Lidov-Kozai 90° Attractor

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Date

1 Equations

1.1 Bin's Papers

Our major references will be Bin's paper with Diego + Dong in 2015 (LML15) and Bin's later paper with Dong on spin-orbit misalignment (LL18). The target of study is §4.3 of LL18, where a 90° attractor in spin-orbit misalignment seems to appear when the octupole effect is negligible.

The easiest formulation is just to express everything in terms of $\mathbf{L}$ and $\mathbf{e}$, following LL18. We drop octupole terms and hold the third perturber constant. These equations come out to be (Eqs. 4–5 w/ substitutions)

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{3}{4t_{LK}}\mu\sqrt{Gm_{12}a}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2) - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2)\right], \tag{1}$$

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \frac{3}{4t_{LK}}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2) + 2\mathbf{j}\times\mathbf{e} - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2)\right]. \tag{2}$$

Note that $\mathbf{j} \equiv \sqrt{1-e^2}\hat{\mathbf{L}} = \frac{\mathbf{L}}{\mu\sqrt{Gm_{12}a}}$. $m_{12} = m_1 + m_2$ and $\mu = m_1 m_2/m_{12}$. We've defined

$$t_{LK} \equiv \frac{L_1}{\mu_1\Phi_0} = \frac{1}{n_1}\left(\frac{m_{12}}{m_3}\right)\left(\frac{a_2}{a}\right)^3\left(1-e_2^2\right)^{3/2}. \tag{3}$$

Here, $n_1 \equiv \sqrt{Gm_{12}/a^3}$. Thus, $1/t_{LK} \propto a^{3/2}$.

GW radiation (Peters 1964) cause decays of $\mathbf{L}$ and $\mathbf{e}$ as

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{GW} = -\frac{32}{5}\frac{G^{7/2}}{c^5}\frac{\mu^2 m_{12}^{5/2}}{a^{7/2}}\frac{1+7e^2/8}{\left(1-e^2\right)^2}\hat{L}, \tag{4}$$

$$\left.\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}\right|_{GW} = -\frac{304}{15}\frac{G^3}{c^5}\frac{\mu m_{12}^2}{a^4\left(1-e^2\right)^{5/2}}\left(1+\frac{121}{304}e^2\right)\mathbf{e}. \tag{5}$$

Here, $m_{12} \equiv m_1 + m_2$, and a is implicitly defined by $\mathbf{L}$ and e. The last GR effect is precession of $\vec{e}$, which acts as

$$\left.\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}\right|_{GR} = \frac{1}{t_{GR}}\hat{\mathbf{L}}\times\mathbf{e}, \tag{6}$$

$$\frac{1}{t_{GR}} \equiv \frac{3Gnm_{12}}{c^2 a\left(1-e^2\right)}. \tag{7}$$

Note that $t_{GR}^{-1} \propto a^{-5/2}$.

Given this system (from LML15 + LL18), we can then add the spin-orbit coupling term (from de Sitter precession), which is given in LL18 to be

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \frac{1}{t_{SL}}\hat{\mathbf{L}} \times \hat{\mathbf{S}}, \tag{8}$$

$$\frac{1}{t_{SL}} \equiv \frac{3Gn\left(m_2 + \mu/3\right)}{2c^2 a\left(1 - e^2\right)}. \tag{9}$$

Note that μ is the reduced mass of the inner binary. We can drop the back-reaction term since $S \ll L$. Thus, $t_{SL}^{-1} \propto a^{-5/2}$ as well.

Finally, an adiabaticity parameter can be defined:

$$\mathscr{A} \equiv \left|\frac{\Omega_{SL}}{\Omega_L}\right|. \tag{10}$$

Here, $\Omega_L \simeq \frac{3(1+4e^2)}{8t_{LK}\sqrt{1-e^2}}\left|\sin 2I\right|$ is an approximate rate of change of L during an LK cycle

It's natural to redimensionalize to the initial LK time such that

$$\frac{1}{t_{LK,0}} \equiv \left(\frac{a}{a_0}\right)^{3/2}\frac{1}{t_{LK}}, \tag{11}$$

since nothing else in t_{LK} is changing. The next natural timescale for gravitational waves is

$$\frac{1}{t_{GW}} \equiv \frac{G^3 \mu m_{12}^2}{c^5 a^4} \equiv \frac{1}{t_{GW,0}}\left(\frac{a_0}{a}\right)^4 \equiv \epsilon_{GW}\frac{1}{t_{LK,0}}\left(\frac{a_0}{a}\right)^4. \tag{12}$$

We can repeat the procedure for the GR precession term and the spin-orbit coupling terms:

$$\frac{1}{t_{GR}} = \epsilon_{GR}\frac{1}{t_{LK,0}}\left(\frac{a_0}{a}\right)^{5/2}, \tag{13}$$

$$\frac{1}{t_{SL}} = \epsilon_{SL}\frac{1}{t_{LK,0}}\left(\frac{a_0}{a}\right)^{5/2}. \tag{14}$$

Thus, finally, if we let $\tau = t/t_{LK,0}$, then we obtain full equations of motion (note that $a_0 = 1$ below)

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}\tau} &= \left(\frac{a}{a_0}\right)^{3/2}\frac{3}{4}\sqrt{a}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2) - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2)\right] \\
&\quad - \epsilon_{GW}\left(\frac{a_0}{a}\right)^4\frac{32}{5}\frac{1 + 7e^2/8}{\left(1-e^2\right)^{5/2}}\mathbf{L},
\end{aligned} \tag{15}$$

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}\tau} &= \left(\frac{a}{a_0}\right)^{3/2}\frac{3}{4}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2) + 2\mathbf{j}\times\mathbf{e} - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2)\right] \\
&\quad - \epsilon_{GW}\left(\frac{a_0}{a}\right)^4\frac{304}{15}\frac{1}{\left(1-e^2\right)^{5/2}}\left(1 + \frac{121}{304}e^2\right)\mathbf{e} \\
&\quad + \epsilon_{GR}\left(\frac{a_0}{a}\right)^{5/2}\frac{1}{1-e^2}\hat{\mathbf{L}}\times\mathbf{e},
\end{aligned} \tag{16}$$

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \epsilon_{SL}\left(\frac{a_0}{a}\right)^{5/2}\frac{1}{1-e^2}\hat{\mathbf{L}}\times\hat{\mathbf{S}}. \tag{17}$$

For reference, we note that $a = |\mathbf{L}|^2/\left(\mu^2 G m_{12}(1-e^2)\right)$, while $\mathbf{j} = \mathbf{L}/\left(\mu\sqrt{G m_{12} a}\right)$. To invert $a(\mathbf{L})$ and $\mathbf{J}(\mathbf{L})$ in this coordinate system where $a_0 = 1$, it is easiest to choose the angular momentum dimensions such that $\mu\sqrt{G m_{12}} = 1$, such that now

$$|\mathbf{L}(t=0)| \equiv \mu\sqrt{G m_{12} a_0 \left(1-e_0^2\right)} = \sqrt{\left(1-e_0^2\right)}, \tag{18}$$

$$a = \frac{|\mathbf{L}|^2}{1-e^2}, \tag{19}$$

$$\mathbf{j} = \frac{\mathbf{L}}{\sqrt{a}} = \hat{\mathbf{L}}\sqrt{1-e^2}. \tag{20}$$

Finally, the timescales are

$$t_{LK,0} = \frac{1}{n}\frac{m_{12}}{m_3}\left(\frac{a_2}{a(t=0)}\right)^3 \left(1-e_2^2\right)^{3/2}, \tag{21}$$

$$\epsilon_{GW} \equiv \frac{t_{LK,0}}{t_{GW,0}} = \frac{1}{n}\frac{m_{12}}{m_3}\frac{a_2^3}{(a(t=0))^7}\left(1-e_2^2\right)^{3/2}\frac{G^3 \mu m_{12}^2}{c^5}, \tag{22}$$

$$\epsilon_{GR} \equiv \frac{t_{LK,0}}{t_{GR,0}} = \frac{m_{12}}{m_3}\frac{a_2^3}{(a(t=0))^4}\left(1-e_2^2\right)^{3/2}\frac{3 G m_{12}}{c^2}, \tag{23}$$

$$\epsilon_{SL} \equiv \frac{t_{SL,0}}{t_{GR,0}} = \frac{m_{12}}{m_3}\frac{a_2^3}{(a(t=0))^4}\left(1-e_2^2\right)^{3/2}\frac{3G\left(m_2+\mu/3\right)}{2c^2}. \tag{24}$$

The adiabacitity parameter

$$\mathscr{A} \equiv \left|\frac{\Omega_{SL}}{\Omega_L}\right| = \frac{\epsilon_{SL}}{t_{LK,0}}\left(\frac{a_0}{a}\right)^{5/2}\frac{1}{1-e^2}\left[\frac{3(1+4e^2)}{8 t_{LK,0}\sqrt{1-e^2}}\left(\frac{a}{a_0}\right)^{3/2}|\sin 2I|\right]^{-1}, \tag{25}$$

(note that Ω_L is a somewhat averaged sense, see LL18) can be evaluated in these units as

$$\mathscr{A} = \epsilon_{SL}\left(\frac{a_0}{a}\right)^4 \frac{1}{\sqrt{1-e^2}}\frac{8}{3\left(1+4e^2\right)|\sin 2I|}. \tag{26}$$

Note also that the Hamiltonian is just

$$H = \Omega_{SL}\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}, = \epsilon_{SL}\left(\frac{a_0}{a}\right)^{5/2}\frac{1}{1-e^2}\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}. \tag{27}$$

1.2 Maximum Eccentricity and Merger Time

Note that, since we are only evolving $\mathbf{L}$ and $\mathbf{e}$, and not $\mathbf{L}_2$ and $\mathbf{e}_2$, we are in the test mass approximation, under which we set $\eta = 0$ in Bin's equations. As such, the maximum eccentricity satisfies (Eq 42 of LL18 with $\eta \to 0$)

$$\frac{3}{8}\frac{j_{\min}^2 - 1}{j_{\min}^2}\left[5\cos I_0^2 - 3 j_{\min}^2\right] + \epsilon_{GR}\left(1 - \frac{1}{j_{\min}}\right) = 0. \tag{28}$$

Note that ϵ_{GR} is exactly as we defined above, incidentally, and that when GR is negligible, this reduces to the classic $j_{\min} \equiv \sqrt{1-e_{\max}^2} = \sqrt{\frac{5}{3}\cos^2 I_0}$. Since ϵ_{GR} is generally very small for most of the evolution, this generally reduces to the well known

$$e_{\max} = \sqrt{1 - \frac{5}{3}\cos^2 I_0}. \tag{29}$$

This only fails to saturate for extremely high eccentricities, so $I_0 \to 90°$.

1.3 Attractor Behavior

Proposal: The reason the 90° attractor appears is that the initial θ_{sb} is roughly stationary for $\mathscr{A} \ll 1$ (only small kicks during each LK cycle, as long as the maximum eccentricity isn't too large), then as we enter the transadiabatic regime, the L-K cycles die down and we simply have conservation of adiabatic invariant.

The latter half of this follows the LL18 claim, where the requirement that $\epsilon_{GR} \lesssim 9/4$ (GR precession of pericenter is slow enough that L-K survives) equates to $\mathscr{A} \lesssim 3$. The former half of this is somewhat tricky, but we can understand what is happening if we consider what is happening in the frame corotating with $\Omega_{SL,e=0}$ about $\hat{z}$: every time that a LK cycle appears, Ω_{SL} becomes much larger, and the axis of precession changes from $\hat{z}$ to the location of $\hat{L}$ very briefly. We can imagine this as a kick in this corotating frame (which is the right frame to consider for $\mathscr{A} \ll 1$). In the limit that I does not change very much between L-K cycles, and the azimuthal angle of $\hat{L}$ is roughly symmetric, the impulses roughly cancel out in the θ_{sb} direction. In other words, after two LK cycles, θ_{sb} does not change much in the corotating frame. This is indeed the picture that we obtain when we observe the plot.

As such, the hypothesis is that if $\mathscr{A} \gtrsim 1$ is satisfied while the kicks are still *small*, then deviations about 90° cannot be very large, and adiabatic invariance tilts us right over. On the other hand, if the kicks have become *large*, then θ_{sb} after any particular LK cycle is far from 90°, and this is frozen in during the adiabatic invariance phase. This explains a few key observations:

- If the initial $\theta_{sb} \approx 90°$, e.g. if it is 70°, then the frozen in θ_{SL} at the end is also $\sim 70°$. Thus, it cannot be a dynamical attraction towards $\theta_{SL} = 90°$, it's just conservation. This is a sensible condition to require though, since one might naively expect $\theta_{SL} \approx 0$ initially, and we know $I_0 \approx 90°$ in order for this mechanism to be active at all.

- Two ways we can test this: if we intentionally weaken Ω_{SL}, we should expect the width near $I_{0,\max}$ where the attractor breaks down to *narrow*, i.e. e should be able to get closer to $I_{0,\max}$ while still seeing the attractor. The converse obviously holds. Secondly, we can do integrations farther away from $I_{0,\max}$ (so we can actually get some substantial $\hat{S}$ evolution) and plot the trajectory of $\hat{S}$ in the corotating frame. Some number tweaking may be necessary...

A j Equations

We define vectors

$$\mathbf{j} = \sqrt{1-e^2}\hat{n}, \tag{30}$$

$$\mathbf{e} = e\hat{u}. \tag{31}$$

Here, $\mathbf{j}$ is the dimensionless angular momentum vector and $\mathbf{e}$ is the eccentricity vector; see LML15 for precise definitions. Note that $\mathbf{j}\cdot\mathbf{e} = 0$, $j^2 + e^2 = 1$. Then, the EOM for the inner and outer vectors satisfy to quadrupolar order

$$\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} = \frac{3}{4t_{LK}}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2) - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2)\right], \tag{32}$$

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \frac{3}{4t_{LK}}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2) + 2\mathbf{j}\times\mathbf{e} - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2)\right]. \tag{33}$$

Let's assume for the time being that $L_1 \ll L_2$, so the system is sufficiently hierarchical that $\mathbf{j}_2$, $\mathbf{e}_2$ are constants. Note for reference that

$$t_{LK} \equiv \frac{L_1}{\mu_1\Phi_0} = \frac{1}{n_1}\left(\frac{m_1+m_2}{m_3}\right)\left(\frac{a_2}{a}\right)^3\left(1-e_2^2\right)^{3/2}. \tag{34}$$

Here, $n_1 \equiv \sqrt{G(m_1+m_2)/a^3}$. Thus, $1/t_{LK} \propto a^{3/2}$.

GW radiation (Peters 1964) cause decays of $\mathbf{L}$ and $\mathbf{e}$ as

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{GW} = -\frac{32}{5}\frac{G^{7/2}}{c^5}\frac{\mu^2 m_{12}^{5/2}}{a^{7/2}}\frac{1+7e^2/8}{\left(1-e^2\right)^2}\hat{L}, \tag{35}$$

$$\left.\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}\right|_{GW} = -\frac{304}{15}\frac{G^3}{c^5}\frac{\mu m_{12}^2}{a^4\left(1-e^2\right)^{5/2}}\left(1+\frac{121}{304}e^2\right)\mathbf{e}, \tag{36}$$

$$\left.\frac{\dot{a}}{a}\right|_{GW} = -\frac{64}{5}\frac{G^3\mu m_{12}^2}{c^5 a^4}\frac{1}{\left(1-e^2\right)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \tag{37}$$

Here, $m_{12} \equiv m_1+m_2$. The last GR effect is precession of $\vec{e}$, which acts as

$$\left.\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}\right|_{GR} = \frac{3Gnm_{12}}{c^2 a\left(1-e^2\right)}. \tag{38}$$

Given this system (from LML15 + LL18), we can then add the spin-orbit coupling term (from de Sitter precession), which is given in LL18 to be

$$\frac{\mathrm{d}\hat{S}}{\mathrm{d}t} = \Omega_{SL}\hat{L}\times\hat{S}, \tag{39}$$

$$\Omega_{SL} \equiv \frac{3Gn\left(m_2+\mu/3\right)}{2c^2 a\left(1-e^2\right)}. \tag{40}$$

Note that μ is the reduced mass of the inner binary. We can drop the back-reaction term since $S \ll L$. Thus, $\Omega_{SL} \propto a^{-5/2}$.

What is observed is that, as this system is evolved forward in time and GR coalesces the inner binary, $\theta_{sl} \equiv \arccos\left(\hat{S}\cdot\hat{L}\right)$ goes to $90°$ consistently. The relevant figure is Fig. 19 of LL18, which shows that for a close-in, low-eccentricity perturber ($\bar{a}_{\mathrm{out,eff}} \propto a_{out}$), the focusing is significantly stronger. Note that initially, $I \equiv \arccos\left(\hat{L}\cdot\hat{L}_2\right) \approx 90°$ while $\theta_{sl} \approx 0$.

In LL18, an adiabaticity parameter is defined:

$$\mathscr{A} \equiv \left|\frac{\Omega_{SL}}{\Omega_L}\right|, \tag{41}$$

where $\Omega_L \simeq \left\langle \frac{\mathrm{d}\hat{L}}{\mathrm{d}t}\right\rangle_{LK}$ to quadrupolar order. As the inner binary coalesces, $\mathscr{A}$ transitions from $\ll 1$ to $\gg 1$ (as Ω_{SL} is a GR effect so ramps up very quickly as orbital separation decreases).

The adiabaticity parameter $\mathscr{A}$ can be plotted upon rescaling in our coordinates. Note that $\Omega_{\mathrm{SL}} = \frac{\delta a_0}{a t_{LK,0}}$, while $\Omega_L \simeq \frac{3(1+4e^2)}{8t_{LK}\sqrt{1-e^2}}|\sin 2I|$ can also be expressed in units of $t_{LK,0}$. This gives us

$$\mathscr{A} = \frac{8\delta\sqrt{1-e^2}}{3\left(1+4e^2\right)}\left(\frac{a_0}{a}\right)^4. \tag{42}$$

A.1 Simulations

First, we run GR-less simulations, so let's take $t_{LK} = 1$ (no semimajor axis evolution), and we reproduce LK oscillations.

Next, when accounting for GR, we should let a evolve as above. Note that since $\mathbf{j}$ and $\vec{e}$ are our dynamical variables, we should use $\mathbf{j} \equiv \sqrt{1-e^2}\hat{L} = \sqrt{1-e^2}\frac{\mathbf{L}}{\mu\sqrt{Gm_{12}a(1-e^2)}}$ and rewrite

$$\left.\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t}\right|_{GW} = \frac{1}{\mu\sqrt{GMa}}\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{GW} - \frac{\mathbf{j}}{2a}\left.\frac{\mathrm{d}a}{\mathrm{d}t}\right|_{GW}. \tag{43}$$

To double check, we should verify that $\frac{\mathrm{d}(j^2+e^2)}{\mathrm{d}t}\Big|_{GW} = 0$, which can be verified as (Let's set $G = M = \mu = a = c = 1$ for convenience)

$$\frac{1}{2}\frac{\mathrm{d}(j^2+e^2)}{\mathrm{d}t} = \mathbf{j}\cdot\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} + \mathbf{e}\cdot\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}, \tag{44}$$

$$= \mathbf{j}\cdot\left[\left(-\frac{32}{5}\frac{1+7e^2/8}{\left(1-e^2\right)^2}\right)\hat{L} - \frac{\mathbf{j}}{2}\left(-\frac{64}{5}\frac{1+73e^2/24+37e^4/96}{\left(1-e^2\right)^{7/2}}\right)\right] + \mathbf{e}\cdot\left(-\frac{304}{15}\frac{1+121e^2/304}{\left(1-e^2\right)^{5/2}}\right)\mathbf{e}, \tag{45}$$

$$= \left(-\frac{32}{5}\frac{1+7e^2/8}{\left(1-e^2\right)^{3/2}}\right) + \left(\frac{32}{5}\frac{1+73e^2/24+37e^4/96}{\left(1-e^2\right)^{5/2}}\right) + e^2\left(-\frac{304}{15}\frac{1+121e^2/304}{\left(1-e^2\right)^{5/2}}\right), \tag{46}$$

$$= \frac{1}{15\left(1-e^2\right)^{5/2}}\left[-96\left(1-e^2\right)\left(1+\frac{7e^2}{8}\right) + 96\left(1+\frac{73e^2}{24}+\frac{37e^4}{96}\right) - 304e^2\left(1+\frac{121e^2}{304}\right)\right]. \tag{47}$$

This can be verified to vanish upon term-by-term examination indeed.

For convenience, let's just define $t_{LK} = t_{LK,0}\frac{a_0^3}{a^3}$ and set $t_{LK,0} = 1$. Furthermore, the timescale of relevance for the GW terms is $t_{GW}^{-1} \sim \frac{G^3\mu m_{12}^2}{c^5 a^4}$. Let's express this as some ratio $t_{GW} = \epsilon t_{LK,0}\frac{a_0^4}{a^4}$. Thus, everything should be nondimensionalized this way.

We lastly add de-Sitter precession of the spin of one of the inner binary components, call this $\hat{S}$. Similarly, let's just define a proportionality constant $t_{SL} = \delta t_{LK,0}\frac{a_0}{a}$, then

$$\frac{\mathrm{d}\hat{S}}{\mathrm{d}(t/t_{LK,0})} = \delta\frac{a_0}{a}\hat{L}\times\hat{S}. \tag{48}$$

Our final simulation equations are thus ($\tau = t/t_{LK,0}$)

$$\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}\tau} = \frac{3}{4}\left(\frac{a_0^3}{a^3}\right)\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2) - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2)\right] - \left(\epsilon\frac{a_0^4}{a^4}\right)\left(\frac{32}{5}\frac{1+7e^2/8}{\left(1-e^2\right)^{5/2}} - \frac{32}{5}\frac{1}{\left(1-e^2\right)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\right)\mathbf{j}, \tag{49}$$

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}\tau} = \frac{3}{4}\left(\frac{a_0^3}{a^3}\right)\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2) + 2\mathbf{j}\times\mathbf{e} - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2)\right] - \left(\epsilon\frac{a_0^4}{a^4}\right)\frac{304}{15}\frac{1}{\left(1-e^2\right)^{5/2}}\left(1+\frac{121}{304}e^2\right)\mathbf{e}, \tag{50}$$

$$\frac{\mathrm{d}\hat{S}}{\mathrm{d}\tau} = \delta\frac{a_0}{a}\frac{\mathbf{j}}{\sqrt{1-e^2}}\times\hat{S}, \tag{51}$$

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -a\left(\epsilon\frac{a_0^4}{a^4}\right)\frac{64}{5}\frac{1}{\left(1-e^2\right)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \tag{52}$$