Computer Science

&

Applied Mathematics

MAA205 - Algorithms for Discrete Mathematics - Fall 2022

Part C - Experimental Mathematics

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Preface

Instructors:

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Prerequisites:

1. MAA106 - Introduction to Numerical Analysis
2. MAA104 - Algebra
3. MAA103 - Discrete Mathematics
4. CSE101 - Computer Programming

Mathematical Toolbox:

1. Dynamics and asymptotics

Python Notebooks:

1. Prime numbers, factorization of integers
2. Arithmetic of square roots. Arithmetic of modulos
3. Numbers and dynamics

Evaluation and Final Grade:

1. 50% Graded Labs (1 for every topic)
2. 50% Individual final project (report due at the end of the last Lab, no oral defense)

Contents

Project

Project from January 3rd to 19th with 3 weeks. More like a long TD with instructions.

1 Asymptotics 渐进分析

1.1 Review of $O(\cdot)$ and $o(\cdot)$ equivalence

Let $(u_n)_n$ and $(v_n)_n$ be arbitrary sequences.

1. $u_n = O(v_n)$ if $\exists c > 0$ such that $|u_n| \leq C|v_n|$.
2. $u_n = o\,(v_n)$. if $\exists\,(\varepsilon_n)$ such that $\varepsilon_n \overset{n\to+\infty}{\longrightarrow} 0$ and $u_n = v_n \cdot \varepsilon_n$. (If $v_n \neq 0$, for large enough n, it means $\frac{u_n}{v_n} \overset{n\to+\infty}{\longrightarrow} 0$)
3. $u_n \overset{n\to+\infty}{\sim} v_n$ if $u_n = v_n + o(v_n) = v_n(1+o(1))$. (if $v_n \neq 0$ for large enough n it means $\frac{u_n}{v_n} \overset{n\to+\infty}{\longrightarrow} 1$)

1.2 Application to Discrete Mathematics

Let $(a_n)_n$ be such $a_n = cn_\alpha \rho^n(1+o(n)) = cn^\alpha\rho^n(1+\frac{\beta}{n}+o(\frac{1}{n}))$ for some $\rho > 0, c, \alpha \in \mathbb{R}$.

How to find ρ numerically?

example. $(F_n)_n$ is Fibonacci Sequences.

$$F_n = \frac{1}{\sqrt{5}} \cdot n^0 \times (\frac{1+\sqrt{5}}{2})^n$$

1st Method:

$$\begin{aligned}\frac{a_{n+1}}{a_n} &= \frac{c(n+1)^\alpha\rho^{n+1}(1+\frac{\beta}{n+1}+\frac{\varepsilon_{n+1}}{n+1})}{cn^\alpha\rho^n(1+\frac{\beta}{n}+\frac{\varepsilon_n}{n})} \quad \text{where } \varepsilon_n \overset{n\to+\infty}{\longrightarrow} 0 \text{ and } \frac{1}{n+1} = 1-n+o(n) \\ &= \rho(1+\frac{1}{n})^\alpha(1+\frac{\beta}{n+1}+\frac{\varepsilon_{n+1}}{n+1})(1-\frac{\beta}{n}+o(\frac{1}{n})) \\ &= \rho(1+\frac{\alpha}{n}+o(\frac{1}{n}))(1+o(\frac{1}{n})) \\ &= \rho(1+\frac{\alpha}{n}+o(\frac{1}{n}))\end{aligned}$$

2nd Method: (too complicated)

$$\begin{aligned}\frac{1}{n}\log a_n &= \frac{\log c}{n} + \frac{\alpha\log n}{n} + \frac{n\log p}{n} + \log\left(1+\frac{\beta}{n}+\frac{\varepsilon_n}{n}\right)\cdot\frac{1}{n} \\ &= \ldots \\ &= \rho(1+\frac{\alpha\log n}{n}+o(\frac{\log n}{n}))\end{aligned}$$

2 Asymptotics & Prime numbers

$p_n = n^{th}$ prime number $(p_1 = 2, p_2 = 3, p_3 = 5, p_4 = 7, ...)$

For $x > 0$, $\Pi(x)$ = number of primes $\leq x$ ($\Pi(8) = 4$ etc.) Therefore we have

$$\begin{cases} \pi(x) \leq x \\ \pi(x) \overset{x \to +\infty}{\longrightarrow} +\infty \end{cases}$$

Lemma: $p_n \leq 2^{2^n}$

Proof: We prove by induction. True for $n = 1$. Then we assume that n is true and we try to prove for $n + 1$ is true.

$$\begin{aligned} p_{n+1} &\leq p_1 ... p_{n+1} \\ &= 2^{2^1} \cdot 2^{2^2} ... 2^{2^n} + 1 \\ &= 2^{2^{n+1}-2} + 1 \\ &= \frac{1}{4} 2^{2^{n+1}} + 1 \\ &\leq 2^{2^{n+1}} \end{aligned}$$

Theorem:

$$\begin{aligned} \forall x \geq 1,\ \Pi(x) &\geq \log_2(\log_2(x)) - 1 \\ \log_2(\log_2(x)) &\leq \Pi(x) + 1 \end{aligned}$$

3 Generating Functions

let $(a_n)_{n\geq 0}$ be a sequence of non-negative reals. The generating function A of $(a_n)_{n\geq 0}$ is the function

$$A : \begin{cases} [0, +\infty) \longrightarrow [0, +\infty) \cup \{+\infty\} \\ \qquad x \longmapsto \sum_{n\geq 0} a_n x^n = a_0 + a_1 x + a_2 x^2 + \dots \end{cases}$$

Example 1. $a_n = (1, 1, 1, ..., 1)$

$$A(x) = \sum_{n\geq 0} 1 \cdot x^n = \begin{cases} \frac{1}{1-x} & \text{if } 0 \leq x \leq 1 \\ +\infty & \text{otherwise} \end{cases}$$

Example 2. $a_n = (1, r, r^2, r^3, r^4, ...)$

$$B(x) = \sum_{n\geq 0} r^n \cdot x^n = \sum_{n\geq 0} r x^n \begin{cases} \frac{1}{1-rx} & \text{if } 0 \leq x \leq \frac{1}{r} \\ +\infty & \text{otherwise} \end{cases}$$

3.1 Basic Properties of Generating Functions

1. $A(0) = a_0$

2. $x \to A(x)$ is non-decreasing and convex.

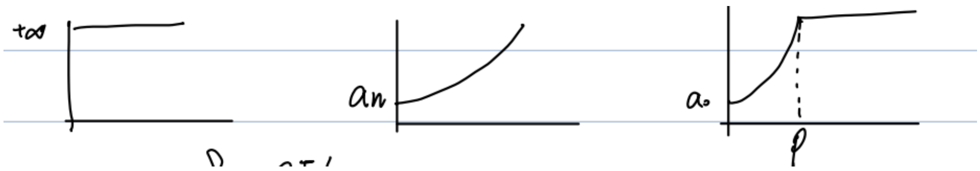

Figure 1: 3 typical situations

3.2 Recurrence & Generating Function

let's study $(g_n)_{n\geq 0}$ defined by $\begin{cases} \mathrm{g}_n = 1 \\ \mathrm{g}_n = 2g_{n-1} + 5 \ \forall n \geq 1 \end{cases}$

goal: find a formula for g_n.

Set $G(x) = \sum g_n x^n$

For $n \geq 1$, $(2g_{n-1} + 5) \cdot x^n$: $g_n x^n = 2g_{n-1}x^n + 5x^n$

$$\begin{aligned} G(x) - g_0 &= \sum_{n\geqslant 1} g_n x^n \\ &= 2\sum_{n\geqslant 1} g_{n-1}x^n + 5\sum_{n\geqslant 1} x^n \\ &= 2x\sum_{n\geqslant 1} g_{n-1}x^{n-1} + 5\sum_{n\geqslant 1} x^n \end{aligned}$$

Then we got

$$G(x) - 1 = 2x\sum_{p\geqslant 0} g_p x^p + 5\left(\frac{1}{1-x} - 1\right)$$

And we can deduce that

$$G(x)(1-2x) = 1 + 5(\frac{1}{1-x} - 1)$$

$$\sum_n x^n = G(x) = \frac{1 + 5\left(\frac{1}{1-x} - 1\right)}{1-2x} = \frac{6}{1-2x} - \frac{5}{1-x} = 6\sum_{n\geqslant 0} 2^n x^n - 5\sum_{n\geqslant 0} x^n$$

We can get

$$g_n = 6 \cdot 2^n - 5 \quad \text{radius of } G: \frac{1}{2}$$

3.3 Generating Function & asymptotics.

Definition: The radius of convergence is defined by $p = sup\{x \geq 0.\ A(x) < +\infty\}$

Broadly speaking. if $A(x)$ has radius ρ, then a_n grows almost like $(\frac{1}{\rho})^n$

Theorem. Assume. (an) is st. $A(x) = \sum a_n x^n$ has finite radius of convergent equal $\forall \varepsilon > 0$. if n is large enough.

$$a_n \leqslant \left(\frac{1}{p} + \varepsilon\right)^n$$