1 Recap

We analyze in the co-rotating frame

$$\left(\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \underbrace{\left(\Omega_{\mathrm{SL}}\left(\sin I\hat{\mathbf{x}} + \cos I\hat{\mathbf{z}}\right) - \dot{\Omega}\hat{\mathbf{z}}\right)}_{\boldsymbol{\Omega}_{\mathrm{eff}}} \cdot \hat{\mathbf{S}}, \tag{1}$$

$$= \boldsymbol{\Omega}_{\mathrm{eff,0}} \times \hat{\mathbf{S}} + \left[\sum_{N=1}^{\infty} \boldsymbol{\Omega}_{\mathrm{eff,N}} \sin\left(2\pi N t/t_{\mathrm{LK}}\right)\right] \times \hat{\mathbf{S}}. \tag{2}$$

where $\boldsymbol{\Omega}_{\mathrm{eff,N}}$ is the N-th component of the vector Fourier transform of $\boldsymbol{\Omega}_{\mathrm{eff}}$.

- In the Paper II regime, and in the Paper I regime near $I_0 = 90°$, we found good conservation of $\theta_{\mathrm{eff,0}}$ where

  $$\cos\theta_{\mathrm{eff,0}} = \hat{\mathbf{S}} \cdot \hat{\boldsymbol{\Omega}}_{\mathrm{eff,0}}. \tag{3}$$

  Note that to estimate the initial $\theta_{\mathrm{eff,0}}$, it is necessary to average over an LK cycle, as the angle is fast-varying.

  We justified this analytically by ignoring the $N \geq 1$ terms in Eq. (2) and assuming the merger is very gentle. However, there are two observed regimes in which this conservation principle breaks down.

- The easiest deviation to understand from conservation of $\theta_{\mathrm{eff,0}}$ is when the merger is fast (Paper II regime, $I_0 - 90° \lesssim 0.4°$); we developed a theory for this in a prior week.

- A trickier one is in the Paper I regime, where, even though the merger is peaceful, when $|I_0 - 90°| \gtrsim 15°$, we numerically find poor conservation of $\theta_{\mathrm{eff,0}}$ (I reproduced this in a single simulation at $I_0 = 70°$).

  This is contrasted with the Paper II regime where a peaceful merger is a sufficient condition for conservation of θ_{eff}.

The question I spent the past two weeks investigating is thus: Why is the peaceful merger condition sometimes but not always sufficient to guarantee conservation of $\theta_{\mathrm{eff,0}}$? I have performed many numerical explorations, and while they shed some more insight on the problem, I do not yet have a precise answer.

2 Ongoing Work

2.1 Angular Dependence

First, we consider whether conservation of $\theta_{\mathrm{eff,0}}$ has any angular dependence. For the Paper II regime (I will try this for the Paper I regime this coming week) and fiducial parameters ($I_0 = 90.5°$), we can sample the initial $\hat{\mathbf{S}}$ uniformly and plot the difference between $\theta^{\mathrm{i}}_{\mathrm{eff,0}}$ and the final $\theta^{\mathrm{f}}_{\mathrm{sl}} = \theta^{\mathrm{f}}_{\mathrm{eff,0}}$. This shows no angular dependence as shown in Fig. 1. NB: I made a coordinate mistake and only covered half the unit sphere of possible initial conditions.

For reference, the behavior of $\theta^{\mathrm{i}}_{\mathrm{eff,0}}$ at late times is given as the black line in Fig. 2.

Finally, we can track the evolution of each of these initial conditions over time, as shown in Fig. 3 in the corotating frame.

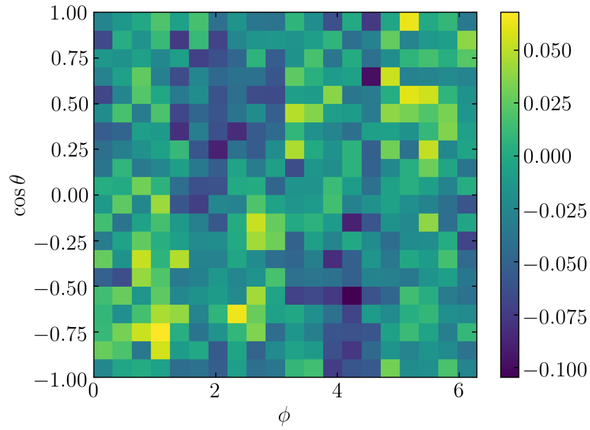

Figure 1: $\theta^{\mathrm{i}}_{\mathrm{eff},0} - \theta^{\mathrm{f}}_{\mathrm{sl}}$ for uniformly sampled $\hat{\mathbf{S}}$. No angular dependence is observed, uniform conservation is observed. Note that the y-axis is actually $\theta^{\mathrm{i}}_{\mathrm{sl}}$, which is not a very physically meaningful angle, but is okay since convergence is so uniform. NB: I made a coordinate mitsake and only half of the hemisphere is covered; nevertheless, the conclusion seems plausibly robust.

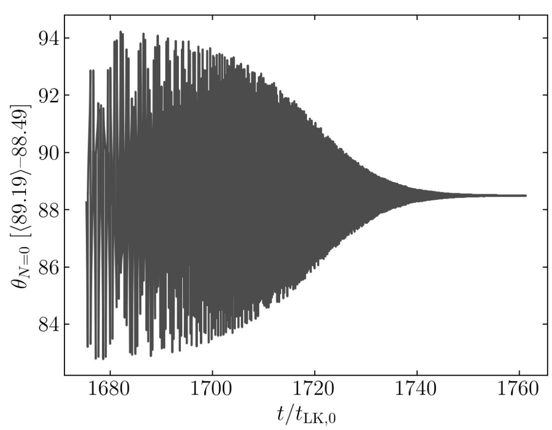

Figure 2: Zoomed in behavior of $\theta_{\mathrm{eff},0}$ at later times in the fiducial Paper II/$I_0 = 90.5°$ simulation.

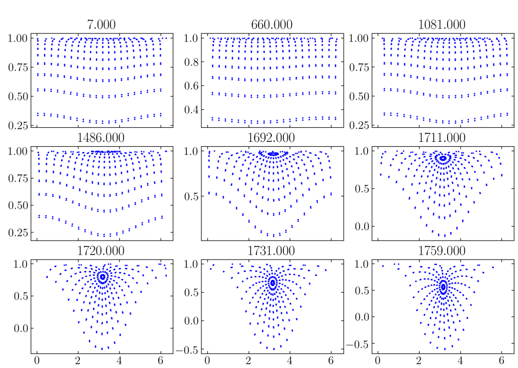

Figure 3: Distribution of spin vector orientations as a function of time; each blue dot is a realization of the Paper II/$I_0 = 90.5°$ simulation for a different initial spin vector. Uniform precession about an effective spin axis is observed (fixed orientation in the corotating frame).

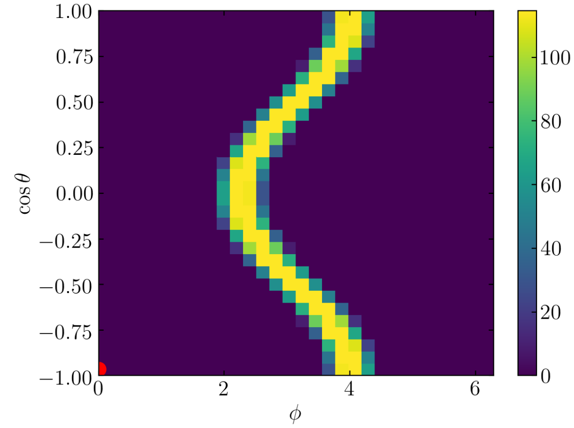

Figure 4: Plot of $\Delta\theta_{\text{eff},0}(\theta,\phi)$ for the locally nondissipative system of the fiducial Paper II simulation at $t = 1692$ (Fig. 2). While much of parameter space has $\Delta\theta_{\text{eff},0} = 0$, a clear resonant zone exists. The width of the zone decreases at later times.

2.2 Locally Nondissipative System

One idea that was developed to attempt to understand whether particular resonances could kick $\theta_{\text{eff},0}$ was to consider the locally nondissipative system. Here, for a given a, $e_{\min}$, and $I_{\min}$ (for $I_0 > 90°$, I is minimized when e is minimized) during inspiral, we solve Eq.(1) for some initial $\hat{\mathbf{S}}$ over 100–500 LK cycles using $\hat{\mathbf{L}}(t)$, $\Omega_{\text{SL}}(t)$, and $\dot{\Omega}(t)$ for a *single* LK cycle, ignoring gravitational radiation.

One useful quantity then to measure is $\Delta\theta_{\text{eff},0}$ for such a locally nondissipative system, the difference between the maximum and minimum $\theta_{\text{eff},0}$ attained. We choose to only measure at each LK cycle, so

$$\Delta\theta_{\text{eff},0} = \max_i \theta_{\text{eff},0}(\tau_i) - \min_j \theta_{\text{eff},0}(\tau_j), \tag{4}$$

where τ_i are the times of *minimum* eccentricity in each LK cycle. If $\Delta\theta_{\text{eff},0}$ is small for the entirety of the fiducial Paper II simulation, then conservation of $\theta_{\text{eff},0}$ can easily be understood.

In reality, it turns out not to be so simple. Note that $\Delta\theta_{\text{eff},0}$ is in general a function of the $\hat{\mathbf{S}}^{\text{i}}$. Parameterize $\hat{\mathbf{S}}^{\text{i}}$ by (θ,ϕ) in the coordinate system of Eq. (1), then a plot of $\Delta\theta_{\text{eff},0}(\theta,\phi)$ is given in Fig. 4.

Comparing to Fig. 2, it is clear that the amplitude of oscillation of $\theta_{\text{eff},0}$ from the GW simulation is not consistent with the prediction of $\Delta\theta_{\text{eff},0}$. But Fig. 2 is not fine-tuned, choosing a different initial spin for the inspiral simulation shows a similar behavior of $\theta_{\text{eff},0}$. Evaluating $\Delta\theta_{\text{eff},0}$ along the inspiral points seems to underpredict variations in $\theta_{\text{eff},0}$, as shown in Fig. 5.

This contrasts with the simulation in the Paper I regime, where the amplitude of oscillation in $\theta_{\text{eff},0}$ matches quite well with $\Delta\theta_{\text{eff},0}$, see Figs. 6 and 7.

This suggests that the resonant kick behavior relies on variations in $\theta_{\text{eff},0}$ being generated by the locally nondissipative dynamics, rather than GW radiation. If this is the case (pending further investigation; the results here are very scattered and are somewhat apples-to-oranges, we should resolve inconsistencies before drawing concrete conclusions), then a simple comparison of timescales over which the locally nondissipative dynamics generate kicks to $\theta_{\text{eff},0}$ to the GW radiation timescale gives the answer to our proposed question.

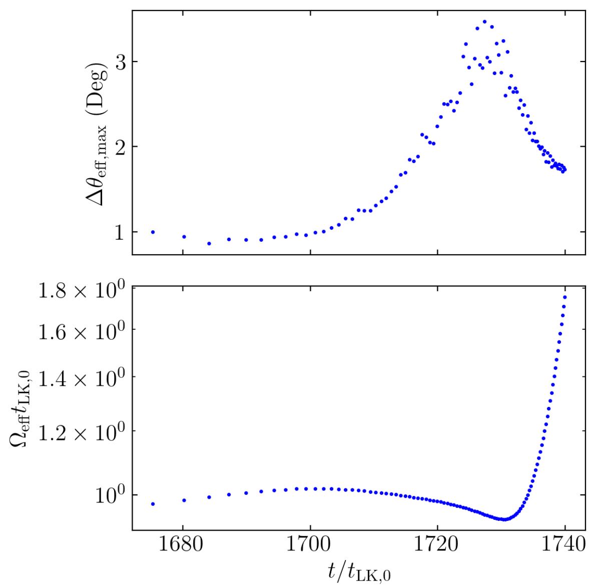

Figure 5: $\Delta\theta_{\rm eff,0}$ evaluated for one realization of the fiducial simulation, using the $\hat{\mathbf{S}}$ at the beginning of each LK cycle as initial conditions for the locally nondissipative simulation. Bottom plot shows $|\mathbf{\Omega}_{\rm eff,0}|$ in time units (LK period is implied by horizontal spacing).

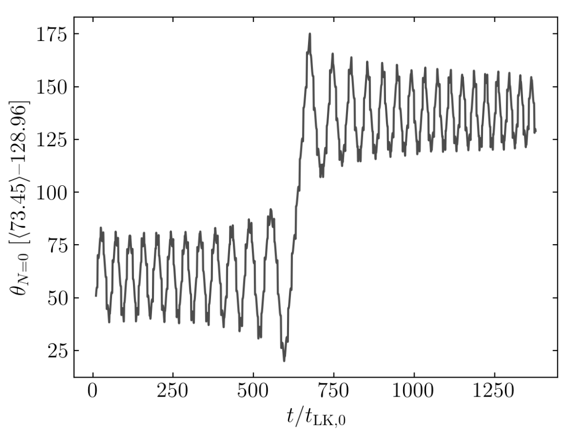

Figure 6: Plot of $\theta_{\rm eff,0}$ near a possible resonant kick for a Paper I/$I_0 = 70^\circ$ realisation. This seems similar in nature to the kicks seen in Fig. 1 of LL17.

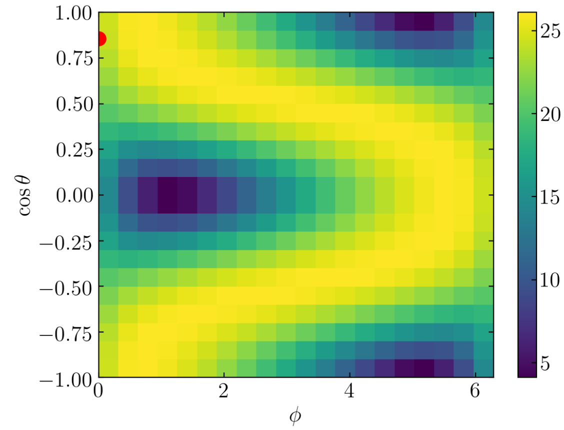

Figure 7: Same as Fig. 4 but for $t = 569$ in Fig. 6.

3 Bin's Response to Line in Hang Yu's Paper

Hang Yu's paper contains a line where they are not sure whether $\theta_{\rm sl}^{\rm f}$ ends up being $\theta_{\rm sb}^{\rm i}$ or $180° - \theta_{\rm sb}^{\rm i}$. With our basic theory, in the corotating frame, this is a very simple insight that we discussed early in this project. Take the limit where $\mathbf{\Omega}_{\rm eff}^{\rm i} = -\dot{\Omega}\hat{\mathbf{L}}_{\rm out}$ for simplicity, and perform analysis in the corotating frame (where $\mathbf{S}$ must always precess in the positive direction about $\mathbf{\Omega}_{\rm eff}$):

- At the end of the dynamics, $\hat{\mathbf{S}}$ precesses around $\mathbf{L}_{\rm in}$ always in the positive direction, and so $\mathbf{\Omega}_{\rm eff}^{\rm f} = \mathbf{L}_{\rm in}$
- Initially, $\hat{\mathbf{S}}$ precesses about $\mathbf{L}_{\rm out}$ in either the positive ($I_0 < 90°$) or negative ($I_0 > 90°$) direction, due to the sign of $\dot{\Omega}$ in the LK EOM. Requiring that $\hat{\mathbf{S}}$ precess around $\mathbf{\Omega}_{\rm eff}^{\rm i}$ in the positive direction shows that $\mathbf{\Omega}_{\rm eff} = \mathbf{L}_{\rm out}$ when $I_0 < 90°$ and $\mathbf{\Omega}_{\rm eff} = -\mathbf{L}_{\rm out}$ when $I_0 > 90°$.

  Thus, when $I_0 < 90°$, we have conservation of $\theta_{\rm eff} = \theta_{\rm sb,i} = \theta_{\rm sl,f}$, while when $I_0 > 90°$ we have conservation of $\theta_{\rm eff} = 180 - \theta_{\rm sb,i} = \theta_{\rm sl,f}$.

This generalizes easily to $I_0 < I_{0,\rm lim}$ and $I_0 > I_{0,\rm lim}$ when η is nonzero.