I checked DL's notes, and our scalings for $\dot{I}_{\rm e}/\Omega_{\rm e}$ as well as $\Omega_{\rm e}$ at $\bar{A} \simeq 1$ agree. I will use DL notation when writing it up.

In this particular parameter regime, $A \simeq 1$ is not in the completely frozen regime, but it is also not in the oscillating regime as defined by DL's notes, as $e_{\max} - e_{\min} \ll 1$. I don't think the distinction ends up mattering for scaling purposes though.

1 Requested Plots

I have made many of these, but I attach just the ones for $I_0 = 90.5^\circ$ below, see Fig. 1. Of note:

- $\dot{\bar{I}}_{\rm e}$ is very smooth.
- $I_{\rm e}$ still nutates rather significantly at $A \simeq 1$.
- $T_{\rm LK}$ is defined even in the e-frozen regime as π/T_ω, where T_ω is the period of the ω orbital element.
- In panel 6, it is clear $\max \dot{\bar{I}}_{\rm e}/\Omega_{\rm e}$ greatly overpredicts the final $\Delta\theta_{\rm e}$ as expected, and that there is significant damping of fluctuations psat the maximum deviation ("narrowing" as described before, and "cancellations" in the DL explicit solution).

  Interestingly, for $I_0 \gtrsim 90.35^\circ$, the shape of $\Delta\theta_{\rm e}$ does not change any more (red dots) even as $\dot{\bar{I}}_{\rm e}/\Omega_{\rm e}$ continues to decrease with increasing I_0. This suggests some other mechanism is sustaining these oscillations. Note though that this does not affect the *final* $\Delta\theta_{\rm e}$, which decreases with increasing I_0.
- Following the results of the next section, the averaging in Panel 6 should be done over multiple LK periods. We average $\theta_{\rm e}$ over $4T_{\rm LK}$, following the approximate ratio in Panel 4.

2 Comment on Averaging Procedure

Consider the full form of the Hamiltonian

$$H = \mathbf{\Omega}_{\rm e} \cdot \hat{\mathbf{S}}. \tag{1}$$

Here, $\mathbf{\Omega}_{\rm e}$ is periodic with period $T_{\rm LK}$. Assume $\hat{\mathbf{S}}$ is also periodic with some period $T_{\rm S}$ (e.g. $\sim 2\pi/\bar{\Omega}_{\rm e}$).

In general, these two periods are irrational, but for sufficiently large integers p, q, there will be a period T satisfying

$$T \approx pT_{\rm LK} \approx qT_{\rm S}. \tag{2}$$

Consider averaging the Hamiltonian over interval T. Writing (note that $\hat{\mathbf{S}}_M$ must be complex, as $\hat{\mathbf{S}}$

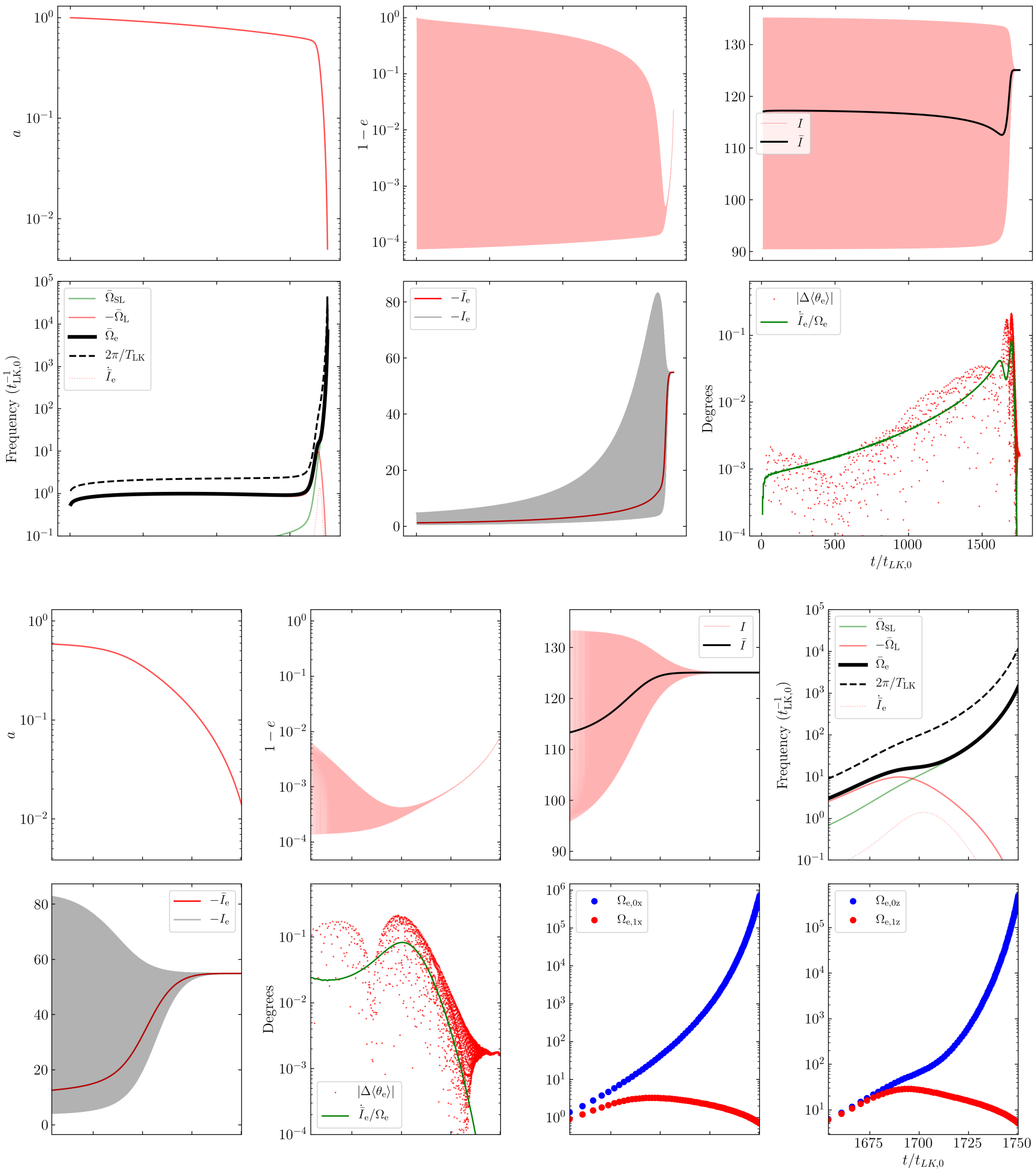

Figure 1: Plot of interesting quantities, top 6 panels are for entire simulation while bottom 8 are zoomed in near $\bar{A} \simeq 1$. For full description, see text.

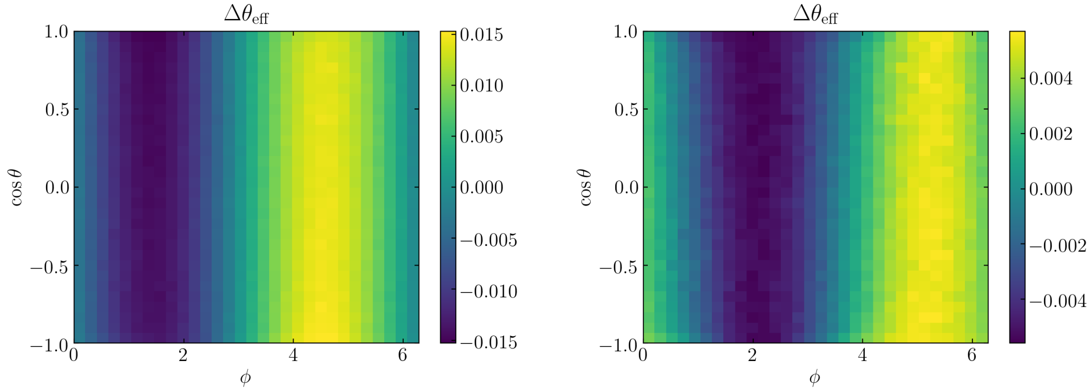

Figure 2: Averaging over $T_{\rm LK}$ and $2T_{\rm LK}$ respectively.

is precessing; while $\mathbf{\Omega}_{\rm e,N}$ can be made real by choice of t)

$$\mathbf{\Omega}_{\rm e} = \bar{\mathbf{\Omega}}_{\rm e} + \sum_{N=1}^{\infty} \mathbf{\Omega}_{\rm e,N} \cos\left(\frac{2\pi N t}{T_{\rm LK}}\right), \tag{3}$$

$$\hat{\mathbf{S}} = \left[\langle\hat{\mathbf{S}}\rangle + \sum_{M=1}^{\infty} \mathbf{S}_{\rm M} \exp\left(i\frac{2\pi M t}{T_{\rm S}}\right)\right], \tag{4}$$

$$\frac{1}{T}\int_0^T H\,\mathrm{d}t = \frac{1}{T}\int_0^T \left[\bar{\mathbf{\Omega}}_{\rm e} + \sum_{N=1}^{\infty} \mathbf{\Omega}_{\rm e,N} \cos\left(\frac{2\pi N q t}{T}\right)\right] \cdot \left[\langle\hat{\mathbf{S}}\rangle + \sum_{M=1}^{\infty} \mathbf{S}_{\rm M} \exp\left(i\frac{2\pi M p t}{T}\right)\right] \mathrm{d}t, \tag{5}$$

$$\langle H\rangle = \bar{\mathbf{\Omega}}_{\rm e} \cdot \langle\hat{\mathbf{S}}\rangle + \frac{1}{2}\sum_{j=1}^{\infty} \mathbf{\Omega}_{\rm e,jp} \cdot \left(\mathrm{Re}\,\mathbf{S}_{jq}\right). \tag{6}$$

When the terms in the summation can be neglected, this reduces to the claim we have made: that $\langle\bar{\mathbf{\Omega}}_{\rm e} \cdot \hat{\mathbf{S}}\rangle$ is an adiabatic invariant, since

$$A \equiv \oint \cos\theta_{\rm e}\,\mathrm{d}\phi_{\rm e} \approx \Omega_{\rm e}\langle\cos\theta_{\rm e}\rangle. \tag{7}$$

This argument suggests that the correct timescale to average over is T, a near-integer multiple of both $T_{\rm LK}$ and $T_{\rm S} \simeq 2\pi/\bar{\mathbf{\Omega}}_{\rm e}$.

Indeed, when using a grid of high-precision $I_0 = 90.5^\circ$ simulations, the maximum $\Delta\theta_{\rm e}$ goes down by a factor of three when using $T = 2T_{\rm LK}$ (see Fig. 2). Initially, $T_{\rm LK} \approx 0.4706\left(2\pi/\bar{\Omega}_{\rm e}\right)$.

I suspect there is a good reason the summed terms in Eq. (6) can be neglected: $\langle\hat{\mathbf{S}}\rangle \parallel \bar{\mathbf{\Omega}}_{\rm e}$, $\mathbf{S}_M \perp \langle\hat{\mathbf{S}}\rangle$ while $\mathbf{\Omega}_{\rm e,N} \parallel \bar{\mathbf{\Omega}}_{\rm e}$ (only when the nutation of $\mathbf{\Omega}_{\rm e}$ is negligible), naively. I haven't been able to check whether this works yet. If the above claim is true, then conservation of $\theta_{\rm e}$ depends on how much $\bar{I}_{\rm e}$ is nutating when $A \simeq 1$.