MNRAS 000, 1–5 (2020) Preprint 22 June 2020 Compiled using MNRAS LATEX style file v3.0

Spin-Orbit Dynamics in Hierarchical Black Hole Triples: Analytical Theory

Yubo Su[1] Dong Lai[1] Bin Liu[1]
[1] *Cornell Center for Astrophysics and Planetary Science, Department of Astronomy, Cornell University, Ithaca, NY 14853, USA*

Accepted XXX. Received YYY; in original form ZZZ

ABSTRACT
Abstract

Key words: keywords

1 INTRODUCTION

This problem is important.

2 ANALYTICAL SETUP

2.1 Orbital Evolution

We study Lidov-Kozai (LK) oscillations due to an external perturber to quadrupole order and include precession of pericenter and gravitational wave radiation due to general relativity. Consider an inner black hole (BH) binary with masses m_1 and m_2 having total mass m_{12} and reduced mass μ orbited by a third BH with mass m_3. Call a_3 the orbital semimajor axis of the third BH from the center of mass of the inner binary, and e_3 the eccentricity of its orbit, and define effective semimajor axis

$$\tilde{a}_3 \equiv a_3\sqrt{1 - e_3^2}. \tag{1}$$

We adopt the test particle approximation such that the orbit of the third mass is fixed. Finally, call $\mathbf{L}_{\rm out} \equiv L_{\rm out}\hat{\mathbf{L}}_{\rm out}$ the fixed angular momentum of the outer BH relative to the center of mass of the inner BH binary, and call $\mathbf{L} \equiv L\hat{\mathbf{L}}$ the orbital angular momentum of the inner BH binary.

We then consider the motion of the inner binary, described by orbital elements Keplerian orbital elements $(a, e, \Omega, I, \omega)$. The equations describing the motion of these orbital elements are then (Peters 1964; Storch & Lai 2015; Liu & Lai 2018)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = \left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_{\rm GW}, \tag{2}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{15}{8t_{\rm LK}} e\sqrt{1-e^2}\sin 2\omega \sin^2 I + \left(\frac{\mathrm{d}e}{\mathrm{d}t}\right)_{\rm GW}, \tag{3}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{3}{4t_{\rm LK}} \frac{\cos I\left(5e^2\cos^2\omega - 4e^2 - 1\right)}{\sqrt{1-e^2}} + \Omega_{\rm GR}, \tag{4}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{15}{16}\frac{e^2 \sin 2\omega \sin 2I}{\sqrt{1-e^2}}, \tag{5}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{3}{4t_{\rm LK}} \frac{2\left(1-e^2\right) + 5\sin^2\omega(e^2 - \sin^2 I)}{\sqrt{1-e^2}}, \tag{6}$$

where we define

$$t_{\rm LK}^{-1} = n\left(\frac{m_3}{m_{12}}\right)\left(\frac{a}{\tilde{a}_3}\right)^3, \tag{7}$$

$$\left(\frac{\mathrm{d}a}{\mathrm{d}t}\right)_{\rm GW} = -\frac{a}{t_{\rm GW}}, \tag{8}$$

$$= \frac{64}{5}\frac{G^3\mu m_{12}^2}{c^5 a^3}\frac{1}{\left(1-e^2\right)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right), \tag{9}$$

$$\left(\frac{\mathrm{d}e}{\mathrm{d}t}\right)_{\rm GW} = -\frac{304}{15}\frac{G^3\mu m_{12}^2}{c^5 a^4}\frac{1}{\left(1-e^2\right)^{5/2}}\left(1 + \frac{121}{304}e^2\right), \tag{10}$$

$$\Omega_{\rm GR} = \frac{3Gnm_{12}}{c^2 a\left(1-e^2\right)}, \tag{11}$$

and $n = \sqrt{Gm_{12}/a^3}$ is the mean motion of the inner binary. We will also sometimes notate $j \equiv \sqrt{1-e^2}$.

The evolution of these orbital elements has been well characterized in previous studies (Anderson et al. 2016; Liu & Lai 2017). We focus on the evolution with LK oscillations, in which case the evolution is approximately broken into two phases [see panels (a) and (b) on both plots in Fig. 1 for reference]:

- In the first phase, e and I vary significantly within each LK period. If $e_{\rm min} \ll e_{\rm max}$, where $e_{\rm min}$ and $e_{\rm max}$ refer to the minimum and maximum e within a LK period, then the system spends a fraction $\sqrt{1-e_{\rm max}^2}$ of the LK period at $e \simeq e_{\rm max}$ (Anderson et al. 2016).

During this phase, $e_{\rm min}$ is excited to larger values under the dual GR effects of gravitational wave radiation and pericenter advance, while a and $e_{\rm max}$ evolve comparatively little .

- In the second phase, $e_{\rm min} \approx e_{\rm max}$, and the system coalesces under gravitational wave radiation with little variation over each LK period.

2.2 Spin Dynamics

We are ultimately interested in the spin orientations of the inner BHs at merger as a function of initial conditions. Since they evolve independently to leading post-Newtonian order, we focus on the dynamics of a single BH spin vector $\mathbf{S} = S\hat{\mathbf{S}}$ Neglecting spin-spin interactions, $\hat{\mathbf{S}}$ undergoes de Sitter precession about $\mathbf{L}$ as

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \Omega_{\mathrm{SL}}\hat{\mathbf{L}} \times \hat{\mathbf{S}}, \tag{12}$$

$$\Omega_{\mathrm{SL}} = \frac{3Gn\left(m_2 + \mu/3\right)}{2c^2 a\left(1 - e^2\right)}. \tag{13}$$

To analyze the dynamics of the spin vector, we go to co-rotating frame with $\hat{\mathbf{L}}$ about $\hat{\mathbf{L}}_{\mathrm{out}}$. Choose $\hat{\mathbf{L}}_{\mathrm{out}} = \hat{\mathbf{z}}$, and choose the $\hat{\mathbf{x}}$ axis such that $\hat{\mathbf{L}}$ lies in the x-z plane. In this 0oordinate system, Eq. (12) becomes

$$\left(\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \left(-\frac{\mathrm{d}\Omega}{\mathrm{d}t}\hat{\mathbf{z}} + \Omega_{\mathrm{SL}}\hat{\mathbf{L}}\right) \times \hat{\mathbf{S}}, \tag{14}$$

$$= \mathbf{\Omega}_{\mathrm{e}} \times \hat{\mathbf{S}}, \tag{15}$$

$$\mathbf{\Omega}_{\mathrm{e}} \equiv \Omega_{\mathrm{L}}\hat{\mathbf{z}} + \Omega_{\mathrm{SL}}\left(\cos I\hat{\mathbf{z}} + \sin I\hat{\mathbf{x}}\right), \tag{16}$$

$$\Omega_{\mathrm{L}} \equiv -\frac{\mathrm{d}\Omega}{\mathrm{d}t}. \tag{17}$$

In general, Eq. (15) is difficult to analyze, since Ω_{L}, Ω_{SL} and I all vary significantly within each LK period, and we are interested in the final outcome after many LK periods. However, if we assume $t_{\mathrm{GW}} \gg t_{\mathrm{LK}}$, then the system can be treated as nearly periodic within each LK cycle. We can then rewrite Eq. (15) in Fourier components

$$\left(\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \left[\overline{\mathbf{\Omega}}_{\mathrm{e}} + \sum_{N=1}^{\infty} \mathbf{\Omega}_{\mathrm{e,N}} \cos\left(\frac{2\pi N t}{T_{\mathrm{LK}}}\right)\right] \times \hat{\mathbf{S}}. \tag{18}$$

The bar denotes an average over an LK cycle. We adopt convention where $t = 0$ is the maximum eccentricity phase of the LK cycle.

We next assume that the $N \geq 1$ harmonics vanish when the equation of motion is averaged over an LK cycle[1], which gives

$$\left(\frac{\mathrm{d}\overline{\hat{\mathbf{S}}}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \overline{\mathbf{\Omega}}_{\mathrm{e}} \times \overline{\hat{\mathbf{S}}}. \tag{19}$$

We accordingly define angle

$$\cos\theta_{\mathrm{e}} \equiv \overline{\hat{\mathbf{S}}} \cdot \hat{\overline{\mathbf{\Omega}}}_{\mathrm{e}}. \tag{20}$$

Eq. (19) suggests that θ_{e} should be a conserved quantity when $\mathbf{\Omega}_{\mathrm{e}}$ varies adiabatically. The adiabaticity condition requires the precession axis evolve slowly compared to the precession frequency at all times:

$$\left|\frac{\mathrm{d}\hat{\overline{\mathbf{\Omega}}}_{\mathrm{e}}}{\mathrm{d}t}\right| \ll \left|\overline{\mathbf{\Omega}}_{\mathrm{e}}\right|. \tag{21}$$

Since the orientation of $\overline{\mathbf{\Omega}}_{\mathrm{e}}$ changes on timescale t_{GW}, we see that the adiabatic assumption is roughly equivalent to assuming the Fourier decomposition [Eq. (18)] within each LK period is valid.

To be more precise, we define an inclination angle I_{e} for $\overline{\mathbf{\Omega}}_{\mathrm{e}}$ as shown in Fig. 2. Denoting also $\overline{\Omega}_{\mathrm{e}} \equiv \left|\overline{\mathbf{\Omega}}_{\mathrm{e}}\right|$, the adiabaticity condition can be expressed as

$$\frac{\mathrm{d}I_{\mathrm{e}}}{\mathrm{d}t} \ll \overline{\Omega}_{\mathrm{e}}. \tag{22}$$

Next, we express I_{e} can be expressed in closed form. When

[1] While this is not strictly accurate and gives incorrect results for certain parameters, it is an intuitively clear picture and makes correct predictions for many physically relevant configurations. A more rigorous discussion is provided in Appendix A.

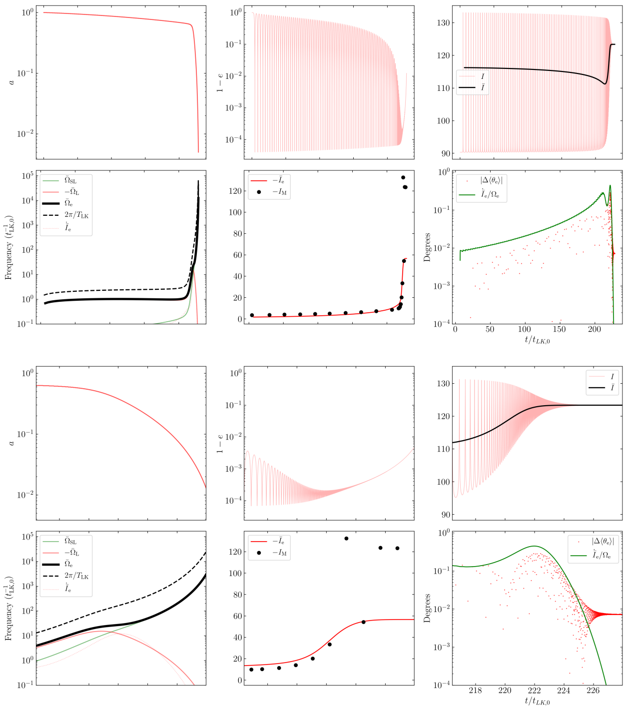

Figure 1. Plot.

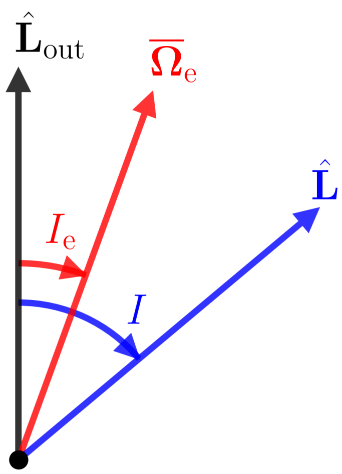

Figure 2. Definition of angles, shown in plane of the two angular momenta $\mathbf{L}_{\mathrm{out}}$ and $\mathbf{L}$, or the $\hat{\mathbf{x}}$–$\hat{\mathbf{z}}$ plane in the corotating frame. Note that for $I > 90°$, $I_{\mathrm{e}} < 0$.

the LK oscillations are strong ("phase one"), we define averaged quantities

$$\overline{\Omega_{\mathrm{SL}} \sin I} \equiv \overline{\Omega}_{\mathrm{SL}} \sin \bar{I}, \tag{23}$$

$$\overline{\Omega_{\mathrm{SL}} \cos I} \equiv \overline{\Omega}_{\mathrm{SL}} \cos \bar{I}. \tag{24}$$

Then, using Eq.(16), we can see that

$$\tan I_{\mathrm{e}} = \frac{\overline{\mathcal{A}} \sin \bar{I}}{1 + \overline{\mathcal{A}} \cos \bar{I}}, \tag{25}$$

where

$$\overline{\mathcal{A}} \equiv \frac{\overline{\Omega}_{\rm SL}}{\overline{\Omega}_{\rm L}}. \tag{26}$$

When LK oscillations are suppressed ("phase two"), we just have $\overline{\Omega}_{\rm SL} = \Omega_{\rm SL}$, $\overline{\Omega}_{\rm L} = \Omega_{\rm L}$, and $\bar{I} = I$.

3 ANALYSIS: DEVIATION FROM ADIABATICITY

In real systems, the particular extent to which $\overline{\theta}_{\rm e}$ is conserved depends on how well Eq. (21) is satisfied. We will first present equations of motion for $\overline{\theta}_{\rm e}$. We will then derive accurate estimates for important quantities in these equations of motion, and use these estimates to derive upper bounds on $\Delta\overline{\theta}_{\rm e}$, the change in $\overline{\theta}_{\rm e}$ over the entire inspiral. Taken together, this calculation estimates the deviation from adiabaticity as a function of initial conditions.

3.1 Equations of Motion

From the corotating frame [Eq. (19)], consider going to the reference frame where $\hat{\mathbf{z}} = \hat{\overline{\mathbf{\Omega}}}_{\rm e}$ by rotation $-\dot{\bar{I}}_{\rm e}\hat{\mathbf{y}}$. In this reference frame, the polar coordinate is just $\theta_{\rm e}$ as defined above in Eq. (20), and we call the azimuthal coordinate $\phi_{\rm e}$. In this reference frame, the equation of motion becomes

$$\left(\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t}\right)' = \hat{\overline{\mathbf{\Omega}}}_{\rm e} \times \hat{\mathbf{S}} - \dot{\bar{I}}_{\rm e}\hat{\mathbf{y}} \times \hat{\mathbf{S}}. \tag{27}$$

If we break $\hat{\mathbf{S}}$ into components $\hat{\mathbf{S}} = S_x\hat{\mathbf{x}} + S_y\hat{\mathbf{y}} + \cos\theta_{\rm e}\hat{\mathbf{z}}$ and define complex variable

$$S_\perp \equiv S_x + iS_y = \sin\theta e^{i\phi_{\rm e}}, \tag{28}$$

we can rewrite Eq. 27 as

$$\frac{\mathrm{d}S_\perp}{\mathrm{d}t} = i\left(\overline{\Omega}_{\rm e}\right) S_\perp - \dot{\bar{I}}_{\rm e}\cos\theta_{\rm e}. \tag{29}$$

Defining

$$\Phi(t) = \int^t \overline{\Omega}_{\rm e}\,\mathrm{d}t, \tag{30}$$

we obtain formal solution

$$e^{-i\Phi}\left[S_\perp(t=\infty) - S_\perp(t=-\infty)\right] = -\int_{-\infty}^{\infty} e^{-i\Phi(\tau)}\dot{\bar{I}}_{\rm e}\cos\theta\,\mathrm{d}\tau. \tag{31}$$

It can be seen that, in the adiabatic limit [Eq. (22)], $|S_\perp|$ (and therefore $\theta_{\rm e}$) is conserved, as the phase of the integrand in the right hand side varies much faster than the magnitude. Furthermore, the deviation from exact conservation of $|S_\perp|$ cannot exceed $\dot{\bar{I}}_{\rm e}/\overline{\Omega}_{\rm e}$ so long as $\dot{\bar{I}}_{\rm e} \lesssim \overline{\Omega}_{\rm e}$[2]. In the following section, we show that this maximum value can be calculated to good accuracy from initial conditions.

[2] Given the complicated evolution of $\overline{\Omega}_{\rm e}$ and $\dot{\bar{I}}_{\rm e}$, it is difficult to give a more exact bound on the deviation from adiabaticity. In practice, deviations $\lesssim 1^\circ$ are astrophysically negligible, so the exact scaling in this regime is of little consequence.

3.2 Estimate of Deviation from Adiabaticity

Towards estimating max $\dot{\bar{I}}_{\rm e}/\overline{\Omega}_{\rm e}$, we first differentiate Eq. (25),

$$\dot{\bar{I}}_{\rm e} = \left(\frac{\dot{\overline{\mathcal{A}}}}{\overline{\mathcal{A}}}\right)\frac{\overline{\mathcal{A}}\sin\bar{I}}{1 + 2\overline{\mathcal{A}}\cos\bar{I} + \overline{\mathcal{A}}^2}. \tag{32}$$

It can be easily hown from Eq. (16) that

$$\overline{\Omega}_{\rm e} = \overline{\Omega}_{\rm L}\left(1 + 2\overline{\mathcal{A}}\cos\bar{I} + \overline{A}^2\right)^{1/2}, \tag{33}$$

from which we obtain

$$\left|\frac{\dot{\bar{I}}_{\rm e}}{\overline{\Omega}_{\rm e}}\right| = \left|\frac{\dot{\overline{\mathcal{A}}}}{\overline{\mathcal{A}}}\right|\frac{1}{\left|\overline{\Omega}_{\rm L}\right|}\frac{\overline{\mathcal{A}}\sin\bar{I}}{\left(1 + 2\overline{\mathcal{A}}\cos\bar{I} + \overline{\mathcal{A}}\right)^{3/2}}. \tag{34}$$

This is maximized when $\overline{\mathcal{A}} \simeq 1$, and so we obtain that the maximum deviation should be bounded by

$$\left|\frac{\dot{\bar{I}}_{\rm e}}{\overline{\Omega}_{\rm e}}\right|_{\max} \simeq \left|\frac{\dot{\overline{\mathcal{A}}}}{\overline{\mathcal{A}}}\right|\frac{1}{\left|\overline{\Omega}_{\rm L}\right|}\frac{\sin\bar{I}}{\left(2 + 2\cos\bar{I}\right)^{3/2}}. \tag{35}$$

To evaluate this, we make two assumptions: (i) $\bar{I}$ is approximately constant, and (ii) $j_{\min} = \sqrt{1 - e_{\max}^2}$ evaluated at $\overline{\mathcal{A}} \simeq 1$ is some constant multiple of the initial $j_{\min}$, so that

$$(j_{\min})_{\overline{\mathcal{A}}\simeq 1} = f\sqrt{\frac{5}{3}\cos^2 I_0}, \tag{36}$$

for some unknown factor $f > 1$. f turns out to be relatively insensitive to I_0, which is unsurprising, as systems with lower $e_{\max}$ values take more cycles to attain $\overline{\mathcal{A}} \simeq 1$ and thus experience a similar amount of decay due to GW radiation.

For simplicity, let's first assume $\overline{\mathcal{A}} \simeq 1$ is satisfied when the LK oscillations are mostly suppressed, and $e \approx e_{\max}$ throughout the LK cycle (we will later see that the scalings are the same in the LK-oscillating regime). Then we can write

$$\overline{\mathcal{A}} = \frac{\overline{\Omega}_{\rm SL}}{\overline{\Omega}_{\rm L}} \tag{37}$$

$$\simeq \frac{3n\,(m_2 + \mu/3)}{2aj^2}\left[\frac{3\cos\bar{I}}{4t_{\rm LK}}\frac{1 + 3e^2/2}{j}\right]^{-1}, \tag{38}$$

$$\propto \frac{a^{-4}}{j}, \tag{39}$$

$$\frac{\dot{\overline{\mathcal{A}}}}{\overline{\mathcal{A}}} = -4\left(\frac{\dot{a}}{a}\right)_{\rm GW} + \frac{e}{j^2}\left(\frac{\mathrm{d}e}{\mathrm{d}t}\right)_{\rm GW}. \tag{40}$$

Approximating $e \lesssim 1$ in Eqs. (9) and (10) gives

$$\frac{\dot{\overline{\mathcal{A}}}}{\overline{\mathcal{A}}} \simeq \frac{64\mu_{12}^2}{5a^4j^7}15, \tag{41}$$

$$\overline{\Omega}_{\rm L} \approx \frac{3\cos\bar{I}}{2t_{\rm LK}j}, \tag{42}$$

$$\left|\frac{\dot{\bar{I}}_{\rm e}}{\overline{\Omega}_{\rm e}}\right| \approx \frac{128\mu_{12}^2}{a^4j^7}\frac{t_{\rm LK}}{\cos\bar{I}}\frac{\sin\bar{I}}{\left(2 + 2\cos\bar{I}\right)^{3/2}}. \tag{43}$$

a and j are evaluated when $\overline{\mathcal{A}} \simeq 1$, providing a further constraint:

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APPENDIX A: EFFECT OF HARMONIC TERMS ON EVOLUTION: AVERAGING REVISITED

Floquet!

APPENDIX B: DEVIATION FROM ADIABATICITY: SIMPLIFIED MODELS

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