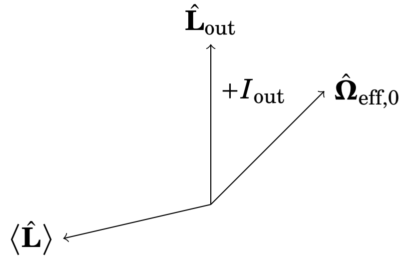

Figure 1: Definition of the angle $I_{\rm out}$. It turns out that for $I_0 > 90°$, $\Omega_{\rm eff,0}$ eventually aligns with $-\langle\hat{\mathbf{L}}\rangle$, hence the sign convention for $I_{\rm out}$.

Throughout this writeup, we will only consider the $N = 0$ harmonic of $\boldsymbol{\Omega}_{\rm eff}$, such that

$$\left(\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t}\right)_{\rm rot} = \langle -\dot{\Omega}\hat{\mathbf{z}} + \Omega_{\rm SL}\hat{\mathbf{L}}_{\rm in}\rangle \times \hat{\mathbf{S}}, \tag{1}$$

$$\equiv \boldsymbol{\Omega}_{\rm eff,0} \times \hat{\mathbf{S}}, \tag{2}$$

in the corotating frame (where $\hat{\mathbf{L}}_{\rm out} = \hat{\mathbf{z}}$ and $\hat{\mathbf{L}}_{\rm in}$ lies in the x-z plane),

1 Adiabaticity Criterion (DL) and Plots

Adiabaticity Criterion (DL): Adiabaticity requires

$$\left|\mathrm{d}\hat{\Omega}_{\rm eff,0}/\mathrm{d}t\right| \ll \Omega_{\rm eff,0} \equiv \left|\boldsymbol{\Omega}_{\rm eff,0}\right|. \tag{3}$$

To parameterize $\left|\mathrm{d}\hat{\Omega}_{\rm eff,0}/\mathrm{d}t\right|$, call $I_{\rm out}$ the angle between $\hat{\boldsymbol{\Omega}}_{\rm eff,0}$ and $\hat{\mathbf{L}}_{\rm out}$, such that

$$\hat{\boldsymbol{\Omega}}_{\rm eff,0} = \cos I_{\rm out}\hat{\mathbf{z}} + \sin I_{\rm out}\hat{\mathbf{x}}, \tag{4}$$

as shown in Fig. 1. Thus,

$$\left|\mathrm{d}\hat{\Omega}_{\rm eff,0}/\mathrm{d}t\right| = \frac{\mathrm{d}I_{\rm out}}{\mathrm{d}t}, \tag{5}$$

and the two rates of change to compare in Eq. (3) are $\dot{I}_{\rm out}$ and $\Omega_{\rm eff,0}$.

To examine how well this works, let's examine two simulations in Fig. 2. Explanation in caption.

2 Yubo's Old Result

I've continually referred to a set of equations of motion without ever explaining them correctly, but they are approximately equivalent to the results of the above section and provide more precise quantitative insights.

Construct a new coordinate system such that $\hat{\mathbf{z}}' = \hat{\boldsymbol{\Omega}}_{\rm eff,0}$, and $\hat{\mathbf{x}}'$ lies in the plane of $\hat{\mathbf{L}}_{\rm in}$ and $\hat{\mathbf{L}}_{\rm out}$ (this corresponds to rotating Fig. 1 counter-clockwise by $I_{\rm out}$). Define a spherical coordinate system $\left(\theta_{\rm eff,0}, \phi_{\rm eff,0}\right)$ for spin vector $\hat{\mathbf{S}}$ in this new coordinate system, then they can be shown to obey equations

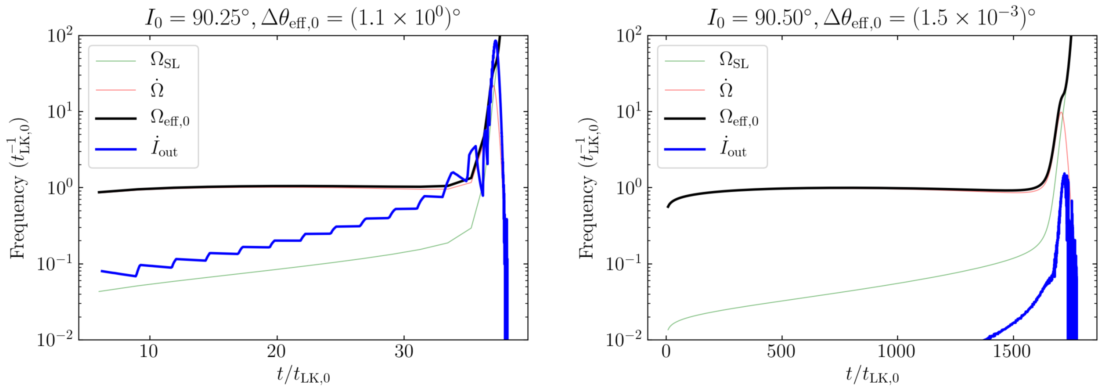

Figure 2: Plot of precession frequencies in two simulations (left: $I_0 = 90.25°$, "fast merger", right: $I_0 = 90.5°$, "slow merger"). The relevant precession frequencies are $\Omega_{\rm eff,0}$ (black) and $\dot{I}_{\rm out} \equiv |{\rm d}\hat{\boldsymbol{\Omega}}_{\rm eff,0}/{\rm d}t|$ (blue). The change in $\theta_{\rm eff,0}$ is shown in the title. Note that in the left plot, $\dot{I}_{\rm out}$ exceeds $\Omega_{\rm eff,0}$, and as a result conservation of $\theta_{\rm eff,0}$ is comparatively poor. In the right plot, $\dot{I}_{\rm out}$ is always much smaller than $\Omega_{\rm eff,0}$, causing conservation of $\theta_{\rm eff,0}$ to improve by three orders of magnitude.

of motion:

$$\frac{{\rm d}\phi_{\rm eff,0}}{{\rm d}t} = \left(-\dot{\Omega}\cos I_{\rm out} + \Omega_{SL}\left(\cos\left(I + I_{\rm out}\right)\right)\right), \tag{6}$$

$$\simeq \Omega_{\rm eff,0}, \tag{7}$$

$$\frac{{\rm d}\theta_{\rm eff,0}}{{\rm d}t} = \dot{I}_{\rm out}\cos\phi_{\rm eff,0}, \tag{8}$$

$$\approx \left|\frac{{\rm d}\hat{\boldsymbol{\Omega}}_{\rm eff,0}}{{\rm d}t}\right| \cos\left(\Omega_{\rm eff,0} t\right). \tag{9}$$

The approximate scaling in Eq. (7) can be seen as follows: $\Omega_{\rm eff,0} \simeq \max\left(\left|\dot{\Omega}\right|, \Omega_{\rm SL}\right)$, while when $\left|\dot{\Omega}\right| \gg$ $[\ll]$$\Omega_{\rm SL}$, $\cos I$ $[\cos(I + I_{\rm out})]$ ≈ 1, so the integrand also satisfies $\frac{{\rm d}\phi_{\rm eff,0}}{{\rm d}t} \simeq \max\left(\left|\dot{\Omega}\right|, \Omega_{\rm SL}\right)$. As such, Eq. (6) can indeed be approximated by Eq. (7).

Now, if we require the usual adiabaticity condition given by Eq. (3), then it is clear why Eq. (9) predicts $\Delta\theta_{\rm eff,0} \to 0$: we are integrating a small quantity multiplied by a fast-varying phase. As such, Eq. (9) *can be thought to be a quantitative prediction of the deviation from adiabaticity.*

2.1 Derivation of Equations of Motion

We provide a very brief derivation of Eqs. (6) and (8). We start with Eq. (1) and rotate about the $\hat{\mathbf{y}}$ axis (pointing into the page) such that $\hat{\boldsymbol{\Omega}}_{\rm eff,0}$ points upwards. This requires rotation by $I_{\rm out}$ satisfying

$$-\dot{\Omega}\sin I_{\rm out} + \Omega_{SL}\sin(I + I_{\rm out}) = 0. \tag{10}$$

Here, I is the angle between $\hat{\mathbf{L}}_{\rm in}$ and $\hat{\mathbf{L}}_{\rm out}$; this equation is general for $I > 90°$ and $I < 90°$. The equation of motion in this frame is then

$$\frac{{\rm d}\hat{\mathbf{S}}}{{\rm d}t} = \left[-\dot{\Omega}\cos I_{\rm out}\hat{\mathbf{z}} + \Omega_{SL}\cos\left(I + I_{\rm out}\right)\hat{\mathbf{z}} - \dot{I}_{\rm out}\hat{\mathbf{y}}\right] \times \hat{\mathbf{S}}. \tag{11}$$

The components of this equation then directly give the equations of motion I used.

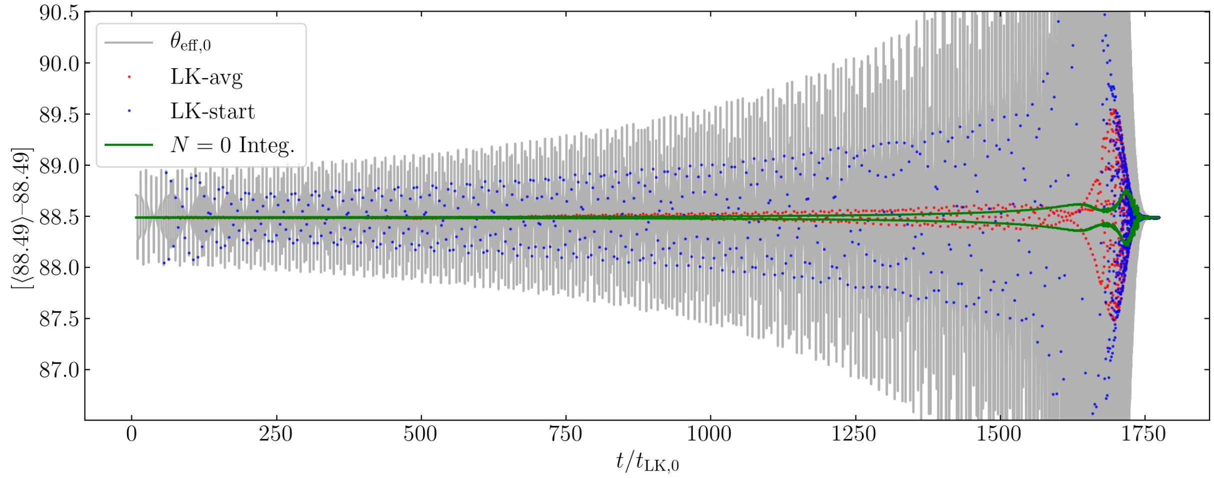

Figure 3: New plot shown today (May 29, 2020) during group meeting. Owing to complexity, the description is given in the text (Section 3).

3 Today's plot

Today, I showed an old plot with some new information, shown in Fig. 3. The plot depicts, for a full inspiral simulation[1]:

Grey Line This shows $\cos^{-1}\left(\hat{\mathbf{S}}\cdot\Omega_{\rm eff,0}\right)(t)$ at all times. Fluctuations are expected since $\hat{\mathbf{S}}$ fluctuates within each Lidov-Kozai (LK) period.

Blue Dots This shows $\cos^{-1}\left(\hat{\mathbf{S}}\cdot\Omega_{\rm eff,0}\right)(T_i)$ where each T_i is the middle of each LK period (maximum eccentricity). Changing to sample the start of each period does not change the plot significantly.

Red Dots This shows the LK-average of the grey line. Specifically, the i-th red dot denotes (for T_i the i-th LK cycle)

$$\theta_{\rm red,i} \equiv \frac{1}{T_{i+1} - T_i} \int\limits_{T_i}^{T_{i+1}} \cos^{-1}\left(\hat{\mathbf{S}}\cdot\Omega_{\rm eff,0}\right)\,\mathrm{d}t. \tag{12}$$

Green Line This is new, and possibly not very useful. A function such as $g(t) = \int^t A\cos(\omega t)\,\mathrm{dt}$ oscillates with amplitude A/ω. Thus, if we know $\dot{I}$ and $\Omega_{\rm eff,0}$ at all times, we can make a prediction about the amplitude of oscillation of $\theta_{\rm red,i}$ if the results of Section 2 are a complete description of $\theta_{\rm red,i}$ (i.e. *if the only thing driving oscillations is nonadiabaticity due to a finite-time merger*).

The green line visibly underpredicts the oscillations of the red dots at $t \sim$ 1650-1700, so the equations given in Section 2 fails to capture the dynamics of $\theta_{\rm eff,0}$ at all times, even though it predicts the final deviations well.

[1]In all quantities shown in this plot, $\Omega_{\rm eff,0}$ (an LK-averaged quantity) is linearly interpolated within each LK cycle.