

Eccentric Tides

Yubo Su

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1 Kushnir et. al., 2016

We coarsely follow the derivation of Kushnir et. al., 2016 (KZ) to express the traveling wave regime of dynamical tides in high-mass stars (convective core, radiative envelope) in analytical form.

1.1 Plane Parallel Case

We will consider IGW in a plane parallel atmosphere in the Boussinesq approximation. Consider buoyancy frequency

$$N^2 = -g \left(\frac{d \ln \rho}{dr} + \frac{g}{c_s^2} \right), \quad (1)$$

where $c_s \rightarrow \infty$ is the sound speed in the fluid. Then the Boussinesq equations can be written in terms of some *buoyancy* variable b

$$\frac{D\vec{u}}{Dt} = \frac{\vec{\nabla}P}{\rho_0} + b\hat{z}, \quad (2a)$$

$$\frac{Db}{Dt} = -N^2 u_z, \quad (2b)$$

$$\vec{\nabla} \cdot \vec{u} = 0. \quad (2c)$$

Note that $b \equiv -\frac{\rho'}{\rho_0}g$, as can be verified via direct substitution into the Euler equations:

$$0 = \frac{D\rho'}{Dt} + \vec{u} \cdot \vec{\nabla}\rho_0 = \frac{D\rho'}{Dt} - u_z \frac{N^2}{g} \rho', \quad (3a)$$

$$\frac{D\vec{u}}{Dt} = \frac{\vec{\nabla}P'}{\rho_0} - \frac{\rho'}{\rho_0^2} \vec{\nabla}P_0 = \frac{\vec{\nabla}P'}{\rho_0} - \frac{\rho'}{\rho_0} g \hat{z}. \quad (3b)$$

These equations can be solved for u_z , or we can just recall the IGW dispersion relation $\omega^2 k^2 = N^2 k_\perp^2$ and write down PDE

$$\frac{\partial^2}{\partial t^2} \nabla_\perp^2 u_z = -N^2 \nabla_\perp^2 u_z. \quad (4)$$

Now, we might recall that in tidally-forced stars, ω the tidal forcing frequency obeys $\omega \ll N$, or $k \gg k_\perp$. But the tidal potential, the quadrupolar expansion of the gravitational perturbation from

the companion, has no quickly-varying directions, or can only excite $k \simeq k_\perp$ modes. Thus, we intuit that waves must be excited where N is much smaller than its typical value, or near the *radiative-convective boundary* (RCB). At the RCB, $N^2 = 0$, and we are concerned with the turning point where $\omega^2 = N^2$. We perform linear expansion about this turning point z_c , and for convenience we set $z_c = 0$, then

$$N^2 \approx \omega^2 + \frac{dN^2}{dz} z, \quad (5)$$

where compared to KZ I've taken $N_0^2 = \omega^2$, there seems to be little harm here. Making then general ansatz $u_z(z, \vec{r}_\perp, t) = \tilde{u}_z(z) e^{i(\vec{k}_\perp \cdot \vec{r}_\perp - \omega t)}$ we obtain

$$-\omega^2(-k_\perp^2 \tilde{u}_z + \tilde{u}_z'') = N^2 k_\perp^2 u_z, \quad (6)$$

$$\tilde{u}_z'' + k_\perp^2 \left(\frac{N^2}{\omega^2} - 1 \right) \tilde{u}_z = 0, \quad (7)$$

$$\tilde{u}_z'' + k_\perp^2 \frac{dN^2}{dz} \frac{z}{\omega^2} \tilde{u}_z = 0. \quad (8)$$

It's easiest now to rescale $\tilde{k}_\perp^2 \equiv k_\perp^2 \frac{dN^2}{dz} \frac{1}{\omega^2}$ so that

$$\tilde{u}_z'' + \tilde{k}_\perp^2 z \tilde{u}_z = 0. \quad (9)$$

The general solution to this ODE is written in terms of Airy functions for arbitrary constants a, b

$$\tilde{u}_z(z) = a \text{Ai}\left(-\frac{z}{\lambda}\right) + b \text{Bi}\left(-\frac{z}{\lambda}\right), \quad (10)$$

where $\lambda = \tilde{k}_\perp^{-2/3}$. For large $-z$, it turns out that

$$\text{Ai}(-z) \sim \frac{\sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)}{z^{1/4}} + \mathcal{O}\left(z^{-7/4}\right), \quad (11)$$

$$\text{Bi}(-z) \sim \frac{\cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)}{z^{1/4}} + \mathcal{O}\left(z^{-7/4}\right). \quad (12)$$

In order for us to get traveling waves with *group velocity* going outwards (towards $z > 0$), we need $\tilde{u}_z(z) \sim e^{-ik_z z}$ such that $u(z, t) \propto e^{i(-k_z z - \omega t)}$ (phase velocity goes inwards, group velocity goes outwards for IGW). Thus, $a = -ib$ in Eq. 13, and we obtain

$$\tilde{u}_z(z) = b \left(-i \text{Ai}\left(-\frac{z}{\lambda}\right) + \text{Bi}\left(-\frac{z}{\lambda}\right) \right), \quad (13)$$

Now we just need to fix b . This is traditionally accomplished by mandating a particular $\frac{d\delta z}{dz}$ the displacement of the mode at the turning point $z = 0$. That the forcing results in a constraint on $\frac{d\delta z}{dz}$ is similar to what I did in my IGW breaking forcing, where forcing induces a jump in the $\frac{du_z}{dz}$ above/below z_c whose magnitude is fixed by the strength of the forcing term. In the stellar problem, it appears the correct way to obtain the δz is to solve the inhomogeneous problem in the

convective zone where $N^2 = 0$ including the tidal potential, so it's not a perfect analogy. But since $\text{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}$, $\text{Bi}'(0) = \frac{3^{1/6}}{\Gamma(1/3)}$, this is not so difficult to evaluate, and I cite the KZ result

$$\frac{d\delta z}{dz} = -\frac{ib}{\lambda\omega} \frac{2}{3^{1/3}\Gamma(1/3)} \frac{3^{1/2} + i}{2}. \quad (14)$$

Finally, we impose one more step: we will compute the luminosity or *energy flux* associated with the wave, since this is the easiest way to get the resulting torque. We can easily write down the energy density of the wave $\frac{\rho_0}{2} \left(v^2 + \frac{b^2}{N^2} \right)$, for which the energy flux is $\vec{F} = \vec{v}P$. Noting furthermore that $\frac{\partial u_z}{\partial z} = -ik_{\perp}u_x = -\frac{ik_{\perp}P}{\rho_0} \frac{k_{\perp}}{\omega}$, we can explicitly express P in terms of u'_z , and so the energy flux density is then simply (I'm not evaluating this, but KZ do)

$$\frac{\delta L}{\delta A} = \frac{1}{2} \text{Re}(Pu_z^*) = \frac{\rho_0\omega}{2k_{\perp}^2} \text{Re}(iu'_z u_z^*), \quad (15)$$

$$= \frac{3^{2/3}\Gamma^2(1/3)\lambda\omega^3\rho_0}{8\pi k_{\perp}^2} \left(\frac{d\delta z}{dz} \right)^2. \quad (16)$$

We would then compute $L = \int \frac{\delta L}{\delta A} dA$, which for us is just $\frac{\delta L}{\delta A} A$ where A is the surface area of the wave.

Finally, we would compute the total torque from $L = \tau\omega$ (the same as $E = \vec{F} \cdot \vec{v}$).

1.2 Spherical Case

To go to the spherical case, we simply replace $z \rightarrow r$ and $k_{\perp}^2 \rightarrow l(l+1)/r^2$, which gives

$$\lambda = \left(\frac{l(l+1)}{r^2\omega^2} \frac{dN^2}{dr} \right)^{-1/3}. \quad (17)$$

Then to get $\frac{d\delta z}{dz} \rightarrow \frac{d\delta r}{dr}$, we use prescription

$$\frac{d\delta r}{dr} = \alpha \frac{\Phi}{gr} \left(1 - \frac{\rho(r)}{\bar{\rho}(r)} \right). \quad (18)$$

Here, $\alpha \sim 1$ depends on the specific stellar structure in the convective zone, while $\bar{\rho}$ is the average density inside r . Finally, instead of getting a clean $L = \frac{\delta L}{\delta A} A$, we have to actually do the integral of $L = \int \frac{\delta L}{\delta A} dA = \int (\dots) |Y_{lm}|^2 r^2 d\cos\theta d\phi = r_c^2 L$ (note that it's not $4\pi r_c^2$, thanks to the Y_{lm} normalization). Finally, the $\ell = 2$ potential is taken to be

$$\Phi_{\text{ext}} = -\sqrt{\frac{6\pi}{5}} \frac{GM_2 R_c^2}{D^3}. \quad (19)$$

I guess the angular dependency is just dropped. With all these things together, we obtain the final KZ result (I omit the derivation, this part is grungy and not very physically interesting)

$$\tau = \dot{J}_z = \frac{GM_2^2 R_c^5}{D^6} \sigma_c^{8/3} \left[\frac{r_c}{g_c} \left(\frac{dN^2}{d \ln R} \right)_{r=r_c} \right]^{-1/3} \frac{\rho_c}{\bar{\rho}_c} \left(1 - \frac{\rho_c}{\bar{\rho}_c} \right)^2 \left[\frac{3}{2} \frac{3^{2/3} \Gamma^2(1/3)}{5 \cdot 6^{4/3}} \frac{3}{4\pi} \alpha^2 \right], \quad (20)$$

$$= \frac{GM_2^2 R_c^5}{D^6} 2\hat{F}(r_c, \sigma_c). \quad (21)$$

Note that \hat{F} follows the convention from Equation 42 of Fuller & Lai's second paper (FL2) and Vick et. al's paper as well (VLF), while $\sigma_c = 2|\Omega - \Omega_s|/\sqrt{GM_c/r_c^3}$ is the ratio of the forcing frequency to the breakup frequency of the core. Finally, I've replaced M_2 the mass of the companion, R_c the radius of the core, and D the separation, while retaining Ω_s spin angular frequency and Ω orbital angular frequency.

NB: The exact definition of \hat{F} for a given m is given in VLF.23 as

$$\dot{J} = G \frac{M_2 R^5}{\alpha^3} \frac{|m|}{2} \hat{F}(\omega) = T_0 \frac{|m|}{2} \hat{F}(\omega). \quad (22)$$

Since the total torque τ has already summed over $m = \pm 2$, we incur the extra factor of 2 above in Eq. 21. That m has already been summed over is visible in the $\sqrt{6\pi/5}$ prefactor used in Φ_{ext} , compared to $W_{2\pm 2} = \sqrt{3\pi/10}$ as seen below.

2 Vick et. al., 2016

We now consider eccentric forcing. We will remove subscript compared to VLF and just call $\vec{r}_i = (r, \theta, \phi + \Omega_s t)$ the position coordinate in the inertial frame. Then the $\ell = 2$ tidal forcing potential is generally a sum over $m \in [-2, 2]$

$$U = \sum_m U_{2m}(\vec{r}, t), \quad (23)$$

$$U_{2m}(\vec{r}) = -\frac{GM_2 W_{2m} r^2}{D(t)^3} e^{-imf(t)} Y_{2m}(\theta, \phi). \quad (24)$$

Note that f is the true anomaly here. Note that W_{2m} is just a constant: $W_{20} = \sqrt{\pi/5}$, $W_{2\pm 1} = 0$, and $W_{2\pm 2} = \sqrt{3\pi/10}$.

This is complicated since $f(t)$ does not evolve uniformly, and also since $D(t)$ is time-varying! The easiest treatment is to decompose

$$U_{2m} = -\frac{GM_2 W_{2m} r^2}{\alpha^3} Y_{2m}(\theta, \phi) \sum_{N=-\infty}^{\infty} F_{Nm} e^{-iN\Omega t}. \quad (25)$$

Note that the F_{Nm} here are *Hansen coefficients* given by

$$F_{Nm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos[N(E - e \sin E) - mf(E)]}{1 - e \cos E} dE. \quad (26)$$

Note E is the eccentric anomaly. This differs from the VLF definition by a factor of 2, such that $F_{Nm} = \delta_{Nm}$ for $e = 0$; I verified against Mardling¹ that this normalization is correct for Hansen coefficients.

Let's explicitly write out the $U_{2\pm 2}$, since they are the only ones that contribute to the tidal torque

$$U_{22} = -\frac{GM_2 \sqrt{\frac{3\pi}{10}} r^2}{a^3} \sum_{N=1}^{\infty} \left[F_{N2} Y_{22}(\theta, \phi) e^{-i(N\Omega - 2\Omega_s)t} + F_{-N2} Y_{22}(\theta, \phi) e^{i(N\Omega + 2\Omega_s)t} \right], \quad (27)$$

$$U_{2-2} = -\frac{GM_2 \sqrt{\frac{3\pi}{10}} r^2}{a^3} \sum_{N=1}^{\infty} \left[F_{-N2} Y_{2-2}(\theta, \phi) e^{-i(N\Omega - 2\Omega_s)t} + F_{N2} Y_{2-2}(\theta, \phi) e^{i(N\Omega + 2\Omega_s)t} \right], \quad (28)$$

$$U_{22} + U_{2-2} = -\frac{GM_2 \sqrt{\frac{3\pi}{10}} r^2}{a^3} \sum_{N=1}^{\infty} \left[F_{N2} Y_{22}(\theta, \phi) e^{-i(N\Omega - 2\Omega_s)t} + c.c. \right] \quad (29)$$

We can verify that if the perturbing orbit is circular $e = 0$, then the Hansen coefficient $F_{Nm} = 2\delta_{Nm}$, and we obtain

$$U_{22} + U_{2-2} = -\frac{GM_2 r^2}{a^3} \sqrt{\frac{6\pi}{5}} \operatorname{Re} \left[Y_{22}(\theta, \phi) e^{-2i(\Omega - \Omega_s)t} \right]. \quad (30)$$

This is the same torque used in KZ. Finally, this yields torque

$$\dot{J} = \tau = T_0 \sum_{N=-\infty}^{\infty} F_{N2}^2 \operatorname{sgn}(N\Omega - 2\Omega_s) \hat{F}(\omega = |N\Omega - 2\Omega_s|). \quad (31)$$

3 Combined Results

We have been somewhat careful in checking the agreement between the VLF and KZ forms. Note now that Eq. 21 has \hat{F} for a single m contribution, just as \hat{F} is defined in VLF. Thus, we should be able to simply plug in

$$\tau = T_0 \sum_{N=-\infty}^{\infty} F_{N2}^2 \frac{\operatorname{sgn}(N\Omega - 2\Omega_s)}{2} \sigma_c^{8/3} \left[\frac{r_c}{g_c} \left(\frac{dN^2}{d \ln R} \right)_{r=r_c} \right]^{-1/3} \frac{\rho_c}{\bar{\rho}_c} \left(1 - \frac{\rho_c}{\bar{\rho}_c} \right)^2 \left[\frac{3}{2} \frac{3^{2/3} \Gamma^2(1/3)}{5 \cdot 6^{4/3}} \frac{3}{4\pi} \alpha^2 \right], \quad (32)$$

$$= T_0 C(r_c) \sum_{N=-\infty}^{\infty} F_{N2}^2 \operatorname{sgn}(N\Omega - 2\Omega_s) \sigma_c^{8/3}. \quad (33)$$

Note that now $\sigma_c = |N\Omega - 2\Omega_s| / \sqrt{GM_c/r_c^3}$, and I've defined $C(r_c)$ to be some constant defined at the RCB and does not change with N .

¹<https://arxiv.org/pdf/1308.0607.pdf>

Thus, the relative significance of each N term is given by the summand of Eq. 33, or

$$\tau_N \equiv F_{N2}^2 \text{sgn}(N\Omega - 2\Omega_s) \sigma_c^{8/3}, \quad (34)$$

where for convenience we reproduce the F_{N2}

$$F_{Nm} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \frac{\cos[N(E - e \sin E) - mf(E)]}{1 - e \cos E} dE. \quad (35)$$

Note that $\tan \frac{f}{2} = \sqrt{\frac{1+e}{1-e}} \tan \frac{E}{2}$ relates the true/eccentric anomalies.

For moderate e , this seems to be easiest to evaluate with a direct integral, rather than tabulating using recursion coefficients and interpolating; I'll move to this if necessary.

One guess is that the sum is dominated by the contribution of the frequency at pericenter. Since pericenter occurs at $a_p = a(1-e)$, using scaling $P^2 \propto r^3 \propto \Omega^{-2}$ we find that $\frac{\Omega_p}{\Omega} = \left(\frac{a}{a_p}\right)^{3/2} = (1-e)^{-3/2}$. Thus, we should expect the dominant term to come at $N \sim (1-e)^{-3/2}$. In reality, this appears to be the maximum F_{Nm} but not the maximum τ_N .