École Polytechnique

Bachelor of Science

CSE202-Design and Algorithms Analysis

Homework 13

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Exercise 1

Question Content

In problem 2-Partition, you are given n positive integers $a_1, \ldots, a_n$ and the question is to decide whether there exists a subset $I \subset \{1, \ldots, n\}$ such that

$$\sum_{i \in I} a_i = \sum_{i \in \{1,\ldots,n\} \setminus I} a_i$$

Show that 2-Partition is NP-complete. [Hint: a non-trivial reduction of the SubsetSum problem can be found.]

We first recall the concept of the **Subset Sum** problem decides whether a set $S = \{s_1, s_2, \ldots, s_n\}$ and $k \in \mathbb{N}_0$ contains a subset of S such that its summation is k or not. This problem is NP-Complete.

In order to prove that 2-Partition is NP-complete. Then we need to prove the following conditions

1. $A \in NP$
2. any NP-Complete problem B can be reduced to A
3. the reduction of B to A works in polynomial time
4. the original problem A has a solution if and only if B has a solution.

Then we try to prove each point

1. 2-Partition is an NP problem since there is not a direct solution that can operate in polynomial time to solve the problem, however, verifying the problem is not difficult since we can compute the sum of each partition that can be operated in polynomial time. Therefore 2-Partition is NP.
2. Then we try to prove that Subset Sum problem can reduce to 2-Partition. Let set $S = \{a_1, \cdots, a_n\}$ and $s = sum(S)$. Then we denote a new set $S' = S \cup \{2k - s\}$ where k is the target number of Subset Sum problem. we choose $2k - s$ since $sum(S') = 2k - s + s = 2k = k + k$ which we want two partitions with equal sizes and equal sum which is k.
3. The reduction is easily to see works in polynomial time.
4. Then we try to prove that Subset(S, k) has solution if and only if Partition2(S') has solution.

   ($\Rightarrow$): We assume there exists a set $X \subset S$ such that $sum(X) = k$, then we have $sum(S \setminus X) = s - k$ then we have in the 2-Partition that $sum(X) = sum((S \setminus X) \cup \{2k - s\}) = k$.

   ($\Leftarrow$): We assume that there exists a 2-Partition solution such that the sum for each partition is k. Then one of the partition contains the element $2k - s$, the other partition $\subset S$. Therefore we have Subset(S, k) is also true.

Then we prove that 2-Partition is NP-complete.