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1.1 Equal Masses: Following Racine

The paper is https://arxiv.org/pdf/0803.1820.pdf. The EOM and definitions in the equal mass case are considerably simplified and we obtain:

$$\frac{d\mathbf{S}_1}{d\psi} = \hat{\mathbf{J}} \times \mathbf{S}_1 - \alpha \mathbf{S}_2 \times \mathbf{S}_1,\tag{1}$$

$$\frac{d\mathbf{S}_2}{d\psi} = \hat{\mathbf{J}} \times \mathbf{S}_2 - \alpha \mathbf{S}_1 \times \mathbf{S}_2,\tag{2}$$

$$\frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{1}{2a_{\text{eff}}^3} \left(7 - \frac{3}{2}\lambda \right) J,\tag{3}$$

$$\alpha = \frac{3}{J} \frac{4 - \lambda}{14 - 3\lambda},\tag{4}$$

$$\lambda = \frac{\mathbf{L}}{L^2} \cdot \left[\left(1 + \frac{M_2}{M_1} \right) \mathbf{S}_1 + \left(1 + \frac{M_1}{M_2} \right) \mathbf{S}_2 \right]. \tag{5}$$

Here, $a_{\text{eff}} = a\sqrt{1-e^2}$, and $\mathbf{J} = J\hat{\mathbf{J}} = \mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2$ is the total angular momentum.

The Hamiltonian is then probably

$$H = -\hat{\mathbf{J}} \cdot (\mathbf{S}_1 + \mathbf{S}_2) + \alpha (\mathbf{S}_1 \cdot \mathbf{S}_2), \tag{6}$$

I think (double check the signs, check that it reproduces Racine's EOM?).

Ansatz: adiabatic invariant, can we set up $\oint p \ dq$? Why is ψ not uniformly advancing and is so complicated? λ is not invariant since $d\lambda/dt$ directly has a GW term.

Up to rotational invariance, equilibria occur when each of θ_1 , θ_2 and $\Delta \phi = \phi_2 - \phi_1$ are stationary. If we go to the corotating frame about $\hat{\mathbf{J}}$, then it's very obvious what is going on:

$$H_{\text{rot}} = \alpha(\mathbf{S}_1 \cdot \mathbf{S}_2). \tag{7}$$

Clearly, the equilibria occur when S_1 and S_2 are aligned/antialigned. I don't know why the phase portrait for them has to be so messy, probably a difficulty with coordinates?

This checks out in coordinate form, even in the inertial frame:

$$H(\theta_1, \theta_2, \Delta\phi) = -S_1 \cos \theta_1 - S_2 \cos \theta_2 + \alpha \left(\cos \theta_1 \cos \theta_2 + \sin \theta_1 \sin \theta_2 \cos \Delta\phi\right). \tag{8}$$

Since centers of libration occur when H attains an extremal value, it's clear that the only extremal values can occur when $\Delta \phi = 0$ or π . If we set that to be the case, then an extremal value can only occur if $\theta_1 \pm \theta_2 = 0$, depending on which ϕ we use, so indeed, we recover alignment/anti-alignment as the two possible centers of libration.

To try and extract an adiabatic invariant, we consider the evolution of the subs and differences of S_i , as pointed out by Racine. It's clear that $S_1 + S_2$ just precesses about \hat{J} (with the opening angle an adiabatic invariant), while $\Delta \equiv S_1 - S_2$ evolves with (S is the total spin)

$$\frac{\mathrm{d}\mathbf{\Delta}}{\mathrm{d}w} = (\hat{\mathbf{J}} - \alpha \mathbf{S}) \times \mathbf{\Delta}.\tag{9}$$

The obvious guess of adiabatic invariant in this case is then just going to be:

$$\cos \theta_{\Lambda} \equiv (\hat{\mathbf{J}} - \alpha \mathbf{S}) \cdot \mathbf{\Delta} \tag{10}$$

With these two adiabatic invariants, what happens during GW emission then? I don't think that the total spin one will do anything interesting (the most pathological case is when $\mathbf{S}_1 \parallel \mathbf{S}_2$: go to the co-rotating frame about \mathbf{L} initially, then the enclosed phase space area is zero, the precession axis just moves). However, during GW decay, we expect α to increase, as J decreases, and so Δ will initially precess about \mathbf{J} but eventually precess about the sum of the total spin and $\hat{\mathbf{J}}$ (note that even at the very late stages of inspiral, I think $\mathbf{L} \gtrsim \mathbf{S}_i$, so it will never be fully ignorable). Nevertheless, this is clearly the mechanism for making Δ tilt

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We first try to find the equilibria of the system before seeking adiabatic invariants. As Schnittman 2004 points out, we should consider the zeros of

$$\frac{\mathrm{d}}{\mathrm{d}t}(\mathbf{S}_1 \cdot \mathbf{S}_2) = \frac{\mathrm{d}\psi}{\mathrm{d}t} \beta_- \hat{\mathbf{J}} \cdot (\mathbf{S}_1 \times \mathbf{S}_2). \tag{11}$$

Here, β_{-} is a constant from Racine's paper, defined to be $\beta_{1} - \beta_{2}$ where

$$\beta_{1,2} = \left(4 + 3\frac{M_{2,1}}{M_{1,2}} - 3\frac{\mu}{M_{1,2}}\lambda\right) / \left(7 - \frac{3\lambda}{2}\right). \tag{12}$$

In order for Eq. (11) to vanish, we need two conditions:

$$\mathbf{S}_2 \cdot (\hat{\mathbf{J}} \times \mathbf{S}_1) = \sin \theta_1 \sin \theta_2 \sin \Delta \phi = 0, \tag{13}$$

$$\frac{\mathbf{d}}{\mathbf{d}t} \left[\mathbf{S}_2 \cdot (\hat{\mathbf{J}} \times \mathbf{S}_1) \right] = 0. \tag{14}$$

To achieve some comparability with the Schnittman equations, we can replace $\hat{\mathbf{J}}$ with \mathbf{J} above without changing the equilibria. Note that this is equivalent to the Schnittman expression

$$\mathbf{S}_2 \cdot [(\mathbf{L} + \mathbf{S}_1 + \mathbf{S}_2) \times \mathbf{S}_1] = \mathbf{S}_2 \cdot [(\mathbf{L} + \mathbf{S}_2) \times \mathbf{S}_1], \tag{15}$$

$$= \mathbf{S}_2 \cdot [\mathbf{L} \times \mathbf{S}_1] + \mathbf{S}_2 \cdot (\mathbf{S}_2 \times \mathbf{S}_1), \tag{16}$$

$$= \mathbf{S}_2 \cdot [\mathbf{L} \times \mathbf{S}_1]. \tag{17}$$

Thus, we find that the Schnittman equilibria are probably correct, but the detailed EOM about them may not necessarily be.

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Oops, I lost this chapter of the writeup, I should probably stop hard reseting hahaha.

We numerically observe that $\Delta \phi$ seems to be oscillating, and eventually begins to narrow as GW radiation shrinks the orbit. Can we understand this analytically?

Call $\mathbf{S} = \mathbf{s}_1 + \mathbf{s}_2$ and $\Delta = \mathbf{s}_1 - \mathbf{s}_2$. The equations of motion are:

$$\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}w} = \hat{\mathbf{J}} \times \mathbf{S},\tag{18}$$

$$\frac{\mathrm{d}\mathbf{\Delta}}{\mathrm{d}\psi} = \hat{\mathbf{J}} \times \mathbf{\Delta} - \alpha \mathbf{S} \times \mathbf{\Delta}.\tag{19}$$

We go to the corotating frame about $\hat{\mathbf{J}}$, which is perfectly legal in the absence of GW radiation, in which case the EOM becom:

$$\left(\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}\psi}\right)_{\mathrm{rot}} = 0,\tag{20}$$

$$\left(\frac{d\mathbf{S}}{d\psi}\right)_{\text{rot}} = 0,$$

$$\left(\frac{d\mathbf{\Delta}}{d\psi}\right)_{\text{rot}} = -\alpha \mathbf{S} \times \mathbf{\Delta}.$$
(20)

It can be shown that, since $\Delta = S - 2s_2 = 2s_1 - S$, that:

$$\left(\frac{\mathrm{d}\mathbf{s}_i}{\mathrm{d}t}\right) = \alpha(\mathbf{S} \times \mathbf{s}_i). \tag{22}$$

Thus, in the absence of GW radiation, both spins just precess about their sum, very clean. This explains why $\Delta \phi$ oscillates very cleanly: if the spins are both on the same side of the chosen $\hat{\mathbf{z}}$ coordinate axis (in our numerical experiments, \propto L the angular momentum axis), then the ϕ angles will be librating very simply. Nothing magical.

However, in our change of reference frame, we changed to be corotating about $\hat{\mathbf{J}}$, which itself is not a fixed axis. Thus, there should really be a time-dependent term that captures the rotation of $\hat{\mathbf{J}}$; what is it? Could it cause the opening angle $\cos \theta_i \propto \mathbf{s}_i \cdot \mathbf{S}$ to decrease (and thus decrease the range of oscillation about $\Delta \phi = 180^{\circ}$)? Only time will tell :0