CSE202 - DESIGN AND ANALYSIS OF ALGORITHMS

Week 4 - Divide & Conquer 3 - Master Theorem, Advanced "Conquer" Example

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EXERCISE 1

We apply the proof of the Master theorem that we have

$$\begin{aligned}C(n) &\le 3C(\frac{n}{2}) + 4n \\ &\le 4n + \frac{3}{2}\cdot 4n + 3^2C(\frac{n}{2^2}) \\ &\le 4n\left(1 + 3/2 + \cdots + (3/2)^{k-1}\right) + 3^kC(1)\end{aligned}$$

Then we use that the depth of the recursion tree is $k = \log_2 n$ and $C(1) = 1$ we got

$$\begin{aligned}C(n) &\le 4n\left(1 + 3/2 + \cdots + (3/2)^{k-1}\right) + 3^kC(1) \\ &\le 4n(\frac{1-(3/2)^k}{1-3/2}) + 3^k \\ &\le 4n\cdot 2(\frac{3^{\log_2 n}}{2^{\log_2 n}} - 1) + 3^k \\ &\le 9n^{\log_2 3} - 8n \\ &\le 9n^{\log_2 3}\end{aligned}$$

EXERCISE 2

We apply a similar method in Exercise 1 we have the inequality

$$C(n) \le 4n\left(1 + 3/2 + \cdots + (3/2)^{k-1}\right) + 3^kC(n/2^k)$$

Since we know that the recursion stops when $n \le s$, therefore we have $s = \frac{n}{2^k}$ and $k = \log_2(n/s)$. For the value of k we got $C(s) = 2s^2$. Then we can compute the boundary of the complexity that

$$\begin{aligned}C(n) &\le 4n\left(1 + 3/2 + \cdots + (3/2)^{k-1}\right) + 3^k 2s^2 \\ &\le 4\cdot 2^k s\cdot(\frac{3}{2})^{k-1}3 + 3^k2s^2 \\ &\le 8s\cdot 3^k + 3^k2s^2 = 3^k(8s+2s^2) \\ &\le 3^{\log_2(n/s)}(8s+2s^2)\end{aligned}$$

And we want to transfer to the form of $f(s)n^{log_2(3)}$ therefore we have

$$3^{\log_2(n/s)}(8s+2s^2) = 3^{\log_2 n - \log_2 s}(8s+2s^2) = \frac{3^{\log_2 n}}{3^{\log_2 s}}(8s+2s^2) = \frac{n^{\log_2 3}}{s^{\log_2 3}}(8s+2s^2)$$

In conclusion,

$$C(n) \le \frac{n^{\log_2 3}}{s^{\log_2 3}}(8s+2s^2)$$

EXERCISE 3

From Question 2 we find that $C(n) \leq \frac{n^{\log_2 3}}{s^{\log_2 3}}(8s + 2s^2)$ in the end. Since we try to Optimize the choice of s and n is the length of the input so here we just ignore it and only consider s as our variable. Then we take the derivative of it.

$$
\begin{aligned}
(\frac{8s+2s^2}{s^{\log_2 3}})' &= \frac{(8+4s)s^{\log_2 3} - (8s+2s^2)log_2 3 \cdot s^{\log_2 3}/s}{(s^{\log_2 3})^2} \\
&= \frac{(8s+4s^2 - \log_2 3(8s+2s^2))}{s(s^{\log_2 3})} = 0
\end{aligned}
$$

Then we can solve the equation that $(4 - 2\log_2 3)s^2 - 8(\log_2 3 - 1)s = 0$. Solving the equation we have

$$
s = \frac{4(\log_2 3 - 1)}{2 - \log_2 3} \simeq 5.64
$$

Therefore since we know that s is power of 2, then we get $s = 4$ or $s = 8$ where we have the constant $\frac{64}{9}$. Compare with the coefficient in Question 1 we got $|\frac{\frac{64}{9} - 9}{9}| \times 100\% = 20.98\%$. We find it's about 20% more efficient compared with the naive Karatsuba's algorithm.