1 Apr 21, 2020

For the 90° attractor in the LK problem, investigated the $N = 0$ dynamics. Recall EOM

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \langle -\dot{\Omega}\hat{\mathbf{z}} + \Omega_{\rm SL}\hat{\mathbf{L}}\rangle_{\rm LK} \times \hat{\mathbf{S}} + \left[\sum_{N=1}^{\infty} \hat{\mathbf{\Omega}}_{\rm eff,N} \exp\left(2\pi i N t/t_{\rm LK}\right)\right] \times \hat{\mathbf{S}}. \tag{1}$$

We ignore $N \geq 1$ for now, assuming resonances are not hit, so we examine

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \langle -\dot{\Omega}\hat{\mathbf{z}} + \Omega_{\rm SL}\hat{\mathbf{L}}\rangle_{\rm LK} \times \hat{\mathbf{S}}. \tag{2}$$

Consider rotation by $I_{\rm out}$ given by Figure 1. In equations, this requires

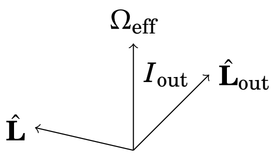

Figure 1: Geometry. I is angle between $\hat{\mathbf{L}}, \hat{\mathbf{L}}_{\rm out}$ while $I_{\rm out}$ is angle between $\hat{\mathbf{L}}_{\rm out}, \Omega_{\rm eff}$.

$$-\dot{\Omega}\sin I_{\rm out} + \Omega_{SL}\sin(I + I_{\rm out}) = 0. \tag{3}$$

We obtain EOM (note that LK-averaged I is almost constant):

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \left[-\dot{\Omega}\cos I_{\rm out} + \Omega_{\rm SL}\cos(I + I_{\rm out})\right]\hat{\mathbf{z}} \times \hat{\mathbf{S}} - \dot{I}_{\rm out}\hat{\mathbf{y}} \times \hat{\mathbf{S}}, \tag{4}$$

$$\Delta\phi(t) = \int_0^t \left[-\dot{\Omega}\cos I_{\rm out}(\tau) + \Omega_{\rm SL}\cos(I + I_{\rm out}(\tau))\right]\,\mathrm{d}\tau, \tag{5}$$

$$\Delta\theta = \int_0^{T_{\rm m}} \dot{I}_{\rm out}\sin\phi\,\mathrm{d}t. \tag{6}$$

Intuition: $\dot{I}_{\rm out} \sim 1/T_{\rm GW}$ while $\dot{\phi} \sim \min(\dot{\Omega}, \Omega_{\rm SL})$. Thus, when sufficiently slow, phases cancel, $\Delta\theta \to 0$. **NB:** in full numerical simulations, $\langle\theta\rangle_{\rm LK}$ is the approximately conserved quantity, θ varies within LK cycles.

How does this do? Figure 2.

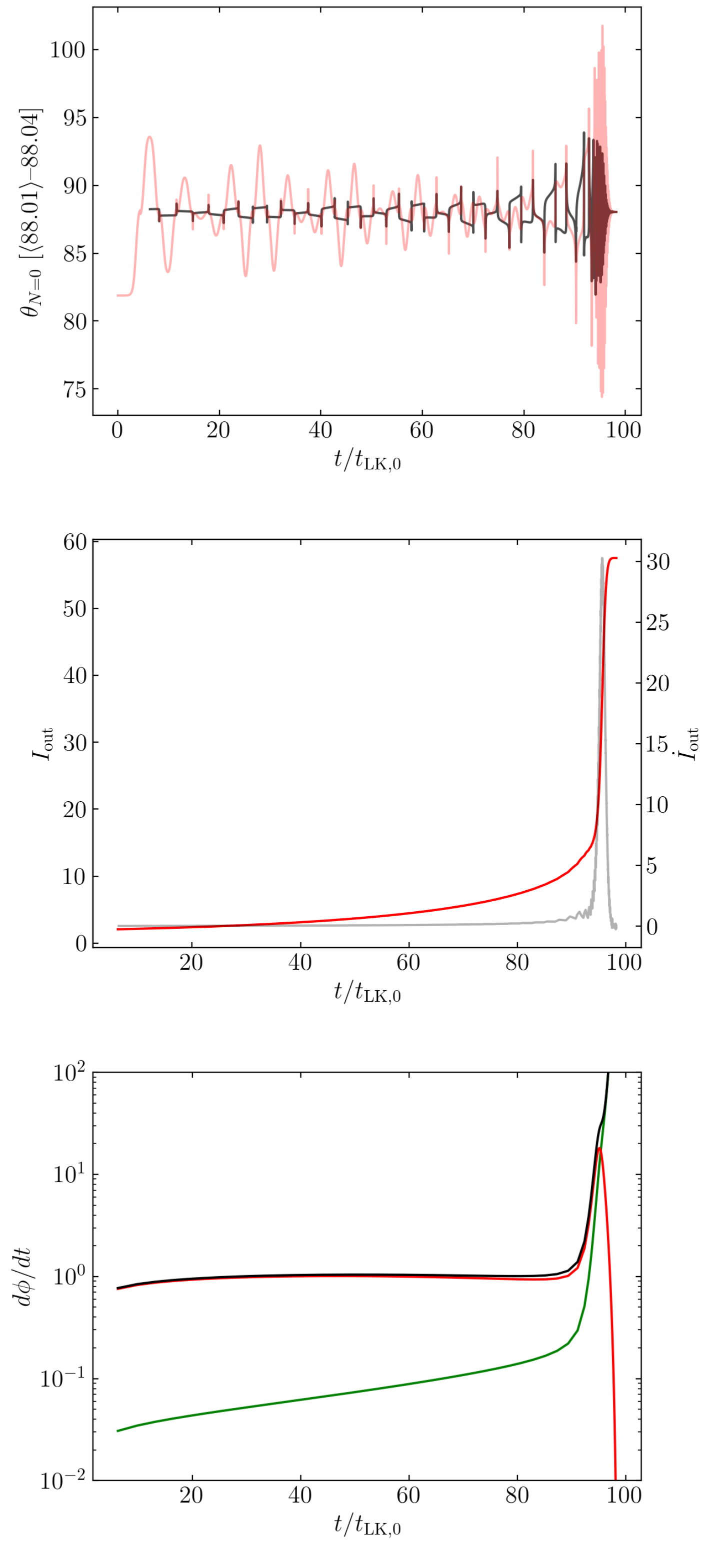

Figure 2: $I = 90.3°$. Top: Comparison of θ (black) vs $\theta_{\rm eff}$ from Bin's paper (red). Middle: $I_{\rm out}$ is red, $\dot{I}_{\rm out}$ is light black. Bottom: Comparison of LK-averaged $\dot{\Omega}$ (red), $\Omega_{\rm SL}$ (green), and $\langle\dot{\Omega} + \Omega_{\rm SL}\rangle_{\rm LK}$ (black).

How well does this do for our ensemble? See Figure 3. Blue is a $\propto \cos^{-9}(I_0)$ slope, dashed green shows the semi-major axis at the end of the Kozai cycle we average over (recall we have to average over a Kozai cycle to determine the initial θ).

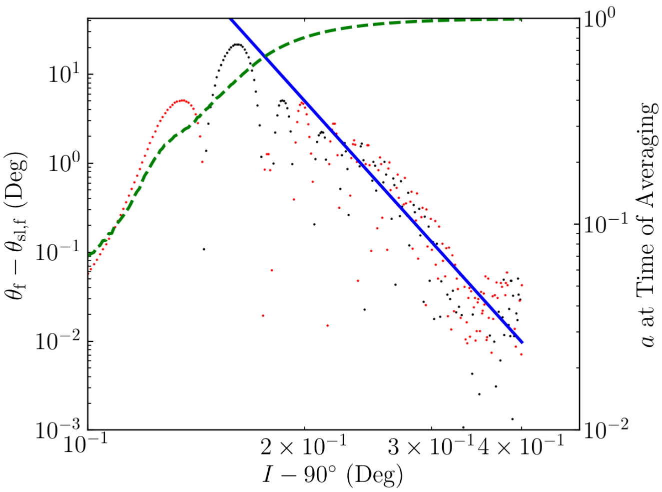

Figure 3: Deviations from exact conservation of θ.

This seems to confirm with expectation: we have calculations that show

$$\dot{I}_{\mathrm{out}}\left(-\cot I_{\mathrm{out}} + \cot(I + I_{\mathrm{out}})\right) = \frac{\mathrm{d}}{\mathrm{d}t} \ln\left(-\dot{\Omega}/\Omega_{\mathrm{SL}}\right). \quad (7)$$

The RHS of the above $\propto 1/T_{\mathrm{m}} \propto \cos^{-6} I_0$, which is the peak of $\dot{I}_{\mathrm{out}}$. Therefore, the width of $\dot{I}_{\mathrm{out}}$ is $\propto I T_{\mathrm{m}}$. Phases cancel better then when I_{out} is broader, so a scaling stronger than -6 seems understandable.