However, when both Eq. (2) is not satisfied and $\varepsilon_{\rm oct}$ is not negligible, the inclination window observed by MLL16 cannot predict whether the inner orbit reaches very high eccentricities. To understand the critical value of $\eta$ below which the prescription of MLL16 is accurate, we simulate the evolution of the inner and outer binary to octupole-order. We also include general relativistic periastron advance and tidal distortion of the Jupiter following LML15. We ignore the orbital decay of the inner Jupiter due to tidal dissipation (which is expected to occur over long timescales). To isolate the impact of different values of $\eta$, we vary $m_{\rm p}$ and $a_{\rm p}$ such that the quadrupole order Lidov-Kozai timescale, given by (see e.g. LML15)

$$t_{\rm LK}^{-1} \equiv \frac{m_{\rm p}}{M_1 + m_{\rm J}} \frac{a_{\rm J}^3}{a_{\rm p}^3 \left(1 - e_{\rm p}^2\right)^{3/2}} n_{\rm J}, \tag{1}$$

is constant. In particular, we fix $e_{\rm p} = 0.6$ and the initial $e_{\rm J0} = 10^{-3}$ and consider six values of $m_{\rm p}$: $m_{\rm p} = \{1, 2, 3, 5, 10\} \times M_{\rm Jup}$ and $m_{\rm p} = M_{\odot}$, while adjusting $a_{\rm p}$ accordingly. For each value of $m_{\rm p}$, we further consider 2000 uniformly spaced initial inclinations $I_0 \in [40^\circ, 140^\circ]$. Then, for each inclination, we run three simulations while choosing $\Omega, \omega \in [0, 2\pi)$ for both the inner and outer orbits, the longitude of the ascending node and argument of periapsis respectively, totaling 6000 simulations per combination of $m_{\rm p}$ and $a_{\rm p}$. We run each simulation for $500 t_{\rm LK}$ and measure the maximum eccentricity attained by the inner planet. Figure 1 shows that the inclination window predicted by MLL16 is accurate when

$$\eta \lesssim 0.1, \tag{2}$$

so we restrict our attention to systems satisfying this criterion.

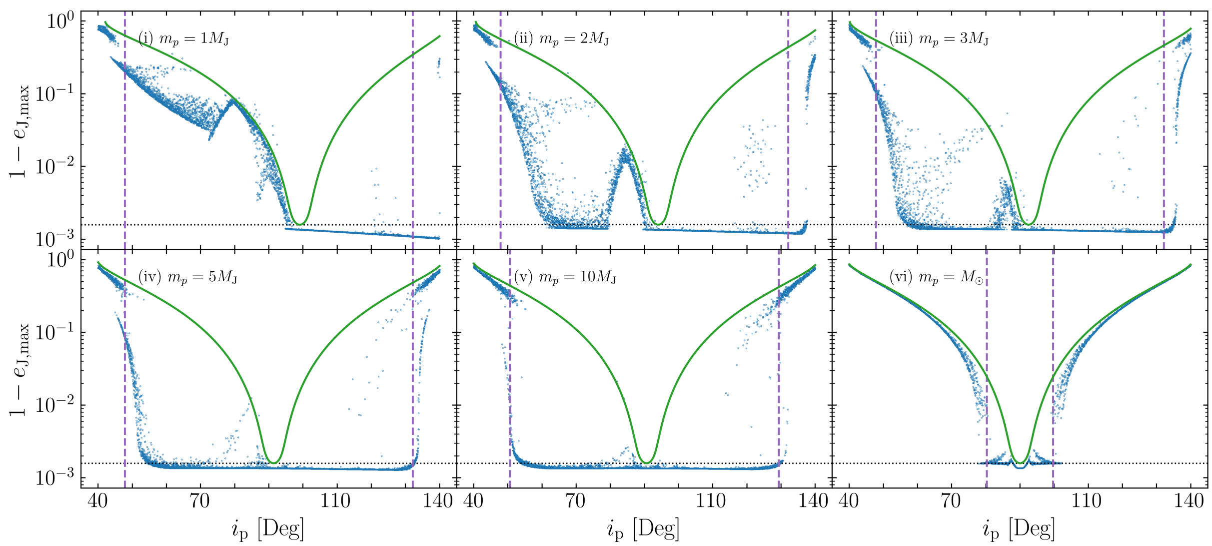


Figure 1: Maximum eccentricity of the Jupiter (as $1 - e_{\mathrm{J,max}}$) versus initial inclination $i_{\mathrm{p}}$ for different values of $m_{\mathrm{p}}$ (labelled) and $a_{\mathrm{p}}$ such that $t_{\mathrm{LK}}$ [Eq. (1)] is held constant. In all cases, we choose $e_{\mathrm{p}} = 0.6$, $M_1 = M_\odot$, $m_{\mathrm{J}} = M_{\mathrm{Jup}}$, $a_{\mathrm{J}} = 5$ AU, and we choose the The longitudes of the ascending node and arguments of pericenter are chosen randomly in $[0, 2\pi)$. The blue dots denote the maximum eccentricities attained by $m_{\mathrm{J}}$ when the vectorial secular equations are integrated for $500 t_{\mathrm{LK}}$. The green line illustrates the analytical $e_{\mathrm{J,max}}(i_{\mathrm{p}})$ curve when the octupole effect is neglected [see Eq. 50 of LML15]. The purple vertical lines denote the inclination window predicted using the fitting formula of MLL16 (see Eq. 7 of MLL16; for the simulated systems, $\epsilon_{\mathrm{oct}} \in [0.01, 0.1]$). The horizontal dashed line denotes $e_{\mathrm{lim}}$ as given by Eq. (6). When $m_{\mathrm{p}} \lesssim 3M_{\mathrm{J}}$, $\eta \gtrsim 0.1$ and is nonnegligible, and it is much more difficult for prograde outer planets ($i_{\mathrm{p}} < 90°$) to excite $e_{\mathrm{J}}$ to $e_{\mathrm{lim}}$ than it is for retrograde outer planets ($i_{\mathrm{p}} > 90°$).