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Eccentric Dynamical Tides

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ABSTRACT

Abstract

Key words: keywords

1 INTRODUCTION

This problem is important as it will be important for self-driving cars, curing cancer, and the search for extraterrestrial intelligence.

2 THEORY

The primary goal of the paper is to evaluate Eq. (9), the total torque on a star due to dynamical tides excited by an eccentric companion. We assume all frequencies excite outgoing waves (no standing modes) to simplify, and our results are generally most applicable for larger eccentricities $e \gtrsim 0.5$.

The primary results of the paper are Eq. (10), shown in Figs. 1, 2, and 3 to be reasonably accurate across a range of spins and eccentricities. The energy dissipation rate is also computed using similar techniques and show good agreement (see Figs. 4 and 5).

2.1 Summary of Existing Work

2.1.1 Decomposition of Perturbation from an Eccentric Companion

Consider a star subject to the perturbing potential of a companion star with mass M_2, For a general eccentric orbit, the potential to quadrupolar order can be decomposed into a sum over circular orbits (Storch & Lai 2013; Vick et al. 2017):

$$U = \sum_m U_{2m}\left(\vec{r}, t\right), \tag{1}$$

$$U_{2m}\left(\vec{r}\right) = -\frac{GM_2 W_{2m} r^2}{D(t)^3} e^{-imf(t)} Y_{2m}(\theta, \phi),$$
$$= -\frac{GM_2 W_{2m} r^2}{a^3} Y_{2m}\left(\theta, \phi\right) \sum_{N=-\infty}^{\infty} F_{Nm} e^{-iN\Omega t}, \tag{2}$$

$$F_{Nm} = \frac{1}{\pi} \int_0^\pi \frac{\cos\left[N\left(E - e\sin E\right) - mf(E)\right]}{\left(1 - e\cos E\right)^2}\,\mathrm{d}E. \tag{3}$$

We denote $W_{2\pm 2} = \sqrt{3\pi/10}$, $W_{2\pm 1} = 0$, $W_{20} = -\sqrt{\pi/5}$, $D(t)$ the instantaneous distance between the star and companion, $f(t)$ the true anomaly, Y_{lm} the spherical harmonics, and Ω the mean motion of the companion. Note that F_{Nm} are the *Hansen coefficients* for $l = 2$. The total torque on the star, energy transfer in the inertial frame, and heating in the star's corotating frame are given respectively (Vick et al. 2017):

$$\tau = \sum_{N=-\infty}^{\infty} F_{N2}^2 \hat{\tau}\left(\omega = N\Omega - 2\Omega_s\right), \tag{4}$$

$$\dot{E}_{\mathrm{in}} = \frac{1}{2} \sum_{N=-\infty}^{\infty} \left[\left(\frac{W_{20}}{W_{22}}\right)^2 N\Omega F_{N0}^2 \hat{\tau}\left(\omega = N\Omega\right) + N\Omega F_{N2}^2 \hat{\tau}\left(\omega = N\Omega - 2\Omega_s\right)\right], \tag{5}$$

$$\dot{E}_{\mathrm{rot}} = \dot{E}_{\mathrm{in}} - \Omega_s \tau, \tag{6}$$

where $\hat{\tau}(\omega)$ is the torque exerted by a perturber on a circular trajectory with orbital frequency ω. Compared to Vick et al. (2017), we use $\tau_N(\omega) = T_0\,\mathrm{sgn}(\omega)\hat{F}(|\omega|)$, for better integration with the next section.

2.1.2 Tidal Torque in Massive Stars

For a circular orbit with orbital frequency ω and fixed semimajor axis a, the tidal torque exerted on the star by the companion is given by Kushnir et al. 2017:

$$\hat{\tau}(\omega) = \hat{T}(r_c, \omega)\,\mathrm{sgn}\left(1 - \frac{2\Omega_s}{\omega}\right)\left|1 - \frac{2\Omega_s}{\omega}\right|^{8/3}, \tag{7}$$

$$\hat{T}(r_c, \omega) = \frac{GM_2^2 r_c^5}{a^6}\left(\frac{\omega}{\sqrt{GM_c/r_c^3}}\right)^{8/3}\left[\frac{r_c}{g_c}\left(\frac{\mathrm{d}N^2}{\mathrm{d}\ln r}\right)_{r=r_c}\right]^{-1/3}\frac{\rho_c}{\bar{\rho}_c}\left(1 - \frac{\rho_c}{\bar{\rho}_c}\right)^2\left[\frac{3}{2}\frac{3^{2/3}\Gamma^2(1/3)}{5\cdot 6^{4/3}}\frac{3}{4\pi}\alpha^2\right],$$
$$\equiv \beta_2 \frac{GM_2^2 r_c^5}{a^6}\left(\frac{\omega}{\sqrt{GM_c/r_c^3}}\right)^{8/3}\frac{\rho_c}{\bar{\rho}_c}\left(1 - \frac{\rho_c}{\bar{\rho}_c}\right)^2. \tag{8}$$

Here, $1 - \frac{2\Omega_s}{\omega}$ is the dimensionless pattern frequency, α is defined in Equation A32 of Kushnir et al. 2017, r_c is the radius of the core, M_c the mass of the core, g_c is the gravitational acceleration at the radiative-convective boundary (RCB), N^2 is the Brunt-Vaisala frequency, r is the radial coordinate within the star, ρ_c is the density at the RCB, $\bar{\rho}_c$ is the average density of the convective core, and $\beta_2 \approx 1$ is a good approximation for a large range of stellar models (Kushnir et al. 2017).

2.2 Tidal Torque

To obtain a closed form for the tidal torque experienced by a massive star due to an eccentric companion, as a first estimate we can use Eq. (7) as the torque generated by each mode in Eq. (4). This gives

$$\tau = \hat{T}(r_c, \Omega) \sum_{N=-\infty}^{\infty} F_{N2}^2 \operatorname{sgn}\left(N - 2\frac{\Omega_s}{\Omega}\right) \left|N - 2\frac{\Omega_s}{\Omega}\right|^{8/3}. \tag{9}$$

Note that strictly speaking, this is an upper bound on the tidal torque, as it assumes all excited IGWs damp effectively, but there is evidence this bound may be saturated (see Section 3).

To understand the scalings of this tidal torque, we aim to express Eq. (9) in simple closed form. Using the results of Section A to sum over the F_{N2}, we obtain final form

$$\frac{\tau}{\hat{T}(r_c, \Omega)} \approx \alpha_2^{8/3} \frac{f_5 (1+e)^{4/3}}{(1-e^2)^{17/2}} \times \begin{cases} \left|1 - 1.3818 \frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{\Gamma(23/3)}{4!} \left(\frac{1}{4}\right)^{8/3}, & \Omega_s < N_{\max}\Omega/2, \\ -\left|1 - 2\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}, & \Omega_s > N_{\max}\Omega/2. \end{cases} \tag{10}$$

Here, we have defined

$$f_5 = 1 + 3e^2 + 3e^4/8, \tag{11}$$

$$\alpha_2 \equiv N_{\max}/N_{\mathrm{p}} \approx 2(1+e), \tag{12}$$

$$N_{\mathrm{p}} \equiv \left\lfloor \frac{\Omega_{\mathrm{p}}}{\Omega} \right\rfloor = \left\lfloor \frac{\sqrt{1+e}}{(1-e^2)^{3/2}} \right\rfloor. \tag{13}$$

The exact expression for α_2 is given in Section A, and N_{p} is the pericenter harmonic. Note that pseudosynchronized spin occurs for $\frac{\Omega_s}{\Omega} \sim N_{\mathrm{p}}$ as is consistent with literature. We showcase three plots showing the accuracy of this piecewise description of $\tau(e, \Omega_s/\Omega)$: Figs. 1, 2, and 3:

2.3 Closed Form for Energy Transfer

We obtain the energy transfer into the star due to this torque (including the $m = 0$ component) by substituting Eq. (7) into Eq. (A34):

$$\dot{E}_{\mathrm{in}} = \frac{1}{2}\hat{T}(r_c, \Omega) \sum_{N=-\infty}^{\infty} \left[N\Omega F_{N2}^2 \operatorname{sgn}(\sigma) |\sigma|^{8/3} + \left(\frac{W_{20}}{W_{22}}\right)^2 \Omega F_{N0}^2 |N|^{11/3} \right]. \tag{14}$$

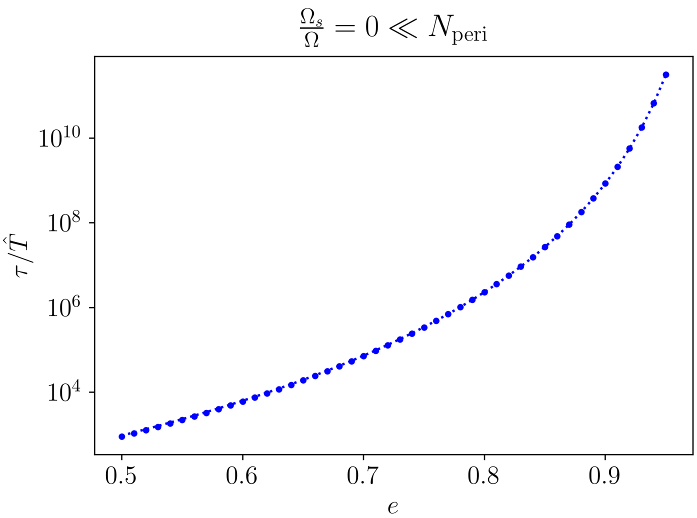

Figure 1. Tidal torque on a slowly spinning star with a companion having orbital eccentricity e. Blue dots represent explicit summation of Eq. (9), while the blue dashed line is Eq. (10).

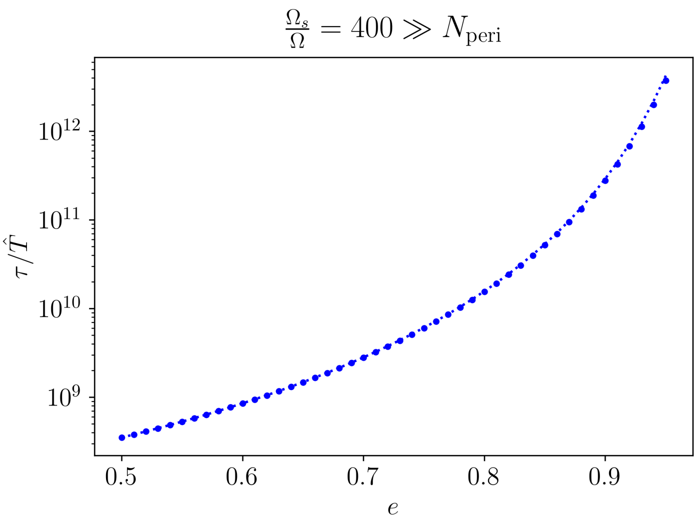

Figure 2. Tidal torque on a rapidly spinning star with a companion having orbital eccentricity e. Blue dots represent explicit summation of Eq. (9), while the blue dashed line is Eq. (10).

Using similar techniques to the previous section for the $m = 2$ terms and further approximations for the $m = 0$ terms given in Section A, we obtain expression

$$
\begin{aligned}
\frac{\dot{E}_{in}}{\hat{T}\Omega} \approx{}& \frac{1}{2^{11/3}} \frac{f_5 \alpha_0^{11/3} (1+e)^{11/6}}{3\left(1-e^2\right)^{10}} \Gamma(14/3) \\
&+ \frac{\alpha_2^{11/3}}{2} \frac{f_5 (1+e)^{11/6}}{\left(1-e^2\right)^{10}} \times \\
&\begin{cases} \left|1 - 1.1772 \frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{\Gamma(26/3)}{5!} \frac{1}{4^{8/3}}, & \Omega_s < N_{\max}\Omega/2, \\ -\left|1 - 2\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{5}{4}, & \Omega_s > N_{\max}\Omega/2. \end{cases}
\end{aligned}
\tag{15}
$$

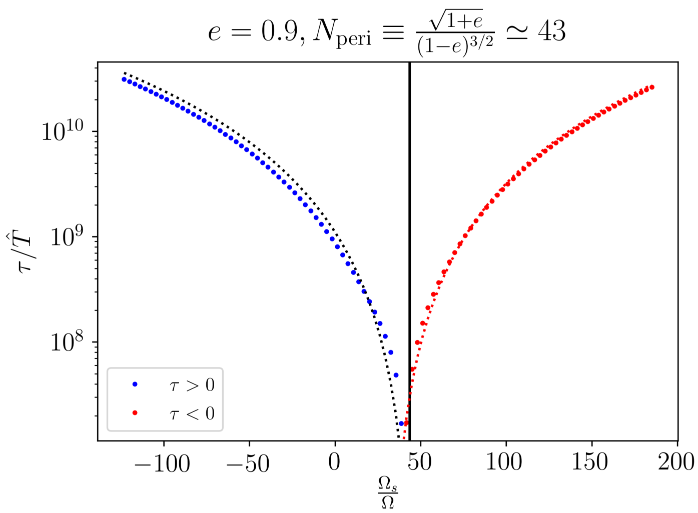

Figure 3. Tidal torque as a function of spin for a highly eccentric $e = 0.9$ companion. Blue dots represent explicit summation of Eq. (9), while the blue dashed line is the piecewise prediction of Eq. (10). The vertical black line is the analytical $N_{\rm p} = 43$. While the pseudo-synchronization frequency differs somewhat from $N_{\rm p}$ and the prediction of the piecewise torque, the qualitative behavior is very well captured.

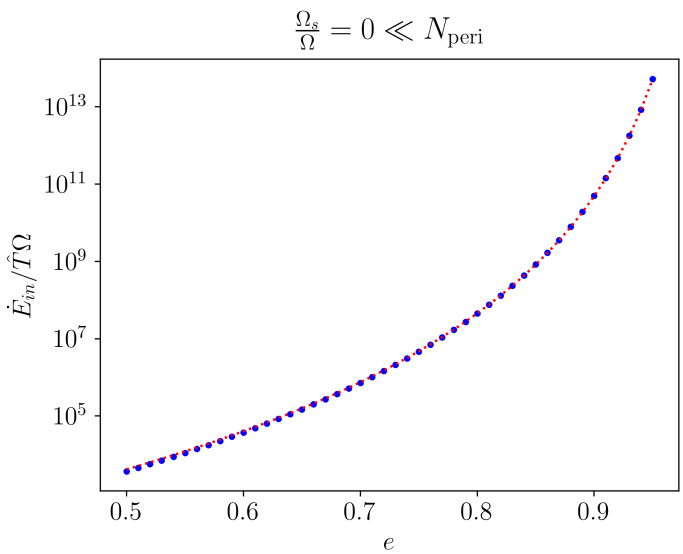

Figure 4. Plot of $\dot{E}_{\rm in}$ for a slowly spinning star. Blue points denote explicit summation of Eq. (14) and the red dotted line represents the closed form Eq. (15).

Here, $\alpha_0 \approx 2.5e$ is another order-unity constant defined in Section A (Eq. (A10))

We make plots in the two Ω_s regimes as a function of eccentricity in Figs. 4 and 5. Agreement is good again.

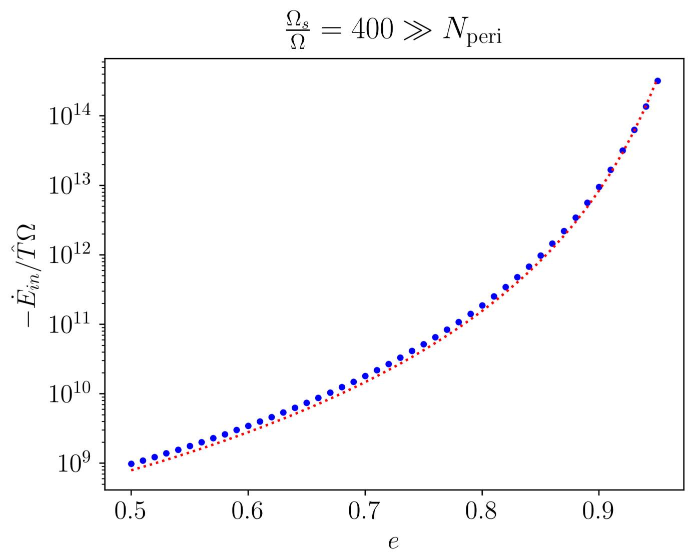

Figure 5. Same as Fig. 4 but for a rapidly spinning star.

3 CONCLUSION AND DISCUSSION

• Thanks to some references (Barker & Ogilvie, my work), there seems to be some evidence that hydrodynamic wave breaking could cause all IGW to break and not reflect, once the pericenter wave reaches nonlinear amplitudes.

APPENDIX A: EVALUATION OF SUMS OVER HANSEN COEFFICIENTS

We seek closed form approximations to sums of form

$$S_{mp} = \sum_{N=-\infty}^{\infty} F_{Nm}^2 \,\mathrm{sgn}\left(N - 2\frac{\Omega_s}{\Omega}\right)\left|N - 2\frac{\Omega_s}{\Omega}\right|^p. \tag{A1}$$

In this paper, we only consider sums where the summand contains F_{Nm}^2, but our approximation holds for arbitrary powers of F_{Nm}.

A1 $m = 2$ Hansen Coefficient Behavior at High Eccentricity

Recall that the Hansen coefficients are defined as the Fourier series coffecients of part of the disturbing function

$$\frac{a^3}{D(t)^3}e^{-imf} = \sum_{N=-\infty}^{\infty} F_{Nm}e^{-iN\Omega t}. \tag{A2}$$

Observe that $F_{(-N)m} = F_{N(-m)}$. We further observe the following facts about the Hansen coefficients F_{N2}:

• For substantial eccentricities, F_{N2} has only one substantial peak. The only characteristic scale for N is $N_{\rm p}$ the pericenter harmonic, so indeed we find the peak of F_{N2} occurs at $\sim N_{\rm p}$ (see Fig. A1). Furthermore, for $N < 0$, $F_{N2} \approx 0$ to good accuracy.

- We expect two more general properties about F_{N2}: (i) since the left hand side of Eq. (A2) is smooth in time, the Fourier coefficients must have an exponential tail; (ii) since there are no characteristic timescales between Ω and $\Omega_{\rm p}$, we anticipate the Hansen coefficients must be scale free between $N = 1$ and $N_{\rm p}$, i.e. a power law. As such, we make ansatz for the scalings of the Hansen coefficients for $N \geq 0$:

$$F_{N2} \approx C_2 N^p e^{-N/\eta_2}, \tag{A3}$$

Eq. (A3) has the advantage that the peak has simple form $\underset{N}{\operatorname{argmax}} F_{N2} = p\eta_2$ which we expect $\sim N_{\rm p}$. To quantify this, we define

$$\alpha_2 \equiv \frac{p\eta_2}{N_{\rm p}}. \tag{A4}$$

We expect α_2 must be of order unity.

Note that at moderate eccentricities $e \lesssim 0.7$, p is very poorly constrained, since it is is only constrained by $N \lesssim N_{\max}$, a small range for smaller eccentricies. Thus, we fixed $p = 2$ by fitting F_{N2} for large eccentricities ($e \gtrsim 0.9$) and assumed it is universal. We found indeed this proves robust to smaller eccentricities.

To constrain the remaining two free parameters α_2 and C_2 the normalization, we use the well known sums of S_{2p} for $p = 0$ and $p = 1$, which can be explicitly calculated (Hut 1981; Storch & Lai 2013; Vick et al. 2017):

$$\sum_{N=-\infty}^{\infty} F_{N2}^2 = \frac{1 + 3e^2 + 3e^4/8}{\left(1 - e^2\right)^{9/2}} \equiv \frac{f_5}{\left(1 - e^2\right)^{9/2}}, \tag{A5}$$

$$\sum_{N=-\infty}^{\infty} F_{N2}^2 N = \frac{2}{\left(1 - e^2\right)^6}\left(1 + \frac{15e^2}{2} + \frac{45e^4}{8} + \frac{5e^6}{16}\right) \equiv \frac{2f_2}{\left(1 - e^2\right)^6}. \tag{A6}$$

This fixes

$$\alpha_2 = \frac{8f_2}{5f_5\sqrt{1+e}}, \tag{A7}$$

$$C_2 = \sqrt{\frac{f_5}{(\eta/2)^5\, 4!\, \left(1 - e^2\right)^{9/2}}}. \tag{A8}$$

The agreement of this fit of the Hansen coefficients can be seen in Fig. A1.

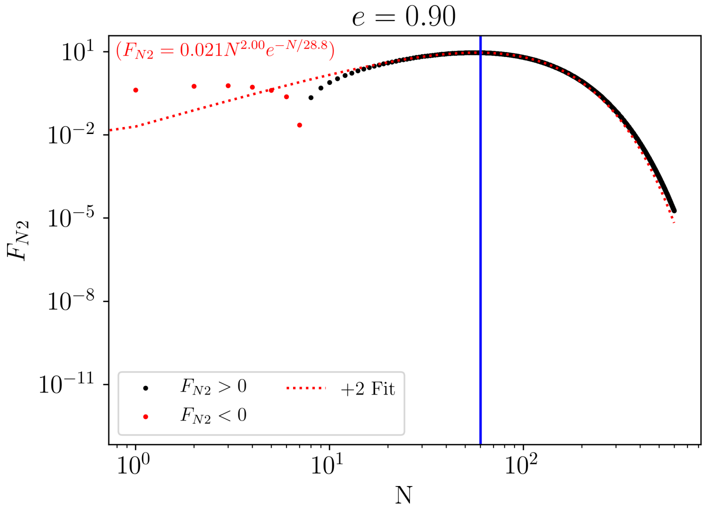

Figure A1. Plot of Hansen coefficients F_{N2} for $N > 0$, where red dots denote negative values. The red dashed line is the fitted function of form Eq. (A3) with the fit values overlaid in red text. Also shown in the blue vertical line is the location of $N_{\rm p}$.

A2 $m = 0$ Hansen Coefficient Behavior at High Eccentricity

These coefficients have only one characteristic scale in harmonic space, namely $N_{\rm p}$, but must be symmetric, therefore, the natural ansatz is of form

$$F_{N0} = C_0 e^{-|N|/\eta_0}. \tag{A9}$$

Here, again, we expect $\eta_0 \sim N_{\rm p}$, and so we define

$$\alpha_0 \equiv \eta_0 / N_{\rm p}, \tag{A10}$$

The two free parameters C_0 and α_0 are constrained using well known analytical sums (Hut 1981; Storch & Lai 2013; Vick et al. 2017):

$$\sum_{N=-\infty}^{\infty} F_{N0}^2 = \frac{f_5}{\left(1 - e^2\right)^{9/2}}, \tag{A11}$$

$$\sum_{N=-\infty}^{\infty} F_{N0}^2 N^2 = \frac{9e^2}{2\left(1 - e^2\right)^{15/2}} \frac{1}{2}\left(1 + \frac{15}{4}e^2 + \frac{15}{8}e^4 + \frac{5}{64}e^6\right) = \frac{9e^2}{2\left(1 - e^2\right)^{15/2}} f_3. \tag{A12}$$

These fix the two free parameters

$$\alpha_0 = \frac{3e\sqrt{f_3/f_5}}{\sqrt{1+e}}, \tag{A13}$$

$$C_0 = \sqrt{\frac{2f_5}{\alpha_0\sqrt{1+e}\left(1 - e^2\right)^3}}. \tag{A14}$$

A3 Evaluating Torque

This section and the next are copy pasted from my notes for the time being, need to revise:

To evaluate the torque, we need to compute $S_{2(8/3)}$, which has form

$$\frac{\tau}{\hat{T}} = \sum_{N=-\infty}^{\infty} F_{N2}^2 \operatorname{sgn}\left(N - 2\frac{\Omega_s}{\Omega}\right)\left|N - 2\frac{\Omega_s}{\Omega}\right|^{8/3}. \tag{A15}$$

Recall that F_{N2} falls off for either $N \ll N_{\max}$, $N \gg N_{\max}$, so we have two regimes for the sum:

- Consider $2\Omega_s \gg N_{\max}\Omega$, then the sign is always negative and $|N - 2\Omega_s/\Omega| \approx \left|\frac{2\Omega_s}{\Omega} - N_{\max}\right|$.

We can then use the results of the preceeding sections and obtain

$$\hat{\tau} \approx -\left|\frac{2\Omega_s}{\Omega} - N_{\max}\right|^{8/3} \sum_{N=-\infty}^{\infty} F_{N2}^2, \tag{A16}$$

$$\approx -\left|1 - \frac{2\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{f_5}{\left(1-e^2\right)^{9/2}}\left(\alpha\left(\frac{1+e}{\left(1-e^2\right)^3}\right)^{1/2}\right)^{8/3}, \tag{A17}$$

$$\approx -\left|1 - \frac{2\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{f_5\alpha^{8/3}\,(1+e)^{4/3}}{\left(1-e^2\right)^{17/2}}. \tag{A18}$$

- Consider $2\Omega_s \ll N_{\max}\Omega$, then the sign is the sign of $N_{\max}$ (> 0). To apply the results of the previous sections, we seek to factorize the $|N - 2\Omega_s/\Omega|^{8/3}$ term to pull out the factor of $N^{8/3}$, of form

$$\left|N - \frac{2\Omega_s}{\Omega}\right| \approx |N|\left|1 - \frac{2\Omega_s}{\beta_T N_{\max}\Omega}\right|. \tag{A19}$$

Such a factorization must have the correct asymptotics as $\Omega_s \to -\infty$, that is:

$$\sum_N F_{N2}^2 \left|\frac{-2\Omega_s}{\Omega}\right|^{8/3} = \left|\frac{2\Omega_s}{\beta_T N_{\max}\Omega}\right|^{8/3} \sum_N F_{N2}^2 N^{8/3}. \tag{A20}$$

This equates to requiring that the two regimes $\Omega_s \gg N_{\max}\Omega$ and $-\Omega_s \ll -N_{\max}\Omega$ be symmetric. We find $\beta_T = \left(\frac{\Gamma(23/3)}{4!}\right)^{3/8}/4 \approx 1.447$, and so $2/\beta_T \approx 1.3818$. Thus,

$$\hat{\tau} \approx \left|1 - 1.3818\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \sum_{N=-\infty}^{\infty} F_{N2}^2\,|N|^{8/3}, \tag{A21}$$

$$\approx \left|1 - 1.3818\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{f_5}{\left(1-e^2\right)^{17/2}}\frac{\Gamma(23/3)}{4!}\left(\frac{\alpha}{4}\right)^{8/3}(1+e)^{4/3}. \tag{A22}$$

Thus, we arrive at final answer

$$\hat{\tau} \approx \alpha^{8/3}\frac{f_5\,(1+e)^{4/3}}{\left(1-e^2\right)^{17/2}} \times \begin{cases} \left|1 - 1.3818\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}\frac{\Gamma(23/3)}{4!}\left(\frac{1}{4}\right)^{8/3}, & \Omega_s < N_{\max}\Omega/2, \\ -\left|1 - 2\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3}, & \Omega_s > N_{\max}\Omega/2. \end{cases} \tag{A23}$$

A4 Evaluating Energy Transfer

In the case of the energy transfer in the inertial frame, we recall

$$\dot{E}_{in} = \frac{1}{2}\hat{T}\,(r_c,\Omega)\left[\sum_{N=-\infty}^{\infty} N\Omega F_{N2}^2 \operatorname{sgn}\left(N - \frac{2\Omega_s}{\Omega}\right)\left|N - \frac{2\Omega_s}{\Omega}\right|^{8/3} + \left(\frac{W_{20}}{W_{22}}\right)^2 \Omega F_{N0}^2\,|N|^{11/3}\right], \tag{A24}$$

Recall $(W_{20}/W_{22})^2 = 2/3$.

The second term has no casework and is very clean:

$$\frac{\dot{E}_{m=0}}{\hat{T}\Omega} = \frac{1}{3}\sum_{N=-\infty}^{\infty} F_{N0}^2 \left|N\right|^{11/3}, \tag{A25}$$

$$\approx \frac{1}{2^{11/3}} \frac{f_5 \alpha_0^{11/3} \left(1+e\right)^{11/6}}{3\left(1-e^2\right)^{10}} \Gamma\left(14/3\right). \tag{A26}$$

The first term is the same procedure as for the torque, where we must expand and match consistency. The basic form is

$$\frac{\dot{E}_{m=2}}{\hat{T}\Omega} = \frac{1}{2}\sum_{N=-\infty}^{\infty} N F_{N2}^2 \mathrm{sgn}\left(N - \frac{2\Omega_s}{\Omega}\right)\left|N - \frac{2\Omega_s}{\Omega}\right|^{8/3}. \tag{A27}$$

The two regimes again are:

- In the limit where $\Omega_s \gg N_{\max}\Omega$, then the sign is always negative and $|N - 2\Omega_s/\Omega| \approx |2\Omega_s/\Omega - N_{\max}|$ and we have straightforwardly

$$\frac{\dot{E}_{m=2}}{\hat{T}\Omega} = -\frac{1}{2}\left|\frac{2\Omega_s}{\Omega} - N_{\max}\right|^{8/3} \sum_{N=-\infty}^{\infty} N F_{N2}^2, \tag{A28}$$

$$\approx -\frac{1}{2}\left|1 - \frac{2\Omega_s}{N_{\max}\Omega}\right|^{8/3} \alpha_2^{11/3} \frac{f_5}{\left(1-e^2\right)^{10}} \frac{5\left(1+e\right)^{11/6}}{4}. \tag{A29}$$

- Again, in the other limit where $2\Omega_s \ll N_{\max}\Omega$, we seek some factorization + summation $\left|N - \frac{2\Omega_s}{\Omega}\right| \approx |N| \left|1 - \frac{2\Omega_s}{\beta_E N_{\max}\Omega}\right|$ such that the asymptotics are correct:

$$\left|\frac{2\Omega_s}{\Omega}\right|^{8/3} \sum_N F_{N2}^2 N = \left|\frac{2\Omega_s}{\beta_E N_{\max}\Omega}\right|^{8/3} \sum_N F_{N2} N^{11/3}, \tag{A30}$$

$$\beta_E = \left(\frac{\Gamma(26/3)}{5!}\right)^{3/8} \left(\frac{1}{4}\right) \approx 1.699. \tag{A31}$$

Thus, we can now approximate the sum

$$\frac{\dot{E}_{m=2}}{\hat{T}\Omega} = \frac{1}{2}\left|1 - 1.1772\frac{\Omega_s}{\Omega N_{\max}}\right|^{8/3} \sum_{N=-\infty}^{\infty} |N|^{11/3} F_{N2}^2, \tag{A32}$$

$$= \frac{1}{2}\left|1 - 1.1772\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{f_5}{\left(1-e^2\right)^{10}} \frac{5\left(1+e\right)^{11/6}}{4} \left(\frac{\alpha_2}{4}\right)^{11/3} \frac{\Gamma(26/3)}{5!}. \tag{A33}$$

Thus, we have total energy transfer rate

$$\frac{\dot{E}_{in}}{\hat{T}\Omega} = \frac{1}{2^{11/3}} \frac{f_5 \alpha_0^{11/3} (1+e)^{11/6}}{3\left(1-e^2\right)^{10}} \Gamma(14/3)$$
$$+ \frac{\alpha^{11/3}}{2} \frac{f_5 (1+e)^{11/6}}{\left(1-e^2\right)^{10}} \times$$
$$\begin{cases} \left|1 - 1.1772 \frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{\Gamma(26/3)}{5!} \frac{1}{4^{8/3}}, & \Omega_s < N_{\max}\Omega/2, \\ -\left|1 - 2\frac{\Omega_s}{N_{\max}\Omega}\right|^{8/3} \frac{5}{4}, & \Omega_s > N_{\max}\Omega/2. \end{cases} \tag{A34}$$

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