

# Lidov-Kozai 90° Attractor

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## 1 Equations

### 1.1 Bin's Papers

Our major references will be Bin's paper with Diego + Dong in 2015 (LML15) and Bin's later paper with Dong on spin-orbit misalignment (LL18). The target of study is §4.3 of LL18, where a 90° attractor in spin-orbit misalignment seems to appear when the octupole effect is negligible.

The easiest formulation is just to express everything in terms of  $\mathbf{L}$  and  $\mathbf{e}$ , following LL18. We drop octupole terms and hold the third perturber constant. These equations come out to be (Eqs. 4–5 w/ substitutions)

$$\frac{d\mathbf{L}}{dt} = \frac{3}{4t_{LK}} \mu \sqrt{Gm_{12}a} [(\mathbf{j} \cdot \hat{n}_2)(\mathbf{j} \times \hat{n}_2) - 5(\mathbf{e} \cdot \hat{n}_2)(\mathbf{e} \times \hat{n}_2)], \quad (1)$$

$$\frac{d\mathbf{e}}{dt} = \frac{3}{4t_{LK}} [(\mathbf{j} \cdot \hat{n}_2)(\mathbf{e} \times \hat{n}_2) + 2\mathbf{j} \times \mathbf{e} - 5(\mathbf{e} \cdot \hat{n}_2)(\mathbf{j} \times \hat{n}_2)]. \quad (2)$$

Note that  $\mathbf{j} \equiv \sqrt{1-e^2} \hat{\mathbf{L}} = \frac{\mathbf{L}}{\mu \sqrt{Gm_{12}a}}$ .  $m_{12} = m_1 + m_2$  and  $\mu = m_1 m_2 / m_{12}$ . We've defined

$$t_{LK} \equiv \frac{L_1}{\mu_1 \Phi_0} = \frac{1}{n_1} \left( \frac{m_{12}}{m_3} \right) \left( \frac{a_2}{a} \right)^3 (1-e_2^2)^{3/2}. \quad (3)$$

Here,  $n_1 \equiv \sqrt{Gm_{12}/a^3}$ . Thus,  $1/t_{LK} \propto a^{3/2}$ .

GW radiation (Peters 1964) cause decays of  $\mathbf{L}$  and  $\mathbf{e}$  as

$$\left. \frac{d\mathbf{L}}{dt} \right|_{GW} = -\frac{32}{5} \frac{G^{7/2}}{c^5} \frac{\mu^2 m_{12}^{5/2}}{a^{7/2}} \frac{1+7e^2/8}{(1-e^2)^2} \hat{\mathbf{L}}, \quad (4)$$

$$\left. \frac{d\mathbf{e}}{dt} \right|_{GW} = -\frac{304}{15} \frac{G^3}{c^5} \frac{\mu m_{12}^2}{a^4 (1-e^2)^{5/2}} \left( 1 + \frac{121}{304} e^2 \right) \mathbf{e}. \quad (5)$$

Here,  $m_{12} \equiv m_1 + m_2$ , and  $a$  is implicitly defined by  $\mathbf{L}$  and  $e$ . The last GR effect is precession of  $\vec{e}$ , which acts as

$$\left. \frac{d\mathbf{e}}{dt} \right|_{GR} = \frac{1}{t_{GR}} \hat{\mathbf{L}} \times \mathbf{e}, \quad (6)$$

$$\frac{1}{t_{GR}} \equiv \frac{3Gnm_{12}}{c^2 a (1-e^2)}. \quad (7)$$

Note that  $t_{GR}^{-1} \propto a^{-5/2}$ .

Given this system (from LML15 + LL18), we can then add the spin-orbit coupling term (from de Sitter precession), which is given in LL18 to be

$$\frac{d\hat{\mathbf{S}}}{dt} = \frac{1}{t_{SL}} \hat{\mathbf{L}} \times \hat{\mathbf{S}}, \quad (8)$$

$$\frac{1}{t_{SL}} \equiv \frac{3Gn(m_2 + \mu/3)}{2c^2 a(1-e^2)}. \quad (9)$$

Note that  $\mu$  is the reduced mass of the inner binary. We can drop the back-reaction term since  $S \ll L$ . Thus,  $t_{SL}^{-1} \propto a^{-5/2}$  as well.

Finally, an adiabaticity parameter can be defined:

$$\mathcal{A} \equiv \left| \frac{\Omega_{SL}}{\Omega_L} \right|. \quad (10)$$

Here,  $\Omega_L \simeq \frac{3(1+4e^2)}{8t_{LK}\sqrt{1-e^2}} |\sin 2I|$  is an approximate rate of change of  $L$  during an LK cycle

It's natural to redimensionalize to the initial LK time such that

$$\frac{1}{t_{LK,0}} \equiv \left( \frac{a}{a_0} \right)^{3/2} \frac{1}{t_{LK}}, \quad (11)$$

since nothing else in  $t_{LK}$  is changing. The next natural timescale for gravitational waves is

$$\frac{1}{t_{GW}} \equiv \frac{G^3 \mu m_{12}^2}{c^5 a^4} \equiv \frac{1}{t_{GW,0}} \left( \frac{a_0}{a} \right)^4 \equiv \epsilon_{GW} \frac{1}{t_{LK,0}} \left( \frac{a_0}{a} \right)^4. \quad (12)$$

We can repeat the procedure for the GR precession term and the spin-orbit coupling terms:

$$\frac{1}{t_{GR}} = \epsilon_{GR} \frac{1}{t_{LK,0}} \left( \frac{a_0}{a} \right)^{5/2}, \quad (13)$$

$$\frac{1}{t_{SL}} = \epsilon_{SL} \frac{1}{t_{LK,0}} \left( \frac{a_0}{a} \right)^{5/2}. \quad (14)$$

Thus, finally, if we let  $\tau = t/t_{LK,0}$ , then we obtain full equations of motion (note that  $a_0 = 1$  below)

$$\begin{aligned} \frac{d\mathbf{L}}{d\tau} &= \left( \frac{a}{a_0} \right)^{3/2} \frac{3}{4} \sqrt{a} [(\mathbf{j} \cdot \hat{n}_2)(\mathbf{j} \times \hat{n}_2) - 5(\mathbf{e} \cdot \hat{n}_2)(\mathbf{e} \times \hat{n}_2)] \\ &\quad - \epsilon_{GW} \left( \frac{a_0}{a} \right)^4 \frac{32}{5} \frac{1+7e^2/8}{(1-e^2)^{5/2}} \mathbf{L}, \end{aligned} \quad (15)$$

$$\begin{aligned} \frac{d\mathbf{e}}{d\tau} &= \left( \frac{a}{a_0} \right)^{3/2} \frac{3}{4} [(\mathbf{j} \cdot \hat{n}_2)(\mathbf{e} \times \hat{n}_2) + 2\mathbf{j} \times \mathbf{e} - 5(\mathbf{e} \cdot \hat{n}_2)(\mathbf{j} \times \hat{n}_2)] \\ &\quad - \epsilon_{GW} \left( \frac{a_0}{a} \right)^4 \frac{304}{15} \frac{1}{(1-e^2)^{5/2}} \left( 1 + \frac{121}{304} e^2 \right) \mathbf{e} \\ &\quad + \epsilon_{GR} \left( \frac{a_0}{a} \right)^{5/2} \frac{1}{1-e^2} \hat{\mathbf{L}} \times \mathbf{e}, \end{aligned} \quad (16)$$

$$\frac{d\hat{\mathbf{S}}}{dt} = \epsilon_{SL} \left( \frac{a_0}{a} \right)^{5/2} \frac{1}{1-e^2} \hat{\mathbf{L}} \times \hat{\mathbf{S}}. \quad (17)$$

For reference, we note that  $a = |\mathbf{L}|^2/(\mu^2 G m_{12}(1 - e^2))$ , while  $\mathbf{j} = \mathbf{L}/(\mu\sqrt{G m_{12}a})$ . To invert  $a(\mathbf{L})$  and  $\mathbf{J}(\mathbf{L})$  in this coordinate system where  $a_0 = 1$ , it is easiest to choose the angular momentum dimensions such that  $\mu\sqrt{G m_{12}} = 1$ , such that now

$$|\mathbf{L}(t=0)| \equiv \mu\sqrt{G m_{12}a_0(1 - e_0^2)} = \sqrt{(1 - e_0^2)}, \quad (18)$$

$$a = \frac{|\mathbf{L}|^2}{1 - e^2}, \quad (19)$$

$$\mathbf{j} = \frac{\mathbf{L}}{\sqrt{a}} = \hat{\mathbf{L}}\sqrt{1 - e^2}. \quad (20)$$

Finally, the timescales are

$$t_{LK,0} = \frac{1}{n} \frac{m_{12}}{m_3} \left( \frac{a_2}{a(t=0)} \right)^3 (1 - e_2^2)^{3/2}, \quad (21)$$

$$\epsilon_{GW} \equiv \frac{t_{LK,0}}{t_{GW,0}} = \frac{1}{n} \frac{m_{12}}{m_3} \frac{a_2^3}{(a(t=0))^7} (1 - e_2^2)^{3/2} \frac{G^3 \mu m_{12}^2}{c^5}, \quad (22)$$

$$\epsilon_{GR} \equiv \frac{t_{LK,0}}{t_{GR,0}} = \frac{m_{12}}{m_3} \frac{a_2^3}{(a(t=0))^4} (1 - e_2^2)^{3/2} \frac{3G m_{12}}{c^2}, \quad (23)$$

$$\epsilon_{SL} \equiv \frac{t_{SL,0}}{t_{GR,0}} = \frac{m_{12}}{m_3} \frac{a_2^3}{(a(t=0))^4} (1 - e_2^2)^{3/2} \frac{3G(m_2 + \mu/3)}{2c^2}. \quad (24)$$

## A $\mathbf{j}$ Equations

We define vectors

$$\mathbf{j} = \sqrt{1 - e^2} \hat{\mathbf{n}}, \quad (25)$$

$$\mathbf{e} = e \hat{\mathbf{u}}. \quad (26)$$

Here,  $\mathbf{j}$  is the dimensionless angular momentum vector and  $\mathbf{e}$  is the eccentricity vector; see LML15 for precise definitions. Note that  $\mathbf{j} \cdot \mathbf{e} = 0$ ,  $j^2 + e^2 = 1$ . Then, the EOM for the inner and outer vectors satisfy to quadrupolar order

$$\frac{d\mathbf{j}}{dt} = \frac{3}{4t_{LK}} [(\mathbf{j} \cdot \hat{\mathbf{n}}_2)(\mathbf{j} \times \hat{\mathbf{n}}_2) - 5(\mathbf{e} \cdot \hat{\mathbf{n}}_2)(\mathbf{e} \times \hat{\mathbf{n}}_2)], \quad (27)$$

$$\frac{d\mathbf{e}}{dt} = \frac{3}{4t_{LK}} [(\mathbf{j} \cdot \hat{\mathbf{n}}_2)(\mathbf{e} \times \hat{\mathbf{n}}_2) + 2\mathbf{j} \times \mathbf{e} - 5(\mathbf{e} \cdot \hat{\mathbf{n}}_2)(\mathbf{j} \times \hat{\mathbf{n}}_2)]. \quad (28)$$

Let's assume for the time being that  $L_1 \ll L_2$ , so the system is sufficiently hierarchical that  $\mathbf{j}_2$ ,  $\mathbf{e}_2$  are constants. Note for reference that

$$t_{LK} \equiv \frac{L_1}{\mu_1 \Phi_0} = \frac{1}{n_1} \left( \frac{m_1 + m_2}{m_3} \right) \left( \frac{a_2}{a} \right)^3 (1 - e_2^2)^{3/2}. \quad (29)$$

Here,  $n_1 \equiv \sqrt{G(m_1 + m_2)/a^3}$ . Thus,  $1/t_{LK} \propto a^{3/2}$ .

GW radiation (Peters 1964) cause decays of  $\mathbf{L}$  and  $\mathbf{e}$  as

$$\left. \frac{d\mathbf{L}}{dt} \right|_{GW} = -\frac{32}{5} \frac{G^{7/2}}{c^5} \frac{\mu^2 m_{12}^{5/2}}{a^{7/2}} \frac{1+7e^2/8}{(1-e^2)^2} \hat{\mathbf{L}}, \quad (30)$$

$$\left. \frac{d\mathbf{e}}{dt} \right|_{GW} = -\frac{304}{15} \frac{G^3}{c^5} \frac{\mu m_{12}^2}{a^4 (1-e^2)^{5/2}} \left( 1 + \frac{121}{304} e^2 \right) \mathbf{e}, \quad (31)$$

$$\left. \frac{\dot{a}}{a} \right|_{GW} = -\frac{64}{5} \frac{G^3 \mu m_{12}^2}{c^5 a^4} \frac{1}{(1-e^2)^{7/2}} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right). \quad (32)$$

Here,  $m_{12} \equiv m_1 + m_2$ . The last GR effect is precession of  $\vec{e}$ , which acts as

$$\left. \frac{d\mathbf{e}}{dt} \right|_{GR} = \frac{3Gnm_{12}}{c^2 a (1-e^2)}. \quad (33)$$

Given this system (from LML15 + LL18), we can then add the spin-orbit coupling term (from de Sitter precession), which is given in LL18 to be

$$\frac{d\hat{\mathbf{S}}}{dt} = \Omega_{SL} \hat{\mathbf{L}} \times \hat{\mathbf{S}}, \quad (34)$$

$$\Omega_{SL} \equiv \frac{3Gn(m_2 + \mu/3)}{2c^2 a (1-e^2)}. \quad (35)$$

Note that  $\mu$  is the reduced mass of the inner binary. We can drop the back-reaction term since  $S \ll L$ . Thus,  $\Omega_{SL} \propto a^{-5/2}$ .

What is observed is that, as this system is evolved forward in time and GR coalesces the inner binary,  $\theta_{sl} \equiv \arccos(\hat{\mathbf{S}} \cdot \hat{\mathbf{L}})$  goes to  $90^\circ$  consistently. The relevant figure is Fig. 19 of LL18, which shows that for a close-in, low-eccentricity perturber ( $\bar{a}_{\text{out,eff}} \propto a_{\text{out}}$ ), the focusing is significantly stronger. Note that initially,  $I \equiv \arccos(\hat{\mathbf{L}} \cdot \hat{\mathbf{L}}_2) \approx 90^\circ$  while  $\theta_{sl} \approx 0$ .

In LL18, an adiabaticity parameter is defined:

$$\mathcal{A} \equiv \left| \frac{\Omega_{SL}}{\Omega_L} \right|, \quad (36)$$

where  $\Omega_L \simeq \left\langle \frac{d\hat{\mathbf{L}}}{dt} \right\rangle_{LK}$  to quadrupolar order. As the inner binary coalesces,  $\mathcal{A}$  transitions from  $\ll 1$  to  $\gg 1$  (as  $\Omega_{SL}$  is a GR effect so ramps up very quickly as orbital separation decreases).

The adiabaticity parameter  $\mathcal{A}$  can be plotted upon rescaling in our coordinates. Note that  $\Omega_{SL} = \frac{\delta a_0}{a t_{LK,0}}$ , while  $\Omega_L \simeq \frac{3(1+4e^2)}{8 t_{LK} \sqrt{1-e^2}} |\sin 2I|$  can also be expressed in units of  $t_{LK,0}$ . This gives us

$$\mathcal{A} = \frac{8\delta\sqrt{1-e^2}}{3(1+4e^2)} \left( \frac{a_0}{a} \right)^4. \quad (37)$$

## A.1 Simulations

First, we run GR-less simulations, so let's take  $t_{LK} = 1$  (no semimajor axis evolution), and we reproduce LK oscillations.

Next, when accounting for GR, we should let  $a$  evolve as above. Note that since  $\mathbf{j}$  and  $\vec{e}$  are our dynamical variables, we should use  $\mathbf{j} \equiv \sqrt{1-e^2} \hat{\mathbf{L}} = \sqrt{1-e^2} \frac{\mathbf{L}}{\mu \sqrt{G m_{12} a (1-e^2)}}$  and rewrite

$$\left. \frac{d\mathbf{j}}{dt} \right|_{GW} = \frac{1}{\mu \sqrt{G M a}} \left. \frac{d\mathbf{L}}{dt} \right|_{GW} - \frac{\mathbf{j}}{2a} \left. \frac{da}{dt} \right|_{GW}. \quad (38)$$

To double check, we should verify that  $\left. \frac{d(j^2 + e^2)}{dt} \right|_{GW} = 0$ , which can be verified as (Let's set  $G = M = \mu = a = c = 1$  for convenience)

$$\frac{1}{2} \frac{d(j^2 + e^2)}{dt} = \mathbf{j} \cdot \frac{d\mathbf{j}}{dt} + \mathbf{e} \cdot \frac{d\mathbf{e}}{dt}, \quad (39)$$

$$= \mathbf{j} \cdot \left[ \left( -\frac{32}{5} \frac{1 + 7e^2/8}{(1 - e^2)^2} \right) \hat{L} - \frac{\mathbf{j}}{2} \left( -\frac{64}{5} \frac{1 + 73e^2/24 + 37e^4/96}{(1 - e^2)^{7/2}} \right) \right] + \mathbf{e} \cdot \left( -\frac{304}{15} \frac{1 + 121e^2/304}{(1 - e^2)^{5/2}} \right) \mathbf{e}, \quad (40)$$

$$= \left( -\frac{32}{5} \frac{1 + 7e^2/8}{(1 - e^2)^{3/2}} \right) + \left( \frac{32}{5} \frac{1 + 73e^2/24 + 37e^4/96}{(1 - e^2)^{5/2}} \right) + e^2 \left( -\frac{304}{15} \frac{1 + 121e^2/304}{(1 - e^2)^{5/2}} \right), \quad (41)$$

$$= \frac{1}{15(1 - e^2)^{5/2}} \left[ -96(1 - e^2) \left( 1 + \frac{7e^2}{8} \right) + 96 \left( 1 + \frac{73e^2}{24} + \frac{37e^4}{96} \right) - 304e^2 \left( 1 + \frac{121e^2}{304} \right) \right]. \quad (42)$$

This can be verified to vanish upon term-by-term examination indeed.

For convenience, let's just define  $t_{LK} = t_{LK,0} \frac{a_0^3}{a^3}$  and set  $t_{LK,0} = 1$ . Furthermore, the timescale of relevance for the GW terms is  $t_{GW}^{-1} \sim \frac{G^3 \mu m_{12}^2}{c^5 a^4}$ . Let's express this as some ratio  $t_{GW} = \epsilon t_{LK,0} \frac{a_0^4}{a^4}$ . Thus, everything should be nondimensionalized this way.

We lastly add de-Sitter precession of the spin of one of the inner binary components, call this  $\hat{S}$ . Similarly, let's just define a proportionality constant  $t_{SL} = \delta t_{LK,0} \frac{a_0}{a}$ , then

$$\frac{d\hat{S}}{d(t/t_{LK,0})} = \delta \frac{a_0}{a} \hat{L} \times \hat{S}. \quad (43)$$

Our final simulation equations are thus ( $\tau = t/t_{LK,0}$ )

$$\begin{aligned} \frac{d\mathbf{j}}{d\tau} &= \frac{3}{4} \left( \frac{a_0^3}{a^3} \right) [(\mathbf{j} \cdot \hat{n}_2)(\mathbf{j} \times \hat{n}_2) - 5(\mathbf{e} \cdot \hat{n}_2)(\mathbf{e} \times \hat{n}_2)] \\ &\quad - \left( \epsilon \frac{a_0^4}{a^4} \right) \left( \frac{32}{5} \frac{1 + 7e^2/8}{(1 - e^2)^{5/2}} - \frac{32}{5} \frac{1}{(1 - e^2)^{7/2}} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right) \right) \mathbf{j}, \end{aligned} \quad (44)$$

$$\frac{d\mathbf{e}}{d\tau} = \frac{3}{4} \left( \frac{a_0^3}{a^3} \right) [(\mathbf{j} \cdot \hat{n}_2)(\mathbf{e} \times \hat{n}_2) + 2\mathbf{j} \times \mathbf{e} - 5(\mathbf{e} \cdot \hat{n}_2)(\mathbf{j} \times \hat{n}_2)] - \left( \epsilon \frac{a_0^4}{a^4} \right) \frac{304}{15} \frac{1}{(1 - e^2)^{5/2}} \left( 1 + \frac{121}{304} e^2 \right) \mathbf{e}, \quad (45)$$

$$\frac{d\hat{S}}{d\tau} = \delta \frac{a_0}{a} \frac{\mathbf{j}}{\sqrt{1 - e^2}} \times \hat{S}, \quad (46)$$

$$\frac{da}{d\tau} = -a \left( \epsilon \frac{a_0^4}{a^4} \right) \frac{64}{5} \frac{1}{(1 - e^2)^{7/2}} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right). \quad (47)$$