Computer Science

&

Applied Mathematics

MAA205 - Algorithms for Discrete Mathematics - Fall 2022

Part A - Graphs and Matrices

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Preface

Instructors:

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Prerequisites:

1. MAA106 - Introduction to Numerical Analysis

2. MAA104 - Algebra

3. MAA103 - Discrete Mathematics

4. CSE101 - Computer Programming

Mathematical Toolbox:

1. Graphs and Matrices

2. Probabilistic graphs

Python Notebooks:

1. Enumeration in graphs

2. Probabilistic graphs

Evaluation and Final Grade:

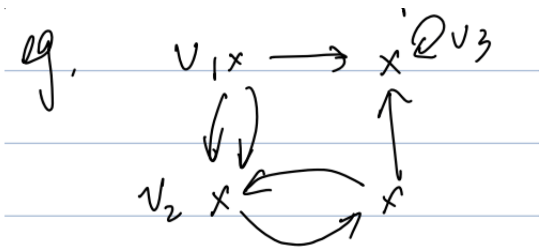
1. 50% Graded Labs (1 for every topic)

2. 50% Individual final project (report due at the end of the last Lab, no oral defense)

Contents

1 Adjacency matrices 邻接矩阵 (review of CSE101)

Let G be a finite graph on a vertex set (V 是顶点，E 是边) $V = \{v_1, ..., v_n\}$



1.1 Definition

The adjacency matrix A_G of G is the n matrix $(A_G) = (a_{ij})_{i,j \leq n}$ where $a_{ij} = Number$ of edges going from v_i to v_j. 从点 v_i 到 v_j 的边的个数

What does $(A_G)^2$ represent?

$(A_G^2)_{i,j} = i \times j = \boxed{a_{i,1}a_{1,j}} + a_{i2}a_{2j} + \dots + a_{i,n}a_{n,j}$

\# paths of length 2.

$V_i \to V_1 \to V_j$

1.2 Proposition

Fot all $k \geqslant 1, (A_G)^k_{i,j}$ is the number of paths from V_i to V_j of length k. A^{100} 是 A 经过 100 次方以后实际的运算结果.

$$Ex : A = \begin{bmatrix} 0 & 2 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \end{bmatrix}$$

$$\Rightarrow A^{100} = \begin{bmatrix} 0 & 0 & 99 & 2 \\ 0 & 1 & 50 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 50 & 1 \end{bmatrix}$$

1.3 (A glimpse of) spectral graph theory 谱系图理论

The maximal out-degree $\underset{\rightarrow}{\Delta}$ of G is

$$\underset{\rightarrow}{\Delta} = \max_i \sum_j a_{ij}$$

Theorem: Let G be a graph with adjacency matrix A. Then

1. If $0 < \varepsilon < \frac{1}{\underset{\rightarrow}{\Delta}}$ then $(Id_n - \varepsilon A)$ is invertible.

2. G is connected $^{(*)}$ 在一个无向图 G 中，若从顶点 $v_i v_i$ 到顶点 $v_j v_j$ 有路径相连（从 $v_j v_j$ 到 $v_i v_i$ 也一定有路径），则称 $v_i v_i$ 和 $v_j v_j$ 是连通的。如果 G 是有向图，那么连接 $v_i v_i$ 和 $v_j v_j$ 的路径中所有的边都必须同向。如果图中任意两点都是连通的，那么图被称作连通图。

   $\Leftrightarrow$ For all $\varepsilon < \frac{1}{\underset{\rightarrow}{\Delta}}$, even coefficient of $(I_d - \varepsilon A)^{-1} > 0$.

   $\Leftrightarrow$ $\exists \varepsilon < \frac{1}{\underset{\rightarrow}{\Delta}}$ such that every coeff. of $(Id - \varepsilon A)^{-1}$ is > 0

$\forall i, j, \quad \exists k$ s.t. $\left(A_G^h\right)_{i,j}$ is > 0.

Proof:

1. Let ϵ be such that $(Id - \varepsilon A)$ is not invertible

   $\Rightarrow$ There exists $\vec{x} = (x_1, \ldots, x_n) \neq \overrightarrow{0} \cdot (0, 0, \ldots, 0)$ st. $(Id - \varepsilon A)\vec{x} = 0$.

   $A\vec{x} = \frac{1}{\varepsilon}\vec{x}$

   $\forall i, \sum_j a_{ij} x_j = \frac{1}{\varepsilon} x_i$ (1)

   Let i^* be such $x_{i*} = \max_{1 \le i \le n} x_i$, we can assume $x_i^* > 0$

   Apply (1) to $i*$ we have

$$x_{i*} \underset{\rightarrow}{\Delta} \geq x_{i*} \sum_j a_{i*,j} = \sum_j a_{i*,j} \cdot a_{i*} \geq \sum_j a_{i*,j} \cdot a_j = \frac{1}{\varepsilon} x_{i*}$$

   We prove that $\Rightarrow \underset{\rightarrow}{\Delta} \geq \frac{1}{\varepsilon}$

2. $(1 - x)\left(1 + x + x^2 + \ldots + x^k\right) = 1 - x^{k+1}$

   Let x be real. $(Id - \varepsilon A)\left(Id + \varepsilon A + \varepsilon^2 +)^2 + \cdots + \varepsilon^k A^k\right) = Id - \varepsilon^{k+1} A^{k+1}$ (every coefficient is $\leq (\varepsilon \underset{\rightarrow}{\Delta})^{k+1}$)

   Let $k \to +\infty \quad (Id - \varepsilon A)(Id + \varepsilon A + \varepsilon^2 A^2 + \ldots) = Id$.

   $(Id - \varepsilon A)^{-1} = Id + \varepsilon A + \varepsilon^2 A^2 + \cdots + \varepsilon^k A^k + \ldots$

   If G is connected, the every coefficient on $RHS > 0 \quad \forall \varepsilon$ Fix ε, Assume every coefficient on RHS > 0.

2 Probabilistic Graphs 概率图谱 (Partly review of MAA103)

$V = \{v_1, ..., v_n\}$

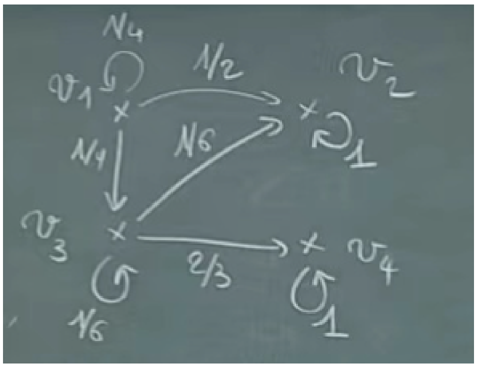
2.1 Definition of Probabilistic Graphs

A transition matrix on V is an $n \times n$ matnix $P = (Pi)_{ij \leq n}$ such that

1. $\forall i, j, \quad 0 \leq p_{i,j} \leq 1$ 个体概率小于 1

2. $\forall i \leq n, \sum_j p_{i,j} = 1$ 每行概率总和等于 1

P_{ij} is interpreted as the probability of going from v_i to v_j in one-time step. For example:



$$P = \begin{bmatrix} \frac{1}{4} & \frac{1}{2} & \frac{1}{4} & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2.2 Definition of Markov Chain 马尔可夫链

1. V be a vertex set $\{v_1, \ldots, v_m\}$

2. P be a transition matrix on V.

3. $S \in V$

A Markov Chain with transition matrix P and starting point S. is a sequence of random variables $X_0, X_1, X_2 \ldots$ with value in V such that $X_0 = S$ and $\forall k \geqslant 1$

$$\mathbb{P}\left(X_1 = x_1, X_2 = x_2 \ldots X_n = x_n\right) = p_{s \cdot x_1} \times p_{x_1 x_2} \times \ldots \times p_{x_n - 1 x_n}$$

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2.3 Proposition of Markov Chain

The coeff $\left(P^k\right)_{i,j}$. is the probability of going from v_i to v_j Th exactly k times steps.

2.4 Absorbing Probability (on an example)

In the mathematical theory of probability, an **absorbing Markov chain** is a Markov chain in which every state can reach an absorbing state. An absorbing state is a state that, once entered, cannot be left.

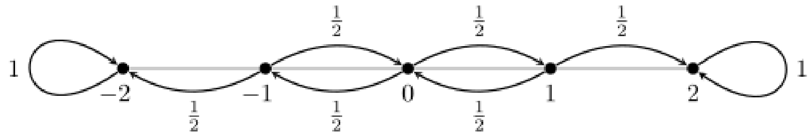

Figure 1: Sample of absorbing Markov chain

Put $\Pi_{v'}^{v} = \mathbb{P}(\text{ starting from } v\text{, I am eventually stuck at } v')$

$\Pi_{v_1}^{v_2} = \mathbb{P}(\text{ stuck at } v_2 | \text{ 1st step is } v_1) \times \mathbb{P}(\text{1st step is } v_1) + \mathbb{P}(\text{ stuck at } v_2 | \text{ 1st step is } v_2) \times \mathbb{P}(\text{1st step is } v_2) + \mathbb{P}(\text{ stuck at } v_2 | \text{ 1st step is } v_3) \times \mathbb{P}(\text{1st step is } v_3) = \Pi_{v_1}^{v_2} \times \frac{1}{4} + 1 \times \frac{1}{2} + \Pi_{v_3}^{v_2} \times \frac{1}{4}$