1 Apr 21, 2020

For the 90° attractor in the LK problem, investigated the $N = 0$ dynamics. Recall EOM

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \langle -\dot{\Omega}\hat{\mathbf{z}} + \Omega_{\mathrm{SL}}\hat{\mathbf{L}}\rangle_{\mathrm{LK}} \times \hat{\mathbf{S}} + \left[\sum_{N=1}^{\infty} \hat{\mathbf{\Omega}}_{\mathrm{eff,N}} \exp\left(2\pi i N t/t_{\mathrm{LK}}\right)\right] \times \hat{\mathbf{S}}. \tag{1}$$

We ignore $N \geq 1$ for now, assuming resonances are not hit, so we examine

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \langle -\dot{\Omega}\hat{\mathbf{z}} + \Omega_{\mathrm{SL}}\hat{\mathbf{L}}\rangle_{\mathrm{LK}} \times \hat{\mathbf{S}}. \tag{2}$$

Consider rotation by I_{out} given by Figure 1. In equa-

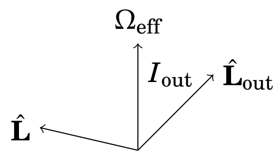

Figure 1: Geometry. I is angle between $\hat{\mathbf{L}}, \hat{\mathbf{L}}_{\mathrm{out}}$ while I_{out} is angle between $\hat{\mathbf{L}}_{\mathrm{out}}, \Omega_{\mathrm{eff}}$.

tions, this requires

$$-\dot{\Omega}\sin I_{\mathrm{out}} + \Omega_{SL}\sin(I + I_{\mathrm{out}}) = 0. \tag{3}$$

We obtain EOM (note that LK-averaged I is almost constant):

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \left[-\dot{\Omega}\cos I_{\mathrm{out}} + \Omega_{\mathrm{SL}}\cos(I + I_{\mathrm{out}})\right]\hat{\mathbf{z}} \times \hat{\mathbf{S}} - \dot{I}_{\mathrm{out}}\hat{\mathbf{y}} \times \hat{\mathbf{S}}, \tag{4}$$

$$\Delta\phi(t) = \int_0^t \left[-\dot{\Omega}\cos I_{\mathrm{out}}(\tau) + \Omega_{\mathrm{SL}}\cos(I + I_{\mathrm{out}}(\tau))\right]\,\mathrm{d}\tau, \tag{5}$$

$$\Delta\theta = \int_0^{T_{\mathrm{m}}} \dot{I}_{\mathrm{out}}\sin\phi\,\mathrm{d}t. \tag{6}$$

Intuition: $\dot{I}_{\mathrm{out}} \sim 1/T_{\mathrm{GW}}$ while $\dot{\phi} \sim \min\left(\dot{\Omega}, \Omega_{\mathrm{SL}}\right)$. Thus, when sufficiently slow, phases cancel, $\Delta\theta \to 0$. **NB:** in full numerical simulations, $\langle\theta\rangle_{\mathrm{LK}}$ is the approximately conserved quantity, θ varies within LK cycles.

How does this do? Figure 2.

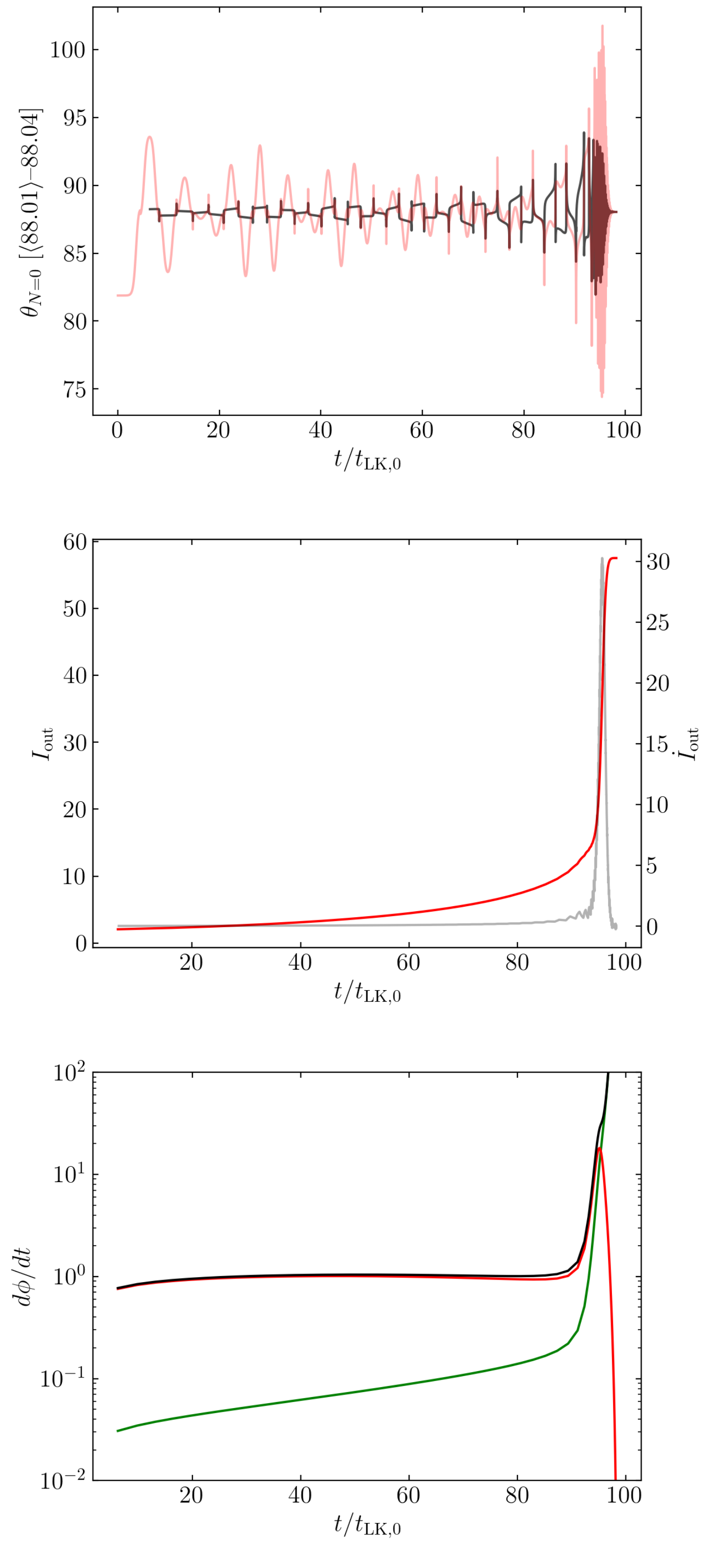

Figure 2: $I = 90.3°$. Top: Comparison of θ (black) vs θ_{eff} from Bin's paper (red). Middle: I_{out} is red, $\dot{I}_{\mathrm{out}}$ is light black. Bottom: Comparison of LK-averaged $\dot{\Omega}$ (red), Ω_{SL} (green), and $\langle\dot{\Omega} + \Omega_{\mathrm{SL}}\rangle_{\mathrm{LK}}$ (black).

How well does this do for our ensemble? See Figure 3. Blue is a $\propto \cos^{-9}(I_0)$ slope, dashed green shows the semi-major axis at the end of the Kozai cycle we average over (recall we have to average over a Kozai cycle to determine the initial θ).

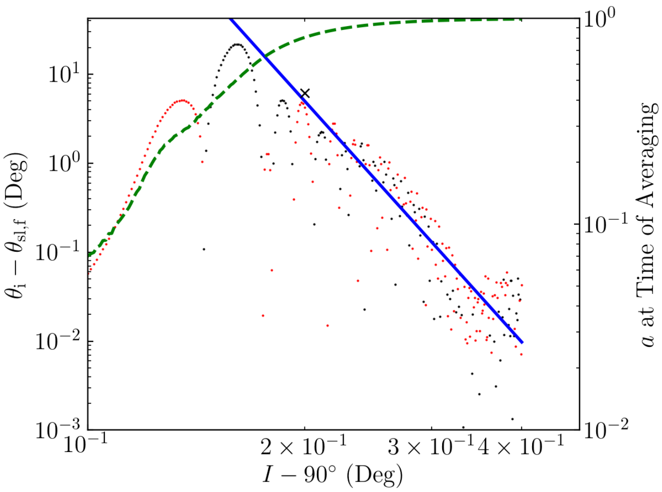

Figure 3: Deviations from exact conservation of θ.

This seems to confirm with expectation: we have calculations that show

$$\dot{I}_{\rm out}\left(-\cot I_{\rm out} + \cot(I + I_{\rm out})\right) = \frac{\mathrm{d}}{\mathrm{d}t}\ln\left(-\dot{\Omega}/\Omega_{\rm SL}\right). \tag{7}$$

The RHS of the above $\propto 1/T_{\rm m} \propto \cos^{-6} I_0$, which is the peak of $\dot{I}_{\rm out}$. Therefore, the width of $\dot{I}_{\rm out}$ is $\propto IT_{\rm m}$. Phases cancel better then when $I_{\rm out}$ is broader, so a scaling stronger than -6 seems understandable.