1 Analytical Results

We consider the EOM in the "co-rotating" frame where $\hat{\mathbf{I}} = \hat{\mathbf{z}}$ is stationary and $\hat{\mathbf{J}}$ is nutating, fixed in the $\hat{\mathbf{x}}$-$\hat{\mathbf{z}}$ plane. Then:

$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\text{rot}} = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{I}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{I}}\right) - \hat{\mathbf{s}}\times\left(\dot{\Omega}\hat{\mathbf{J}} + \dot{I}\hat{\mathbf{y}}\right), \tag{1}$$

where

$$\hat{\mathbf{J}} = \cos I\hat{\mathbf{z}} - \sin I\hat{\mathbf{x}}, \tag{2}$$

$$\dot{I} \approx -I_2\Delta g\sin(\Delta g t), \tag{3}$$

$$\dot{\Omega} \approx g_1 + \Delta g\frac{I_2}{I_1}\cos(\Delta g t). \tag{4}$$

However, we seek an equilibrium with the resonant angle $\phi_{\text{res}} = \phi_{\text{sl}} + (\Delta g/2)t$. Thus, we perform another rotation into the $\hat{\mathbf{x}}'$-$\hat{\mathbf{y}}'$-$\hat{\mathbf{z}}$ frame (where $\hat{\mathbf{x}} = \cos(\Delta g t/2)\hat{\mathbf{x}}' - \sin(\Delta g t/2)\hat{\mathbf{y}}'$ and $\hat{\mathbf{y}} = \sin(\Delta g t/2)\hat{\mathbf{x}}' + \cos(\Delta g t/2)\hat{\mathbf{y}}'$), then

$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\text{res}} = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{I}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{I}}\right) - \hat{\mathbf{s}}\times\left(\dot{\Omega}\hat{\mathbf{J}} + \dot{I}\hat{\mathbf{y}}\right) - \hat{\mathbf{s}}\times\frac{\Delta g}{2}\hat{\mathbf{I}}, \tag{5}$$

$$\approx \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{I}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{I}}\right) - \hat{\mathbf{s}}\times\left[g_1(\hat{\mathbf{z}} - \sin I_1\hat{\mathbf{x}}) + \Delta g\frac{I_2}{I_1}\cos(\Delta g t)\hat{\mathbf{z}} - \Delta g I_2\cos(\Delta g t)\hat{\mathbf{x}} - I_2\Delta g\sin(\Delta g t)\hat{\mathbf{y}} + \frac{\Delta g}{2}\hat{\mathbf{z}}\right]. \tag{6}$$

Now, we consider expansion in successive orders of I_2, so we decompose $\hat{\mathbf{s}} = \mathbf{s}_0 + \mathbf{s}_1 + \dots$ where $\mathbf{s}_k \sim \mathcal{O}(I_2^k)$. Then, suppressing the "res" subscript:

$$\frac{\mathrm{d}\mathbf{s}_0}{\mathrm{d}t} = \alpha(\mathbf{s}_0\cdot\hat{\mathbf{z}})(\mathbf{s}_0\times\hat{\mathbf{z}}) - \mathbf{s}_0\times\left(\frac{g_1+g_2}{2}\right)\hat{\mathbf{z}}, \tag{7}$$

$$\begin{aligned}\frac{\mathrm{d}\mathbf{s}_1}{\mathrm{d}t} &= \alpha(\mathbf{s}_1\cdot\hat{\mathbf{z}})(\mathbf{s}_0\times\hat{\mathbf{z}}) + \cancel{\alpha(\mathbf{s}_0\cdot\hat{\mathbf{z}})(\mathbf{s}_1\times\hat{\mathbf{z}}) - \mathbf{s}_1\times\left(\frac{g_1+g_2}{2}\hat{\mathbf{z}}\right)} \\ &\quad - \mathbf{s}_0\times\left[-\cancel{g_1\sin I_1\hat{\mathbf{x}}} \overset{?}{+} \Delta g\frac{I_2}{I_1}\cos(\Delta g t)\hat{\mathbf{z}} - I_2\Delta g\left[\sin(\Delta g t)\hat{\mathbf{y}} + \cos(\Delta g t)\hat{\mathbf{x}}\right]\right], \end{aligned} \tag{8}$$

$$\begin{aligned} &= \alpha(\mathbf{s}_1\cdot\hat{\mathbf{z}})[\mathbf{s}_0\times\hat{\mathbf{z}}] - \mathbf{s}_0\times\left\{\cancel{-g_1\sin I_1\left(\cos\left(\frac{\Delta g t}{2}\right)\hat{\mathbf{x}}' + \sin\left(\frac{\Delta g t}{2}\right)\hat{\mathbf{y}}'\right)}\right. \\ &\quad \left. + \Delta g\frac{I_2}{I_1}\cos(\Delta g t)\hat{\mathbf{z}} + I_2\Delta g\left[-\sin\left(\frac{\Delta g t}{2}\right)\hat{\mathbf{y}}' + \cos\left(\frac{\Delta g t}{2}\right)\hat{\mathbf{x}}'\right]\right\}. \end{aligned} \tag{9}$$

Eq. (7) reduces to the expression we already know,

$$\mathbf{s}_0\cdot\hat{\mathbf{z}} = \frac{g_1+g_2}{2\alpha}. \tag{10}$$

At the next order, we can cancel one of the terms in Eq. (8) using Eq. (7) (shown), arriving at Eq. (9). If we ignore the two $\mathbf{s}_0\times\hat{\mathbf{z}}$ terms (they will act purely in the ϕ direction and are at the Δg frequency, so may be responsible for the Δg harmonic in the $\phi(t)$ graph), then we arrive at

$$\frac{\mathrm{d}\mathbf{s}_1}{\mathrm{d}t} = \mathbf{s}_0\times\left\{I_2\Delta g\left[\sin\left(\frac{\Delta g t}{2}\right)\hat{\mathbf{y}}' - \cos\left(\frac{\Delta g t}{2}\right)\hat{\mathbf{x}}'\right]\right\}. \tag{11}$$

Thus, if there is no transient behavior (due to the alignment torque), we find that the oscillation amplitude in θ should be given by

$$\Delta\theta \sim 2I_0\sin\theta_0. \tag{12}$$

NB: I had to ignore the $-g_1$ term to make this work; perhaps it is because it isn't an I_2-dependent term? It may be luck, or maybe this is the correct calculation.

How well does this do? We consult Fig. 1. The agreement seems qualitatively correct, though there is yet still work to be done.

2 Numerics

Numerical outcomes for $I = 3^\circ$ and a variety of g_2 values are shown in Fig. 2. Visible are: (i) the decrease in M1-CS2 (mode 1, CS2) as η_2 is increased from zero, since chaos begins to eject trajectories near the edge of the separatrix, and (ii) the appearance of a substantial mixed mode for $\eta_2 > 1.0$.

Perhaps further simulations would be interesting to explore the regime $\eta_2 \lesssim 1$.

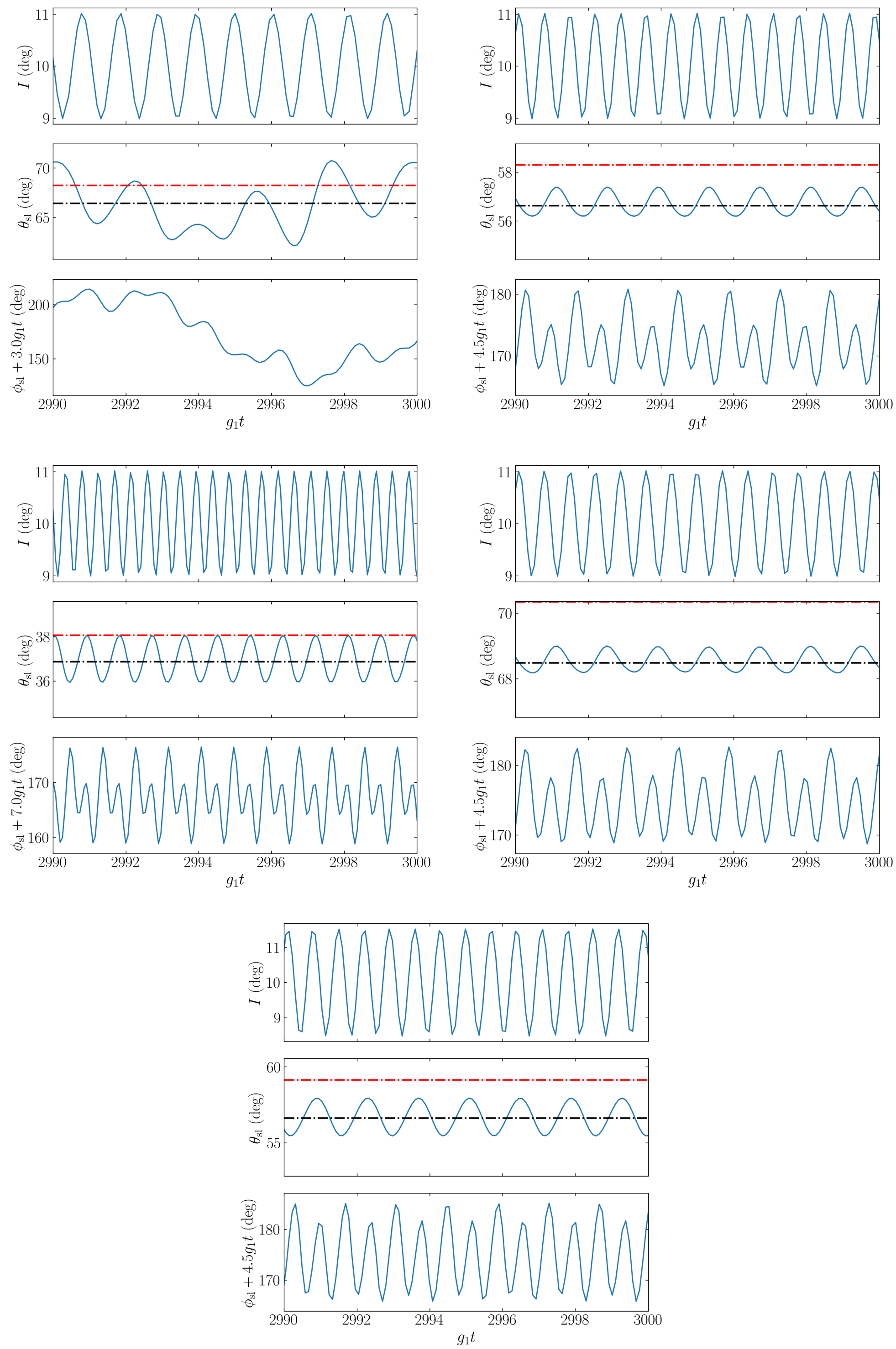

Figure 1: Some examples of zoomed-in evolution for (unless otherwise noted, $\alpha = 10g_1$ and $I = 1°$): (i) $g_2 = 7g_1$, (ii) $g_2 = 10g_1$, (iii) $g_2 = 15g_1$, (iv) $\alpha = 15g_1$, (v) $I_2 = 1.5°$. Black line is Eq. (10) and red line is Eq. (12).

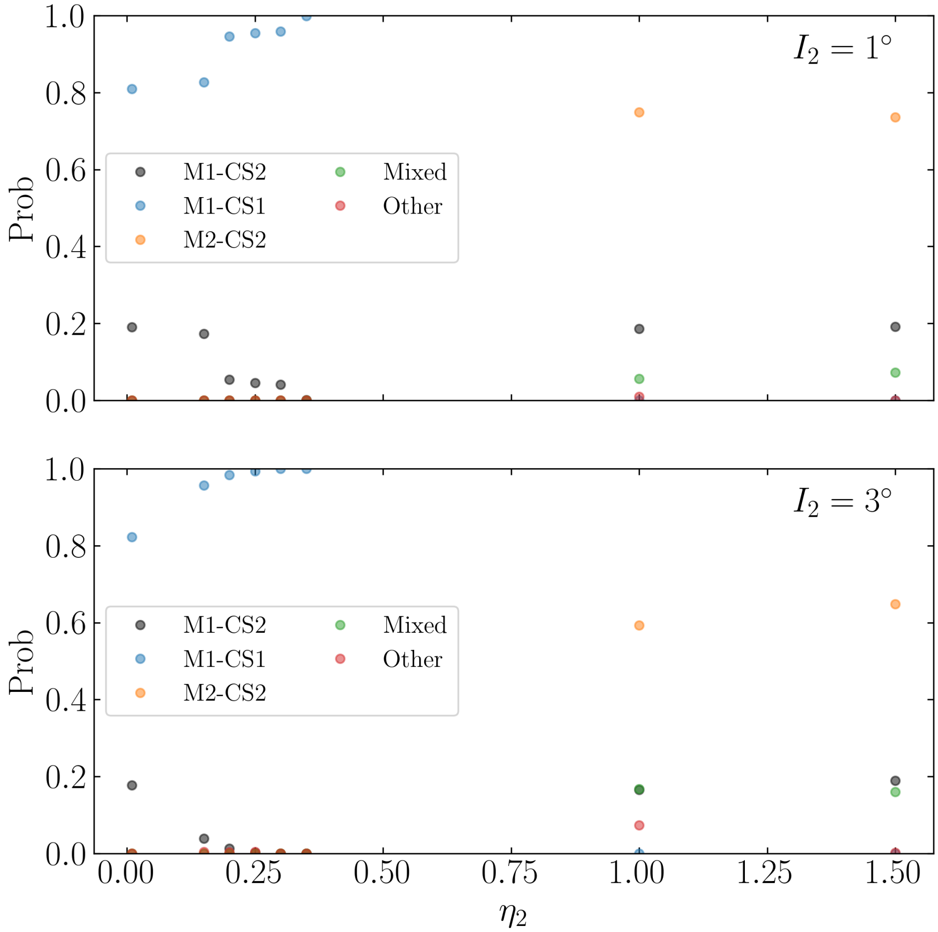

Figure 2: Probability of reaching various CSs for each mode as a function of η_2, with inclinations labeled. $\alpha = 10g_1$.