Eccentric Tides

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As usual, this is kind of a scattered document. It isn't written linearly, so notation evolves somewhat as it converges to the published version. Hopefully things make sense if one jumps around a bit.

1 Kushnir et. al., 2016

We coarsely follow the derivation of Kushnir et. al., 2016 (KZ) to express the traveling wave regime of dynamical tides in high-mass stars (convective core, radiative envelope) in analytical form.

1.1 Plane Parallel Case

We will consider IGW in a plane parallel atmosphere in the Boussinesq approximation. Consider buoyancy frequency

$$N^2 = -g\left(\frac{\mathrm{d}\ln\rho}{\mathrm{d}r} + \frac{g}{c_s^2}\right), \tag{1}$$

where $c_s \to \infty$ is the sound speed in the fluid. Then the Boussinesq equations can be written in terms of some *buoyancy* variable b

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \frac{\vec{\nabla}P}{\rho_0} + b\hat{z}, \tag{2a}$$

$$\frac{\mathrm{D}b}{\mathrm{D}t} = -N^2 u_z, \tag{2b}$$

$$\vec{\nabla}\cdot\vec{u} = 0. \tag{2c}$$

Note that $b \equiv -\frac{\rho'}{\rho_0}g$, as can be verified via direct substitution into the Euler equations:

$$0 = \frac{\mathrm{D}\rho'}{\mathrm{D}t} + \vec{u}\cdot\vec{\nabla}\rho_0 = \frac{\mathrm{D}\rho'}{\mathrm{D}t} - u_z\frac{N^2}{g}\rho', \tag{3a}$$

$$\frac{\mathrm{D}\vec{u}}{\mathrm{D}t} = \frac{\vec{\nabla}P'}{\rho_0} - \frac{\rho'}{\rho_0^2}\vec{\nabla}P_0 = \frac{\vec{\nabla}P'}{\rho_0} - \frac{\rho'}{\rho_0}g\hat{z}. \tag{3b}$$

These equations can be solved for u_z, or we can just recall the IGW dispersion relation $\omega^2 k^2 =$

$N^2k_\perp^2$ and write down PDE

$$\frac{\partial^2}{\partial t^2}\nabla^2 u_z = -N^2\nabla_\perp^2 u_z. \tag{4}$$

Now, we might recall that in tidally-forced stars, ω the tidal forcing frequency obeys $\omega \ll N$, or $k \gg k_\perp$. But the tidal potential, the quadrupolar expansion of the gravitational perturbation from the companion, has no quickly-varying directions, or can only excite $k \simeq k_\perp$ modes. Thus, we intuit that waves must be excited where N is much smaller than its typical value, or near the *radiative-convective boundary* (RCB). At the RCB, $N^2 = 0$, and we are concerned with the turning point where $\omega^2 = N^2$. We perform linear expansion about this turning point z_c, and for convenience we set $z_c = 0$, then

$$N^2 \approx \omega^2 + \frac{\mathrm{d}N^2}{\mathrm{d}z}z, \tag{5}$$

where compared to KZ I've taken $N_0^2 = \omega^2$, there seems to be little harm here. Making then general ansatz $u_z(z,\vec{r}_\perp,t) = \tilde{u}_z(z)e^{i\left(\vec{k}_\perp\cdot\vec{r}_\perp - \omega t\right)}$ we obtain

$$-\omega^2\left(-k_\perp^2\tilde{u}_z + \tilde{u}_z''\right) = N^2k_\perp^2 u_z, \tag{6}$$

$$\tilde{u}_z'' + k_\perp^2\left(\frac{N^2}{\omega^2} - 1\right)\tilde{u}_z = 0, \tag{7}$$

$$\tilde{u}_z'' + k_\perp^2\frac{\mathrm{d}N^2}{\mathrm{d}z}\frac{z}{\omega^2}\tilde{u}_z = 0. \tag{8}$$

It's easiest now to rescale $\tilde{k}_\perp^2 \equiv k_\perp^2 3\frac{\mathrm{d}N^2}{\mathrm{d}z}\frac{1}{\omega^2}$ so that

$$\tilde{u}_z'' + \tilde{k}_\perp z\tilde{u}_z = 0. \tag{9}$$

The general solution to this ODE is written in terms of Airy functions for arbitrary constants a, b

$$\tilde{u}_z(z) = a\mathrm{Ai}\left(-\frac{z}{\lambda}\right) + b\mathrm{Bi}\left(-\frac{z}{\lambda}\right), \tag{10}$$

where $\lambda = \tilde{k}_\perp^{-2/3}$. For large $-z$, it turns out that

$$\mathrm{Ai}(-z) \sim \frac{\sin\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)}{z^{1/4}} + \mathcal{O}\left(z^{-7/4}\right), \tag{11}$$

$$\mathrm{Bi}(-z) \sim \frac{\cos\left(\frac{2}{3}z^{3/2} + \frac{\pi}{4}\right)}{z^{1/4}} + \mathcal{O}\left(z^{-7/4}\right). \tag{12}$$

In order for us to get traveling waves with *group velocity* going outwards (towards $z > 0$), we need $\tilde{u}_z(z) \sim e^{-ik_zz}$ such that $u(z,t) \propto e^{i(-k_zz-\omega t)}$ (phase velocity goes inwards, group velocity goes outwards for IGW). Thus, $a = -ib$ in Eq. 13, and we obtain

$$\tilde{u}_z(z) = b\left(-i\mathrm{Ai}\left(-\frac{z}{\lambda}\right) + \mathrm{Bi}\left(-\frac{z}{\lambda}\right)\right), \tag{13}$$

Now we just need to fix b. This is traditionally accomplished by mandating a particular $\frac{\mathrm{d}\delta z}{\mathrm{d}z}$ the displacement of the mode at the turning point $z = 0$. That the forcing results in a constraint on $\frac{\mathrm{d}\delta z}{\mathrm{d}z}$ is similar to what I did in my IGW breaking forcing, where forcing induces a jump in the $\frac{\mathrm{d}u_z}{\mathrm{d}z}$ above/below z_c whose magnitude is fixed by the strength of the forcing term. In the stellar problem, it appears the correct way to obtain the δz is to solve the inhomogeneous problem in the convective zone where $N^2 = 0$ including the tidal potential, so it's not a perfect analogy. But since $\mathrm{Ai}'(0) = -\frac{1}{3^{1/3}\Gamma(1/3)}, \mathrm{Bi}'(0) = \frac{3^{1/6}}{\Gamma(1/3)}$, this is not so difficult to evaluate, and I cite the KZ result

$$\frac{\mathrm{d}\delta z}{\mathrm{d}z} = -\frac{ib}{\lambda\omega}\frac{2}{3^{1/3}\Gamma(1/3)}\frac{3^{1/2}+i}{2}. \tag{14}$$

Finally, we impose one more step: we will compute the luminosity or *energy flux* associated with the wave, since this is the easiest way to get the resulting torque. We can easily write down the energy density of the wave $\frac{\rho_0}{2}\left(v^2 + \frac{b^2}{N^2}\right)$, for which the energy flux is $\vec{F} = \vec{v}P$. Noting furthermore that $\frac{\partial u_z}{\partial z} = -ik_\perp u_x = -\frac{ik_\perp P}{\rho_0}\frac{k_\perp}{\omega}$, we can explicitly express P in terms of u_z', and so the energy flux density is then simply (I'm not evaluating this, but KZ do)

$$\frac{\delta L}{\delta A} = \frac{1}{2}\mathrm{Re}\left(Pu_z^*\right) = \frac{\rho_0\omega}{2k_\perp^2}\mathrm{Re}\left(iu_z'u_z^*\right), \tag{15}$$

$$= \frac{3^{2/3}\Gamma^2(1/3)\lambda\omega^3\rho_0}{8\pi k_\perp^2}\left(\frac{\mathrm{d}\delta z}{\mathrm{d}z}\right)^2. \tag{16}$$

We would then compute $L = \int \frac{\delta L}{\delta A}\,\mathrm{d}A$, which for us is just $\frac{\delta L}{\delta A}A$ where A is the surface area of the wave.

Finally, we would compute the total torque from $L = \tau\omega$ (the same as $E = \vec{F}\cdot\vec{v}$).

1.2 Spherical Case

To go to the spherical case, we simply replace $z \to r$ and $k_\perp^2 \to l(l+1)/r^2$, which gives

$$\lambda = \left(\frac{l(l+1)}{r^2\omega^2}\frac{\mathrm{d}N^2}{\mathrm{d}r}\right)^{-1/3}. \tag{17}$$

Then to get $\frac{\mathrm{d}\delta z}{\mathrm{d}z} \to \frac{\mathrm{d}\delta r}{\mathrm{d}r}$, we use prescription

$$\frac{\mathrm{d}\delta r}{\mathrm{d}r} = \alpha\frac{\Phi}{gr}\left(1 - \frac{\rho(r)}{\bar{\rho}(r)}\right). \tag{18}$$

Here,

$$\alpha = \left(\frac{r_c}{R}\right)^{-5}\left(\frac{M_c}{M}\right)\left(1 - \frac{\rho}{\bar{\rho}}\right)^{-1}H_2, \tag{19}$$

while $\bar{\rho}$ is the average density inside r. Finally, instead of getting a clean $L = \frac{\delta L}{\delta A}A$, we have to actually do the integral of $L = \int \frac{\delta L}{\delta A}\,\mathrm{d}A = \int (\dots)|Y_{lm}|^2 r^2\,\mathrm{d}\cos\theta\,\mathrm{d}\phi = r_c^2 L$ (note that it's not $4\pi r_c^2$, thanks to the Y_{lm} normalization). The $\ell = 2$ potential is taken to be

$$\Phi_{\text{ext}} = -\sqrt{\frac{6\pi}{5}}\frac{GM_2R_c^2}{D^3}. \tag{20}$$

I guess the angular dependency is just dropped. With all these things together, we obtain the final KZ result (I omit the derivation, this part is grungy and not very physically interesting)

$$\begin{aligned}\tau = \dot{J}_z &= \frac{GM_2^2R_c^5}{D^6}\sigma_c^{8/3}\left[\frac{r_c}{g_c}\left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln R}\right)_{r=r_c}\right]^{-1/3}\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2\left[\frac{3}{2}\frac{3^{2/3}\Gamma^2(1/3)}{5\cdot 6^{4/3}}\frac{3}{4\pi}\alpha^2\right], && (21)\\ &= \frac{GM_2^2R_c^5}{D^6}2\hat{F}(r_c,\sigma_c). && (22)\end{aligned}$$

Note that $\hat{F}$ follows the convention from Equation 42 of Fuller & Lai's second paper (FL2) and Vick et. al's paper as well (VLF), while $\sigma_c = 2|\Omega - \Omega_s|/\sqrt{GM_c/r_c^3}$ is the ratio of the forcing frequency to the breakup frequency *of the core*. Finally, I've replaced M_2 the mass of the companion, R_c the radius of the core, and D the separation, while retaining Ω_s spin angular frequency and Ω orbital angular frequency.

NB: The exact definition of $\hat{F}$ for a given m is given in VLF.23 as

$$\dot{J} = G\frac{M_2^2R^5}{a^3}\frac{|m|}{2}\hat{F}(\omega) = T_0\frac{|m|}{2}\hat{F}(\omega). \tag{23}$$

Since the total torque τ has already summed over $m = \pm 2$, we incur the extra factor of 2 above in Eq. 22. That m has already been summed over is visible in the $\sqrt{6\pi/5}$ prefactor used in Φ_{ext}, compared to $W_{2\pm 2} = \sqrt{3\pi/10}$ as seen below.

2 Vick et. al., 2016

We now consider eccentric forcing. We will remove subscript compared to VLF and just call $\vec{r}_i = (r,\theta,\phi+\Omega_s t)$ the position coordinate in the inertial frame. Then the $\ell = 2$ tidal forcing potential is generally a sum over $m \in [-2,2]$

$$\begin{aligned}U &= \sum_m U_{2m}(\vec{r},t), && (24)\\ U_{2m}(\vec{r}) &= -\frac{GM_2W_{2m}r^2}{D(t)^3}e^{-imf(t)}Y_{2m}(\theta,\phi). && (25)\end{aligned}$$

Note that f is the true anomaly here. Note that W_{2m} is just a constant: $W_{20} = \sqrt{\pi/5}$, $W_{2\pm 1} = 0$, and $W_{2\pm 2} = \sqrt{3\pi/10}$.

This is complicated since $f(t)$ does not evolve uniformly, ando also since $D(t)$ is time-varying! The easiest treatment is to decompose

$$U_{2m} = -\frac{GM_2 W_{2m} r^2}{a^3} Y_{2m}\left(\theta,\phi\right) \sum_{N=-\infty}^{\infty} F_{Nm} e^{-iN\Omega t}. \tag{26}$$

Note that the F_{Nm} here are *Hansen coefficients* given by

$$F_{Nm} = \frac{1}{\pi}\int\limits_0^\pi \frac{\cos\left[N(E - e\sin E) - mf(E)\right]}{(1-e\cos E)^2}\,\mathrm{d}E. \tag{27}$$

Note E is the eccentric anomaly. This differs from the VLF definition in a few places but is in agreement with Natalia's paper w/ Dong (SD), such that $F_{Nm} = \delta_{Nm}$ for $e = 0$. It bears noting that VLF's formula normalizes to $F_{Nm} = 2\delta_{Nm}$, so we use the restricted domain of integration for numerical speed (the integrand is symmetric since the argument of the cosine is antisymmetric in E, so both the numerator/denominator are even in E).

Let's explicitly write out the $U_{2\pm 2}$, since they are the only ones that contribute to the tidal torque

$$U_{22} = -\frac{GM_2\sqrt{\frac{3\pi}{10}}r^2}{a^3}\sum_{N=1}^{\infty}\left[F_{N2}Y_{22}\left(\theta,\phi\right)e^{-i(N\Omega - 2\Omega_s)t} + F_{-N2}Y_{22}\left(\theta,\phi\right)e^{i(N\Omega+2\Omega_s)t}\right], \tag{28}$$

$$U_{2-2} = -\frac{GM_2\sqrt{\frac{3\pi}{10}}r^2}{a^3}\sum_{N=1}^{\infty}\left[F_{-N2}Y_{2-2}\left(\theta,\phi\right)e^{-i(N\Omega - 2\Omega_s)t} + F_{N2}Y_{2-2}\left(\theta,\phi\right)e^{i(N\Omega+2\Omega_s)t}\right], \tag{29}$$

$$U_{22} + U_{2-2} = -\frac{GM_2\sqrt{\frac{3\pi}{10}}r^2}{a^3}\sum_{N=1}^{\infty}\left[F_{N2}Y_{22}\left(\theta,\phi\right)e^{-i(N\Omega - 2\Omega_s)t} + c.c.\right] \tag{30}$$

We can verify that if the perturbing orbit is circular $e = 0$, then the Hansen coefficient $F_{Nm} = \delta_{Nm}$, and we obtain

$$U_{22} + U_{2-2} = -\frac{GM_2 r^2}{a^3}\sqrt{\frac{6\pi}{5}}\,\mathrm{Re}\left[Y_{22}\left(\theta,\phi\right)e^{-2i(\Omega-\Omega_s)t}\right]. \tag{31}$$

This is the same torque used in KZ. Finally, this yields torque

$$\dot{J} = \tau = T_0\sum_{N=-\infty}^{\infty} F_{N2}^2\,\mathrm{sgn}\left(N\Omega - 2\Omega_s\right)\hat{F}\left(\omega = |N\Omega - 2\Omega_s|\right). \tag{32}$$

2.1 Hansen Coefficients

Maybe someday follow `https://arxiv.org/pdf/1308.0607.pdf` and get the derivation of the Hansen coefficients? One fast way to calculate them is to take an FFT of the $F^{lm} = \left(\frac{r}{a}\right)^l e^{imf}$, per `https://www.aanda.org/articles/aa/pdf/2014/11/aa24211-14.pdf`(CBLR). Basically, the Hansen coefficients are just the FT of the disturbing function. Consider that we want to make jump from

$$U(r,t) = -GM_r r^2\sum_m \frac{W_{2m}}{D(t)^3}e^{-imf(t)}Y_{2m}\left(\theta,\phi\right), \tag{33}$$

to

$$U(r,t) = -\frac{GM_2 r^2}{a^3} \sum_{m,N} W_{2m} F_{Nm}(e) Y_{2m}\left(\theta,\phi\right) e^{-in\Omega t}. \tag{34}$$

Thus, we seek coefficients such that

$$\frac{a^3}{D(t)^3} e^{-imf} = \left(\frac{1+e\cos f}{1-e^2}\right)^3 e^{-imf} = \sum_N F_{Nm} e^{-iN\Omega t}. \tag{35}$$

Thus, it's clear the Hansen coefficients are defined by computing Fourier series coefficients (NB: In hindsight, using $r = a(1-e\cos E)$ probably would have been much faster/easier)

$$F_{Nm} \equiv \frac{1}{T}\int_0^T \frac{e^{-imf}}{\left(1-e^2\right)^3}(1+e\cos f)^3 e^{iN\Omega t}\,\mathrm{d}t, \tag{36}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{e^{-imf}}{\left(1-e^2\right)^3}(1+e\cos f)^3 e^{iN\Omega t}\,\mathrm{d}M \tag{37}$$

We have notated T the period, and M the mean anomaly. Then one just evaluates using $\cos f = \frac{\cos E - e}{1-e\cos E}$ and $M = E - e\sin E$ or more usefully $\mathrm{d}M = (1-e\cos E)\mathrm{d}E$ and obtains

$$F_{Nm} = \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1+e\cos f}{1-e^2}\right)^3 e^{-imf+iN\Omega t}\,\mathrm{d}M, \tag{38}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \left(\frac{1}{1-e\cos E}\right)^3 e^{-imf+iNM}(1-e\cos E)\,\mathrm{d}E, \tag{39}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{\exp\left[i\left(N(E-e\sin E)-mf\right)\right]}{(1-e\cos E)^2}\,\mathrm{d}E. \tag{40}$$

Now, as we observed above, the integrand is symmetric with respect to E, but it had to be, since in an elliptical orbit the first half and second half are obviously symmetric. Thus, we arrive at final expression as promised

$$F_{Nm} = \frac{1}{\pi}\int_0^{\pi} \frac{\cos\left[N(E-e\sin E)-mf(E)\right]}{(1-e\cos E)^2}\,\mathrm{d}E. \tag{41}$$

3 Combined Results

We have been somewhat careful in checking the agreement between the VLF and KZ forms. Note now that Eq. 22 has $\hat{F}$ for a single m contribution, just as $\hat{F}$ is defined in VLF. Thus, we should be able to simply plug in

$$\tau = T_0 \sum_{N=-\infty}^{\infty} F_{N2}^2 \frac{\mathrm{sgn}(N\Omega - 2\Omega_s)}{2} \sigma_c^{8/3} \left[\frac{r_c}{g_c}\left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln R}\right)_{r=r_c}\right]^{-1/3} \frac{\rho_c}{\bar{\rho}_c}\left(1 - \frac{\rho_c}{\bar{\rho}_c}\right)^2 \left[\frac{3}{2}\frac{3^{2/3}\Gamma^2(1/3)}{5\cdot 6^{4/3}}\frac{3}{4\pi}\alpha^2\right], \tag{42}$$

$$= T_0 C(r_c) \sum_{N=-\infty}^{\infty} F_{N2}^2 \,\mathrm{sgn}(N\Omega - 2\Omega_s) \left|N\Omega - 2\Omega_s\right|^{8/3}. \tag{43}$$

Note that now $\sigma_c = |N\Omega - 2\Omega_s| / \sqrt{GM_c/r_c^3}$, and I've defined $C(r_c)$ to be some (dimensional) constant defined at the RCB and does not change with N.

Thus, the relative significance of each N term is given by the summand of Eq. 43, or

$$\tau_N \equiv F_{N2}^2 \,\mathrm{sgn}(N\Omega - 2\Omega_s)\sigma_c^{8/3}, \tag{44}$$

F_{N2} is easiest to evaluate via an integral for now, but can probably be done via a sampling + FFT when speed is necessary (CBLR).

One guess is that the sum is dominated by the contribution of the frequency at pericenter. We can compute pericenter frequency as follows:.

$$r_p^2 \Omega_p = \sqrt{GMa\left(1 - e^2\right)}, \tag{45}$$

$$\Omega_p = \sqrt{\frac{GMa(1+e)(1-e)}{a^4(1-e)^4}}, \tag{46}$$

$$= \Omega \frac{\sqrt{1+e}}{(1-e)^{3/2}}. \tag{47}$$

Thus, we should expect the dominant term to come at $N \sim \frac{\Omega_p}{\Omega} = \frac{\sqrt{1+e}}{(1-e)^{3/2}}$. This indeed very nearly maximizes F_{N2} but probably won't maximize τ_N.

NB: It appears that the maximum N to sum to is (according to Michelle/Chris)

$$N_{\mathrm{max}} = 10\Omega_p. \tag{48}$$

To understand exactly to what N we should sum, we should recall $|\tau_N| \propto |F_{N2}|^2 \left|N\Omega - 2\Omega_s\right|^{8/3}$. If $\Omega_s \gg \Omega_p$, then the latter term σ_c^2 is roughly independent of N and indeed we get that τ_N turns over similarly to where F_{N2} turns over. On the other hand, if $\Omega_s \lesssim \Omega_p$, then $\tau_N \propto |F_{N2}|^2 N^{8/3}$. Plots of these two are given in Fig. 1.

A plot of the actual maximum $F_{N\pm 2}$ is provided below in Fig. 2. Note that F_{N2} seems to be a constant multiple of $(1-e)^{-3/2}$ our prediction, while F_{N-2} seems to be impacted by the shallowness of the fit at smaller eccentricities. That F_{N2} has its maximum at a slight multiple of $(1-e)^{-3/2}$ should not be surprising, since the actual pericenter passage time can be computed by conservation of angular momentum

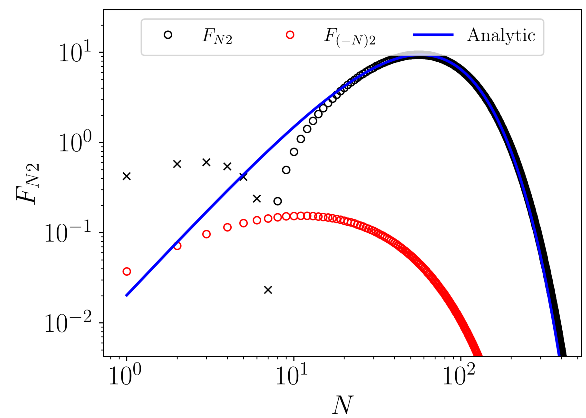

Figure 1: F_{N2} and $F_{N2}N^{8/3}$ as we integrate straightforwardly. Note the vertical blue line is $N = (1-e)^{-3/2}$ while the vertical black/green line are the actual $\mathrm{argmax}_N F_{N\pm2}$ resrpectively.

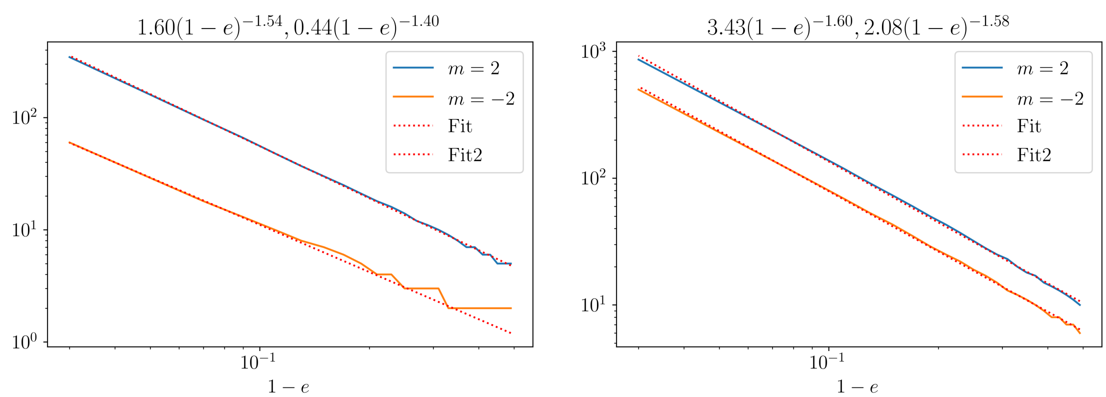

Figure 2: Maxima of $F_{N\pm2}$ and $N^{8/3}F_{N\pm2}$ with fits.

3.1 $m = 2$ Hansen Coefficient Fit

To understand the behavior of how multiplying by $N^{8/3}$ changes the peak of F_{N2}, let's consider the simplest model for the scaling of F_{N2}. Upon examination, it decays as $F_{N2} \propto e^{-aN}$, where $a \approx -\frac{1}{75}$. If we allow a simple model for $F_{N2} \propto N^q e^{-aN}$ (this conforms very coarsely with the plotted F_{N2} and allows for a maximum and an exponential tail), then we can identify that its maximum is at $N_{\max} = q/a$. On the other hand, if we seek the maximum of $N^p F_{N2}$, we find its maximum is instead at $N_{\max}^{(p)} = (p+q)/a$. Comparing the two, we expect the maximum to be shifted by roughly factor $\frac{p+q}{q}$. Since the actual shift is $\gtrsim 2$ for $p = 8/3$, we can guess $q \approx 2$, which is plausible gauging from our loglog plot.

Empirically, we find that the $F_{N\pm2} = CN^p e^{-N/a}$ is actually a surprisingly good fit. This is not surprising: the coefficients must be small for $N \ll N_{\max}$, but must fall off exponentially for $N \gg N_{\max}$. Note that simple calculus shows us that $\mathrm{argmax}_N F_{N\pm2} = p/a$, and so $a = p/N_{\max} \simeq \frac{p\Omega_0}{\Omega_p}$. Furthermore, between Ω, Ω_p, there is no preferred timescale, so a power law dependence between Ω, Ω_p is expected (and $\Omega \ll \Omega_p \approx 0$ for large eccentricities).

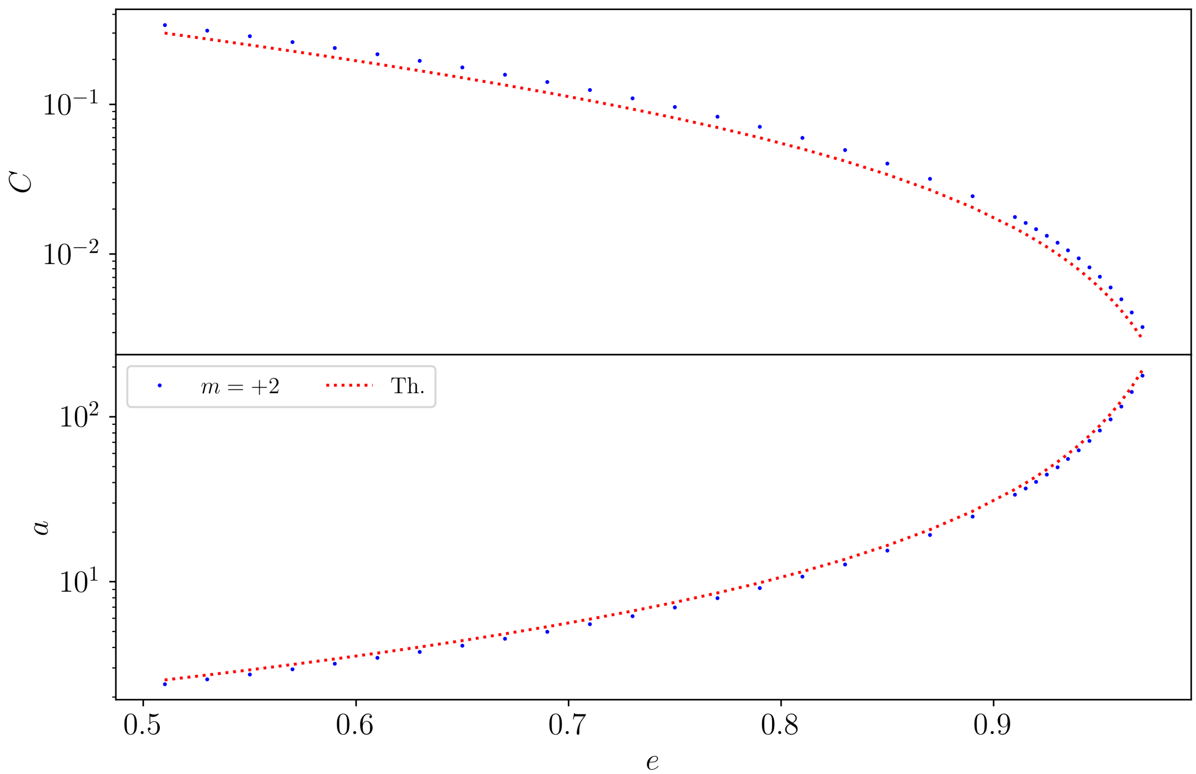

Figure 3: Params of the $P_{N\pm 2} = CN^p e^{-N/a}$ fit as function of eccentricity.

3.1.1 Parameter Scalings

The next thing to check is how robust these parameters are, and whether they have analytical forms. We present such a comparison in Fig. 3. The good news is that p is relatively independent of eccentricity! This gives obvious scaling for

$$a = \frac{N_{\max}}{p} = \frac{\sqrt{1+e}}{p(1-e)^{3/2}}, \tag{49}$$

since I changed convention $P_{n\pm 2} \propto e^{-N/a}$ whoops.

The final difficulty is understanding how C scales. To understand this, the easiest thing to do seems to be to invoke Parseval's Theorem, which will let us claim that

$$\frac{1}{T}\int\limits_0^T \left|\frac{a^3}{D(t)^3}e^{-imf}\right|^2 \mathrm{d}t = \sum_{N=-\infty}^{\infty} |F_{Nm}|^2. \tag{50}$$

Let's assume for now that $F_{N-2} \ll F_{N2}$[1]. Let's further assume that $\mathrm{Re}\,F_{N2} \gg \mathrm{Im}\,F_{N2}$ (no idea if this is

[1]This is empirically true, but also, we can examine the F_{Nm} integral and observe that for large e, $f(E)$ will only be nonzero if $\delta E \lesssim \sqrt{\delta e}$ ($\delta e = 1-e$; examine the arctan relation, but also, $f(E) = \pi$ at apoapsis), then the argument of the cosine is $\left(N\delta e - m/\sqrt{\delta e}\right)\delta E$. Thus, it's quickly oscillating for most N unless $N \sim m/\delta e^{3/2}$, roughly our $N_{\max}$ criterion. Also,

the case yet), then the sum over coefficients is just dominated the contribution from the $C_+ N^{p_+} e^{-N/a_+}$ parts, which we can approximate with an integral analytically.

What about the time-domain integral? Well, this seems to evaluate cleanly

$$\frac{1}{T}\int_0^T \frac{a^6}{D^6}\,\mathrm{d}t = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1+e\cos f}{1-e^2}\right)^6 \frac{\left(1-e^2\right)^{3/2}}{(1+e\cos f)^2}\,\mathrm{d}f, \tag{51}$$

$$= \frac{1}{2\pi\left(1-e^2\right)^{9/2}}\int_0^{2\pi}(1+e\cos f)^4\,\mathrm{d}f, \tag{52}$$

$$= \frac{1}{2\pi\left(1-e^2\right)^{9/2}}\int_0^{2\pi} 1+4e\cos f+6e^2\cos^2 f+4e^3\cos^3 f+e^4\cos^4 f\,\mathrm{d}f, \tag{53}$$

$$= \frac{1}{\left(1-e^2\right)^{9/2}}\left[1+3e^2+\frac{3e^4}{8}\right]. \tag{54}$$

This is analytic and correct (I double checked numerically). The sum of coefficients can be approximated (see later)

$$\sum_{N=-\infty}^{\infty}|F_{Nm}|^2 \approx \sum_{N=0}^{\infty}\left(\mathrm{Re}F_{Nm}\right)^2, \tag{55}$$

$$\approx C_+^2\,(a_+/2)^{2p_++1}\,\Gamma(2p_++1). \tag{56}$$

This agreement is remarkable (`check_ft_parsevals.png`). Thus, explicitly we may write down

$$C_+ \approx \sqrt{\frac{1+3e^2+\frac{3e^4}{8}}{\left(1-e^2\right)^{9/2}}\frac{1}{(a_+/2)^{2p_++1}\,\Gamma(2p_++1)}}. \tag{57}$$

Thus, if we simply take $p_+ = 2$, we have analytical predictions for each of the params. It turns out that we want closer to $p_+ = 2, N_{\max} = \sqrt{2}\Omega_p/\Omega$, but this gives us a very good fit, as seen in Fig. 3.

3.2 $m = 0$ Hansen Coefficients

Consider the $m = 0$ Hansen coefficients, which are given by

$$\frac{a^3}{D(t)^3} = \sum_{N=-\infty}^{\infty} F_{N0}e^{-iN\Omega t}. \tag{58}$$

Again, the tail must be exponential, but since there is obviously symmetry about $N = 0$, the peak is at $N = 0$. We might have naively expected a Gaussian, thanks to the tight peaking in the time domain about $t = 0$, but instead what we observe is something like $F_{N0} \propto e^{-\sqrt{2}N/N_{\rm peri}}$ for some reason.

it is quickly oscillating when N, m have opposite signs.

Assuming this is the case, we can again invoke Parseval's and find

$$\frac{1}{\left(1-e^2\right)^{9/2}}\left[1+3e^2+\frac{3e^4}{8}\right] \approx 2\int_0^\infty C^2 e^{-\frac{N2\sqrt{2}}{N_{\mathrm{peri}}}}\,\mathrm{d}N, \tag{59}$$

$$\approx 2C^2 N_{\mathrm{peri}}\sqrt{2}, \tag{60}$$

$$C^2 = \frac{1+3e^2+\frac{3e^4}{8}}{\left(1-e^2\right)^{9/2}}\frac{1}{2N_{\mathrm{peri}}\sqrt{2}}. \tag{61}$$

3.3 Effective Torque

With this approximate closed form, it is very easy to sum over all N by approximating as an integral:

$$\sum_{N=1}^{\infty} N^q F_{N\pm 2} \approx \int_0^\infty C N^{p+q} e^{-N/a}\,\mathrm{d}N \tag{62}$$

The RHS is almost a Gamma function though!! Thus

$$\sum_{N=1}^{\infty} N^q F_{N\pm 2} \approx C a^{p+q+1}\int_0^\infty (N/a)^{p+q} e^{-N/a}\,\mathrm{d}(N/a), \tag{63}$$

$$\approx C a^{p+q+1}\Gamma(p+q+1). \tag{64}$$

Thus, we return to our total torque. Call $F_{N\pm 2} = C_\pm N^{p_\pm} e^{-N/a_\pm} = F_{\pm N2}$, then

$$\tau = T_0 C(r_c)\sum_{N=-\infty}^{\infty} F_{N2}^2 \,\mathrm{sgn}(N\Omega - 2\Omega_s)\sigma_c^{8/3}, \tag{65}$$

$$= T_0 C(r_c)\frac{\Omega^{8/3}}{\left(GM_c/r_c^3\right)^{4/3}}\sum_{N=-\infty}^{\infty} F_{N2}^2 \,\mathrm{sgn}\left(N - 2\frac{\Omega_s}{\Omega}\right)\left|N - 2\frac{\Omega_s}{\Omega}\right|^{8/3}, \tag{66}$$

$$= T_0 \hat{C}(r_c)\sum_{N=0}^{\infty}\left[F_{N2}^2 \,\mathrm{sgn}\left(N - 2\frac{\Omega_s}{\Omega}\right)\left|N - 2\frac{\Omega_s}{\Omega}\right|^{8/3} - F_{-N2}^2\left|-N - 2\frac{\Omega_s}{\Omega}\right|^{8/3}\right], \tag{67}$$

$$\approx T_0 \hat{C}(r_c)\left[\int_0^\infty C_+^2 N^{2p_+} e^{-2N/a_+}\,\mathrm{sgn}\left(N - 2\frac{\Omega_s}{\Omega}\right)\left|N - 2\frac{\Omega_s}{\Omega}\right|^{8/3}\mathrm{d}N - \int_0^\infty C_-^2 N^{2p_-} e^{-2N/a_-}\left|-N - 2\frac{\Omega_s}{\Omega}\right|^{8/3}\mathrm{d}N\right]. \tag{68}$$

Note that the summation should not double count the F_{02} term; this is resolved correctly in the integral, where the $N=0$ contribution is not double counted. It's convenient to call

$$\hat{\tau}_N \equiv F_{N2}^2 \,\mathrm{sgn}\left(N - 2\frac{\Omega_s}{\Omega}\right)\left|N - 2\frac{\Omega_s}{\Omega}\right|^{8/3}. \tag{69}$$

At this point, let's specialize to two regimes:

- Let $2\Omega_s \ll N_{\max}\Omega$, then the sign term is just the sign of N, and $\left|\pm N - \frac{2\Omega_s}{\Omega}\right| \sim |N|\left(1 - \frac{\Omega_s/\Omega}{N_{peri}}\right)$.

This gives (dropping the C_- terms since we've seen they're unimportant and we don't have fits for them)

$$\tau \approx T_0 \hat{C}(r_c)\left(1 - \frac{\Omega_s/\Omega}{N_{peri}}\right)^{8/3}\left[C_+^2 \int_0^\infty N^{2p_+ + 8/3} e^{-2N/a_+}\,\mathrm{d}N\right], \tag{70}$$

$$\approx T_0 \hat{C}(r_c)\left[C_+^2 (a_+/2)^{2p_+ + 11/3}\Gamma(2p_+ + 11/3)\right]. \tag{71}$$

- Alternatively, let $2\Omega_s \gg N_{\max}\Omega$, then the sign is just always negative and $\left|N - \frac{2\Omega_s}{\Omega}\right| = \frac{2\Omega_s}{\Omega} - N_{\max}$, and we find

$$\tau \approx T_0 \hat{C}(r_c)\left(\frac{2\Omega_s}{\Omega} - N_{\max}\right)^{8/3}\left[-C_+^2 \int_0^\infty N^{2p_+} e^{-2N/a_+}\,\mathrm{d}N\right], \tag{72}$$

$$\approx T_0 \hat{C}(r_c)\left(\frac{2\Omega_s}{\Omega} - N_{\max}\right)^{8/3}\left[-C_+^2 (a_+/2)^{2p_+ + 1}\Gamma(2p_+ + 1)\right]. \tag{73}$$

Indeed, for $2\Omega_s \ll N_{\max}\Omega$, we find $\tau > 0$, while for $2\Omega_s \gg N_{\max}\Omega$, we obtain $\tau < 0$, which obeys intuition and suggests some synchronization frequency around $2\Omega_s \simeq \Omega_p$.

We make some plots as in Fig. 4 and these agree. What remains is to identify the eccentricity scaling, which will require understanding how the fit coefficients depend on e, which we've also plotted but don't quite yet understand so don't include.

3.4 Heating

Heating in the inertial frame is given (absorbed the second N into the absolute value exponent)

$$\dot{E}_{in} = \frac{1}{2}\hat{T}(r_c, \Omega)\left[\sum_{N=-\infty}^{\infty} N\Omega F_{N2}^2 \,\mathrm{sgn}(\sigma)|\sigma|^{8/3} + \left(\frac{W_{20}}{W_{22}}\right)^2 \Omega F_{N0}^2 |N|^{11/3}\right]. \tag{74}$$

I'm lazy so $\sigma = N - \frac{2\Omega_s}{\Omega}$, and $(W_{20}/W_{22})^2 = 2/3$.

The former term follows our work above, for instance in the $\Omega_s \approx 0$ limit, we just have the first term (subscript)

$$\dot{E}_1 = \frac{\hat{T}\Omega}{2} C^2 \left(\frac{\eta}{2}\right)^{26/3}\Gamma(26/3), \tag{75}$$

or in the $\Omega_s \gg 0$ case

$$\dot{E}_1 = \frac{\hat{T}\Omega}{2} C^2 \left(\frac{2\Omega_s}{\Omega} - N_{\max}\right)^{8/3} (\eta/2)^6 \Gamma(6). \tag{76}$$

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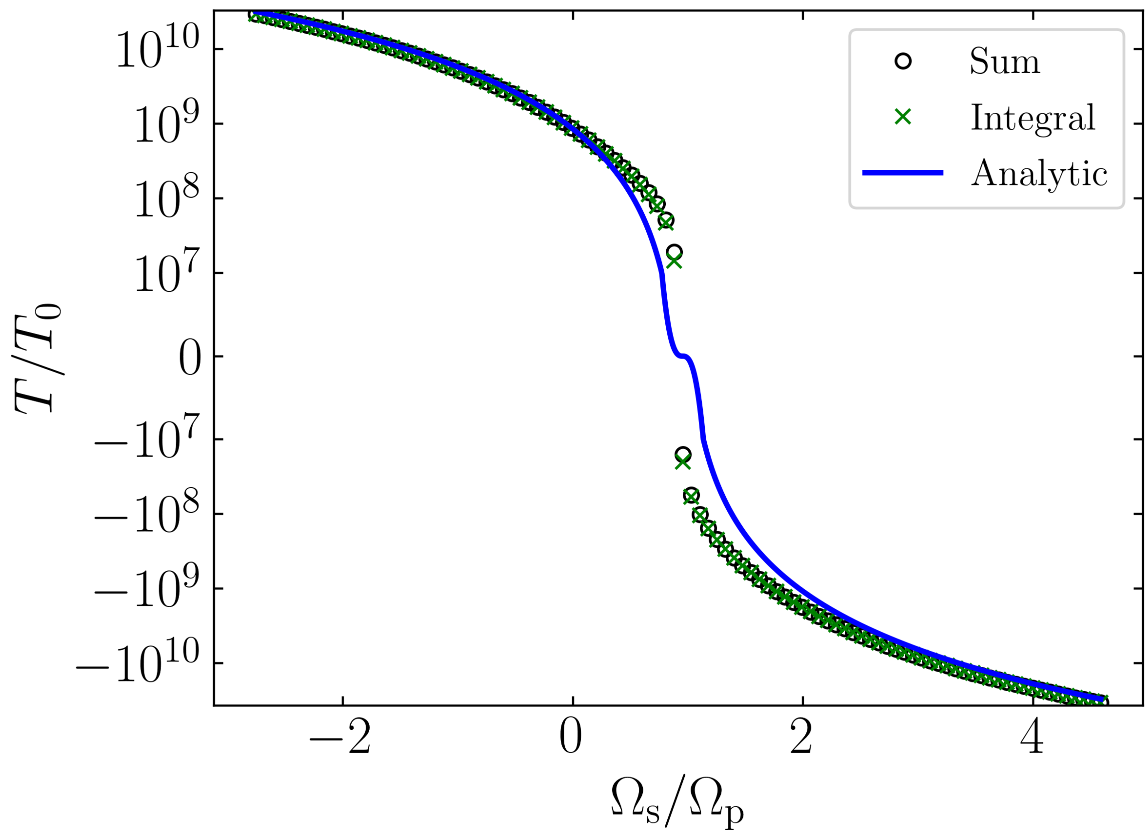

Figure 4: Plot of $\frac{\tau}{T_0 \hat{C}(r_c)}$. The predictions in the two regimes are the black horizontal line $\Omega_s = 0$ and red dashed line $2\Omega_s \gg N_{\max}\Omega$. Good agreement!

For the latter term, it is symmetric about N, so we can integrate about just $N \geq 0$:

$$\dot{E}_2 = \hat{T}\Omega \frac{2}{3} \int\limits_0^\infty C_0^2 e^{-\frac{N}{N_{\rm peri}} 2\sqrt{2}} N^{11/3}\, \mathrm{d}N, \tag{77}$$

$$= \hat{T}\Omega \frac{2}{3} C_0^2 \left(\frac{N_{\rm peri}}{2\sqrt{2}}\right)^{14/3} \Gamma(14/3). \tag{78}$$

4 Comparison with Existing Work, Moving Past Massive Stars

Key results are Eq. (101), Eq. (124), and Eq. (137). Note that the technique used to constrain the prefactor on the $\Omega_s \ll N_{\max}\Omega$ is general and necessary for any of these results where we need to expand $|N\Omega - m\Omega_s|^p$ in two regimes.

4.1 Weak Tides w/ Hut, Storch and Vick: Updating F_{N2}

In the aforementioned papers, results of form

$$\frac{1}{2}\sum_N F_{N2}^2\left(N - m\frac{\Omega_s}{\Omega}\right) = \frac{1}{\left(1-e^2\right)^6}\left[f_2 - \left(1-e^2\right)^{3/2} f_5\frac{\Omega_s}{\Omega}\right], \tag{79}$$

are obtained, where $m = 2$, where $f_5 = 1 + 3e^2 + 3e^4/8$ and $f_2 = 1 + 15e^2/2 + 45e^4/8 + 5e^6/16$ are the relevant numbers. In the latter term, the Parseval's result is simply reproduced exactly and is in agreement with above:

$$\sum_N -F_{N2}^2\frac{\Omega_s}{\Omega} = \left(-\frac{\Omega_s}{\Omega}\right)\left(\frac{1+3e^2+3e^4/8}{\left(1-e^2\right)^{9/2}}\right). \tag{80}$$

Recall that this means that

$$C^2 = \frac{f_5}{\left(1-e^2\right)^{9/2}}\frac{1}{4!\left(\eta/2\right)^5} \tag{81}$$

The more interesting case is the linear term. An extended application of Parseval's Theorem is in play, where the two functions are not the same. We then must use

$$\frac{\mathrm{d}}{\mathrm{d}(-i\Omega t)}\frac{a^3}{D^3}e^{-imf} = \sum_N F_{Nm}Ne^{-iN\Omega t}, \tag{82}$$

as then (star denotes conjugate)

$$\sum_N F_{N2}^2 N = \frac{1}{T}\int \frac{a^3}{D^3}e^{-imf}\left(\frac{\mathrm{d}}{\mathrm{d}(-i\Omega t)}\left(\frac{a^3}{D^3}e^{-imf}\right)\right)^* \mathrm{d}t, \tag{83}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{a^3}{D^3}e^{-imf}\left(i\frac{\mathrm{d}}{\mathrm{d}f}\left(\frac{a^3}{D^3}e^{-imf}\right)\right)^* \mathrm{d}f, \tag{84}$$

$$= \frac{1}{2\pi}\int_0^{2\pi} \frac{a^6}{D^6}\left(m + \frac{3e\sin f}{1+e\cos f}\right)\mathrm{d}f. \tag{85}$$

Recall that $a/D = (1+e\cos f)/\left(1-e^2\right)$, so

$$\sum_N F_{N2}^2 N = \frac{1}{2\pi\left(1-e^2\right)^6}\int_0^{2\pi}(1+e\cos f)^6\left(m + i\frac{3e\sin f}{1+e\cos f}\right)\mathrm{d}f. \tag{86}$$

The $\sin f$ term integrates to zero, and setting $m = 2$ cancels with the prefactor, and we can explicitly integrate (only even powers survive)

$$\frac{1}{2}\sum_N F_{N2}^2 N = \frac{1}{2\pi\left(1-e^2\right)^6}\int_0^{2\pi}\left(1+15e^2\cos^2 f+15e^4\cos^4 f+e^6\cos^6 f\right)\,\mathrm{d}f, \tag{87}$$

$$= \frac{1}{\left(1-e^2\right)^6}\left(1+\frac{15e^2}{2}+\frac{45e^4}{8}+\frac{5e^6}{16}\right). \tag{88}$$

Note that we have used the following identities:

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^2 x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+\cos 2x}{2}\,\mathrm{d}x = \frac{1}{2}, \tag{89}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^4 x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1+\cos 2x}{2}\right)^2\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+\cos^2 2x}{4}+(\dots)\,\mathrm{d}x = \frac{3}{8}, \tag{90}$$

$$\frac{1}{2\pi}\int_0^{2\pi}\cos^6 x\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\left(\frac{1+\cos 2x}{2}\right)^3\,\mathrm{d}x = \frac{1}{2\pi}\int_0^{2\pi}\frac{1+3\cos^2 2x}{8}+(\dots) = \frac{5}{16}. \tag{91}$$

Now, how does our fitted formula compare to this? Well, we really only have to compare the $F_{N2}^2 N$ term, so

$$\sum_N F_{N2}^2 N \approx \int_0^\infty C^2 N^4 e^{-2N/\eta} N\,\mathrm{d}N, \tag{92}$$

$$= \frac{f_5}{\left(1-e^2\right)^{9/2}}\frac{1}{(\eta/2)^5\,4!}\int_0^\infty N^5 e^{-2N/\eta}\,\mathrm{d}N, \tag{93}$$

$$= \frac{f_5}{\left(1-e^2\right)^{9/2}}\frac{1}{(\eta/2)^5\,4!}\left(\frac{\eta}{2}\right)^6\int_0^\infty\left(\frac{2N}{\eta}\right)^5 e^{-2N/\eta}\,\mathrm{d}\frac{2N}{\eta}, \tag{94}$$

$$= \frac{f_5}{\left(1-e^2\right)^{9/2}}\frac{1}{(\eta/2)^5\,4!}\left(\frac{\eta}{2}\right)^6\int_0^\infty\left(\frac{2N}{\eta}\right)^5 e^{-2N/\eta}\,\mathrm{d}\frac{2N}{\eta}, \tag{95}$$

$$= \frac{f_5}{\left(1-e^2\right)^{9/2}}\frac{1}{(\eta/2)^5\,4!}\left(\frac{\eta}{2}\right)^6 5!, \tag{96}$$

$$= \frac{5f_5\eta}{2\left(1-e^2\right)^{9/2}}. \tag{97}$$

So we match forms for this first moment we find

$$\eta = \frac{4f_2}{5f_5\left(1-e^2\right)^{3/2}} \tag{98}$$

Recall that we guessed $\eta = \frac{N_{\max}}{2} = \frac{\sqrt{1+e}}{2(1-e)^{3/2}}\alpha$ in subsection 3.1, where we guessed $\alpha \approx \sqrt{2}$. Note that η

indeed diverges as $(1-e)^{-3/2}$, and the prefactor is of order unity (e is bound).

Note that the power of our approximation comes in evaluating expressions as follows (recall $\alpha \approx 2(1+e)$)

$$\sum_{N=-\infty}^{\infty} F_{N2}^2 N^q \approx \int\limits_0^{\infty} C^2 N^4 e^{-2N/\eta} N^q \, \mathrm{d}N, \tag{99}$$

$$= C^2 \left(\frac{\eta}{2}\right)^{5+q} \Gamma(5+q), \tag{100}$$

$$= \frac{f_5}{\left(1-e^2\right)^{9/2}} \frac{\Gamma(5+q)}{4!} \left(\frac{N_{\max}}{4}\right)^q, \tag{101}$$

4.2 Also Updating F_{N0}

Before, we postulated that $F_{N0} \propto e^{-\sqrt{2}|N|/N_{peri}}$, but we can do better now, with updated knowledge. Note that

$$\sum_N F_{Nm}^2 = \frac{1}{T} \int_0^T \frac{a^3}{D^3} e^{-imf} \frac{a^3}{D^3} e^{+imf} \, \mathrm{d}t. \tag{102}$$

This makes it obvious that $\sum\limits_N F_{N2}^2 = \sum\limits_N F_{N0}^2 = f_5 / \left(1-e^2\right)^{9/2}$. Taking our fitting formula to be of form $F_{N0} = Ce^{-|N|/\eta}$, we find

$$\int\limits_{-\infty}^{\infty} F_{N0}^2 \, \mathrm{d}N = C^2 \left(\frac{\eta}{2}\right). \tag{103}$$

Furthermore, the first moment is also easier to write down

$$\sum_N F_{N0}^2 N = \frac{1}{T} \int \frac{a^3}{D^3} \left(\frac{\mathrm{d}}{\mathrm{d}(-i\Omega t)} \frac{a^3}{D^3}\right) \mathrm{d}t, \tag{104}$$

$$= \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{a^6}{D^6} \frac{3e \sin f}{1+e\cos f} \, \mathrm{d}f. \tag{105}$$

This vanishes, per our earlier analysis, and this is no suprise, since we found that F_{N0} are symmetric, so there can be no first moment. The second moment is easiest to compute via

$$\sum_N F_{N0}^2 N^2 = \frac{1}{2\pi} \int\limits_0^{2\pi} \frac{a^6}{D^6} \left(\frac{3e\sin f}{1+e\cos f}\right)^2 \frac{\mathrm{d}f}{\mathrm{d}t} \, \mathrm{d}f, \tag{106}$$

$$= \frac{1}{2\pi \left(1-e^2\right)^6} \int\limits_0^{2\pi} (1+e\cos f)^4 (3e\sin f)^2 \frac{(1+e\cos f)^2}{\left(1-e^2\right)^{3/2}} \, \mathrm{d}f, \tag{107}$$

$$= \frac{1}{2\pi \left(1-e^2\right)^{15/2}} \int\limits_0^{2\pi} \left(1 + 15e^2\cos^2 f + 15e^4 \cos^4 f + e^6 \cos^6 f\right) (3e\sin f)^2 \, \mathrm{d}f, \tag{108}$$

$$= \frac{9e^2}{\left(1-e^2\right)^{15/2}} \left(\frac{1}{2} + \frac{15e^2}{8} + \frac{15e^4}{16} + \frac{5e^6}{128}\right) \equiv \frac{9e^2}{2\left(1-e^2\right)^{15/2}} f_3. \tag{109}$$

I just Mathematica'd the trig identities, nothing magical going on. This is numerically checked and in agreement w/ Michelle's result as well.

We can also look at the second moment of our fitting formula. Integrating twice by parts lets us evaluate

$$\int_{-\infty}^{\infty} C^2 e^{-2|N|/\eta} N^2 \, \mathrm{d}N = 2C^2 \int_0^{\infty} e^{-2N/\eta} N^2 \, \mathrm{d}N, \tag{110}$$

$$= 4C^2 \left(\frac{\eta}{2}\right)^3. \tag{111}$$

The latter follows from the gamma function definition of course (didn't see this at first lol).

We now have two constraints (0th and 2nd moment of F_{N0}) for the two variables C, η, and therefore we find that

$$C^2\eta = f_5 / \left(1 - e^2\right)^{9/2}, \tag{112}$$

$$C^2 \frac{\eta^3}{2} = \frac{9e^2}{2\left(1 - e^2\right)^{15/2}} f_3, \tag{113}$$

$$\eta^2 = \frac{9e^2 f_3}{\left(1 - e^2\right)^3 f_5}, \tag{114}$$

$$C^2 = \frac{f_5}{\eta \left(1 - e^2\right)^{9/2}}. \tag{115}$$

We can verify that as $e \to 1$, we have the scaling $\eta \propto (1-e)^{3/2}$, which probably is what told us $\eta \sim N_{peri}$.

4.3 New Closed Forms for Torque

We can now update our torque formula. Recall we want to sum

$$\tau = T_0 \hat{C}(r_c) \sum_{N=-\infty}^{\infty} F_{N2}^2 \operatorname{sgn}\left(N - 2\frac{\Omega_s}{\Omega}\right) \left|N - 2\frac{\Omega_s}{\Omega}\right|^{8/3}. \tag{116}$$

Note that $\hat{C}(r_c)$ was defined a while ago to be dimensionless quantity

$$\hat{C}(r_c) \equiv \frac{1}{2} \left[\frac{r_c}{g_c} \left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln R}\right)_{r=r_c}\right]^{-1/3} \frac{\rho_c}{\bar{\rho}_c} \left(1 - \frac{\rho_c}{\bar{\rho}_c}\right)^2 \left[\frac{3}{2} \frac{3^{2/3}\Gamma^2(1/3)}{5 \cdot 6^{4/3}} \frac{3}{4\pi} \alpha^2\right]. \tag{117}$$

We can drop everything but the 1/2 and the density terms under Kushnir's suggestion.

Let's just consider the summation, which we can call $\hat{\tau}$

$$\hat{\tau} \equiv \sum_{N=-\infty}^{\infty} F_{N2}^2 \operatorname{sgn}\left(N - 2\frac{\Omega_s}{\Omega}\right) \left|N - 2\frac{\Omega_s}{\Omega}\right|^{8/3}. \tag{118}$$

Again, let's consider the two limits (recall $\eta = N_{\max}/2$):

- Consider $2\Omega_s \gg N_{\max}\Omega$, then the sign is always negative and $|N - 2\Omega_s/\Omega| \approx \left|\frac{2\Omega_s}{\Omega} - N_{\max}\right|$ and

$$\hat{\tau} \approx -\left|\frac{2\Omega_s}{\Omega} - N_{\max}\right|^{8/3} \sum_{N=-\infty}^{\infty} F_{N2}^2, \tag{119}$$

$$\approx -(N_{\max})^{8/3}\left|1 - \frac{2\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{f_5}{\left(1-e^2\right)^{9/2}}. \tag{120}$$

- Consider $2\Omega_s \ll N_{\max}\Omega$, then the sign is the sign of $N_{\max}$ (> 0). We choose a factorization for $\left|N - \frac{2\Omega_s}{\Omega}\right| \approx |N|\left|1 - \frac{\beta\Omega_s}{N_{\max}\Omega}\right|$ such that we can express as a prefactor and a summation. We must choose it such that in the limit $\Omega_s \to -\infty$, the asymptotics are correct, that is:

$$\sum_N F_{N2}^2 |2\Omega_s|^{8/3} = \left|\frac{\beta\Omega_s}{N_{\max}\Omega}\right|^{8/3} \sum_N F_{N2}^2 \left(\frac{N_{\max}}{4}\right)^{8/3}. \tag{121}$$

  This can be solved to find (I did this on scratch paper, see the calculation in heating for a solved example) that $\beta = 2^3/(\Gamma(23/3)/4!)^{3/8} \approx 1.38$. Thus,

$$\hat{\tau} \approx \left|1 - 1.38\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \sum_{N=-\infty}^{\infty} F_{N2}^2 |N|^{8/3}, \tag{122}$$

$$\approx \left|1 - 1.38\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{f_5}{\left(1-e^2\right)^{9/2}} \frac{\Gamma(23/3)}{4!}\left(\frac{N_{\max}}{4}\right)^{8/3}. \tag{123}$$

Thus, we arrive at final answer

$$\hat{\tau} \approx \frac{f_5}{\left(1-e^2\right)^{9/2}} (N_{\max})^{8/3} \times \begin{cases} \left|1 - 1.38\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{\Gamma(23/3)}{4!}\left(\frac{1}{4}\right)^{8/3}, & \Omega_s < N_{\max}\Omega/2, \\ -\left|1 - 2\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}, & \Omega_s > N_{\max}\Omega/2. \end{cases} \tag{124}$$

Note that the reason we have these two different forms is basically because I didn't want three components to the piecewise function. But it should be obvious that the two regimes $|\Omega_s| \gg |N_{\max}\Omega|$ should be symmetric, and this is what fixes the prefactor on the $\Omega_s/N_{\max}\Omega$ term inside the absolute value... no need to fully compute the asymptotic behavior.

4.4 And Heating

In the case of the heating, we recall

$$\dot{E}_{in} = \frac{1}{2}\hat{T}(r_c, \Omega)\left[\sum_{N=-\infty}^{\infty} N\Omega F_{N2}^2 \operatorname{sgn}(\sigma)|\sigma|^{8/3} + \left(\frac{W_{20}}{W_{22}}\right)^2 \Omega F_{N0}^2 |N|^{11/3}\right], \tag{125}$$

where we write $\sigma = N - \frac{2\Omega_s}{\Omega}$ and know $(W_{20}/W_{22})^2 = 2/3$.

Let's handle the second term first, since it's on our minds; subscript

$$\frac{\dot{E}_2}{\hat{T}\Omega} = \frac{1}{3}\sum_{N=-\infty}^{\infty} F_{N0}^2 \,|N|^{11/3}, \tag{126}$$

$$= \frac{2}{3}\int_0^\infty C^2 e^{-2N/\eta} N^{11/3}\,\mathrm{d}N, \tag{127}$$

$$= \frac{2C^2}{3}\left(\frac{\eta}{2}\right)^{14/3}\Gamma(14/3), \tag{128}$$

$$= \frac{f_5\Gamma(14/3)}{3\left(1-e^2\right)^{9/2}}\left(\frac{9e^2 f_3}{2\left(1-e^2\right)^3 f_5}\right)^{11/6}. \tag{129}$$

The first term is the same procedure as for the torque, where we must expand and match consistency. The basic form is

$$\frac{\dot{E}_1}{\hat{T}\Omega} = \frac{1}{2}\sum_{N=-\infty}^{\infty} N F_{N2}^2 \mathrm{sgn}(\sigma)\,|\sigma|^{8/3}. \tag{130}$$

Recall $\sigma \equiv N - \frac{2\Omega_s}{\Omega}$. Thus, we calculate:

- In the limit where $\Omega_s \gg N_{\max}\Omega$, then the sign is always negative and $|N - 2\Omega_s/\Omega| \approx |2\Omega_s/\Omega - N_{\max}|$ and we have straightforwardly

$$\frac{\dot{E}_1}{\hat{T}\Omega} = -\frac{1}{2}\left|\frac{2\Omega_s}{\Omega} - N_{\max}\right|^{8/3}\sum_{N=-\infty}^{\infty} N F_{N2}^2, \tag{131}$$

$$= -\frac{1}{2}\left|1 - \frac{2\Omega_s}{N_{\max}\Omega}\right|^{8/3} N_{\max}^{8/3}\frac{f_5}{\left(1-e^2\right)^{9/2}} 5\left(\frac{N_{\max}}{4}\right). \tag{132}$$

- Again, in the other limit where $2\Omega_s \ll N_{\max}\Omega$, we seek some factorization + summation $\left|N - \frac{2\Omega_s}{\Omega}\right| \approx |N|\left|1 - \frac{\beta\Omega_s}{N_{\max}\Omega}\right|$ such that the asymptotics are correct:

$$\left|\frac{2\Omega_s}{\Omega}\right|^{8/3}\sum_N F_{N2}^2 N = \left|\frac{2\Omega_s}{\beta N_{\max}\Omega}\right|^{8/3}\sum_N F_{N2} N^{11/3}. \tag{133}$$

Using Eq. 101 and keeping only the q dependent parts, we obtain (recall $N_{\max} = \alpha\frac{\sqrt{1+e}}{(1-e)^{3/2}}$)

$$\beta = 8/\left(\frac{\Gamma(26/3)}{5!}\right)^{3/8} \approx 1.1772 \tag{134}$$

Thus, we can now approximate the sum

$$\frac{\dot{E}_1}{\hat{T}\Omega} = \frac{1}{2}\left|1 - 1.1772\frac{\Omega_s}{\Omega N_{\max}}\right|^{8/3}\sum_{N=-\infty}^{\infty}|N|^{11/3}F_{N2}^2, \tag{135}$$

$$= \frac{1}{2}\left|1 - 1.1772\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}\frac{f_5}{\left(1-e^2\right)^{9/2}}\frac{\Gamma(26/3)}{4!}\left(\frac{N_{\max}}{4}\right)^{11/3}. \tag{136}$$

Thus, we have total energy transfer rate (in the inertial frame, this indeed can be signed)

$$\begin{aligned}\frac{\dot{E}_{in}}{\hat{T}\Omega} &= \frac{f_5\Gamma(14/3)}{3\left(1-e^2\right)^{10}}\left(\frac{9e^2 f_3}{2f_5}\right)^{11/6}\\ &+\frac{N_{\max}^{11/3}}{8}\frac{f_5}{\left(1-e^2\right)^{9/2}}\Gamma(14/3)\times\\ &\begin{cases}\left|1-1.1772\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}\frac{\Gamma(26/3)}{4!}\left(\frac{1}{4}\right)^{8/3} & \Omega_s < N_{\max}\Omega/2,\\ -\left|1-2\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}5. & \Omega_s > N_{\max}\Omega/2.\end{cases}\end{aligned}\tag{137}$$

4.5 Being Brave and Useless: Expanding Non-integer Powers

Let's just drop $\alpha \approx 4$ for the time being, so

$$\sum_{N=-\infty}^{\infty} F_{N2}^2 N^q \approx \frac{f_5}{\left(1-e^2\right)^{(9+3q)/2}}\frac{\Gamma(5+q)}{4!}(1+e)^{q/2}.\tag{138}$$

One fun question is: can we do better than the piecewise approximation that we proposed above for the $N = 8/3$ case, via the generalized binomial theorem? For reference, this is:

$$(a+b)^n = \sum_{k=0}^{\infty}\binom{n}{k}a^k b^{n-k},\tag{139}$$

where $\binom{n}{k} \equiv \frac{n(n-1)\dots(n-k+1)}{k!}$ for integer k and not necessarily integer n (and thus, not necessarily positive $n-k+1$).

Let's drop all the signs and absolute value nonsense, in which case we are attempting to evaluate (where $m' \equiv m\Omega_s/\Omega$ for convenience)

$$\begin{aligned}\smiley \equiv& \sum_{N=-\infty}^{\infty} F_{N2}^2\left(N-m'\right)^{8/3}, &(140)\\ =& \sum_{N=-\infty}^{\infty} F_{N2}^2\left(N^{8/3}+\frac{8}{3}N^{5/3}m'+\frac{(8/3)(5/3)}{2}N^{2/3}(m')^2+\dots\right), &(141)\\ =& \frac{f_5}{\left(1-e^2\right)^{9/2}4!}\left[\frac{\Gamma(5+8/3)(1+e)^{(8/3)/2}}{\left(1-e^2\right)^{8/2}}+\frac{\frac{8}{3}\Gamma(5+5/3)(1+e)^{(5/3)/2}m'}{\left(1-e^2\right)^{5/2}}+\frac{\frac{(8/3)(5/3)}{2}\Gamma(5+2/3)(1+e)^{(2/3)/2}(m')^2}{\left(1-e^2\right)^{2/2}}+\dots\right]. &(142)\end{aligned}$$

This distinction only makes sense obviously when $m' \approx N_{\max} \sim \frac{2\sqrt{1+e}}{\left(1-e^2\right)^{3/2}}$, so let's go ahead and set $m' = N_{\max}\beta$; we want eventually to examine asymptotic behavior for $\beta \to^{\pm} 1$ or some similarly critical value. This gives

$$☺ = \frac{f_5}{\left(1-e^2\right)^{9/2} 4!} \sum_{k=0}^{\infty} \binom{8/3}{k} \Gamma\left(\frac{23}{3}-k\right) \frac{(1+e)^{(8/3-k)/2}}{\left(1-e^2\right)^{(8-3k)/2}} \left(-2\beta\right)^k \frac{(1+e)^{k/2}}{\left(1-e^2\right)^{3k/2}}, \tag{143}$$

$$= \frac{f_5 (1+e)^{8/3}}{\left(1-e^2\right)^{17/2} 4!} \sum_{k=0}^{\infty} \binom{8/3}{k} \Gamma\left(\frac{23}{3}-k\right) \left(-2\beta\right)^k. \tag{144}$$

The trick to proceed here is to handle negative arguments to the Gamma function with the Euler reflection formula $\Gamma(z)\Gamma(1-z) = \pi/\sin(\pi z)$. Dropping the first few terms that don't have this problem (they're finite, we can return to them), we then have

$$☺ - ☺_6 = \frac{f_5 (1+e)^{8/3}}{\left(1-e^2\right)^{17/2} 4!} \sum_{k=7}^{\infty} \binom{8/3}{k} \frac{\pi}{\sin(\pi(23/3-k))\Gamma(k-20/3)} \left(-2\beta\right)^k, \tag{145}$$

$$= \frac{f_5 (1+e)^{8/3}}{\left(1-e^2\right)^{17/2} 4!} \sum_{k=7}^{\infty} \binom{8/3}{k} \frac{\pi}{-\sqrt{3}/2(-1)^k \Gamma(k-20/3)} \left(-2\beta\right)^k, \tag{146}$$

$$= -\frac{f_5 (1+e)^{8/3}}{\left(1-e^2\right)^{17/2} 4!} \sum_{k=7}^{\infty} \binom{8/3}{k} \frac{2\pi/\sqrt{3}}{\Gamma(k-20/3)} \left(2\beta\right)^k. \tag{147}$$

Asymptotically, $\binom{8/3}{k}$ flips signs for every k, but each successive term is being multiplied by a number that gets closer to unity. The convergence is dominated by the gamma function in the denominator, which scales like a factorial. The sum thus converges to something in the vicinity of

$$\sum_{k=7}^{\infty} \binom{8/3}{k} \frac{2\pi/\sqrt{3}}{\Gamma(k-20/3)} \left(2\beta\right)^k \lesssim 2\pi/\sqrt{3}(2\beta)^7 e^{-2\beta}. \tag{148}$$

Is this enlightening at all? Well, there is some critical $\beta_c \sim 1$ (corresponding to some critical $\Omega_{s,c}$) for which ☺ and thus the torque is zero, the pseudosynchronizated spin rate. For β just below/above this β_c, I guess we can see that the torque grows exponentially, before the dominant power law behavior takes over, from the piecewise approximations. And this was thoroughly useless, albeit pretty fun. Note that the signs do work out, a pretty big surprise to me at least.

The original hope would have been to determine how near pseudosynchronization our piecewise approximation breaks down, but this seems somewhat hard. We probably can guesstimate by looking at the values of β for which the above is either half of its maximum value. We know it is maximized for $\beta \sim 3.5$, and the half max apparently happens around 2.2 or 5.2, so our power law approximation should only be good to within about 70% of the pseudosynchronization frequency.

4.6 Discrete Modes

This is very tenuous, let's give it a shot. When we have discrete modes (cf. Michelle's paper 2019, VL19), we generally have torques of form

$$\tau \propto \sum_{\alpha N} \frac{\gamma F_{Nm}^2 \omega_{Nm}^2}{(\omega_\alpha - \omega_{Nm})^2 + \Gamma_\alpha^2}. \tag{149}$$

Consider just the torque on a single mode for now, so take the sum only over N not α. Let's furthermore assume that the eccentricity is sufficiently large such that Ω is much slower than all other scales of interest, and we have a continuum N of driving frequencies. Then ($\hat{\tau}_\alpha$ drops all proportionality factors)

$$\tau_\alpha \approx \int\limits_{-\infty}^{\infty} \frac{\gamma F_{Nm}^2 \omega_{Nm}^2}{(\omega_\alpha - \omega_{Nm})^2 + \Gamma_\alpha^2}\,\mathrm{d}N. \tag{150}$$

Note that $\omega_{Nm} \equiv N\Omega - m\Omega_s$.

If we further assume that Γ_α is small, the Lorentzion in the denominator might be approximated by a delta function. Its height is fixed by normalization

$$\int\limits_{-\infty}^{\infty} \frac{1}{(\omega_\alpha - \omega_{Nm})^2 + \Gamma_\alpha^2}\,\mathrm{d}\omega_{Nm} = \int\limits_{-\infty}^{\infty} \frac{1}{\Gamma}\frac{1}{(x/\Gamma)^2 + 1}\,\mathrm{d}(x/\Gamma), \tag{151}$$

$$= \frac{\pi}{\Gamma}, \tag{152}$$

$$\frac{1}{(\omega_\alpha - \omega_{Nm})^2 + \Gamma_\alpha^2} \approx \frac{\pi}{\Gamma_\alpha}\delta(\omega_\alpha - \omega_{Nm}), \tag{153}$$

$$= \frac{\delta(N - N_\alpha)\pi}{\Gamma_\alpha \Omega}, \tag{154}$$

where $\omega_\alpha - (N_\alpha\Omega - m\Omega_s) = 0$. Thus, we are able to turn the Lorentzion into a delta function in terms of N, which allows us to write

$$\tau_\alpha \approx \left[\gamma F_{Nm}^2 \omega_{Nm}^2\right]_{N=N_\alpha} \frac{\pi}{\Gamma_\alpha \Omega}. \tag{155}$$

With our approximation for F_{Nm}^2, we can easily evaluate for any arbitrary N, so maybe given a particular spectrum of modes ω_α it would be possible to sum τ in closed form? Seems somewhat unlikely.

In an alternative world, when Γ is much *larger* than Ω_{peri}, the Lorentzian is fat and we can probably just set it equal to its peak value $1/\Gamma^2$ and perform the integration as usual?

5 J0045–7319

5.1 Preliminary Calculations

Let's try to plug in some realistic numbers. Consider J0045+7319, which has parameters $e = 0.808$, $M_t = 10.2M_\odot$, $a = 126R_\odot$, $\Omega_o = 1.42 \times 10^{-6}$ rad/s, $\Omega_p/2\pi = 3.61\mu\mathrm{Hz}$. We can assume the mass of one star is $1.4M_\odot$, the NS, so the other is $8.8M_\odot$.

Some fiducial numbers from KQ include that the B star has RCB at $0.23R_*$, where the star is taken to be roughly $6.4R_\odot$, and the mass of the convective core is $3M_\odot$, the density and BV frequency outside the core ar 2 g/cm^3 and 100μHz respectively. Finally, the star is observed to have $\frac{\dot{a}}{a} \sim 2 \times 10^{-6}$ /yr.

Let's first evaluate $\hat{T}$, the circular orbit limit:

$$\hat{T}(r_c,\omega) \equiv \frac{GM_2^2 r_c^5}{a^6}\left(\frac{\omega}{\sqrt{GM_c/r_c^3}}\right)^{8/3}\left[\frac{r_c}{g_c}\left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln r}\right)_{r=r_c}\right]^{-1/3}\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2\left[\frac{3}{2}\frac{3^{2/3}\Gamma^2(1/3)}{5\cdot 6^{4/3}}\frac{3}{4\pi}\alpha^2\right]. \tag{156}$$

Let's take Kushnir's prescription and set $\beta_2 = 1$, so $\hat{T}$ is now

$$\hat{T}(r_c,\omega) \approx \beta_2\frac{GM_2^2 r_c^5}{a^6}\left(\frac{\omega}{\sqrt{GM_c/r_c^3}}\right)^{8/3}\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2. \tag{157}$$

We evaluate in pieces:

- Note that $\frac{GM_2^2 r_c^5}{a^6} = 1.284 \times 10^{30}\,\mathrm{N\cdot m}$ for the given parameters.
- Then evaluating $\hat{T}$ at the orbital frequency, we can evaluate the ω term, which is 9.563×10^{-8}.
- Let's just take $\rho_c/\bar{\rho}_c \sim 0.76$, then we can evaluate the final piece 0.0438.

This tells us that $\hat{T} \approx 5.38 \times 10^{21}\,\mathrm{N\cdot m}$, and so

$$\hat{T}\Omega \approx 7.636 \times 10^{15}\,\mathrm{W}, \tag{158}$$

$$E \approx 1.85 \times 10^{40}\,\mathrm{J}, \tag{159}$$

$$\dot{E}_{obs} = -\frac{\dot{a}}{a}E = 1.173 \times 10^{27}\,\mathrm{W}. \tag{160}$$

Thus, we need an enhancement $\dot{E}_{obs}/\left(\hat{T}\Omega\right) = 1.536 \times 10^{11}$. The breakup frequency in units of the orbital frequency is $428.822\Omega_o$, and it turns out the two values for the spin of the core evaluate to be $-306\Omega_o$ or $341\Omega_o$ depending on whether retrograde or prograde rotation is used. This is quite a large fraction of breakup!

For reference, the heating in the inertial frame for these two values (given $\dot{E}_{\mathrm{rot}} = \dot{E}_{\mathrm{in}} - \Omega_s\tau$) is $\sim 4 \times 10^{12}\hat{T}\Omega = 3 \times 10^{28}\,\mathrm{W} = 80L_\odot$. This is still somewhat negligible compared to the luminosity of the host star…

5.2 MESA Sims Take 1

Let's try running this via MESA. It looks like the metallicity should be somewhere around $0.1\times$ or $0.2\times$ solar[2].It seems the metallicity tends to be low, so let's generously assume solar is 0.02 and thus SMC can be as high as 0.004 (we tried quite a few parameters with 0.0012 with very little success).

Hilariously, the old mass estimate comes from requiring a $1.4M_\odot$ NS, which is not accurate; Thorsett & Chakrabatty 1999 give a new estimate $\sim 10M_\odot$. We want a star of the correct metallicity to reproduce the only observational constraints, $L \sim 1.2 \times 10^4 L_\odot$ and $T \sim 24000 \pm 100\,\mathrm{K}$. This seems to be hard to satisfy, my original MESA simulations have higher temperatures and lower luminosities, at least on the MS down to a central hydrogen abundance ~ 0.5. This seems to suggest possible signatures of tidal heating, which could puff up the star (lowering the Teff and raising the luminosity), but this seems somewhat unlikely from the $\dot{E}_{\rm rot}$ calculation using the OG values above. See plots for comparison of MESA data.

Taking $M = 11.4M_\odot$ (which matches at least the luminosity) and companion mass $1.81M_\odot$, we can plug in some values $GM_2^2 r_c^5/a^6 = 1.137 \times 10^{29}\,\mathrm{N\cdot m}$, $\omega^{8/3} \approx 8.9181 \times 10^{-9}$, and $\rho_c/\bar{\rho}_c \sim 0.742$ so the final piece comes out 0.0494. This again gives substantially lower

$$\hat{T}\Omega \approx 7.113 \times 10^{13}\,\mathrm{W}. \tag{161}$$

The new breakup frequency is $\Omega_*/\Omega_o = 1043.87$. The new required enhancement $\dot{E}_{\rm obs}/\hat{T}\Omega = 1.649 \times 10^{13}$. The core must rotate faster than breakup for this to be the case though!

5.3 Bound on Convective Core Mass

`https://arxiv.org/pdf/2003.08982.pdf` concludes that convective core masses should be larger. Let's try to estimate what sort of parameters we need for our host star. Let's start the calculation over, to expose all dependencies on system parameters.

Consider a measurement $\dot{P}/P$. Since $P^2 \propto a^3$, this means $\dot{a}/a = 2\dot{P}/3P$. But then

$$\dot{E} = \frac{GM_t^2}{2a^2}\dot{a}, \tag{162}$$

$$\frac{GM_t^2}{3a}\frac{\dot{P}}{P} = \dot{E}_{\rm in}, \tag{163}$$

$$\approx \hat{T}\Omega\left(\frac{f_5 N_{\max}}{8\left(1-e^2\right)^{9/2}} 5\Gamma(14/3)2^{8/3}\right)\left|\frac{\Omega_s}{\Omega}\right|^{8/3}. \tag{164}$$

We have dropped the leading constant term since it seems small. Let's further define $\omega_c \equiv \sqrt{GM_c/R_c^3}$ the core breakup rotation rate. For our problem, only the stellar parameters themselves are uncer-

[2] `https://academic.oup.com/mnras/article/422/2/1109/1032973#18429474`, `https://iopscience.iop.org/article/10.1088/0004-637X/741/1/12/meta`

tain, so we will place all measurable parameters in curly braces going forwards:

$$-\frac{GM_t^2}{3a}\frac{\dot{P}}{P} \leq \beta_2 \frac{GM_2^2 r_c^5}{a^6}\left(\frac{\Omega}{\omega_c}\right)^{8/3} \frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2 \Omega\left(\frac{f_5 N_{\max}}{8\left(1-e^2\right)^{9/2}} 5\Gamma(14/3)2^{8/3}\right)\left|\frac{\Omega_s}{\Omega}\right|^{8/3}, \tag{165}$$

$$-\frac{(1+q)^2}{3}\frac{\dot{P}}{2\pi} \leq \beta_2 \left(\frac{r_c}{a}\right)^5 \left(\frac{\Omega_s}{\omega_c}\right)^{8/3} \left\{\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2 \left(\frac{f_5 N_{\max}}{2^{1/3}\left(1-e^2\right)^{9/2}} 5\Gamma(14/3)\right)\right\}. \tag{166}$$

Using this, we can relate r_c and Ω_s, assuming all IGW break. Surprisingly, r_c does *not* depend on the mass of the star. We evaluate numerically and find for the given values:

$$\dot{P} = -\frac{4421040.6(4)\,\mathrm{s}}{0.5\,\mathrm{Myr}} \approx -2.8\times 10^{-7}, \tag{167}$$

$$q = 6.3 \pm 1.2, \tag{168}$$

$$e = 0.807949(3), \tag{169}$$

$$\frac{\rho_c}{\bar{\rho}_c}\left(1-\frac{\rho_c}{\bar{\rho}_c}\right)^2 \approx 0.0438, \tag{170}$$

$$a \approx 126 R_\odot, \tag{171}$$

$$N_{\max} = \frac{8 f_2}{5 f_5 \left(1-e^2\right)^{3/2}} \approx 21.016, \tag{172}$$

$$\frac{r_c}{a} \geq \left[-\frac{(1+q)^2}{6\pi}\frac{\dot{P}}{\beta_2\{19550.5\}}\right]^{1/5}\left(\frac{\Omega_s}{\omega_c}\right)^{-8/15}, \tag{173}$$

$$\geq 8.346\times 10^{-3}\left(\frac{\Omega_s}{\omega_c}\right)^{-8/15}\left(\frac{1+q}{7.3}\right)^{2/5}\left(-\frac{\dot{P}}{2.8\times 10^{-7}}\right)^{1/5}, \tag{174}$$

$$r_c \gtrsim 1.052 R_\odot \left(\frac{\Omega_s}{\omega_c}\right)^{-8/15}\left(\frac{1+q}{7.3}\right)^{2/5}\left(-\frac{\dot{P}}{2.8\times 10^{-7}}\right)^{1/5}. \tag{175}$$

This is a surprisingly simple relation that relates r_c and Ω_s... and the scaling seems correct, since the accepted parameters have $r_c \approx 1.6 R_\odot$ while many of my MESA simulations have r_c near 5×10^{10} cm, smaller than $R_\odot = 6.96\times 10^{10}$ cm.

5.4 MESA Sims Take 2

We seek a $r_c \gtrsim 1.052 R_\odot$. It seems that using $\alpha_{\rm ov} \sim 0.6$ (from the "higher convective core masses" paper) and a substantial ZAMS $v_{\rm surf} = 500$ km/s is able to generate $v_{\rm surf} \approx 160$ km/s, as expected. If we further inflate the Z and use $Z = 0.004$ ($Z = 0.0012$ has no chance, which was our previous number), then we can run MESA sims that fit $L, T_{\rm eff}$ and see what the resultant r_c is.

Note that it seems all SMC metallicies are reported in solar fractions, since it seems it all has the same biases. Recent papers[3] seem to suggest higher solar metallicities again, which backs up our $Z \approx 0.004$ for the SMC.

[3]https://www.mdpi.com/2218-2004/7/2/41

6 Paper Results

Let's check a few things regarding circ/inspiral times:

$$\frac{\dot{a}}{a} = -\frac{\dot{E}_g}{E_g} = \frac{\dot{E}_{\rm in}}{E_g} = -\frac{\dot{E}_{\rm in}}{GMM_2/2a}, \tag{176}$$

$$\frac{\dot{L}}{L} = -\frac{T}{L_{\rm orb}} = \frac{\dot{a}}{2a} + \frac{\mathrm{d}(1-e^2)/\mathrm{d}t}{2(1-e^2)} = \frac{\dot{a}}{2a} - \frac{\dot{e}e}{1-e^2}, \tag{177}$$

$$\frac{\dot{e}}{e} = \left(\frac{\dot{a}}{2a} + \frac{T}{L_{\rm orb}}\right)\left(\frac{1-e^2}{e^2}\right). \tag{178}$$

6.1 Checking against Kumar & Quataert

By back-of-the-envelope calculations, we can drop a bunch of constants and find that, when the stellar spin is small:

$$\dot{E}_{\rm in}^{(\rm m=2)} \sim \frac{T_0\Omega\left(N_p\right)^{11/3}}{(1-e)^{9/2}} \times 10. \tag{179}$$

Here, the $\left(4/\gamma_E\right)^{8/3} \sim 160$ while $(1+e)^{11/2} \sim 40$ and $f_5 \sim 3$, so the extra factor is of order 10 indeed. Recall that

$$T_0 \sim \frac{GM_2^2 r_{\rm c}^5}{a^6}\left(\frac{\Omega}{\sqrt{GM_{\rm c}/r_{\rm c}^3}}\right)^{8/3}. \tag{180}$$

So

$$\dot{E}_{\rm in} \sim \frac{GM_2^2 r_{\rm c}^5}{a^6}\Omega_{\rm p}\left(\frac{\Omega_{\rm p}}{\sqrt{GM_{\rm c}/r_{\rm c}^3}}\right)^{8/3}\frac{1}{(1-e)^{9/2}} \times 10. \tag{181}$$

We could instead arrange $\Omega_{\rm p}/\left(a^6(1-e)^{9/2}\right) = \Omega\sqrt{1+e}/r_p^6$, where r_p is the pericenter distance.

We can verify by hand that $GM_2^2 r_{\rm c}^5/a^6 \sim 10^{30}\ \mathrm{J} = 10^{37}$ erg, while $\Omega_{\rm p} \sim 2\pi/(3.5\ \mathrm{day}) \sim 10^{-5}\ \mathrm{s}^{-1}$ and $\Omega_{\rm p}/\Omega_{\rm s,c} \sim 0.03$, so the three dimensionless quantities come out to be ~ 1 interestingly, and

$$\dot{E}_{\rm in} \sim 10^{32}\ \mathrm{erg/s}. \tag{182}$$

By Kumar, the orbital energy is 10^{47} erg, while the characteritic rate of evolution of the system is $\sim 10^6\ \mathrm{yr} \sim 10^{13}$ s, so the required dissipation rate is $\sim 10^{34}$ erg/s, and we are two orders of magnitude short. This can indeed be accomplished by stellar spin at a few times the pericenter frequency (since the power is $\sim N_{\rm p}^{8/3}$ that is getting changed), which is what our plots show. For reference, stellar breakup is also a few times $\Omega_{\rm p}$, so it's quite tight, but our calculations seem a bit off?

How does this differ from the Kumar estimate? In their fiducial case, $\dot{E}_{\rm orb} = -\Gamma E_{\rm tide}$, where Γ is radiative damping and $E_{\rm tide}$ is the "energy of the dynamical tide", which turns out to amount to $\dot{L}_{\rm tide}$, the angular momentum luminosity of the dynamical tide (where the pattern frequency is set to $\Omega_{\rm p} - \Omega_{\rm s}$, i.e. forcing at pericenter vs stellar spin). This is because the energy is $E_{\rm tide} \sim L_{\rm tide}\Omega_{\rm p}$, if the star spins at characteristic spin $\Omega_{\rm p}$, and $L_{\rm tide} = \dot{L}_{\rm tide} T_{\rm p}$ where $T_{\rm p} \equiv 2\pi/\Omega_{\rm p}$ is the pericenter period (this is also not very consistent, since it assumes $L_{\rm tide}$ is fully dissipated away in a single encounter, else $L_{\rm tide} \gtrsim \dot{L}_{\rm tide} T_{\rm p}$).

However, if differential rotation is admitted in order to effect complete damping, then $\dot{E}_{\rm orb} \sim \dot{L}_{\rm tide}\Omega$. I will assume that both Kumar's $\dot{L}_{\rm tide}$ and KZ16's $\dot{L}_{\rm tide}$ are in agreement (supposedly so, with GN89). The angular momentum flux is just the total angular momentum deposited though, i.e. the torque, and so Kumar & Quataert effectively predict in their §3 that $\dot{E}_{\rm orb} \sim T\Omega$ is exactly sufficient to match observations.

But what is $T\Omega$ for us? We can evaluate T_0 for $\Omega = \Omega_{\rm p}$ and $a \Rightarrow a(1-e)$, the pericenter separation. This gives the same scaling as above, and we are off by about two orders of magnitude.

6.2 Plotting J0045-7319

We have $g(e,\Omega) \equiv \dot{E}_{\rm in}/T_0\Omega$ evaluated explicitly (named `get_energies`), so we just need to evaluate $\dot{P}$ in terms of this. Note that

$$\dot{E}_{\rm in} = \frac{GqM_2^2}{3a}\frac{\dot{P}}{P}, \tag{183}$$

$$\dot{P} = \frac{3\dot{E}_{\rm in} aP}{GqM_2^2}, \tag{184}$$

$$= \frac{6a\pi}{GqM_2^2\Omega} T_0 \Omega g(e,\Omega), \tag{185}$$

$$= \frac{6a\pi}{GqM_2^2}\frac{GM_2^2 r_c^5}{a^6} f(\rho_c)\left(\frac{\Omega}{\sqrt{GM_c/r_c^3}}\right)^{8/3} g(e,\Omega), \tag{186}$$

$$= \frac{6\pi}{q}\left(\frac{r_c}{a}\right)^5 f(\rho_c)\left(\frac{\Omega}{\sqrt{GM_c/r_c^3}}\right)^{8/3} g(e,\Omega). \tag{187}$$

For fiducial $M_c = 3M_\odot$ and $r_c = 1.38R_\odot$, and $a = 126R_\odot$, we obtain indeed

$$\dot{P} = 1.1 \times 10^{-18}\left(\frac{r_c}{1.38R_\odot}\right)^9 \left(\frac{M_c}{3M_\odot}\right)^{-4/3} g(e,\Omega). \tag{188}$$

We have taken $f\left(\rho_c/\bar{\rho}_c = 0.76\right)$.

Does this agree with the back-of-the-envelope calculations we did before? In there, we took $g(e,\Omega) \approx 10\left(N_p\right)^{11/3}/(1-e)^{9/2}$, or

$$g(e,\Omega) \approx 10\frac{(1+e)^{11/6}}{(1-e)^{10}}. \tag{189}$$

For $e = 0.8$, $\Omega \approx 0$, this is $\sim 3\times 10^8$, and we end up with $\dot{P} \sim 3 \times 10^{-10}$, or $\dot{P}/P \sim \dot{E}/E \sim 6 \times 10^{-17}$, implying $\dot{E} \sim 6 \times 10^{30}$ erg/s. The discrepancy comes because we ignored the $\rho_c/\bar{\rho}_c$ factor in the earlier calculation, which is about a factor of 30 with $\rho_c/\bar{\rho}_c = 0.76$ and about a factor of 10 for $\rho_c/\bar{\rho}_c = 1/3$.