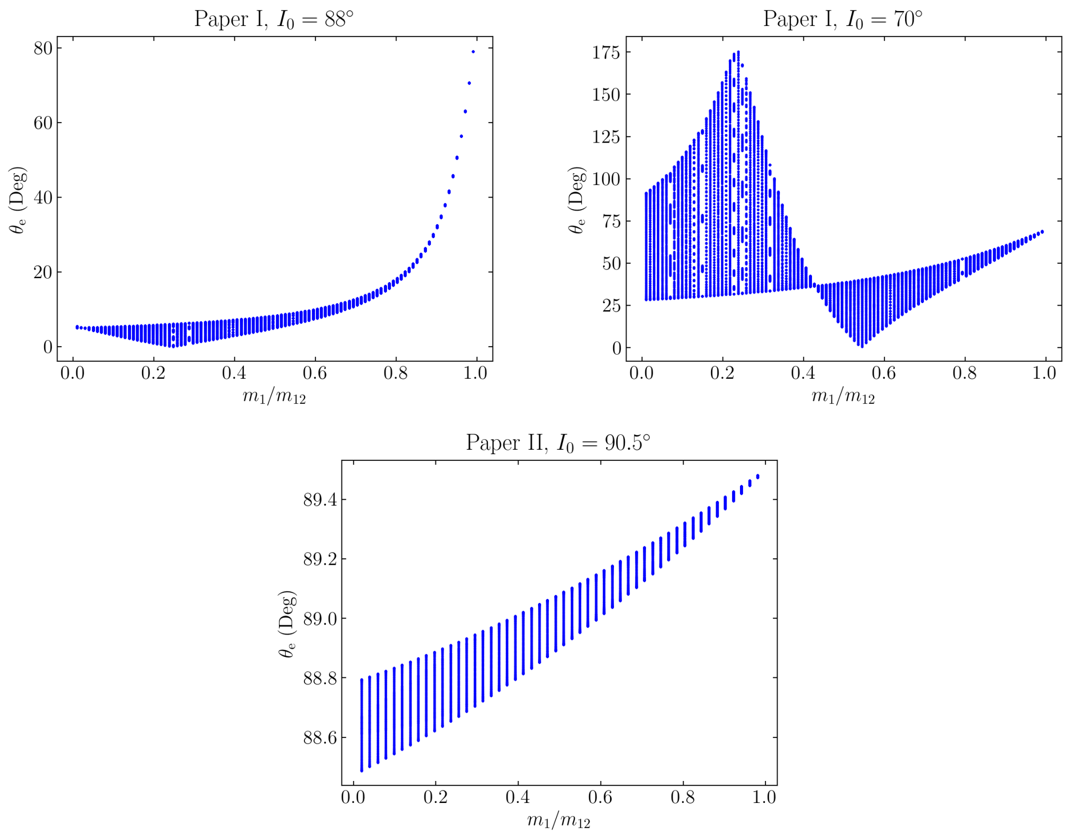

Figure 1: Bifurcation diagrams for the indicated parameters, where $\theta_{\rm e}$ is sampled at every eccentricity maximum for 200 LK cycles.

Spent most of the week writing the paper draft, so just a few plots below.

1 Bifurcation Diagram

We take $\boldsymbol{\Omega}_{\rm e}$ from a full LK simulation, and go to the co-rotating frame where $\boldsymbol{\Omega}_{\rm e} \cdot \hat{\mathbf{y}} = 0$. We then evolve

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \boldsymbol{\Omega}_{\rm e} \times \hat{\mathbf{S}}, \tag{1}$$

using the periodic solution for $\boldsymbol{\Omega}_{\rm e}$ over $200 T_{\rm LK}$ (I ran for 500 but with a bug, so I ran for 200 to have it done in time for the meeting). Then, at each eccentricity maximum, we measure

$$\cos\theta_{\rm e} \equiv \hat{\mathbf{S}} \cdot \hat{\boldsymbol{\Omega}}_{\rm e}. \tag{2}$$

We vary the mass ratio m_1/m_{12}, which changes $\Omega_{\rm SL}$ and thus $\boldsymbol{\Omega}_{\rm e}$. We run this for three parameter sets: the Paper I regime for $I_0 = 88°$ and $e_0 = 10^{-3}$, the Paper I regime for $I_0 = 70°$ and $e_0 = 10^{-3}$, and the Paper II regime for $I_0 = 90.5°$ and $e_0 = 10^{-3}$. These are shown in Fig. 1. We start from initial alignment, for convenience, so $\hat{\mathbf{S}} = \hat{\mathbf{L}}_{\rm in}$.

2 Individual Trajectories and Comparison to Floquet Theory

Consider a single trajectory, evaluated when $m_1/m_{12} = 0.5$. We consider plotting $\hat{\mathbf{S}}$ in the $\hat{\mathbf{x}}$–$\hat{\mathbf{z}}$ plane. We can evaluate $\hat{\mathbf{S}}$ at all times (that are returned by the integrator) and at eccentricity maxima.

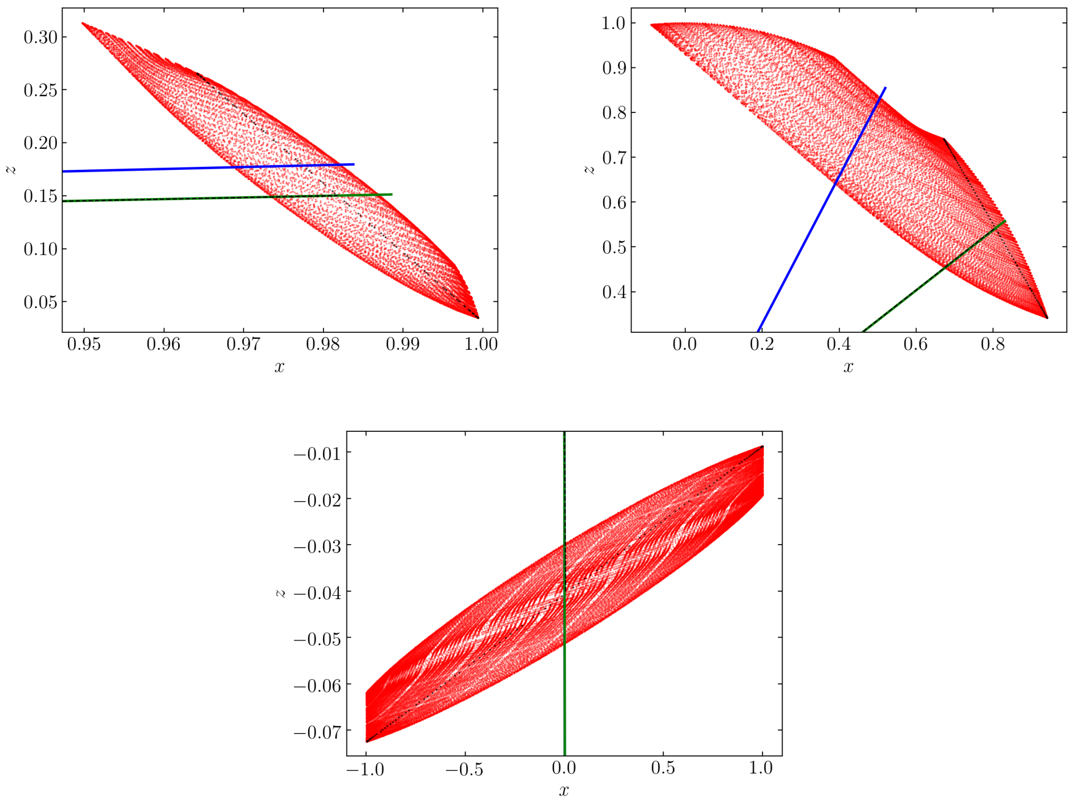

Figure 2: Single $\hat{\mathbf{S}}$ trajectories plotted in the $\hat{\mathbf{x}}$–$\hat{\mathbf{z}}$ plane. Red dots are evaluated at all times returned by the integrator, black dots are those at eccentricity maxima, blue is $\mathbf{\Omega}_{\mathrm{e}}$, green is the eigenvector of the monodromy matrix, and the black dashed line is the numerically computed axis of rotation of the black dots. Good agreement is observed between the numerical axis of rotation and the monodromy matrix eigenvector.

Three such trajectories are presented in Fig. 2.

In Fig. 2, we have included three axes: (blue) is the proposed $\mathbf{\Omega}_{\mathrm{e}}$, which is our proposed axis of rotation, (green) is the axis of the so-called *monodromy matrix*, described later, and (black dotted line) is the numerically calculated axis of rotation for the Poincaré section. Note that the aspect ratio isn't 1:1, so perpendicular lines do not appear perpendicular. It can be seen that the green line and black dotted line are in excellent agreement.

2.1 Floquet Theory

The basis of Floquet theory is that, for any linear system with periodic coefficients, the *monodromy matrix* gives the evolution over integer multiples of the period. In our problem, the monodromy matrix is constructed as follows:

- Evolve Eq. (1) using the linearly independent solutions $\hat{\mathbf{x}}$, $\hat{\mathbf{y}}$, and $\hat{\mathbf{z}}$. Call these solutions $\phi_1(t)$, $\phi_2(t)$, and $\phi_3(t)$.

- The monodromy matrix is defined ($T = T_{\mathrm{LK}}$ period):

$$\tilde{\mathbf{M}} = \begin{bmatrix}\phi_1(0) & \phi_2(0) & \phi_3(0)\end{bmatrix}^T \begin{bmatrix}\phi_1(T) & \phi_2(T) & \phi_3(T)\end{bmatrix}. \tag{3}$$

  Note that $\tilde{\mathbf{M}}$ must be a proper orthogonal matrix, so it is a rotation matrix about some rotation axis $\hat{\mathbf{R}}$ by some angle ψ. $\hat{\mathbf{R}}$ is the eigenvalue of $\tilde{\mathbf{M}}$ with eigenvalue 1.

The general solution can then be written

$$\hat{\mathbf{S}}(NT) = \tilde{\mathbf{M}}^N \hat{\mathbf{S}}(0). \tag{4}$$

Thus, every T, $\hat{\mathbf{S}}$ must rotate about $\hat{\mathbf{R}}$. $\hat{\mathbf{R}}$ is the green line in Fig. 2.

Thus, deviation from exact conservation of θ_{e} occur when $\hat{\mathbf{R}}$ differs significantly from $\mathbf{\Omega}_{\mathrm{e}}$. At late times, it's obvious that $\hat{\mathbf{R}} = \hat{\mathbf{\Omega}}_{\mathrm{e}} = \hat{\mathbf{L}}_{\mathrm{in}}$.

I still haven't checked whether the deviation from θ_{e} is due to:

- The true conserved angle should be $\hat{\mathbf{S}} \cdot \hat{\mathbf{R}}$, evaluated at each eccentricity maximum.
- Nonadiabatic passage through large changes in $\hat{\mathbf{R}}$ could kick θ_{e}.

Finally, note that we generally expect $\hat{\mathbf{R}} = \hat{\mathbf{\Omega}}_{\mathrm{e}}$ when the $N \geq 1$ components are neglected in Eq. (1), so this is consistent with our earlier picture.