Evection Resonances in BH Triples

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1 03/15/21—Basics & Introduction

1.1 Writing Down the Hamiltonian

We assume a triple system $m_{1,2,3}$ and a, $a_{\rm out}$ with mutual inclination I. The 1PN apsidal precession of the inner binary has energy/Hamiltonian

$$H_{\rm GR} = -\frac{3G^2 m_1 m_2 m_{12}}{c^2 a^2 j(e)}, \tag{1}$$

while the external companion has averaged energy

$$H_{\rm out} = -\frac{G m_3 \mu_{12} a^2}{a_{\rm out}^3}\left[\frac{1}{16}\left[\left(6+9e^2\right)\cos^2 I - \left(2+3e^2\right)\right] + \frac{15}{32}\left(1+\cos I\right)^2 e^2 \cos\left(2\varpi - 2\lambda_{\rm out}\right)\right]. \tag{2}$$

Here, we have averaged over: $\varpi = \Omega + \omega$ is the longitude of pericenter of the inner orbit, so $\hat{\mathbf{e}} = \cos\varpi\hat{\mathbf{x}} + \sin\varpi\hat{\mathbf{y}}$, and $\lambda_{\rm out} = \varpi_{\rm out} + M_{\rm out}$ is the mean longitude of m_3, where $M_{\rm out}$ is the outer mean anomaly. Recall that $\Omega_{\rm out} = \dot{\lambda}_{\rm out} = \dot{M}_{\rm out}$, and the useful component form

$$\hat{\mathbf{r}}_{\rm out} = \cos\lambda_{\rm out}\hat{\mathbf{x}} + \sin\lambda_{\rm out}\cos I\hat{\mathbf{y}} + \sin\lambda_{\rm out}\sin I\hat{\mathbf{z}}. \tag{3}$$

Why is this interesting? Well, let's write $\epsilon \equiv \frac{Gm_3\mu_{12}a^2}{a_{\rm out}^3}/H_{\rm GR,0}$, where $H_{\rm GR,0} = [H_{\rm GR}]_{e=0}$, or

$$\epsilon = \frac{m_3\mu_{12}a^2c^2a^2}{3G^2m_1m_2m_{12}a_{\rm out}^3}, \tag{4}$$

$$= \frac{m_3a^4c^2}{3Gm_{12}^2a_{\rm out}^3}. \tag{5}$$

This is like $\epsilon_{\rm GR}^{-1}$ from our previous LK work. We are interested in the regime where $\epsilon \ll 1$. The total Hamiltonian of the system is then

$$\frac{H}{H_{\rm GR,0}} = -\frac{1}{j(e)} - \epsilon\left[\frac{1}{16}\left[\left(6+9e^2\right)\cos^2 I - \left(2+3e^2\right)\right] + \frac{15}{32}(1+\cos I)^2 e^2 \cos(2\varpi - 2\lambda_{\rm out})\right]. \tag{6}$$

We will eventually expand this Hamiltonian in terms of the conjugate variables $-\varpi$ and $1-\left(1-e^2\right)^{1/2} \approx e^2/2$ and obtain a separatrix'd Hamiltonian [Xu & Lai (26)]. But for now, we can be satisfied that some sort of separatrix might appear at $\epsilon \sim 1$? It's not clear yet.

1.2 Timescale Comparison

This section mostly follows Dong's notes, for completeness.

We need $\Omega_{\rm out} \equiv \sqrt{Gm_{123}/a_{\rm out}^3}$ to be of order $\dot\omega \equiv 3Gnm_{12}/(c^2aj^2)$. Assuming the eccentricity is already mostly damped (when $\epsilon_{\rm GR} \gg 1$, we expect this), then this gives

$$\frac{3Gm_{12}}{c^2a} \simeq \frac{\Omega_{\rm out}}{n} = \sqrt{\frac{m_{123}}{m_{12}}\frac{a^3}{a_{\rm out}^3}}, \tag{7}$$

$$\left(\frac{a}{a_{\rm out}}\right)^{5/2} \simeq \frac{3Gm_{12}}{c^2a_{\rm out}}\sqrt{\frac{m_{12}}{m_{123}}}. \tag{8}$$

Indeed, since everything is fixed, as a decays, the evection resonance will be crossed.

Will there be enough time to excite eccentricity? The eccentricity growth rate due to the evection resonance must of order $t_{\rm ZLK}^{-1} \sim n(m_3/m_{12})(a/a_{\rm out})^3$. On the other hand, orbital decay due to GW is of order

$$t_{\rm GW}^{-1} \simeq \frac{64}{5}\frac{G^3m_{12}^2\mu}{c^5a^4} = \frac{64}{5}n\frac{G^{5/2}m_{12}^{3/2}\mu}{c^5a^{5/2}}. \tag{9}$$

Thus, the resonance has time to grow if (in the third line, we invoke the resonance condition above)

$$t_{\rm GW}^{-1} \ll t_{\rm ZLK}^{-1}, \tag{10}$$

$$\frac{64}{5}\frac{G^{5/2}m_{12}^{3/2}\mu}{c^5 a^{5/2}} \ll \frac{m_3}{m_{12}}\left(\frac{a}{a_{\rm out}}\right)^3, \tag{11}$$

$$\frac{64}{5}\frac{G^{5/2}m_{12}^{3/2}\mu}{c^5 a_{\rm out}^{5/2}}\frac{c^2 a_{\rm out}}{3Gm_{12}}\sqrt{\frac{m_{123}}{m_{12}}} \ll \tag{12}$$

$$\frac{64}{15}\frac{G^{3/2}m_{123}^{1/2}\mu}{c^3 a_{\rm out}^{3/2}} \ll \tag{13}$$

$$\frac{64}{15}\left(\frac{v_{\rm out}}{c}\right)^3\frac{m_{12}/4}{m_{123}} \ll \tag{14}$$

$$\left(\frac{v_{\rm out}}{c}\right)^3\left(\frac{a_{\rm out}}{a}\right)^3\frac{m_{12}^2}{m_{123}m_3} \ll 1. \tag{15}$$

Indeed, this must be the case. Another check requires

$$t_{\rm ZLK}^{-1} \ll \dot{\omega}, \tag{16}$$

$$\frac{m_3}{m_{12}}\left(\frac{a}{a_{\rm out}}\right)^3 \ll \frac{3Gm_{12}}{c^2 a} \sim \frac{\Omega_{\rm out}}{n}, \tag{17}$$

$$\ll \left(\frac{m_{123}}{m_{12}}\right)^{1/2}\left(\frac{a}{a_{\rm out}}\right)^{3/2}, \tag{18}$$

$$\frac{m_3}{m_{12}}\left(\frac{m_{12}}{m_{123}}\right)^{1/2}\left(\frac{a}{a_{\rm out}}\right)^{3/2} \ll 1. \tag{19}$$

This is also satisfied. Thus, resonance excitation should be possible.

What are the kinds of systems that are interacting? If we want LISA band, we need $n/\pi \sim 10^{-3}$ Hz, and:

$$\Omega_{\rm out} \simeq \frac{3Gnm_{12}}{c^2 a}, \tag{20}$$

$$\simeq \frac{3n^3 a^2}{c^2}, \tag{21}$$

$$\simeq \frac{3n^3}{c^2}\left(\frac{Gm_{12}}{n^2}\right)^{2/3}, \tag{22}$$

$$\simeq 10^{-7}\left(\frac{P}{10^3\,{\rm s}}\right)^{-5/3}\left(\frac{m_{12}}{2M_\odot}\right)^{2/3}\,{\rm s}^{-1}, \tag{23}$$

$$a_{\rm out} = \left(\frac{Gm_{123}}{\Omega_{\rm out}^2}\right)^{1/3}, \tag{24}$$

$$= 2.4\left(\frac{m_{123}}{3M_\odot}\right)^{1/3}\left(\frac{P}{10^3\,{\rm s}}\right)^{10/9}\left(\frac{m_{12}}{2M_\odot}\right)^{-4/9}\,{\rm AU}. \tag{25}$$

Indeed then, this is not going to be super useful unless m_3 is a SMBH, in which case $a_{\rm out} \sim 100$–1000 AU. Note that $a \sim 3\times 10^8$ m.

The other scenario then is that we cross this resonance, get a large eccentricity, and it doesn't completely damp by the time it crosses the LISA band? Well, we saw above that $(a/a_{\rm out})^{5/2} \propto a_{\rm out}^{-1}$, so if we fix the masses then increasing $a_{\rm out}$ by a factor of 4 increases a by a factor of 32, i.e. $a \sim 0.05$ AU while $a_{\rm out} = 10$ AU, somewhat believable. Since the rates of change of $\ln a$ and $\ln e$ differ only by a factor of $j^2(e)$, if e is only modest, then it will have to also decay by ~ 30 by the time a enters the LISA band. However, if we can excite a substantial e like $j^2(e) = 0.1$ (corresponding to $e = 0.95$), then e will only decay by ~ 3 upon entering the LISA band, which leaves us with an $e = 0.3$. Wenrui's paper suggests evection isn't quite this strong, but maybe some sort of scenario is possible.

The final solution is to use evection to pump an existing large eccentricity up a little bit. But it's becoming clear that we aren't going to cleanly get excitation in the LISA band, and that we will really need to consider dynamics during *and after* the resonance.

1.3 Hamiltonian Level Curves

Let's try to nondimensionalize the Hamiltonian now, like Wenrui's paper. Call $\gamma = -\varpi$ and $\Gamma = 1 - \sqrt{1-e^2}$, so that $j(e) = 1 - \Gamma$ and $e^2 = 1 - (1-\Gamma)^2 = 2\Gamma + \Gamma^2$, then

$$\begin{aligned}
\frac{H}{H_{\mathrm{GR},0}} &= -\frac{1}{1-\Gamma} - \frac{\epsilon}{16}\left[\left(6 + 9\left(2\Gamma + \Gamma^2\right)\right)\cos^2 I - \left(2 + 3\left(2\Gamma + \Gamma^2\right)\right)\right]\\
&\quad - \frac{15\epsilon}{32}(1+\cos I)^2\left(2\Gamma+\Gamma^2\right)\cos\left(2\gamma + 2\lambda_{\mathrm{out}}\right), &(26)\\
H' &= -\frac{1}{1-\Gamma} - \frac{\epsilon}{16}\left[\left(9\cos^2 I - 3\right)\left(2\Gamma+\Gamma^2\right)\right] - \frac{15\epsilon}{32}(1+\cos I)^2\left(2\Gamma+\Gamma^2\right)\cos\left(2\gamma+2\lambda_{\mathrm{out}}\right), &(27)\\
&\approx \Gamma(-1-2\epsilon A) + \Gamma^2(-1+\epsilon A) - \Gamma\epsilon B\cos\theta + \mathcal{O}\left(\Gamma^3, \Gamma^2\cos\theta\right), &(28)\\
A &= \frac{9\cos^2 I - 3}{16}, &(29)\\
B &= \frac{15}{16}(1+\cos I)^2, &(30)\\
\theta &= 2\varpi - 2\lambda_{\mathrm{out}}. &(31)
\end{aligned}$$

This is not quite as clean as Wenrui's form, but it has the advantage (for me) that the ϵ dependence is still explicit, while A, B are almost always positive (except for when $\cos^2 I < 1/3$).

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2.1 Deriving the Hamiltonian Carefully: Circular Perturber

After doing some simulations, it is pretty clear that we will need a precise derivation of the Hamiltonian. We start with the full Hamiltonian

$$H = -\frac{3G^2 m_1 m_2 m_{12}}{c^2 a^2 j(e)} - \frac{G m_3 \mu_{12} a^2}{r_{\mathrm{out}}^3}\left[\frac{1}{16}\left[\left(6+9e^2\right)\cos^2 I - \left(2+3e^2\right)\right] + \frac{15}{32}(1+\cos I)^2 e^2 \cos(2\varpi - 2\lambda_{\mathrm{out}})\right]. \tag{32}$$

Note that when the outer orbit is circular, $r_{\mathrm{out}} = a_{\mathrm{out}}$ and $\lambda_{\mathrm{out}} = f_{\mathrm{out}}$. I can't seem to figure out how to non-dimensionalize the Hamiltonian and the time right now, so let's just factor out $H_{\mathrm{GR},0} \equiv 3G^2 m_1 m_2 m_{12}/c^2 a^2$, and drop constant terms

$$H = -H_{\mathrm{GR},0}\left\{\frac{1}{j(e)} + \epsilon\left[\frac{(6+9e^2)\cos^2 I - 3e^2}{16} + \frac{15}{32}(1+\cos I)^2 e^2 \cos(2\varpi - 2\lambda_{\mathrm{out}})\right]\right\}. \tag{33}$$

Here, again, ϵ was given above

$$\epsilon = \frac{m_3 a^4 c^2}{3G m_{12}^2 a_{\mathrm{out}}^3}. \tag{34}$$

Okay, I give up, we non-dimensionalize the Hamiltonian by dividing by $H_{\mathrm{GR},0}$ and scale time via

$$\tau \equiv \dot{\omega}_{\mathrm{GR}} t = \frac{3Gnm_{12}}{c^2 a j^2} t. \tag{35}$$

We now seek the appropriate canonical transformation for our rescaled H. We can first directly recast H in terms of the modified Delaunay variables; since the generating function is time-independent, we just need to re-express H in terms of the new variables ($\theta_1 = \lambda$ does not appear in the original H, so $L_D = J_1 = \sqrt{Gm_{12}a}$ is conserved, which we've just used in renormalizing the Hamiltonian and by convention set $L_D = J_1 = 1$)

$$J_2 = 1 - \sqrt{1-e^2} \qquad \theta_2 = -\varpi, \tag{36}$$

$$J_3 = \sqrt{1-e^2}(1-\cos I) \qquad \theta_3 = -\Omega, \tag{37}$$

and so $\sqrt{1-e^2} = 1 - J_2$, $e^2 = 1-(1-J_2)^2 = 2J_2 - J_2^2$, and $\cos I = 1 - [J_3/(1-J_2)]$ (note that J_3 is also a constant, but since e is not constant, neither is $\cos I$, strictly speaking)

$$H(J_2,\theta_2,J_3,\theta_3) = -\frac{1}{1-J_2} - \epsilon[A + B\cos(-2\theta_2 - 2\lambda_{\mathrm{out}})], \tag{38}$$

$$A = \frac{3\cos^2 I - 1}{16} 3e^2 + \frac{3}{8}\cos^2 I = \frac{3}{16}\left(2J_2 - J_2^2\right)\left(2 - 6\frac{J_3}{1-J_2} + 3\left(\frac{J_3}{1-J_2}\right)^2\right) + \frac{3}{8}\cos^2 I, \tag{39}$$

$$B = \frac{15}{32}(1+\cos I)^2 e^2 = \frac{15}{32}\left[4 - 4\frac{J_3}{1-J_2} + \left(\frac{J_3}{1-J_2}\right)^2\right]\left(2J_2 - J_2^2\right). \tag{40}$$

We further want to transform the Hamiltonian for some canonical variable $\phi = -2\theta_2 - 2\lambda_{\mathrm{out}}$. The canonical variable conjugate to ϕ is just $\Gamma = -J_2/2$, as we can verify via the Poisson bracket:

$$\{\phi, \Gamma\} = \frac{\partial\phi}{\partial\theta_2}\frac{\partial\Gamma}{\partial J_2} - \frac{\partial\phi}{\partial J_2}\frac{\partial\Gamma}{\partial\theta_2} = (-2)\left(-\frac{1}{2}\right) = 1. \tag{41}$$

The generating function for this canonical transformation is just $S(p,Q) = -J_2\phi/2$, so the Hamiltonian for the resonant angle ϕ becomes

$$H(\Gamma,\phi;J_3) = -\frac{1}{1+2\Gamma} - \epsilon\left(A + B\cos\phi\right) + \frac{\partial S}{\partial t}, \tag{42}$$

$$= -\frac{1}{1+2\Gamma} - \epsilon\left(A + B\cos\phi\right) - \frac{J_2}{2}\left(-2\frac{\mathrm{d}\lambda_{\mathrm{out}}}{\mathrm{d}t}\right), \tag{43}$$

$$= -\frac{1}{1+2\Gamma} - \epsilon\left(A + B\cos\phi\right) - 2\Gamma\frac{\Omega_{\mathrm{out}}}{\Omega_{\mathrm{GR},0}}, \tag{44}$$

$$A = \frac{3}{16}\left(-4\Gamma - 4\Gamma^2\right)\left(2 - 6\frac{J_3}{1+2\Gamma} + 3\left(\frac{J_3}{1+2\Gamma}\right)^2\right) + \frac{3}{8}\left(1 - \frac{J_3}{1+2\Gamma}\right)^2, \tag{45}$$

$$B = \frac{15}{32}\left[4 - 4\frac{J_3}{1+2\Gamma} + \left(\frac{J_3}{1+2\Gamma}\right)^2\right]\left(-4\Gamma - 4\Gamma^2\right). \tag{46}$$

Note that $\lambda_{\mathrm{out}} = \omega_{\mathrm{out}} + \Omega_{\mathrm{out}} + \mathcal{M}_{\mathrm{out}}$, so the time derivative is just the mean motion (at the current order of approximation) in nondimensional units.

Up until here, everything is still exact; we can now compute the Hamiltonian to leading order in Γ, Γ^2 and $\cos\phi$ (we drop constant terms in ϵ and J_3)

$$H(\Gamma,\phi) = 2\Gamma - 4\Gamma^2 - \epsilon\left(A + B\cos\phi\right) - 2\Gamma\frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} + \mathcal{O}\left(\Gamma^3\right), \tag{47}$$

$$= 2\Gamma\left(1 - \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}}\right) - 4\Gamma^2 - \epsilon\left(A + B\cos\phi\right) + \mathcal{O}\left(\Gamma^3\right), \tag{48}$$

$$A = -\frac{3\left(\Gamma + \Gamma^2\right)}{4}(2 - 6J_3) + \frac{3}{8}(4J_3\Gamma) + \mathcal{O}\left(\Gamma^3, \Gamma^2 J_3\right), \tag{49}$$

$$= \frac{3}{2}\left(4J_3\Gamma - \Gamma - \Gamma^2\right) + \mathcal{O}\left(\Gamma^3, \Gamma^2 J_3\right), \tag{50}$$

$$B = -\frac{15}{2}\Gamma + \mathcal{O}\left(\Gamma^2, J_3\Gamma\right), \tag{51}$$

$$H(\Gamma,\phi) = 2\Gamma\left(1 - \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}}\right) - 4\Gamma^2 - \frac{\epsilon}{2}\left[3\left(4J_3\Gamma - \Gamma - \Gamma^2\right) - 15\Gamma\cos\phi\right] + \mathcal{O}\left(\Gamma^3, \Gamma^2 J_3, \Gamma^2\cos\phi\right), \tag{52}$$

$$= 2\Gamma\left(1 - \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} - \frac{\epsilon(12J_3 - 3)}{4}\right) - \Gamma^2\left(4 - \frac{3\epsilon}{2}\right) + \frac{\epsilon}{2}15\Gamma\cos\phi + \mathcal{O}\left(\Gamma^3, \Gamma^2 J_3, \Gamma^2\cos\phi\right). \tag{53}$$

Note that, to leading order,

$$J_3 \approx (1 - J_2)(1 - \cos I) = (1 + 2\Gamma)(1 - \cos I) \approx 1 + 2\Gamma - \cos I. \tag{54}$$

This substitution cannot be made into the Hamiltonian directly, however, since J_3 is an independent momentum from Γ and is conserved. Thus, we obtain

$$H(\Gamma,\phi) \approx \Gamma P - \Gamma^2 Q + R\Gamma\cos\phi, \tag{55}$$

$$P \approx 2\left[1 - \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} - \frac{\epsilon(12J_3 - 3)}{4}\right], \qquad Q \approx 4 - \frac{3\epsilon}{2} \approx 4,$$

$$R \approx \frac{15\epsilon}{2}, \tag{56}$$

$$\epsilon = \frac{m_3 a^4 c^2}{3Gm_{12}^2 a_{\rm out}^3}, \qquad \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} = \frac{(a/a_{\rm out})^{3/2}(m_{123}/m_{12})^{1/2}}{3Gm_{12}/(c^2 a)} \tag{57}$$

Here, $\Gamma \in [-0.5, 0]$, and $\Gamma \simeq -e^2/4$ for $e \ll 1$; these are checked with `sympy`.

This seems to largely agree with Wenrui's Hamiltonian, except he omitted the ϵ contribution to the Γ^2 coefficient (XL Eq. 17). There are then three questions to answer about this Hamiltonian:

- Are there any bifurcations ([dis]appearances of equilibria)?
- What does the phase portrait look like, qualitatively?
- What is the resonance width?

To answer these, we use Hamilton's equations to compute the EOM and equilibria:

$$\dot{\phi} = \frac{\partial H}{\partial \Gamma} = P - 2\Gamma Q + R\cos\phi, \tag{58}$$

$$\dot{\Gamma} = -\frac{\partial H}{\partial \phi} = -R\Gamma\sin\phi. \tag{59}$$

From the second equation, there are only zeros when $\phi = 0, \pi$, and from the first equation, these occur at

$$\Gamma_{\phi=0,\pi} = \frac{P \pm R}{2Q}, \tag{60}$$

where the positive sign corresponds to $\phi = 0$. Recalling that $\Gamma \in [-0.5, 0]$, we see that the solutions only exist when $P \pm 15\epsilon/2 \in [-Q, 0]$. Ps the inspiral progresses, P is increasing, so we see that the $\Gamma_{\phi=0}$ equilibrium will disappear when $P = -15\epsilon/2$ and the $\Gamma_{\phi=\pi}$ equilibrium will disappear when $P = 15\epsilon/2$.

TODO: Resonance width.

2.2 Eccentric Perturber

The goal here is to find the correct way to average the single-averaged expression

$$\tilde{H}_{\rm out} \equiv \frac{H_{\rm out}}{Gm_3\mu_{12}a^2/a_{\rm out}^3} = \frac{1}{4}\left(\frac{a_{\rm out}}{r_{\rm out}}\right)^3 \left[-1 + 6e^2 + 3\left(1-e^2\right)(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}_{\rm out})^2 - 15e^2(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2\right]. \tag{61}$$

The coordinate system we choose matches XL16, where $\hat{\mathbf{L}}_{\rm in} \propto \hat{\mathbf{z}}$ while $\Omega_{\rm out} = 0$. This gives component form (see MD 2.20 and 2.122)

$$r_{\rm out} = \frac{a_{\rm out}\left(1-e_{\rm out}^2\right)}{1+e_{\rm out}\cos f_{\rm out}} \qquad\qquad \hat{\mathbf{r}}_{\rm out} = \begin{pmatrix}\cos v_{\rm out}\\ \sin v_{\rm out}\cos I\\ \sin v_{\rm out}\sin I\end{pmatrix}. \tag{62}$$

Here, $v_{\rm out} = \Omega_{\rm out} + \omega_{\rm out} + f_{\rm out} = \omega_{\rm out} + f_{\rm out}$ is the *true longitude*. Averaging is done via the identities

$$\left\langle\frac{\cos^2 f_{\rm out}}{r_{\rm out}^3}\right\rangle = \left\langle\frac{\sin^2 f_{\rm out}}{r_{\rm out}^3}\right\rangle = \frac{1}{2a_{\rm out}^3\left(1-e_{\rm out}^2\right)^{3/2}}, \tag{63}$$

$$\left\langle\frac{1}{r_{\rm out}^3}\right\rangle = \frac{1}{a_{\rm out}^3\left(1-e_{\rm out}^2\right)^{3/2}}, \tag{64}$$

$$\left\langle\frac{\cos f_{\rm out}\sin f_{\rm out}}{r_{\rm out}^3}\right\rangle = 0. \tag{65}$$

2.2.1 Circular Averaging

Just to review the averaging procedure, let's consider the case where $e_{\rm out} = 0$, then $v_{\rm out} = \lambda_{\rm out}$ the *mean longitude* and we can write

$$\tilde{H}_{\rm out} = \frac{1}{4}\left[-1 + 6e^2 + 3\left(1-e^2\right)\sin^2\lambda_{\rm out}\sin^2 I - 15e^2(\cos\varpi\cos\lambda_{\rm out} + \sin\varpi\sin\lambda_{\rm out}\cos I)^2\right], \tag{66}$$

$$\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out} = \cos(\varpi - \lambda_{\rm out})\left(\frac{1+\cos I}{2}\right) + \cos(\varpi - \lambda_{\rm out})\left(\frac{1-\cos I}{2}\right), \tag{67}$$

$$\left\langle(\ldots)^2\right\rangle = \cos^2(\varpi - \lambda_{\rm out})\left(\frac{1+\cos I}{2}\right)^2 + \frac{(1-\cos I)^2}{8}, \tag{68}$$

$$\left\langle\tilde{H}_{\rm out}\right\rangle = \frac{1}{4}\left[-1 + 6e^2 + \frac{3}{2}\left(1-e^2\right)\left(1-\cos^2 I\right) - \frac{15}{2}e^2(1+\cos(2\varpi - 2\lambda_{\rm out}))\left(\frac{1+\cos I}{2}\right)^2 - \frac{15e^2(1-\cos I)^2}{8}\right], \tag{69}$$

$$= \frac{1}{16}\left[-4 + 24e^2 + 6\left(1 - e^2 - \cos^2 I + e^2\cos^2 I\right) - 15e^2\left(1+\cos I^2\right) - \frac{15}{2}e^2\cos\phi(1+\cos I)^2\right], \tag{70}$$

$$= \frac{1}{16}\left[2 + 3e^2 - 6\cos^2 I - 9e^2\cos^2 I - \frac{15}{2}(1+\cos I)^2 e^2\cos\phi\right]. \tag{71}$$

Success!

2.3 Eccentric Averaging

We cannot just substitute $a_{\rm out} \Rightarrow a_{\rm out,eff} \equiv a_{\rm out}\sqrt{1-e_{\rm out}^2}$ because ϕ is no longer a resonant angle. All of the other terms are the same, however, so we effectively just need to compute the expression

$$a_{\rm out}\left\langle\frac{(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2}{r_{\rm out}^3}\right\rangle \tag{72}$$

Consider a Fourier decomposition of $\hat{\mathbf{r}}_{\rm out}/r_{\rm out}^{3/2}$, where $\Omega_{\rm out}$ is the outer mean motion (just in the x–y plane), then (we assume $\omega_{\rm out}(t=0)=0$ and give an arbitrary phase offset ϖ_0 to the inner eccentricity vector)

$$\begin{aligned}
\hat{\mathbf{e}}(t) &= \cos(\Omega_{\rm GR}t+\varpi_0)\hat{\mathbf{x}}+\sin(\Omega_{\rm GR}t+\varpi_0)\hat{\mathbf{y}}, &(73)\\
\frac{\hat{\mathbf{r}}_{\rm out}(t)_\perp}{r_{\rm out}^{3/2}} &= \frac{1}{r_{\rm out}^{3/2}}\left[\cos v_{\rm out}\hat{\mathbf{x}}+\sin v_{\rm out}\cos I\hat{\mathbf{y}}\right], &(74)\\
&= \sum_{N=1}^{\infty}\frac{c_N}{a_{\rm out}^{3/2}}\left\{\cos(N\Omega_{\rm out}t)\hat{\mathbf{x}}+\sin(N\Omega_{\rm out}t)\cos I\hat{\mathbf{y}}\right\}, &(75)\\
\hat{\mathbf{e}}\cdot\frac{\hat{\mathbf{r}}_{\rm out}}{r_{\rm out}^{3/2}} &= \sum_{N=1}^{\infty}\frac{c_N}{a_{\rm out}^{3/2}}\left\{\cos(N\Omega_{\rm out}t)\cos(\Omega_{\rm GR}t+\varpi_0)+\sin(N\Omega_{\rm out}t)\cos I\sin(\Omega_{\rm GR}t+\varpi_0)\right\}, &(76)\\
&= \sum_{N=1}^{\infty}\frac{c_N}{a_{\rm out}^{3/2}}\left\{\cos((N\Omega_{\rm out}-\Omega_{\rm GR})t-\varpi_0)\left(\frac{1+\cos I}{2}\right)+\cos((N\Omega_{\rm out}+\Omega_{\rm GR})t+\varpi_0)\left(\frac{1-\cos I}{2}\right)\right\}, &(77)\\
\left\langle\frac{(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2}{r_{\rm out}^3}\right\rangle &= \sum_{M=0}^{N-1} f_{NM}\frac{c_{N_{\rm GR}+M}c_{N_{\rm GR}-M}}{2a_{\rm out}^3}\left(\frac{1+\cos I}{2}\right)^2\cos((2N_{\rm GR}\Omega_{\rm out}-2\Omega_{\rm GR})t-2\varpi_0). &(78)
\end{aligned}$$

Here, $N_{\rm GR}\equiv\lfloor\Omega_{\rm GR}/\Omega_{\rm out}$, and $f_{NM}=1$ if $M=N/2$ else 2 (double counting factor). Since the c_N should fall off for $N\gtrsim N_{\rm p}$ where

$$N_{\rm p}\equiv\frac{\sqrt{1+e}}{(1-e_{\rm out})^{3/2}}, \tag{79}$$

we see that there will generally be resonances for all $N\Omega_{\rm out}\sim\Omega_{\rm GR}$ as long as $N\lesssim N_{\rm p}$. Furthermore, we guess that the c_N are expected to scale like N^2 for $N\lesssim N_{\rm p}$, so the sum is dominated by the $M=0$ contribution.

Does this agree with simulations yet? Well, we do observe resonances for both $N\approx 1$ and $N\approx N_{\rm p}$, todo more in depth exploration.

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3.1 Eccentric Averaging: Coplanar

Dong pointed out the correct way to do eccentric averaging, and I think I mixed up the coordinate systems a bit in my earlier work, so let's redo this. Let's start with the coplanar case, so

$$\tilde{H}_{\rm out}\equiv\frac{H_{\rm out}}{Gm_3\mu_{12}a^2/a_{\rm out}^3}=\frac{1}{4}\left(\frac{a_{\rm out}}{r_{\rm out}}\right)^3\left[-1+6e^2-15e^2(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2\right], \tag{80}$$

where we choose $t=0$ to be $v_{\rm out}=0$:

$$\hat{\mathbf{e}}(t)=\cos(\Omega_{\rm GR}t+\varpi_0)\hat{\mathbf{x}}+\sin(\Omega_{\rm GR}t+\varpi_0)\hat{\mathbf{y}},\qquad r_{\rm out}=\frac{a_{\rm out}\left(1-e_{\rm out}^2\right)}{1+e_{\rm out}\cos f_{\rm out}},\qquad \hat{\mathbf{r}}_{\rm out}=\cos v_{\rm out}\hat{\mathbf{x}}+\sin v_{\rm out}\hat{\mathbf{y}}. \tag{81}$$

Then we can try averaging (let's set $\varpi_0 = 0$, it's just an IC)

$$\left\langle\left(\frac{a_{\rm out}}{r_{\rm out}}\right)^3(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2\right\rangle = \left\langle\left(\frac{a_{\rm out}}{r_{\rm out}}\right)^3\left[\cos^2(\Omega_{\rm GR}t)\cos^2 v_{\rm out} + 2\cos(\Omega_{\rm GR}t)\cos v_{\rm out}\sin(\Omega_{\rm GR}t)\sin v_{\rm out} + \sin^2(\Omega_{\rm GR}t)\sin^2 v_{\rm out}\right]\right\rangle, \tag{82}$$

$$= \frac{1}{4}\left\langle\left[\left(\frac{a_{\rm out}}{r_{\rm out}}\right)^3(1+\cos(2\Omega_{\rm GR}t))(1+\cos(2v_{\rm out})) + 2\sin(2\Omega_{\rm GR}t)\sin(2v_{\rm out}) + (1-\cos(2\Omega_{\rm GR}t))(1-\cos(2v_{\rm out}))\right]\right\rangle, \tag{83}$$

$$= \frac{1}{4}\left\langle\left(\frac{a_{\rm out}}{r_{\rm out}}\right)^3\left[2 + 2\cos(2\Omega_{\rm GR}t)\cos(2v_{\rm out}) + 2\sin(2\Omega_{\rm GR}t)\sin(2v_{\rm out})\right]\right\rangle, \tag{84}$$

$$= \frac{1}{4}\mathrm{Re}\left\langle\left(\frac{a_{\rm out}}{r_{\rm out}}\right)^3\left[2 + 2e^{2i\Omega_{\rm GR}t - 2iv_{\rm out}}\right]\right\rangle, \tag{85}$$

$$= \frac{1}{2} + \frac{1}{2}\mathrm{Re}\sum_{N=-\infty}^{\infty} F_{N2}\left\langle e^{2i\Omega_{\rm GR}t - iN\Omega t}\right\rangle, \tag{86}$$

$$= \frac{1}{2} + \frac{1}{2}\sum_{N=-\infty}^{\infty} F_{N2}\left\langle\cos\left[(2\Omega_{\rm GR} - N\Omega)t\right]\right\rangle. \tag{87}$$

Indeed then, when there are no commensurabilities, this averages to $\frac{1}{2}$ as expected, but this is nice and clean. Thus, the total Hamiltonian becomes

$$\bar{H}_{\rm out}(e_{\rm out}) = \frac{1}{4}\left[-1 + 6e^2 - \frac{15}{2}e^2\left(1 + F_{N2;e_{\rm out}}\cos(2\Omega_{\rm GR} - N\Omega)t\right)\right]. \tag{88}$$

I've spelled out the explicit dependence of F_{N2} on $e_{\rm out}$. The circular case is recovered by setting $F_{N2} = 1$ and $N = 2$.

3.2 Eccentric Averaging: Inclined

If the perturber is inclined, this calculation becomes a bit more complicated. Can we do it? We choose the coordinate system aligned with the invariable plane $\hat{\mathbf{z}} \propto L_{\rm tot} = \mathbf{L} + \mathbf{L}_{\rm out}$. Define the inner Keplerian orbital elements $(a, e, i, \Omega, \omega)$, and osculating outer Keplerian orbital elements $(a_{\rm out}, e_{\rm out}, i_{\rm out}, \Omega_{\rm out}, \omega_{\rm out}, f_{\rm out})$. By conservation of angular momentum, $\Omega_{\rm out} = \Omega + \pi$ while $L\sin i = L_{\rm out}\sin i_{\rm out}$. The SA Hamiltonian and various components of vectors are then [LML15 (22–23)]:

$$\tilde{H}_{\rm out} = \frac{1}{4}\left(\frac{a_{\rm out}}{r_{\rm out}}\right)^3\left[-1 + 6e^2 + 3\left(1-e^2\right)(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}_{\rm out})^2 - 15e^2(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2\right], \tag{89}$$

$$\hat{\mathbf{n}} = \begin{pmatrix} \sin i\sin\Omega \\ -\sin i\cos\Omega \\ \cos i \end{pmatrix}, \tag{90}$$

$$\hat{\mathbf{e}} = \begin{pmatrix} \cos\omega\cos\Omega - \sin\omega\cos i\sin\Omega \\ \cos\omega\sin\Omega + \sin\omega\cos i\cos\Omega \\ \sin\omega\sin i \end{pmatrix}, \tag{91}$$

$$\hat{\mathbf{r}}_{\rm out} = \begin{pmatrix} \cos\Omega\cos(\omega_{\rm out} + f_{\rm out}) + \sin\Omega\sin(\omega_{\rm out} + f)\cos i_{\rm out} \\ -\sin\Omega\cos(\omega_{\rm out} + f_{\rm out}) + \cos\Omega\sin(\omega_{\rm out} + f)\cos i_{\rm out} \\ \sin(\omega_{\rm out} + f_{\rm out})\sin I_{\rm out} \end{pmatrix}. \tag{92}$$

Eventually, we need to use that $\dot{\varpi} = \Omega_{\rm GR,0}/j(e)$ to get a resonance. We can come back to this some other time, it shouldn't be too hard if we only consider angles that will contribute to the resonance.

3.3 Maximum Eccentricity Excitation

This shouldn't be hard, right? We start with the simplified Hamiltonian (setting $J_3 = 0$ for the coplanar case)

$$H\left(\Gamma,\phi\right) \approx \Gamma P - \Gamma^2 Q + R\Gamma\cos\phi, \tag{93}$$

$$P \approx 2\left[1 - \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} + \frac{3\epsilon}{4}\right], \tag{94}$$

$$Q \approx 4, \tag{95}$$

$$R \approx \frac{15\epsilon}{2}. \tag{96}$$

Recall that $\Gamma \approx -e^2/4$. Indeed, it's immediately obvious that if $\Gamma \gg \epsilon$ then the Γ^2 term will dominate and H will be independent of ϕ. We want to find the largest eccentricity excitation, so either $H(\phi=\pi)/H(\phi=0)$ or the ratio of the two solutions to $H\left(\phi=\pi\right)$.

- If the solution is librating, the maximum eccentricity excitation is just the ratio between the two roots to $Q\Gamma^2 - \Gamma P + R\Gamma + H_0 = 0$, which is immediately:

$$\frac{\Gamma_2}{\Gamma_1} = \frac{(P-R) - \sqrt{(P-R)^2 - 4QH_0}}{(P-R) + \sqrt{(P-R)^2 - 4QH_0}}, \tag{97}$$

$$= \frac{-1 - \sqrt{1 - 4QH_0/(P-R)^2}}{-1 + \sqrt{1 - 4QH_0/(P-R)^2}}, \tag{98}$$

$$= \frac{-1-\alpha}{-1+\alpha} = \frac{1+\alpha^2+2\alpha}{1-\alpha^2}, \tag{99}$$

$$= 1 + \frac{2\alpha+2\alpha^2}{1-\alpha^2}. \tag{100}$$

$$= 1 + \frac{2\alpha}{1-\alpha} \tag{101}$$

  Obviously this ratio is singular as $\alpha \to 1$ which is when $H_0 \to 0$ which is when the smaller root gives $e = 0$. Whoops. Not the best way of quantifying this.

  What about the difference between the roots? This is just

$$\Gamma_2 - \Gamma_1 = -\frac{\sqrt{(P-R)^2 - 4QH_0}}{Q}. \tag{102}$$

  Note that if $H_0 < 0$, then $\Gamma_1 > 0$. Thus, again, the difference is also maximized as $H_0 \to 0$. Note that the only way to get $H_0 > 0$ is for $P < 0$, i.e. fairly off resonance.

- If the solution is circulating, then we want to solve for the difference between the two negative (smallest) solutions to $Q\Gamma^2 - \Gamma P \mp R\Gamma + H_0 = 0$. The ratio is then given by

$$\frac{\Gamma_2}{\Gamma_1} = \frac{P - R - \sqrt{(P-R)^2 - 4QH_0}}{P + R - \sqrt{(P+R)^2 - 4QH_0}}. \tag{103}$$

  This function diverges as $H_0 \to 0^-$ by L'Hôpital's rule, and goes to 1 monotonically as $H_0 \to -\infty$. Thus, again, we see that small H_0 gives the largest eccentricity excitation.

Finally, to calculate the difference:

$$\Gamma_2 - \Gamma_1 = \frac{-2R - \sqrt{(P-R)^2 - 4QH_0} + \sqrt{(P+R)^2 - 4QH_0}}{2Q}. \tag{104}$$