1 Rapid Mergers

Last week, Dong sent out some notes describing how to estimate the important frequencies in understanding the adiabatic criterion:

$$\left|\frac{\mathrm{d}\langle\hat{\boldsymbol{\Omega}}_{\mathrm{e}}\rangle}{\mathrm{d}t}\right|_{\star} \ll |\langle\Omega_{\mathrm{e}}\rangle|_{\star}. \tag{1}$$

The $\star$ subscript denotes evaluation at $\mathscr{A} \simeq 1$, where

$$\mathscr{A} \equiv \frac{\Omega_{\mathrm{SL}}}{|\dot{\Omega}|}. \tag{2}$$

I use $\langle\hat{\boldsymbol{\Omega}}_{\mathrm{e}}\rangle$ to denote the LK-averaged ($N = 0$) component of the effective spin angular frequency.

A useful insight is that the eccentricity during the LK-driven inspiral has two phases of evolution:

Phase I The eccentricity undergoes significant excursions over the course of an LK cycle. During this phase, $e_{\max}$ and a decrease very slowly under GW, while $e_{\min}$ increases relatively quickly due to combined GR effects.

Phase II The eccentricity e is *frozen* at a fixed value over the course of an LK cycle ("suppressed by GR"), and both e and a decrease under GW, while I is fixe.

In the Paper II regime, $\mathscr{A} \simeq 1$ occurs in Phase II, where the eccentricity is already frozen and only undergoes GW-driven evolution. In this limit, we can make very accurate predictions about the relevant frequencies towards evaluating Eq. (1).

1.1 Estimating Frequencies assuming Phase II Inspiral

We adopt the same notation as Dong but derive much tighter scaling relations and bounds. Define $j \equiv \sqrt{1-e^2}$, and $j_\star$ is evaluated at $\mathscr{A} \simeq 1$. All quantities in this section are LK-averaged, so we omit brackets.

Note that a simple and symmetric form may be derived for I_{e} as follows:

$$\begin{aligned}
\boldsymbol{\Omega}_{\mathrm{e}} &= -\dot{\Omega}\hat{\mathbf{z}} + \Omega_{\mathrm{SL}} L_{\mathrm{in}}, &(3)\\
&= -\dot{\Omega}\hat{\mathbf{z}} + \Omega_{\mathrm{SL}}\cos I\hat{\mathbf{z}} + \Omega_{\mathrm{SL}}\sin I\hat{\mathbf{x}}, &(4)\\
\hat{\boldsymbol{\Omega}}_{\mathrm{e}} &\equiv \cos I_{\mathrm{e}}\hat{\mathbf{z}} + \sin I_{\mathrm{e}}\hat{\mathbf{x}}, &(5)\\
0 &= \left|\boldsymbol{\Omega}_{\mathrm{e}} \times \hat{\boldsymbol{\Omega}}_{\mathrm{e}}\right|, &(6)\\
&= -\dot{\Omega}\sin I_{\mathrm{e}} + \Omega_{\mathrm{SL}}\sin(I + I_{\mathrm{e}}). &(7)
\end{aligned}$$

Assuming $\dot{I} = 0$ in Phase II (numerically confirmed), this can be differentiated:

$$\begin{aligned}
\dot{I}_{\mathrm{e}}\left(-\dot{\Omega}\cos I_{\mathrm{e}} + \Omega_{\mathrm{SL}}\cos(I + I_{\mathrm{e}})\right) &= \ddot{\Omega}\sin I_{\mathrm{e}} - \dot{\Omega}_{\mathrm{SL}}\sin(I + I_{\mathrm{e}}), &(8)\\
\dot{I}_{\mathrm{e}}(-\cot I_{\mathrm{e}} + \cot(I + I_{\mathrm{e}})) &= \frac{\mathrm{d}\left(\ln\left|\dot{\Omega}\right|\right)}{\mathrm{d}t} - \frac{\mathrm{d}(\ln\Omega_{\mathrm{SL}})}{\mathrm{d}t}, &(9)\\
&= -\frac{\mathrm{d}}{\mathrm{d}t}(\ln\mathscr{A}) < 0. &(10)
\end{aligned}$$

The sign convention on I_{e} depends on whether $I < 90^\circ$ or $I > 90^\circ$, but it is clear that $\dot{I}_{\mathrm{e}}$ is almost always zero except when $\mathscr{A} = \mathscr{A}_0 / j_\star = 1$ as anticipated.

For instance, specialize to $\dot{\Omega} > 0$, which corresponds to $I > 90°$. Then, I_e evolves from 0[1] to $180° - I < 90°$, $\cot I_e > 0$ while $\cot(I + I_e) < 0$, and indeed $\dot{I}_e > 0$. Finally, evaluating at $\mathscr{A} \simeq 1$ just amounts to evaluating at $I_e \simeq (180° - I)/2$, and we can say

$$\dot{I}_{e,\max} \simeq \frac{1}{2\cot(I_e/2)}\left(\frac{\mathrm{d}\ln\mathscr{A}}{\mathrm{d}t}\right)_{\mathscr{A}\simeq 1}. \tag{11}$$

Since, during Phase II, only a and e change under GW radiation, and $\dot{I}_e$ only scales with the *log* of these quantities, we can make quite a robust prediction for $\dot{I}_e$. In particular, taking the logarithm of $\Omega_{SL}/\left|\dot{\Omega}\right|$ and only keeping scalings with a and $j = \sqrt{1-e^2}$ [units of time are $t_{LK,0}$, the initial LK time, and a is normalized to its initial value ($a_0 = 1$)]:

$$\left(\frac{\mathrm{d}(\ln a)}{\mathrm{d}t}\right)_{GW} = -\epsilon_{GW}\frac{64}{5}\frac{1 + 73e^2/24 + 37e^4/96}{a^4\left(1-e^2\right)^{7/2}}, \tag{12}$$

$$\approx -\epsilon_{GW}\frac{64}{5}\frac{4}{a^4 j^{7/2}}, \tag{13}$$

$$\left(\frac{\mathrm{d}\left(\ln(1-e^2)\right)}{\mathrm{d}t}\right)_{GW} = \epsilon_{GW}\frac{608}{15}\frac{e^2\left(1 + 121e^2/304\right)}{a^4(1-e^2)^{7/2}}, \tag{14}$$

$$\approx \epsilon_{GW}\frac{608}{15}\frac{1}{a^4 j^{7/2}}, \tag{15}$$

$$\Omega_{SL} = \frac{\epsilon_{SL}}{a^{5/2}j^2}, \tag{16}$$

$$\dot{\Omega} = \frac{3a^{3/2}}{4}\frac{\cos I\left(5e^2\cos^2\omega - 4e^2 - 1\right)}{\sqrt{1-e^2}}, \tag{17}$$

$$\approx \frac{3a^{3/2}}{2j}, \tag{18}$$

$$\dot{I}_{e,\max} \simeq \frac{1}{2\cot(I_e/2)}\left(-4\frac{\mathrm{d}(\ln a)}{\mathrm{d}t} - \frac{1}{2}\frac{\mathrm{d}\left(\ln(1-e^2)\right)}{\mathrm{d}t}\right)_{\mathscr{A}=1}, \tag{19}$$

$$\approx 200\frac{\epsilon_{GW}}{a_\star^4 j_\star^7}. \tag{20}$$

To simplify this, we next impose two constraints: $\Omega_{SL}(a_\star, j_\star) = \left|\dot{\Omega}(a_\star, j_\star)\right|$, and $j_\star \propto j_{\min} \equiv f j_{\min}$, some scaling ansatz. These two constraints together give

$$\mathscr{A} = 1 \approx \frac{3}{2}\frac{a_\star^4 j_\star}{\epsilon_{SL}}, \tag{21}$$

$$a_\star \approx \left(\frac{2\epsilon_{SL}/3}{f j_{\min}}\right)^{1/4}. \tag{22}$$

This gives us the two primary predictions for this calculation:

$$\dot{I}_{e,\max} = 200\frac{1}{2\cot(I_e/2)}\frac{\epsilon_{GW}}{\epsilon_{SL}}\frac{1}{(f j_{\min})^6}, \tag{23}$$

$$\Omega_{e,\star} = \left|\dot{\Omega}\right|\cos(I/2) + \Omega_{SL}\cos(I/2) = \frac{\epsilon_{SL}^{3/8}}{(f j_{\min})^{11/8}}2\cos(I/2). \tag{24}$$

Note that $j_{\min} = \sqrt{\frac{5\cos^2 I_0}{3}}$. Numerically, we find that $f \approx 2.2$ fits the data well. While there is a very steep dependence on f, note that $f \geq 1$, so it is a reasonably well constrained parameter. We present the agreement below in Fig. 1, for which I_0 is varied and $I \approx 125°$ is used throughout.

[1] I_e is not exactly zero originally, as for any nonzero Ω_{SL}, $\mathbf{\Omega}_e$ is not exactly coincident with $\hat{\mathbf{z}}$.

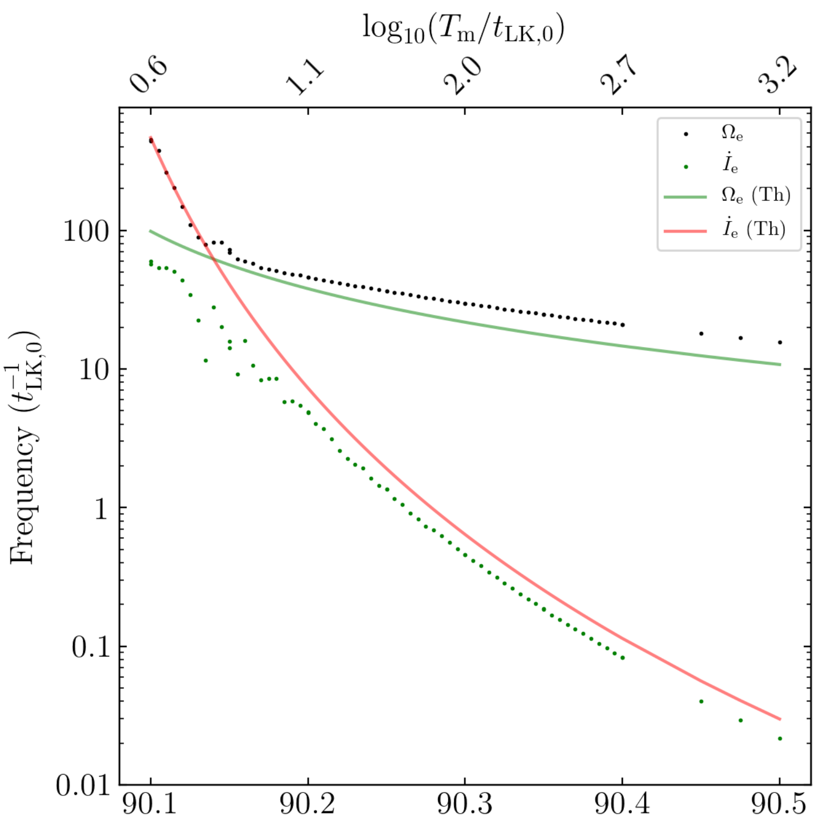

Figure 1: Plot of Ω_e and $\dot{I}_\mathrm{e}$ at $\mathscr{A} \simeq 1$, with Eqs. (23) and (24) overlaid. While the constant prefactors are not a perfect match, the scaling is very good. Top axis labels merger time, for reference.

1.2 Attempt at Predicting $\Delta\theta_e$

Since we know $\dot{I}_{e,\max}$, as well as the total range over which I_e evolves ($0 \to 180 - I$, or, for our particular simulations, $10 \to 55$), we can model $\dot{I}_e(t)$ as a Gaussian with width

$$\sigma_e = \frac{\Delta I_e}{\dot{I}_{e,\max}\sqrt{2\pi}}, \tag{25}$$

where ΔI_e is the total change in I_e.

Following the calculation given in the last week, this can yield an approximate estimate for $\Delta\theta_e$ for some arbitrary initial phase ϕ_0 as:

$$\Delta\theta_e \approx \int\limits_{-\infty}^{\infty} \dot{I}_{e,\max} \exp\left[-\frac{(t-t_\star)^2}{2\sigma_e^2}\right] \cos\left(\Omega_e t + \phi_0\right)\, \mathrm{d}t. \tag{26}$$

If we assume $\Omega_e \approx \Omega_{e,\star}$, this can easily be evaluated

$$\Delta\theta_e \approx \Delta I_e e^{-\Omega_e^2\sigma_e^2/2} \cos\phi_0. \tag{27}$$

Using Eqs. (23) and (24), the scaling with I_0 can be evaluated; it's important to see that the deviation from adiabaticity decays exponentially, in agreement with Landau & Lifshitz (Vol 1, Eq. 51.6).

This does *not* reproduce the simulation results, however, as the data suggest a power law decay, see Fig. 2. Blue dots are explicit integrations of the Gaussian approximation to $\dot{I}_e$ against the real ϕ_e, i.e.

$$\Delta\theta_{e,\mathrm{blue}} = \int\limits_{0}^{t_f} \dot{I}_{e,\max} \exp\left[-\frac{(t-t_\star)^2}{2\sigma_e^2}\right] \cos\left(\phi_e\right)\, \mathrm{d}t. \tag{28}$$

It is clear that this semi-analytic intergation yields the correct scalings, albeit off by a constant factor, and so treating Ω_e as a constant predicts much better conservation than expected (though the deviation is $\lesssim 1^\circ$). Note that I use the real ϕ_e, as any sort of smoothing seems to introduce too much numerical noise, somehow.

A second attempt can be made by allowing a $\dot{\Omega}_e$ term; this produces estimate

$$\begin{aligned}
\Delta\theta_e &\approx \mathrm{Re} \int\limits_{-\infty}^{\infty} \dot{I}_{e,\max} \exp\left[-\frac{(t-t_\star)^2}{2\sigma_e^2} + i\left(\Omega_{e,\star} + \dot{\Omega}_e(t-t_\star)\right)t + \phi_0\right] \mathrm{d}t, && (29)\\
&\equiv \mathrm{Re} \int\limits_{-\infty}^{\infty} \dot{I}_{e,\max} \exp\left[-a\tau^2 + i\Omega_{e,\star} + \phi_0\right] \mathrm{d}\tau, && (30)\\
&\approx \frac{\Delta I}{\sqrt{2\sigma^2 a}} \exp\left[-\left(8\sigma^2 \frac{\mathrm{d}\ln\Omega_e}{\mathrm{d}t}\right)^{-1}\right], && (31)\\
&\sim \frac{\Delta I}{\sqrt{2\sigma_e\Omega_{e,\star} f'}} \exp\left[-\left(8\left(1 - 1/f'\right)\right)^{-1}\right]. && (32)
\end{aligned}$$

The last approximation is in the limit $\dot{\Omega}\sigma^2 \gg 1$, and approximating $\sigma_e \mathrm{d}\ln\Omega_e/\mathrm{d}t \simeq f'$, where $f' \sim 1/8$ according to our estimates. It's clear f' scales as a constant, since

$$\begin{aligned}
\sigma\frac{\mathrm{d}\ln\Omega_e}{\mathrm{d}t} &\simeq \frac{\Delta I}{\sqrt{2\pi}} \left(\frac{1}{2\cot I_e} \frac{\mathrm{d}\left(\ln(\Omega_{SL}/|\dot{\Omega}|)\right)}{\mathrm{d}t}\right)^{-1} \frac{\mathrm{d}\ln\Omega_e}{\mathrm{d}t}, && (33)\\
&\sim \frac{\Delta I(2\cot I_e)}{\sqrt{2\pi}} \left[\frac{\mathrm{d}\ln\Omega_e/\mathrm{d}t}{\mathrm{d}\left(\ln(\Omega_{SL}/|\dot{\Omega}|)\right)/\mathrm{d}t}\right]. && (34)
\end{aligned}$$

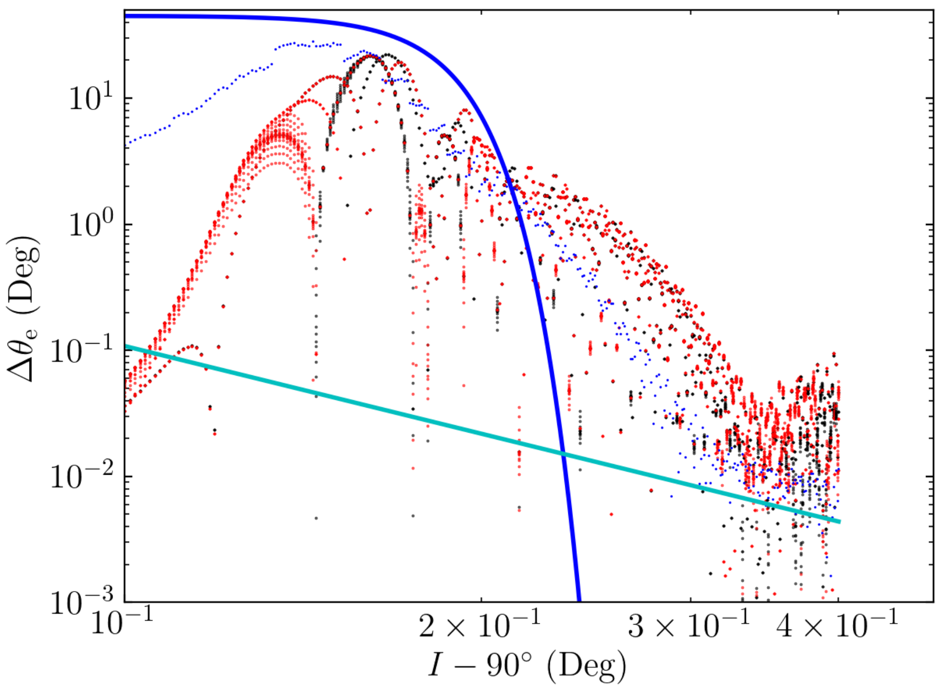

Figure 2: As a function of $I_0 - 90^\circ$: (red/black dots) $\Delta\theta_e$ for different spin realizations at a fixed I_0 (spin initial condition is varied), where red denotes negative. Blue line is the maximum permitted $\Delta\theta_e$ permitted by Eq. (27), which can be clearly seen to be a poor fit to the data. Blue dots are integrations using the true phases from the data, as given by Eq. (28). Agreement is poor for $I_0 \lesssim 90.2^\circ$ as the system evolves significantly within LK cycles. Note that the right end of the plot is dominated by numerical noise, as simulations towards the right merge more slowly and accumulate more numerical error; a low precision was used for these simulations.

The first term is a constant, and the second term is a quotient of a bunch of GW-decay terms with identical scalings.

Obviously, the result is a highly sensitive function of f', but the qualitative behavior may be in agreement with the tail of the distribution? See Fig. 2. This model correctly predicts a power law, though there is an intermediate regime with a clear power law dependence that falls under neither of the two models considered here. The integration using the exact ϕ_{e} seems to capture the correct power law scope, surprisingly, despite its very jumpy nature, so maybe that is responsible for the actual decay law. In any case, it is clear that it is easy to break exponential convergence to zero, so we shouldn't be troubled by the "unexpectedly poor" conservation of adiabatic invariant.