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Spin-Orbit Dynamics in Hierarchical Black Hole Triples: Analytical Theory

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ABSTRACT
Abstract

Key words: keywords

1 INTRODUCTION

This problem is important.

2 ANALYTICAL SETUP

2.1 Equations of Motion

We study Lidov-Kozai (LK) oscillations due to an external perturber to quadrupole order and include precession of pericenter and gravitational wave radiation due to general relativity. Consider an inner black hole (BH) binary with masses m_1 and m_2 having total mass m_{12} and reduced mass μ orbited by a third BH with mass m_3. Call a_3 the orbital semimajor axis of the third BH from the center of mass of the inner binary, and e_3 the eccentricity of its orbit, and define effective semimajor axis

$$\bar{a}_3 \equiv a_3\sqrt{1 - e_3^2}. \tag{1}$$

We adopt the test particle approximation such that the orbit of the third mass is fixed. Finally, call $\mathbf{L}_{\rm out} \equiv L_{\rm out}\hat{\mathbf{L}}_{\rm out}$ the fixed angular momentum of the outer BH relative to the center of mass of the inner BH binary, and call $\mathbf{L} \equiv L\hat{\mathbf{L}}$ the orbital angular momentum of the inner BH binary.

We then consider the motion of the inner binary, described by orbital elements Keplerian orbital elements $(a, e, \Omega, I, \omega)$. The equations describing the motion of these orbital elements is then (Peters 1964; Storch & Lai 2015; Liu & Lai 2018)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{a}{t_{\rm GW}}, \tag{2}$$

$$\frac{\mathrm{d}e}{\mathrm{d}t} = \frac{15}{8t_{\rm LK}} e\sqrt{1 - e^2}\sin 2\omega \sin^2 I, \tag{3}$$

$$\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{3}{4t_{\rm LK}}\frac{\cos I\left(5e^2\cos^2\omega - 4e^2 - 1\right)}{\sqrt{1 - e^2}} + \Omega_{\rm GR}, \tag{4}$$

$$\frac{\mathrm{d}I}{\mathrm{d}t} = \frac{15}{16}\frac{e^2\sin 2\omega \sin 2I}{\sqrt{1 - e^2}}, \tag{5}$$

$$\frac{\mathrm{d}\omega}{\mathrm{d}t} = \frac{3}{4t_{\rm LK}}\frac{2\left(1 - e^2\right) + 5\sin^2\omega(e^2 - \sin^2 I)}{\sqrt{1 - e^2}}. \tag{6}$$

Here, we have defined

$$t_{\rm LK}^{-1} = n\left(\frac{m_3}{m_{12}}\right)\left(\frac{a}{\bar{a}_3}\right)^3, \tag{7}$$

$$t_{\rm GW}^{-1} = \frac{64}{5}\frac{G^3\mu m_{12}^2}{c^5a^4}\frac{1}{\left(1 - e^2\right)^{7/2}}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right), \tag{8}$$

$$\Omega_{\rm GR} = \frac{3Gnm_{12}}{c^2a\left(1 - e^2\right)}, \tag{9}$$

and $n = \sqrt{Gm_{12}/a^3}$ is the mean motion of the inner binary.

The evolution of these orbital elements has been well characterized in previous studies (Anderson et al. 2016; Liu & Lai 2017). We focus on the evolution with LK oscillations, in which case the evolution is approximately broken into two phases:

- In the first phase, a and $e_{\max}$ change very slowly, while $e_{\min}$ increases to $e_{\max}$ due to the effect of pericenter precession.
- In the second phase, $e_{\min} \approx e_{\max}$, and a and $e = e_{\min} = e_{\max}$ decay together due to GW radiation.

2.2 Spin Dynamics

We are ultimately interested in the evolution of the spin angular momentum of the inner BHs. Since they evolve independently to leading post-Newtonian order, we focus on the dynamics of a single BH spin vector $\mathbf{S} = S\hat{\mathbf{S}}$ Neglecting spin-spin interactions, $\hat{\mathbf{S}}$ undergoes de Sitter precession about $\mathbf{L}$ as

$$\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t} = \Omega_{\rm SL}\hat{\mathbf{L}} \times \hat{\mathbf{S}}, \tag{10}$$

$$\Omega_{\rm SL} = \frac{3Gn\left(m_2 + \mu/3\right)}{2c^2a\left(1 - e^2\right)}. \tag{11}$$

We next go to the co-rotating frame with $\hat{\mathbf{L}}$ about $\hat{\mathbf{L}}_{\rm out}$. Choose $\hat{\mathbf{L}}_{\rm out} = \hat{\mathbf{z}}$, and choose the $\hat{\mathbf{x}}$ axis such that $\hat{\mathbf{L}}$ lies in the x-z plane. In this coordinate system, Eq. (10) becomes

$$\left(\frac{\mathrm{d}\mathbf{S}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \left(-\frac{\mathrm{d}\Omega}{\mathrm{d}t}\hat{\mathbf{z}} + \Omega_{\mathrm{SL}}\hat{\mathbf{L}}\right) \times \hat{\mathbf{S}}, \tag{12}$$

$$= \boldsymbol{\Omega}_{\mathrm{e}} \times \hat{\mathbf{S}}, \tag{13}$$

$$\boldsymbol{\Omega}_{\mathrm{e}} \equiv \Omega_{\mathrm{PL}}\hat{\mathbf{z}} + \Omega_{\mathrm{SL}}\left(\cos I\hat{\mathbf{z}} + \sin I\hat{\mathbf{x}}\right), \tag{14}$$

$$\Omega_{\mathrm{PL}} \equiv -\frac{\mathrm{d}\Omega}{\mathrm{d}t}. \tag{15}$$

In general, Eq. (13) is difficult to analyze, since Ω_{PL}, Ω_{SL} and I all vary significantly within every LK cycle. However, since these quantities are all periodic with period T_{LK}, we can rewrite Eq. (13) in Fourier components

$$\left(\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \left[\langle\boldsymbol{\Omega}_{\mathrm{e}}\rangle + \sum_{N=1}^{\infty} \boldsymbol{\Omega}_{\mathrm{e,N}} \cos\left(\frac{2\pi N t}{T_{\mathrm{LK}}}\right)\right] \times \hat{\mathbf{S}}. \tag{16}$$

The angle brackets denote an average over an LK cycle. We adopt convention where $t = 0$ is the maximum eccentricity phase of the LK cycle.

To simplify, we assume the $N \geq 1$ terms have no significant contribution to the dynamics (see Appendix A). We thus analyze.

$$\left(\frac{\mathrm{d}\hat{\mathbf{S}}}{\mathrm{d}t}\right)_{\mathrm{rot}} = \langle\boldsymbol{\Omega}_{\mathrm{e}}\rangle \times \hat{\mathbf{S}}. \tag{17}$$

We accordingly define angle

$$\hat{\mathbf{S}} \cdot \langle\hat{\boldsymbol{\Omega}}_{\mathrm{e}}\rangle \equiv \cos\theta_{\mathrm{e}}. \tag{18}$$

Intuitively, Eq. (17) suggests that θ_{e} should be a conserved quantity when $\boldsymbol{\Omega}_{\mathrm{e}}$ varies adiabatically. The adiabaticity condition requires the precession axis evolve slowly compared to the precession frequency, or

$$\left|\frac{\mathrm{d}\langle\hat{\boldsymbol{\Omega}}_{\mathrm{e}}\rangle}{\mathrm{d}t}\right| \ll \left|\langle\boldsymbol{\Omega}_{\mathrm{e}}\rangle\right|. \tag{19}$$

Of course, over timescales $\lesssim T_{\mathrm{LK}}$, $\hat{\mathbf{S}}$ will vary under the influence of the $N \geq 1$ harmonics. Thus, we expect that the quantity that will be well-conserved under adiabatic evolution is actually the LK-averaged $\langle\theta_{\mathrm{e}}\rangle$. A more careful analysis (Appendix A) indicates that the correct averaging interval to use is generally be an integer multiple of T_{LK}.

2.3 Analysis: Deviation from Adiabaticity

In real systems, the particular extent to which $\langle\theta_{\mathrm{e}}\rangle$ is conserved depends on how well Eq. (19) is satisfied. We will derive analytical expressions for the relevant frequencies in the problem, then use these to estimate $\Delta\langle\theta_{\mathrm{e}}\rangle$, the change in $\langle\theta_{\mathrm{e}}\rangle$ over the full inspiral, a purely non-adiabatic effect.

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APPENDIX A: EFFECT OF HARMONIC TERMS ON EVOLUTION: AVERAGING REVISITED

In order for the harmonic terms in Eq. (16) to be nonzero, the system must be in the LK-oscillating regime. Consider the Hamiltonian corresponding to Eq. (16) (see e.g. Kinoshita 1993; Storch & Lai 2015):

$$H = \left[\langle \boldsymbol{\Omega}_{\rm e}\rangle + \sum_{N=1}^{\infty} \boldsymbol{\Omega}_{\rm e,N}\cos\left(\frac{2\pi N t}{T_{\rm LK}}\right)\right]\cdot \hat{\mathbf{S}}. \tag{A1}$$

Assume for simplicity that the system is not evolving, such that all Fourier components $\langle \boldsymbol{\Omega}_{\rm e}\rangle, \boldsymbol{\Omega}_{\rm e,N}$ are constant in time.

The objective is now to average H over a suitable interval of time. Assume that $\hat{\mathbf{S}}$ is also periodic with some period $T_{\rm S}$, such that

$$\hat{\mathbf{S}} = \left[\langle \hat{\mathbf{S}}\rangle + \sum_{M=1}^{\infty} \mathbf{S}_{\rm M}\exp\left(i\frac{2\pi M p t}{T}\right)\right]. \tag{A2}$$

Note that $\mathbf{S}_{\rm M}$ must generally be complex, to ensure that $\hat{\mathbf{S}}\cdot\hat{\mathbf{S}}$ does not vary in time. Note also that $T_{\rm S}/T_{\rm LK}$ is generally *irrational*. Nevertheless, consider averaging over some interval of time T that is a near-integer multiple of both $T_{\rm S}$ and $T_{\rm LK}$:

$$T \approx pT_{\rm S} \approx qT_{\rm LK}, \tag{A3}$$

$$\frac{1}{T}\int_0^T H\,{\rm d}t = \frac{1}{T}\int_0^T \left[\langle \boldsymbol{\Omega}_{\rm e}\rangle + \sum_{N=1}^{\infty} \boldsymbol{\Omega}_{\rm e,N}\cos\left(\frac{2\pi N q t}{T}\right)\right]\cdot\left[\langle \hat{\mathbf{S}}\rangle + \sum_{M=1}^{\infty} \mathbf{S}_{\rm M}\exp\left(i\frac{2\pi M p t}{T}\right)\right]\,{\rm d}t, \tag{A4}$$

$$\langle H\rangle = \langle \boldsymbol{\Omega}_{\rm e}\rangle\cdot\langle \hat{\mathbf{S}}\rangle + \frac{1}{2}\sum_{j=1}^{\infty} \boldsymbol{\Omega}_{\rm e,jp}\cdot\left({\rm Re}\,\mathbf{S}_{jq}\right). \tag{A5}$$

Note that $\langle H\rangle$ is a conserved quantity of the evolution of the system, absent GW dissipation. When the summation in this equation can be neglected, this implies $\theta_{\rm e}$ is a conserved quantity, as claimed in the main text. To understand when the summation can be neglected, we make the following observations:

- The magnitudes of the coefficients $\boldsymbol{\Omega}_{\rm e,N}$ fall off exponentially with characteristic scale $N_{\rm scale}\sim\frac{\Delta t}{T_{\rm LK}}$, where Δt is the width of the LK eccentricity peak. When $e_{\rm min}\approx 0$ and $e_{\rm max}\to 1$, this is $N_\Omega \sim j_{\rm min}^{1/2}$ (Anderson et al. 2016). At later times, N_Ω decreases as $e_{\rm min}$ is excited and the LK cycle becomes less violent.
- In general, $\hat{\mathbf{S}}$ tends to precess around $\langle \boldsymbol{\Omega}_{\rm e}\rangle$, albeit not uniformly in time. Therefore, $\langle \hat{\mathbf{S}}\rangle$ is expected to be approximately aligned with $\langle \boldsymbol{\Omega}_{\rm e}\rangle$, while the $\mathbf{S}_{\rm M}$ are expected to be *perpendicular* to $\langle \boldsymbol{\Omega}_{\rm e}\rangle$.
- The degree of misalignment of each of the $\boldsymbol{\Omega}_{\rm e,N}$ from $\langle \boldsymbol{\Omega}_{\rm e}\rangle$ is related to the amplitude of nutation of $\boldsymbol{\Omega}_{\rm e}$ over an LK cycle. In particular, the x and z components of $\boldsymbol{\Omega}_{\rm e,N}$ will exhibit order-unity variations in cutoff harmonic $N_{\rm scale}$ when $I_{\rm e}$ is strongly nutating.
- Finally, since $\hat{\mathbf{S}}$ is driven by $\boldsymbol{\Omega}_{\rm e}$, it must have a similar frequency spectrum, implying $M_{\rm scale}\sim N_{\rm scale}q/p$.

Thus, non-negligible contributions from the summation in Eq. (A5) arise when: (i) $\boldsymbol{\Omega}_{\rm e}$ is nutating substantially, and (ii) q, p are sufficiently small that a substantial number of terms $\mathbf{S}_{jq}$ dot against the non-aligned $\boldsymbol{\Omega}_{\rm e,jp}$.

Of course, the only astrophysically relevant triples are those that merge within a Hubble time, and the systems that best satisfy the assumptions laid out in this study are those with well-separated inner binaries $a_{\rm in}\gtrsim 10$ AU. Such systems, which merge within just a few thousands of $T_{\rm LK}$, evolve very quickly relative to $T_{\rm LK}$ at early times, and exhibit very little $I_{\rm e}$ nutation at later times, and so the contribution from such terms in Eq. (A5) can be neglected.

For systems with more compact binaries, such as those studied in (Liu & Lai 2017), the contributions from such higher harmonics cannot be neglected. This explains the breakdown of conservation of $\theta_{\rm e}$ in these systems.

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