the forcing results in a constraint on $\frac{\mathrm{d}\delta z}{\mathrm{d}z}$ is similar to what I did in my IGW breaking forcing, where forcing induces a jump in the $\frac{\mathrm{d}u_z}{\mathrm{d}z}$ above/below z_c whose magnitude is fixed by the strength of the forcing term. In the stellar problem, it appears the correct way to obtain the δz is to solve the inhomogeneous problem in the convective zone where $N^2 = 0$ including the tidal potential, so it's not a perfect analogy. But since $\mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}, \mathrm{Bi}'(0) = \frac{3^{1/6}}{\Gamma(1/3)}$, this is not so difficult to evaluate, and I cite the KZ result

$$\frac{\mathrm{d}\delta z}{\mathrm{d}z} = -\frac{ib}{\lambda\omega}\frac{2}{3^{1/3}\Gamma(1/3)}\frac{3^{1/2}+i}{2}. \tag{14}$$

Finally, we impose one more step: we will compute the luminosity or *energy flux* associated with the wave, since this is the easiest way to get the resulting torque. We can easily write down the energy density of the wave $\frac{\rho_0}{2}\left(v^2 + \frac{b^2}{N^2}\right)$, for which the energy flux is $\vec{F} = \vec{v}P$. Noting furthermore that $\frac{\partial u_z}{\partial z} = -ik_\perp u_x = -\frac{ik_\perp P}{\rho_0}\frac{k_\perp}{\omega}$, we can explicitly express P in terms of u_z', and so the energy flux density is then simply (I'm not evaluating this, but KZ do)

$$\frac{\delta L}{\delta A} = \frac{1}{2}\mathrm{Re}\left(Pu_z^*\right) = \frac{\rho_0\omega}{2k_\perp^2}\mathrm{Re}\left(iu_z'u_z^*\right), \tag{15}$$

$$= \frac{3^{2/3}\Gamma^2(1/3)\lambda\omega^3\rho_0}{8\pi k_\perp^2}\left(\frac{\mathrm{d}\delta z}{\mathrm{d}z}\right)^2. \tag{16}$$

We would then compute $L = \int \frac{\delta L}{\delta A}\,\mathrm{d}A$, which for us is just $\frac{\delta L}{\delta A}A$ where A is the surface area of the wave.

Finally, we would compute the total torque from $L = \tau\omega$ (the same as $E = \vec{F}\cdot\vec{v}$).

1.2 Spherical Case

To go to the spherical case, we simply replace $z \to r$ and $k_\perp^2 \to l(l+1)/r^2$, which gives

$$\lambda = \left(\frac{l(l+1)}{r^2\omega^2}\frac{\mathrm{d}N^2}{\mathrm{d}r}\right)^{-1/3}. \tag{17}$$

Then to get $\frac{\mathrm{d}\delta z}{\mathrm{d}z} \to \frac{\mathrm{d}\delta r}{\mathrm{d}r}$, we use prescription

$$\frac{\mathrm{d}\delta r}{\mathrm{d}r} = \alpha\frac{\Phi}{gr}\left(1 - \frac{\rho(r)}{\bar{\rho}(r)}\right). \tag{18}$$

Here, $\alpha \sim 1$ depends on the specific stellar structure in the convective zone, while $\bar{\rho}$ is the average density inside r. Finally, instead of getting a clean $L = \frac{\delta L}{\delta A}A$, we have to actually do the integral of $L = \int \frac{\delta L}{\delta A}\,\mathrm{d}A = \int(\ldots)|Y_{lm}|^2 r^2\,\mathrm{d}\cos\theta\mathrm{d}\phi = r_c^2 L$ (note that it's not $4\pi r_c^2$, thanks to the Y_{lm} normalization). Finally, the $\ell = 2$ potential is taken to be

$$\Phi_{\mathrm{ext}} = -\sqrt{\frac{6\pi}{5}}\frac{GM_2R_c^2}{D^3}. \tag{19}$$

I guess the angular dependency is just dropped. With all these things together, we obtain the final KZ result (I omit the derivation, this part is grungy and not very physically interesting)

$$\tau = \dot{J}_z = \frac{GM_2^2R_c^5}{D^6}\sigma_c^{8/3}\left[\frac{r_c}{g_c}\left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln R}\right)_{r=r_c}\right]^{-1/3}\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2\left[\frac{3}{2}\frac{3^{2/3}\Gamma^2(1/3)}{5\cdot 6^{4/3}}\frac{3}{4\pi}\alpha^2\right], \tag{20}$$

$$= \frac{GM_2^2R_c^5}{D^6}2\hat{F}(r_c,\sigma_c). \tag{21}$$

Note that $\hat{F}$ follows the convention from Equation 42 of Fuller & Lai's second paper (FL2) and Vick et. al's paper as well (VLF), while $\sigma_c = 2|\Omega - \Omega_s|/\sqrt{GM_c/r_c^3}$ is the ratio of the forcing frequency to the breakup frequency *of the core*. Finally, I've replaced M_2 the mass of the companion, R_c the radius of the core, and D the separation, while retaining Ω_s spin angular frequency and Ω orbital angular frequency.

NB: The exact definition of $\hat{F}$ for a given m is given in VLF.23 as

$$\dot{J} = G\frac{M_2R^5}{a^3}\frac{|m|}{2}\hat{F}(\omega) = T_0\frac{|m|}{2}\hat{F}(\omega). \tag{22}$$

Since the total torque τ has already summed over $m = \pm 2$, we incur the extra factor of 2 above in Eq. 21. That m has already been summed over is visible in the $\sqrt{6\pi/5}$ prefactor used in Φ_{ext}, compared to $W_{2\pm 2} = \sqrt{3\pi/10}$ as seen below.

2 Vick et. al., 2016

We now consider eccentric forcing. We will remove subscript compared to VLF and just call $\vec{r}_i = (r, \theta, \phi + \Omega_s t)$ the position coordinate in the inertial frame. Then the $\ell = 2$ tidal forcing potential is generally a sum over $m \in [-2, 2]$

$$U = \sum_m U_{2m}(\vec{r}, t), \tag{23}$$

$$U_{2m}(\vec{r}) = -\frac{GM_2W_{2m}r^2}{D(t)^3}e^{-imf(t)}Y_{2m}(\theta,\phi). \tag{24}$$

Note that f is the true anomaly here. Note that W_{2m} is just a constant: $W_{20} = \sqrt{\pi/5}$, $W_{2\pm 1} = 0$, and $W_{2\pm 2} = \sqrt{3\pi/10}$.

This is complicated since $f(t)$ does not evolve uniformly, ando also since $D(t)$ is time-varying! The easiest treatment is to decompose

$$U_{2m} = -\frac{GM_2W_{2m}r^2}{a^3}Y_{2m}(\theta,\phi)\sum_{N=-\infty}^{\infty}F_{Nm}e^{-iN\Omega t}. \tag{25}$$

Note that the F_{Nm} here are *Hansen coefficients* given by

$$F_{Nm} = \frac{1}{\pi}\int_0^\pi \frac{\cos[N(E - e\sin E) - mf(E)]}{(1 - e\cos E)^2}\,\mathrm{d}E. \tag{26}$$

Note E is the eccentric anomaly. This differs from the VLF definition in a few places but is in agreement with Natalia's paper w/ Dong (SD), such that $F_{Nm} = \delta_{Nm}$ for $e = 0$. It bears noting that VLF's formula normalizes to $F_{Nm} = 2\delta_{Nm}$, so we use the restricted domain of integration for numerical speed (the integrand is symmetric since the argument of the cosine is antisymmetric in E, so both the numerator/denominator are even in E).

Let's explicitly write out the $U_{2\pm 2}$, since they are the only ones that contribute to the tidal torque

$$U_{22} = -\frac{GM_2\sqrt{\frac{3\pi}{10}}r^2}{a^3}\sum_{N=1}^{\infty}\left[F_{N2}Y_{22}(\theta,\phi)e^{-i(N\Omega - 2\Omega_s)t} + F_{-N2}Y_{22}(\theta,\phi)e^{i(N\Omega+2\Omega_s)t}\right], \tag{27}$$

$$U_{2-2} = -\frac{GM_2\sqrt{\frac{3\pi}{10}}r^2}{a^3}\sum_{N=1}^{\infty}\left[F_{-N2}Y_{2-2}(\theta,\phi)e^{-i(N\Omega - 2\Omega_s)t} + F_{N2}Y_{2-2}(\theta,\phi)e^{i(N\Omega+2\Omega_s)t}\right], \tag{28}$$

$$U_{22} + U_{2-2} = -\frac{GM_2\sqrt{\frac{3\pi}{10}}r^2}{a^3}\sum_{N=1}^{\infty}\left[F_{N2}Y_{22}(\theta,\phi)e^{-i(N\Omega - 2\Omega_s)t} + c.c.\right] \tag{29}$$

We can verify that if the perturbing orbit is circular $e = 0$, then the Hansen coefficient $F_{Nm} = \delta_{Nm}$, and we obtain

$$U_{22} + U_{2-2} = -\frac{GM_2 r^2}{a^3}\sqrt{\frac{6\pi}{5}}\,\mathrm{Re}\left[Y_{22}(\theta,\phi)e^{-2i(\Omega-\Omega_s)t}\right]. \tag{30}$$

This is the same torque used in KZ. Finally, this yields torque

$$\dot{J} = \tau = T_0\sum_{N=-\infty}^{\infty} F_{N2}^2\,\mathrm{sgn}(N\Omega - 2\Omega_s)\hat{F}(\omega = |N\Omega - 2\Omega_s|). \tag{31}$$

2.1 Hansen Coefficients

Maybe someday follow `https://arxiv.org/pdf/1308.0607.pdf` and get the derivation of the Hansen coefficients? One fast way to calculate them is to take an FFT of the $F^{lm} = \left(\frac{r}{a}\right)^l e^{imf}$, per `https://www.aanda.org/articles/aa/pdf/2014/11/aa24211-14.pdf`(CBLR). Basically, the Hansen coefficients are just the FT of the disturbing function. Consider that we want to make jump from

$$U(r,t) = -GM_r r^2\sum_m \frac{W_{2m}}{D(t)^3}e^{-imf(t)}Y_{2m}(\theta,\phi), \tag{32}$$

to

$$U(r,t) = -\frac{GM_2 r^2}{a^3}\sum_{m,N} W_{2m}F_{Nm}(e)Y_{2m}(\theta,\phi)e^{-in\Omega t}. \tag{33}$$

Thus, we seek coefficients such that

$$\frac{a^3}{D(t)^3} e^{-imf} = \left(\frac{1 + e\cos f}{1 - e^2}\right)^3 e^{-imf} = \sum_N F_{Nm} e^{-iN\Omega t}. \tag{34}$$

Thus, it's clear the Hansen coefficients are defined by computing Fourier series coefficients (NB: In hindsight, using $r = a(1 - e\cos E)$ probably would have been much faster/easier)

$$F_{Nm} \equiv \frac{1}{T}\int_0^T \frac{e^{-imf}}{\left(1-e^2\right)^3}(1 + e\cos f)^3 e^{iN\Omega t}\,\mathrm{d}t, \tag{35}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-imf}}{\left(1-e^2\right)^3}(1 + e\cos f)^3 e^{iN\Omega t}\,\mathrm{d}M \tag{36}$$

We have notated T the period, and M the mean anomaly. Then one just evaluates using $\cos f = \frac{\cos E - e}{1 - e\cos E}$ and $M = E - e\sin E$ or more usefully $\mathrm{d}M = (1 - e\cos E)\mathrm{d}E$ and obtains

$$F_{Nm} = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1 + e\cos f}{1 - e^2}\right)^3 e^{-imf + iN\Omega t}\,\mathrm{d}M, \tag{37}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{1 - e\cos E}\right)^3 e^{-imf + iNM}(1 - e\cos E)\,\mathrm{d}E, \tag{38}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\exp[i(N(E - e\sin E) - mf)]}{(1 - e\cos E)^2}\,\mathrm{d}E. \tag{39}$$

Now, as we observed above, the integrand is symmetric with respect to E, but it had to be, since in an elliptical orbit the first half and second half are obviously symmetric. Thus, we arrive at final expression as promised

$$F_{Nm} = \frac{1}{\pi}\int_0^{\pi} \frac{\cos[N(E - e\sin E) - mf(E)]}{(1 - e\cos E)^2}\,\mathrm{d}E. \tag{40}$$

3 Combined Results

We have been somewhat careful in checking the agreement between the VLF and KZ forms. Note now that Eq. 21 has $\hat{F}$ for a single m contribution, just as $\hat{F}$ is defined in VLF. Thus, we should be able to simply plug in

$$\tau = T_0 \sum_{N=-\infty}^{\infty} F_{N2}^2 \frac{\operatorname{sgn}(N\Omega - 2\Omega_s)}{2}\sigma_c^{8/3}\left[\frac{r_c}{g_c}\left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln R}\right)_{r=r_c}\right]^{-1/3}\frac{\rho_c}{\bar{\rho}_c}\left(1 - \frac{\rho_c}{\bar{\rho}_c}\right)^2\left[\frac{3}{2}\frac{3^{2/3}\Gamma^2(1/3)}{5\cdot 6^{4/3}}\frac{3}{4\pi}\alpha^2\right], \tag{41}$$

$$= T_0 C(r_c)\sum_{N=-\infty}^{\infty} F_{N2}^2 \operatorname{sgn}(N\Omega - 2\Omega_s)\sigma_c^{8/3}. \tag{42}$$

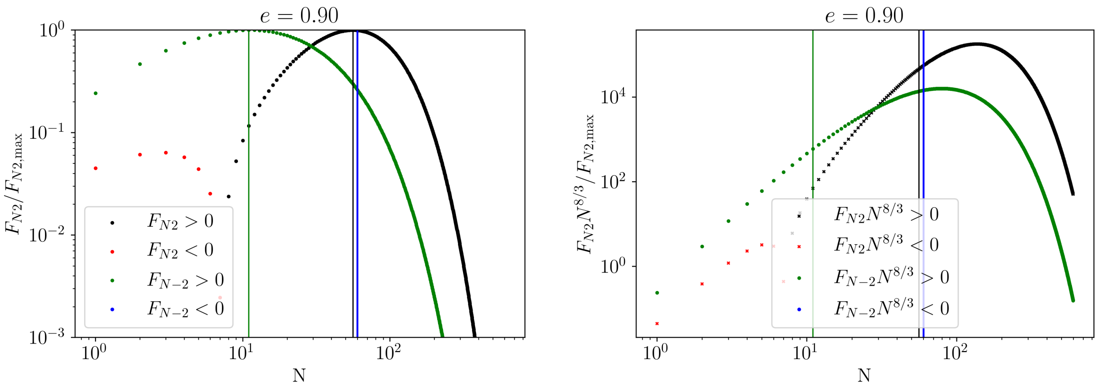

Figure 1: F_{N2} and $F_{N2}N^{8/3}$ as we integrate straightforwardly. Note the vertical blue line is $N = (1-e)^{-3/2}$ while the vertical black/green line are the actual $\mathrm{argmax}_N F_{N\pm 2}$ resrpectively.

Note that now $\sigma_c = |N\Omega - 2\Omega_s|/\sqrt{GM_c/r_c^3}$, and I've defined $C(r_c)$ to be some constant defined at the RCB and does not change with N.

Thus, the relative significance of each N term is given by the summand of Eq. 42, or

$$\tau_N \equiv F_{N2}^2 \mathrm{sgn}(N\Omega - 2\Omega_s)\sigma_c^{8/3}, \tag{43}$$

F_{N2} is easiest to evaluate via an integral for now, but can probably be done via a sampling + FFT when speed is necessary (CBLR).

One guess is that the sum is dominated by the contribution of the frequency at pericenter. We can compute pericenter frequency as follows:.

$$r_p^2\Omega_p = \sqrt{GMa\left(1-e^2\right)}, \tag{44}$$

$$\Omega_p = \sqrt{\frac{GMa(1+e)(1-e)}{a^4(1-e)^4}}, \tag{45}$$

$$= \Omega\frac{\sqrt{1+e}}{(1-e)^{3/2}}. \tag{46}$$

Thus, we should expect the dominant term to come at $N \sim \Omega_p$. This indeed very nearly maximizes F_{N2} but probably won't maximize τ_N.

NB: It appears that the maximum N to sum to is (according to Michelle/Chris)

$$N_{\max} = 10\Omega_p. \tag{47}$$

To understand exactly to what N we should sum, we should recall $|\tau_N| \propto |F_{N2}|^2|N\Omega - 2\Omega_s|^{8/3}$. If $\Omega_s \gg \Omega_p$, then the latter term σ_c^2 is roughly independent of N and indeed we get that τ_N turns over similarly to where F_{N2} turns over. On the other hand, if $\Omega_s \lesssim \Omega_p$, then $\tau_N \propto |F_{N2}|^2 N^{8/3}$. Plots of these two are given in Fig. 1.

A plot of the actual maximum $F_{N\pm 2}$ is provided below in Fig. 2. Note that F_{N2} seems to be a

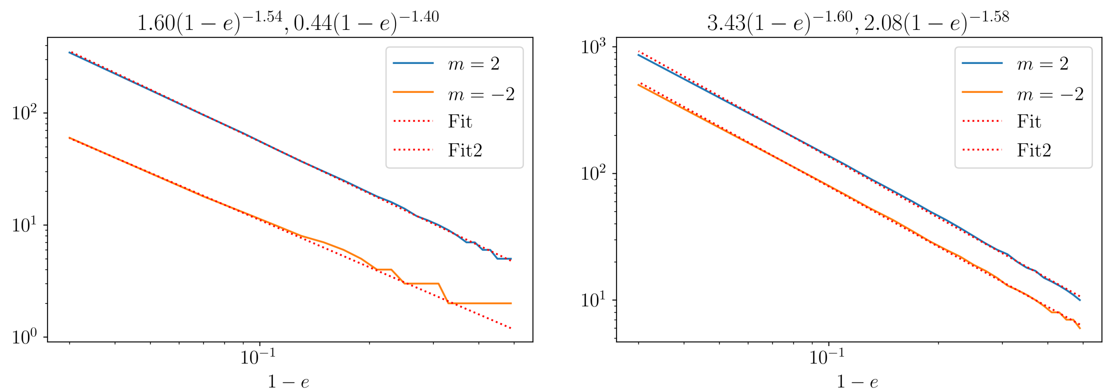

Figure 2: Maxima of $F_{N\pm 2}$ and $N^{8/3}F_{N\pm 2}$ with fits.

constant multiple of $(1-e)^{-3/2}$ our prediction, while F_{N-2} seems to be impacted by the shallowness of the fit at smaller eccentricities. That F_{N2} has its maximum at a slight multiple of $(1-e)^{-3/2}$ should not be surprising, since the actual pericenter passage time can be computed by conservation of angular momentum

To understand the behavior of how multiplying by $N^{8/3}$ changes the peak of F_{N2}, let's consider the simplest model for the scaling of F_{N2}. Upon examination, it decays as $F_{N2} \propto e^{-aN}$, where $a \approx -\frac{1}{75}$. If we allow a simple model for $F_{N2} \propto N^q e^{-aN}$ (this conforms very coarsely with the plotted F_{N2} and allows for a maximum and an exponential tail), then we can identify that its maximum is at $N_{\max} = q/a$. On the other hand, if we seek the maximum of $N^p F_{N2}$, we find its maximum is instead at $N_{\max}^{(p)} = (p+q)/a$. Comparing the two, we expect the maximum to be shifted by roughly factor $\frac{p+q}{q}$. Since the actual shift is $\gtrsim 2$ for $p = 8/3$, we can guess $q \approx 2$, which is plausible gauging from our loglog plot.