

Octupole-order Lidov-Kozai Population Statistics

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I. 10/22/20—INITIAL THOUGHTS

A. Equations

The equations of motion we want to study come from LML15. Describe the inner binary by $(a, e, I, \delta\Omega, \omega)$ and the outer binary with “out” subscripts, and denote $I_{\text{tot}} = I + I_{\text{out}}$. Call the inner binary component masses m_1, m_2 , and the tertiary mass m_3 , define the inner binary total and reduced masses $m_{12} = m_1 + m_2$ and $\mu = m_1 m_2 / m_{12}$, and define the tertiary orbit total and reduced masses $m_{123} = m_{12} + m_3$ and $\mu_{\text{out}} = m_{12} m_3 / m_{123}$. The equations of motion are $(j(e) = \sqrt{1 - e^2})$

$$\frac{da}{dt} = -\frac{64}{5} \frac{a}{t_{\text{GW}} j^7(e)} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right), \quad (1)$$

$$\begin{aligned} \frac{de}{dt} = & \frac{j(e)}{64 t_{\text{LK}}} \left\{ 120 e \sin^2 I_{\text{tot}} \sin(2\omega) \right. \\ & + \frac{15 \epsilon_{\text{oct}}}{8} \cos \omega_{\text{out}} [(4 + 3e^2) (3 + 5 \cos(2I_{\text{tot}})) \\ & \times \sin \omega + 210 e^2 \sin^2 I_{\text{tot}} \sin 3\omega] \\ & - \frac{15 \epsilon_{\text{oct}}}{4} \cos I_{\text{tot}} \cos \omega [15(2 + 5e^2) \cos(2I_{\text{tot}}) \\ & + 7(30e^2 \cos(2\omega) \sin^2 I_{\text{tot}} - 2 - 9e^2)] \sin \omega_{\text{out}} \Big\} \\ & - \frac{304}{15} \frac{e}{t_{\text{GW}} j^5(e)} \left(1 + \frac{121}{304} e^2 \right), \end{aligned} \quad (2)$$

$$\begin{aligned} \frac{dI}{dt} = & -\frac{3e}{32 t_{\text{LK}} j(e)} \left\{ 10 \sin(2I_{\text{tot}}) [e \sin(2\omega) \right. \\ & + \frac{5 \epsilon_{\text{oct}}}{8} (2 + 5e^2 + 7e^2 \cos(2\omega)) \cos \omega_{\text{out}} \sin \omega] \\ & + \frac{5 \epsilon_{\text{oct}}}{8} \cos \omega [26 + 37e^2 - 35e^2 \cos(2\omega) \\ & - 15 \cos(2I_{\text{tot}}) (7e^2 \cos(2\omega) - 2 - 5e^2)] \\ & \times \sin I_{\text{tot}} \sin \omega_{\text{out}} \Big\} \end{aligned} \quad (3)$$

$$\begin{aligned} \frac{d\delta\Omega}{dt} = \frac{d\delta\Omega_{\text{out}}}{dt} = & -\frac{3 \csc I}{32 t_{\text{LK}} j(e)} \left\{ 2 [(2 + 3e^2 - 5e^2 \cos(2\omega)) \right. \\ & + \frac{25 \epsilon_{\text{oct}} e}{8} \cos \omega (2 + 5e^2 - 7e^2 \cos(2\omega)) \cos \omega_{\text{out}}] \\ & \times \sin(2I_{\text{tot}}) - \frac{5 \epsilon_{\text{oct}} e}{8} [35e^2 (1 + 3 \cos(2I_{\text{tot}})) \cos 2\omega \\ & - 46 - 17e^2 - 15(6 + e_1^2) \cos(2I_{\text{tot}})] \\ & \times \sin I_{\text{tot}} \sin \omega \sin \omega_{\text{out}} \Big\}, \end{aligned} \quad (4)$$

$$\begin{aligned} \frac{d\omega}{dt} = & \frac{3}{8 t_{\text{LK}}} \left\{ \frac{1}{j(e)} [4 \cos^2 I_{\text{tot}} + (5 \cos(2\omega) - 1) \right. \\ & \times (1 - e^2 - \cos^2 I_{\text{tot}})] + \frac{L \cos I_{\text{tot}}}{L_{\text{out}} j(e_{\text{out}})} [2 + e^2 (3 \\ & - 5 \cos(2\omega))] \Big\} + \frac{15 \epsilon_{\text{oct}}}{64 t_{\text{LK}}} \left\{ \left(\frac{L}{L_{\text{out}} j(e_{\text{out}})} + \frac{\cos I_{\text{tot}}}{j(e)} \right) \right. \\ & \times e [\sin \omega \sin \omega_{\text{out}} [10(3 \cos^2 I_{\text{tot}} - 1)(1 - e^2) + A] \\ & - 5B \cos I_{\text{tot}} \cos \Theta] - \frac{j(e)}{e} [10 \sin \omega \sin \omega_{\text{out}} \cos I_{\text{tot}} \\ & \times \sin^2 I_{\text{tot}} (1 - 3e^2) + \cos \Theta (3A - 10 \cos^2 I_{\text{tot}} + 2)] \Big\} \\ & + \Omega_{\text{GR}}, \end{aligned} \quad (5)$$

$$\begin{aligned} \frac{de_{\text{out}}}{dt} = & \frac{15eL j(e_{\text{out}}) \epsilon_{\text{oct}}}{256 t_{\text{LK}} e_{\text{out}} L_{\text{out}}} \left\{ \cos \omega [6 - 13e^2 \right. \\ & + 5(2 + 5e^2) \cos(2I_{\text{tot}}) + 70e^2 \cos(2\omega) \sin^2 I_{\text{tot}}] \\ & \times \sin \omega_{\text{out}} - \cos I_{\text{tot}} \cos \omega_{\text{out}} [5(6 + e^2) \cos(2I_{\text{tot}}) \\ & + 7(10e^2 \cos(2\omega) \sin^2 I_{\text{tot}} - 2 + e^2)] \sin \omega \Big\}, \end{aligned} \quad (6)$$

$$\begin{aligned} \frac{dI_{\text{out}}}{dt} = & -\frac{3eL}{32 t_{\text{LK}} j(e_{\text{out}}) L_{\text{out}}} \left\{ 10 [2e \sin I_{\text{tot}} \sin(2\omega) \right. \\ & + \frac{5 \epsilon_{\text{oct}}}{8} \cos \omega (2 + 5e^2 - 7e^2 \cos(2\omega)) \sin(2I_{\text{tot}}) \sin \omega_{\text{out}}] \\ & + \frac{5 \epsilon_{\text{oct}}}{8} [26 + 107e^2 + 5(6 + e^2) \cos(2I_{\text{tot}}) \\ & - 35e^2 (\cos(2I_{\text{tot}}) - 5) \cos(2\omega)] \cos \omega_{\text{out}} \sin I_{\text{tot}} \sin \omega \Big\}, \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{d\omega_{\text{out}}}{dt} = & \frac{3}{16 t_{\text{LK}}} \left\{ \frac{2 \cos I_{\text{tot}}}{j(e)} [2 + e^2 (3 - 5 \cos(2\omega))] \right. \\ & + \frac{L}{L_{\text{out}} j(e_{\text{out}})} [4 + 6e^2 + (5 \cos^2 I_{\text{tot}} - 3) \\ & \times [2 + e^2 (3 - 5 \cos(2\omega))]] \Big\} - \frac{15 \epsilon_{\text{oct}} e}{64 t_{\text{LK}} e_{\text{out}}} \\ & \times \left\{ \sin \omega \sin \omega_{\text{out}} \left[\frac{L(4e_{\text{out}}^2 + 1)}{e_{\text{out}} L_{\text{out}} j(e_{\text{out}})} 10 \cos I_{\text{tot}} \sin^2 I_{\text{tot}} \right. \right. \\ & \times (1 - e^2) - e_{\text{out}} \left(\frac{1}{j(e)} + \frac{L \cos I_{\text{tot}}}{L_{\text{out}} j(e_{\text{out}})} \right) \\ & \times [A + 10(3 \cos^2 I_{\text{tot}} - 1)(1 - e^2)] + \cos \Theta \\ & \times [5B \cos I_{\text{tot}} e_{\text{out}} \left(\frac{1}{j(e)} + \frac{L \cos I_{\text{tot}}}{L_{\text{out}} j(e_{\text{out}})} \right) \\ & \left. \left. + \frac{L(4e_{\text{out}}^2 + 1)}{e_{\text{out}} L_{\text{out}} j(e_{\text{out}})} A \right] \right\}. \end{aligned} \quad (8)$$

where $n = \sqrt{Gm_{12}/a^3}$ is the mean motion, $L = \mu \sqrt{Gm_{12}a}$ and $L_{\text{out}} = \mu_{\text{out}} \sqrt{Gm_{123}a_{\text{out}}}$ are the circular

angular momenta, and

$$t_{\text{LK}}^{-1} = n \left(\frac{m_3}{m_{12}} \right) \left(\frac{a}{a_{\text{out}} j(e_{\text{out}})} \right)^3, \quad (9)$$

$$t_{\text{GW}}^{-1} = \frac{G^3 \mu m_{12}^2}{c^5 a^4}, \quad (10)$$

$$\Omega_{\text{GR}} = \frac{3Gnm_{12}}{c^2 a j^2(e)}, \quad (11)$$

$$\epsilon_{\text{oct}} = \frac{m_2 - m_1}{m_{12}} \frac{a}{a_{\text{out}}} \frac{e_{\text{out}}}{1 - e_{\text{out}}^2}, \quad (12)$$

$$A \equiv 4 + 3e^2 - \frac{5}{2} B \sin^2 I_{\text{tot}}, \quad (13)$$

$$B \equiv 2 + 5e^2 - 7e^2 \cos(2\omega), \quad (14)$$

$$\cos \Theta \equiv -\cos \omega \cos \omega_{\text{out}} - \cos I_{\text{tot}} \sin \omega \sin \omega_{\text{out}}. \quad (15)$$

These equations can be nondimensionalized via the following steps (I won't rewrite the equations): (i) multiply through by $t_{\text{LK},0}$ ($a = a_0$ and $e_{\text{out}} = 0$), and call $\tau \equiv t/t_{\text{LK},0}$ the new variable of differentiation, (ii) re-express all of the timescales as

$$\frac{t_{\text{LK},0}}{t_{\text{LK}}} = \left(\frac{a}{a_0} \right)^{3/2} j^{-3}(e_{\text{out}}), \quad (16)$$

$$\begin{aligned} \frac{t_{\text{LK},0}}{t_{\text{GW}}} &= \frac{G^3 \mu m_{12}^3}{m_3 c^5 a^4} \frac{1}{n_0} \left(\frac{a_{\text{out}}}{a_0} \right)^3, \\ &= \epsilon_{\text{GW}} \left(\frac{a_0}{a} \right)^4, \end{aligned} \quad (17)$$

$$\epsilon_{\text{GW}} \equiv \frac{G^3 \mu m_{12}^3 a_{\text{out}}^3}{m_3 c^5 a_0^7 n_0}, \quad (18)$$

$$\begin{aligned} \Omega_{\text{GR}} t_{\text{LK},0} &= \frac{3Gnm_{12}^2}{m_3 c^2 a} \frac{1}{n_0} \left(\frac{a_{\text{out}}}{a_0} \right)^3, \\ &= \epsilon_{\text{GR}} \left(\frac{a_0}{a} \right)^{5/2}, \end{aligned} \quad (19)$$

$$\epsilon_{\text{GR}} = \frac{3Gm_{12}^2 a_{\text{out}}^3}{m_3 c^2 a_0^4}. \quad (20)$$

(iii) re-express da/dt as

$$\frac{d(a/a_0)}{d\tau} = -\frac{64}{5} \frac{\epsilon_{\text{GW}}}{j^{7/2}(e)} \left(\frac{a_0}{a} \right)^3 \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right). \quad (21)$$

As such, the natural unit of length is $a_0 = 1$, the natural unit of time is $t_{\text{LK},0} = 1$, and everything else is dimensionless. When computing these ϵ , I use convention where $1M_{\odot} = 1 \text{ AU} = c = 1$, under which $G = 9.87 \times 10^{-9}$.

B. Points of Inquiry

The goal is to understand how the merger window varies as a function of $q \equiv m_1/m_2$ ($m_1 < m_2$) when the octupole order LK effects are important.

- First, let's set $\epsilon_{\text{GW}} = 0$. It is well known that the octupole order LK is nonintegrable. What does the Fourier spectrum of the eccentricity look like? Will this help us get a delay time distribution between high- e phases?

When the octupole effect is unimportant, the spectrum falls off exponentially over scales $\tau \simeq P_{\text{LK}} j^{-1}(e_{\text{max}})$, where P_{LK} is the quadrupole LK period. One imagines the tail of the spectrum gets heavier when ϵ_{oct} is increased, and this might help us get the delay time distribution.

A second way we can postprocess this is to take a histogram of $e(t)$. If there is some regular structure, it's likely this will allow us to compute the average rate of binary coalescence due to GW radiation.

- The goal is to understand the size of the merger window, ΔI , as a function of q . To do this, we numerically sample the merger time function $T_{\text{m}}(I_0, q)$. At each I_0 , the natural thing to do would be to try for $\sim 5 - 10$ random Ω, ω , and define the merger window to be where $T_{\text{m}} \leq 10^{10} \text{ yr}$.