Octupole-order Lidov-Kozai Population Statistics

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I. 10/22/20—INITIAL THOUGHTS

A. Equations

The equations of motion we want to study come from LML15. Describe the inner binary by $(a, e, I, \Omega, \omega)$ and the outer binary with "out" subscripts, and denote $I_{\rm tot} = I + I_{\rm out}$. Call the inner binary component masses m_1, m_2, and the tertiary mass m_3, define the inner binary total and reduced masses $m_{12} = m_1 + m_2$ and $\mu = m_1 m_2 / m_{12}$, and define the tertiary orbit total and reduced masses $m_{123} = m_{12} + m_3$ and $\mu_{\rm out} = m_{12} m_3 / m_{123}$. The equations of motion are ($j(e) = \sqrt{1 - e^2}$)

$$\frac{\mathrm{d}a}{\mathrm{d}t} = -\frac{64}{5}\frac{a}{t_{\rm GW} j^{7/2}(e)}\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right), \tag{1}$$

$$\begin{aligned}\frac{\mathrm{d}e}{\mathrm{d}t} ={}& \frac{j(e)}{64 t_{\rm LK}}\Big\{120 e \sin^2 I_{\rm tot}\sin(2\omega)\\ &+ \frac{15\epsilon_{\rm oct}}{8}\cos\omega_{\rm out}\big[\left(4 + 3e^2\right)\left(3 + 5\cos(2I_{\rm tot})\right)\\ &\times \sin\omega + 210 e^2 \sin^2 I_{\rm tot}\sin 3\omega\big]\\ &- \frac{15\epsilon_{\rm oct}}{4}\cos I_{\rm tot}\cos\omega\big[15(2 + 5e^2)\cos(2I_{\rm tot})\\ &+ 7\left(30e^2\cos(2\omega)\sin^2 I_{\rm tot} - 2 - 9e^2\right)\big]\sin\omega_{\rm out}\Big\}\\ &- \frac{304}{15}\frac{e}{t_{\rm GW} j^{5/2}(e)}\left(1 + \frac{121}{304}e^2\right),\end{aligned} \tag{2}$$

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} ={}& -\frac{3e}{32 t_{\rm LK} j(e)}\Big\{10\sin(2I_{\rm tot})\left[e\sin(2\omega)\right.\\ &+ \frac{5\epsilon_{\rm oct}}{8}\left(2 + 5e^2 + 7e^2\cos(2\omega)\right)\cos\omega_{\rm out}\sin\omega\big]\\ &+ \frac{5\epsilon_{\rm oct}}{8}\cos\omega\big[26 + 37e^2 - 35e^2\cos(2\omega)\\ &- 15\cos(2I_{\rm tot})\left(7e^2\cos(2\omega) - 2 - 5e^2\right)\big]\\ &\times \sin I_{\rm tot}\sin\omega_{\rm out}\Big\}\end{aligned} \tag{3}$$

$$\begin{aligned}\frac{\mathrm{d}\Omega}{\mathrm{d}t} = \frac{\mathrm{d}\Omega_{\rm out}}{\mathrm{d}t} ={}& -\frac{3\csc I}{32 t_{\rm LK} j(e)}\Big\{2\big[(2 + 3e^2 - 5e^2\cos(2\omega))\\ &+ \frac{25\epsilon_{\rm oct} e}{8}\cos\omega\left(2 + 5e^2 - 7e^2\cos(2\omega)\right)\cos\omega_{\rm out}\big]\\ &\times \sin(2I_{\rm tot}) - \frac{5\epsilon_{\rm oct} e}{8}\big[35e^2(1 + 3\cos(2I_{\rm tot}))\cos 2\omega\\ &- 46 - 17e^2 - 15\left(6 + e_1^2\right)\cos(2I_{\rm tot})\big]\\ &\times \sin I_{\rm tot}\sin\omega\sin\omega_{\rm out}\Big\},\end{aligned} \tag{4}$$

$$\begin{aligned}\frac{\mathrm{d}\omega}{\mathrm{d}t} ={}& \frac{3}{8 t_{\rm LK}}\Big\{\frac{1}{j(e)}\big[4\cos^2 I_{\rm tot} + (5\cos(2\omega) - 1)\\ &\times \left(1 - e^2 - \cos^2 I_{\rm tot}\right)\big] + \frac{L\cos I_{\rm tot}}{L_{\rm out} j(e_{\rm out})}\big[2 + e^2(3\\ &- 5\cos(2\omega))\big]\Big\} + \frac{15\epsilon_{\rm oct}}{64 t_{\rm LK}}\Big\{\left(\frac{L}{L_{\rm out} j(e_{\rm out})} + \frac{\cos I_{\rm tot}}{j(e)}\right)\\ &\times e\big[\sin\omega\sin\omega_{\rm out}\left[10(3\cos^2 I_{\rm tot} - 1)(1 - e^2) + A\right]\\ &- 5B\cos I_{\rm tot}\cos\Theta\big] - \frac{j(e)}{e}\big[10\sin\omega\sin\omega_{\rm out}\cos I_{\rm tot}\\ &\times \sin^2 I_{\rm tot}\left(1 - 3e^2\right) + \cos\Theta\left(3A - 10\cos^2 I_{\rm tot} + 2\right)\big]\Big\}\\ &+ \Omega_{\rm GR},\end{aligned} \tag{5}$$

$$\begin{aligned}\frac{\mathrm{d}e_{\rm out}}{\mathrm{d}t} ={}& \frac{15 e L\, j(e_{\rm out})\epsilon_{\rm oct}}{256 t_{\rm LK} e_{\rm out} L_{\rm out}}\Big\{\cos\omega\big[6 - 13e^2\\ &+ 5(2 + 5e^2)\cos(2I_{\rm tot}) + 70e^2\cos(2\omega)\sin^2 I_{\rm tot}\big]\\ &\times \sin\omega_{\rm out} - \cos I_{\rm tot}\cos\omega_{\rm out}\big[5(6 + e^2)\cos(2I_{\rm tot})\\ &+ 7\left(10e^2\cos(2\omega)\sin^2 I_{\rm tot} - 2 + e^2\right)\big]\sin\omega\Big\},\end{aligned} \tag{6}$$

$$\begin{aligned}\frac{\mathrm{d}I_{\rm out}}{\mathrm{d}t} ={}& -\frac{3eL}{32 t_{\rm LK} j(e_{\rm out}) L_{\rm out}}\Big\{10\big[2e\sin I_{\rm tot}\sin(2\omega)\\ &+ \frac{5\epsilon_{\rm oct}}{8}\cos\omega\left(2 + 5e^2 - 7e^2\cos(2\omega)\right)\sin(2I_{\rm tot})\sin\omega_{\rm out}\big]\\ &+ \frac{5\epsilon_{\rm oct}}{8}\big[26 + 107e^2 + 5(6 + e^2)\cos(2I_{\rm tot})\\ &- 35e^2\left(\cos 2(I_{\rm tot}) - 5\right)\cos(2\omega)\big]\cos\omega_{\rm out}\sin I_{\rm tot}\sin\omega\Big\},\end{aligned} \tag{7}$$

$$\begin{aligned}\frac{\mathrm{d}\omega_{\rm out}}{\mathrm{d}t} ={}& \frac{3}{16 t_{\rm LK}}\Big\{\frac{2\cos I_{\rm tot}}{j(e)}\left[2 + e^2(3 - 5\cos(2\omega))\right]\\ &+ \frac{L}{L_{\rm out} j(e_{\rm out})}\big[4 + 6e^2 + (5\cos^2 I_{\rm tot} - 3)\\ &\times \left[2 + e^2(3 - 5\cos(2\omega))\right]\big]\Big\} - \frac{15\epsilon_{\rm oct} e}{64 t_{\rm LK} e_{\rm out}}\\ &\times \Big\{\sin\omega\sin\omega_{\rm out}\Big[\frac{L(4e_{\rm out}^2 + 1)}{e_{\rm out} L_{\rm out} j(e_{\rm out})}10\cos I_{\rm tot}\sin^2 I_{\rm tot}\\ &\times (1 - e^2) - e_{\rm out}\left(\frac{1}{j(e)} + \frac{L\cos I_{\rm tot}}{L_{\rm out} j(e_{\rm out})}\right)\\ &\times \left[A + 10\left(3\cos^2 I_{\rm tot} - 1\right)\left(1 - e^2\right)\right]\Big] + \cos\Theta\\ &\times \Big[5B\cos I_{\rm tot} e_{\rm out}\left(\frac{1}{j(e)} + \frac{L\cos I_{\rm tot}}{L_{\rm out} j(e_{\rm out})}\right)\\ &+ \frac{L\left(4e_{\rm out}^2 + 1\right)}{e_{\rm out} L_{\rm out} j(e_{\rm out})}A\Big]\Big\}.\end{aligned} \tag{8}$$

where $n = \sqrt{Gm_{12}/a^3}$ is the mean motion, $L = \mu\sqrt{Gm_{12}a}$ and $L_{\rm out} = \mu_{\rm out}\sqrt{Gm_{123}a_{\rm out}}$ are the circular angular momenta, and

$$t_{\rm LK}^{-1} = n\left(\frac{m_3}{m_{12}}\right)\left(\frac{a}{a_{\rm out} j(e_{\rm out})}\right)^3, \tag{9}$$

$$t_{\rm GW}^{-1} = \frac{G^3 \mu m_{12}^2}{c^5 a^4}, \tag{10}$$

$$\Omega_{\rm GR} = \frac{3Gnm_{12}}{c^2 a}, \tag{11}$$

$$\epsilon_{\rm oct} = \frac{m_2 - m_1}{m_{12}}\frac{a}{a_{\rm out}}\frac{e_{\rm out}}{1 - e_{\rm out}^2}, \tag{12}$$

$$A \equiv 4 + 3e^2 - \frac{5}{2}B\sin^2 I_{\rm tot}, \tag{13}$$

$$B \equiv 2 + 5e^2 - 7e^2\cos(2\omega), \tag{14}$$

$$\cos\Theta \equiv -\cos\omega\cos\omega_{\rm out} - \cos I_{\rm tot}\sin\omega\sin\omega_{\rm out}. \tag{15}$$

These equations can be nondimensionalized via the following steps (I won't rewrite the equations): (i) multiply through by $t_{\rm LK,0}$ ($a = a_0$ and $e_{\rm out} = 0$), and call $\tau \equiv t/t_{\rm LK,0}$ the new variable of differentiation, (ii) re-express all of the timescales as

$$\frac{t_{\rm LK,0}}{t_{\rm LK}} = \left(\frac{a}{a_0}\right)^{3/2} j^{-3}\left(e_{\rm out}\right), \tag{16}$$

$$\begin{aligned}\frac{t_{\rm LK,0}}{t_{\rm GW}} &= \frac{G^3\mu m_{12}^3}{m_3 c^5 a^4}\frac{1}{n_0}\left(\frac{a_{\rm out}}{a_0}\right)^3, \\ &= \epsilon_{\rm GW}\left(\frac{a_0}{a}\right)^4,\end{aligned} \tag{17}$$

$$\epsilon_{\rm GW} \equiv \frac{G^3\mu m_{12}^3 a_{\rm out}^3}{m_3 c^5 a_0^7 n_0}, \tag{18}$$

$$\begin{aligned}\Omega_{\rm GR} t_{\rm LK,0} &= \frac{3Gnm_{12}^2}{m_3 c^2 a}\frac{1}{n_0}\left(\frac{a_{\rm out}}{a_0}\right)^3, \\ &= \epsilon_{\rm GR}\left(\frac{a_0}{a}\right)^{5/2},\end{aligned} \tag{19}$$

$$\epsilon_{\rm GR} = \frac{3Gm_{12}^2 a_{\rm out}^3}{m_3 c^2 a_0^4}. \tag{20}$$

(iii) re-express $\mathrm{d}a/\mathrm{d}t$ as

$$\frac{\mathrm{d}(a/a_0)}{\mathrm{d}\tau} = -\frac{64}{5}\frac{\epsilon_{\rm GW}}{j^{7/2}(e)}\left(\frac{a_0}{a}\right)^3\left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4\right). \tag{21}$$

As such, the natural unit of length is $a_0 = 1$, the natural unit of time is $t_{\rm LK,0} = 1$, and everything else is dimensionless. When computing these ϵ, I use convention where $1M_\odot = 1$ AU $= c = 1$, under which $G = 9.87 \times 10^{-9}$.

B. Points of Inquiry

The goal is to understand how the merger window varies as a function of $q \equiv m_1/m_2$ ($m_1 < m_2$) when the octupole order LK effects are important.

- First, let's set $\epsilon_{\rm GW} = 0$. It is well known that the octupole order LK is nonintegrable. What does the Fourier spectrum of the eccentricity look like? Will this help us get a delay time distribution between high-e phases?

  When the octupole effect is unimportant, the spectrum falls off exponentially over scales $\tau \simeq P_{\rm LK} j^{-1}(e_{\max})$, where $P_{\rm LK}$ is the quadrupole LK period. One imagines the tail of the spectrum gets heavier when $\epsilon_{\rm oct}$ is increased, and this might help us get the delay time distribution.

  A second way we can postprocess this is to take a histogram of $e(t)$. If there is some regular structure, it's likely this will allow us to compute the average rate of binary coalescence due to GW radiation.

- The goal is to understand the size of the merger window, ΔI, as a function of q. To do this, we numerically sample the merger time function $T_{\rm m}(I_0, q)$. At each I_0, the natural thing to do would be to try for $\sim 5 - 10$ random Ω, ω, and define the merger window to be where $T_{\rm m} \leq 10^10$ yr.