1 Examination of Resonant Terms

We study the equation of motion

$$\frac{\mathrm{d}S_\perp}{\mathrm{d}t} = i\overline{\Omega}_{\mathrm{e}}S_\perp + \sum_{N=1}^{\infty}\left[\cos(\Delta I_{\mathrm{eN}})S_\perp - i\cos\bar{\theta}_{\mathrm{e}}\sin(\Delta I_{\mathrm{eN}})\right]\Omega_{\mathrm{eN}}\cos(N\Omega_{\mathrm{LK}}t). \tag{1}$$

1.1 Summary of Existing Results in My Paper

For simplicity, we consider only a single N. In the paper, we obtained the result

$$S_\perp(t) = S_\perp(t_{\mathrm{i}})\exp\left[i\overline{\Omega}_{\mathrm{e}}(t_{\mathrm{f}}-t_{\mathrm{i}}) + \frac{\cos(\Delta I_{\mathrm{eN}})\Omega_{\mathrm{eN}}}{N\Omega_{\mathrm{LK}}}\left[\sin(N\Omega_{\mathrm{LK}}t_{\mathrm{f}}) - \sin(N\Omega_{\mathrm{LK}}t_{\mathrm{i}})\right]\right]. \tag{2}$$

when neglecting the driving term, and the result

$$e^{-i\overline{\Omega}_{\mathrm{e}}t}S_\perp\Big|_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} = -\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}\frac{i\sin(\Delta I_{\mathrm{eN}})\Omega_{\mathrm{eN}}}{2}e^{-i\overline{\Omega}_{\mathrm{e}}t+iN\Omega_{\mathrm{LK}}t}\cos\bar{\theta}_{\mathrm{e}}\,\mathrm{d}t, \tag{3}$$

when neglecting the parametric term.

1.2 General Result for a Single Resonance

In full generality, if we define

$$\begin{aligned}\Phi &\equiv \int^{t} i\overline{\Omega}_{\mathrm{e}} + \cos(N\Omega t)\cos(\Delta I_{\mathrm{N}})\Omega_{\mathrm{eN}}\,\mathrm{d}t,\\ &= i\overline{\Omega}_{\mathrm{e}}t - \frac{\cos\Delta I_{\mathrm{N}}\Omega_{\mathrm{eN}}}{N\Omega}\sin(N\Omega t),\\ &\equiv i\overline{\Omega}_{\mathrm{e}}t + \eta\sin(N\Omega t),\end{aligned} \tag{4}$$

where $\eta \equiv (\cos\Delta I_{\mathrm{N}}\Omega_{\mathrm{eN}})/(N\Omega)$, then it is easy to obtain solution

$$\begin{aligned}e^{-\Phi(t)}S_\perp(t) - e^{-\Phi(t_{\mathrm{i}})}S_\perp(t_{\mathrm{i}}) &= \int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}} -e^{-\Phi(\tau)}i\cos\bar{\theta}_{\mathrm{e}}\cos(N\Omega\tau)(\sin\Delta I_{\mathrm{N}})\Omega_{\mathrm{eN}}\,\mathrm{d}\tau,\\ &\equiv A\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}\cos(N\Omega\tau)e^{-\Phi(\tau)}\,\mathrm{d}\tau,\end{aligned} \tag{5}$$

where $A = -i\cos\bar{\theta}_{\mathrm{e}}\sin\Delta I_{\mathrm{N}}\Omega_{\mathrm{eN}}$. We note in passing that if $A = 0$, we recover Eq. (2). We further expand

$$\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}\cos(N\Omega\tau)e^{-\Phi(\tau)}\,\mathrm{d}\tau = \frac{1}{2}\int_{t_{\mathrm{i}}}^{t_{\mathrm{f}}}\left(e^{iN\Omega\tau} + e^{-iN\Omega\tau}\right)\exp\left(i\overline{\Omega}_{\mathrm{e}}\tau + \eta\sin(N\Omega\tau)\right)\,\mathrm{d}\tau. \tag{6}$$

If we define $\omega_\pm = \overline{\Omega}_\mathrm{e} \pm N\Omega$, understanding the behavior of the term above requires we understand the integral

$$\int\limits_{t_\mathrm{i}}^{t_\mathrm{f}} \exp\left(i\omega_\pm \tau + \eta \sin(N\Omega\tau)\right)\,\mathrm{d}\tau = \frac{1}{\omega_\pm}\int\limits_{x_\mathrm{i}}^{x_\mathrm{f}} \exp\left[-ix' - \eta\sin\beta x\right]\,\mathrm{d}x', \tag{7}$$

where $x' \equiv \omega_\pm \tau$ and $x_{\mathrm{i,f}} = \omega_\pm t_{\mathrm{i,f}}$, and we've defined $\beta \equiv N\Omega/\omega_\pm$. We also note at this point that if $\eta = 0$, the above integral oscillates between $\pm\frac{1}{\omega_\pm}$, so the total amplitude of oscillation of $S_\perp$ is $A/\min(\omega_+, \omega_-) = A/\left(\overline{\Omega}_\mathrm{e} - N\Omega\right)$, since $\overline{\Omega}_\mathrm{e} > 0$, and we recover Eq. (3).

At this point, we have recovered both limits considered in the paper, but the η-dependent term in the integrand of Eq. (7) is new to this treatment. The effect of this term can easily be calculated analytically, however. We take $\omega_\pm \neq 0$, since this is the resonance already well understood in the paper, then

$$I(x_\mathrm{f}) = \int\limits_{x_\mathrm{i}}^{x_\mathrm{f}} \exp\left[-ix' - \eta\sin\beta x\right]\,\mathrm{d}x' = \int\limits_{x_\mathrm{i}}^{x_\mathrm{f}} \left(\cos x' - i\sin x'\right)\sum_{k=0}^{\infty}\frac{\left(-\eta\sin(\beta x')\right)^k}{k!}\,\mathrm{d}x'. \tag{8}$$

We next examine the general power-reduction trigonometric identities[1]:

$$\sin^{2n} y = \frac{1}{2^{2n}}\binom{2n}{n} + \frac{(-1)^n}{2^{2n-1}}\sum_{k=0}^{n-1}(-1)^k\binom{2n}{k}\cos[2(n-k)y], \tag{9}$$

$$\sin^{2n+1} y = \frac{(-1)^n}{4^n}\sum_{k=0}^{n}(-1)^k\binom{2n+1}{k}\sin[(2n+1-2k)y]. \tag{10}$$

We consider three cases for β:

- If β is irrational, $\sin^k\left(\beta x'\right)$ decomposes into trigonometric functions with irrational frequency, which when integrated by $\cos x'$ or $\sin x'$ will always be bounded.
- If $\beta = 1/q$ for some integer q, then let's evaluate Eq. (8) over interval $x_\mathrm{f} - x_\mathrm{i} = 2\pi q$. Then many terms will vanish since the trigonometric functions satisfy orthogonality conditions:

$$\int\limits_0^{2\pi}\cos mx\cos nx\,\mathrm{d}x = \delta_{mn}, \tag{11}$$

$$\int\limits_0^{2\pi}\sin mx\sin nx\,\mathrm{d}x = \delta_{mn}, \tag{12}$$

$$\int\limits_0^{2\pi}\sin mx\cos nx\,\mathrm{d}x = 0, \tag{13}$$

where δ_{mn} is the Kronecker delta. However, $\sin^q\left(x'/q\right)$ will contain either a $\sin\left(x'\right)$ or $\cos\left(x'\right)$

[1] Zwillinger, Daniel, ed. CRC standard mathematical tables and formulae. CRC press, 2002.

term if q is odd or even respectively. The coefficient of this term is given by either Eq. (9) or (10), and we conclude

$$|I(x_i + 2\pi m q)| \approx 2\pi m q \frac{\eta^q}{q!} \frac{1}{2^q}. \tag{14}$$

Note that this formula is approximate, because the in Eq. (8) of form $\sin^{q+2k}\left(x'/q\right)$ for positive integer k will also contain factors of $\sin\left(x'\right)$ or $\cos\left(x'\right)$, but these are higher order corrections. We conclude that when $\beta = 1/q$, $S_\perp(t)$ grows without bound, and the growth rate is estimated by

$$\frac{\mathrm{d}\,|I(x)|}{\mathrm{d}x} \approx \frac{\eta^q}{q!2^q}. \tag{15}$$

- If $\beta = p/q$ for integers p, q, we only get unbounded growth for $I(x)$ if either $2(n-k)p/q = 1$ or $(2(n-k)+1)p/q = 1$ for any integers n, k. Since both $2(n-k)$ and $(2(n-k)+1)$ are integers, their product with p/q can only equal 1 if $p = 1$, which is the case studied above. Otherwise, there is also no unbounded growth.

In conclusion, the resonance condition is, for nonzero integer q,

$$\beta = \frac{1}{q} = \frac{1}{\overline{\Omega}_{\mathrm{e}}/N\Omega \pm 1}. \tag{16}$$

This is satisfied when $\overline{\Omega}_{\mathrm{e}}/N\Omega$ is an integer, except when $\overline{\Omega}_{\mathrm{e}} = N\Omega$ as we already have studied in the paper. However, the growth rate of this instability falls off very quickly for large q, see Eq. (15).

1.3 Numerical Simulations

We numerically compute the two integrals

$$F(t) = \int_0^t \cos x' e^{-\eta \sin \beta x'}\,\mathrm{d}x', \tag{17}$$

$$G(t) = \int_0^t \sin x' e^{-\eta \sin \beta x'}\,\mathrm{d}x'. \tag{18}$$

We choose $\eta = 1$ for simplicity. We expect $F(t)$ to have resonant growth when $\beta = 1/2n$, and $G(t)$ to have resonant growth when $\beta = 1/(2n+1)$ for $n \in \mathbb{Z}_+$. These are plotted in Fig. 1.

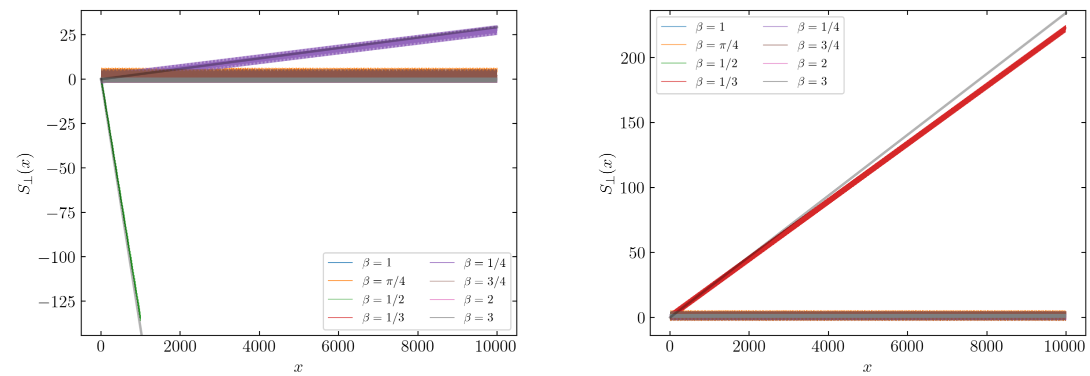

Figure 1: Plot of $F(t)$ [Eq. (17), left] and plot of $G(t)$ [Eq. (18), right]. Our analysis suggests resonant growth when $\beta = 1/2n$ for $F(t)$ and when $\beta = 1/(2n+1)$ for $G(t)$, where $n \in \mathbb{Z}_+$, which agrees with the simulation. The thick grey lines are the analytic growth rates predicted by Eq. (15), illustrating good agreement.