Evection Resonances in BH Triples

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1 03/15/21—Basics & Introduction

1.1 Writing Down the Hamiltonian

We assume a triple system $m_{1,2,3}$ and a, $a_{\rm out}$ with mutual inclination I. The 1PN apsidal precession of the inner binary has energy/Hamiltonian

$$H_{\rm GR} = -\frac{3G^2 m_1 m_2 m_{12}}{c^2 a^2 j(e)}, \tag{1}$$

while the external companion has averaged energy

$$H_{\rm out} = -\frac{G m_3 \mu_{12} a^2}{a_{\rm out}^3}\left[\frac{1}{16}\left[\left(6+9e^2\right)\cos^2 I - (2+3e^2)\right] + \frac{15}{32}(1+\cos I)^2 e^2 \cos(2\varpi - 2\lambda_{\rm out})\right]. \tag{2}$$

Here, we have averaged over: $\varpi = \Omega + \omega$ is the longitude of pericenter of the inner orbit, so $\hat{\mathbf{e}} = \cos\varpi\hat{\mathbf{x}} + \sin\varpi\hat{\mathbf{y}}$, and $\lambda_{\rm out} = \varpi_{\rm out} + M_{\rm out}$ is the mean longitude of m_3, where $M_{\rm out}$ is the outer mean anomaly. Recall that $n_{\rm out} = \dot{\lambda}_{\rm out} = \dot{M}_{\rm out}$, and the useful component form

$$\hat{\mathbf{r}}_{\rm out} = \cos\lambda_{\rm out}\hat{\mathbf{x}} + \sin\lambda_{\rm out}\cos I\hat{\mathbf{y}} + \sin\lambda_{\rm out}\sin I\hat{\mathbf{z}}. \tag{3}$$

Why is this interesting? Well, let's write $\epsilon \equiv \frac{G m_3 \mu_{12} a^2}{a_{\rm out}^3}/H_{\rm GR,0}$, where $H_{\rm GR,0} = [H_{\rm GR}]_{e=0}$, or

$$\epsilon = \frac{m_3 \mu_{12} a^2 c^2 a^2}{3G^2 m_1 m_2 m_{12} a_{\rm out}^3}, \tag{4}$$

$$= \frac{m_3 a^4 c^2}{3G^2 m_{12}^2 a_{\rm out}^3}. \tag{5}$$

This is like $\epsilon_{\rm GR}^{-1}$ from our previous LK work. We are interested in the regime where $\epsilon \ll 1$. The total Hamiltonian of the system is then

$$\frac{H}{H_{\rm GR,0}} = -\frac{1}{j(e)} - \epsilon\left[\frac{1}{16}\left[\left(6+9e^2\right)\cos^2 I - (2+3e^2)\right] + \frac{15}{32}(1+\cos I)^2 e^2\cos(2\varpi - 2\lambda_{\rm out})\right]. \tag{6}$$

We will eventually expand this Hamiltonian in terms of the conjugate variables $-\varpi$ and $1-\left(1-e^2\right)^{1/2} \approx e^2/2$ and obtain a separatrix'd Hamiltonian [Xu & Lai (26)]. But for now, we can be satisfied that some sort of separatrix might appear at $\epsilon \sim 1$? It's not clear yet.

1.2 Timescale Comparison

This section mostly follows Dong's notes, for completeness.

We need $n_{\rm out} \equiv \sqrt{Gm_{123}/a_{\rm out}^3}$ to be of order $\dot{\omega} \equiv 3Gnm_{12}/(c^2 a j^2)$. Assuming the eccentricity is already mostly damped (when $\epsilon_{\rm GR} \gg 1$, we expect this), then this gives

$$\frac{3Gm_{12}}{c^2 a} \simeq \frac{n_{\rm out}}{n} = \sqrt{\frac{m_{123}}{m_{12}} \frac{a^3}{a_{\rm out}^3}}, \tag{7}$$

$$\left(\frac{a}{a_{\rm out}}\right)^{5/2} \simeq \frac{3Gm_{12}}{c^2 a_{\rm out}} \sqrt{\frac{m_{12}}{m_{123}}}. \tag{8}$$

Indeed, since everything is fixed, as a decays, the evection resonance will be crossed.

Will there be enough time to excite eccentricity? The eccentricity growth rate due to the evection resonance must of order $t_{\rm ZLK}^{-1} \sim n(m_3/m_{12})(a/a_{\rm out})^3$. On the other hand, orbital decay due to GW is of order

$$t_{\rm GW}^{-1} \simeq \frac{64}{5} \frac{G^3 m_{12}^2 \mu}{c^5 a^4} = \frac{64}{5} n \frac{G^{5/2} m_{12}^{3/2} \mu}{c^5 a^{5/2}}. \tag{9}$$

Thus, the resonance has time to grow if (in the third line, we invoke the resonance condition above)

$$t_{\rm GW}^{-1} \ll t_{\rm ZLK}^{-1}, \tag{10}$$

$$\frac{64}{5} \frac{G^{5/2} m_{12}^{3/2} \mu}{c^5 a^{5/2}} \ll \frac{m_3}{m_{12}} \left(\frac{a}{a_{\rm out}}\right)^3, \tag{11}$$

$$\frac{64}{5} \frac{G^{5/2} m_{12}^{3/2} \mu}{c^5 a_{\rm out}^{5/2}} \frac{c^2 a_{\rm out}}{3Gm_{12}} \sqrt{\frac{m_{123}}{m_{12}}} \ll \tag{12}$$

$$\frac{64}{15} \frac{G^{3/2} m_{123}^{1/2} \mu}{c^3 a_{\rm out}^{3/2}} \ll \tag{13}$$

$$\frac{64}{15} \left(\frac{v_{\rm out}}{c}\right)^3 \frac{m_{12}/4}{m_{123}} \ll \tag{14}$$

$$\left(\frac{v_{\rm out}}{c}\right)^3 \left(\frac{a_{\rm out}}{a}\right)^3 \frac{m_{12}^2}{m_{123} m_3} \ll 1. \tag{15}$$

Indeed, this must be the case. Another check requires

$$t_{\rm ZLK}^{-1} \ll \dot{\omega}, \tag{16}$$

$$\frac{m_3}{m_{12}} \left(\frac{a}{a_{\rm out}}\right)^3 \ll \frac{3Gm_{12}}{c^2 a} \sim \frac{n_{\rm out}}{n}, \tag{17}$$

$$\ll \left(\frac{m_{123}}{m_{12}}\right)^{1/2} \left(\frac{a}{a_{\rm out}}\right)^{3/2}, \tag{18}$$

$$\frac{m_3}{m_{12}} \left(\frac{m_{12}}{m_{123}}\right)^{1/2} \left(\frac{a}{a_{\rm out}}\right)^{3/2} \ll 1. \tag{19}$$

This is also satisfied. Thus, resonance excitation should be possible.

What are the kinds of systems that are interacting? If we want LISA band, we need $n/\pi \sim 10^{-3}$ Hz, and:

$$\begin{aligned}
n_{\rm out} &\simeq \frac{3Gnm_{12}}{c^2 a}, &(20)\\
&\simeq \frac{3n^3a^2}{c^2}, &(21)\\
&\simeq \frac{3n^3}{c^2}\left(\frac{Gm_{12}}{n^2}\right)^{2/3}, &(22)\\
&\simeq 10^{-7}\left(\frac{P}{10^3\,\mathrm{s}}\right)^{-5/3}\left(\frac{m_{12}}{2M_\odot}\right)^{2/3}\ \mathrm{s}^{-1}, &(23)\\
a_{\rm out} &= \left(\frac{Gm_{123}}{n_{\rm out}^2}\right)^{1/3}, &(24)\\
&= 2.4\left(\frac{m_{123}}{3M_\odot}\right)^{1/3}\left(\frac{P}{10^3\,\mathrm{s}}\right)^{10/9}\left(\frac{m_{12}}{2M_\odot}\right)^{-4/9}\ \mathrm{AU}. &(25)
\end{aligned}$$

Indeed then, this is not going to be super useful unless m_3 is a SMBH, in which case $a_{\rm out} \sim 100$–1000 AU. Note that $a \sim 3 \times 10^8$ m.

The other scenario then is that we cross this resonance, get a large eccentricity, and it doesn't completely damp by the time it crosses the LISA band? Well, we saw above that $(a/a_{\rm out})^{5/2} \propto a_{\rm out}^{-1}$, so if we fix the masses then increasing $a_{\rm out}$ by a factor of 4 increases a by a factor of 32, i.e. $a \sim 0.05$ AU while $a_{\rm out} = 10$ AU, somewhat believable. Since the rates of change of $\ln a$ and $\ln e$ differ only by a factor of $j^2(e)$, if e is only modest, then it will have to also decay by ~ 30 by the time a enters the LISA band. However, if we can excite a substantial e like $j^2(e) = 0.1$ (corresponding to $e = 0.95$), then e will only decay by ~ 3 upon entering the LISA band, which leaves us with an $e = 0.3$. Wenrui's paper suggests evection isn't quite this strong, but maybe some sort of scenario is possible.

The final solution is to use evection to pump an existing large eccentricity up a little bit. But it's becoming clear that we aren't going to cleanly get excitation in the LISA band, and that we will really need to consider dynamics during *and after* the resonance.

1.3 Hamiltonian Level Curves

Let's try to nondimensionalize the Hamiltonian now, like Wenrui's paper. Call $\gamma = -\varpi$ and $\Gamma = 1 - \sqrt{1-e^2}$, so that $j(e) = 1 - \Gamma$ and $e^2 = 1 - (1-\Gamma)^2 = 2\Gamma + \Gamma^2$, then

$$\begin{aligned}
\frac{H}{H_{\mathrm{GR},0}} &= -\frac{1}{1-\Gamma} - \frac{\epsilon}{16}\left[\left(6 + 9\left(2\Gamma + \Gamma^2\right)\right)\cos^2 I - \left(2 + 3\left(2\Gamma + \Gamma^2\right)\right)\right] \\
&\quad - \frac{15\epsilon}{32}(1+\cos I)^2\left(2\Gamma + \Gamma^2\right)\cos\left(2\gamma + 2\lambda_{\mathrm{out}}\right), && (26) \\
H' &= -\frac{1}{1-\Gamma} - \frac{\epsilon}{16}\left[\left(9\cos^2 I - 3\right)\left(2\Gamma + \Gamma^2\right)\right] - \frac{15\epsilon}{32}(1+\cos I)^2\left(2\Gamma + \Gamma^2\right)\cos\left(2\gamma + 2\lambda_{\mathrm{out}}\right), && (27) \\
&\approx \Gamma(-1 - 2\epsilon A) + \Gamma^2(-1 + \epsilon A) - \Gamma\epsilon B\cos\theta + \mathcal{O}\left(\Gamma^3, \Gamma^2\cos\theta\right), && (28) \\
A &= \frac{9\cos^2 I - 3}{16}, && (29) \\
B &= \frac{15}{16}(1+\cos I)^2, && (30) \\
\theta &= 2\varpi - 2\lambda_{\mathrm{out}}. && (31)
\end{aligned}$$

This is not quite as clean as Wenrui's form, but it has the advantage (for me) that the ϵ dependence is still explicit, while A, B are almost always positive (except for when $\cos^2 I < 1/3$).