Evection Resonances in BH Triples

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1 03/15/21—Basics & Introduction

1.1 Writing Down the Hamiltonian

We assume a triple system $m_{1,2,3}$ and a, $a_{\rm out}$ with mutual inclination I. The 1PN apsidal precession of the inner binary has energy/Hamiltonian

$$H_{\rm GR} = -\frac{3G^2 m_1 m_2 m_{12}}{c^2 a^2 j(e)}, \tag{1}$$

while the external companion has averaged energy

$$H_{\rm out} = -\frac{G m_3 \mu_{12} a^2}{a_{\rm out}^3}\left[\frac{1}{16}\left[\left(6+9e^2\right)\cos^2 I - \left(2+3e^2\right)\right] + \frac{15}{32}\left(1+\cos I\right)^2 e^2 \cos\left(2\varpi - 2\lambda_{\rm out}\right)\right]. \tag{2}$$

Here, we have averaged over: $\varpi = \Omega + \omega$ is the longitude of pericenter of the inner orbit, so $\hat{\mathbf{e}} = \cos\varpi\hat{\mathbf{x}} + \sin\varpi\hat{\mathbf{y}}$, and $\lambda_{\rm out} = \varpi_{\rm out} + M_{\rm out}$ is the mean longitude of m_3, where $M_{\rm out}$ is the outer mean anomaly. Recall that $\Omega_{\rm out} = \dot{\lambda}_{\rm out} = \dot{M}_{\rm out}$, and the useful component form

$$\hat{\mathbf{r}}_{\rm out} = \cos\lambda_{\rm out}\hat{\mathbf{x}} + \sin\lambda_{\rm out}\cos I\hat{\mathbf{y}} + \sin\lambda_{\rm out}\sin I\hat{\mathbf{z}}. \tag{3}$$

Why is this interesting? Well, let's write $\epsilon \equiv \frac{G m_3 \mu_{12} a^2}{a_{\rm out}^3} / H_{\rm GR,0}$, where $H_{\rm GR,0} = [H_{\rm GR}]_{e=0}$, or

$$\epsilon = \frac{m_3 \mu_{12} a^2 c^2 a^2}{3G^2 m_1 m_2 m_{12} a_{\rm out}^3}, \tag{4}$$

$$= \frac{m_3 a^4 c^2}{3G m_{12}^2 a_{\rm out}^3}. \tag{5}$$

This is like $\epsilon_{\rm GR}^{-1}$ from our previous LK work. We are interested in the regime where $\epsilon \ll 1$. The total Hamiltonian of the system is then

$$\frac{H}{H_{\rm GR,0}} = -\frac{1}{j(e)} - \epsilon\left[\frac{1}{16}\left[\left(6+9e^2\right)\cos^2 I - (2+3e^2)\right] + \frac{15}{32}(1+\cos I)^2 e^2 \cos(2\varpi - 2\lambda_{\rm out})\right]. \tag{6}$$

We will eventually expand this Hamiltonian in terms of the conjugate variables $-\varpi$ and $1-\left(1-e^2\right)^{1/2} \approx e^2/2$ and obtain a separatrix'd Hamiltonian [Xu & Lai (26)]. But for now, we can be satisfied that some sort of separatrix might appear at $\epsilon \sim 1$? It's not clear yet.

1.2 Timescale Comparison

This section mostly follows Dong's notes, for completeness.

We need $\Omega_{\rm out} \equiv \sqrt{Gm_{123}/a_{\rm out}^3}$ to be of order $\dot{\varpi} \equiv 3Gnm_{12}/(c^2 a j^2)$. Assuming the eccentricity is already mostly damped (when $\epsilon_{\rm GR} \gg 1$, we expect this), then this gives

$$\frac{3Gm_{12}}{c^2 a} \simeq \frac{\Omega_{\rm out}}{n} = \sqrt{\frac{m_{123}}{m_{12}}\frac{a^3}{a_{\rm out}^3}}, \tag{7}$$

$$\left(\frac{a}{a_{\rm out}}\right)^{5/2} \simeq \frac{3Gm_{12}}{c^2 a_{\rm out}}\sqrt{\frac{m_{12}}{m_{123}}}. \tag{8}$$

Indeed, since everything is fixed, as a decays, the evection resonance will be crossed.

Will there be enough time to excite eccentricity? The eccentricity growth rate due to the evection resonance must of order $t_{\rm ZLK}^{-1} \sim n(m_3/m_{12})(a/a_{\rm out})^3$. On the other hand, orbital decay due to GW is of order

$$t_{\rm GW}^{-1} \simeq \frac{64}{5}\frac{G^3 m_{12}^2 \mu}{c^5 a^4} = \frac{64}{5} n \frac{G^{5/2} m_{12}^{3/2} \mu}{c^5 a^{5/2}}. \tag{9}$$

Thus, the resonance has time to grow if (in the third line, we invoke the resonance condition above)

$$t_{\rm GW}^{-1} \ll t_{\rm ZLK}^{-1}, \tag{10}$$

$$\frac{64}{5}\frac{G^{5/2} m_{12}^{3/2}\mu}{c^5 a^{5/2}} \ll \frac{m_3}{m_{12}}\left(\frac{a}{a_{\rm out}}\right)^3, \tag{11}$$

$$\frac{64}{5}\frac{G^{5/2} m_{12}^{3/2}\mu}{c^5 a_{\rm out}^{5/2}}\frac{c^2 a_{\rm out}}{3Gm_{12}}\sqrt{\frac{m_{123}}{m_{12}}} \ll \tag{12}$$

$$\frac{64}{15}\frac{G^{3/2} m_{123}^{1/2}\mu}{c^3 a_{\rm out}^{3/2}} \ll \tag{13}$$

$$\frac{64}{15}\left(\frac{v_{\rm out}}{c}\right)^3 \frac{m_{12}/4}{m_{123}} \ll \tag{14}$$

$$\left(\frac{v_{\rm out}}{c}\right)^3 \left(\frac{a_{\rm out}}{a}\right)^3 \frac{m_{12}^2}{m_{123} m_3} \ll 1. \tag{15}$$

Indeed, this must be the case. Another check requires

$$t_{\mathrm{ZLK}}^{-1} \ll \dot{\omega}, \tag{16}$$

$$\frac{m_3}{m_{12}}\left(\frac{a}{a_{\mathrm{out}}}\right)^3 \ll \frac{3Gm_{12}}{c^2 a} \sim \frac{\Omega_{\mathrm{out}}}{n}, \tag{17}$$

$$\ll \left(\frac{m_{123}}{m_{12}}\right)^{1/2}\left(\frac{a}{a_{\mathrm{out}}}\right)^{3/2}, \tag{18}$$

$$\frac{m_3}{m_{12}}\left(\frac{m_{12}}{m_{123}}\right)^{1/2}\left(\frac{a}{a_{\mathrm{out}}}\right)^{3/2} \ll 1. \tag{19}$$

This is also satisfied. Thus, resonance excitation should be possible.

What are the kinds of systems that are interacting? If we want LISA band, we need $n/\pi \sim 10^{-3}$ Hz, and:

$$\Omega_{\mathrm{out}} \simeq \frac{3Gnm_{12}}{c^2 a}, \tag{20}$$

$$\simeq \frac{3n^3 a^2}{c^2}, \tag{21}$$

$$\simeq \frac{3n^3}{c^2}\left(\frac{Gm_{12}}{n^2}\right)^{2/3}, \tag{22}$$

$$\simeq 10^{-7}\left(\frac{P}{10^3\,\mathrm{s}}\right)^{-5/3}\left(\frac{m_{12}}{2M_\odot}\right)^{2/3}\ \mathrm{s}^{-1}, \tag{23}$$

$$a_{\mathrm{out}} = \left(\frac{Gm_{123}}{\Omega_{\mathrm{out}}^2}\right)^{1/3}, \tag{24}$$

$$= 2.4\left(\frac{m_{123}}{3M_\odot}\right)^{1/3}\left(\frac{P}{10^3\,\mathrm{s}}\right)^{10/9}\left(\frac{m_{12}}{2M_\odot}\right)^{-4/9}\ \mathrm{AU}. \tag{25}$$

Indeed then, this is not going to be super useful unless m_3 is a SMBH, in which case $a_{\mathrm{out}} \sim 100$–$1000$ AU. Note that $a \sim 3\times10^8$ m.

The other scenario then is that we cross this resonance, get a large eccentricity, and it doesn't completely damp by the time it crosses the LISA band? Well, we saw above that $(a/a_{\mathrm{out}})^{5/2} \propto a_{\mathrm{out}}^{-1}$, so if we fix the masses then increasing a_{out} by a factor of 4 increases a by a factor of 32, i.e. $a \sim 0.05$ AU while $a_{\mathrm{out}} = 10$ AU, somewhat believable. Since the rates of change of $\ln a$ and $\ln e$ differ only by a factor of $j^2(e)$, if e is only modest, then it will have to also decay by ~ 30 by the time a enters the LISA band. However, if we can excite a substantial e like $j^2(e) = 0.1$ (corresponding to $e = 0.95$), then e will only decay by ~ 3 upon entering the LISA band, which leaves us with an $e = 0.3$. Wenrui's paper suggests evection isn't quite this strong, but maybe some sort of scenario is possible.

The final solution is to use evection to pump an existing large eccentricity up a little bit. But it's becoming clear that we aren't going to cleanly get excitation in the LISA band, and that we will really need to consider dynamics during *and after* the resonance.

1.3 Hamiltonian Level Curves

Let's try to nondimensionalize the Hamiltonian now, like Wenrui's paper. Call $\gamma = -\varpi$ and $\Gamma = 1 - \sqrt{1-e^2}$, so that $j(e) = 1 - \Gamma$ and $e^2 = 1 - (1-\Gamma)^2 = 2\Gamma + \Gamma^2$, then

$$\begin{aligned}
\frac{H}{H_{\rm GR,0}} ={}& -\frac{1}{1-\Gamma} - \frac{\epsilon}{16}\left[\left(6 + 9\left(2\Gamma + \Gamma^2\right)\right)\cos^2 I - \left(2 + 3\left(2\Gamma + \Gamma^2\right)\right)\right] \\
&- \frac{15\epsilon}{32}\left(1 + \cos I\right)^2\left(2\Gamma + \Gamma^2\right)\cos\left(2\gamma + 2\lambda_{\rm out}\right), \qquad (26)\\
H' ={}& -\frac{1}{1-\Gamma} - \frac{\epsilon}{16}\left[\left(9\cos^2 I - 3\right)\left(2\Gamma + \Gamma^2\right)\right] - \frac{15\epsilon}{32}\left(1 + \cos I\right)^2\left(2\Gamma + \Gamma^2\right)\cos\left(2\gamma + 2\lambda_{\rm out}\right), \qquad (27)\\
\approx{}& \Gamma\left(-1 - 2\epsilon A\right) + \Gamma^2\left(-1 + \epsilon A\right) - \Gamma\epsilon B\cos\theta + \mathcal{O}\left(\Gamma^3, \Gamma^2\cos\theta\right), \qquad (28)\\
A ={}& \frac{9\cos^2 I - 3}{16}, \qquad (29)\\
B ={}& \frac{15}{16}\left(1 + \cos I\right)^2, \qquad (30)\\
\theta ={}& 2\varpi - 2\lambda_{\rm out}. \qquad (31)
\end{aligned}$$

This is not quite as clean as Wenrui's form, but it has the advantage (for me) that the ϵ dependence is still explicit, while A, B are almost always positive (except for when $\cos^2 I < 1/3$).

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2.1 Deriving the Hamiltonian Carefully: Circular Perturber

After doing some simulations, it is pretty clear that we will need a precise derivation of the Hamiltonian. We start with the full Hamiltonian

$$H = -\frac{3G^2 m_1 m_2 m_{12}}{c^2 a^2 j(e)} - \frac{G m_3 \mu_{12} a^2}{r_{\rm out}^3}\left[\frac{1}{16}\left[\left(6 + 9e^2\right)\cos^2 I - \left(2 + 3e^2\right)\right] + \frac{15}{32}\left(1 + \cos I\right)^2 e^2 \cos\left(2\varpi - 2\lambda_{\rm out}\right)\right]. \qquad (32)$$

Note that when the outer orbit is circular, $r_{\rm out} = a_{\rm out}$ and $\lambda_{\rm out} = f_{\rm out}$. I can't seem to figure out how to non-dimensionalize the Hamiltonian and the time right now, so let's just factor out $H_{\rm GR,0} \equiv 3G^2 m_1 m_2 m_{12}/c^2 a^2$, and drop constant terms

$$H = -H_{\rm GR,0}\left\{\frac{1}{j(e)} + \epsilon\left[\frac{(6 + 9e^2)\cos^2 I - 3e^2}{16} + \frac{15}{32}\left(1 + \cos I\right)^2 e^2 \cos\left(2\varpi - 2\lambda_{\rm out}\right)\right]\right\}. \qquad (33)$$

Here, again, ϵ was given above

$$\epsilon = \frac{m_3 a^4 c^2}{3G m_{12}^2 a_{\rm out}^3}. \qquad (34)$$

Okay, I give up, we non-dimensionalize the Hamiltonian by dividing by $H_{\rm GR,0}$ and scale time via

$$\tau \equiv \dot{\omega}_{\rm GR} t = \frac{3Gn m_{12}}{c^2 a j^2} t. \qquad (35)$$

We now seek the appropriate canonical transformation for our rescaled H. We can first directly recast H in terms of the modified Delaunay variables; since the generating function is time-independent, we just need to re-express H in terms of the new variables ($\theta_1 = \lambda$ does not appear in the original H, so $L_D = J_1 = \sqrt{G m_{12} a}$ is conserved, which we've just used in renormalizing the Hamiltonian and by convention set $L_D = J_1 = 1$)

$$J_2 = 1 - \sqrt{1-e^2} \qquad \theta_2 = -\varpi, \tag{36}$$

$$J_3 = \sqrt{1-e^2}(1-\cos I) \qquad \theta_3 = -\Omega, \tag{37}$$

and so $\sqrt{1-e^2} = 1 - J_2$, $e^2 = 1-(1-J_2)^2 = 2J_2 - J_2^2$, and $\cos I = 1 - [J_3/(1-J_2)]$ (note that J_3 is also a constant, but since e is not constant, neither is $\cos I$, strictly speaking)

$$H(J_2,\theta_2,J_3,\theta_3) = -\frac{1}{1-J_2} - \epsilon[A + B\cos(-2\theta_2 - 2\lambda_{\rm out})], \tag{38}$$

$$A = \frac{3\cos^2 I - 1}{16}3e^2 + \frac{3}{8}\cos^2 I = \frac{3}{16}\left(2J_2 - J_2^2\right)\left(2 - 6\frac{J_3}{1-J_2} + 3\left(\frac{J_3}{1-J_2}\right)^2\right) + \frac{3}{8}\cos^2 I, \tag{39}$$

$$B = \frac{15}{32}(1+\cos I)^2 e^2 = \frac{15}{32}\left[4 - 4\frac{J_3}{1-J_2} + \left(\frac{J_3}{1-J_2}\right)^2\right]\left(2J_2 - J_2^2\right). \tag{40}$$

We further want to transform the Hamiltonian for some canonical variable $\phi = -2\theta_2 - 2\lambda_{\rm out}$. The canonical variable conjugate to ϕ is just $\Gamma = -J_2/2$, as we can verify via the Poisson bracket:

$$\{\phi,\Gamma\} = \frac{\partial\phi}{\partial\theta_2}\frac{\partial\Gamma}{\partial J_2} - \frac{\partial\phi}{\partial J_2}\frac{\partial\Gamma}{\partial\theta_2} = (-2)\left(-\frac{1}{2}\right) = 1. \tag{41}$$

The generating function for this canonical transformation is just $S(p,Q) = -J_2\phi/2$, so the Hamiltonian for the resonant angle ϕ becomes

$$H\left(\Gamma,\phi;J_3\right) = -\frac{1}{1+2\Gamma} - \epsilon\left(A + B\cos\phi\right) + \frac{\partial S}{\partial t}, \tag{42}$$

$$= -\frac{1}{1+2\Gamma} - \epsilon\left(A + B\cos\phi\right) - \frac{J_2}{2}\left(-2\frac{\mathrm{d}\lambda_{\rm out}}{\mathrm{d}t}\right), \tag{43}$$

$$= -\frac{1}{1+2\Gamma} - \epsilon\left(A + B\cos\phi\right) - 2\Gamma\frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}}, \tag{44}$$

$$A = \frac{3}{16}\left(-4\Gamma - 4\Gamma^2\right)\left(2 - 6\frac{J_3}{1+2\Gamma} + 3\left(\frac{J_3}{1+2\Gamma}\right)^2\right) + \frac{3}{8}\left(1 - \frac{J_3}{1+2\Gamma}\right)^2, \tag{45}$$

$$B = \frac{15}{32}\left[4 - 4\frac{J_3}{1+2\Gamma} + \left(\frac{J_3}{1+2\Gamma}\right)^2\right]\left(-4\Gamma - 4\Gamma^2\right). \tag{46}$$

Note that $\lambda_{\rm out} = \omega_{\rm out} + \Omega_{\rm out} + \mathscr{M}_{\rm out}$, so the time derivative is just the mean motion (at the current order of approximation) in nondimensional units.

Up until here, everything is still exact; we can now compute the Hamiltonian to leading order in Γ, Γ^2 and $\cos\phi$ (we drop constant terms in ϵ and J_3)

$$H\left(\Gamma,\phi\right) = 2\Gamma - 4\Gamma^2 - \epsilon\left(A + B\cos\phi\right) - 2\Gamma\frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} + \mathcal{O}\left(\Gamma^3\right), \tag{47}$$

$$= 2\Gamma\left(1 - \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}}\right) - 4\Gamma^2 - \epsilon\left(A + B\cos\phi\right) + \mathcal{O}\left(\Gamma^3\right), \tag{48}$$

$$A = -\frac{3\left(\Gamma+\Gamma^2\right)}{4}(2 - 6J_3) + \frac{3}{8}(4J_3\Gamma) + \mathcal{O}\left(\Gamma^3, \Gamma^2 J_3\right), \tag{49}$$

$$= \frac{3}{2}\left(4J_3\Gamma - \Gamma - \Gamma^2\right) + \mathcal{O}\left(\Gamma^3, \Gamma^2 J_3\right), \tag{50}$$

$$B = -\frac{15}{2}\Gamma + \mathcal{O}\left(\Gamma^2, J_3\Gamma\right), \tag{51}$$

$$H\left(\Gamma,\phi\right) = 2\Gamma\left(1 - \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}}\right) - 4\Gamma^2 - \frac{\epsilon}{2}\left[3\left(4J_3\Gamma - \Gamma - \Gamma^2\right) - 15\Gamma\cos\phi\right] + \mathcal{O}\left(\Gamma^3, \Gamma^2 J_3, \Gamma^2\cos\phi\right), \tag{52}$$

$$= 2\Gamma\left(1 - \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} - \frac{\epsilon(12J_3 - 3)}{4}\right) - \Gamma^2\left(4 - \frac{3\epsilon}{2}\right) + \frac{\epsilon}{2}15\Gamma\cos\phi + \mathcal{O}\left(\Gamma^3, \Gamma^2 J_3, \Gamma^2\cos\phi\right). \tag{53}$$

Note that, to leading order,

$$J_3 \approx (1 - J_2)(1 - \cos I) = (1 + 2\Gamma)(1 - \cos I) \approx 1 + 2\Gamma - \cos I. \tag{54}$$

This substitution cannot be made into the Hamiltonian directly, however, since J_3 is an independent momentum from Γ and is conserved. Thus, we obtain

$$H\left(\Gamma,\phi\right) \approx \Gamma P - \Gamma^2 Q + R\Gamma\cos\phi, \tag{55}$$

$$P \approx 2\left[1 - \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} - \frac{\epsilon(12J_3 - 3)}{4}\right], \qquad Q \approx 4 - \frac{3\epsilon}{2} \approx 4,$$

$$R \approx \frac{15\epsilon}{2}, \tag{56}$$

$$\epsilon = \frac{m_3 a^4 c^2}{3Gm_{12}^2 a_{\rm out}^3}, \qquad \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} = \frac{(a/a_{\rm out})^{3/2}(m_{123}/m_{12})^{1/2}}{3Gm_{12}/(c^2 a)} \tag{57}$$

Here, $\Gamma \in [-0.5, 0]$, and $\Gamma \simeq -e^2/4$ for $e \ll 1$; these are checked with `sympy`.

This seems to largely agree with Wenrui's Hamiltonian, except he omitted the ϵ contribution to the Γ^2 coefficient (XL Eq. 17). There are then three questions to answer about this Hamiltonian:

- Are there any bifurcations ([dis]appearances of equilibria)?
- What does the phase portrait look like, qualitatively?
- What is the resonance width?

To answer these, we use Hamilton's equations to compute the EOM and equilibria:

$$\dot\phi = \frac{\partial H}{\partial\Gamma} = P - 2\Gamma Q + R\cos\phi, \tag{58}$$

$$\dot\Gamma = -\frac{\partial H}{\partial\phi} = -R\Gamma\sin\phi. \tag{59}$$

From the second equation, there are only zeros when $\phi = 0, \pi$, and from the first equation, these occur at

$$\Gamma_{\phi=0,\pi} = \frac{P \pm R}{2Q}, \tag{60}$$

where the positive sign corresponds to $\phi = 0$. Recalling that $\Gamma \in [-0.5, 0]$, we see that the solutions only exist when $P \pm 15\epsilon/2 \in [-Q, 0]$. Ps the inspiral progresses, P is increasing, so we see that the $\Gamma_{\phi=0}$ equilibrium will disappear when $P = -15\epsilon/2$ and the $\Gamma_{\phi=\pi}$ equilibrium will disappear when $P = 15\epsilon/2$.

TODO: Resonance width.

2.2 Eccentric Perturber

The goal here is to find the correct way to average the single-averaged expression

$$\tilde{H}_{\rm out} \equiv \frac{H_{\rm out}}{Gm_3\mu_{12}a^2/a_{\rm out}^3} = \frac{1}{4}\left(\frac{a_{\rm out}}{r_{\rm out}}\right)^3\left[-1 + 6e^2 + 3\left(1-e^2\right)(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}_{\rm out})^2 - 15e^2(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2\right]. \tag{61}$$

The coordinate system we choose matches XL16, where $\hat{\mathbf{L}}_{\rm in} \propto \hat{\mathbf{z}}$ while $\Omega_{\rm out} = 0$. This gives component form (see MD 2.20 and 2.122)

$$r_{\rm out} = \frac{a_{\rm out}\left(1-e_{\rm out}^2\right)}{1+e_{\rm out}\cos f_{\rm out}} \qquad \hat{\mathbf{r}}_{\rm out} = \begin{pmatrix}\cos v_{\rm out}\\ \sin v_{\rm out}\cos I\\ \sin v_{\rm out}\sin I\end{pmatrix}. \tag{62}$$

Here, $v_{\rm out} = \Omega_{\rm out} + \omega_{\rm out} + f_{\rm out} = \omega_{\rm out} + f_{\rm out}$ is the *true longitude*. Averaging is done via the identities

$$\left\langle \frac{\cos^2 f_{\rm out}}{r_{\rm out}^3}\right\rangle = \left\langle \frac{\sin^2 f_{\rm out}}{r_{\rm out}^3}\right\rangle = \frac{1}{2a_{\rm out}^3\left(1-e_{\rm out}^2\right)^{3/2}}, \tag{63}$$

$$\left\langle \frac{1}{r_{\rm out}^3}\right\rangle = \frac{1}{a_{\rm out}^3\left(1-e_{\rm out}^2\right)^{3/2}}, \tag{64}$$

$$\left\langle \frac{\cos f_{\rm out}\sin f_{\rm out}}{r_{\rm out}^3}\right\rangle = 0. \tag{65}$$

2.2.1 Circular Averaging

Just to review the averaging procedure, let's consider the case where $e_{\rm out} = 0$, then $v_{\rm out} = \lambda_{\rm out}$ the *mean longitude* and we can write

$$\tilde{H}_{\rm out} = \frac{1}{4}\left[-1 + 6e^2 + 3\left(1-e^2\right)\sin^2\lambda_{\rm out}\sin^2 I - 15e^2\left(\cos\varpi\cos\lambda_{\rm out} + \sin\varpi\sin\lambda_{\rm out}\cos I\right)^2\right], \tag{66}$$

$$\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out} = \cos(\varpi - \lambda_{\rm out})\left(\frac{1+\cos I}{2}\right) + \cos(\varpi - \lambda_{\rm out})\left(\frac{1-\cos I}{2}\right), \tag{67}$$

$$\left\langle (\dots)^2\right\rangle = \cos^2(\varpi - \lambda_{\rm out})\left(\frac{1+\cos I}{2}\right)^2 + \frac{(1-\cos I)^2}{8}, \tag{68}$$

$$\begin{aligned}
\left\langle \tilde{H}_{\rm out}\right\rangle &= \frac{1}{4}\left[-1 + 6e^2 + \frac{3}{2}\left(1-e^2\right)\left(1-\cos^2 I\right) - \frac{15}{2}e^2\left(1+\cos(2\varpi - 2\lambda_{\rm out})\right)\left(\frac{1+\cos I}{2}\right)^2 - \frac{15e^2(1-\cos I)^2}{8}\right], && (69)\\
&= \frac{1}{16}\left[-4 + 24e^2 + 6\left(1 - e^2 - \cos^2 I + e^2\cos^2 I\right) - 15e^2\left(1+\cos I^2\right) - \frac{15}{2}e^2\cos\phi(1+\cos I)^2\right], && (70)\\
&= \frac{1}{16}\left[2 + 3e^2 - 6\cos^2 I - 9e^2\cos^2 I - \frac{15}{2}(1+\cos I)^2 e^2\cos\phi\right]. && (71)
\end{aligned}$$

Success!

2.3 Eccentric Averaging

We cannot just substitute $a_{\rm out} \Rightarrow a_{\rm out,eff} \equiv a_{\rm out}\sqrt{1-e_{\rm out}^2}$ because ϕ is no longer a resonant angle. All of the other terms are the same, however, so we effectively just need to compute the expression

$$a_{\rm out}\left\langle \frac{(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2}{r_{\rm out}^3}\right\rangle \tag{72}$$

Consider a Fourier decomposition of $\hat{\mathbf{r}}_{\rm out}/r_{\rm out}^{3/2}$, where $\Omega_{\rm out}$ is the outer mean motion (just in the x–y plane), then (we assume $\omega_{\rm out}(t=0)=0$ and give an arbitrary phase offset ϖ_0 to the inner eccentricity vector)

$$\begin{aligned}
\hat{\mathbf{e}}(t) &= \cos(\Omega_{\rm GR}t+\varpi_0)\hat{\mathbf{x}} + \sin(\Omega_{\rm GR}t+\varpi_0)\hat{\mathbf{y}}, && (73)\\
\frac{\hat{\mathbf{r}}_{\rm out}(t)_\perp}{r_{\rm out}^{3/2}} &= \frac{1}{r_{\rm out}^{3/2}}\left[\cos v_{\rm out}\hat{\mathbf{x}} + \sin v_{\rm out}\cos I\hat{\mathbf{y}}\right], && (74)\\
&= \sum_{N=1}^{\infty}\frac{c_N}{a_{\rm out}^{3/2}}\left\{\cos(N\Omega_{\rm out}t)\hat{\mathbf{x}} + \sin(N\Omega_{\rm out}t)\cos I\hat{\mathbf{y}}\right\}, && (75)\\
\hat{\mathbf{e}}\cdot\frac{\hat{\mathbf{r}}_{\rm out}}{r_{\rm out}^{3/2}} &= \sum_{N=1}^{\infty}\frac{c_N}{a_{\rm out}^{3/2}}\left\{\cos(N\Omega_{\rm out}t)\cos(\Omega_{\rm GR}t+\varpi_0) + \sin(N\Omega_{\rm out}t)\cos I\sin(\Omega_{\rm GR}t+\varpi_0)\right\}, && (76)\\
&= \sum_{N=1}^{\infty}\frac{c_N}{a_{\rm out}^{3/2}}\left\{\cos((N\Omega_{\rm out}-\Omega_{\rm GR})t-\varpi_0)\left(\frac{1+\cos I}{2}\right) + \cos((N\Omega_{\rm out}+\Omega_{\rm GR})t+\varpi_0)\left(\frac{1-\cos I}{2}\right)\right\}, && (77)\\
\left\langle\frac{(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2}{r_{\rm out}^3}\right\rangle &= \sum_{M=0}^{N-1} f_{NM}\frac{c_{N_{\rm GR}+M}c_{N_{\rm GR}-M}}{2a_{\rm out}^3}\left(\frac{1+\cos I}{2}\right)^2\cos((2N_{\rm GR}\Omega_{\rm out}-2\Omega_{\rm GR})t-2\varpi_0). && (78)
\end{aligned}$$

Here, $N_{\rm GR}\equiv\lfloor\Omega_{\rm GR}/\Omega_{\rm out}$, and $f_{NM}=1$ if $M=N/2$ else 2 (double counting factor). Since the c_N should fall off for $N\gtrsim N_{\rm p}$ where

$$N_{\rm p}\equiv\frac{\sqrt{1+e}}{(1-e_{\rm out})^{3/2}}, \tag{79}$$

we see that there will generally be resonances for all $N\Omega_{\rm out}\sim\Omega_{\rm GR}$ as long as $N\lesssim N_{\rm p}$. Furthermore, we guess that the c_N are expected to scale like N^2 for $N\lesssim N_{\rm p}$, so the sum is dominated by the $M=0$ contribution.

Does this agree with simulations yet? Well, we do observe resonances for both $N\approx1$ and $N\approx N_{\rm p}$, todo more in depth exploration.