Eccentric Tides

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As usual, this is kind of a scattered document. It isn't written linearly, so notation evolves somewhat as it converges to the published version. Hopefully things make sense if one jumps around a bit.

1 Kushnir et. al., 2016

We coarsely follow the derivation of Kushnir et. al., 2016 (KZ) to express the traveling wave regime of dynamical tides in high-mass stars (convective core, radiative envelope) in analytical form.

1.1 Plane Parallel Case

We will consider IGW in a plane parallel atmosphere in the Boussinesq approximation. Consider buoyancy frequency

$$N^2 = -g\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{g}{c_s^2}\right), \tag{1}$$

where $c_s \to \infty$ is the sound speed in the fluid. Then the Boussinesq equations can be written in terms of some *buoyancy* variable b

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \frac{\vec{\nabla}P}{\rho_0} + b\hat{z}, \tag{2a}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} = -N^2 u_z, \tag{2b}$$

$$\vec{\nabla}\cdot\vec{u} = 0. \tag{2c}$$

Note that $b \equiv -\frac{\rho'}{\rho_0}g$, as can be verified via direct substitution into the Euler equations:

$$0 = \frac{\mathrm{D}\rho'}{\mathrm{D}t} + \vec{u}\cdot\vec{\nabla}\rho_0 = \frac{\mathrm{D}\rho'}{\mathrm{D}t} - u_z\frac{N^2}{g}\rho', \tag{3a}$$

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \frac{\vec{\nabla}P'}{\rho_0} - \frac{\rho'}{\rho_0^2}\vec{\nabla}P_0 = \frac{\vec{\nabla}P'}{\rho_0} - \frac{\rho'}{\rho_0}g\hat{z}. \tag{3b}$$

These equations can be solved for u_z, or we can just recall the IGW dispersion relation $\omega^2 k^2 =$

$N^2k_\perp^2$ and write down PDE

$$\frac{\partial^2}{\partial t^2}\nabla^2 u_z = -N^2\nabla_\perp^2 u_z. \tag{4}$$

Now, we might recall that in tidally-forced stars, ω the tidal forcing frequency obeys $\omega \ll N$, or $k \gg k_\perp$. But the tidal potential, the quadrupolar expansion of the gravitational perturbation from the companion, has no quickly-varying directions, or can only excite $k \simeq k_\perp$ modes. Thus, we intuit that waves must be excited where N is much smaller than its typical value, or near the *radiative-convective boundary* (RCB). At the RCB, $N^2 = 0$, and we are concerned with the turning point where $\omega^2 = N^2$. We perform linear expansion about this turning point z_c, and for convenience we set $z_c = 0$, then

$$N^2 \approx \omega^2 + \frac{\mathrm{d}N^2}{\mathrm{d}z}z, \tag{5}$$

where compared to KZ I've taken $N_0^2 = \omega^2$, there seems to be little harm here. Making then general ansatz $u_z(z, \vec{r}_\perp, t) = \tilde{u}_z(z)e^{i\left(\vec{k}_\perp\cdot\vec{r}_\perp - \omega t\right)}$ we obtain

$$-\omega^2\left(-k_\perp^2\tilde{u}_z + \tilde{u}_z''\right) = N^2k_\perp^2 u_z, \tag{6}$$

$$\tilde{u}_z'' + k_\perp^2\left(\frac{N^2}{\omega^2} - 1\right)\tilde{u}_z = 0, \tag{7}$$

$$\tilde{u}_z'' + k_\perp^2\frac{\mathrm{d}N^2}{\mathrm{d}z}\frac{z}{\omega^2}\tilde{u}_z = 0. \tag{8}$$

It's easiest now to rescale $\tilde{k}_\perp^2 \equiv k_\perp^2 3\frac{\mathrm{d}N^2}{\mathrm{d}z}\frac{1}{\omega^2}$ so that

$$\tilde{u}_z'' + \tilde{k}_\perp z\tilde{u}_z = 0. \tag{9}$$

The general solution to this ODE is written in terms of Airy functions for arbitrary constants a, b

$$\tilde{u}_z(z) = a\mathrm{Ai}\left(-\frac{z}{\lambda}\right) + b\mathrm{Bi}\left(-\frac{z}{\lambda}\right), \tag{10}$$

where $\lambda = \tilde{k}_\perp^{-2/3}$. For large $-z$, it turns out that

$$\mathrm{Ai}(-z) \sim \frac{\sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)}{z^{1/4}} + \mathcal{O}\left(z^{-7/4}\right), \tag{11}$$

$$\mathrm{Bi}(-z) \sim \frac{\cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)}{z^{1/4}} + \mathcal{O}\left(z^{-7/4}\right). \tag{12}$$

In order for us to get traveling waves with *group velocity* going outwards (towards $z > 0$), we need $\tilde{u}_z(z) \sim e^{-ik_z z}$ such that $u(z,t) \propto e^{i(-k_z z - \omega t)}$ (phase velocity goes inwards, group velocity goes outwards for IGW). Thus, $a = -ib$ in Eq. 13, and we obtain

$$\tilde{u}_z(z) = b\left(-i\mathrm{Ai}\left(-\frac{z}{\lambda}\right) + \mathrm{Bi}\left(-\frac{z}{\lambda}\right)\right), \tag{13}$$

Now we just need to fix b. This is traditionally accomplished by mandating a particular $\frac{\mathrm{d}\delta z}{\mathrm{d}z}$ the displacement of the mode at the turning point $z=0$. That the forcing results in a constraint on $\frac{\mathrm{d}\delta z}{\mathrm{d}z}$ is similar to what I did in my IGW breaking forcing, where forcing induces a jump in the $\frac{\mathrm{d}u_z}{\mathrm{d}z}$ above/below z_c whose magnitude is fixed by the strength of the forcing term. In the stellar problem, it appears the correct way to obtain the δz is to solve the inhomogeneous problem in the convective zone where $N^2=0$ including the tidal potential, so it's not a perfect analogy. But since $\mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}, \mathrm{Bi}'(0) = \frac{3^{1/6}}{\Gamma(1/3)}$, this is not so difficult to evaluate, and I cite the KZ result

$$\frac{\mathrm{d}\delta z}{\mathrm{d}z} = -\frac{ib}{\lambda\omega}\frac{2}{3^{1/3}\Gamma(1/3)}\frac{3^{1/2}+i}{2}. \tag{14}$$

Finally, we impose one more step: we will compute the luminosity or *energy flux* associated with the wave, since this is the easiest way to get the resulting torque. We can easily write down the energy density of the wave $\frac{\rho_0}{2}\left(v^2+\frac{b^2}{N^2}\right)$, for which the energy flux is $\vec{F}=\vec{v}P$. Noting furthermore that $\frac{\partial u_z}{\partial z} = -ik_\perp u_x = -\frac{ik_\perp P}{\rho_0}\frac{k_\perp}{\omega}$, we can explicitly express P in terms of u_z', and so the energy flux density is then simply (I'm not evaluating this, but KZ do)

$$\frac{\delta L}{\delta A} = \frac{1}{2}\mathrm{Re}\left(Pu_z^*\right) = \frac{\rho_0\omega}{2k_\perp^2}\mathrm{Re}\left(iu_z'u_z^*\right), \tag{15}$$

$$= \frac{3^{2/3}\Gamma^2(1/3)\lambda\omega^3\rho_0}{8\pi k_\perp^2}\left(\frac{\mathrm{d}\delta z}{\mathrm{d}z}\right)^2. \tag{16}$$

We would then compute $L=\int\frac{\delta L}{\delta A}\,\mathrm{d}A$, which for us is just $\frac{\delta L}{\delta A}A$ where A is the surface area of the wave.

Finally, we would compute the total torque from $L=\tau\omega$ (the same as $E=\vec{F}\cdot\vec{v}$).

1.2 Spherical Case

To go to the spherical case, we simply replace $z\to r$ and $k_\perp^2\to l(l+1)/r^2$, which gives

$$\lambda = \left(\frac{l(l+1)}{r^2\omega^2}\frac{\mathrm{d}N^2}{\mathrm{d}r}\right)^{-1/3}. \tag{17}$$

Then to get $\frac{\mathrm{d}\delta z}{\mathrm{d}z}\to\frac{\mathrm{d}\delta r}{\mathrm{d}r}$, we use prescription

$$\frac{\mathrm{d}\delta r}{\mathrm{d}r} = \alpha\frac{\Phi}{gr}\left(1-\frac{\rho(r)}{\bar{\rho}(r)}\right). \tag{18}$$

Here,

$$\alpha = \left(\frac{r_c}{R}\right)^{-5}\left(\frac{M_c}{M}\right)\left(1-\frac{\rho}{\bar{\rho}}\right)^{-1}H_2, \tag{19}$$

while $\bar{\rho}$ is the average density inside r. Finally, instead of getting a clean $L=\frac{\delta L}{\delta A}A$, we have to actually do the integral of $L=\int\frac{\delta L}{\delta A}\,\mathrm{d}A = \int(\ldots)|Y_{lm}|^2r^2\,\mathrm{d}\cos\theta\mathrm{d}\phi = r_c^2L$ (note that it's not $4\pi r_c^2$, thanks to the Y_{lm} normalization). The $\ell = 2$ potential is taken to be

$$\Phi_{\text{ext}} = -\sqrt{\frac{6\pi}{5}} \frac{GM_2 R_c^2}{D^3}. \tag{20}$$

I guess the angular dependency is just dropped. With all these things together, we obtain the final KZ result (I omit the derivation, this part is grungy and not very physically interesting)

$$\tau = \dot{J}_z = \frac{GM_2^2 R_c^5}{D^6} \sigma_c^{8/3} \left[\frac{r_c}{g_c} \left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln R} \right)_{r=r_c} \right]^{-1/3} \frac{\rho_c}{\bar{\rho}_c} \left(1 - \frac{\rho_c}{\bar{\rho}_c} \right)^2 \left[\frac{3}{2} \frac{3^{2/3}\Gamma^2(1/3)}{5 \cdot 6^{4/3}} \frac{3}{4\pi} \alpha^2 \right], \tag{21}$$

$$= \frac{GM_2^2 R_c^5}{D^6} 2\hat{F}(r_c, \sigma_c). \tag{22}$$

Note that $\hat{F}$ follows the convention from Equation 42 of Fuller & Lai's second paper (FL2) and Vick et. al's paper as well (VLF), while $\sigma_c = 2|\Omega - \Omega_s| / \sqrt{GM_c/r_c^3}$ is the ratio of the forcing frequency to the breakup frequency *of the core*. Finally, I've replaced M_2 the mass of the companion, R_c the radius of the core, and D the separation, while retaining Ω_s spin angular frequency and Ω orbital angular frequency.

NB: The exact definition of $\hat{F}$ for a given m is given in VLF.23 as

$$\dot{J} = G \frac{M_2^2 R^5}{a^3} \frac{|m|}{2} \hat{F}(\omega) = T_0 \frac{|m|}{2} \hat{F}(\omega). \tag{23}$$

Since the total torque τ has already summed over $m = \pm 2$, we incur the extra factor of 2 above in Eq. 22. That m has already been summed over is visible in the $\sqrt{6\pi/5}$ prefactor used in Φ_{ext}, compared to $W_{2\pm2} = \sqrt{3\pi/10}$ as seen below.

2 Vick et. al., 2016

We now consider eccentric forcing. We will remove subscript compared to VLF and just call $\vec{r}_i = (r, \theta, \phi + \Omega_s t)$ the position coordinate in the inertial frame. Then the $\ell = 2$ tidal forcing potential is generally a sum over $m \in [-2, 2]$

$$U = \sum_m U_{2m}(\vec{r}, t), \tag{24}$$

$$U_{2m}(\vec{r}) = -\frac{GM_2 W_{2m} r^2}{D(t)^3} e^{-imf(t)} Y_{2m}(\theta, \phi). \tag{25}$$

Note that f is the true anomaly here. Note that W_{2m} is just a constant: $W_{20} = \sqrt{\pi/5}$, $W_{2\pm1} = 0$, and $W_{2\pm2} = \sqrt{3\pi/10}$.

This is complicated since $f(t)$ does not evolve uniformly, ando also since $D(t)$ is time-varying! The easiest treatment is to decompose

$$U_{2m} = -\frac{GM_2 W_{2m} r^2}{a^3} Y_{2m}(\theta,\phi) \sum_{N=-\infty}^{\infty} F_{Nm} e^{-iN\Omega t}. \tag{26}$$

Note that the F_{Nm} here are *Hansen coefficients* given by

$$F_{Nm} = \frac{1}{\pi} \int_0^{\pi} \frac{\cos[N(E - e\sin E) - mf(E)]}{(1 - e\cos E)^2}\,\mathrm{d}E. \tag{27}$$

Note E is the eccentric anomaly. This differs from the VLF definition in a few places but is in agreement with Natalia's paper w/ Dong (SD), such that $F_{Nm} = \delta_{Nm}$ for $e = 0$. It bears noting that VLF's formula normalizes to $F_{Nm} = 2\delta_{Nm}$, so we use the restricted domain of integration for numerical speed (the integrand is symmetric since the argument of the cosine is antisymmetric in E, so both the numerator/denominator are even in E).

Let's explicitly write out the $U_{2\pm 2}$, since they are the only ones that contribute to the tidal torque

$$U_{22} = -\frac{GM_2\sqrt{\frac{3\pi}{10}}r^2}{a^3} \sum_{N=1}^{\infty} \left[F_{N2} Y_{22}(\theta,\phi) e^{-i(N\Omega - 2\Omega_s)t} + F_{-N2} Y_{22}(\theta,\phi) e^{i(N\Omega + 2\Omega_s)t}\right], \tag{28}$$

$$U_{2-2} = -\frac{GM_2\sqrt{\frac{3\pi}{10}}r^2}{a^3} \sum_{N=1}^{\infty} \left[F_{-N2} Y_{2-2}(\theta,\phi) e^{-i(N\Omega - 2\Omega_s)t} + F_{N2} Y_{2-2}(\theta,\phi) e^{i(N\Omega + 2\Omega_s)t}\right], \tag{29}$$

$$U_{22} + U_{2-2} = -\frac{GM_2\sqrt{\frac{3\pi}{10}}r^2}{a^3} \sum_{N=1}^{\infty} \left[F_{N2} Y_{22}(\theta,\phi) e^{-i(N\Omega - 2\Omega_s)t} + c.c.\right] \tag{30}$$

We can verify that if the perturbing orbit is circular $e = 0$, then the Hansen coefficient $F_{Nm} = \delta_{Nm}$, and we obtain

$$U_{22} + U_{2-2} = -\frac{GM_2 r^2}{a^3}\sqrt{\frac{6\pi}{5}}\,\mathrm{Re}\left[Y_{22}(\theta,\phi) e^{-2i(\Omega - \Omega_s)t}\right]. \tag{31}$$

This is the same torque used in KZ. Finally, this yields torque

$$\dot{J} = \tau = T_0 \sum_{N=-\infty}^{\infty} F_{N2}^2 \,\mathrm{sgn}(N\Omega - 2\Omega_s) \hat{F}(\omega = |N\Omega - 2\Omega_s|). \tag{32}$$

2.1 Hansen Coefficients

Maybe someday follow `https://arxiv.org/pdf/1308.0607.pdf` and get the derivation of the Hansen coefficients? One fast way to calculate them is to take an FFT of the $F^{lm} = \left(\frac{r}{a}\right)^l e^{imf}$, per `https://www.aanda.org/articles/aa/pdf/2014/11/aa24211-14.pdf`(CBLR). Basically, the Hansen coefficients are just the FT of the disturbing function. Consider that we want to make jump from

$$U(r,t) = -GM_r r^2 \sum_m \frac{W_{2m}}{D(t)^3} e^{-imf(t)} Y_{2m}(\theta,\phi), \tag{33}$$

to

$$U(r,t) = -\frac{GM_2 r^2}{a^3} \sum_{m,N} W_{2m} F_{Nm}(e) Y_{2m}\left(\theta,\phi\right) e^{-in\Omega t}. \tag{34}$$

Thus, we seek coefficients such that

$$\frac{a^3}{D(t)^3} e^{-imf} = \left(\frac{1+e\cos f}{1-e^2}\right)^3 e^{-imf} = \sum_N F_{Nm} e^{-iN\Omega t}. \tag{35}$$

Thus, it's clear the Hansen coefficients are defined by computing Fourier series coefficients (NB: In hindsight, using $r = a(1-e\cos E)$ probably would have been much faster/easier)

$$F_{Nm} \equiv \frac{1}{T}\int_0^T \frac{e^{-imf}}{\left(1-e^2\right)^3}(1+e\cos f)^3 e^{iN\Omega t}\,\mathrm{d}t, \tag{36}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-imf}}{\left(1-e^2\right)^3}(1+e\cos f)^3 e^{iN\Omega t}\,\mathrm{d}M \tag{37}$$

We have notated T the period, and M the mean anomaly. Then one just evaluates using $\cos f = \frac{\cos E - e}{1 - e\cos E}$ and $M = E - e\sin E$ or more usefully $\mathrm{d}M = (1-e\cos E)\mathrm{d}E$ and obtains

$$F_{Nm} = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1+e\cos f}{1-e^2}\right)^3 e^{-imf+iN\Omega t}\,\mathrm{d}M, \tag{38}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{1-e\cos E}\right)^3 e^{-imf+iNM}(1-e\cos E)\,\mathrm{d}E, \tag{39}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\exp[i(N(E-e\sin E)-mf)]}{(1-e\cos E)^2}\,\mathrm{d}E. \tag{40}$$

Now, as we observed above, the integrand is symmetric with respect to E, but it had to be, since in an elliptical orbit the first half and second half are obviously symmetric. Thus, we arrive at final expression as promised

$$F_{Nm} = \frac{1}{\pi}\int_0^{\pi} \frac{\cos[N(E-e\sin E)-mf(E)]}{(1-e\cos E)^2}\,\mathrm{d}E. \tag{41}$$

3 Combined Results

We have been somewhat careful in checking the agreement between the VLF and KZ forms. Note now that Eq. 22 has $\hat{F}$ for a single m contribution, just as $\hat{F}$ is defined in VLF. Thus, we should be able to simply plug in

$$\tau = T_0 \sum_{N=-\infty}^{\infty} F_{N2}^2 \frac{\mathrm{sgn}(N\Omega - 2\Omega_s)}{2} \sigma_c^{8/3} \left[\frac{r_c}{g_c}\left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln R}\right)_{r=r_c}\right]^{-1/3} \frac{\rho_c}{\bar{\rho}_c}\left(1 - \frac{\rho_c}{\bar{\rho}_c}\right)^2 \left[\frac{3}{2}\frac{3^{2/3}\Gamma^2(1/3)}{5\cdot 6^{4/3}}\frac{3}{4\pi}\alpha^2\right], \tag{42}$$

$$= T_0 C(r_c) \sum_{N=-\infty}^{\infty} F_{N2}^2 \,\mathrm{sgn}(N\Omega - 2\Omega_s) |N\Omega - 2\Omega_s|^{8/3}. \tag{43}$$

Note that now $\sigma_c = |N\Omega - 2\Omega_s|/\sqrt{GM_c/r_c^3}$, and I've defined $C(r_c)$ to be some (dimensional) constant defined at the RCB and does not change with N.

Thus, the relative significance of each N term is given by the summand of Eq. 43, or

$$\tau_N \equiv F_{N2}^2 \,\mathrm{sgn}(N\Omega - 2\Omega_s)\sigma_c^{8/3}, \tag{44}$$

F_{N2} is easiest to evaluate via an integral for now, but can probably be done via a sampling + FFT when speed is necessary (CBLR).

One guess is that the sum is dominated by the contribution of the frequency at pericenter. We can compute pericenter frequency as follows:.

$$r_p^2 \Omega_p = \sqrt{GMa\left(1 - e^2\right)}, \tag{45}$$

$$\Omega_p = \sqrt{\frac{GMa(1+e)(1-e)}{a^4(1-e)^4}}, \tag{46}$$

$$= \Omega \frac{\sqrt{1+e}}{(1-e)^{3/2}}. \tag{47}$$

Thus, we should expect the dominant term to come at $N \sim \frac{\Omega_p}{\Omega} = \frac{\sqrt{1+e}}{(1-e)^{3/2}}$. This indeed very nearly maximizes F_{N2} but probably won't maximize τ_N.

NB: It appears that the maximum N to sum to is (according to Michelle/Chris)

$$N_{\max} = 10\Omega_p. \tag{48}$$

To understand exactly to what N we should sum, we should recall $|\tau_N| \propto |F_{N2}|^2 |N\Omega - 2\Omega_s|^{8/3}$. If $\Omega_s \gg \Omega_p$, then the latter term σ_c^2 is roughly independent of N and indeed we get that τ_N turns over similarly to where F_{N2} turns over. On the other hand, if $\Omega_s \lesssim \Omega_p$, then $\tau_N \propto |F_{N2}|^2 N^{8/3}$. Plots of these two are given in Fig. 1.

A plot of the actual maximum $F_{N\pm 2}$ is provided below in Fig. 2. Note that F_{N2} seems to be a constant multiple of $(1-e)^{-3/2}$ our prediction, while F_{N-2} seems to be impacted by the shallowness of the fit at smaller eccentricities. That F_{N2} has its maximum at a slight multiple of $(1-e)^{-3/2}$ should not be surprising, since the actual pericenter passage time can be computed by conservation of angular momentum

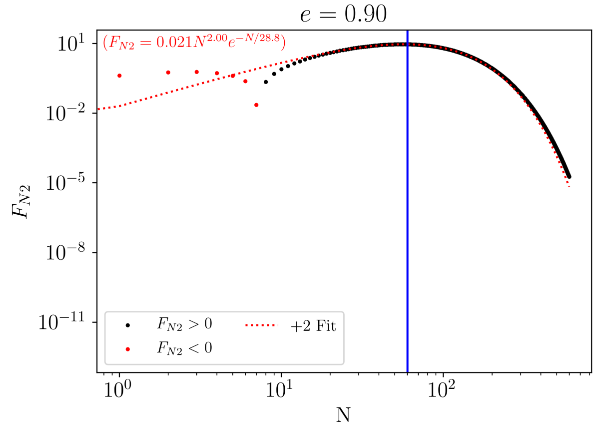

Figure 1: F_{N2} and $F_{N2}N^{8/3}$ as we integrate straightforwardly. Note the vertical blue line is $N = (1-e)^{-3/2}$ while the vertical black/green line are the actual $\mathrm{argmax}_N F_{N\pm2}$ resrpectively.

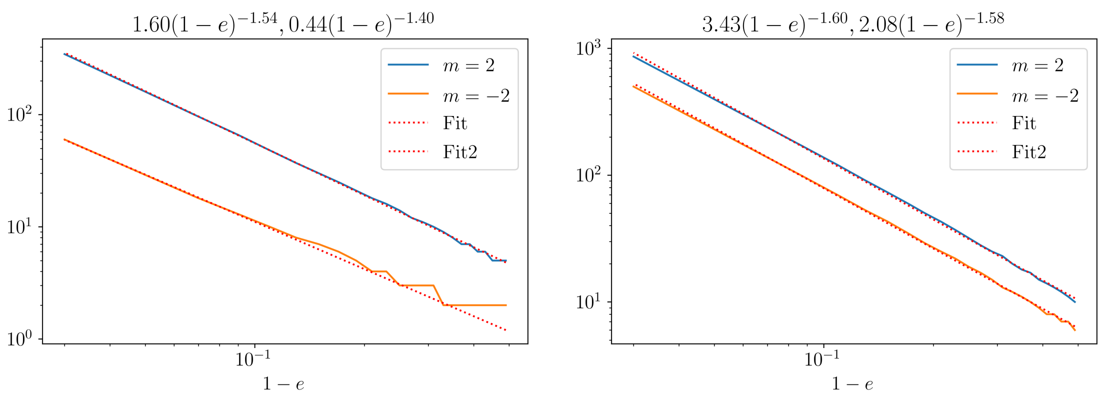

Figure 2: Maxima of $F_{N\pm2}$ and $N^{8/3}F_{N\pm2}$ with fits.

3.1 $m = 2$ Hansen Coefficient Fit

To understand the behavior of how multiplying by $N^{8/3}$ changes the peak of F_{N2}, let's consider the simplest model for the scaling of F_{N2}. Upon examination, it decays as $F_{N2} \propto e^{-aN}$, where $a \approx -\frac{1}{75}$. If we allow a simple model for $F_{N2} \propto N^q e^{-aN}$ (this conforms very coarsely with the plotted F_{N2} and allows for a maximum and an exponential tail), then we can identify that its maximum is at $N_{\max} = q/a$. On the other hand, if we seek the maximum of $N^p F_{N2}$, we find its maximum is instead at $N_{\max}^{(p)} = (p+q)/a$. Comparing the two, we expect the maximum to be shifted by roughly factor $\frac{p+q}{q}$. Since the actual shift is $\gtrsim 2$ for $p = 8/3$, we can guess $q \approx 2$, which is plausible gauging from our loglog plot.

Empirically, we find that the $F_{N\pm2} = CN^p e^{-N/a}$ is actually a surprisingly good fit. This is not surprising: the coefficients must be small for $N \ll N_{\max}$, but must fall off exponentially for $N \gg N_{\max}$. Note that simple calculus shows us that $\mathrm{argmax}_N F_{N\pm2} = p/a$, and so $a = p/N_{\max} \simeq \frac{p\Omega_0}{\Omega_p}$. Furthermore, between Ω, Ω_p, there is no preferred timescale, so a power law dependence between Ω, Ω_p is expected (and $\Omega \ll \Omega_p \approx 0$ for large eccentricities).

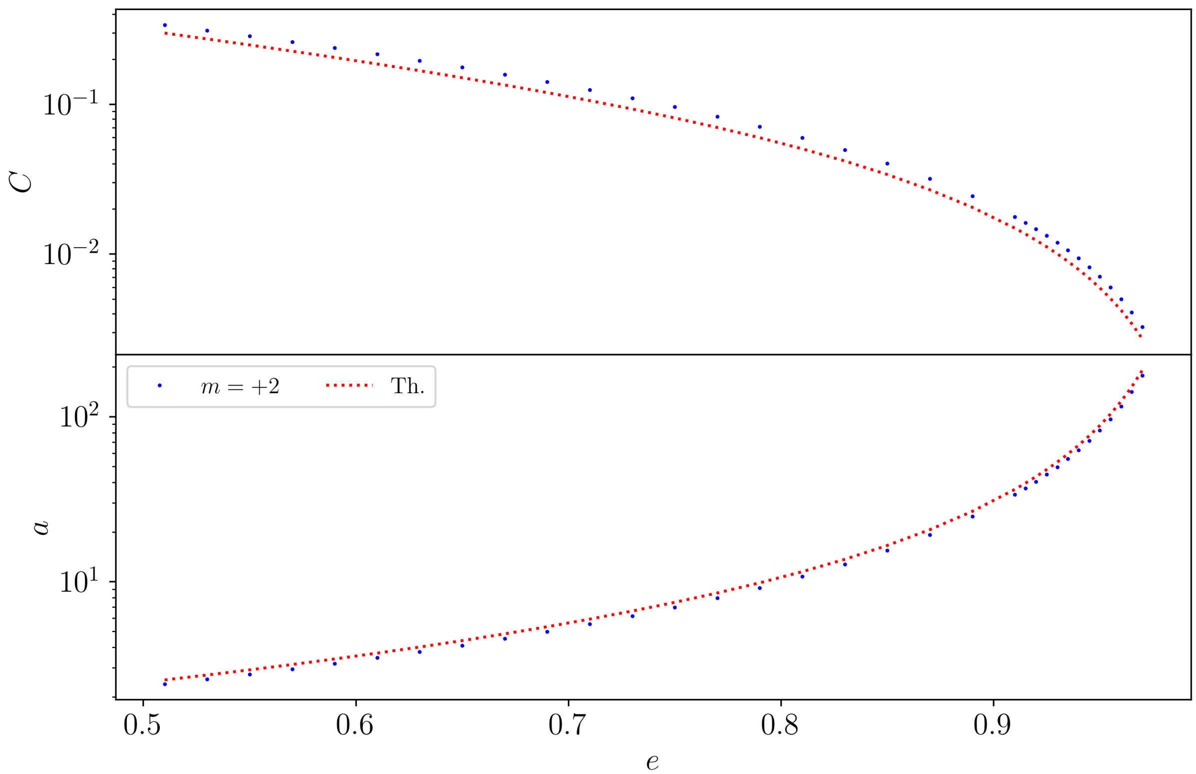

Figure 3: Params of the $P_{N\pm 2} = CN^p e^{-N/a}$ fit as function of eccentricity.

3.1.1 Parameter Scalings

The next thing to check is how robust these parameters are, and whether they have analytical forms. We present such a comparison in Fig. 3. The good news is that p is relatively independent of eccentricity! This gives obvious scaling for

$$a = \frac{N_{\max}}{p} = \frac{\sqrt{1+e}}{p(1-e)^{3/2}}, \tag{49}$$

since I changed convention $P_{n\pm 2} \propto e^{-N/a}$ whoops.

The final difficulty is understanding how C scales. To understand this, the easiest thing to do seems to be to invoke Parseval's Theorem, which will let us claim that

$$\frac{1}{T}\int_0^T \left|\frac{a^3}{D(t)^3} e^{-imf}\right|^2 \mathrm{d}t = \sum_{N=-\infty}^{\infty} |F_{Nm}|^2. \tag{50}$$

Let's assume for now that $F_{N-2} \ll F_{N2}$[1]. Let's further assume that $\mathrm{Re}\, F_{N2} \gg \mathrm{Im}\, F_{N2}$ (no idea if this is

[1]This is empirically true, but also, we can examine the F_{Nm} integral and observe that for large e, $f(E)$ will only be nonzero if $\delta E \lesssim \sqrt{\delta e}$ ($\delta e = 1 - e$; examine the arctan relation, but also, $f(E) = \pi$ at apoapsis), then the argument of the cosine is $\left(N\delta e - m/\sqrt{\delta e}\right)\delta E$. Thus, it's quickly oscillating for most N unless $N \sim m/\delta e^{3/2}$, roughly our $N_{\max}$ criterion. Also,

the case yet), then the sum over coefficients is just dominated the contribution from the $C_+N^{p_+}e^{-N/a_+}$ parts, which we can approximate with an integral analytically.

What about the time-domain integral? Well, this seems to evaluate cleanly

$$\frac{1}{T}\int_0^T \frac{a^6}{D^6}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1+e\cos f}{1-e^2}\right)^6 \frac{\left(1-e^2\right)^{3/2}}{(1+e\cos f)^2}\,\mathrm{d}f, \tag{51}$$

$$= \frac{1}{2\pi\left(1-e^2\right)^{9/2}}\int_0^{2\pi}(1+e\cos f)^4\,\mathrm{d}f, \tag{52}$$

$$= \frac{1}{2\pi\left(1-e^2\right)^{9/2}}\int_0^{2\pi} 1+4e\cos f+6e^2\cos^2 f+4e^3\cos^3 f+e^4\cos^4 f\,\mathrm{d}f, \tag{53}$$

$$= \frac{1}{\left(1-e^2\right)^{9/2}}\left[1+3e^2+\frac{3e^4}{8}\right]. \tag{54}$$

This is analytic and correct (I double checked numerically). The sum of coefficients can be approximated (see later)

$$\sum_{N=-\infty}^{\infty}\left|F_{Nm}\right|^2 \approx \sum_{N=0}^{\infty}\left(\mathrm{Re}F_{Nm}\right)^2, \tag{55}$$

$$\approx C_+^2(a_+/2)^{2p_++1}\Gamma(2p_++1). \tag{56}$$

This agreement is remarkable (`check_ft_parsevals.png`). Thus, explicitly we may write down

$$C_+ \approx \sqrt{\frac{1+3e^2+\frac{3e^4}{8}}{\left(1-e^2\right)^{9/2}}\frac{1}{(a_+/2)^{2p_++1}\Gamma(2p_++1)}}. \tag{57}$$

Thus, if we simply take $p_+=2$, we have analytical predictions for each of the params. It turns out that we want closer to $p_+=2, N_{\max}=\sqrt{2}\Omega_p/\Omega$, but this gives us a very good fit, as seen in Fig. 3.

3.2 $m=0$ Hansen Coefficients

Consider the $m=0$ Hansen coefficients, which are given by

$$\frac{a^3}{D(t)^3} = \sum_{N=-\infty}^{\infty} F_{N0}e^{-iN\Omega t}. \tag{58}$$

Again, the tail must be exponential, but since there is obviously symmetry about $N=0$, the peak is at $N=0$. We might have naively expected a Gaussian, thanks to the tight peaking in the time domain about $t=0$, but instead what we observe is something like $F_{N0}\propto e^{-\sqrt{2}N/N_{\mathrm{peri}}}$ for some reason.

it is quickly oscillating when N,m have opposite signs.

Assuming this is the case, we can again invoke Parseval's and find

$$\frac{1}{\left(1-e^2\right)^{9/2}}\left[1+3e^2+\frac{3e^4}{8}\right] \approx 2\int_0^\infty C^2 e^{-\frac{N2\sqrt{2}}{N_{\rm peri}}}\,\mathrm{d}N, \tag{59}$$

$$\approx 2C^2 N_{\rm peri}\sqrt{2}, \tag{60}$$

$$C^2 = \frac{1+3e^2+\frac{3e^4}{8}}{\left(1-e^2\right)^{9/2}}\frac{1}{2N_{\rm peri}\sqrt{2}}. \tag{61}$$

3.3 Effective Torque

With this approximate closed form, it is very easy to sum over all N by approximating as an integral:

$$\sum_{N=1}^{\infty} N^q F_{N\pm 2} \approx \int_0^\infty C N^{p+q} e^{-N/a}\,\mathrm{d}N \tag{62}$$

The RHS is almost a Gamma function though!! Thus

$$\sum_{N=1}^{\infty} N^q F_{N\pm 2} \approx C a^{p+q+1}\int_0^\infty (N/a)^{p+q} e^{-N/a}\,\mathrm{d}(N/a), \tag{63}$$

$$\approx C a^{p+q+1}\Gamma(p+q+1). \tag{64}$$

Thus, we return to our total torque. Call $F_{N\pm 2} = C_\pm N^{p_\pm} e^{-N/a_\pm} = F_{\pm N2}$, then

$$\tau = T_0 C(r_c)\sum_{N=-\infty}^{\infty} F_{N2}^2\,\mathrm{sgn}(N\Omega - 2\Omega_s)\sigma_c^{8/3}, \tag{65}$$

$$= T_0 C(r_c)\frac{\Omega^{8/3}}{\left(GM_c/r_c^3\right)^{4/3}}\sum_{N=-\infty}^{\infty} F_{N2}^2\,\mathrm{sgn}\left(N-2\frac{\Omega_s}{\Omega}\right)\left|N-2\frac{\Omega_s}{\Omega}\right|^{8/3}, \tag{66}$$

$$= T_0\hat{C}(r_c)\sum_{N=0}^{\infty}\left[F_{N2}^2\,\mathrm{sgn}\left(N-2\frac{\Omega_s}{\Omega}\right)\left|N-2\frac{\Omega_s}{\Omega}\right|^{8/3} - F_{-N2}^2\left|-N-2\frac{\Omega_s}{\Omega}\right|^{8/3}\right], \tag{67}$$

$$\approx T_0\hat{C}(r_c)\left[\int_0^\infty C_+^2 N^{2p_+} e^{-2N/a_+}\,\mathrm{sgn}\left(N-2\frac{\Omega_s}{\Omega}\right)\left|N-2\frac{\Omega_s}{\Omega}\right|^{8/3}\mathrm{d}N - \int_0^\infty C_-^2 N^{2p_-} e^{-2N/a_-}\left|-N-2\frac{\Omega_s}{\Omega}\right|^{8/3}\mathrm{d}N\right]. \tag{68}$$

Note that the summation should not double count the F_{02} term; this is resolved correctly in the integral, where the $N=0$ contribution is not double counted. It's convenient to call

$$\hat{\tau}_N \equiv F_{N2}^2\,\mathrm{sgn}\left(N-2\frac{\Omega_s}{\Omega}\right)\left|N-2\frac{\Omega_s}{\Omega}\right|^{8/3}. \tag{69}$$

At this point, let's specialize to two regimes:

- Let $2\Omega_s \ll N_{\rm max}\Omega$, then the sign term is just the sign of N, and $\left|\pm N - \frac{2\Omega_s}{\Omega}\right| \sim |N|\left(1-\frac{\Omega_s/\Omega}{N_{peri}}\right)$. This gives (dropping the C_- terms since we've seen they're unimportant and we don't have fits for them)

$$\tau \approx T_0 \hat{C}(r_c)\left(1 - \frac{\Omega_s/\Omega}{N_{peri}}\right)^{8/3} \left[C_+^2 \int_0^\infty N^{2p_+ + 8/3} e^{-2N/a_+}\,\mathrm{d}N\right], \tag{70}$$

$$\approx T_0 \hat{C}(r_c)\left[C_+^2 (a_+/2)^{2p_+ + 11/3} \Gamma(2p_+ + 11/3)\right]. \tag{71}$$

- Alternatively, let $2\Omega_s \gg N_{\max}\Omega$, then the sign is just always negative and $\left|N - \frac{2\Omega_s}{\Omega}\right| = \frac{2\Omega_s}{\Omega} - N_{\max}$, and we find

$$\tau \approx T_0 \hat{C}(r_c)\left(\frac{2\Omega_s}{\Omega} - N_{\max}\right)^{8/3} \left[-C_+^2 \int_0^\infty N^{2p_+} e^{-2N/a_+}\,\mathrm{d}N\right], \tag{72}$$

$$\approx T_0 \hat{C}(r_c)\left(\frac{2\Omega_s}{\Omega} - N_{\max}\right)^{8/3} \left[-C_+^2 (a_+/2)^{2p_+ + 1} \Gamma(2p_+ + 1)\right]. \tag{73}$$

Indeed, for $2\Omega_s \ll N_{\max}\Omega$, we find $\tau > 0$, while for $2\Omega_s \gg N_{\max}\Omega$, we obtain $\tau < 0$, which obeys intuition and suggests some synchronization frequency around $2\Omega_s \simeq \Omega_p$.

We make some plots as in Fig. 4 and these agree. What remains is to identify the eccentricity scaling, which will require understanding how the fit coefficients depend on e, which we've also plotted but don't quite yet understand so don't include.

3.4 Heating

Heating in the inertial frame is given (absorbed the second N into the absolute value exponent)

$$\dot{E}_{in} = \frac{1}{2}\hat{T}(r_c, \Omega)\left[\sum_{N=-\infty}^{\infty} N\Omega F_{N2}^2 \,\mathrm{sgn}(\sigma)|\sigma|^{8/3} + \left(\frac{W_{20}}{W_{22}}\right)^2 \Omega F_{N0}^2 |N|^{11/3}\right]. \tag{74}$$

I'm lazy so $\sigma = N - \frac{2\Omega_s}{\Omega}$, and $(W_{20}/W_{22})^2 = 2/3$.

The former term follows our work above, for instance in the $\Omega_s \approx 0$ limit, we just have the first term (subscript)

$$\dot{E}_1 = \frac{\hat{T}\Omega}{2} C^2 \left(\frac{\eta}{2}\right)^{26/3} \Gamma(26/3), \tag{75}$$

or in the $\Omega_s \gg 0$ case

$$\dot{E}_1 = \frac{\hat{T}\Omega}{2} C^2 \left(\frac{2\Omega_s}{\Omega} - N_{\max}\right)^{8/3} (\eta/2)^6 \Gamma(6). \tag{76}$$

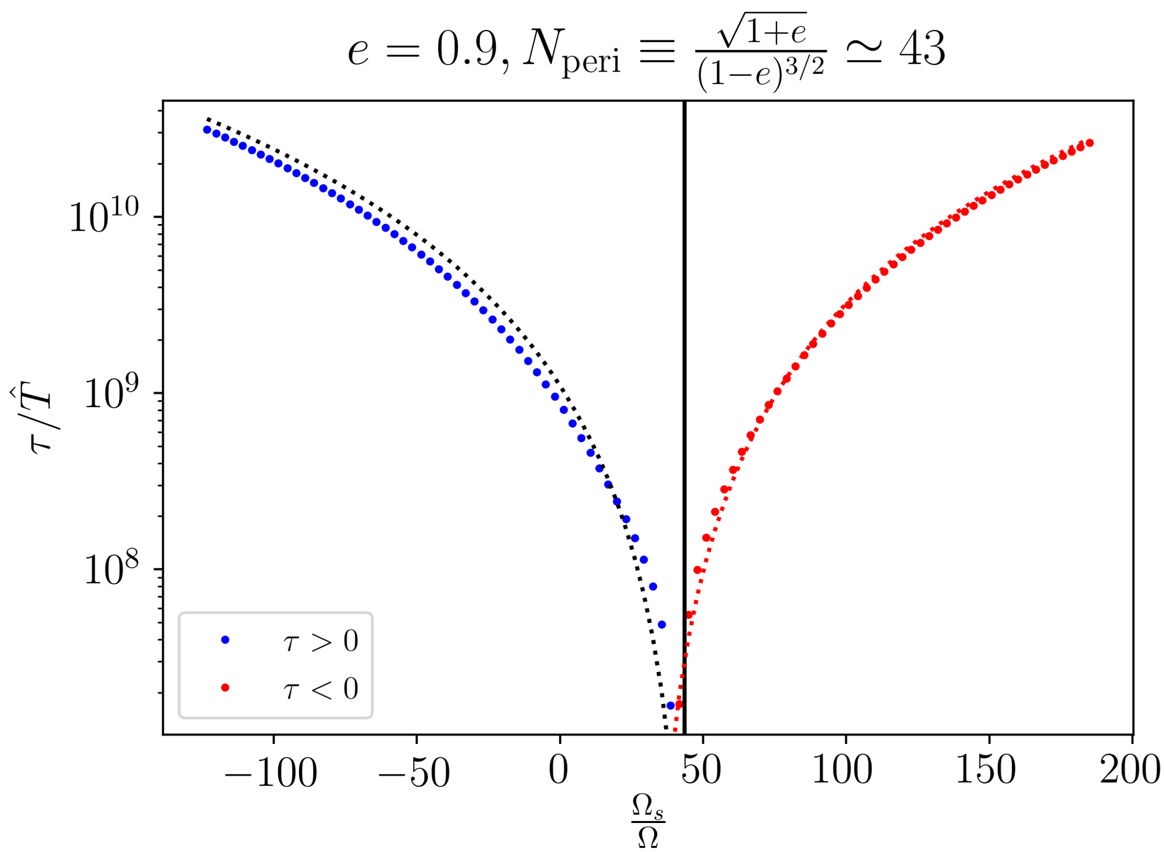

Figure 4: Plot of $\frac{\tau}{T_0 \hat{C}(r_c)}$. The predictions in the two regimes are the black horizontal line $\Omega_s = 0$ and red dashed line $2\Omega_s \gg N_{\max}\Omega$. Good agreement!

For the latter term, it is symmetric about N, so we can integrate about just $N \geq 0$:

$$\dot{E}_2 = \hat{T}\Omega\frac{2}{3}\int\limits_0^\infty C_0^2 e^{-\frac{N}{N_{\rm peri}}2\sqrt{2}} N^{11/3}\,\mathrm{d}N, \tag{77}$$

$$= \hat{T}\Omega\frac{2}{3}C_0^2\left(\frac{N_{\rm peri}}{2\sqrt{2}}\right)^{14/3}\Gamma(14/3). \tag{78}$$

4 J0045–7319

Let's try to plug in some realistic numbers. Consider J0045+7319, which has parameters $e = 0.808$, $M_t = 10.2M_\odot$, $a = 126R_\odot$, $\Omega_o = 1.42 \times 10^{-6}$ rad/s, $\Omega_p/2\pi = 3.61\mu$Hz. We can assume the mass of one star is $1.4M_\odot$, the NS, so the other is $8.8M_\odot$.

Some fiducial numbers from KQ include that the B star has RCB at $0.23R_*$, where the star is taken to be roughly $6.4R_\odot$, and the mass of the convective core is $3M_\odot$, the density and BV frequency outside the core ar 2 g/cm^3 and 100μHz respectively. Finally, the star is observed to have $\frac{\dot{a}}{a} \sim 5 \times$

10^5 /yr.

Let's first evaluate $\hat{T}$, the circular orbit limit:

$$\hat{T}(r_c,\omega) \equiv \frac{GM_2^2 r_c^5}{a^6}\left(\frac{\omega}{\sqrt{GM_c/r_c^3}}\right)^{8/3}\left[\frac{r_c}{g_c}\left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln r}\right)_{r=r_c}\right]^{-1/3}\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2\left[\frac{3}{2}\frac{3^{2/3}\Gamma^2(1/3)}{5\cdot 6^{4/3}}\frac{3}{4\pi}\alpha^2\right]. \tag{79}$$

Let's take Kushnir's prescription and set $\beta_2 = 1$, so $\hat{T}$ is now

$$\hat{T}(r_c,\omega) \approx \beta_2\frac{GM_2^2 r_c^5}{a^6}\left(\frac{\omega}{\sqrt{GM_c/r_c^3}}\right)^{8/3}\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2. \tag{80}$$

We evaluate in pieces:

- Note that $\frac{GM_2^2 r_c^5}{a^6} = 9.3\times 10^{29}\,\mathrm{N\cdot m}$ for the given parameters.
- Then evaluating $\hat{T}$ at the orbital frequency, we can evaluate the ω term, which is 1.56×10^{-7}.
- Let's just take $\rho_c/\bar{\rho}_c \sim 0.3$, then we can evaluate the final piece 0.147.

This tells us that $\hat{T} \approx 2.13\times 10^{22}\,\mathrm{N\cdot m}$. Since we know $\dot{E} = \frac{\hat{T}\Omega}{2}$ in a circular orbit, then we have

$$\frac{\hat{T}\Omega}{2} \approx 3.1\times 10^{16}\,\mathrm{W}, \tag{81}$$

$$E \approx 1.85\times 10^{40}\,\mathrm{J}, \tag{82}$$

$$\left.\frac{\dot{E}}{E}\right|_{circ} = -\frac{\dot{a}}{a} \sim 0.56\times 10^{24}\,/\mathrm{s} \sim 1.7\times 10^{16}\,/\mathrm{yr}. \tag{83}$$

What about the eccentricity enhancement? It turns out that $\frac{\dot{E}}{\hat{T}\Omega} \sim 1.1\times 10^8$ when directly summed (according to `get_energies` in my routine, if I take the spin to be zero), which is still a bit slow. But of course, we should expect the spin to be substantially larger than Ω, and retrograde. We need an enhancement closer to 3×10^{10}, which is satisfied by $\frac{\Omega_s}{\Omega} \sim -160$. The breakup frequency for the core using the above parameters is 6.089×10^{-4} rad/s $\approx 428\Omega_o$, so our calculation proposes we need $\frac{\Omega_s}{\Omega_c} = -0.37$, which seems quite fast (and is in fact faster than the breakup rotation rate of the star!).

Let's try running this via MESA. It looks like the metallicity should be somewhere around $0.1\times$ or $0.2\times$ solar[2]. Hilariously, the old mass estimate comes from forcing a $1.4M_\odot$ NS, which is not accurate; Thorsett & Chakrabatty 1999 give a new estimate $\sim 10M_\odot$. We want a star of the correct metallicity to reproduce the only observational constraints, $L \sim 1.2\times 10^4 L_\odot$ and $T \sim 24000$ K. This seems to be hard to satisfy, my original MESA simulations have higher temperatures and lower luminosities, at least on the MS down to a central hydrogen abundance ~ 0.5. Taking $M = 10.3M_\odot$, we can plug in some values and obtain $\hat{T} \approx 2.51\times 10^{19}$. For a new breakup frequency $\Omega_* = 0.001609 \approx 1133\Omega_o$, even at critical rotation we only have an enhancement 4.37×10^{12}, not enough to explain the tidal decay.

[2] https://academic.oup.com/mnras/article/422/2/1109/1032973#18429474

The principal difficulty seems to come in that the MESA star is always much smaller, both in the total stellar radius and the core radius.

5 Comparison with Existing Work, Moving Past Massive Stars

Key results are Eq. (135).

5.1 Weak Tides w/ Hut, Storch and Vick: Updating F_{N2}

In the aforementioned papers, results of form

$$\frac{1}{2}\sum_N F_{N2}^2\left(N - m\frac{\Omega_s}{\Omega}\right) = \frac{1}{\left(1-e^2\right)^6}\left[f_2 - \left(1-e^2\right)^{3/2} f_5 \frac{\Omega_s}{\Omega}\right], \tag{84}$$

are obtained, where $m = 2$, where $f_5 = 1 + 3e^2 + 3e^4/8$ and $f_2 = 1 + 15e^2/2 + 45e^4/8 + 5e^6/16$ are the relevant numbers. In the latter term, the Parseval's result is simply reproduced exactly and is in agreement with above:

$$\sum_N -F_{N2}^2\frac{\Omega_s}{\Omega} = \left(-\frac{\Omega_s}{\Omega}\right)\left(\frac{1+3e^2+3e^4/8}{\left(1-e^2\right)^{9/2}}\right). \tag{85}$$

The more interesting case is the linear term. An extended application of Parseval's Theorem is in play, where the two functions are not the same. We then must use

$$\frac{\mathrm{d}}{\mathrm{d}(-i\Omega t)}\frac{a^3}{D^3}e^{-imf} = \sum_N F_{Nm} N e^{-iN\Omega t}, \tag{86}$$

as then (star denotes conjugate)

$$\begin{aligned}
\sum_N F_{N2}^2 N &= \frac{1}{T}\int \frac{a^3}{D^3}e^{-imf}\left(\frac{\mathrm{d}}{\mathrm{d}(-i\Omega t)}\left(\frac{a^3}{D^3}e^{-imf}\right)\right)^* \mathrm{d}t, && (87)\\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{a^3}{D^3}e^{-imf}\left(i\frac{\mathrm{d}}{\mathrm{d}f}\left(\frac{a^3}{D^3}e^{-imf}\right)\right)^* \mathrm{d}f, && (88)\\
&= \frac{1}{2\pi}\int_0^{2\pi} \frac{a^6}{D^6}\left(m + \frac{3e\sin f}{1+e\cos f}\right)\mathrm{d}f. && (89)
\end{aligned}$$

Recall that $a/D = (1+e\cos f)/\left(1-e^2\right)$, so

$$\sum_N F_{N2}^2 N = \frac{1}{2\pi\left(1-e^2\right)^6}\int_0^{2\pi}(1+e\cos f)^6\left(m + i\frac{3e\sin f}{1+e\cos f}\right)\mathrm{d}f. \tag{90}$$

The $\sin f$ term integrates to zero, and setting $m = 2$ cancels with the prefactor, and we can explicitly integrate (only even powers survive)

$$\frac{1}{2}\sum_N F_{N2}^2 N = \frac{1}{2\pi\left(1-e^2\right)^6}\int_0^{2\pi}\left(1+15e^2\cos^2 f + 15e^4\cos^4 f + e^6\cos^6 f\right)\mathrm{d}f, \tag{91}$$

$$= \frac{1}{\left(1-e^2\right)^6}\left(1+\frac{15e^2}{2}+\frac{45e^4}{8}+\frac{5e^6}{16}\right). \tag{92}$$

Note that we have used the following identities:

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2 x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+\cos 2x}{2}\,\mathrm{d}x = \frac{1}{2}, \tag{93}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^4 x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1+\cos 2x}{2}\right)^2\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+\cos^2 2x}{4} + (\ldots)\,\mathrm{d}x = \frac{3}{8}, \tag{94}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^6 x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1+\cos 2x}{2}\right)^3\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+3\cos^2 2x}{8} + (\ldots) = \frac{5}{16}. \tag{95}$$

Now, how does our fitted formula compare to this? Well, we really only have to compare the $F_{N2}^2 N$ term, so

$$\sum_N F_{N2}^2 N \approx \int_0^\infty C^2 N^4 e^{-2N/\eta} N\,\mathrm{d}N, \tag{96}$$

$$= \frac{f_5}{\left(1-e^2\right)^{9/2}}\frac{1}{(\eta/2)^5 4!}\int_0^\infty N^5 e^{-2N/\eta}\,\mathrm{d}N, \tag{97}$$

$$= \frac{f_5}{\left(1-e^2\right)^{9/2}}\frac{1}{(\eta/2)^5 4!}\left(\frac{\eta}{2}\right)^6\int_0^\infty\left(\frac{2N}{\eta}\right)^5 e^{-2N/\eta}\,\mathrm{d}\frac{2N}{\eta}, \tag{98}$$

$$= \frac{f_5}{\left(1-e^2\right)^{9/2}}\frac{1}{(\eta/2)^5 4!}\left(\frac{\eta}{2}\right)^6\int_0^\infty\left(\frac{2N}{\eta}\right)^5 e^{-2N/\eta}\,\mathrm{d}\frac{2N}{\eta}, \tag{99}$$

$$= \frac{f_5}{\left(1-e^2\right)^{9/2}}\frac{1}{(\eta/2)^5 4!}\left(\frac{\eta}{2}\right)^6 5!. \tag{100}$$

Recall that $\eta = \frac{N_{\max}}{2} = \frac{\sqrt{1+e}}{2\left(1-e^2\right)^{3/2}}\alpha$ per the results of subsection 3.1, where we guessed $\alpha \approx \sqrt{2}$, and so everything all together evaluates to

$$\sum_N F_{N2}^2 N \approx \frac{f_5}{\left(1-e^2\right)^6}\frac{5\sqrt{1+e}}{4}\alpha. \tag{101}$$

We thus want to compare

$$2f_2 = 2\left(1 + \frac{15e^2}{2} + \frac{45e^4}{8} + \frac{5e^6}{16}\right) \approx \alpha\frac{5}{4}\sqrt{1+e}\left(1 + 3e^2 + \frac{3e^4}{8}\right). \tag{102}$$

Our approximation is valid when $e \to 1$, so let's first set $e = 1$ and evaluate. This fixes α, and indeed when we do so we find $\frac{231}{8} \approx \frac{175\sqrt{2}}{32}\alpha$, so $\alpha = \frac{462\sqrt{2}}{175} \approx 3.7335$. This is all verified numerically.

Now, what about when $\delta e \equiv 1 - e \ll 1$? Let's expand

$$2\left(1 + \frac{15(1-2\delta e)}{2} + \frac{45(1-4\delta e)}{8} + \frac{5(1-6\delta e)}{16}\right) \approx \frac{5\alpha\sqrt{2}}{4}\left(1 - \frac{\delta e}{4}\right)\left(1 + 3(1-2\delta e) + \frac{3}{8}(1-4\delta e)\right), \tag{103}$$

$$\frac{231}{8} - \frac{315}{4}\delta e \approx \frac{5\alpha\sqrt{2}}{4}\left(\frac{35}{8} - \frac{275}{32}\delta e\right) = \frac{231}{8} - \frac{1815}{32}\delta e \tag{104}$$

The leading order correction on the LHS evaluates to be 78.75, while the LHS is 56.71. We could enforce exact agreement with this by relaxing $p = 2$, but then we wouldn't be able to match $\sum F_{N2}N^2$ anyway, so this seems to be acceptable.

Of note: we only set α by setting the expressions equal at $e \to 1$, but what about in general? Let's make a coarse plot, and of course it's linear and not constant... see Fig. 5. It's approximately linear between $\alpha(0) = 8/5$ and $\alpha(1) \approx 3.7335$. We should probably just generally use the exact value of $\alpha(e)$

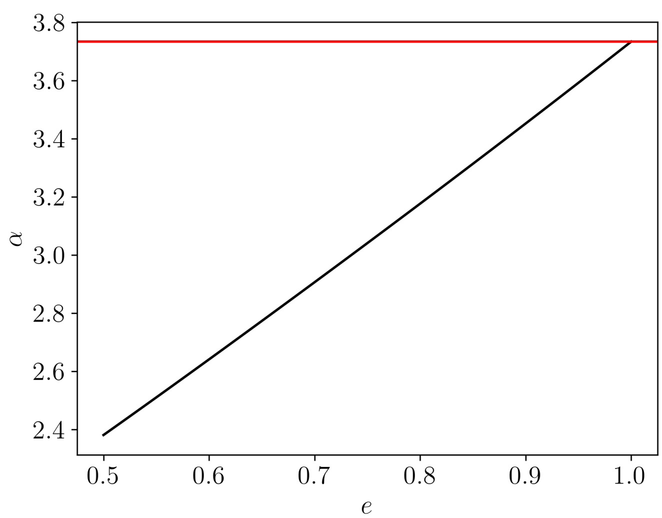

Figure 5: Plot of $\alpha(e)$...

then; it only changes η and not C or $p = 2$, so it's not too bad.

5.2 Also Updating F_{N0}

Before, we postulated that $F_{N0} \propto e^{-\sqrt{2}|N|/N_{peri}}$, but we can do better now, with updated knowledge. Note that

$$\sum_N F_{Nm}^2 = \frac{1}{T}\int_0^T \frac{a^3}{D^3}e^{-imf}\frac{a^3}{D^3}e^{+imf}\,\mathrm{d}t. \tag{105}$$

This makes it obvious that $\sum_N F_{N2}^2 = \sum_N F_{N0}^2 = f_5/\left(1-e^2\right)^{9/2}$. Taking our fitting formula to be of form $F_{N0} = Ce^{-|N|/\eta}$, we find

$$\int_{-\infty}^{\infty} F_{N0}^2 \,\mathrm{d}N = C^2\left(\frac{\eta}{2}\right). \tag{106}$$

Furthermore, the first moment is also easier to write down

$$\sum_N F_{N0}^2 N = \frac{1}{T}\int \frac{a^3}{D^3}\left(\frac{\mathrm{d}}{\mathrm{d}(-i\Omega t)}\frac{a^3}{D^3}\right)\mathrm{d}t, \tag{107}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{a^6}{D^6}\frac{3e\sin f}{1+e\cos f}\,\mathrm{d}f. \tag{108}$$

This vanishes, per our earlier analysis, and this is no suprise, since we found that F_{N0} are symmetric, so there can be no first moment. The second moment is easiest to compute via

$$\sum_N F_{N0}^2 N^2 = \frac{1}{2\pi}\int_0^{2\pi} \frac{a^6}{D^6}\left(\frac{3e\sin f}{1+e\cos f}\right)^2 \frac{\mathrm{d}f}{\mathrm{d}t}\,\mathrm{d}f, \tag{109}$$

$$= \frac{1}{2\pi\left(1-e^2\right)^6}\int_0^{2\pi} (1+e\cos f)^4 (3e\sin f)^2 \frac{(1+e\cos f)^2}{\left(1-e^2\right)^{3/2}}\,\mathrm{d}f, \tag{110}$$

$$= \frac{1}{2\pi\left(1-e^2\right)^{15/2}}\int_0^{2\pi} \left(1+15e^2\cos^2 f+15e^4\cos^4 f+e^6\cos^6 f\right)(3e\sin f)^2\,\mathrm{d}f, \tag{111}$$

$$= \frac{9e^2}{\left(1-e^2\right)^{15/2}}\left(\frac{1}{2}+\frac{15e^2}{8}+\frac{15e^4}{16}+\frac{5e^6}{128}\right) \equiv \frac{9e^2}{2\left(1-e^2\right)^{15/2}} f_3. \tag{112}$$

I just Mathematica'd the trig identities, nothing magical going on. This is numerically checked and in agreement w/ Michelle's result as well.

We can also look at the second moment of our fitting formula. Integrating twice by parts lets us evaluate

$$\frac{1}{2}\int_{-\infty}^{\infty} C^2 e^{-2|N|/\eta} N^2\,\mathrm{d}N = C^2\int_0^{\infty} e^{-2N/\eta}N^2\,\mathrm{d}N, \tag{113}$$

$$= 2C^2\left(\frac{\eta}{2}\right)^3. \tag{114}$$

The latter follows from the gamma function definition of course (didn't see this at first lol).

We now have two constraints (0th and 2nd moment of F_{N0}) for the two variables C, η, and therefore we find that

$$C^2\left(\frac{\eta}{2}\right) = f_5/\left(1-e^2\right)^{9/2}, \tag{115}$$

$$C^2\frac{\eta^3}{2} = \frac{9e^2}{2\left(1-e^2\right)^{15/2}} f_3, \tag{116}$$

$$\eta^2 = \frac{9e^2 f_3}{2\left(1-e^2\right)^3 f_5}, \tag{117}$$

$$C^2 = \frac{2f_5}{\eta\left(1-e^2\right)^{9/2}} = \frac{(2f_5)^{3/2}}{3e\left(1-e^2\right)^3\sqrt{f_3}}. \tag{118}$$

We can verify that as $e \to 1$, we have the scaling $\eta \propto \left(1-e^2\right)^{3/2}$, which probably is what told us $\eta \sim N_{peri}$.

5.3 New Closed Forms for Torque and Energy

Note that the power of our approximation comes in evaluating expressions as follows (recall $\alpha = \frac{462\sqrt{2}}{175} \approx 3.7335$)

$$\sum_{N=-\infty}^{\infty} F_{N2}^2 N^q \approx \int_0^\infty C^2 N^4 e^{-2N/\eta} N^q \,\mathrm{d}N, \tag{119}$$

$$\approx C^2 \int_0^\infty N^{4+q} e^{-2N/\eta}\,\mathrm{d}N, \tag{120}$$

$$\approx C^2\left(\frac{\eta}{2}\right)^{5+q}\Gamma(5+q), \tag{121}$$

$$\approx \frac{1+3e^2+3e^4/8}{\left(1-e^2\right)^{9/2}}\frac{\Gamma(5+q)}{4!}\left(\frac{\eta}{2}\right)^q, \tag{122}$$

$$\approx \frac{1+3e^2+3e^4/8}{\left(1-e^2\right)^{9/2}}\frac{\Gamma(5+q)}{4!}\left(\alpha\frac{\sqrt{1+e}}{4\left(1-e^2\right)^{3/2}}\right)^q, \tag{123}$$

$$\approx \frac{f_5}{\left(1-e^2\right)^{(9+3q)/2}}\frac{\Gamma(5+q)}{4!}\left(\frac{\alpha}{4}\right)^q(1+e)^{q/2}. \tag{124}$$

It bears noting that $\alpha \approx 2(1+e)$, in which case we can simply approximate

$$\sum_{N=-\infty}^{\infty} F_{N2}^2 N^q \approx \frac{f_5}{\left(1-e^2\right)^{(9+3q)/2}}\frac{\Gamma(5+q)}{4!}2^{-q}(1+e)^{3q/2}. \tag{125}$$

This lets us update our torque formula. Recall we want to sum

$$\tau = T_0\hat{C}(r_c)\sum_{N=-\infty}^{\infty} F_{N2}^2\,\mathrm{sgn}\left(N-2\frac{\Omega_s}{\Omega}\right)\left|N-2\frac{\Omega_s}{\Omega}\right|^{8/3}. \tag{126}$$

Note that $\hat{C}(r_c)$ was defined a while ago to be dimensionless quantity

$$\hat{C}(r_c) \equiv \frac{1}{2}\left[\frac{r_c}{g_c}\left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln R}\right)_{r=r_c}\right]^{-1/3}\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2\left[\frac{3}{2}\frac{3^{2/3}\Gamma^2(1/3)}{5\cdot 6^{4/3}}\frac{3}{4\pi}\alpha^2\right]. \tag{127}$$

We can drop everything but the 1/2 and the density terms under Kushnir's suggestion.

Let's just consider the summation, which we can call $\hat{\tau}$

$$\hat{\tau} \equiv \sum_{N=-\infty}^{\infty} F_{N2}^2 \operatorname{sgn}\left(N - 2\frac{\Omega_s}{\Omega}\right)\left|N - 2\frac{\Omega_s}{\Omega}\right|^{8/3}. \tag{128}$$

Again, let's consider the two limits (recall $N_{\max} = \alpha\frac{\sqrt{1+e}}{(1-e)^{3/2}}$ for $\alpha \approx 2(1+e)$, while $f_5 = 1 + 3e^2 + 3e^4/8$):

- Consider $2\Omega_s \gg N_{\max}\Omega$, then the sign is always negative and $|N - 2\Omega_s/\Omega| \approx \left|\frac{2\Omega_s}{\Omega} - N_{\max}\right|$ and

$$\begin{aligned}
\hat{\tau} &\approx -\left|\frac{2\Omega_s}{\Omega} - N_{\max}\right|^{8/3}\sum_{N=-\infty}^{\infty} F_{N2}^2, && (129)\\
&\approx -\left|1 - \frac{2\Omega_s}{N_{\max}\Omega}\right|^{8/3}\frac{f_5}{\left(1-e^2\right)^{9/2}}\left(2\left(\frac{1+e}{1-e}\right)^{3/2}\right)^{8/3}, && (130)\\
&\approx -\left|1 - \frac{2\Omega_s}{N_{\max}\Omega}\right|^{8/3}\frac{f_5 2^{8/3}(1+e)^4}{\left(1-e^2\right)^{17/2}}. && (131)
\end{aligned}$$

- Consider $2\Omega_s \ll N_{\max}\Omega$, then the sign is the sign of $N_{\max}$ (> 0). We choose a factorization for $\left|N - \frac{2\Omega_s}{\Omega}\right| \approx |N|\left|1 - \frac{2\Omega_s}{\beta N_{\max}\Omega}\right|$ such that we can express as a prefactor and a summation. We must choose it such that in the limit $\Omega_s \to -\infty$, the asymptotics are correct, that is:

$$\sum_N F_{N2}^2 |2\Omega_s|^{8/3} = \left|\frac{2\Omega_s}{\beta N_{\max}\Omega}\right|^{8/3}\sum_N F_{N2}^2 N^{8/3}. \tag{132}$$

  This can be solved to find that $\beta = \left(\frac{\Gamma(23/3)}{4!}\right)^{3/8}/4 \approx 1.38$. Thus,

$$\begin{aligned}
\hat{\tau} &\approx \left|1 - 1.38\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}\sum_{N=-\infty}^{\infty} F_{N2}^2 |N|^{8/3}, && (133)\\
&\approx \left|1 - 1.38\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}\frac{f_5}{\left(1-e^2\right)^{17/2}}\frac{\Gamma(23/3)}{4!}2^{-8/3}(1+e)^4. && (134)
\end{aligned}$$

Thus, we arrive at final answer

$$\hat{\tau} \approx \frac{f_5(1+e)^4}{\left(1-e^2\right)^{17/2}} \times \begin{cases} \left|1 - 1.38\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}\frac{\Gamma(23/3)}{4!}2^{-8/3}, & \Omega_s < N_{\max}\Omega/2, \\ -\left|1 - 2\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}2^{8/3}, & \Omega_s > N_{\max}\Omega/2. \end{cases} \tag{135}$$

TODO we also update the energy expression as well.

5.4 Being Brave and Useless: Expanding Non-integer Powers

Let's just drop $\alpha \approx 4$ for the time being, so

$$\sum_{N=-\infty}^{\infty} F_{N2}^2 N^q \approx \frac{f_5}{\left(1-e^2\right)^{(9+3q)/2}} \frac{\Gamma(5+q)}{4!}(1+e)^{q/2}. \tag{136}$$

One fun question is: can we do better than the piecewise approximation that we proposed above for the $N = 8/3$ case, via the generalized binomial theorem? For reference, this is:

$$(a+b)^n = \sum_{k=0}^{\infty} \binom{n}{k} a^k b^{n-k}, \tag{137}$$

where $\binom{n}{k} \equiv \frac{n(n-1)\dots(n-k+1)}{k!}$ for integer k and not necessarily integer n (and thus, not necessarily positive $n-k+1$).

Let's drop all the signs and absolute value nonsense, in which case we are attempting to evaluate (where $m' \equiv m\Omega_s/\Omega$ for convenience)

$$\begin{aligned}
☺ &\equiv \sum_{N=-\infty}^{\infty} F_{N2}^2 \left(N-m'\right)^{8/3}, && (138)\\
&= \sum_{N=-\infty}^{\infty} F_{N2}^2 \left(N^{8/3} + \frac{8}{3}N^{5/3}m' + \frac{(8/3)(5/3)}{2}N^{2/3}(m')^2 + \dots\right), && (139)\\
&= \frac{f_5}{\left(1-e^2\right)^{9/2}4!}\left[\frac{\Gamma(5+8/3)(1+e)^{(8/3)/2}}{\left(1-e^2\right)^{8/2}} + \frac{\frac{8}{3}\Gamma(5+5/3)(1+e)^{(5/3)/2}m'}{\left(1-e^2\right)^{5/2}} + \frac{\frac{(8/3)(5/3)}{2}\Gamma(5+2/3)(1+e)^{(2/3)/2}(m')^2}{\left(1-e^2\right)^{2/2}} + \dots\right]. && (140)
\end{aligned}$$

This distinction only makes sense obviously when $m' \approx N_{\max} \sim \frac{2\sqrt{1+e}}{(1-e^2)^{3/2}}$, so let's go ahead and set $m' = N_{\max}\beta$; we want eventually to examine asymptotic behavior for $\beta \rightarrow^{\pm} 1$ or some similarly critical value. This gives

$$\begin{aligned}
☺ &= \frac{f_5}{\left(1-e^2\right)^{9/2}4!} \sum_{k=0}^{\infty} \binom{8/3}{k} \Gamma\left(\frac{23}{3}-k\right) \frac{(1+e)^{(8/3-k)/2}}{\left(1-e^2\right)^{(8-3k)/2}} \left(-2\beta\right)^k \frac{(1+e)^{k/2}}{\left(1-e^2\right)^{3k/2}}, && (141)\\
&= \frac{f_5(1+e)^{8/3}}{\left(1-e^2\right)^{17/2}4!} \sum_{k=0}^{\infty} \binom{8/3}{k} \Gamma\left(\frac{23}{3}-k\right)\left(-2\beta\right)^k. && (142)
\end{aligned}$$

The trick to proceed here is to handle negative arguments to the Gamma function with the Euler reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$. Dropping the first few terms that don't have this problem (they're finite, we can return to them), we then have

$$☺ - ☺_6 = \frac{f_5(1+e)^{8/3}}{\left(1-e^2\right)^{17/2}4!}\sum_{k=7}^{\infty}\binom{8/3}{k}\frac{\pi}{\sin(\pi(23/3-k))\Gamma(k-20/3)}\left(-2\beta\right)^k, \tag{143}$$

$$= \frac{f_5(1+e)^{8/3}}{\left(1-e^2\right)^{17/2}4!}\sum_{k=7}^{\infty}\binom{8/3}{k}\frac{\pi}{-\sqrt{3}/2(-1)^k\Gamma(k-20/3)}\left(-2\beta\right)^k, \tag{144}$$

$$= -\frac{f_5(1+e)^{8/3}}{\left(1-e^2\right)^{17/2}4!}\sum_{k=7}^{\infty}\binom{8/3}{k}\frac{2\pi/\sqrt{3}}{\Gamma(k-20/3)}\left(2\beta\right)^k. \tag{145}$$

Asymptotically, $\binom{8/3}{k}$ flips signs for every k, but each successive term is being multiplied by a number that gets closer to unity. The convergence is dominated by the gamma function in the denominator, which scales like a factorial. The sum thus converges to something in the vicinity of

$$\sum_{k=7}^{\infty}\binom{8/3}{k}\frac{2\pi/\sqrt{3}}{\Gamma(k-20/3)}\left(2\beta\right)^k \lesssim 2\pi/\sqrt{3}(2\beta)^7 e^{-2\beta}. \tag{146}$$

Is this enlightening at all? Well, there is some critical $\beta_c \sim 1$ (corresponding to some critical $\Omega_{s,c}$) for which ☺ and thus the torque is zero, the pseudosynchronizated spin rate. For β just below/above this β_c, I guess we can see that the torque grows exponentially, before the dominant power law behavior takes over, from the piecewise approximations. And this was thoroughly useless, albeit pretty fun. Note that the signs do work out, a pretty big surprise to me at least.

The original hope would have been to determine how near pseudosynchronization our piecewise approximation breaks down, but this seems somewhat hard. We probably can guesstimate by looking at the values of β for which the above is either half of its maximum value. We know it is maximized for $\beta \sim 3.5$, and the half max apparently happens around 2.2 or 5.2, so our power law approximation should only be good to within about 70% of the pseudosynchronization frequency.

Next: what about in non-all-traveling-wave approximation?