Octupole-order Lidov-Kozai Population Statistics

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Today

1 10/22/20—Initial Thoughts

1.1 Equations

The equations of motion we want to study come from LML15. Describe the inner binary by $(a, e, I, \Omega, \omega)$ and the outer binary with "out" subscripts, and denote $I_{\rm tot} = I + I_{\rm out}$. Call the inner binary component masses m_1, m_2, and the tertiary mass m_3, define the inner binary total and reduced masses $m_{12} = m_1 + m_2$ and $\mu = m_1 m_2 / m_{12}$, and define the tertiary orbit total and reduced masses $m_{123} = m_{12} + m_3$ and $\mu_{\rm out} = m_{12} m_3 / m_{123}$. The equations of motion are $(j(e) = \sqrt{1 - e^2})$

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{64}{5}\frac{a}{t_{\rm GW} j^7(e)}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right), \tag{1}$$

$$\begin{aligned}\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{j(e)}{64 t_{\rm LK}}\Big\{&120 e \sin^2 I_{\rm tot} \sin(2\omega) \\ &+ \frac{15\epsilon_{\rm oct}}{8}\cos\omega_{\rm out}\left[\left(4 + 3e^2\right)\left(3 + 5\cos(2I_{\rm tot})\right)\right. \\ &\left.\times \sin\omega + 210 e^2 \sin^2 I_{\rm tot} \sin 3\omega\right] \\ &- \frac{15\epsilon_{\rm oct}}{4}\cos I_{\rm tot}\cos\omega\left[15(2 + 5e^2)\cos(2I_{\rm tot})\right. \\ &\left.+ 7\left(30e^2\cos(2\omega)\sin^2 I_{\rm tot} - 2 - 9e^2\right)\right]\sin\omega_{\rm out}\Big\} \\ &- \frac{304}{15}\frac{e}{t_{\rm GW} j^5(e)}\left(1 + \frac{121}{304}e^2\right),\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{3e}{32 t_{\rm LK} j(e)}\Big\{&10\sin(2I_{\rm tot})\left[e\sin(2\omega)\right. \\ &\left.+ \frac{5\epsilon_{\rm oct}}{8}\left(2 + 5e^2 + 7e^2\cos(2\omega)\right)\cos\omega_{\rm out}\sin\omega\right] \\ &+ \frac{5\epsilon_{\rm oct}}{8}\cos\omega\left[26 + 37e^2 - 35e^2\cos(2\omega)\right. \\ &\left.- 15\cos(2I_{\rm tot})\left(7e^2\cos(2\omega) - 2 - 5e^2\right)\right] \\ &\times \sin I_{\rm tot}\sin\omega_{\rm out}\Big\}\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{\mathrm{d}\Omega_{\rm out}}{\mathrm{d}t} = -\frac{3\csc I}{32 t_{\rm LK} j(e)}\Big\{&2\left[(2 + 3e^2 - 5e^2\cos(2\omega))\right. \\ &\left.+ \frac{25\epsilon_{\rm oct} e}{8}\cos\omega\left(2 + 5e^2 - 7e^2\cos(2\omega)\right)\cos\omega_{\rm out}\right] \\ &\times \sin(2I_{\rm tot}) - \frac{5\epsilon_{\rm oct} e}{8}\left[35e^2(1 + 3\cos(2I_{\rm tot}))\cos 2\omega\right. \\ &\left.- 46 - 17e^2 - 15\left(6 + e_1^2\right)\cos(2I_{\rm tot})\right] \\ &\times \sin I_{\rm tot}\sin\omega\sin\omega_{\rm out}\Big\},\end{aligned} \tag{4}$$

$$
\begin{aligned}
\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{3}{8t_{\rm LK}}\Big\{ & \frac{1}{j(e)}\big[4\cos^2 I_{\rm tot} + (5\cos(2\omega)-1) \\
& \times\big(1-e^2-\cos^2 I_{\rm tot}\big)\big] + \frac{L\cos I_{\rm tot}}{L_{\rm out}j(e_{\rm out})}\big[2+e^2(3 \\
& -5\cos(2\omega))\big]\Big\} + \frac{15\epsilon_{\rm oct}}{64t_{\rm LK}}\Big\{\left(\frac{L}{L_{\rm out}j(e_{\rm out})}+\frac{\cos I_{\rm tot}}{j(e)}\right) \\
& \times e\big[\sin\omega\sin\omega_{\rm out}\big[10(3\cos^2 I_{\rm tot}-1)(1-e^2)+A\big] \\
& -5B\cos I_{\rm tot}\cos\Theta\big] - \frac{j(e)}{e}\big[10\sin\omega\sin\omega_{\rm out}\cos I_{\rm tot} \\
& \times\sin^2 I_{\rm tot}\left(1-3e^2\right)+\cos\Theta\left(3A-10\cos^2 I_{\rm tot}+2\right)\big]\Big\} \\
& +\Omega_{\rm GR},
\end{aligned} \tag{5}
$$

$$
\begin{aligned}
\frac{\mathrm{d}e_{\rm out}}{\mathrm{d}t} = \frac{15eL\,j(e_{\rm out})\epsilon_{\rm oct}}{256t_{\rm LK}e_{\rm out}L_{\rm out}}\Big\{ & \cos\omega\big[6-13e^2 \\
& +5(2+5e^2)\cos(2I_{\rm tot})+70e^2\cos(2\omega)\sin^2 I_{\rm tot}\big] \\
& \times\sin\omega_{\rm out}-\cos I_{\rm tot}\cos\omega_{\rm out}\big[5(6+e^2)\cos(2I_{\rm tot}) \\
& +7\left(10e^2\cos(2\omega)\sin^2 I_{\rm tot}-2+e^2\right)\big]\sin\omega\Big\},
\end{aligned} \tag{6}
$$

$$
\begin{aligned}
\frac{\mathrm{d}I_{\rm out}}{\mathrm{d}t} = -\frac{3eL}{32t_{\rm LK}j(e_{\rm out})L_{\rm out}}\Big\{ & 10\big[2e\sin I_{\rm tot}\sin(2\omega) \\
& +\frac{5\epsilon_{\rm oct}}{8}\cos\omega\left(2+5e^2-7e^2\cos(2\omega)\right)\sin(2I_{\rm tot})\sin\omega_{\rm out}\big] \\
& +\frac{5\epsilon_{\rm oct}}{8}\big[26+107e^2+5(6+e^2)\cos(2I_{\rm tot}) \\
& -35e^2\left(\cos 2(I_{\rm tot})-5\right)\cos(2\omega)\big]\cos\omega_{\rm out}\sin I_{\rm tot}\sin\omega\Big\},
\end{aligned} \tag{7}
$$

$$
\begin{aligned}
\frac{\mathrm{d}\omega_{\rm out}}{\mathrm{d}t} = \frac{3}{16t_{\rm LK}}\Big\{ & \frac{2\cos I_{\rm tot}}{j(e)}\left[2+e^2(3-5\cos(2\omega))\right] \\
& +\frac{L}{L_{\rm out}j(e_{\rm out})}\big[4+6e^2+(5\cos^2 I_{\rm tot}-3) \\
& \times[2+e^2(3-5\cos(2\omega))]\big]\Big\} - \frac{15\epsilon_{\rm oct}e}{64t_{\rm LK}e_{\rm out}} \\
& \times\Big\{\sin\omega\sin\omega_{\rm out}\Big[\frac{L(4e_{\rm out}^2+1)}{e_{\rm out}L_{\rm out}j(e_{\rm out})}10\cos I_{\rm tot}\sin^2 I_{\rm tot} \\
& \times(1-e^2)-e_{\rm out}\left(\frac{1}{j(e)}+\frac{L\cos I_{\rm tot}}{L_{\rm out}j(e_{\rm out})}\right) \\
& \times\left[A+10\left(3\cos^2 I_{\rm tot}-1\right)\left(1-e^2\right)\right]\Big]+\cos\Theta \\
& \times\Big[5B\cos I_{\rm tot}e_{\rm out}\left(\frac{1}{j(e)}+\frac{L\cos I_{\rm tot}}{L_{\rm out}j(e_{\rm out})}\right) \\
& +\frac{L\left(4e_{\rm out}^2+1\right)}{e_{\rm out}L_{\rm out}j(e_{\rm out})}A\Big]\Big\}.
\end{aligned} \tag{8}
$$

where $n = \sqrt{Gm_{12}/a^3}$ is the mean motion, $L = \mu\sqrt{Gm_{12}a}$ and $L_{\rm out} = \mu_{\rm out}\sqrt{Gm_{123}a_{\rm out}}$ are the circular angular momenta, and

$$t_{\rm LK}^{-1} = n\left(\frac{m_3}{m_{12}}\right)\left(\frac{a}{a_{\rm out}j(e_{\rm out})}\right)^3, \tag{9}$$

$$t_{\rm GW}^{-1} = \frac{G^3\mu m_{12}^2}{c^5a^4}, \tag{10}$$

$$\Omega_{\rm GR} = \frac{3Gnm_{12}}{c^2aj^2(e)}, \tag{11}$$

$$\epsilon_{\rm oct} = \frac{m_2 - m_1}{m_{12}}\frac{a}{a_{\rm out}}\frac{e_{\rm out}}{1 - e_{\rm out}^2}, \tag{12}$$

$$A \equiv 4 + 3e^2 - \frac{5}{2}B\sin^2 I_{\rm tot}, \tag{13}$$

$$B \equiv 2 + 5e^2 - 7e^2\cos(2\omega), \tag{14}$$

$$\cos\Theta \equiv -\cos\omega\cos\omega_{\rm out} - \cos I_{\rm tot}\sin\omega\sin\omega_{\rm out}. \tag{15}$$

These equations can be nondimensionalized via the following steps (I won't rewrite the equations): (i) multiply through by $t_{\rm LK,0}$ ($a = a_0$ and $e_{\rm out} = 0$), and call $\tau \equiv t/t_{\rm LK,0}$ the new variable of differentiation, (ii) re-express all of the timescales as

$$\frac{t_{\rm LK,0}}{t_{\rm LK}} = \left(\frac{a}{a_0}\right)^{3/2} j^{-3}\left(e_{\rm out}\right), \tag{16}$$

$$\begin{aligned}\frac{t_{\rm LK,0}}{t_{\rm GW}} &= \frac{G^3\mu m_{12}^3}{m_3c^5a^4}\frac{1}{n_0}\left(\frac{a_{\rm out}}{a_0}\right)^3,\\ &= \epsilon_{\rm GW}\left(\frac{a_0}{a}\right)^4,\end{aligned} \tag{17}$$

$$\epsilon_{\rm GW} \equiv \frac{G^3\mu m_{12}^3a_{\rm out}^3}{m_3c^5a_0^7n_0}, \tag{18}$$

$$\begin{aligned}\Omega_{\rm GR}t_{\rm LK,0} &= \frac{3Gnm_{12}^2}{m_3c^2a}\frac{1}{n_0}\left(\frac{a_{\rm out}}{a_0}\right)^3,\\ &= \epsilon_{\rm GR}\left(\frac{a_0}{a}\right)^{5/2},\end{aligned} \tag{19}$$

$$\epsilon_{\rm GR} = \frac{3Gm_{12}^2a_{\rm out}^3}{m_3c^2a_0^4}. \tag{20}$$

(iii) re-express $\mathrm{d}a/\mathrm{d}t$ as

$$\frac{\mathrm{d}(a/a_0)}{\mathrm{d}\tau} = -\frac{64}{5}\frac{\epsilon_{\rm GW}}{j^{7/2}(e)}\left(\frac{a_0}{a}\right)^3\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right). \tag{21}$$

As such, the natural unit of length is $a_0 = 1$, the natural unit of time is $t_{\rm LK,0} = 1$, and everything else is dimensionless. When computing these ϵ, I use convention where $1M_\odot = 1$ AU $= c = 1$, under which $G = 9.87 \times 10^{-9}$.

1.2 Points of Inquiry

The goal is to understand how the merger window varies as a function of $q \equiv m_1/m_2$ ($m_1 < m_2$) when the octupole order LK effects are important.

- First, let's set $\epsilon_{\rm GW} = 0$. It is well known that the octupole order LK is nonintegrable. What does the Fourier spectrum of the eccentricity look like? Will this help us get a delay time distribution between high-e phases?

When the octupole effect is unimportant, the spectrum falls off exponentially over scales $\tau \simeq P_{\rm LK} j^{-1}(e_{\max})$, where $P_{\rm LK}$ is the quadrupole LK period. One imagines the tail of the spectrum gets heavier when $\epsilon_{\rm oct}$ is increased, and this might help us get the delay time distribution.

A second way we can postprocess this is to take a histogram of $e(t)$. If there is some regular structure, it's likely this will allow us to compute the average rate of binary coalescence due to GW radiation.

- The goal is to understand the size of the merger window, ΔI, as a function of q. To do this, we numerically sample the merger time function $T_{\rm m}(I_0, q)$. At each I_0, the natural thing to do would be to try for $\sim 5 - 10$ random Ω, ω, and define the merger window to be where $T_{\rm m} \leq 10^1 0$ yr.

2 11/02/20

We've done a lot more inquiry on this, read the Nov 3 weekly for a recap. In summary though, there should roughly be two ways to have LK-induced mergers:

- Quadrupole LK-induced mergers. The $e_{\max}$ of these systems is well understood, however, and the merger fraction should be easy to calculate: we can get the merger time using the $T_m \propto (1 - e_{\max})^{-3}$ physically-justified fitting law from LL18 in the LK-induced regime, and we just have to evaluate where it crosses 10 Gyr.

- Octupole LK-induced mergers. For these, there is a characteristic initial inclination range for which orbit flipping occurs, which is a function of $\epsilon_{\rm oct}$. This can likely be calculated analytically, but I'm not sure yet.

  For these systems, there is a characteristic orbit flipping timescale that is robust up to a factor of a few (since K oscillates on a fixed timescale, and orbit flips occur whenever K crosses $-\eta/2$), call this $t_{\rm ELK}$. Thus, octupole LK-induced mergers occur over characteristic times $t_{\rm ELK}$ within the desired inclination window (as once the system reaches an orbit flipping eccentricity, it executes a one-shot merger, approximately). It is well known that $t_{\rm ELK}$ depends on $\epsilon_{\rm oct}$, see e.g. Antognini 2015.

  There is some small variability, however, in this picture, according to my plots. This can likely be attributed to the exact history of eccentricity maxima prior to the one-shot maximum, since particularly dissipative sequences (i.e. many large eccentricity maxima prior to undergoing "the big one") can shrink the orbit and change the LK cycle pattern.

Armed with this, we should have enough information to compute the indicator function $\mathbb{I}_{\rm merge}(a_{\rm in}, I_0, t_{\rm LK}, \epsilon_{\rm oct})$, whether a system will merge, by simply computing the two ranges of I_0 from above. This should be valid wherever we are in a strongly LK-induced regime, i.e. very large eccentricities are necessary to merge.

In fact, there's a possibility the critical $\epsilon_{\rm oct}$ can be calculated analytically, see Katz et. al. 2011 for the test particle case. Is this generalizable?

2.1 The Quadrupole Conserved Quantity

This is a quick derivation. To quadrupolar order, $L_{\rm out}$ is conserved, as is the total angular momentum, so

$$L_{\rm tot}^2 = L_{\rm out}^2 + L_{\rm in}^2 + 2L_{\rm out}L_{\rm in}\cos I, \tag{22}$$

$$\frac{L_{\rm tot}^2 - L_{\rm out}^2}{L_{\rm out}L_{\rm in,0}} = \eta j_{\rm in}^2 + 2j_{\rm in}\cos I, \tag{23}$$

$$\frac{1}{2}\left[\frac{L_{\rm tot}^2 - L_{\rm out}^2}{L_{\rm out}L_{\rm in,0}} - \eta\right] = j_{\rm in}\cos I - \eta\frac{e_{\rm in}^2}{2}. \tag{24}$$

This is the form of the constant given in LL18. Note that $\eta = \eta_0/j_{\rm out}$, where η_0 is evaluated at $e_{\rm out} = 0$ as well.

3 11/03/20

We found out that the critical inclination window has a clean fitting formula in the test particle limit, MLL16 (which agrees with the leading order expansion of Katz et al 2011). If we take $\epsilon_{\rm oct}$ to be small, then

$$\cos^2 I_0 \leq \frac{\epsilon_{\rm oct}}{0.4}, \tag{25}$$

$$\Delta I_0 \leq \sqrt{\frac{\epsilon_{\rm oct}}{0.4}}. \tag{26}$$

This marks the inclination window in which the octupole-order effects are expected to be dominant, if the inclination window is similar to the test particle case (it's not exactly, since with finite η the window is offset to larger I_0 than $I_{0,\rm lim}$)

We will make two simplifying assumptions:

- DA is valid.
- $e_{\rm lim}$ is sufficiently large that one-shot mergers occur.

If we assume that DA is valid, then

$$t_{\rm LK}\sqrt{1 - e_{\max}^2} \gtrsim \frac{1}{n_{\rm out}}, \tag{27}$$

$$\frac{a^{3/2}}{a_{\rm out}^{3/2}}\frac{m_{12}}{m_3}\left(\frac{a_{\rm out}}{a}\right)^3\sqrt{1 - e_{\max}^2}\left(1 - e_{\rm out}^2\right)^{3/2} \gtrsim 1, \tag{28}$$

$$\frac{a_{\rm out}}{a} \gtrsim \frac{1}{j_{\min}^{2/3}}. \tag{29}$$

In the last line, we take $e_{\rm out} \sim 0.6$ for which $\left(1 - e_{\rm out}^2\right)^{3/2} = 0.5$, and $m_{12}/m_3 = 2$. For some reference $e_{\max} = 10^{-4}$, $j_{\min}^{2/3} = 0.058$ (it's okay for DA to break down at $e_{\rm lim}$, since one shot mergers should occur). TODO this should probably be evaluated for $e_{\max}$ near the edge of the quadrupole merger window.

Recall also that

$$\epsilon_{\rm oct} = \frac{1-q}{1+q}\frac{a}{a_{\rm out}}\frac{e_{\rm out}}{1 - e_{\rm out}^2}. \tag{30}$$

Thus, in the DA regime, we can explicitly write down the merger window for ELK-induced mergers

$$f_{\rm oct} \equiv \frac{2\Delta I_0}{\pi} \lesssim \sqrt{\frac{1-q}{1+q}\frac{1}{2}\frac{e_{\rm out}}{1 - e_{\rm out}^2}}. \tag{31}$$

This isn't great, since my data suggest that ΔI_0 should be linear in $(1-q)/(1+q)$, where $\epsilon_{\rm oct} \simeq 0.01$ should be small enough to satisfy the assumptions. Bin's data also agree with this, at least up to $e_{\rm out} = 0.6$ (25°, 15°, 10° degree merger windows for $q = 0.2$, 0.4, 0.6 respectively).

So it's clear that the ELK-active window should be revised for the finite-η case.

3.1 Differentiating K

We showed earlier that the quadrupole-conserved quantity is

$$K = j\cos I + j^2\eta = \frac{1}{2}\frac{L_{\rm tot}^2 - L_{\rm out}^2}{L_{\rm out}L_{\rm in,0}}. \tag{32}$$

Differentiating the second expression, we obtain that

$$\frac{\mathrm{d}K}{\mathrm{d}t} = \frac{1}{2L_{\in,0}}\left(\frac{-L_{\rm tot}^2}{L_{\rm out}^2}\frac{\mathrm{d}L_{\rm out}}{\mathrm{d}t} - \frac{\mathrm{d}L_{\rm out}}{\mathrm{d}t}\right), \tag{33}$$

$$= -\frac{1 + L_{\rm tot}^2/L_{\rm out}^2}{2L_{\rm in,0}}\hat{\mathbf{L}}_{\rm out}\cdot\frac{\mathrm{d}\mathbf{L}_{\rm out}}{\mathrm{d}t}. \tag{34}$$

Indeed, from LML15, we see that this term vanishes to quadrupolar order, and that the only terms that survive are (I use 1 for in and 2 for out somewhat interchangeably, due to the source materials I'm pulling from)

$$\frac{\hat{\mathbf{L}}_{\rm out}}{L_{\rm in}} \cdot \frac{\mathrm{d}\mathbf{L}_{\rm out}}{\mathrm{d}t} = -\frac{75\epsilon_{\rm oct}}{64}\left[2\left(\mathbf{e}_1 \cdot \hat{\mathbf{n}}_2\right)\left(\mathbf{j}_1 \cdot \hat{\mathbf{n}}_2\right)\hat{\mathbf{L}}_2 \cdot \left(\hat{\mathbf{u}}_2 \times \mathbf{j}_1\right) + \left[\frac{8}{5}e_1^2 - \frac{1}{5} - 7\left(\mathbf{e}_1 \cdot \hat{\mathbf{n}}_2\right)^2 + \left(\mathbf{j}_1 \cdot \hat{\mathbf{n}}_2\right)^2\right]\hat{\mathbf{L}}_2 \cdot \left(\hat{\mathbf{u}}_2 \times \mathbf{e}_1\right)\right]. \tag{35}$$

Triple products let us simplify a little bit to be in terms of $\hat{\mathbf{v}}_2 = \hat{\mathbf{L}}_2 \times \hat{\mathbf{e}}_2$ as in LML15. But this won't be useful: if we don't have the second derivative as well, we can't compute a characteristic oscillation frequency.

We can make one more approximation though: all of the $\mathbf{j}_1 \cdot \hat{\mathbf{n}}_2$ terms are generally small, since they are of order j_{1z} which is not conserved exactly but remains small (this is similar to Katz's approximation). This means we're left with

$$\frac{\mathrm{d}K}{\mathrm{d}t} \approx \left(1 + \frac{\eta^2}{4}\right)\frac{75\epsilon_{\rm oct}}{64}\left[\frac{8}{5}e_1^2 - \frac{1}{5} - 7\left(\mathbf{e}_1 \cdot \hat{\mathbf{n}}_2\right)^2\right]\mathbf{e}_1 \cdot \hat{\mathbf{v}}_2. \tag{36}$$

We've assumed $L_{\rm tot}^2/L_{\rm out}^2 \sim 1 + \eta^2/2$, since typically $\mathbf{L}_{\rm in}$ and $\mathbf{L}_{\rm out}$ are misaligned by $I \simeq 90^\circ$. This is again in agreement with the Katz formula except now $\mathbf{n}_2$ and $\mathbf{v}_2$ are permitted to vary (and our prefactor).

Now, their $\Omega_{\rm e}$ is such that $e_z/e = \cos\Omega_{\rm e}$. This is a bit more difficult to generalize for us, but it shouldn't be impossible. We will return later if it proves interesting.

3.2 Timescale Analysis

Let's suppose we only consider systems with $T_{\rm m,0} > 10$ Gyr, that cannot merge in a Hubble time. For simplicity, we also require DA hold, and all three masses be comparable ($m_{12} = 2m_3$, say). Then this places constraint

$$\frac{5c^5a_0^4}{256G^3m_{12}^2\mu} > 10\ \text{Gyr}. \tag{37}$$

If we fix $m_3 = 30M_\odot$, so $m_{12}^2\mu = q(1+q)\left(30M_\odot\right)^3$, then

$$\frac{a_0}{0.202\ \text{AU}}\left(\frac{2}{q(1+q)}\right)^{1/4} > 1. \tag{38}$$

This is a very weak constraint.

Nevertheless, if we are firmly in the LK-induced regime, LL18 can be used to easily compute the quadrupole merger window, $I_{0,\rm merger}^+ - I_{0,\rm merger}^-$ by just using their Eq. (42) to compute $e_{\max}$. We know that in the LK-induced regime, $\epsilon_{\rm GR}$ is very weak except for at very large $e_{\max}$, so we can likely omit it when computing the $I_{0,\rm merger}^\pm$ since these are very smooth mergers. Then, since $j_{\min}^6 = 10\ \text{Gyr}/T_{\rm m,0}$, we can obtain

$$\cos I_{0,\rm merger}^- - \cos I_{0,\rm merger}^+ = \frac{1}{5}\sqrt{\left(5\eta - 4\eta j_{\min}^2\right)^2 - 20\left(\frac{5\eta^2}{4} - j_{\min}^2\left(3 + \frac{9\eta^2}{4}\right) + \eta^2 j_{\min}^4\right)}, \tag{39}$$

$$\approx \frac{j_{\min}\sqrt{60}}{5} + \mathcal{O}\left(\eta j_{\min}\right), \tag{40}$$

$$\approx \frac{\sqrt{60}}{5}\left(\frac{10\ \text{Gyr}}{T_{m,0}}\right)^{1/6} + \mathcal{O}\left(\eta j_{\min}\right), \tag{41}$$

$$\approx \frac{\sqrt{60}}{5}\left(\frac{0.202\ \text{AU}}{a_0}\right)^{2/3}\left(\frac{q(1+q)}{2}\right)^{1/6} + \mathcal{O}\left(\eta j_{\min}\right). \tag{42}$$

For $I_{0,\rm merger} \sim 90^\circ$, we can replace $\cos(x) \approx 90^\circ - x$, and so the difference of these cosines is just the negative difference of their arguments. For $q = 2/3$ and $a_0 = 100$ AU, this gives $\Delta I_{0,\rm merger} = 1.231^\circ$, while LL18 obtain 1.20°. Thus, this is the right scaling.