Lidov-Kozai 90° Attractor

Yubo Su

Date

1 Equations

1.1 Bin's Papers

Our major references will be Bin's paper with Diego + Dong in 2015 (LML15) and Bin's later paper with Dong on spin-orbit misalignment (LL18). The target of study is §4.3 of LL18, where a 90° attractor in spin-orbit misalignment seems to appear when the octupole effect is negligible.

The easiest formulation is just to express everything in terms of $\mathbf{L}$ and $\mathbf{e}$, following LL18. We drop octupole terms and hold the third perturber constant. These equations come out to be (Eqs. 4–5 w/ substitutions)

$$\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t} = \frac{3}{4t_{LK}}\mu\sqrt{Gm_{12}a}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2) - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2)\right], \tag{1}$$

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} = \frac{3}{4t_{LK}}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2) + 2\mathbf{j}\times\mathbf{e} - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2)\right]. \tag{2}$$

Note that $\mathbf{j} \equiv \sqrt{1-e^2}\hat{\mathbf{L}} = \frac{\mathbf{L}}{\mu\sqrt{Gm_{12}a}}$. $m_{12} = m_1 + m_2$ and $\mu = m_1 m_2/m_{12}$. We've defined

$$t_{LK} \equiv \frac{L_1}{\mu_1\Phi_0} = \frac{1}{n_1}\left(\frac{m_{12}}{m_3}\right)\left(\frac{a_2}{a}\right)^3\left(1-e_2^2\right)^{3/2}. \tag{3}$$

Here, $n_1 \equiv \sqrt{Gm_{12}/a^3}$. Thus, $1/t_{LK} \propto a^{3/2}$.

GW radiation (Peters 1964) cause decays of $\mathbf{L}$ and $\mathbf{e}$ as

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{GW} = -\frac{32}{5}\frac{G^{7/2}}{c^5}\frac{\mu^2 m_{12}^{5/2}}{a^{7/2}}\frac{1+7e^2/8}{\left(1-e^2\right)^2}\hat{L}, \tag{4}$$

$$\left.\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}\right|_{GW} = -\frac{304}{15}\frac{G^3}{c^5}\frac{\mu m_{12}^2}{a^4\left(1-e^2\right)^{5/2}}\left(1+\frac{121}{304}e^2\right)\mathbf{e}. \tag{5}$$

Here, $m_{12} \equiv m_1 + m_2$, and a is implicitly defined by $\mathbf{L}$ and e. The last GR effect is precession of $\vec{e}$, which acts as

$$\left.\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}\right|_{GR} = \frac{1}{t_{GR}}\hat{\mathbf{L}}\times\mathbf{e}, \tag{6}$$

$$\frac{1}{t_{GR}} \equiv \frac{3Gnm_{12}}{c^2 a\left(1-e^2\right)}. \tag{7}$$

Note that $t_{GR}^{-1} \propto a^{-5/2}$.

Given this system (from LML15 + LL18), we can then add the spin-orbit coupling term (from de Sitter precession), which is given in LL18 to be

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \frac{1}{t_{SL}}\hat{\mathbf{L}} \times \hat{\mathbf{S}}, \tag{8}$$

$$\frac{1}{t_{SL}} \equiv \frac{3Gn\left(m_2 + \mu/3\right)}{2c^2 a\left(1-e^2\right)}. \tag{9}$$

Note that μ is the reduced mass of the inner binary. We can drop the back-reaction term since $S \ll L$. Thus, $t_{SL}^{-1} \propto a^{-5/2}$ as well.

Finally, an adiabaticity parameter can be defined:

$$\mathscr{A} \equiv \left|\frac{\Omega_{SL}}{\Omega_L}\right|. \tag{10}$$

Here, $\Omega_L \simeq \frac{3(1+4e^2)}{8t_{LK}\sqrt{1-e^2}}\left|\sin 2I\right|$ is an approximate rate of change of L during an LK cycle

It's natural to redimensionalize to the initial LK time such that

$$\frac{1}{t_{LK,0}} \equiv \left(\frac{a}{a_0}\right)^{3/2}\frac{1}{t_{LK}}, \tag{11}$$

since nothing else in t_{LK} is changing. The next natural timescale for gravitational waves is

$$\frac{1}{t_{GW}} \equiv \frac{G^3\mu m_{12}^2}{c^5 a^4} \equiv \frac{1}{t_{GW,0}}\left(\frac{a_0}{a}\right)^4 \equiv \epsilon_{GW}\frac{1}{t_{LK,0}}\left(\frac{a_0}{a}\right)^4. \tag{12}$$

We can repeat the procedure for the GR precession term and the spin-orbit coupling terms:

$$\frac{1}{t_{GR}} = \epsilon_{GR}\frac{1}{t_{LK,0}}\left(\frac{a_0}{a}\right)^{5/2}, \tag{13}$$

$$\frac{1}{t_{SL}} = \epsilon_{SL}\frac{1}{t_{LK,0}}\left(\frac{a_0}{a}\right)^{5/2}. \tag{14}$$

Thus, finally, if we let $\tau = t/t_{LK,0}$, then we obtain full equations of motion (note that $a_0 = 1$ below)

$$\begin{aligned}\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}\tau} ={}& \left(\frac{a}{a_0}\right)^{3/2}\frac{3}{4}\sqrt{a}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2) - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2)\right] \\ &- \epsilon_{GW}\left(\frac{a_0}{a}\right)^4\frac{32}{5}\frac{1+7e^2/8}{\left(1-e^2\right)^{5/2}}\mathbf{L},\end{aligned} \tag{15}$$

$$\begin{aligned}\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}\tau} ={}& \left(\frac{a}{a_0}\right)^{3/2}\frac{3}{4}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2) + 2\mathbf{j}\times\mathbf{e} - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2)\right] \\ &- \epsilon_{GW}\left(\frac{a_0}{a}\right)^4\frac{304}{15}\frac{1}{\left(1-e^2\right)^{5/2}}\left(1+\frac{121}{304}e^2\right)\mathbf{e} \\ &+ \epsilon_{GR}\left(\frac{a_0}{a}\right)^{5/2}\frac{1}{1-e^2}\hat{\mathbf{L}}\times\mathbf{e},\end{aligned} \tag{16}$$

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \epsilon_{SL}\left(\frac{a_0}{a}\right)^{5/2}\frac{1}{1-e^2}\hat{\mathbf{L}}\times\hat{\mathbf{S}}. \tag{17}$$

For reference, we note that $a = |\mathbf{L}|^2/\left(\mu^2 G m_{12}\left(1-e^2\right)\right)$, while $\mathbf{j} = \mathbf{L}/\left(\mu\sqrt{Gm_{12}a}\right)$. To invert $a(\mathbf{L})$ and $\mathbf{J}(\mathbf{L})$ in this coordinate system where $a_0 = 1$, it is easiest to choose the angular momentum dimensions such that $\mu\sqrt{Gm_{12}} = 1$, such that now

$$|\mathbf{L}(t=0)| \equiv \mu\sqrt{Gm_{12}a_0\left(1-e_0^2\right)} = \sqrt{\left(1-e_0^2\right)}, \tag{18}$$

$$a = \frac{|\mathbf{L}|^2}{1-e^2}, \tag{19}$$

$$\mathbf{j} = \frac{\mathbf{L}}{\sqrt{a}} = \hat{\mathbf{L}}\sqrt{1-e^2}. \tag{20}$$

Finally, the timescales are

$$t_{LK,0} = \frac{1}{n}\frac{m_{12}}{m_3}\left(\frac{a_2}{a(t=0)}\right)^3\left(1-e_2^2\right)^{3/2}, \tag{21}$$

$$\epsilon_{GW} \equiv \frac{t_{LK,0}}{t_{GW,0}} = \frac{1}{n}\frac{m_{12}}{m_3}\frac{a_2^3}{(a(t=0))^7}\left(1-e_2^2\right)^{3/2}\frac{G^3\mu m_{12}^2}{c^5}, \tag{22}$$

$$\epsilon_{GR} \equiv \frac{t_{LK,0}}{t_{GR,0}} = \frac{m_{12}}{m_3}\frac{a_2^3}{(a(t=0))^4}\left(1-e_2^2\right)^{3/2}\frac{3Gm_{12}}{c^2}, \tag{23}$$

$$\epsilon_{SL} \equiv \frac{t_{SL,0}}{t_{GR,0}} = \frac{m_{12}}{m_3}\frac{a_2^3}{(a(t=0))^4}\left(1-e_2^2\right)^{3/2}\frac{3G\left(m_2+\mu/3\right)}{2c^2}. \tag{24}$$

The adiabacitity parameter

$$\mathscr{A} \equiv \left|\frac{\Omega_{SL}}{\Omega_L}\right| = \frac{\epsilon_{SL}}{t_{LK,0}}\left(\frac{a_0}{a}\right)^{5/2}\frac{1}{1-e^2}\left[\frac{3(1+4e^2)}{8t_{LK,0}\sqrt{1-e^2}}\left(\frac{a}{a_0}\right)^{3/2}|\sin 2I|\right]^{-1}, \tag{25}$$

(note that Ω_L is a somewhat averaged sense, see LL18) can be evaluated in these units as

$$\mathscr{A} = \epsilon_{SL}\left(\frac{a_0}{a}\right)^4\frac{1}{\sqrt{1-e^2}}\frac{8}{3\left(1+4e^2\right)|\sin 2I|}. \tag{26}$$

Note also that the Hamiltonian is just

$$H = \Omega_{SL}\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}, = \epsilon_{SL}\left(\frac{a_0}{a}\right)^{5/2}\frac{1}{1-e^2}\hat{\mathbf{L}}\cdot\hat{\mathbf{S}}. \tag{27}$$

1.2 Maximum Eccentricity and Merger Time

Note that, since we are only evolving $\mathbf{L}$ and $\mathbf{e}$, and not $\mathbf{L}_2$ and $\mathbf{e}_2$, we are in the test mass approximation, under which we set $\eta = 0$ in Bin's equations. As such, the maximum eccentricity satisfies (Eq 42 of LL18 with $\eta \to 0$)

$$\frac{3}{8}\frac{j_{\min}^2 - 1}{j_{\min}^2}\left[5\cos I_0^2 - 3j_{\min}^2\right] + \epsilon_{GR}\left(1 - \frac{1}{j_{\min}}\right) = 0. \tag{28}$$

Note that ϵ_{GR} is exactly as we defined above, incidentally, and that when GR is negligible, this reduces to the classic $j_{\min} \equiv \sqrt{1-e_{\max}^2} = \sqrt{\frac{5}{3}\cos^2 I_0}$. Since ϵ_{GR} is generally very small for most of the evolution, this generally reduces to the well known

$$e_{\max} = \sqrt{1 - \frac{5}{3}\cos^2 I_0}. \tag{29}$$

This only fails to saturate for extremely high eccentricities, so $I_0 \to 90°$.

1.3 Attractor Behavior

Proposal: The reason the 90° attractor appears is that the initial θ_{sb} is roughly stationary for $\mathscr{A} \ll 1$ (only small kicks during each LK cycle, as long as the maximum eccentricity isn't too large), then as we enter the transadiabatic regime, the L-K cycles die down and we simply have conservation of adiabatic invariant.

The latter half of this follows the LL18 claim, where the requirement that $\epsilon_{GR} \lesssim 9/4$ (GR precession of pericenter is slow enough that L-K survives) equates to $\mathscr{A} \lesssim 3$. The former half of this is somewhat tricky, but we can understand what is happening if we consider what is happening in the frame corotating with $\Omega_{SL,e=0}$ about $\hat{z}$: every time that a LK cycle appears, Ω_{SL} becomes much larger, and the axis of precession changes from $\hat{z}$ to the location of $\hat{L}$ very briefly. We can imagine this as a kick in this corotating frame (which is the right frame to consider for $\mathscr{A} \ll 1$). In the limit that I does not change very much between L-K cycles, and the azimuthal angle of $\hat{L}$ is roughly symmetric, the impulses roughly cancel out in the θ_{sb} direction. In other words, after two LK cycles, θ_{sb} does not change much in the corotating frame. This is indeed the picture that we obtain when we observe the plot.

As such, the hypothesis is that if $\mathscr{A} \gtrsim 1$ is satisfied while the kicks are still *small*, then deviations about 90° cannot be very large, and adiabatic invariance tilts us right over. On the other hand, if the kicks have become *large*, then θ_{sb} after any particular LK cycle is far from 90°, and this is frozen in during the adiabatic invariance phase. This explains the key observation that the initial θ_{sb} eventually becomes the final θ_{sl}, regardless of whether it is 90°. Furthermore, it explains why the kicks to θ_{eff} become larger over time, but peak smaller for larger I_0.

The natural way to think about this is to consider the evolution of the trajectory in (a,e) space. There are two curves that can be drawn on here, $\mathscr{A} \sim 1$ and $|\Delta\theta_{sb}| \sim 1$ (the kick size), and then we can see which one gets crossed first. The hypothesis is that the first always gets crossed first, but if $e_{\max}$ is too large, then the second gets crossed in the same LK cycle, and we get kicked far away from the starting θ, and have this frozen into θ_{sl}. We need to find out how to draw these boundaries in (a,e) space. Drawing $\mathscr{A}$ is very easy, since we have the explicit formulla for it.

To get the kick size, we have to integrate one of the LK peaks. This is easiest done by considering the evolution of the $\delta e \equiv 1-e$ variable by dotting $\vec{e}$ into $\frac{\mathrm{d}\vec{e}}{\mathrm{d}t}$, such that

$$
\begin{aligned}
2e\frac{\mathrm{d}e}{\mathrm{d}t} &= \frac{\mathrm{d}(\vec{e}\cdot\vec{e})}{\mathrm{d}t} = 2\vec{e}\cdot\frac{\mathrm{d}\vec{e}}{\mathrm{d}t}, && (30)\\
&= -\frac{15}{2t_{LK}}(\mathbf{e}\cdot\hat{n}_2)(\hat{n}_2\cdot(\mathbf{e}\times\mathbf{j})), && (31)\\
&\lesssim \pm\frac{15}{2t_{LK}}e^2\sqrt{1-e^2}, && (32)\\
\frac{\mathrm{d}e}{\mathrm{d}t} &\sim -\frac{15}{4t_{LK}}e\sqrt{1-e^2}, && (33)\\
\frac{\mathrm{d}(\delta e)}{\mathrm{d}t} &\sim \frac{15}{4t_{LK}}\sqrt{2\delta e}, && (34)\\
\delta e(t) &\sim \left(\frac{15t}{4\sqrt{2}t_{LK}}\right)^2. && (35)
\end{aligned}
$$

The finding of a power law/quadratic seems in accordance w/ my simulations, though I have to plot

$\delta e - \delta e_{\rm min}$. Then, we can simply integrate

$$\begin{aligned}
\Delta\theta_{sb} &\sim \oint\limits_{LK} \Omega_{SL}\,\mathrm{d}t, && (36)\\
&\sim \frac{\epsilon_{SL}}{2}\left(\frac{a_0}{a}\right)^{5/2} \oint\limits_{LK} \frac{1}{\delta e}\,\mathrm{d}t, && (37)\\
&\sim \frac{\epsilon_{SL}}{2}\left(\frac{a_0}{a}\right)^{5/2} \oint\limits_{LK} \frac{1}{\delta e_{\rm min} + \left(\frac{15}{4\sqrt{2}t_{LK}}\right)^2 t^2}\,\mathrm{d}t, && (38)\\
&\sim \frac{\epsilon_{SL}}{2\delta e_{\rm min}}\left(\frac{a_0}{a}\right)^{5/2} \pi \frac{4t_{LK}\sqrt{2\delta e_{\rm min}}}{15}, && (39)\\
&\sim \frac{\epsilon_{SL}}{\sqrt{2\delta e_{\rm min}}}\frac{a_0}{a}\pi\frac{4}{15}. && (40)
\end{aligned}$$

In the last few steps, we've just taken the bounds of integration to be $t \in [-\infty, \infty]$ for simplicity (they contribute negligibly), and used $t_{LK} = (a/a_0)^{3/2}$ since $t_{LK,0} = 0$.

If we now explicitly write down the criteria where $\mathscr{A} \sim 1$ and $\Delta\theta_{sb} \sim 1$ in the (a, e) plane, then we obtain

$$\begin{aligned}
a_{c,\theta} &\sim \frac{\epsilon_{SL}}{\sqrt{2\delta e_{\rm min}}}\frac{4\pi}{15}, && (41)\\
a_{c,\mathscr{A}} &\sim \left[\epsilon_{SL}\frac{8/3}{\sqrt{1-e_{\rm min}^2}\left(1+4e_{\rm min}^2\right)\left|\sin 2I\right|}\right]^{1/4}. && (42)
\end{aligned}$$

The key difference between the two is that kicks occur at $e_{\rm max}$ or $\delta e_{\rm min}$, while the adiabaticity parameter is moreso evaluated at $e_{\rm min}$.

A j Equations

We define vectors

$$\begin{aligned}
\mathbf{j} &= \sqrt{1-e^2}\hat{n}, && (43)\\
\mathbf{e} &= e\hat{u}. && (44)
\end{aligned}$$

Here, $\mathbf{j}$ is the dimensionless angular momentum vector and $\mathbf{e}$ is the eccentricity vector; see LML15 for precise definitions. Note that $\mathbf{j} \cdot \mathbf{e} = 0$, $j^2 + e^2 = 1$. Then, the EOM for the inner and outer vectors satisfy to quadrupolar order

$$\begin{aligned}
\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} &= \frac{3}{4t_{LK}}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2) - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2)\right], && (45)\\
\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t} &= \frac{3}{4t_{LK}}\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2) + 2\mathbf{j}\times\mathbf{e} - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2)\right]. && (46)
\end{aligned}$$

Let's assume for the time being that $L_1 \ll L_2$, so the system is sufficiently hierarchical that $\mathbf{j}_2$, $\mathbf{e}_2$ are constants. Note for reference that

$$t_{LK} \equiv \frac{L_1}{\mu_1\Phi_0} = \frac{1}{n_1}\left(\frac{m_1+m_2}{m_3}\right)\left(\frac{a_2}{a}\right)^3\left(1-e_2^2\right)^{3/2}. \tag{47}$$

Here, $n_1 \equiv \sqrt{G(m_1+m_2)/a^3}$. Thus, $1/t_{LK} \propto a^{3/2}$.

GW radiation (Peters 1964) cause decays of $\mathbf{L}$ and $\mathbf{e}$ as

$$\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{GW} = -\frac{32}{5}\frac{G^{7/2}}{c^5}\frac{\mu^2 m_{12}^{5/2}}{a^{7/2}}\frac{1+7e^2/8}{\left(1-e^2\right)^2}\hat{L}, \tag{48}$$

$$\left.\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}\right|_{GW} = -\frac{304}{15}\frac{G^3}{c^5}\frac{\mu m_{12}^2}{a^4\left(1-e^2\right)^{5/2}}\left(1+\frac{121}{304}e^2\right)\mathbf{e}, \tag{49}$$

$$\left.\frac{\dot{a}}{a}\right|_{GW} = -\frac{64}{5}\frac{G^3\mu m_{12}^2}{c^5 a^4}\frac{1}{\left(1-e^2\right)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \tag{50}$$

Here, $m_{12} \equiv m_1+m_2$. The last GR effect is precession of $\vec{e}$, which acts as

$$\left.\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}\right|_{GR} = \frac{3Gnm_{12}}{c^2 a\left(1-e^2\right)}. \tag{51}$$

Given this system (from LML15 + LL18), we can then add the spin-orbit coupling term (from de Sitter precession), which is given in LL18 to be

$$\frac{\mathrm{d}\hat{S}}{\mathrm{d}t} = \Omega_{SL}\hat{L}\times\hat{S}, \tag{52}$$

$$\Omega_{SL} \equiv \frac{3Gn\left(m_2+\mu/3\right)}{2c^2 a\left(1-e^2\right)}. \tag{53}$$

Note that μ is the reduced mass of the inner binary. We can drop the back-reaction term since $S \ll L$. Thus, $\Omega_{SL} \propto a^{-5/2}$.

What is observed is that, as this system is evolved forward in time and GR coalesces the inner binary, $\theta_{sl} \equiv \arccos\left(\hat{S}\cdot\hat{L}\right)$ goes to $90°$ consistently. The relevant figure is Fig. 19 of LL18, which shows that for a close-in, low-eccentricity perturber ($\bar{a}_{\mathrm{out,eff}} \propto a_{out}$), the focusing is significantly stronger. Note that initially, $I \equiv \arccos\left(\hat{L}\cdot\hat{L}_2\right) \approx 90°$ while $\theta_{sl} \approx 0$.

In LL18, an adiabaticity parameter is defined:

$$\mathscr{A} \equiv \left|\frac{\Omega_{SL}}{\Omega_L}\right|, \tag{54}$$

where $\Omega_L \simeq \left\langle \frac{\mathrm{d}\hat{L}}{\mathrm{d}t}\right\rangle_{LK}$ to quadrupolar order. As the inner binary coalesces, $\mathscr{A}$ transitions from $\ll 1$ to $\gg 1$ (as Ω_{SL} is a GR effect so ramps up very quickly as orbital separation decreases).

The adiabaticity parameter $\mathscr{A}$ can be plotted upon rescaling in our coordinates. Note that $\Omega_{\mathrm{SL}} = \frac{\delta a_0}{a t_{LK,0}}$, while $\Omega_L \simeq \frac{3(1+4e^2)}{8t_{LK}\sqrt{1-e^2}}|\sin 2I|$ can also be expressed in units of $t_{LK,0}$. This gives us

$$\mathscr{A} = \frac{8\delta\sqrt{1-e^2}}{3\left(1+4e^2\right)}\left(\frac{a_0}{a}\right)^4. \tag{55}$$

A.1 Simulations

First, we run GR-less simulations, so let's take $t_{LK} = 1$ (no semimajor axis evolution), and we reproduce LK oscillations.

Next, when accounting for GR, we should let a evolve as above. Note that since $\mathbf{j}$ and $\vec{e}$ are our dynamical variables, we should use $\mathbf{j} \equiv \sqrt{1-e^2}\hat{L} = \sqrt{1-e^2}\frac{\mathbf{L}}{\mu\sqrt{Gm_{12}a(1-e^2)}}$ and rewrite

$$\left.\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t}\right|_{GW} = \frac{1}{\mu\sqrt{GMa}}\left.\frac{\mathrm{d}\mathbf{L}}{\mathrm{d}t}\right|_{GW} - \frac{\mathbf{j}}{2a}\left.\frac{\mathrm{d}a}{\mathrm{d}t}\right|_{GW}. \tag{56}$$

To double check, we should verify that $\left.\frac{\mathrm{d}(j^2+e^2)}{\mathrm{d}t}\right|_{GW} = 0$, which can be verified as (Let's set $G = M = \mu = a = c = 1$ for convenience)

$$\frac{1}{2}\frac{\mathrm{d}(j^2+e^2)}{\mathrm{d}t} = \mathbf{j}\cdot\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}t} + \mathbf{e}\cdot\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}t}, \tag{57}$$

$$= \mathbf{j}\cdot\left[\left(-\frac{32}{5}\frac{1+7e^2/8}{\left(1-e^2\right)^2}\right)\hat{L} - \frac{\mathbf{j}}{2}\left(-\frac{64}{5}\frac{1+73e^2/24+37e^4/96}{\left(1-e^2\right)^{7/2}}\right)\right] + \mathbf{e}\cdot\left(-\frac{304}{15}\frac{1+121e^2/304}{\left(1-e^2\right)^{5/2}}\right)\mathbf{e}, \tag{58}$$

$$= \left(-\frac{32}{5}\frac{1+7e^2/8}{\left(1-e^2\right)^{3/2}}\right) + \left(\frac{32}{5}\frac{1+73e^2/24+37e^4/96}{\left(1-e^2\right)^{5/2}}\right) + e^2\left(-\frac{304}{15}\frac{1+121e^2/304}{\left(1-e^2\right)^{5/2}}\right), \tag{59}$$

$$= \frac{1}{15\left(1-e^2\right)^{5/2}}\left[-96\left(1-e^2\right)\left(1+\frac{7e^2}{8}\right) + 96\left(1+\frac{73e^2}{24}+\frac{37e^4}{96}\right) - 304e^2\left(1+\frac{121e^2}{304}\right)\right]. \tag{60}$$

This can be verified to vanish upon term-by-term examination indeed.

For convenience, let's just define $t_{LK} = t_{LK,0}\frac{a_0^3}{a^3}$ and set $t_{LK,0} = 1$. Furthermore, the timescale of relevance for the GW terms is $t_{GW}^{-1} \sim \frac{G^3\mu m_{12}^2}{c^5 a^4}$. Let's express this as some ratio $t_{GW} = \epsilon t_{LK,0}\frac{a_0^4}{a^4}$. Thus, everything should be nondimensionalized this way.

We lastly add de-Sitter precession of the spin of one of the inner binary components, call this $\hat{S}$. Similarly, let's just define a proportionality constant $t_{SL} = \delta t_{LK,0}\frac{a_0}{a}$, then

$$\frac{\mathrm{d}\hat{S}}{\mathrm{d}(t/t_{LK,0})} = \delta\frac{a_0}{a}\hat{L}\times\hat{S}. \tag{61}$$

Our final simulation equations are thus ($\tau = t/t_{LK,0}$)

$$\frac{\mathrm{d}\mathbf{j}}{\mathrm{d}\tau} = \frac{3}{4}\left(\frac{a_0^3}{a^3}\right)\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2) - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2)\right]$$
$$-\left(\epsilon\frac{a_0^4}{a^4}\right)\left(\frac{32}{5}\frac{1+7e^2/8}{\left(1-e^2\right)^{5/2}} - \frac{32}{5}\frac{1}{\left(1-e^2\right)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right)\right)\mathbf{j}, \tag{62}$$

$$\frac{\mathrm{d}\mathbf{e}}{\mathrm{d}\tau} = \frac{3}{4}\left(\frac{a_0^3}{a^3}\right)\left[(\mathbf{j}\cdot\hat{n}_2)(\mathbf{e}\times\hat{n}_2) + 2\mathbf{j}\times\mathbf{e} - 5(\mathbf{e}\cdot\hat{n}_2)(\mathbf{j}\times\hat{n}_2)\right] - \left(\epsilon\frac{a_0^4}{a^4}\right)\frac{304}{15}\frac{1}{\left(1-e^2\right)^{5/2}}\left(1+\frac{121}{304}e^2\right)\mathbf{e}, \tag{63}$$

$$\frac{\mathrm{d}\hat{S}}{\mathrm{d}\tau} = \delta\frac{a_0}{a}\frac{\mathbf{j}}{\sqrt{1-e^2}}\times\hat{S}, \tag{64}$$

$$\frac{\mathrm{d}a}{\mathrm{d}\tau} = -a\left(\epsilon\frac{a_0^4}{a^4}\right)\frac{64}{5}\frac{1}{\left(1-e^2\right)^{7/2}}\left(1+\frac{73}{24}e^2+\frac{37}{96}e^4\right). \tag{65}$$