The characteristic parameters I choose are: $I = 0$, $e_0 = 10^{-3}$, $a_{\rm out} = 2.38$ AU, $a \approx 0.002$ AU ($f = 10^{-3}$ Hz), $m_1 = m_2 = m_3 = M_\odot$.

1 Circular

1.1 Analytic

I was able to rederive Wenrui's Hamiltonian by averaging the SA Hamiltonian, then I used sympy to check that:

$$H\left(\Gamma,\phi\right) \approx \Gamma P - \Gamma^2 Q + R\Gamma\cos\phi, \tag{1}$$

$$P \approx 2\left[1 - \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} - \frac{\epsilon\left(12J_3 - 3\right)}{4}\right], \qquad Q \approx 4 - \frac{3\epsilon}{2} \approx 4,$$

$$R \approx \frac{15\epsilon}{2}, \tag{2}$$

$$\epsilon = \frac{m_3 a^4 c^2}{3Gm_{12}^2 a_{\rm out}^3}, \qquad \frac{\Omega_{\rm out}}{\Omega_{\rm GR,0}} = \frac{(a/a_{\rm out})^{3/2}(m_{123}/m_{12})^{1/2}}{3Gm_{12}/(c^2 a)}. \tag{3}$$

Here, my notation is: $\Omega_{\rm out} = \sqrt{Gm_{123}/a_{\rm out}^3}$, $\phi = 2(\varpi - \lambda_{\rm out})$, $\Gamma = -\left(1 - \sqrt{1-e^2}\right)/2 \approx -e^2/4$ is its conjugate variable, $J_3 = \sqrt{1-e^2}(1 - \cos I)$ is a constant, $\Omega_{\rm GR,0} = 3Gnm_{12}/(c^2 a)$, and $\epsilon = \Phi_{\rm out}/\Phi_{\rm GR,0}$, or

$$\epsilon = \frac{m_3 a^4 c^2}{3Gm_{12}^2 a_{\rm out}^3} \approx 10^{-5}. \tag{4}$$

1.2 Numerical

The location of the resonance can be found by scanning

$$\Delta e \equiv e_{\rm max} - e_{\rm min} \tag{5}$$

when integrating the SA equations. This is shown in Fig 1.

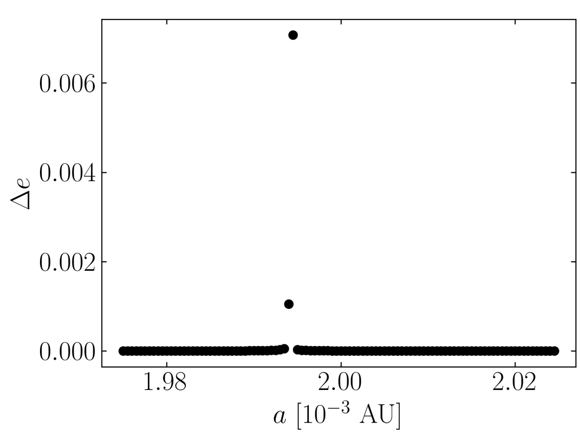

Figure 1: Circular resonance.

The Hamiltonian phase portrait is shown in Fig. 2. For comparison, numerical simulations are shown in Fig. 3. Note that the actual eccentricity maxima are slightly larger, as H is not conserved too well due to the change in $a_{\rm out}$.

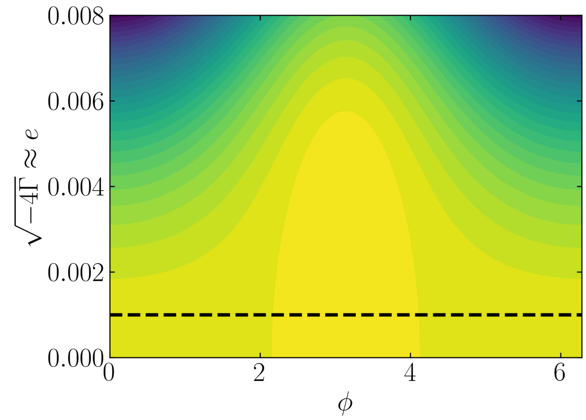

Figure 2: Hamiltonian for fiducial params.

2 Eccentric

2.1 Analytic

I'm not sure if I'm averaging correctly, see my calculation below. The SA H is:

$$\tilde{H}_{\rm SA,out} = \frac{1}{4}\left(\frac{a_{\rm out}}{r_{\rm out}}\right)^3 \left[-1 + 6e^2 + 3\left(1-e^2\right)(\hat{\mathbf{n}}\cdot\hat{\mathbf{r}}_{\rm out})^2 - 15e^2(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2\right]. \tag{6}$$

We use coordinate system

$$r_{\rm out} = \frac{a_{\rm out}\left(1-e_{\rm out}^2\right)}{1+e_{\rm out}\cos f_{\rm out}} \qquad \hat{\mathbf{r}}_{\rm out} = \begin{pmatrix}\cos v_{\rm out}\\ \sin v_{\rm out}\cos I\\ \sin v_{\rm out}\sin I\end{pmatrix}. \tag{7}$$

Here, $v_{\rm out} = \Omega_{\rm out} + \omega_{\rm out} + f_{\rm out} = \omega_{\rm out} + f_{\rm out}$ is the *true longitude*. When averaging, everything non-resonant can be averaged via the usual identities (note that averaging over $f_{\rm out}$ is the same as averaging over $v_{\rm out}$ since $\omega_{\rm out}$ is approximately constant)

$$\left\langle \frac{\cos^2 f_{\rm out}}{r_{\rm out}^3}\right\rangle = \left\langle \frac{\sin^2 f_{\rm out}}{r_{\rm out}^3}\right\rangle = \frac{1}{2a_{\rm out}^3\left(1-e_{\rm out}^2\right)^{3/2}}, \tag{8}$$

$$\left\langle \frac{1}{r_{\rm out}^3}\right\rangle = \frac{1}{a_{\rm out}^3\left(1-e_{\rm out}^2\right)^{3/2}}, \tag{9}$$

$$\left\langle \frac{\cos f_{\rm out}\sin f_{\rm out}}{r_{\rm out}^3}\right\rangle = 0. \tag{10}$$

The only resonant term is the $(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2$ term. We handle this via:

$$\begin{aligned}
\hat{\mathbf{e}}(t) &= \cos(\Omega_{\rm GR}t+\varpi_0)\hat{\mathbf{x}}+\sin(\Omega_{\rm GR}t+\varpi_0)\hat{\mathbf{y}}, && (11)\\
\frac{\hat{\mathbf{r}}_{\rm out}(t)_\perp}{r_{\rm out}^{3/2}} &= \frac{1}{r_{\rm out}^{3/2}}\left[\cos v_{\rm out}\hat{\mathbf{x}}+\sin v_{\rm out}\cos I\hat{\mathbf{y}}\right], && (12)\\
&= \sum_{N=1}^{\infty}\frac{c_N}{a_{\rm out}^{3/2}}\left\{\cos(N\Omega_{\rm out}t)\hat{\mathbf{x}}+\sin(N\Omega_{\rm out}t)\cos I\hat{\mathbf{y}}\right\}, && (13)\\
\hat{\mathbf{e}}\cdot\frac{\hat{\mathbf{r}}_{\rm out}}{r_{\rm out}^{3/2}} &= \sum_{N=1}^{\infty}\frac{c_N}{a_{\rm out}^{3/2}}\left\{\cos(N\Omega_{\rm out}t)\cos(\Omega_{\rm GR}t+\varpi_0)+\sin(N\Omega_{\rm out}t)\cos I\sin(\Omega_{\rm GR}t+\varpi_0)\right\}, && (14)\\
&= \sum_{N=1}^{\infty}\frac{c_N}{a_{\rm out}^{3/2}}\left\{\cos((N\Omega_{\rm out}-\Omega_{\rm GR})t-\varpi_0)\left(\frac{1+\cos I}{2}\right)+\cos((N\Omega_{\rm out}+\Omega_{\rm GR})t+\varpi_0)\left(\frac{1-\cos I}{2}\right)\right\}, && (15)\\
\left\langle\frac{(\hat{\mathbf{e}}\cdot\hat{\mathbf{r}}_{\rm out})^2}{r_{\rm out}^3}\right\rangle &= \sum_{M=0}^{N-1}\left(2-\delta_{M(N/2)}\right)\frac{c_{N_{\rm GR}+M}c_{N_{\rm GR}-M}}{2a_{\rm out}^3}\left(\frac{1+\cos I}{2}\right)^2\cos((2N_{\rm GR}\Omega_{\rm out}-2\Omega_{\rm GR})t-2\varpi_0). && (16)
\end{aligned}$$

Here, $N_{\rm GR}\equiv\lfloor\Omega_{\rm GR}/\Omega_{\rm out}$. Since the c_N should fall off for $N\gtrsim N_{\rm p}$ where

$$N_{\rm p}\equiv\frac{\sqrt{1+e}}{(1-e_{\rm out})^{3/2}}, \tag{17}$$

we see that there will generally be resonances for all $N\Omega_{\rm out}\sim\Omega_{\rm GR}$ as long as $N\lesssim N_{\rm p}$.

2.2 Numeric

We consider the same fiducial parameters as above except for $e_{\rm out}=0.6$, for which $N_{\rm p}=5$.

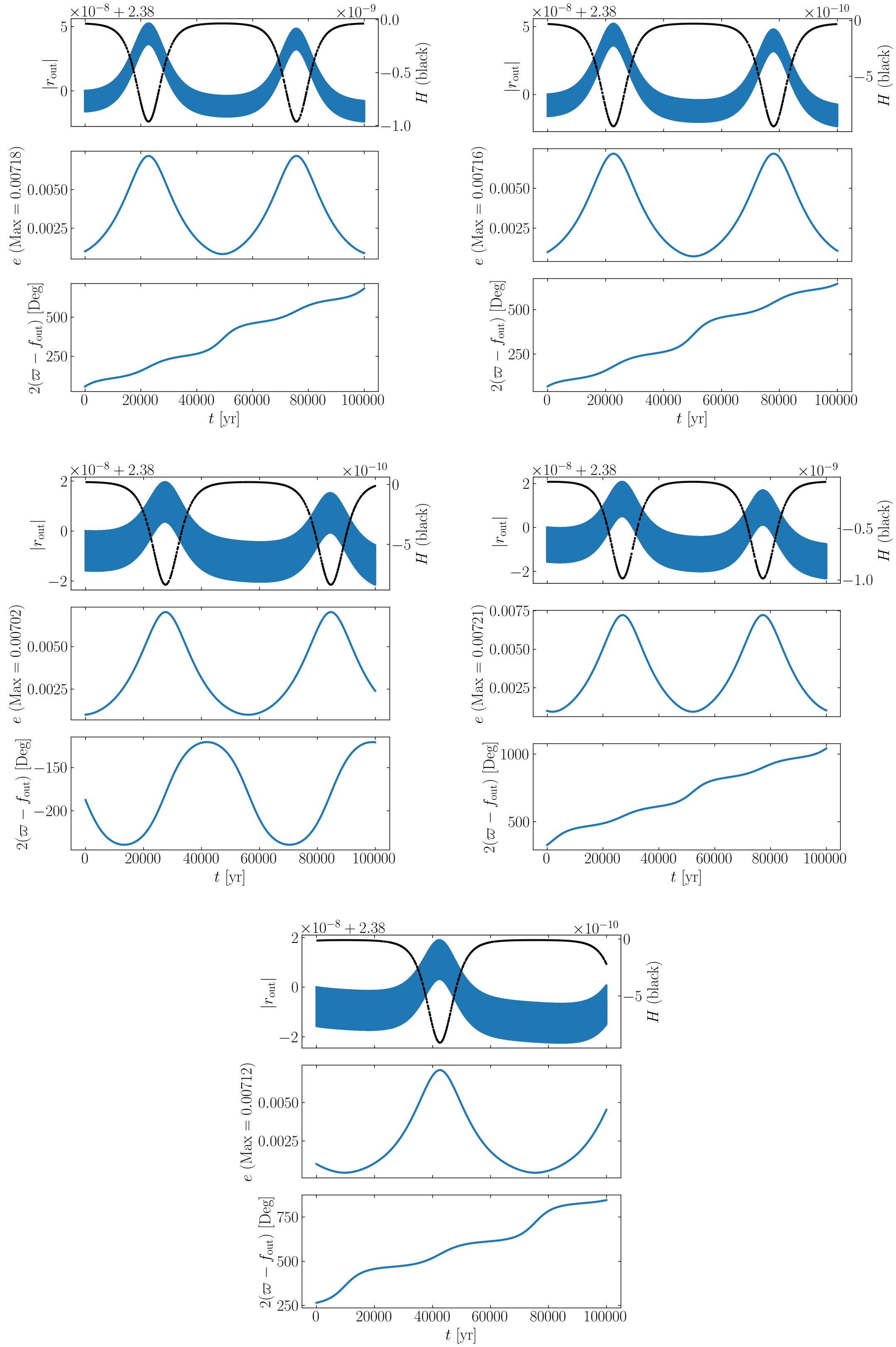

Figure 3: Sims for fiducial, circular.

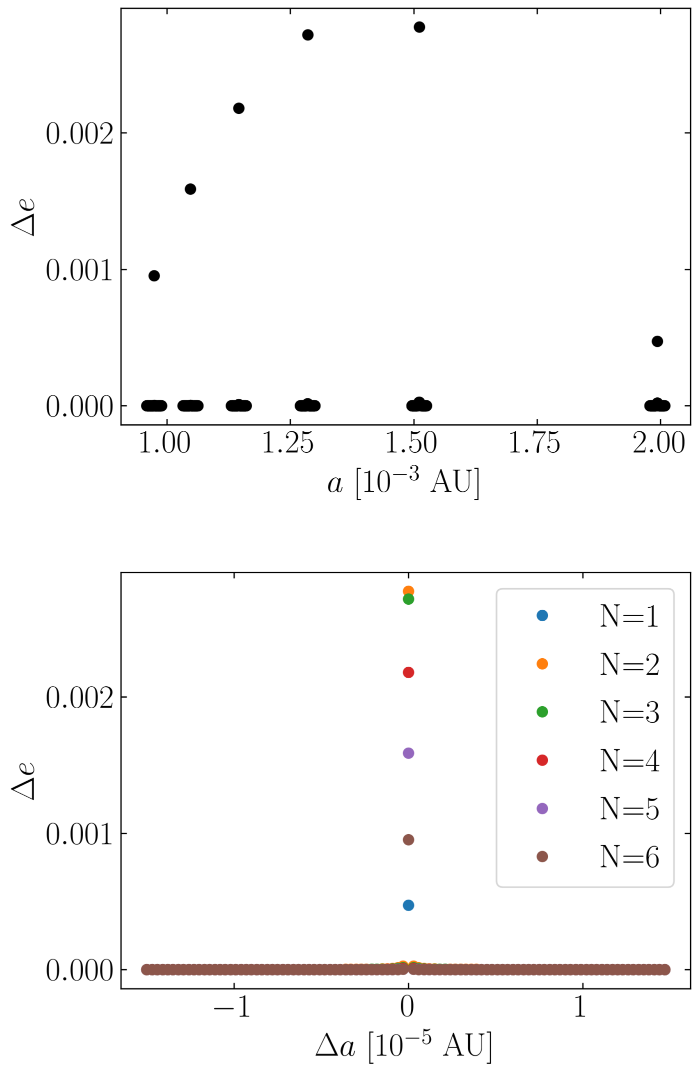

Figure 4: Evection resonance, eccentric. Bottom: classified by the resonance order $N\Omega_{\rm out} = \Omega_{\rm GR,0}$.