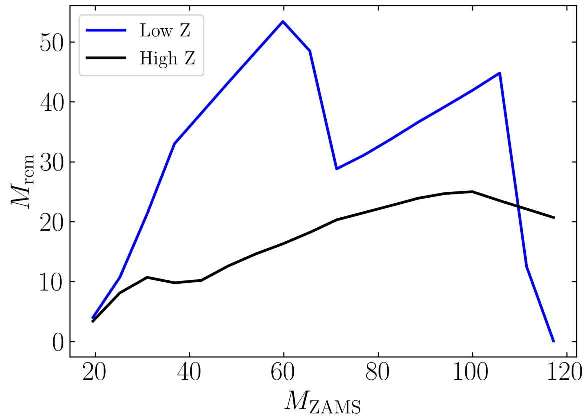

Figure 1: SNe mass transfer function

We want to answer what the primordial BH q distribution is in a few simplified cases if:

- The ZAMS masses are randomly drawn Salpeter IMF $P(M) \propto M^{-2.35}$, then go supernova following `https://ui.adsabs.harvard.edu/abs/2017MNRAS.470.4739S/abstract` (bounded by large/small Z)
- The ZAMS mass ratio is uniform.
- The ZAMS mass ratio is uniform in $\log q$.

For reference, the supernova mass transfer function is shown in Fig. 1

1 Corrections to Appendix A

I found Appendix A is wrong: $P(q) \propto q^{-p}$ using the convention $q \geq 1$, but not in our convention! See Fig. 2. To draw the distributions, I use either

$$q = \min\left(\frac{m_2}{m_1}, \frac{m_1}{m_2}\right) \leq 1, \tag{1}$$

or max and ≥ 1, where $m_{1,2}$ are drawn from $P(m) \propto m^{-2.35}$. I double checked the Moe & di Stefano paper, and under their (2) they really assume that $P(q \leq 1) \propto q^{-p}$ as well, so I think this might be a misconception in their paper as well?

Note that in my Appendix, the calculation doesn't change if we take $m_2 \geq m_1$, i.e. originally

$$\begin{aligned} P\left(\frac{m_{\rm min}}{m_{\rm max}} \leq q \leq 1\right) &= \int\limits_{m_{\rm min}}^{m_{\rm max}} \mathrm{d}m_1 \int\limits_{m_{\rm min}}^{m_1} \mathrm{d}m_2\, \delta\left(\frac{m_2}{m_1} - q\right) P(m_1)P(m_2), & (2)\\ &= \int\limits_{m_{\rm min}}^{m_{\rm max}} \mathrm{d}m_1 m_1 P(m_1)P(qm_1), & (3)\\ &\propto q^{-p}, & (4) \end{aligned}$$

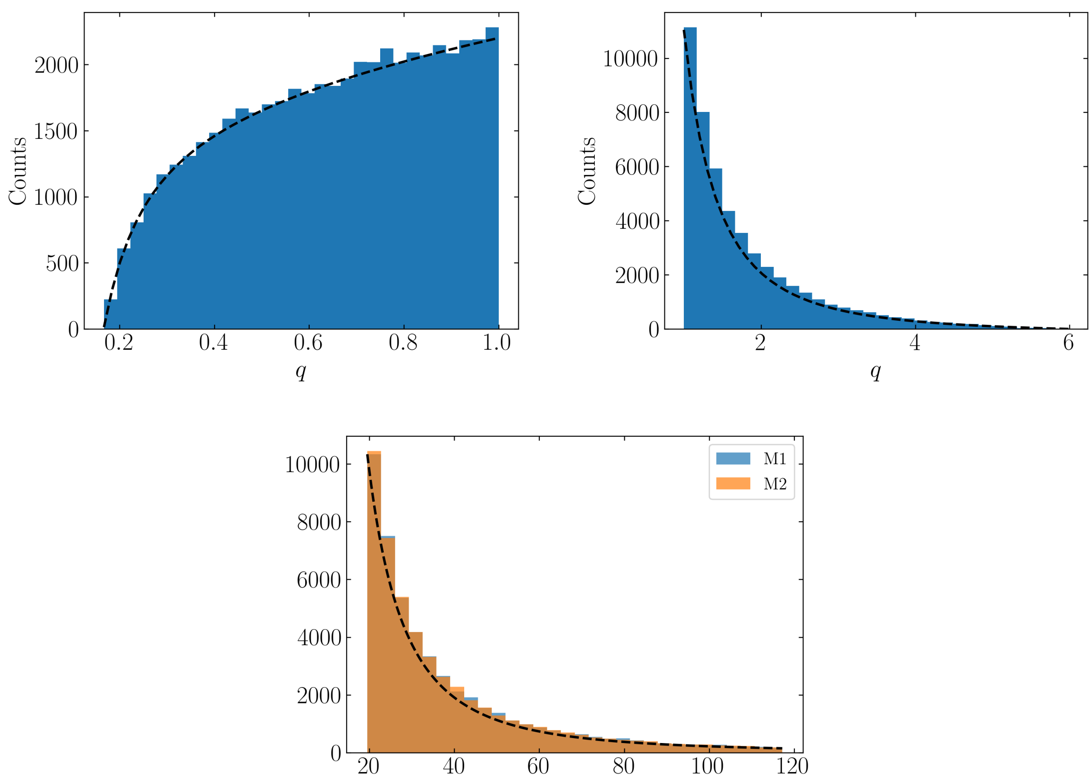

Figure 2: (i) Histogram of $q \leq 1$ with random pairings from Salpeter IMF, (ii) histogram of $q \geq 1$ with random pairings from Salpeter IMF, with $q^{-2.35}$ overlaid, and (iii) histogram of masses, with $M^{-2.35}$ power law overlaid, as a sanity check.

but also

$$P\left(1 \le q \le \frac{m_{\max}}{m_{\min}}\right) = \int_{m_{\min}}^{m_{\max}} \mathrm{d}m_1 \int_{m_1}^{m_{\max}} \mathrm{d}m_2\, \delta\left(\frac{m_2}{m_1} - q\right) P(m_1)P(m_2), \tag{5}$$

$$= \int_{m_{\min}}^{m_{\max}} \mathrm{d}m_1 m_1 P(m_1)P(qm_1), \tag{6}$$

$$\propto q^{-p}. \tag{7}$$

Note the different bounds of integration on the second integral. Clearly the first of these two derivations is wrong (the one that is in the paper), but I am not sure why yet. I will hopefully have an answer by the time of the meeting.

1.1 Resolution

In fact, Tout 1991 `https://doi.org/10.1093/mnras/250.4.701` has solved this problem, and in particular, "notice that it is sharply peaked at $q = 1$ and does not have the form $n(q) \propto q^{-\alpha}$ when $q < 1$ as many authors, following Warner (1961), have assumed". Both integrals I wrote above are incorrect, since the bounds of integration on the first integral should change based on the value of q ($m_{\min} \Rightarrow m_{\min}/q$ in the former, and $m_{\max} \Rightarrow m_{\max}/q$ in the latter). However, the error on the second one is nearly negligible, while the error on the former is significant and changes the asymptotic behavior! The correct distributions are then:

$$P\left(q_{\min} \le q \le 1\right) = \int_{m_{\min}/q}^{m_{\max}} \mathrm{d}m_1 \left(-m_1\right)\left(m_1\right)^{-2p} q^{-p}, \tag{8}$$

$$\propto q^{-p}\left[-m_1^{-2p+2}\right]_{m_{\min}/q}^{m_{\max}}, \tag{9}$$

$$\propto \left(\frac{m_{\min}^{2-2p}}{q^{2-p}}\right) - m_{\max}^{2-2p} q^{-p}, \tag{10}$$

$$\propto q^{p-2}\left[1 - \left(\frac{q_{\min}}{q}\right)^{2p-2}\right], \tag{11}$$

and

$$P\left(1 \le q \le q_{\max}\right) = \int_{m_{\min}}^{m_{\max}/q} \mathrm{d}m_1 \left(-m_1\right)\left(m_1\right)^{-2p} q^{-p}, \tag{12}$$

$$= q^{-p}\left[-m_1^{-2p+2}\right]_{m_{\min}}^{m_{\max}/q}, \tag{13}$$

$$= m_{\min}^{2-2p} q^{-p} - m_{\max}^{2-2p} q^{p-2}, \tag{14}$$

$$\propto q^{-p}\left[1 - \left(\frac{q}{q_{\max}}\right)^{2p-2}\right]. \tag{15}$$

2 Histograms

The three requested plots are shown in Fig. 3. For (i), I just took the masses from the previous section and sent them through the SNe transfer function (Fig. 1). For (ii) and (iii), the procedure is somewhat

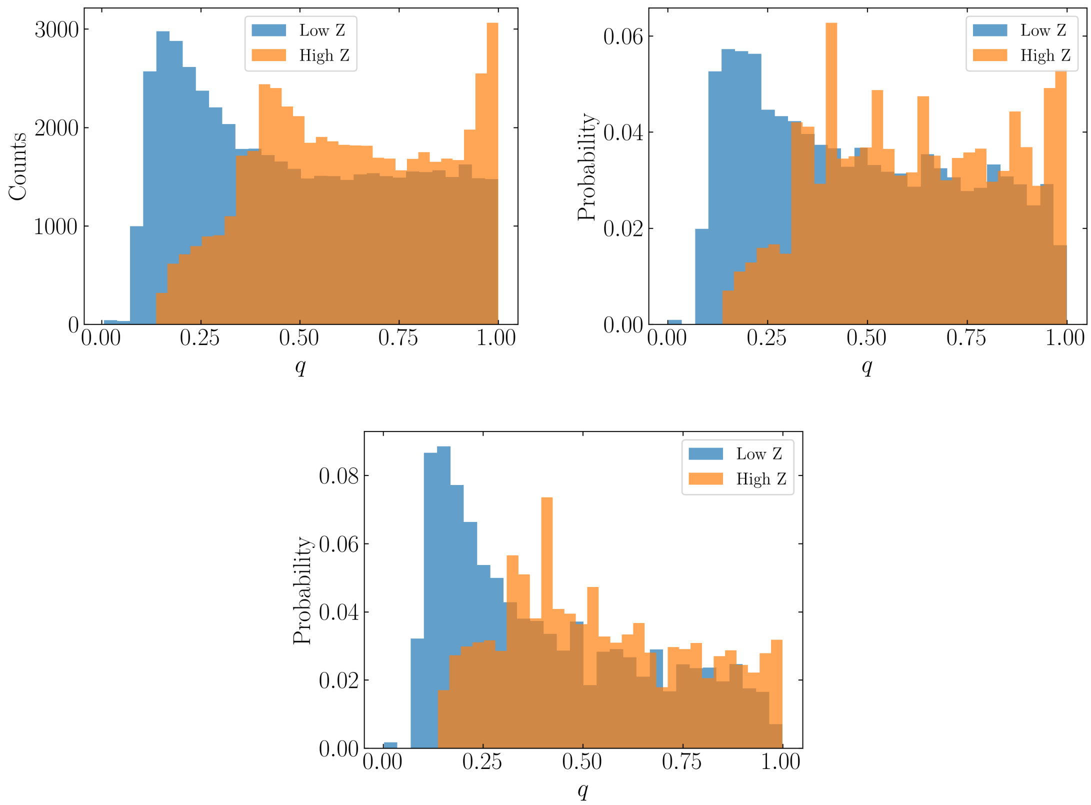

Figure 3: Distribution of q after (i) random pairings Salpeter IMF + supernovae, (ii) uniform $q_{\rm ZAMS}$, and (iii) uniform $\log\left(q_{\rm ZAMS}\right)$. Black dashed lines are Eqs. (11, 15), and $P(m) \propto m^{-2.35}$ in the three plots respectively.

more complicated; for each value of q at ZAMS: choose $m_2 \in [M_{\min}, qM_{\max}]$ and $m_1 = m_2/q$. Compute the BH value of $q_{\rm BH}$ by sending it through the SNe transfer function, and weight it by $P(m_1)P(m_2)$. Repeat for a grid of q and m_2, and histogram it all.

3 Checking Formula from Paper

I double checked Equations (20-21) in the new draft. I actually had the correct $j(e_{\rm lim})$ expression that Dong left in comments in the paper, but saw LML15 Eq. (52) and convinced myself that I had made an algebra mistake. It appears I must have misread LML15.

The derivation of the formula in the paper was omitted for its ugliness, but I give it below for verification. The equations we have are:

$$j^6(e_{\rm os}) = \frac{842}{15}\frac{G^3\mu m_{12}^3}{m_3 c^5 a^4 n}\left(\frac{a_{\rm out,eff}}{a}\right)^3, \tag{16}$$

$$j(e_{\rm lim}) \approx \frac{8\epsilon_{\rm GR}}{9+3\eta^2/4}. \tag{17}$$

We re-express the condition $j(e_{\rm os}) \gtrsim j(e_{\rm lim})$ (i.e. the limiting eccentricity is sufficiently extreme to execute one-shot mergers), so

$$842\frac{G^{5/2}a_{\rm out,eff}^3 m_{12}^{5/2}\mu}{15a^{11/2}c^5 m_3} \gtrsim \left(\frac{8}{9+3\eta^2/4}\frac{3Gm_{12}^2 a_{\rm out,eff}^3}{c^2 a^4 m_3}\right)^6, \tag{18}$$

$$a^{37/2} \gtrsim \frac{2^{18}\cdot 15}{842}\frac{G^{7/2}a_{\rm out,eff}^{15} m_{12}^{19/2}}{c^7 m_3^5 \mu \left(3+\eta^2/4\right)^6}, \tag{19}$$

$$\left(\frac{a}{a_{\rm out,eff}}\right)^{37/2} \gtrsim \frac{2^{18}\cdot 15}{842}\frac{G^{7/2}m_{12}^{17/2}}{c^7 m_3^5 a_{\rm out,eff}^{7/2}[q/(1+q)^2]\left(3+\eta^2/4\right)^6}, \tag{20}$$

$$\gtrsim 0.0186\left(\frac{a_{\rm out,eff}}{3600\,{\rm AU}}\right)^{-7/37}\left(\frac{m_{12}}{50M_\odot}\right)^{17/37}\left(\frac{30M_\odot}{m_3}\right)^{10/37}\left(\frac{q/(1+q)^2}{1/4}\right)^{-2/37}. \tag{21}$$

We have used that $\mu = m_{12}\left[q/(1+q)^2\right]$. The final numerical evaluation was done using WolframAlpha, and the URL linking to the evaluation is provided here.