1 $\bar{g}$ Equilibrium

Recall that we showed that, in the co-rotating frame with frequency $\bar{g} = (g_1 + g_2)/2$:

$$\left(\frac{\mathrm{d}\hat{\mathbf{s}}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\hat{\mathbf{s}}\times\hat{\mathbf{l}}\right) - \bar{g}\left(\hat{\mathbf{s}}\times\hat{\mathbf{z}}\right), \tag{1}$$

$$\hat{\mathbf{l}}(t) = \begin{bmatrix} (I_1 + I_2)\cos\left(\frac{\Delta g t - \phi_0}{2}\right) \\ (I_1 - I_2)\sin\left(\frac{\Delta g t - \phi_0}{2}\right) \\ 1 \end{bmatrix} + \mathcal{O}\left[(I_1 + I_2)^2\right]. \tag{2}$$

We will try to seek a set of conditions for an $\mathbf{s}$ to be exactly stationary under these equations. We will find a leading-order prediction of the location of an equilibrium, though an exact equilibrium is impossible.

We first examine the $\hat{\mathbf{z}}$ component of the equation of motion for $\hat{\mathbf{s}}$, suppressing the "rot" subscript. Denote $\mathbf{s}_\perp$ and $\mathbf{l}_\perp$ to be the $\hat{\mathbf{x}}$-$\hat{\mathbf{y}}$ plane components of the two vectors, then we obtain

$$\frac{\mathrm{d}s_z}{\mathrm{d}t} = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\hat{\mathbf{s}}_\perp\times\hat{\mathbf{l}}_\perp\right). \tag{3}$$

An easy way for this to vanish is if

$$\hat{\mathbf{s}}_\perp \parallel \hat{\mathbf{l}}_\perp. \tag{4}$$

We next examine the in-plane component of the equation of motion:

$$\frac{\mathrm{d}\mathbf{s}_\perp}{\mathrm{d}t} = \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\mathbf{s}_\perp\times\hat{\mathbf{z}} + s_z\hat{\mathbf{z}}\times\mathbf{l}_\perp\right) - \bar{g}\left(\mathbf{s}_\perp\times\hat{\mathbf{z}}\right). \tag{5}$$

Since $\mathbf{s}_\perp \parallel \mathbf{l}_\perp$, we can express everything in terms of $\mathbf{l}_\perp$ and the two magnitudes $s_\perp$ and $l_\perp$. This gives the following manipulation:

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{s}_\perp}{\mathrm{d}t} &= \alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(\mathbf{s}_\perp\times\hat{\mathbf{z}} - s_z\frac{l_\perp}{s_\perp}\mathbf{s}_\perp\times\hat{\mathbf{z}}\right) - \bar{g}\left(\mathbf{s}_\perp\times\hat{\mathbf{z}}\right), &(6)\\
&= \left[\alpha\left(\hat{\mathbf{s}}\cdot\hat{\mathbf{l}}\right)\left(1 - s_z\frac{l_\perp}{s_\perp}\right) - \bar{g}\right]\left(\mathbf{s}_\perp\times\hat{\mathbf{z}}\right), &(7)\\
&= \left\{\left[\alpha s_z - \bar{g}\right] + \alpha\left[s_\perp l_\perp - s_z^2\frac{l_\perp}{s_\perp}\right] - s_z l_\perp^2\right\}\left(\mathbf{s}_\perp\times\hat{\mathbf{z}}\right). &(8)
\end{aligned}$$

I've grouped the terms in order of $\mathcal{O}(l_\perp)$, since $l_\perp \ll 1$. If an exact equilibrium exists, the expression in the curly brackets must vanish, as well as Eq. (4) be satisfied exactly. In particular, if $l_\perp \ll 1$, we recover the relation I claimed in the writeup earlier:

$$s_z \approx \frac{\bar{g}}{\alpha} + \mathcal{O}(l_\perp). \tag{9}$$

However, since $l_\perp$ is in general *not* constant in time (ranging from $I_1 + I_2$ to $I_1 - I_2$), an exact equilibrium does not exist: $l_\perp(t)$ has a $\Delta g/2$ harmonic, which means $s_\perp(t)$ will also have a $\Delta g/2$ harmonic.

1.1 Perturbation Theory

We try to do perturbation theory in I_2, since nonzero I_2 gives rise to the harmonic terms.