1 04/28/20

1.1 Adiabatic Invariance, LL17 Reproduction

In panel 1 of LL17 Fig. 4, the result of applying adiabatic invariance assuming $e = 0$ is presented. We have an updated prescription for adiabatic invariance though, so I have regenerated the figure using the original data.

In the original paper, the adiabatic invariant is (all precession frequencies evaluated at $e = 0$)

$$\cos\theta_{\rm eff,S1} = \hat{\mathbf{S}} \cdot \hat{\boldsymbol{\Omega}}_{\rm eff}, \tag{1}$$

$$\boldsymbol{\Omega}_{\rm eff} = \Omega_{\rm SL}\hat{\mathbf{L}} + \Omega_{\rm L}\frac{J}{L_{\rm out}}\hat{\mathbf{J}}, \tag{2}$$

$$\mathbf{J} = \mathbf{L} + \mathbf{L}_{\rm out}. \tag{3}$$

The updated adiabatic invariant is (angle brackets denote period averages for a given I, allowing e to vary)

$$\cos\theta_{\rm eff,YS} = \hat{\mathbf{S}} \cdot \hat{\boldsymbol{\Omega}}_{\rm eff}, \tag{4}$$

$$\boldsymbol{\Omega}_{\rm eff} = \langle \Omega_{\rm SL}\hat{\mathbf{L}}\rangle - \langle \dot{\Omega}\hat{\mathbf{J}}\rangle. \tag{5}$$

Reusing the data from LL17, the agreement can be seen to improve, see Fig. 1. **NB:** The agreement near $I = 90^\circ$ can be seen to be slightly off-center, since $\eta \neq 0$ (neither my numerically-averaged prediction nor the $e = 0$ prediction can capture this). Rerunning the simulations for $\eta = 0$ would produce a minimum at $I_0 = 90^\circ$ as expected. In lieu of this, I have temporarily manually offset the data to be centered at 90° to investigate the scaling near $I_{0,\rm lim}$, and my prediction is very good.

Vertical dotted lines denote where $\langle\Omega_{\rm eff}\rangle = \pi/P_{\rm LK}$. It was suspected this was the cutoff where a resonance could be hit (precession period is *half* the Kozai period). We investigate this below

1.2 Resonances: Toy Problem

Recall EOM

$$\begin{aligned}\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} &= \langle \Omega_{\rm SL}\hat{\mathbf{L}} - \dot{\Omega}\hat{\mathbf{z}}\rangle \times \hat{\mathbf{S}} \\ &\quad + \left[\sum_{N=1}^{\infty}\hat{\boldsymbol{\Omega}}_{\rm eff,N}\exp(2\pi i N t/t_{\rm LK})\right] \times \hat{\mathbf{S}}.\end{aligned} \tag{6}$$

Ignore GR, such that $\boldsymbol{\Omega}_{\rm eff}$ is a constant, and orient it along $\hat{\mathbf{z}}$. Further consider including only the $N = 1$ term, then we can write down the fundamental toy problem ($\boldsymbol{\Omega}_0 \propto \hat{\mathbf{z}}$)

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \boldsymbol{\Omega}_0 \times \hat{\mathbf{S}} + \epsilon\sin(\omega t)\hat{\boldsymbol{\Omega}}_1 \times \hat{\mathbf{S}}. \tag{7}$$

1.2.1 Simple Resonance

When $\omega \approx \Omega_0$ and $\hat{\boldsymbol{\Omega}}_1 = \hat{\mathbf{x}}$ (for simplicity), we have seen how this is solved, transforming to the frame rotating as $\omega\hat{\mathbf{z}}$ gives the following EOM (in rotating frame)

$$\frac{\mathrm{d}\theta}{\mathrm{d}t} = -\epsilon\left[\sin\phi - \sin\omega t\cos\left(\phi - \omega t\right)\right], \tag{8a}$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}t} = \Omega_0 - \omega + \epsilon\cos\omega t\cot\theta\cos\left(\phi - \omega t\right). \tag{8b}$$

If $\Omega_0 \approx \omega$, then assume ωt terms can be dropped/averaged out, this equates to EOM in rotating frame of form (after some work)

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = (\Omega_0 - \omega)\hat{\mathbf{z}} \times \hat{\mathbf{S}} + \epsilon\hat{\mathbf{x}} \times \hat{\mathbf{S}}. \tag{9}$$

Given initial θ, ϕ, we can easily compute the range of θ (precession about a fixed axis). This can be compared to simulations and yields plausible agreement (Fig. 2).

Of particular interest in the figure is the behavior of the peak at $\Omega_0/\omega = 0.5$, and I have discovered the following properties (cited without evidence):

- Amplitude & inverse period scales with ϵ.
- Depends on the angle of $\hat{\boldsymbol{\Omega}}_1$.

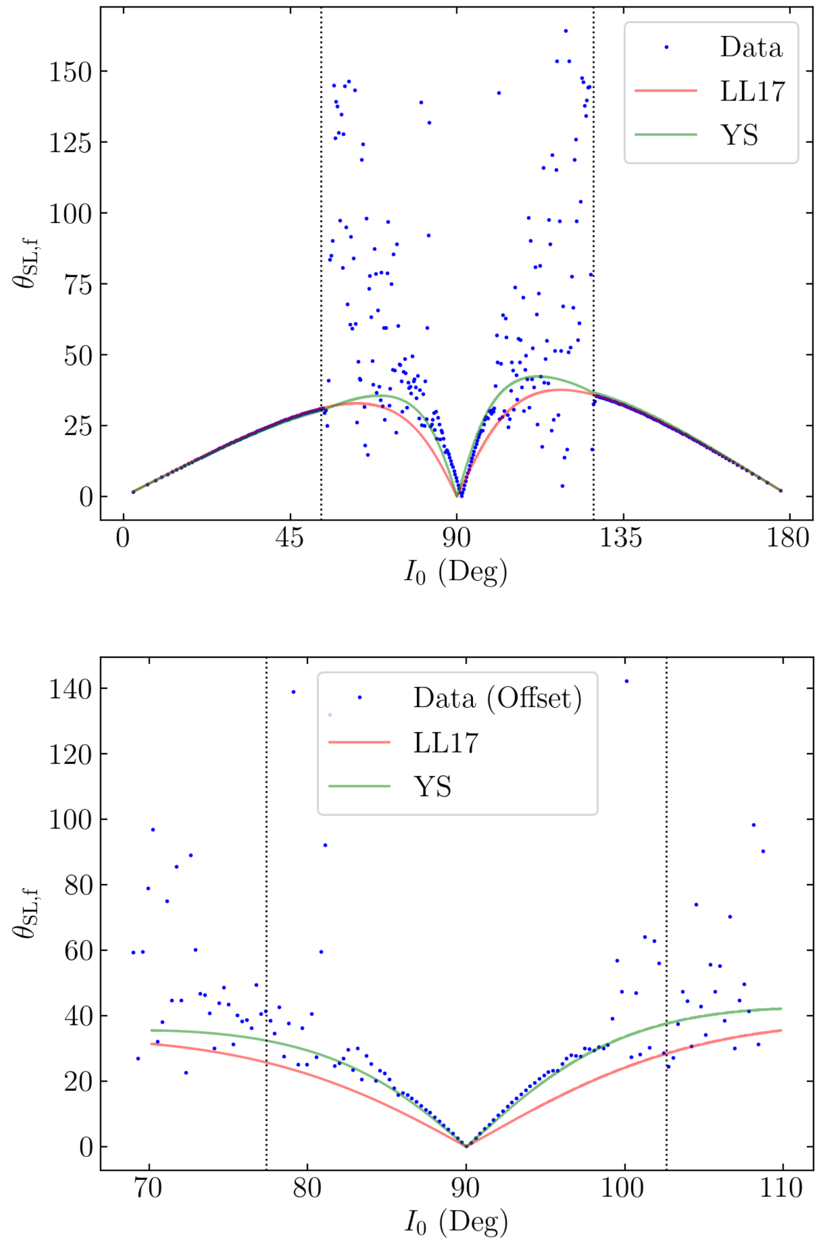

Figure 1: Top: comparison of predicted final spin-orbit misalignment angles between Eqs. (1) and (4). Bottom: Zoomed in on $I_0 = 90^\circ$ with an artificial offset introduced to illustrate comparison.

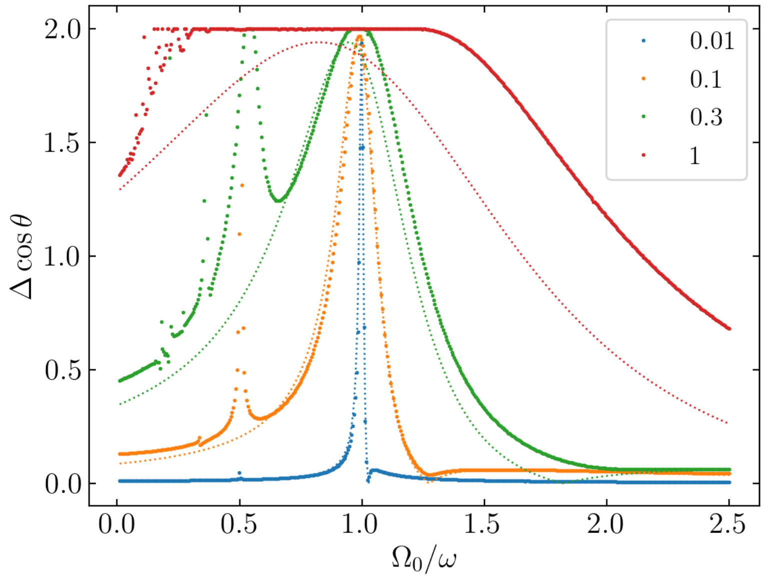

Figure 2: Range of $\cos\theta$ excited when simulating Eq. (7), starting with $\theta = 20^\circ$ (here, $\hat{\mathbf{\Omega}}_1$ is pointed at 60°). Legend shows different values of ϵ. Dotted lines are analytical predictions from analysis of Eq. (9).

- Smaller peaks seem to exist for $\Omega_0/\omega = 1/N$.
- Requires amplitude modulation of perturbation (i.e. not excited for perturbation $\hat{\mathbf{x}}\sin(\omega t) + \hat{\mathbf{y}}\cos(\omega t)$).

The first two points suggest some sort of parametric instability, but I have not solved it yet.