I show four simulations in phase space coordinates $(\theta_{\rm sl}, \phi_{\rm sl})$ showing the resonant angles that we see, with an alignment timescale $g_1 t_{\rm al} = 2500$. Here, $\cos\theta_{\rm sl} = \hat{\mathbf{s}} \cdot \hat{\mathbf{l}}$ while $\phi_{\rm sl}$ is defined relative to the $\hat{\mathbf{l}}$–$\hat{\mathbf{l}}_{\rm p}$ plane. For all of these, $I_1 = 10^\circ$, $I_2 = 1^\circ$, and $g_2 = 10 g_1$.

- In Fig 1, we show a case where $\alpha = 10 g_1$ (so $\eta_1 = 0.1$ and $\eta_1 = 1$). We start with $\theta_{\rm sl,i} = 10^\circ$. The system librates with resonant angle $\phi_{\rm sl} + \Delta g t$, where $\Delta g = g_2 - g_1 = 9 g_1$. The obliquity oscillates about the mode II CS2 obliquity in the limit where $I_1 = 0^\circ$, the horizontal line (i.e. the CS2 obliquity for $\eta = \eta_2 = 1$).

- In Fig 2, we use the same parameters as for Fig. 1 but starting with $\theta_{\rm sl,i} = 90^\circ$. The system librates with resonant angle $\phi_{\rm sl}$. The obliquity oscillates about the mode I CS2 obliquity in the limit where $I_2 = 0^\circ$, the horizontal line (i.e. the CS2 obliquity for $\eta = \eta_1 = 0.1$).

- In Fig 3, we again use the same parameters but start with $\theta_{\rm sl,i} = 55^\circ$. The system librates with resonant angle $\phi_{\rm sl} + \Delta g t/2$. Furthermore, the oscillation frequency in $\theta_{\rm sl}$ is exactly twice that of the oscillation frequency in I (and we showed that the oscillation frequency of I is Δg). The obliquity oscillates about
$$\theta_{\rm sl,res} \equiv \frac{g_1 + g_2}{2\alpha}, \tag{1}$$
shown in the horizontal line.

  Recall that the argument for this was very simplistic: assuming $\hat{\mathbf{l}} \approx \hat{\mathbf{l}}_{\rm p}$ (i.e. to zeroth order in I_1 and I_2), then we can work in the inertial frame where $\hat{\mathbf{l}} = \hat{\mathbf{l}}_{\rm p} = \hat{\mathbf{z}}$, and $\phi_{\rm sl} \approx \phi_{\rm inertial} + g_1 t$. Then, if we go to the co-rotating frame where $\phi_{\rm sl} + \Delta g t/2 = \phi_{\rm inertial} + (g_1 + g_2) t/2$ is librating, then:
$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\rm rot} = 0 \approx \alpha (\hat{\mathbf{s}} \cdot \hat{\mathbf{z}})(\hat{\mathbf{s}} \times \hat{\mathbf{z}}) - \frac{g_1 + g_2}{2} (\hat{\mathbf{s}} \times \hat{\mathbf{z}}), \tag{2}$$
$$= \left[\alpha (\hat{\mathbf{s}} \cdot \hat{\mathbf{z}}) - \frac{g_1 + g_2}{2}\right] (\hat{\mathbf{s}} \times \hat{\mathbf{z}}). \tag{3}$$

  Thus, $\hat{\mathbf{s}} \cdot \hat{\mathbf{z}} = (g_1 + g_2)/(2\alpha)$ is an equilibrium. Again, this is accurate to zeroth order in I_1 and I_2.

- In Fig 4, we show a case where $\alpha = 15 g_1$ and starting with $\theta_{\rm sl,i} = 68^\circ$. The system again librates with resonant angle $\phi_{\rm sl} + \Delta g t/2$. The obliquity still oscillates about $\theta_{\rm sl,res} \equiv (g_1 + g_2)/(2\alpha)$ (horizontal line). Same for Figs. 5–6.

According to my suspicion/intuition, this new equilibrium is a parametric-type resonance and is unlikely to have the same phase space structure as a Cassini State (so Natalia's formalism may not work, since the character of the resonance is different). Some analytical insight can be obtained by considering the equation that Dong wrote down, in the "co-rotating" frame where $\hat{\mathbf{l}} = \hat{\mathbf{z}}$ is fixed and $\hat{\mathbf{l}}_{\rm p}$ is only allowed to nutate:
$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\rm rot} = \alpha (\hat{\mathbf{s}} \cdot \hat{\mathbf{z}})(\hat{\mathbf{s}} \times \hat{\mathbf{z}}) - \hat{\mathbf{R}} \times \hat{\mathbf{s}}, \tag{4}$$
where $\hat{\mathbf{R}}$ is the "rotation" matrix with that has a time dependency with frequency Δg. This is clearly a parametrically driven, nonlinear system. But maybe if the periodic component of $\hat{\mathbf{R}}$ is treated as a small perturbation, the leading order time-varying component of $\hat{\mathbf{s}}$ may be able to be solved for, like:
$$\hat{\mathbf{s}}(t) = \mathbf{s}_0 + \mathbf{s}_1 \cos\left(\frac{\Delta g}{2} t\right) + \ldots, \tag{5}$$
$$\hat{\mathbf{R}} = \mathbf{R}_0 + \mathbf{R}_1 \cos(\Delta g t) + \ldots \tag{6}$$
$$\text{At the frequency} \cos\left(\frac{\Delta g}{2} t\right): \quad 0 = \alpha (\hat{\mathbf{s}}_1 \cdot \hat{\mathbf{z}})(\hat{\mathbf{s}}_0 \times \hat{\mathbf{z}}) + \alpha (\hat{\mathbf{s}}_0 \cdot \hat{\mathbf{z}})(\hat{\mathbf{s}}_1 \times \hat{\mathbf{z}}) - \frac{\hat{\mathbf{R}}_1}{2} \times \hat{\mathbf{s}}_1. \tag{7}$$

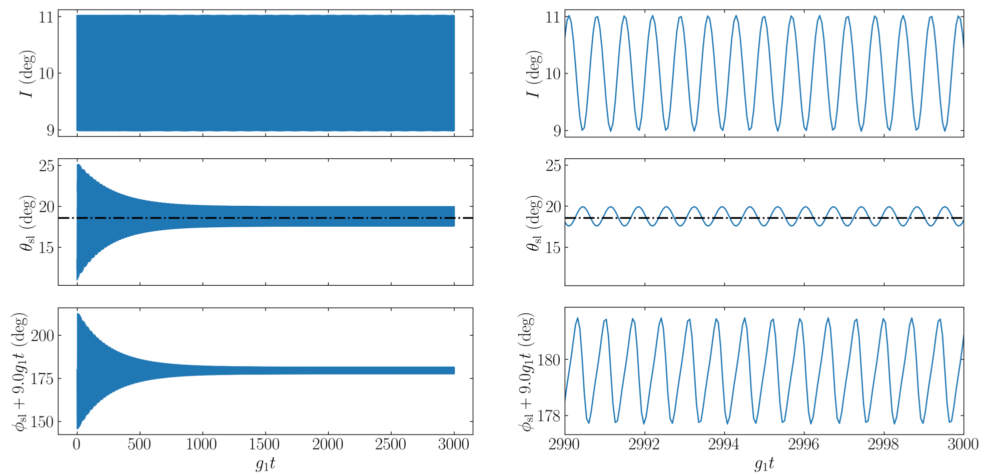

Figure 1: Plot of the evolution of an initially low-obliquity system. Horizontal line is CS2 of mode 2.

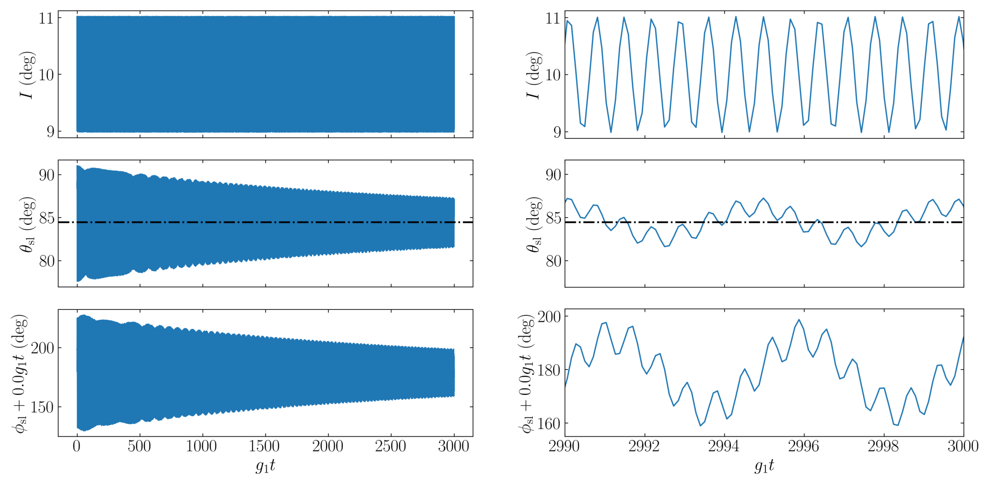

Figure 2: Plot of the evolution of an initially high-obliquity system. Horizontal line is CS2 of mode 1.

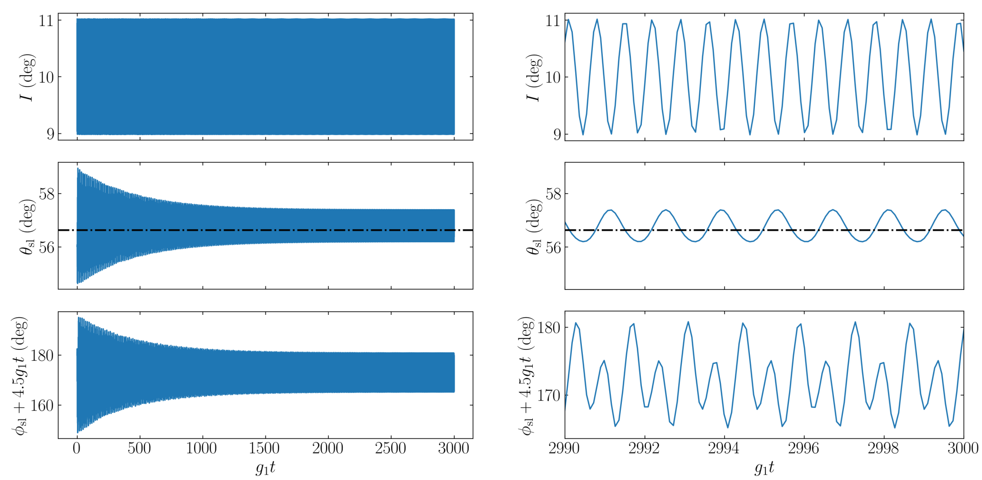

Figure 3: Plot of the evolution of an initially intermediate-obliquity system. Horizontal line is Eq. (1).

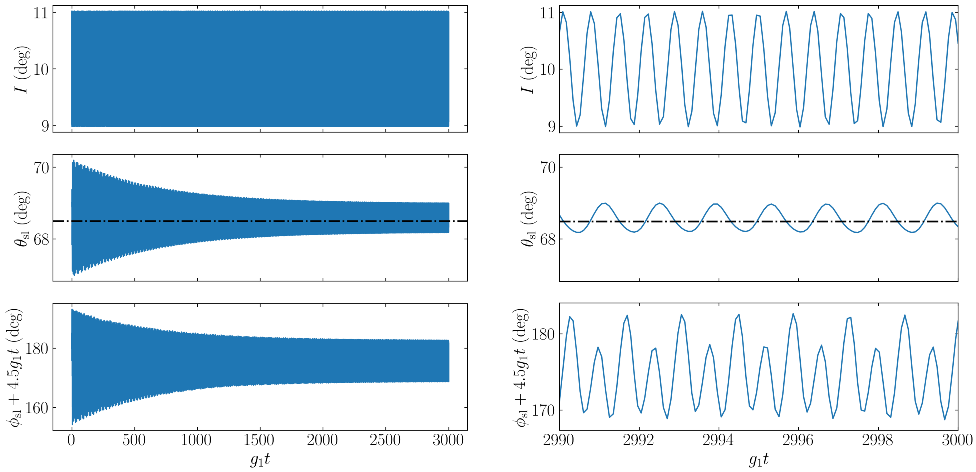

Figure 4: Same as Fig. 3 but for $\alpha = 15g_1$. Horizontal line is Eq. (1).

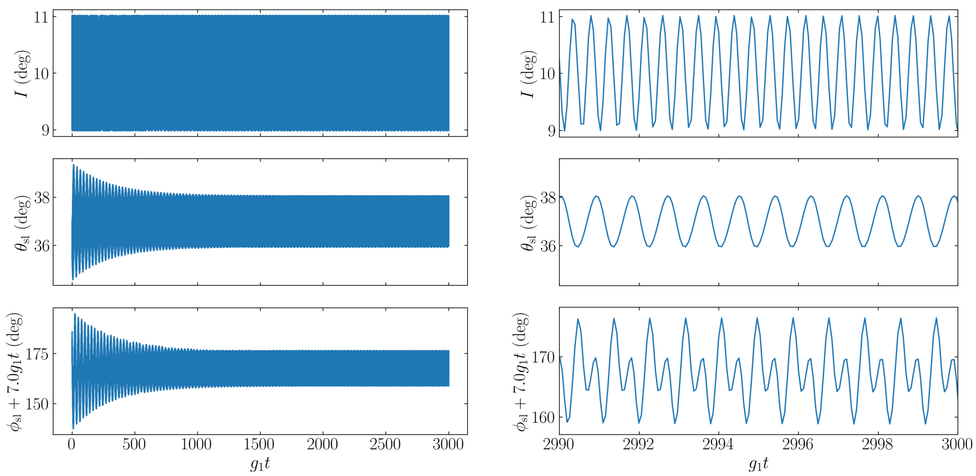

Figure 5: Same as Fig. 3 but for $g_2 = 15g_1$. Horizontal line is Eq. (1).

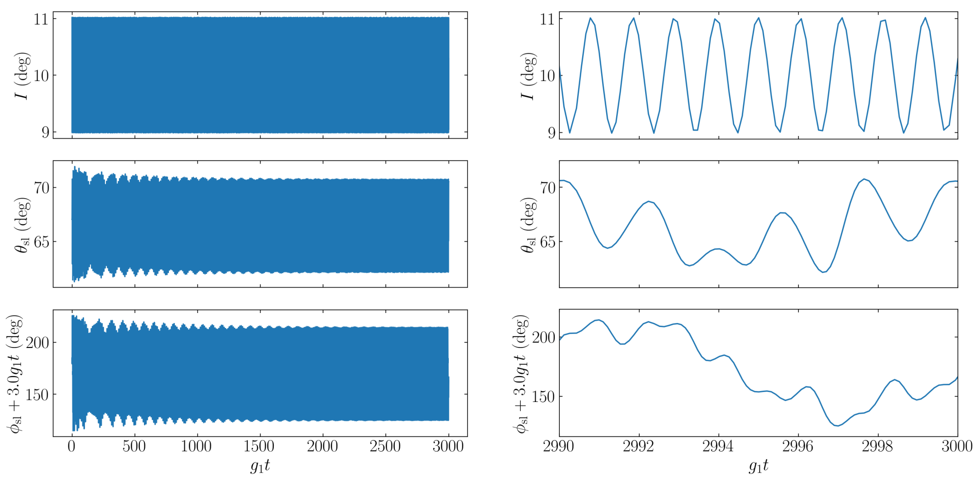

Figure 6: Same as Fig. 3 but for $g_2 = 7g_1$. Horizontal line is Eq. (1). Being near the separatrix of mode 1 gives extra perturbation harmonics..

1 Possible Analytical Work?

I've only thought about this briefly, but in our traditional "co-rotating" frame with fixed $\hat{\mathbf{l}}$ and nutating $\hat{\mathbf{J}}$, we have

$$\dot{\Omega} = g_1 + \frac{I_2}{I_1}\Delta g \cos(\Delta g t), \tag{8}$$

$$\dot{I} = -I_2 \Delta g \sin(\Delta g t), \tag{9}$$

$$\hat{\mathbf{J}} = \cos I \hat{\mathbf{z}} - \sin I \hat{\mathbf{x}}. \tag{10}$$

Consider the EOM in the frame $\phi_{\rm sl} + \Delta g/2$, where $\Delta g = g_2 - g_1$, then the new coordinate vectors are $\hat{\mathbf{x}}' = \hat{\mathbf{x}}\cos(\Delta g t/2) + \hat{\mathbf{y}}\sin(\Delta g t/2)$ and similarly for $\hat{\mathbf{y}}'$, and:

$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\rm rot} = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}\right) - \hat{\mathbf{s}}\times\left(\dot{\Omega}\hat{\mathbf{J}} + \dot{I}\hat{\mathbf{y}} + \frac{\Delta g}{2}\hat{\mathbf{l}}\right). \tag{11}$$

Let's work to leading order in I_2. Then, to zeroth order (assuming $\hat{\mathbf{J}} \approx \hat{\mathbf{l}} = \hat{\mathbf{z}}$):

$$\frac{\mathrm{d}\hat{\mathbf{s}}_0}{\mathrm{d}t} \approx \alpha(\hat{\mathbf{s}}_0\cdot\hat{\mathbf{z}})(\hat{\mathbf{s}}_0\times\hat{\mathbf{z}}) - \hat{\mathbf{s}}_0\times\left(\frac{g_1+g_2}{2}\hat{\mathbf{z}}\right). \tag{12}$$

Asking that this vanish exactly recovers Eq. (1). At the next order, we have

$$\frac{\mathrm{d}\hat{\mathbf{s}}_1}{\mathrm{d}t} \approx \alpha\left[(\hat{\mathbf{s}}_1\cdot\hat{\mathbf{z}})(\hat{\mathbf{s}}_0\times\hat{\mathbf{z}}) + (\hat{\mathbf{s}}_0\cdot\hat{\mathbf{z}})(\hat{\mathbf{s}}_1\times\hat{\mathbf{z}})\right] - \hat{\mathbf{s}}_0\times\left(\frac{I_2}{I_1}\Delta g\cos(\Delta g t)\hat{\mathbf{J}} - I_2\Delta g\sin(\Delta g t)\hat{\mathbf{y}}\right). \tag{13}$$

If we expand this in $\hat{\mathbf{x}}'$, $\hat{\mathbf{y}}'$ coordinates, there should be a forcing term with frequency $(\Delta g t)/2$?