LAB151 - Vector and Fouier Analysis
Ecole Polytechnique, Spring 2022
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Homework 3

Exercise 1

1.1

We find that the function is single-valued and we want to testify whether it's able to apply with the *Dirichlet conditions.*On one period it has three discontinuities which is a finite number (for example on the period$[-\pi, \pi]$, we have discontinuities at $-\pi$, π, and 0. Also the function has a finite number of maximum and minimum values and the function is bounded, which implies that $\int_{-\pi}^{\pi} |f(x)|\, dx$ is finite. Therefore the conditions of applicability of the Fourier series are respected.

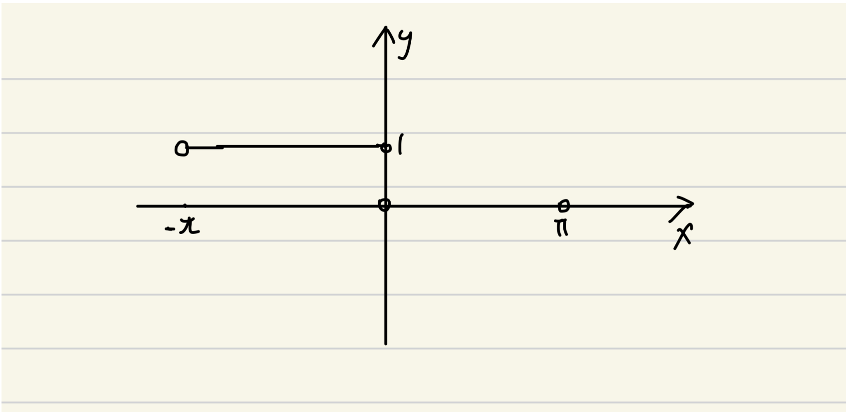

Figure 1: plot of the function $f(x)$ in the period $[-\pi, \pi]$

1.2

We have $f(x) = \sum_{n=-\infty}^{+\infty} C_n e^{inx}$. We first compute C_0, we have $C_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx = \frac{1}{2\pi} \cdot \pi = \frac{1}{2}$. Then we try to compute C_n with the formula $c_n = \frac{1}{2L}\int_{-L}^{L} f(x)e^{-in\pi x/L}dx = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-inx}dx = \frac{1}{2\pi}\int_{-\pi}^{0} e^{-inx}dx = \frac{1}{2\pi}\left[\frac{1}{-in}e^{-inx}\right]_{-\pi}^{0} = \frac{i}{2n\pi}(1 - (\cos(n\pi) + i\sin n\pi))$

Therefore, when $n = 2m+1$, we have $\frac{i}{2n\pi}(1-(\cos(n\pi)+i\sin n\pi)) = \frac{i}{2(2m+1)\pi}(1+1) = \frac{i}{(2m+1)\pi}$.
when $n = 2m$, we have $\frac{i}{2n\pi}(1 - (\cos(n\pi) + i\sin n\pi)) = 0$

In conclusion, $C_0 = \frac{1}{2}$, $C_{2m(m\neq 0)} = 0$ and $C_{2m+1} = \frac{i}{(2m+1)\pi}$

1.3

We have $f(x) = \sum_{n=-\infty}^{+\infty} c_n e^{i\frac{n\pi x}{L}} = \sum_{n=-\infty}^{+\infty} c_n = \frac{1}{2} + \sum_{m=0}^{\infty}\left(\frac{i}{(2m+1)\pi} + \frac{i}{(-2m-1)\pi}\right) = \frac{1}{2}$

1.4

We directly apply the *Parseval's theorem*, we have $\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx = \sum_{-\infty}^{+\infty} C_n^2 = \frac{1}{4} + 2\sum_{m=0}^{+\infty}\frac{1}{((2m+1)\pi)^2} = \frac{1}{2}$, therefore we can solve that$\sum_{m=0}^{+\infty}\frac{1}{(2m+1)^2} = \frac{\pi^2}{8}$

Exercise 2

2.1

Since we have $\vec{F} = \left(z^2\vec{e}_\theta + \vec{e}_r\right)$ and $\vec{\nabla}\times\vec{A} = \vec{e}_r\left(\frac{\partial_\theta A_z}{r} - \partial_z A_\theta\right) + \vec{e}_\theta\left(\partial_z A_r - \partial_r A_z\right) + \vec{e}_z\left(\frac{1}{r}\partial_r\left(rA_\theta\right) - \frac{1}{r}\partial_\theta A_r\right)$ we can just plugin $\vec{A}$ with $\vec{F}$ we have

$$\vec{\nabla}\times\vec{F} = \vec{e}_r\left(0 - \partial_z z^2\right) + \vec{e}_\theta(0) + \vec{e}_z(\frac{z^2}{r} - 0) = -2z\vec{e}_r + \frac{z^2}{r}\vec{e}_z$$

2.2

$I = \iint_\Sigma(\vec{\nabla}\times\vec{F})\cdot\vec{n}d^{(2)}\sigma = \iint_\Sigma(-2z\vec{e}_r + \frac{z^2}{r}\vec{e}_z)\cdot\vec{n}d^{(2)}\sigma$. Since $\vec{n}$ is the outwards-oriented, unit vector normal to the surface, we have $\vec{n} = \vec{e}_r$ and $r = 1$, then we have

$$I = \int_{\theta=0}^{2\pi}\int_{z=0}^{H}\left(-2z\vec{e}_r + \frac{z^2}{r}\vec{e}_z\right)\vec{e}_r d\theta dz = -2\int_{\theta=0}^{2\pi}\int_{-\frac{H}{2}}^{\frac{H}{2}}(z)d\theta dz = -4\pi\int_{-\frac{H}{2}}^{\frac{H}{2}} zdz = 0$$

2.3

$$\iiint_V \operatorname{div}\mathbf{F}d^{(3)}\tau = \iint_A \mathbf{F}\cdot\mathbf{n}d^{(2)}\sigma$$

Here, $d^{(3)}\tau$ represent a small volume, say $d^{(3)}\tau = dxdydz$ in cartesian basis, $\mathbf{n}$ is a normal vector to a surface and $d^{(2)}\sigma$ is the element of area for each surface enclosing the volume V. This A is a closed surface for the column.

2.4

We first apply the divergence theorem in three dimensions that $\iiint_V \operatorname{div}(\vec{\nabla}\times\vec{F})d^{(3)}\tau = \iint_A \mathbf{F}\cdot\mathbf{n}d^{(2)}\sigma = 0$ and since for $\iint_{\text{top disk}}(\vec{\nabla}\times\vec{F})\cdot\vec{n}d^{(2)}\sigma$ we have $\vec{n} = \vec{e_z}$ and for $\iint_{\text{bottom disk}}(\vec{\nabla}\times\vec{F})\cdot\vec{n}d^{(2)}\sigma$ we have $\vec{n} = -\vec{e_z}$. Therefore, we have

$$I + \iint_{\text{bottom disk}}(\vec{\nabla}\times\vec{F})\cdot\vec{n}d^{(2)}\sigma + \iint_{\text{top disk}}(\vec{\nabla}\times\vec{F})\cdot\vec{n}d^{(2)}\sigma = 0$$

2.5

We have $d^{(2)}\sigma = rdrd\theta$
Therefore $\iint_{\text{top disk}}(\vec{\nabla}\times\vec{F})\cdot\vec{n}d^{(2)}\sigma = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}(z^2)d\theta dr = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}(\frac{H^2}{4})d\theta dr = \frac{\pi H^2}{2}$

similarly we can use the same method to compute $\iint_{\text{bottom disk}}(\vec{\nabla}\times\vec{F})\cdot\vec{n}d^{(2)}\sigma$ we have

$\iint_{\text{bottom disk}}(\vec{\nabla}\times\vec{F})\cdot\vec{n}d^{(2)}\sigma = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}(-z^2)d\theta dr = \int_{r=0}^{1}\int_{\theta=0}^{2\pi}(-\frac{H^2}{4})d\theta dr = \frac{-\pi H^2}{2}$

Since we have the result from 2.2 and 2.4 that $I = 0$ and $I + \iint_{\text{bottom disk}}(\vec{\nabla}\times\vec{F})\cdot\vec{n}d^{(2)}\sigma + \iint_{\text{top disk}}(\vec{\nabla}\times\vec{F})\cdot\vec{n}d^{(2)}\sigma = 0$ therefore the result is verified.