Spin-Orbit Misalignment Dynamics in Black Hole Triples

Group Meeting

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Equations:

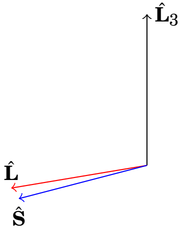
$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \Omega_{SL}\hat{\mathbf{L}} \times \hat{\mathbf{S}}, \tag{1}$$

$$\Omega_{SL} = \frac{3Gn\left(m_2 + \mu/3\right)}{2c^2 a\left(1 - e^2\right)}, \tag{2}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = -\frac{15}{16t_{LK}}\frac{e^2 \sin(2\omega)\sin(2I)}{\sqrt{1-e^2}}, \tag{3}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{3}{4t_{LK}}\frac{\cos i\left(5e^2\cos^2\omega - 4e^2 - 1\right)}{\sqrt{1-e^2}}, \tag{4}$$

$$\frac{1}{t_{LK}} = n\left(\frac{m_3}{m_1+m_2}\right)\left(\frac{a}{\bar{a}_{3,\mathrm{eff}}}\right)^3. \tag{5}$$



GW radiation narrows range of e oscillations.
$\theta_{\mathrm{sl}}^f = \cos^{-1}(\hat{\mathbf{L}} \cdot \hat{\mathbf{S}})$?

- Go to corotating frame with $\dot{\Omega}$, so that $\hat{\mathbf{L}}$ nutates and $\hat{\mathbf{L}}_3 = \hat{\mathbf{z}}$:

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \left(\Omega_{SL}\hat{\mathbf{L}} - \dot{\Omega}\hat{\mathbf{L}}_3\right) \times \hat{\mathbf{S}} \equiv \hat{\mathbf{\Omega}}_{\mathrm{eff}} \times \hat{\mathbf{S}}. \tag{6}$$

- Intuition: If no LK, Ω_{SL}, $\dot{\Omega}$ slowly vary, $\theta_{\mathrm{sl}}^f = \theta_{\mathrm{s3}}^i$.
- $\Omega_{SL}\hat{\mathbf{L}}$ and $\dot{\Omega}\hat{\mathbf{L}}_3$ periodic with T_{LK} (varying), so decompose about the mean value:

$$\begin{aligned}\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} &= \langle \Omega_{SL}\hat{\mathbf{L}} - \dot{\Omega}\hat{\mathbf{L}_3}\rangle_{LK} \times \hat{\mathbf{S}} \\ &\quad - \mathbf{S} \times \sum_{N=1}^{\infty} \hat{\mathbf{A}}_N \cos\left(\frac{2\pi N t}{T_{LK}}\right).\end{aligned} \tag{7}$$

- Formally, can average (or WKBJ) **unless** $\Omega_{\mathrm{eff}} = \frac{2\pi\left(N'/2\right)}{T_{LK}}$.

- What are these **linear** resonances? Consider toy model, $\epsilon \to 0$:

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \left[\omega_0\hat{\mathbf{z}} + \epsilon\left(\cos\omega t\hat{\mathbf{x}} + \sin\omega t\hat{\mathbf{y}}\right)\right] \times \hat{\mathbf{S}}, \tag{8}$$

$$\left(\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \left[(\omega_0 - \omega)\hat{\mathbf{z}} + \epsilon\hat{\mathbf{x}}\right] \times \hat{\mathbf{S}}. \tag{9}$$

- In resonances, the precession axis tilts away from $\hat{\mathbf{z}}$ plane when $\omega_0 - \omega \sim \epsilon$. Then $\dot{\theta} \simeq \epsilon$.
- Thus, when crossing:
  - If slow, adiabatic invariance, $\theta^f = \theta^i$.
  - If fast, no time to rotate, $\theta^f \approx \theta^i$.
  - Only when $\frac{\mathrm{d}\ln\omega}{\mathrm{d}t} \simeq \epsilon$ will $\theta^f \neq \theta^i$.

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \langle \Omega_{SL}\hat{\mathbf{L}} - \dot{\Omega}\hat{\mathbf{L}_3}\rangle_{LK} \times \hat{\mathbf{S}} - \frac{\hat{\mathbf{S}}}{2} \times \sum_{N=1}^{N'} \hat{\mathbf{A}}_N \cos\left(\frac{2\pi N t}{T_{LK}}\right) - \frac{\hat{\mathbf{S}}}{2} \times \sum_{N=1}^{\infty} \hat{\mathbf{B}}_N \cos\left(\Omega_{\rm eff} t\right) - \hat{\mathbf{C}}_N \sin\left(\Omega_{\rm eff} t\right), \tag{10}$$

$$\hat{\mathbf{B}}_N(t) = \left(\hat{\mathbf{A}}_N + \hat{\mathbf{A}}_{N+N'}\right)\cos\left(\frac{\pi\left(N+N'\right)t}{T_{LK}}\right), \tag{11}$$

$$\hat{\mathbf{C}}_N(t) = \left(\hat{\mathbf{A}}_N - \hat{\mathbf{A}}_{N+N'}\right)\sin\left(\frac{\pi\left(N+N'\right)t}{T_{LK}}\right). \tag{12}$$

- In reality, $\hat{\mathbf{A}}_N$ are small (Fourier coefficients of $\left(\Omega_{SL}(t)\hat{\mathbf{L}}(t) - \dot{\Omega}(t)\hat{\mathbf{L}}_3\right)$), so seek maximum resonance width. If too small, *all* fast encounter, conservation.