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1.1 Equal Masses: Following Racine

The paper is `https://arxiv.org/pdf/0803.1820.pdf`. The EOM and definitions in the equal mass case are considerably simplified and we obtain:

$$\frac{\mathrm{d}\mathbf{S}_1}{\mathrm{d}\psi} = \hat{\mathbf{J}} \times \mathbf{S}_1 - \alpha \mathbf{S}_2 \times \mathbf{S}_1, \tag{1}$$

$$\frac{\mathrm{d}\mathbf{S}_2}{\mathrm{d}\psi} = \hat{\mathbf{J}} \times \mathbf{S}_2 - \alpha \mathbf{S}_1 \times \mathbf{S}_2, \tag{2}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{1}{2a_{\mathrm{eff}}^3}\left(7 - \frac{3}{2}\lambda\right) J, \tag{3}$$

$$\alpha = \frac{3}{J}\frac{4-\lambda}{14-3\lambda}, \tag{4}$$

$$\lambda = \frac{\mathbf{L}}{L^2} \cdot \left[\left(1 + \frac{M_2}{M_1}\right)\mathbf{S}_1 + \left(1 + \frac{M_1}{M_2}\right)\mathbf{S}_2\right]. \tag{5}$$

Here, $a_{\mathrm{eff}} = a\sqrt{1-e^2}$, and $\mathbf{J} = J\hat{\mathbf{J}} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2$ is the total angular momentum.

The Hamiltonian is then probably

$$H = -\hat{\mathbf{J}} \cdot (\mathbf{S}_1 + \mathbf{S}_2) + \alpha(\mathbf{S}_1 \cdot \mathbf{S}_2), \tag{6}$$

I think (double check the signs, check that it reproduces Racine's EOM?).

Ansatz: adiabatic invariant, can we set up $\oint p\,dq$? Why is ψ not uniformly advancing and is so complicated? λ is not invariant since $d\lambda/dt$ directly has a GW term.

Up to rotational invariance, equilibria occur when each of θ_1, θ_2 and $\Delta\phi = \phi_2 - \phi_1$ are stationary. If we go to the corotating frame about $\hat{\mathbf{J}}$, then it's very obvious what is going on:

$$H_{\mathrm{rot}} = \alpha(\mathbf{S}_1 \cdot \mathbf{S}_2). \tag{7}$$

Clearly, the equilibria occur when $\mathbf{S}_1$ and $\mathbf{S}_2$ are aligned/antialigned. I don't know why the phase portrait for them has to be so messy, probably a difficulty with coordinates?

This checks out in coordinate form, even in the inertial frame:

$$H\left(\theta_1, \theta_2, \Delta\phi\right) = -S_1\cos\theta_1 - S_2\cos\theta_2 + \alpha\left(\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2\cos\Delta\phi\right). \tag{8}$$

Since centers of libration occur when H attains an extremal value, it's clear that the only extremal values can occur when $\Delta\phi = 0$ or π. If we set that to be the case, then an extremal value can only occur if $\theta_1 \pm \theta_2 = 0$, depending on which ϕ we use, so indeed, we recover alignment/anti-alignment as the two possible centers of libration.

To try and extract an adiabatic invariant, we consider the evolution of the subs and differences of $\mathbf{S}_i$, as pointed out by Racine. It's clear that $\mathbf{S}_1 + \mathbf{S}_2$ just precesses about $\hat{\mathbf{J}}$ (with the opening angle an adiabatic invariant), while $\mathbf{\Delta} \equiv \mathbf{S}_1 - \mathbf{S}_2$ evolves with ($\mathbf{S}$ is the total spin)

$$\frac{\mathrm{d}\mathbf{\Delta}}{\mathrm{d}\psi} = \left(\hat{\mathbf{J}} - \alpha\mathbf{S}\right) \times \mathbf{\Delta}. \tag{9}$$

The obvious guess of adiabatic invariant in this case is then just going to be:

$$\cos\theta_\Delta \equiv \left(\hat{\mathbf{J}} - \alpha\mathbf{S}\right) \cdot \mathbf{\Delta} \tag{10}$$

With these two adiabatic invariants, what happens during GW emission then? I don't think that the total spin one will do anything interesting (the most pathological case is when $\mathbf{S}_1 \parallel \mathbf{S}_2$: go to the co-rotating frame about $\mathbf{L}$ initially, then the enclosed phase space area is zero, the precession axis just moves). However, during GW decay, we expect α to increase, as J decreases, and so $\boldsymbol{\Delta}$ will initially precess about $\mathbf{J}$ but eventually precess about the sum of the total spin and $\hat{\mathbf{J}}$ (note that even at the very late stages of inspiral, I think $\mathbf{L} \gtrsim \mathbf{S}_i$, so it will never be fully ignorable). Nevertheless, this is clearly the mechanism for making $\boldsymbol{\Delta}$ tilt

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We first try to find the equilibria of the system before seeking adiabatic invariants. As Schnittman 2004 points out, we should consider the zeros of

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\mathbf{S}_1 \cdot \mathbf{S}_2\right) = \frac{\mathrm{d}\psi}{\mathrm{d}t}\beta_- \hat{\mathbf{J}} \cdot \left(\mathbf{S}_1 \times \mathbf{S}_2\right). \tag{11}$$

Here, β_- is a constant from Racine's paper, defined to be $\beta_1 - \beta_2$ where

$$\beta_{1,2} = \left(4 + 3\frac{M_{2,1}}{M_{1,2}} - 3\frac{\mu}{M_{1,2}}\lambda\right) \Big/ \left(7 - \frac{3\lambda}{2}\right). \tag{12}$$

In order for Eq. (11) to vanish, we need two conditions:

$$\mathbf{S}_2 \cdot \left(\hat{\mathbf{J}} \times \mathbf{S}_1\right) = \sin\theta_1 \sin\theta_2 \sin\Delta\phi = 0, \tag{13}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[\mathbf{S}_2 \cdot \left(\hat{\mathbf{J}} \times \mathbf{S}_1\right)\right] = 0. \tag{14}$$

To achieve some comparability with the Schnittman equations, we can replace $\hat{\mathbf{J}}$ with $\mathbf{J}$ above without changing the equilibria. Note that this is equivalent to the Schnittman expression

$$\begin{aligned}
\mathbf{S}_2 \cdot \left[(\mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2) \times \mathbf{S}_1\right] &= \mathbf{S}_2 \cdot \left[(\mathbf{L} + \mathbf{S}_2) \times \mathbf{S}_1\right], && (15)\\
&= \mathbf{S}_2 \cdot \left[\mathbf{L} \times \mathbf{S}_1\right] + \mathbf{S}_2 \cdot \left(\mathbf{S}_2 \times \mathbf{S}_1\right), && (16)\\
&= \mathbf{S}_2 \cdot \left[\mathbf{L} \times \mathbf{S}_1\right]. && (17)
\end{aligned}$$