

Lidov-Kozai 90° Attractor

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Date

1 Equations

1.1 Bin's Papers

Our major references will be Bin's paper with Diego + Dong in 2015 (LML15) and Bin's later paper with Dong on spin-orbit misalignment (LL18). The target of study is §4.3 of LL18, where a 90° attractor in spin-orbit misalignment seems to appear when the octupole effect is negligible.

When the octupole effect is negligible, we define vectors

$$\mathbf{j} = \sqrt{1 - e^2} \hat{n}, \quad (1)$$

$$\mathbf{e} = e \hat{u}. \quad (2)$$

Here, \mathbf{j} is the dimensionless angular momentum vector and \mathbf{e} is the eccentricity vector; see LML15 for precise definitions. Note that $\mathbf{j} \cdot \mathbf{e} = 0$, $j^2 + e^2 = 1$. Then, the EOM for the inner and outer vectors satisfy to quadrupolar order

$$\frac{d\mathbf{j}}{dt} = \frac{3}{4t_{LK}} [(\mathbf{j} \cdot \hat{n}_2)(\mathbf{j} \times \hat{n}_2) - 5(\mathbf{e} \cdot \hat{n}_2)(\mathbf{e} \times \hat{n}_2)], \quad (3)$$

$$\frac{d\mathbf{e}}{dt} = \frac{3}{4t_{LK}} [(\mathbf{j} \cdot \hat{n}_2)(\mathbf{e} \times \hat{n}_2) + 2\mathbf{j} \times \mathbf{e} - 5(\mathbf{e} \cdot \hat{n}_2)(\mathbf{j} \times \hat{n}_2)]. \quad (4)$$

Let's assume for the time being that $L_1 \ll L_2$, so the system is sufficiently hierarchical that \mathbf{j}_2 , \mathbf{e}_2 are constants. Note for reference that

$$t_{LK} \equiv \frac{L_1}{\mu_1 \Phi_0} = \frac{1}{n_1} \left(\frac{m_0 + m_1}{m_2} \right) \left(\frac{a_2}{a} \right)^3 (1 - e_2^2)^{3/2}. \quad (5)$$

Here, $n_1 \equiv \sqrt{G(m_0 + m_1)/a^3}$. Finally, the GR effects (Peters 1964) cause decays of \mathbf{L} and \mathbf{e} as

$$\left. \frac{d\mathbf{L}}{dt} \right|_{GW} = -\frac{32}{5} \frac{G^{7/2}}{c^5} \frac{\mu^2 m_{12}^{5/2}}{a^{7/2}} \frac{1 + 7e^2/8}{(1-e^2)^2} \hat{\mathbf{L}}, \quad (6)$$

$$\left. \frac{d\mathbf{e}}{dt} \right|_{GW} = -\frac{304}{15} \frac{G^3}{c^5} \frac{\mu m_{12}^2}{a^4 (1-e^2)^{5/2}} \left(1 + \frac{121}{304} e^2 \right) \mathbf{e}, \quad (7)$$

$$\left. \frac{\dot{a}}{a} \right|_{GW} = -\frac{64}{5} \frac{G^3}{c^5 a^4} \frac{\mu m_{12}^2}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right). \quad (8)$$

Here, $m_{12} \equiv m_1 + m_2$.

Given this system (from LML15), we can then add the spin-orbit coupling term, which is given in LL18 to be

$$\frac{d\hat{\mathbf{S}}}{dt} = \Omega_{SL} \hat{\mathbf{L}} \times \hat{\mathbf{S}}, \quad (9)$$

$$\Omega_{SL} \equiv \frac{3Gn(m_2 + \mu/3)}{2c^2 a (1-e^2)}. \quad (10)$$

Note that μ is the reduced mass of the inner binary. We can drop the back-reaction term since $S \ll L$. What is observed is that, as this system is evolved forward in time and GR coalesces the inner binary, $\theta_{sl} \equiv \arccos(\hat{\mathbf{S}} \cdot \hat{\mathbf{L}})$ goes to 90° consistently. The relevant figure is Fig. 19 of LL18, which shows that for a close-in, low-eccentricity perturber ($\bar{a}_{\text{out,eff}} \propto a_{\text{out}}$), the focusing is significantly stronger. Note that initially, $I \equiv \arccos(\hat{\mathbf{L}} \cdot \hat{\mathbf{L}}_2) \approx 90^\circ$ while $\theta_{sl} \approx 0$.

In LL18, an adiabaticity parameter is defined:

$$\mathcal{A} \equiv \left| \frac{\Omega_{SL}}{\Omega_L} \right|, \quad (11)$$

where $\Omega_L \simeq \left\langle \frac{d\hat{\mathbf{L}}}{dt} \right\rangle_{LK}$ to quadrupolar order. As the inner binary coalesces, \mathcal{A} transitions from $\ll 1$ to $\gg 1$ (as Ω_{SL} is a GR effect so ramps up very quickly as orbital separation decreases).

1.2 Simulations

First, we run GR-less simulations, so let's take $t_{LK} = 1$ (no semimajor axis evolution), and we reproduce LK oscillations.

Next, when accounting for GR, we should let a evolve as above. Note that since \mathbf{j} and \vec{e} are our dynamical variables, we should use $\mathbf{j} \equiv \sqrt{1-e^2} \hat{\mathbf{L}} = \sqrt{1-e^2} \frac{\mathbf{L}}{\mu \sqrt{Gm_{12}a(1-e^2)}}$ and rewrite

$$\left. \frac{d\mathbf{j}}{dt} \right|_{GW} = \frac{1}{\mu \sqrt{GMa}} \left. \frac{d\mathbf{L}}{dt} \right|_{GW} - \frac{\hat{\mathbf{L}}}{2a} \left. \frac{da}{dt} \right|_{GW}. \quad (12)$$

For convenience, let's just define $t_{LK} = t_{LK,0} \frac{a_0^3}{a^3}$ and set $t_{LK,0} = 1$. Furthermore, the timescale of relevance for the GW terms is $t_{GW}^{-1} \sim \frac{G^3 \mu m_{12}^2}{c^5 a^4}$. Let's express this as some ratio $t_{GW} = \epsilon t_{LK,0} \frac{a_0^4}{a^4}$. Thus,

everything should be nondimensionalized this way.

We lastly add de-Sitter precession of the spin of one of the inner binary components, call this \hat{S} . Similarly, let's just define a proportionality constant $t_{SL} = \delta t_{LK,0} \frac{a_0}{a}$, then

$$\frac{d\hat{S}}{d(t/t_{LK,0})} = \delta \frac{a_0}{a} \hat{L} \times \hat{S}. \quad (13)$$

Our final simulation equations are thus ($\tau = t/t_{LK,0}$)

$$\begin{aligned} \frac{d\mathbf{j}}{d\tau} = & \frac{3}{4} \left(\frac{a_0^3}{a^3} \right) [(\mathbf{j} \cdot \hat{n}_2)(\mathbf{j} \times \hat{n}_2) - 5(\mathbf{e} \cdot \hat{n}_2)(\mathbf{e} \times \hat{n}_2)] \\ & - \left(\epsilon \frac{a_0^4}{a^4} \right) \left(\frac{32}{5} \frac{1+7e^2/8}{(1-e^2)^2} - \frac{32}{5} \frac{1}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \right) \right) \frac{\mathbf{j}}{\sqrt{1-e^2}}, \end{aligned} \quad (14)$$

$$\frac{d\mathbf{e}}{d\tau} = \frac{3}{4} \left(\frac{a_0^3}{a^3} \right) [(\mathbf{j} \cdot \hat{n}_2)(\mathbf{e} \times \hat{n}_2) + 2\mathbf{j} \times \mathbf{e} - 5(\mathbf{e} \cdot \hat{n}_2)(\mathbf{j} \times \hat{n}_2)] - \left(\epsilon \frac{a_0^4}{a^4} \right) \frac{304}{15} \frac{1}{(1-e^2)^{5/2}} \left(1 + \frac{121}{304}e^2 \right) \mathbf{e}, \quad (15)$$

$$\frac{d\hat{S}}{d\tau} = \delta \frac{a_0}{a} \frac{\mathbf{j}}{\sqrt{1-e^2}} \times \hat{S}, \quad (16)$$

$$\frac{da}{d\tau} = -a \left(\epsilon \frac{a_0^4}{a^4} \right) \frac{64}{5} \frac{1}{(1-e^2)^{7/2}} \left(1 + \frac{73}{24}e^2 + \frac{37}{96}e^4 \right). \quad (17)$$