Spin-Orbit Misalignment Dynamics in Black Hole Triples

Group Meeting?

Yubo Su

Apr. 4, 2020?

Equations:

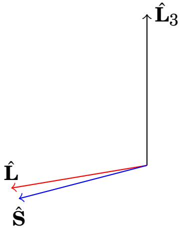
$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \Omega_{SL}\hat{\mathbf{L}} \times \hat{\mathbf{S}}, \tag{1}$$

$$\Omega_{SL} = \frac{3Gn\left(m_2 + \mu/3\right)}{2c^2 a\left(1 - e^2\right)}, \tag{2}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{15}{16t_{LK}}\frac{e^2 \sin(2\omega)\sin(2I)}{\sqrt{1-e^2}}, \tag{3}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{3}{4t_{LK}}\frac{\cos i\left(5e^2\cos^2\omega - 4e^2 - 1\right)}{\sqrt{1-e^2}}, \tag{4}$$

$$\frac{1}{t_{LK}} = n\left(\frac{m_3}{m_1+m_2}\right)\left(\frac{a}{\bar{a}_{3,\mathrm{eff}}}\right)^3. \tag{5}$$



GW radiation narrows range of e oscillations.
$\theta_{\mathrm{sl}}^f = \cos^{-1}(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}})$?

- Go to corotating frame with $\dot{\Omega}$, so that $\hat{\mathbf{L}}$ nutates and $\hat{\mathbf{L}}_3 = \hat{\mathbf{z}}$:

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \left(\Omega_{SL}\hat{\mathbf{L}} - \dot{\Omega}\hat{\mathbf{L}}_3\right) \times \hat{\mathbf{S}} \equiv \hat{\boldsymbol{\Omega}}_{\mathrm{eff}} \times \hat{\mathbf{S}}. \tag{6}$$

- Intuition: If no LK, Ω_{SL}, $\dot{\Omega}$ slowly vary, $\theta_{\mathrm{sl}}^f = \theta_{\mathrm{s3}}^i$.
- $\Omega_{SL}\hat{\mathbf{L}}$ and $\dot{\Omega}\hat{\mathbf{L}}_3$ periodic with T_{LK} (varying), so decompose about the mean value:

$$\begin{aligned}\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} &= \langle\hat{\boldsymbol{\Omega}}_{\mathrm{eff}}\rangle_{LK} \times \hat{\mathbf{S}} \\ &\quad - \mathbf{S} \times \sum_{N=1}^{\infty} \hat{\mathbf{A}}_N \cos\left(\frac{2\pi N t}{T_{LK}}\right).\end{aligned} \tag{7}$$

- Formally, can average (or WKBJ) **unless** $\Omega_{\mathrm{eff}} = \frac{2\pi(N'/2)}{T_{LK}}$.

- What are these **linear** resonances? Consider toy model, $\epsilon \to 0$:

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \left[\omega_0\hat{\mathbf{z}} + \epsilon(\cos\omega t\hat{\mathbf{x}} + \sin\omega t\hat{\mathbf{y}})\right] \times \hat{\mathbf{S}}, \tag{8}$$

$$\left(\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \left[(\omega_0 - \omega)\hat{\mathbf{z}} + \epsilon\hat{\mathbf{x}}\right] \times \hat{\mathbf{S}}. \tag{9}$$

- In resonances, the precession axis tilts away from $\hat{\mathbf{z}}$ plane when $\omega_0 - \omega \sim \epsilon$. Then $\dot{\theta} \simeq \epsilon$.
- Thus, when crossing:
  - If slow, adiabatic invariance, $\theta^f = \theta^i$.
  - If fast, no time to rotate, $\theta^f \approx \theta^i$.
  - Only when $\frac{\mathrm{d}\ln\omega}{\mathrm{d}t} \simeq \epsilon$ will $\theta^f \neq \theta^i$.

- If resonance $\Omega_{\mathrm{eff}} = \pi N'/T_{LK}$:

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \langle \hat{\mathbf{\Omega}}_{\mathrm{eff}} \rangle_{LK} \times \hat{\mathbf{S}} - \frac{\hat{\mathbf{S}}}{2} \times \sum_{N=1}^{N'} \hat{\mathbf{A}}_N \cos\left(\frac{2\pi N t}{T_{LK}}\right) - \frac{\hat{\mathbf{S}}}{2} \times \sum_{N=1}^{\infty} \hat{\mathbf{B}}_N \cos\left(\Omega_{\mathrm{eff}} t\right) - \hat{\mathbf{C}}_N \sin\left(\Omega_{\mathrm{eff}} t\right), \tag{10}$$

$$\hat{\mathbf{B}}_N(t) = \left(\hat{\mathbf{A}}_N + \hat{\mathbf{A}}_{N+N'}\right) \cos\left(\frac{\pi\left(N+N'\right)t}{T_{LK}}\right), \tag{11}$$

$$\hat{\mathbf{C}}_N(t) = \left(\hat{\mathbf{A}}_N - \hat{\mathbf{A}}_{N+N'}\right) \sin\left(\frac{\pi\left(N+N'\right)t}{T_{LK}}\right). \tag{12}$$

- Equivalently, consider using Hamiltonian and going to coordinates (sum every N and $N+N'$ term to generate multiples of $N'/2$). Note $\sin\left(\phi - \frac{\pi N' t}{T_{LK}}\right)$ dependencies.

$$H = \langle \hat{\mathbf{\Omega}}_{\mathrm{eff}} \rangle_{LK} \cdot \hat{\mathbf{S}} + \sum_{N=1}^{\infty} \left(A_{N,z} \cos\theta + A_{N,x} \sin\theta \cos\phi\right) \cos\left(\frac{2\pi N t}{T_{LK}}\right), \tag{13}$$

$$= \langle \hat{\mathbf{\Omega}}_{\mathrm{eff}} \rangle_{LK} \cdot \hat{\mathbf{S}} + \frac{\hat{\mathbf{S}}}{2} \cdot \sum_{N=1}^{N'} \hat{\mathbf{A}}_N \cos\left(\frac{2\pi N t}{T_{LK}}\right) + \frac{\hat{\mathbf{S}}}{2} \cdot \sum_{N=1}^{\infty} \left(B_{N,z} \cos\theta + B_{N,x} \sin\theta \cos\phi\right) \cos\left(\frac{\pi N' t}{T_{LK}}\right) - \left(C_{N,z} \cos\theta + C_{N,x} \sin\theta \cos\phi\right) \sin\left(\frac{\pi N' t}{T_{LK}}\right). \tag{14}$$

- Note: If $\Omega_{SL} \gtrsim \dot{\Omega}$, LK suppressed. Thus, consider primarily $\langle\dot{\Omega}\rangle_{LK}\hat{\mathbf{z}}$ dominating $\hat{\mathbf{\Omega}}_{\mathrm{eff}}$.
- We want $\langle\dot{\Omega}\rangle = \pi N'/T_{LK}$, which is equivalent to:

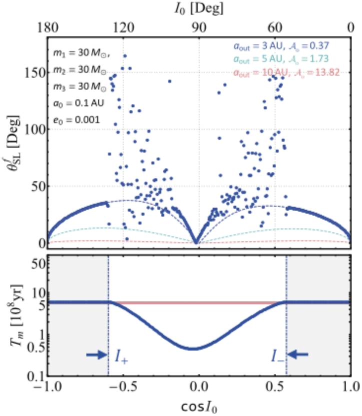
$$\pi N' = \Delta\Omega = \int_0^{T_{LK}} \dot{\Omega}\,\mathrm{d}t. \tag{15}$$

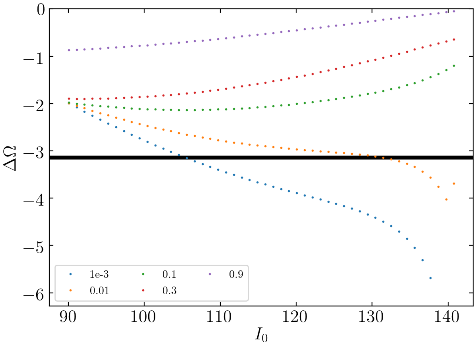
- It turns out that for a wide range of parameters, $|\Delta\Omega| < \pi$, thus, no resonance can be hit.

Averaged Scenario

When are Resonances Hit?

- As GW radiates, e_0 increases, *decreases* $\Delta\Omega$ (right; numerical). Never resonance crossing ($\Delta\Omega < -\pi$, black line) when $e_0 = 10^{-3}$ for $I_0 \in [75, 105]$. Seems about right! Fig. 4 from Liu & Lai 2017.





- Consider again toy problem

$$\left(\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \left[(\omega_0 - \omega)\hat{\mathbf{z}} + \epsilon\hat{\mathbf{x}}\right] \times \hat{\mathbf{S}}. \tag{16}$$

- Allow $\omega = \omega_0 + \dot{\omega}t$. What is $\theta_f - \theta_i$? Exact:

$$\Delta\theta = \epsilon\sqrt{\frac{2\pi}{\dot{\omega}}}\sin\left(\phi_0 + \frac{\pi}{4}\right). \tag{17}$$

- Here, $\dot{\omega} \sim 1/T_{GW}$, while $\epsilon \propto 1/T_{LK}$, $\Rightarrow \Delta\theta \gg 1$?
- Caveat: ϵ is Fourier coefficient of $\dot{\Omega}$, has $1/T_{LK}$ scaling but *is probably substantially smaller*. So can probably predict width $(\sim \epsilon)$?
  - Might explain why deviations from the blue curve in Bin's figure seem to appear closer to $I_0 = 90^\circ$ than expected (100° vs 105°).
  - Might also predict the amplitude of the deviations (which appear bounded closer to 90°).

Resonance-Free Dynamics

Few-Shot Merger Solution

- When no resonances, just

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \langle \Omega_{SL}\hat{\mathbf{L}} - \dot{\Omega}\hat{\mathbf{L_3}}\rangle_{LK} \times \hat{\mathbf{S}}, \tag{18}$$

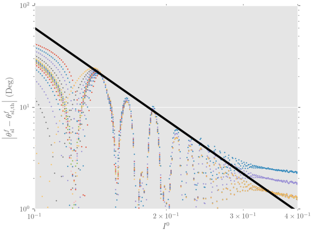
$$\frac{1}{\dot{\Omega}}\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} \simeq \left[\mathscr{A}_0\left(\sin I\hat{\mathbf{x}} + \cos I\hat{\mathbf{z}}\right) \mp \hat{\mathbf{z}}\right] \times \hat{\mathbf{S}}. \tag{19}$$

- Here, I is the inclination of $\langle \Omega_{SL}\hat{\mathbf{L}}\rangle$, shouldn't change much, $\sim 120^\circ$?
- Crosses resonance at $\mathscr{A} \simeq 1/\cos I \simeq 1$, and the jump in angle is

$$\Delta\theta \simeq \sin I \sqrt{\frac{2\pi}{\dot{\mathscr{A}}_0}} \sin\left(\phi_0 + \pi/4\right). \tag{20}$$

- Explains why for fixed ϕ, $\Delta\theta$ overlaps.

- Sweeped some stuff under the rug, but does this make qualitative sense?
- $T_{\mathrm{m}} \sim \cos^6 I_0$, so if $\dot{\mathscr{A}}_0 \propto 1/T_{\mathrm{m}}$ then $\Delta\theta \propto 1/\cos^3 I_0$. Seems to be the right scaling (black is fit by eye, correct power law index):



- (Ignore the nonzero tail at the right, I made a lazy approximation in my code)