eccentric_tides

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1 1. Formalism

According to KZ16, when the companion is on a circular orbit with pattern frequency $\sigma = 2(\Omega - \Omega_s)$ (where Ω_s is the spin frequency of the stellar core and Ω is the orbital frequency), the torque is given

$$\tau = \beta_2 \frac{GM_2^2 r_c^5}{a^6} \mathrm{sgn}(\sigma) \left| \frac{\sigma}{\sqrt{GM_c/r_c^3}} \right|^{8/3} \frac{\rho_c}{\bar{\rho}_c} \left(1 - \frac{\rho_c}{\bar{\rho}_c}\right)^2.$$

where $\beta_2 \approx 1$ for most stars, and r_c is the core radius.

VLF17 furthermore give the total torque for an eccentric mode

$$\tau = \sum_{N=-\infty}^{\infty} F_{N2}^2 \tau_N$$

where the F_{N2} are Hansen coefficients, and the τ_N are the contributions from each harmonic N. If we let $\sigma = N\Omega - 2\Omega_s$ in the KZ16 torque, then we can rewrite the torque exerted by each harmonic as

$$\tau_N(r_c) = \hat{\tau}(r_c) \mathrm{sgn}\left(N - 2\Omega_s/\Omega\right) \left|N - 2\Omega_s/\Omega\right|^{8/3}$$

where

$$\hat{\tau}(r_c) = \beta_2 \frac{GM_2^2 r_c^5}{a^6} \left(\frac{\Omega}{\sqrt{GM_c/r_c^3}} \right)^{8/3} \frac{\rho_c}{\bar{\rho}_c} \left(1 - \frac{\rho_c}{\bar{\rho}_c}\right)^2.$$

This implies we can write

$$\tau = \hat{\tau}(r_c) \sum_{N=-\infty}^{\infty} F_{N2}^2 \mathrm{sgn}\left(N - 2\Omega_s/\Omega\right) \left|N - 2\Omega_s/\Omega\right|^{8/3}$$

The energy term is similar:

$$\dot{E}_{\mathrm{in}} = \frac{\hat{\tau}(r_c, \Omega)}{2} \sum_{N=-\infty}^{\infty} \left[N\Omega F_{N2}^2 \mathrm{sgn}\left(N - 2\Omega_s/\Omega\right) \left|N - 2\Omega_s/\Omega\right|^{8/3} + \left(\frac{W_{20}}{W_{22}}\right)^2 \Omega F_{N0}^2 |N|^{11/3} \right]$$

and

$$\dot{E}_{\mathrm{rot}} = \dot{E}_{\mathrm{in}} - \Omega_s \tau$$

[2]:
```python
import sympy as sp
import numpy as np
```

```python
from scipy.special import gamma
from IPython.display import display
sp.init_printing(use_latex=True)

N, e = sp.symbols('N e ', positive=True)
f2, f3, f5, j = sp.symbols('f_2 f_3 f_5 j', positive=True)
ecc_subdict = {
    f2: 1 + e**2 * sp.Rational(15, 2) + e**4 * sp.Rational(45, 8) + e**6 * sp.
 ↪Rational(5, 16),
    f3: 1 + e**2 * sp.Rational(15, 4) + e**4 * sp.Rational(15, 8) + e**6 * sp.
 ↪Rational(5, 64),
    f5: 1 + e**2 * 3 + e**4 * sp.Rational(3, 8)
}
jsub = { j: sp.sqrt(1 - e**2) }
def my_display(expr, other_subs={}):
    all_subs = {**jsub, **other_subs}
    if type(expr) == list:
        expr = [e.subs(all_subs) for e in expr]
    else:
        expr = expr.subs(all_subs)
    display(expr)
display([f2, f3, f5])
my_display([f2, f3, f5], ecc_subdict)

C2, eta2 = sp.symbols(r'C_2 \eta_2', positive=True)
p = 2 # ansatz
C0, eta0 = sp.symbols(r'C_0 \eta_0', positive=True)

F_N2 = C2 * N**p * sp.E**(-N / eta2)
F_N0 = C0 * sp.E**(-N / eta0)
def get_fn2_integral(p):
    return sp.Integral(F_N2**2 * N**p, (N, 0, sp.oo))
def get_fn0_integral(p):
    return sp.Integral(2 * F_N0**2 * N**p, (N, 0, sp.oo))
```

$$[f_2,\ f_3,\ f_5]$$

$$\left[\frac{5e^6}{16} + \frac{45e^4}{8} + \frac{15e^2}{2} + 1,\ \frac{5e^6}{64} + \frac{15e^4}{8} + \frac{15e^2}{4} + 1,\ \frac{3e^4}{8} + 3e^2 + 1\right]$$

2 2. Fitting Formulas

2.1 a. F__N2

First, we aim to calculate the fitting formulas for the two sets of Hansen coefficients, F_{N0} and F_{N2}. Below we first do so for F_{N2}. The constraints are

$$\sum_{N=-\infty}^{\infty} F_{N2}^2 = \frac{f_5}{(1-e^2)^{9/2}}, \tag{1}$$

$$\sum_{N=-\infty}^{\infty} F_{N2}^2 N = \frac{2f_2}{(1-e^2)^6}. \tag{2}$$

These are analytic. We approximate

$$F_{N2} = \begin{cases} 0 & N \le 0 \\ C_2 N^p e^{-N/\eta_2} & N > 0. \end{cases}$$

We take universal $p = 2$ based on numerical comparison. We then fit for C_2, η_2 by setting both of the below equal to zero:

[3]:
```
fn2_moment0 = get_fn2_integral(0) - f5 / j**9
fn2_moment1 = get_fn2_integral(1)  - 2 * f2 / j**12
my_display(fn2_moment0)
my_display(fn2_moment1)
```

$$-\frac{f_5}{(1-e^2)^{\frac{9}{2}}} + \int\limits_0^\infty C_2^2 N^4 e^{-\frac{2N}{\eta_2}}\,dN$$

$$-\frac{2f_2}{(1-e^2)^6} + \int\limits_0^\infty C_2^2 N^5 e^{-\frac{2N}{\eta_2}}\,dN$$

Setting both of the above equal to zero gives solution for η_2, C_2:

[4]:
```
[[c2sol, eta2sol]] = sp.solve([
    fn2_moment0.doit(),
    fn2_moment1.doit(),
], [C2, eta2])
sol2subs = { C2: c2sol, eta2: eta2sol }
my_display([eta2, C2**2 * (eta2 / 2)**5])
print('=')
my_display([sp.simplify(eta2sol), sp.simplify(c2sol**2 * (eta2sol / 2)**5)])
```

$$\left[\eta_2, \frac{C_2^2\eta_2^5}{32}\right]$$

=

$$\left[\frac{4f_2}{5f_5\,(1-e^2)^{\frac{3}{2}}}, \frac{f_5}{24\,(1-e^2)^{\frac{9}{2}}}\right]$$

Note that the N for which F_{N2}^2 peaks is just $p\eta_2 = 2\eta_2$. Naively, we expect this to scale with the pericenter harmonic $N_{\rm peri} = \sqrt{1+e}(1-e^2)^{3/2}$. For $e \to 1$, the fitted formula peaks at $N = 132/25(1-e^2)^{-3/2}$, which is the right scaling.

[5]:
```
nmax = eta2sol.subs(ecc_subdict).subs(e, 1)
my_display(sp.simplify(2 * nmax))
```

$$\frac{132}{25\left(1-e^2\right)^{\frac{3}{2}}}$$

2.2 b. F_N0

Next, we fit F_{N0}. The constraints are

$$\sum_{N=-\infty}^{\infty} F_{N0}^2 = \frac{f_5}{(1-e^2)^{9/2}}, \tag{3}$$

$$\sum_{N=-\infty}^{\infty} F_{N0}^2 N^2 = \frac{9e^2 f_3}{2(1-e^2)^{15/2}}. \tag{4}$$

We pick fitting formula

$$F_{N0} = C_0 e^{-|N|/\eta_0}.$$

This produces fit

[6]:
```
fn0_moment0 = get_fn0_integral(0) - f5 / j**9
fn0_moment2 = get_fn0_integral(2) - sp.Rational(9, 2) * e**2 / j**15 * f3
my_display(fn0_moment0)
my_display(fn0_moment2)
[[c0sol, eta0sol]] = sp.solve([
    fn0_moment0.doit(),
    fn0_moment2.doit(),
], [C0, eta0])
sol0subs = { C0: c0sol, eta0: eta0sol }
my_display([eta0**2, C0**2 * eta0])
print('=')
my_display([sp.simplify(eta0sol**2), sp.simplify(c0sol**2 * eta0sol)])
```

$$-\frac{f_5}{\left(1-e^2\right)^{\frac{9}{2}}} + \int\limits_0^\infty 2C_0^2 e^{-\frac{2N}{\eta_0}}\, dN$$

$$-\frac{9e^2 f_3}{2\left(1-e^2\right)^{\frac{15}{2}}} + \int\limits_0^\infty 2C_0^2 N^2 e^{-\frac{2N}{\eta_0}}\, dN$$

$$\left[\eta_0^2,\ C_0^2\eta_0\right]$$

=

$$\left[\frac{9e^2 f_3}{f_5\left(1-e^2\right)^3},\ \frac{f_5}{\left(1-e^2\right)^{\frac{9}{2}}}\right]$$

3 3. Results

3.1 a. Torque

Now, we evaluate the torque,

$$\tau = \hat{\tau} \sum_{N=-\infty}^{\infty} F_{N2}^2 \mathrm{sgn}\left(N - 2\Omega_s/\Omega\right) \left|N - 2\Omega_s/\Omega\right|^{8/3}, \tag{5}$$

$$= \hat{\tau} \int_0^\infty C_2^2 N^4 e^{-2N/\eta_2} \mathrm{sgn}\left(N - 2\Omega_s/\Omega\right) \left|N - 2\Omega_s/\Omega\right|^{8/3}\ \mathrm{d}N. \tag{6}$$

Call N_{max} the N for which the summand is maximized (we will determine this a posterori); note $N_{\mathrm{max}} > 0$. First, we will approximate $|\Omega_s/\Omega| \gg N_{\mathrm{max}}$, then we will show that the accuracy of the prediction can be improved via yet another ansatz

3.1.1 Case 1: Asymptotic

First, consider $N_{\mathrm{max}} \ll |2\Omega_s/\Omega|$. The sign is just $-\mathrm{sgn}(\Omega_s)$, and the rest simplifies easily

$$\tau = -\hat{\tau}\ \mathrm{sgn}(\Omega_s)\ \left|2\Omega_s/\Omega\right|^{8/3} \sum_{N=-\infty}^{N=\infty} F_{N2}^2 \tag{7}$$

$$= -\hat{\tau}\ \mathrm{sgn}(\Omega_s)\ \left|2\Omega_s/\Omega\right|^{8/3} \frac{f_5}{(1-e^2)^{9/2}} \tag{8}$$

3.1.2 Case 2: Approaching Pseudosynchronization

In the limit where $N_{\mathrm{max}} \simeq 2\Omega_s/\Omega$, the largest terms in the summation have opposite signs, and we must be more careful. We make ansatz for unknown α

$$N - 2\Omega_s/\Omega \simeq \frac{N}{N_{\mathrm{max}}}\left(N_{\mathrm{max}} - \frac{2\alpha\Omega_s}{\Omega}\right),$$

which gives torque (see below)

$$\tau = \hat{\tau}\ \mathrm{sgn}\left(N_{\mathrm{max}} - \frac{2\alpha\Omega_s}{\Omega}\right)\left|N_{\mathrm{max}} - \frac{2\alpha\Omega_s}{\Omega}\right|^{8/3} \sum_{N=-\infty}^{\infty} F_{N2}^2 \left(\frac{N}{N_{\mathrm{max}}}\right)^{8/3}, \tag{9}$$

$$= \hat{\tau}\ \mathrm{sgn}\left(\frac{10}{3}\eta_2 - \frac{2\alpha\Omega_s}{\Omega}\right)\left|\frac{10}{3}\eta_2 - \frac{2\alpha\Omega_s}{\Omega}\right|^{8/3} \frac{1}{4!}\frac{\Gamma(23/3)}{(20/3)^{8/3}}\frac{f_5}{(1-e^2)^{9/2}}, \tag{10}$$

$$= \hat{\tau}\ \mathrm{sgn}\left(1 - \frac{3}{5\eta_2}\frac{\alpha\Omega_s}{\Omega}\right)\left|1 - \frac{3}{5\eta_2}\frac{\alpha\Omega_s}{\Omega}\right|^{8/3} \frac{\Gamma(23/3)}{4!}\left(\frac{\eta_2}{2}\right)^{8/3}\frac{f_5}{(1-e^2)^{9/2}}. \tag{11}$$

We have gone ahead and used $N_{\mathrm{max}} = (2 + 4/3)\eta_2$, since the summand $\propto N^(4 + 8/3)e^{-2N/\eta_2}$. The integral can be checked below:

[7]:
```
t_case2_intexpr = get_fn2_integral(sp.Rational(8, 3))
print('Integral is:')
my_display(t_case2_intexpr)
t_case2 = t_case2_intexpr.subs(sol2subs).doit()
```

```
t_case2simpdiv = sp.simplify(
    t_case2 * 24 / sp.gamma(sp.Rational(23, 3)) / (eta2sol * sp.Rational(10,␣
↪3))**(sp.Rational(8, 3))
    * (sp.Rational(20, 3))**sp.Rational(8, 3))
print('Integral * 4! / Gamma(23/3) / (eta2 * 10 / 3)**(8/3) * (20/3)**(8/3) =')
my_display(t_case2simpdiv)
```

```
Integral is:
```

$$\int_0^\infty C_2^2 N^{\frac{20}{3}} e^{-\frac{2N}{\eta_2}}\, dN$$

```
Integral * 4! / Gamma(23/3) / (eta2 * 10 / 3)**(8/3) * (20/3)**(8/3) =
```

$$\frac{f_5}{(1-e^2)^{\frac{9}{2}}}$$

α is fixed by requiring $\tau(|\Omega_s| \to \infty)$ have the correct asymptotic behavior. This then requires

$$1 = \left(\frac{3\alpha}{20}\right)^{8/3} \frac{\Gamma(23/3)}{4!}, \tag{12}$$

$$\frac{3\alpha}{5} = 4\left(\frac{4!}{\Gamma(23/3)}\right)^{3/8} \approx 0.691. \tag{13}$$

[8]:
```
(gamma(5) / gamma(23/3))**(3/8) * 4
```

[8]: 0.690919908604587

Thus, we arrive at final answer

$$\tau = \hat{\tau}\frac{f_5\eta_2^{8/3}}{(1-e^2)^{9/2}}\,\mathrm{sgn}\left(1 - 0.691\frac{\Omega_s}{\eta_2\Omega}\right)\left|1 - 0.691\frac{\Omega_s}{\eta_2\Omega}\right|^{8/3}\frac{\Gamma(23/3)}{4!}\frac{1}{2^{8/3}}.$$

This gives a very clear prediction for the pseudosynchronization frequency, i.e. $\tau(\Omega_s = \eta_2\Omega/0.691) = 0$. This matches the pseudosynchronization frequency calculated using the full integral approximation.

3.2 b. Energy

We can also write down the integral approximation for the energy dissipation

$$\dot{E}_{\mathrm{in}} = \frac{\hat{\tau}(r_c,\Omega)}{2}\sum_{N=-\infty}^{\infty}\left[N\Omega F_{N2}^2\mathrm{sgn}\left(N - 2\Omega_s/\Omega\right)|N - 2\Omega_s/\Omega|^{8/3} + \left(\frac{W_{20}}{W_{22}}\right)^2 \Omega F_{N0}^2|N|^{11/3}\right] \tag{14}$$

$$= \frac{\hat{\tau}(r_c,\Omega)\Omega}{2}\int_0^\infty\left[C_2^2N^5e^{-2N/\eta_2}\mathrm{sgn}\left(N - 2\Omega_s/\Omega\right)|N - 2\Omega_s/\Omega|^{8/3} + 2\frac{2}{3}C_0^2e^{-2N/\eta_0}N^{11/3}\right]\,\mathrm{d}N \tag{15}$$

3.2.1 Term 1: m=2 Term

We first consider the first term in the integral, which is handled very similarly to the above. The high spin limit is just

$$\dot{E}_{\text{in}}^{(m=2)} = -\frac{\hat{\tau}\Omega}{2}\,\mathrm{sgn}(\Omega_s)|2\Omega_s/\Omega|^{8/3}\frac{2f_2}{(1-e^2)^6}$$

[9]:
```
e_case1base = get_fn2_integral(1)
my_display(e_case1base)
my_display(e_case1base.doit().subs(sol2subs))
```

$$\int\limits_0^\infty C_2^2N^5e^{-\frac{2N}{\eta_2}}\,dN$$

$$\frac{2f_2}{(1-e^2)^6}$$

Making the same ansatz as above, we end up with

[10]:
```
e_case1_intexpr = get_fn2_integral(sp.Rational(11, 3))
print('Integral is:')
my_display(e_case1_intexpr)
e_case1 = e_case1_intexpr.subs(sol2subs).doit()
e_case1simpdiv = sp.simplify(
    e_case1 * 24 / sp.gamma(sp.Rational(26, 3)) / (eta2sol * sp.Rational(13,␣
↪3))**(sp.Rational(11, 3))
    * (sp.Rational(26, 3))**sp.Rational(11, 3))
print('Integral * 4! / Gamma(26/3) / (eta2 * 13 / 3)**(11/3) * (26/3)**(11/3)␣
↪=')
my_display(e_case1simpdiv)
```

```
Integral is:
```

$$\int\limits_0^\infty C_2^2N^{\frac{23}{3}}e^{-\frac{2N}{\eta_2}}\,dN$$

```
Integral * 4! / Gamma(26/3) / (eta2 * 13 / 3)**(11/3) * (26/3)**(11/3) =
```

$$\frac{f_5}{(1-e^2)^{\frac{9}{2}}}$$

Noting that now $N_{\max} \simeq \frac{5+8/3}{2}\eta_2$, our constraint becomes

$$\lim_{|\Omega_s|\to\infty} \left|1 - \frac{2\alpha\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{f_5}{(1-e^2)^{9/2}} \frac{\Gamma(26/3)}{4!} \left(\frac{\eta_2}{2}\right)^{11/3} = |2\Omega_s/\Omega|^{8/3} \frac{2f_2}{(1-e^2)^6}, \tag{16}$$

$$\left(\frac{6\alpha}{23\eta_2}\right)^{8/3} \frac{\Gamma(26/3)}{4!} \left(\frac{\eta_2}{2}\right)^{11/3} \frac{f_5(1-e^2)^{3/2}}{2f_2} = 1, \tag{17}$$

$$\frac{12\alpha}{23} = \left(\frac{5!2^{16/3}}{\Gamma(26/3)}\right)^{3/8} \approx 0.5886 \tag{18}$$

and so

$$\dot{E}_{\rm in}^{(m=2)} = -\frac{\hat{\tau}\Omega}{2}\,\mathrm{sgn}(\Omega_s) \left|1 - 0.5886\frac{\Omega_s}{\eta_2\Omega}\right|^{8/3} \frac{f_5}{(1-e^2)^{9/2}} \frac{\Gamma(26/3)}{4!} \left(\frac{\eta_2}{2}\right)^{11/3}$$

[11]:
```
alpha = sp.symbols('alpha', positive=True)
expr = (
    (sp.Rational(6, 23) * alpha / eta2)**(sp.Rational(8, 3)) * sp.gamma(sp.
 ↪Rational(26, 3))
            / sp.gamma(5) * (eta2 / 2)**(sp.Rational(11, 3))
        * f5 * j**3 / (2 * f2)) - 1
my_display(expr)
[root] = sp.solve(expr.subs(eta2, eta2sol), alpha)
print(12 * root.evalf() / 23)
print((gamma(5) * 2**(16/3) * 5 / gamma(26/3))**(3/8))
```

$$\frac{3\sqrt[3]{23}\cdot 3^{\frac{2}{3}}\eta_2\alpha^{\frac{8}{3}} f_5\left(1-e^2\right)^{\frac{3}{2}}\Gamma\left(\frac{26}{3}\right)}{389344 f_2} - 1$$

```
0.588591506083493
0.5885915060834931
```

3.2.2 Term 2: m=0 Sum

This integral is trivial to evaluate with our fitting formulas above:

$$\dot{E}_{\rm in}^{(m=0)} = \frac{\hat{\tau}(r_c,\Omega)\Omega}{2} \int_0^\infty \left[\frac{4}{3}C_0^2 e^{-2N/\eta_0} N^{11/3}\right] \mathrm{d}N, \tag{19}$$

$$= \frac{\hat{\tau}\Omega}{2} \frac{f_5\Gamma(14/3)}{(1-e^2)^{10}} \left(\frac{3}{2}\right)^{8/3} \left(\frac{e^2 f_3}{f_5}\right)^{11/6} \tag{20}$$

[12]:
```
# guess is above, simplify is too messy
m0intexpr = sp.Integral(sp.Rational(4, 3) * C0**2 * sp.E**(-2 * N / eta0) *␣
 ↪N**(sp.Rational(11, 3)), (N, 0, sp.oo))
my_display(m0intexpr)
res = m0intexpr.doit().subs(sol0subs)
guess = sp.simplify(
    f5 * sp.gamma(sp.Rational(14, 3)) / j**20 * (
```

```
        e**2 * f3 / f5)**(sp.Rational(11, 6))) * (sp.Rational(3, 2))**(sp.
↪Rational(8, 3))
display(res / guess)
```

$$\int_0^\infty \frac{4C_0^2 N^{\frac{11}{3}} e^{-\frac{2N}{\eta_0}}}{3}\, dN$$

1

3.2.3 Final edot

Putting it all together, we obtain

$$\dot{E}_{\text{in}} = \frac{\hat{\tau}\Omega}{2}\left[\operatorname{sgn}\left(1 - 0.5886\frac{\Omega_s}{\eta_2\Omega}\right)\left|1 - 0.5886\frac{\Omega_s}{\eta_2\Omega}\right|^{8/3}\frac{f_5}{(1-e^2)^{9/2}}\frac{\Gamma(26/3)}{4!}\left(\frac{\eta_2}{2}\right)^{11/3} + \frac{f_5\Gamma(14/3)}{(1-e^2)^{10}}\left(\frac{3}{2}\right)^{8/3}\left(\frac{e^2 f_3}{f_5}\right)^{1}\right.$$

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While the expression for $\dot{E}_{\text{in}}$ is rather complex, it is well separated. Denote the entire bracketed term $g(e, \Omega_s/\Omega)$, then

$$\dot{E}_{\text{in}} = \hat{\tau}\Omega g\left(e, \frac{\Omega_s}{\Omega}\right).$$

If $\dot{P}/P$ is measured for a system, we can write the change in gravitational binding energy

$$\dot{E}_g = \frac{GM_1M_2}{2a}\frac{\dot{a}}{a} = \frac{GqM_2^2}{3a}\frac{\dot{P}}{P}, \tag{21}$$

$$\dot{E}_{\text{in}} = \hat{\tau}\Omega\, g\left(e, \frac{\Omega_s}{\Omega}\right) = -\dot{E}_g \tag{22}$$

[20]:
```
Eg, G, M, q, a, P, rhoterm, g, beta2, spin, rc, W = sp.symbols(r'Eg G M_2 q a P␣
↪f(\rho_{c}) g beta_2 \omega r_c \Omega')
adot, Pdot, g = sp.symbols(r'\dot{a} \dot{P} g(e)')
Eg = -G * q * M**2 / (2 * a)
dEg_dt = sp.Derivative(Eg, a).doit() * adot
# P^2 \propto a^3, so 2\dot{P} / P = 3\dot{a} / a, adot = 2 * Pdot * a / 3 * P
dEg_dt_pdot = dEg_dt.subs(adot, (2 * Pdot * a) / (3 * P))
display(dEg_dt_pdot)

tau = beta2 * G * M**2 * rc**5 / a**6 * W**(sp.Rational(8, 3)) * rhoterm
dEin_dt = tau * W * g
[res] = sp.solve(dEg_dt_pdot + dEin_dt, Pdot)
display(res.subs(P, 2 * sp.pi / W))
display(
    -6 * sp.pi / q
        * beta2 * (rc / a)**5
        * W**(sp.Rational(8, 3))
```

```
        * rhoterm * g
)
```

$$\frac{GM_2^2\dot{P}q}{3Pa}$$

$$-\frac{6\pi\Omega^{\frac{8}{3}}\beta_2 f(\rho_c)g(e)r_c^5}{a^5 q}$$

$$-\frac{6\pi\Omega^{\frac{8}{3}}\beta_2 f(\rho_c)g(e)r_c^5}{a^5 q}$$

Substituting in $\hat{\tau}$, we obtain ($q = M_1/M_2$, and $\Omega_{s,c} = \sqrt{GM_c/r_c^3}$)

$$-\dot{P} = \frac{6\pi}{q}\beta_2\left(\frac{r_c}{a}\right)^5\left(\frac{\Omega}{\Omega_{s,c}}\right)^{8/3}\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2 g\left(e,\frac{\Omega_s}{\Omega}\right). \tag{23}$$

Note that g increases for faster spins, but $|\Omega_s| \lesssim \sqrt{GM_c/r_c^3}$, the breakup spin frequency. Otherwise, taking $\beta_2 \approx 1$, the only unconstrained quantities are ρ_c, $\bar{\rho}_c$, and r_c. Generally, the density terms seem constant for a range of stellar models we simulated using MESA. Thus, if we assume $\dot{P}/P$ solely comes from dissipation of the dynamical tide, that the $m = 2$ components dominate the $m = 0$ contribution (i.e. the critical rotation rate is fast), and that all harmonics are dissipated completely in the envelope, we arrive at constraint for r_c:

$$g\left(e,\frac{\Omega_s}{\Omega}\right) \lesssim \frac{1}{2}|2\Omega_s/\Omega|^{8/3}\frac{2f_2}{(1-e^2)^6}, \tag{24}$$

$$-\dot{P} \lesssim \frac{6\pi}{q}\beta_2\left(\frac{r_c}{a}\right)^5\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2 2^{8/3}\frac{f_2}{(1-e^2)^6}. \tag{25}$$

If we take $\rho_c/\bar{\rho}_c \approx 0.76$, which is what we see in our MESA models, the rest of the parameters can be explicitly evaluated:

[19]:
```
pdot = 3.03e-7
rho_c_over_bar_rho_c = 0.76
e_val = 0.808
f5_val = f5.subs(ecc_subdict).subs(e, e_val).evalf()
f2_val = f2.subs(ecc_subdict).subs(e, e_val).evalf()
eta2_val = eta2sol.subs(ecc_subdict).subs(jsub).subs(e, e_val).evalf()
q = 6.3
a = 126 # solar radii

rc_min1 = (
    pdot / (2 * np.pi)
    / (
        3 / q
        * (rho_c_over_bar_rho_c * (1 - rho_c_over_bar_rho_c)**2)
        * 2**(8/3) * f2_val / (1 - e_val**2)**6
    )
)**(1/5)
```

```python
print(rc_min1 * a)
```

```
1.19054726670891
```