Exercises

Exercise 1. Consider a matrix $A \in \mathbb{R}^{N\times N}$ diagonally strictly dominant.

1. Prove that it is invertible.
2. Suppose furthermore that A is symmetric such that all the entries are non-negative $A_{i,j} \geq 0$. Prove that A is positive definite.

Solution.

- If A is singular, then $\exists V \in \mathbb{R}^N \setminus \{0_{\mathbb{R}^N}\}$, such that $AV = 0_{\mathbb{R}^N}$. In such case,

$$-A_{i,i}V_i = \sum_{j\neq i} A_{i,j}V_j.$$

Thus

$$|A_{i,i}||V_i| \leq \sum_{j\neq i} |A_{i,j}||V_j|.$$

Choose the index $i = \arg\max_j(|V_j|)$ leads to

$$|A_{i,i}| \leq \sum_{j\neq i} |A_{i,j}|,$$

then A is not strictly diagonally dominant.

- Suppose $V \neq 0_{\mathbb{R}^N}$, compute

$$\begin{aligned} V^TAV &= \sum_i \left(A_{i,i}V_i^2 + \sum_{j\neq i} A_{i,j}V_iV_j \right) \\ &> \sum_i \sum_{j\neq i} A_{i,j}(V_i^2 + V_iV_j) = \sum_i \sum_{j\neq i} A_{i,j}V_i(V_i+V_j). \end{aligned}$$

By inverting the indices (since A is symmetric), we have

$$\begin{aligned} \sum_i \sum_{j\neq i} A_{i,j}V_i(V_i+V_j) &= \sum_j \sum_{i\neq j} A_{j,i}V_j(V_j+V_i) \\ &= \sum_i \sum_{j\neq i} A_{j,i}V_j(V_j+V_i) \\ &= \sum_i \sum_{j\neq i} A_{i,j}V_j(V_j+V_i). \end{aligned}$$

Then together,

$$V^TAV > \sum_i \sum_{j\neq i} A_{i,j}(V_i+V_j)V_i = \sum_i \sum_{j\neq i} A_{i,j}\frac{(V_i+V_j)^2}{2} \geq 0.$$

Exercise 2. Consider the matrices

$$A = \begin{pmatrix} 1 & 0 & 2 & 0 \\ 1 & 1 & 1 & 1 \\ 3 & 1 & 6 & 0 \\ 0 & 2 & 0 & 1 \end{pmatrix}, \qquad B = \begin{pmatrix} 1 & 2 & 3 & 0 & 1 & 1 & 2 \\ 1 & 1 & 1 & 0 & 2 & 1 & 1 \\ 1 & 3 & 2 & 0 & 1 & 1 & 3 \\ 1 & 4 & 1 & 1 & 2 & 3 & 2 \\ 1 & 1 & 4 & 0 & 1 & 1 & 3 \\ 1 & 2 & 2 & 0 & 2 & 1 & -1 \\ 1 & -1 & 1 & 0 & 1 & 1 & 1 \end{pmatrix}.$$

1. Compute the determinants of A and B.
2. When it is possible, compute its inverse.

Solution.

$$\det(A) = 1, \qquad A^{-1} = \frac{1}{1}\begin{pmatrix} -7 & 2 & 2 & -2 \\ -3 & 0 & 1 & 0 \\ 4 & -1 & -1 & 1 \\ 6 & 0 & -2 & 1 \end{pmatrix}, \quad \det(B) = (-1)^{4+4} 1 \begin{vmatrix} 1 & 2 & 3 & 1 & 1 & 2 \\ 1 & 1 & 1 & 2 & 1 & 1 \\ 1 & 3 & 2 & 1 & 1 & 3 \\ 1 & 1 & 4 & 1 & 1 & 3 \\ 1 & 2 & 2 & 2 & 1 & -1 \\ 1 & -1 & 1 & 1 & 1 & 1 \end{vmatrix} = 0$$

Exercise 3. Prove Cramer's formula: The solution V of a system $AV = b$ with $\det(A) \neq 0$ satisfies

$$V_i = \frac{\det(C^1, \dots, C^{i-1}, b, C^{i+1}, \dots, C^N)}{\det(A)},$$

where C^i is the i-th column of A.

Solution. Remark that $b = AV$, then

$$\begin{aligned} \det(C^1, \dots, C^{i-1}, b, C^{i+1}, \dots, C^N) &= \det(C^1, \dots, C^{i-1}, AV, C^{i+1}, \dots, C^N) \\ &= \sum_{j=1}^{N} V_j \det(C^1, \dots, C^{i-1}, C^j, C^{i+1}, \dots, C^N) \\ &= V_i \det(C^1, \dots, C^{i-1}, C^i, C^{i+1}, \dots, C^N) \\ &= V_i \det(A). \end{aligned}$$

Exercises

Exercise 1. We seek an interpolation polynomials of degree $N-1$ decomposed in a certain basis $(b_1, \dots, b_N)$ as

$$p(x) = \lambda_1 b_1(x) + \lambda_2 b_2(x) + \cdots + \lambda_N b_N(x)$$

and passing at N points $(x_i, y_i)_{i=1,\dots,N}$, *i.e.* such that

$$y_i = p(x_i) \quad \text{for all} \quad i = 1, \dots, N.$$

1. Write this problem under a matrix form $M\lambda = y$ where $\lambda \in \mathbb{R}^N$ are the coefficients of p in the basis $(b_1, \dots, b_N)$. Define M_{can} when $(b_1(x), \dots, b_N(x)) = (1, \dots, x^{N-1})$ are the canonical basis.
2. Using Gaussian elimination, compute the determinant of M_{can}.
3. Give a condition for M_{can} to be invertible.
4. Rewrite this problem in Lagrange basis M_{Lag} (based on the points x_i) instead of the canonical basis. How would you deduce the coefficients of the matrix M_{can}^{-1}?

Solution.

1. $M_{i,j} = b_i(x_j)$
   M_{can} is a Vandermonde matrix
   $$M_{can} = \begin{pmatrix} 1 & x_1 & \dots & x_1^{n-1} \\ 1 & x_2 & \dots & x_2^{n-1} \\ \vdots & & & \vdots \\ 1 & x_n & \dots & x_n^{n-1} \end{pmatrix}.$$
2. One must remember that adding one row (or column) to another in a matrix does not change its determinant (see determinant of a product with a row addition matrix $\det(B) = \det(A^{\lambda,i,j}B)$ in the lecture) and that multiplying one row of a matrix by a coefficient λ provides a matrix with a determinant λ times larger (see determinant of a product with a row multiplication matrix $\lambda \det(B) = \det(M^{\lambda,i}B)$ in the lecture).
   The following is not exactly the Gaussian elimination, but it uses the same technique, i.e. exploiting the elementary matrices to eliminate terms in the matrix.
   Write C_i the i-th column of the matrix.
   - Then adding $-x_1 C_{n-1}$ to C_n provides the transformation
   $$\det(M_{can}) = \det \begin{pmatrix} 1 & x_1 & \dots & x_1^{n-2} & 0 \\ 1 & x_2 & \dots & x_2^{n-2} & x_2^{n-2}(x_2 - x_1) \\ \vdots & & & \vdots & \\ 1 & x_n & \dots & x_n^{n-2} & x_n^{n-2}(x_n - x_1) \end{pmatrix}.$$

- Reproducing it, i.e. adding $-x_1 C_{n-2}$ to C_{n-1}, then $-x_1 C_{n-3}$ to C_{n-2}... provides

$$\det(M_{can}) = \det \begin{pmatrix} 1 & 0 & \dots & 0 & 0 \\ 1 & (x_2 - x_1) & \dots & x_2^{n-3}(x_2 - x_1) & x_2^{n-2}(x_2 - x_1) \\ \vdots & & & & \vdots \\ 1 & (x_n - x_1) & \dots & x_n^{n-3}(x_n - x_1) & x_n^{n-2}(x_n - x_1) \end{pmatrix}.$$

- Then using Laplace expansion w.r.t. the 1st row provides

$$\det(M_{can}) = \det \begin{pmatrix} (x_2 - x_1) & \dots & x_2^{n-3}(x_2 - x_1) & x_2^{n-2}(x_2 - x_1) \\ \vdots & & & \vdots \\ (x_n - x_1) & \dots & x_n^{n-3}(x_n - x_1) & x_n^{n-2}(x_n - x_1) \end{pmatrix}$$

- Multiplying the first row by $(x_2 - x_1)^{-1}$ provides

$$\det(M_{can}) = (x_2 - x_1) \det \begin{pmatrix} 1 & \dots & x_2^{n-3} & x_2^{n-2} \\ (x_3 - x_1) & \dots & x_3^{n-3}(x_3 - x_1) & x_3^{n-2}(x_3 - x_1) \\ \vdots & & & \vdots \\ (x_n - x_1) & \dots & x_n^{n-3}(x_n - x_1) & x_n^{n-2}(x_n - x_1) \end{pmatrix}$$

  Then multiplying each i-th row by $(x_i - x_1)^{-1}$ provides

$$\det(M_{can}) = \prod_{i=2}^{N} (x_i - x_1) \det \begin{pmatrix} 1 & x_2 & \dots & x_2^{n-3} & x_2^{n-2} \\ 1 & x_3 & \dots & x_3^{n-3} & x_3^{n-2} \\ \vdots & & & \vdots & \\ 1 & x_n & \dots & x_n^{n-3} & x_n^{n-2} \end{pmatrix}$$

  where the remaining matrix is again a Vandermonde matrix with $N-1$ positions x_i.

  Reproducing it provides $\det(M_{can}) = \prod_{1 \le i < j \le N} (x_j - x_i)$.

3. M_{can} is invertible if and only if the interpolation points x_i are different.

4. In Lagrange basis, $M_{Lag} = Id$ which considerably simplifies the problem. We have a unique interpolation polynomials

$$P(x) = (M_{can}^{-1} y) b_{can}(x) = (M_{Lag}^{-1} . y) b_{Lag}(x) = y . b_{Lag}(x)$$

   for all y. Especially, by chosing $y = e_i$, we have $(M_{can}^{-1})_{i,:} . b_{can}(x) = b_{Lag,i}(x)$. And one identifies the coefficients $(M_{can}^{-1})_{i,j}$ as the coefficients before x^j in $b_{Lag,i}(x)$.

Exercises

Exercise 1. *LU decomposition with partial pivoting.*

1. Consider the matrix
$$A = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$
Prove that that there exists no $L \in \mathbb{R}^{2\times 2}$ lower triangular and $U \in \mathbb{R}^{2\times 2}$ upper triangular such that $A = LU$.

2. Recall the iteration of Doolitle's algorithm (the one from the lecture) for the construction of the LU decomposition of a matrix A. Under what condition on the computed components of U can you perform the i-th iteration of this algorithm ?

Suppose that this condition is not satisfied at step $i+1$. Then, instead of looking for L and U such that $LU = A$, we want $\mathcal{L}$ and $\mathcal{U}$ to satisfy $\mathcal{L}\mathcal{U} = TA$ where T is a transposition matrix exchanging the $(i+1)$-th row of A with a latter one.

This decomposition leads to the so-called PLU decomposition, where P is a permutation matrix (product of transposition matrices). This also corresponds to *partial pivoting* in the Gaussian elimination algorithm.

3. Suppose that T is the transposition that exchanges the $(i+1)$-th and $(i+2)$-th rows of A. Write A, L and U under the following block form
$$A = \left(\begin{array}{c:c} B & C \\ \hdashline D & E \end{array}\right), \quad L = \left(\begin{array}{c:c} H & 0_{\mathbb{R}^{i\times(N-i)}} \\ \hdashline J & K \end{array}\right), \quad U = \left(\begin{array}{c:c} V & W \\ \hdashline 0_{\mathbb{R}^{(N-i)\times i}} & X \end{array}\right),$$
and write (in caligraphic) the same decomposition for the matrix TA
$$TA = \mathcal{A} = \left(\begin{array}{c:c} \mathcal{B} & \mathcal{C} \\ \hdashline \mathcal{D} & \mathcal{E} \end{array}\right), \quad \mathcal{L} = \left(\begin{array}{c:c} \mathcal{H} & 0_{\mathbb{R}^{i\times(N-i)}} \\ \hdashline \mathcal{J} & \mathcal{K} \end{array}\right), \quad \mathcal{U} = \left(\begin{array}{c:c} \mathcal{V} & \mathcal{W} \\ \hdashline 0_{\mathbb{R}^{(N-i)\times i}} & \mathcal{X} \end{array}\right).$$
Prove that $H = \mathcal{H}$ and $V = \mathcal{V}$.

4. Write $\mathcal{W}$ as a function of W and $\mathcal{J}$ as a function of J.

5. Propose a choice of permutation P such that this PLU decomposition exists for all invertible matrices.

6. Consider the matrix
$$A = \begin{pmatrix} 0 & 2 & 1 \\ -1 & 1 & -1 \\ 2 & 2 & 1 \end{pmatrix}.$$
Write the result P, $\mathcal{L}$ and $\mathcal{U}$ of this algorithm applied to A by choosing at each iteration the transposition T such that the diagonal coefficient $\mathcal{U}_{i,i}$ is maximum in norm.

Remarks:

- For a better stability, one commonly exchanges the i-th row with one below, e.g. j-th row with $j > i$, such that $|A_{j,i}| = \max_{k \geq i} |A_{k,i}|$ has the highest value in norm on this column. This makes the division $\frac{\tilde{A}_{j,i}}{\tilde{A}_{i,i}}$ of smaller amplitude and diminishes the round-off errors produced by this division.

- For an even better stability, one may chose to exchange both the rows and the columns to obtain a diagonal coefficient $A_{i,j}$ such that $|A_{i,j}| = \max_{i',j' \geq i} |A_{i',j'}|$. This corresponds to look for a $PLUP'$ decomposition. The algorithm is then called LU with complete pivoting.

- If these techniques offer a better algorithmic stability, they have an additional cost (more complicated implementation and additional operations) and change the structure of the matrix. Therefore the improvement of the stability needs to be put in balance with the higher complexity of the algorithm.

Solution. 1. Assume it exists, then $LU = \begin{pmatrix} L_{1,1}U_{1,1} & L_{1,1}U_{1,2} \\ L_{1,2}U_{1,1} & L_{1,2}U_{1,2} + L_{2,2}U_{2,2} \end{pmatrix} = A$. Thus, $A_{1,1} = 0$ implies $U_{1,1} = 0$ or $L_{1,1} = 0$ which makes impossible $A_{1,2} \neq 0 \neq A_{2,1}$.

2. See course.

3. By construction T impacts the $n+1$ and $n+2$-th rows. Thus, the n firsts are the same. Especially, $B = \mathcal{B}$. Since the LU decomposition is unique, $H = \mathcal{H}$ and $V = \mathcal{V}$.

4. Similarily, $C = \mathcal{C} = HW = \mathcal{H}\mathcal{W}$. Since $H = \mathcal{H}$ is invertible (otherwise LU is not available), then multiplying the equality $HW = \mathcal{H}\mathcal{W}$ from the left by H^{-1} reads $W = \mathcal{W}$.

   The matrix $\mathcal{D}$ corresponds to the matrix D with its first two rows exchanged. Write $\tilde{T}$ this transposition such that $\tilde{T}D = \mathcal{D}$, then

$$\mathcal{D} = \tilde{T}D = \tilde{T}JV = \tilde{T}J\mathcal{V}.$$

   Again as $V = \mathcal{V}$ is invertible, multiplying from the right, one obtains $\mathcal{J} = \tilde{T}J$.

5. At each iteration n, we chose the permutation exchanging n-th and n'-th rows of A with $n' \geq n$ such that $U_{n,n}$ is maximum in norm (and especially non-zero). If there is no row such that $\mathcal{U}_{n,n} \neq 0$ then at that step all the remaining line are the same and the matrix is not invertible.

6.

$$P^1 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad (P^1A) = \begin{pmatrix} 2 & 2 & 1 \\ -1 & 1 & -1 \\ 0 & 2 & 1 \end{pmatrix},$$

$$U^1 = \begin{pmatrix} 2 & 2 & 1 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad L^1 = \begin{pmatrix} 1 & 0 & 0 \\ -\frac{1}{2} & * & * \\ 0 & * & * \end{pmatrix},$$

However, $U^1_{2,2} = (P^1A)_{2,2} - L^1_{2,1}U^1_{1,2} = 0$, thus this pivot is necessary.

$$P^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}, \quad P^2P^1 = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \quad (P^2P^1A) = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & -1 \end{pmatrix},$$

$$U^1 = \begin{pmatrix} 2 & 2 & 1 \\ 0 & * & * \\ 0 & * & * \end{pmatrix}, \quad P^2L^1 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & * & * \\ -\frac{1}{2} & * & * \end{pmatrix},$$

and we compute

$$U^2 = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & * \end{pmatrix}, \quad L^2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 1 & * \end{pmatrix}.$$

Finally,

$$U^3 = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ 0 & 0 & \frac{-3}{2} \end{pmatrix}, \quad L^3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -\frac{1}{2} & 1 & 1 \end{pmatrix}.$$

And we verify that

$$L^3U^3 = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & 1 \\ -1 & 1 & -1 \end{pmatrix} = P^2P^1A$$

Exercises

Exercise 1. Consider the problem $AV = b$ with

$$A = \begin{pmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 4 & 5 & 6 \end{pmatrix}, \qquad b = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

1. Compute the solution V using Cramer's rule.
2. Compute the solution V using Gaussian elimination.
3. Compute the LU decomposition of A using Doolittle algorithm.
4. Implement those methods in Python and verify your computations.

Solution.

1. $det(A) = 18 + 10 + 4 - 12 - 12 - 5 = 3$

$$\begin{vmatrix} 1 & 1 & 1 \\ 1 & 3 & 1 \\ 1 & 5 & 6 \end{vmatrix} = 18 + 1 + 5 - 3 - 5 - 6 = 10$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 1 & 1 \\ 4 & 1 & 6 \end{vmatrix} = 6 + 2 + 4 - 4 - 12 - 1 = -5$$

$$\begin{vmatrix} 1 & 1 & 1 \\ 2 & 3 & 1 \\ 4 & 5 & 1 \end{vmatrix} = 3 + 10 + 4 - 12 - 2 - 5 = -2$$

Thus $V = \frac{1}{3}(10, -5, -2)^T$

2. 1st step: $A \leftarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & 2 \end{pmatrix} \qquad b \leftarrow \begin{pmatrix} 1 \\ -1 \\ -3 \end{pmatrix}$

2nd step: $A \leftarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix} \qquad b \leftarrow \begin{pmatrix} 1 \\ -1 \\ -2 \end{pmatrix}$

Using e.g. back substitution here provide $V_3 = \frac{-2}{3}$, $V_2 = -1 - \frac{2}{3} = -\frac{5}{3}$, $V_3 = 1 - \frac{-2}{3} - \frac{-5}{3} = \frac{10}{3}$

3. 1st step: $U \leftarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad L \leftarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 0 & 1 \end{pmatrix}$

2nd step: $U \leftarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{pmatrix} \qquad L \leftarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix}$

3rd step: $U \leftarrow \begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{pmatrix} \qquad L \leftarrow \begin{pmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 4 & 1 & 1 \end{pmatrix}$

4. ...

Exercises

Exercise 1.

Consider the matrix

$$A := \begin{pmatrix} 1 & 0 & 1 \\ 2 & 2 & 1 \\ 8 & 0 & 3 \end{pmatrix}$$

1. Find a trivial pair of eigenvalue-eigenvector of the matrix A.
2. Compute the characteristic polynomials of A and deduce the other eigenvalues.
3. Provide an eigenvector associated to each of them.

Solution. 1. The vector $(0,1,0)^T$ is a trivial eigenvector of A associated to the eigenvalue 2.

2.

$$\begin{aligned}\chi_A(x) &= (2-x)\left[(1-x)(3-x)-a\right] \\ &= (2-x)(x^2-4x+3-a) \\ &= (2-x)(2-\sqrt{4+a-3}-x)(2+\sqrt{4+a-3}-x) \\ &= (2-x)(2-\sqrt{1+a}-x)(2+\sqrt{1+a}-x)\end{aligned}$$

then -1 and 5 are the other eigenvalues.

3.

$$A + Id = \begin{pmatrix} 2 & 0 & 1 \\ 2 & 3 & 1 \\ 8 & 0 & 4 \end{pmatrix}$$

then $(1,0,-1)^T$ is an eigenvector associated to -1.

$$A - 5Id = \begin{pmatrix} -4 & 0 & 1 \\ 2 & -3 & 1 \\ 8 & 0 & -2 \end{pmatrix}$$

then $V = (x_1, 1, x_2)^T$ is an eigenvector if $-4x_1 + x_2 = 0$ (1) and $2x_1 - 3 \times 1 + x_2 = 0$ (2). Computing (1) - (2) provides $x_1 = 1/2$, and reinjecting in (1) provides $x_2 = 2$. Therefore $(\frac{1}{2}, 1, 2)^T$ is an eigenvector of A associated with 5. One verifies

$$AV = (5/2, 5, 10)^T = 5V.$$

Exercise 2.

Consider the matrices

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 2 & 2 & 1 \\ 2 & \alpha & 2 \end{pmatrix}$$

1. Compute the characteristic polynomials of A.

2. Deduce the eigenvalues of A depending on α.
   Hints: there are not 3 eigenvalues for all values of α.

3. In the case $\alpha = 1$, compute the algebraic and arithmetic multiplicities of each eigenvalues. Is A diagonalizable? Trigonalizable?

4. In the case $\alpha = 0$, compute the algebraic and arithmetic multiplicities of each eigenvalues. Is A diagonalizable? Trigonalizable?

Solution. 1.

$$\begin{aligned}\chi_A(x) &= (1-x)\left[(2-x)(2-x)-\alpha\right] \\ &= (1-x)(x^2-4x+4-\alpha) \\ &= (1-x)(2-\sqrt{4-(4-\alpha)}-x)(2+\sqrt{4-(4-\alpha)}-x) \\ &= (1-x)(2-\sqrt{\alpha}-x)(2+\sqrt{\alpha}-x)\end{aligned}$$

2. 1 is always an eigenvalue and $2 \pm \sqrt{\alpha}$ when $\alpha \geq 0$

3. With $\alpha = 1$, then 1 is an eigenvalue with arithmetic multiplicity 2 and 3 is an eigenvalue with arithmetic multiplicity 1. Let us seek 2 eigenvectors associated to 1 and 1 associated to 3

$$A - Id = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 2 & 1 & 1 \end{pmatrix}, \qquad A - 3Id = \begin{pmatrix} -2 & 0 & 0 \\ 2 & -1 & 1 \\ 2 & 1 & -1 \end{pmatrix},$$

then $(2,-1,0)^T$ and $(0,1,-1)^T$ are two independent eigenvectors associated to 1. Then the algebraic multiplicity of 1 is 2.
$(0,1,1)^T$ is an eigenvector associated to 3, then its algebraic multiplicity is 1. We have $\sum_i \mu_i = 3 = dim(A)$ then A is diagonalizable.

4. With $\alpha = 0$, then 1 is an eigenvalue of arithmetic multiplicity 1 and 2 is an eigenvalue of arithmetic multiplicity 2. Let us seek 1 eigenvector associated to 1 and 2 associated to 2

$$A - Id = \begin{pmatrix} 0 & 0 & 0 \\ 2 & 1 & 1 \\ 2 & 0 & 1 \end{pmatrix}, \qquad A - 2Id = \begin{pmatrix} -1 & 0 & 0 \\ 2 & 0 & 1 \\ 2 & 0 & 0 \end{pmatrix},$$

then $(1,0,-2)^T$ is an eigenvector associated to 1.
$(0,1,0)^T$ is an eigenvector associated to 2 and there is no other (linearly independent). Then its algebraic multiplicity is 1. Then $\sum_i \mu_i = 2 < 3 = dim(A)$, and this matrix is not diagonalizable but only trigonalizable.

Exercise 3. *Jordan form.* Consider the triangular matrix

$$A = \begin{pmatrix} 1 & 2 & -4 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{pmatrix}.$$

1. What are its eigenvalues and their algebraic multiplicities ?
2. Find one eigenvector V^i of norm 1 associated to every eigenvalue λ_i.
3. Show that one of the eigenvalue λ_1 has a double algebraic multiplicity $m_1 = 1$, but a single geometric multiplicity $\mu_1 = 1$.
4. Show that there exists a non-zero vector V such that $AV = \lambda_1 V + V^1$.
5. Deduce that the matrix A can be put under Jordan form, i.e.

$$A = PDiag(J_1, J_2)P^{-1} = P \begin{pmatrix} \lambda_1 & 1 & 0 \\ 0 & \lambda_1 & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix} P^{-1}, \qquad J_1 = \begin{pmatrix} \lambda_1 & 1 \\ 0 & \lambda_1 \end{pmatrix}, \qquad J_2 = \lambda_2.$$

Solution.

1. $\lambda_1 = 1$ $\lambda_2 = 2$,
   $m_1 = 2$, $m_2 = 1$

2. $$Id - A = \begin{pmatrix} 0 & -2 & 4 \\ 0 & 0 & -2 \\ 0 & 0 & -1 \end{pmatrix} \Rightarrow V^1 = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
   $$2Id - A = \begin{pmatrix} 1 & -2 & 4 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{pmatrix} \Rightarrow V^1 = \begin{pmatrix} 0 \\ \frac{2}{\sqrt{5}} \\ \frac{1}{\sqrt{5}} \end{pmatrix}$$

3. $Ker(Id - A) = Span(V^1)$ and $\mu_1 = 1 \neq m_1$

4. $$AV = V + \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \Rightarrow \begin{pmatrix} 0 & 2 & -4 \\ 0 & 0 & 2 \\ 0 & 0 & 1 \end{pmatrix} V = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}$$
   One can for instance use $V = (\frac{\sqrt{3}}{2}, \frac{1}{2}, 0)^T$

5. Defining the matrix $P = (V^1, V, V^2)$ provides $AP = (\lambda_1 V^1, \lambda_1 V + V^1, V^2) = PDiag(J_1, J_2)$

Exercises

Exercise 1. Consider the matrix of the Laplacian $L \in \mathbb{R}^{N\times N}$ defined by $L_{i,i} = 2$, $L_{i,i\pm1} = -1$ and $L_{i,j} = 0$ for all $|i-j| > 2$.

1. Write the iteration matrix M^J of Jacobi algorithms.
2. Show that its spectral radius $\rho(M^J) \leq 1$.
3. Show that L is symmetric positive definite.
4. Deduce that $\rho(M^J) \neq 1$ and that Jacobi method converges.

Solution.

1. $(M^J)_{i,j} = -\frac{1}{2}\delta_{i,j\pm1}$
2. The Gershgorin discs are the unit closed ball.
3. One computes

$$\begin{aligned} V^T LV &= (2V_1 - V_2)V_1 + \sum_{i=2}^{N-1} \left(-V_{i-1} + 2V_i - V_{i+1}\right) V_i + (2V_N - V_{N-1})V_N \\ &= V_1^2 + (V_1 - V_2)V_1 + \sum_{i=2}^{N-1} \left[(V_i - V_{i-1})V_i - (V_{i+1} - V_i)V_i\right] + (V_N - V_{N-1})V_N + V_N^2 \\ &= V_1^2 + \sum_{i=2}^{N}(V_i - V_{i-1})V_i - \sum_{i=1}^{N-1}(V_{i+1} - V_i)V_i + V_N^2 \\ &= V_1^2 + \sum_{i=2}^{N}(V_i - V_{i-1})(V_i - V_{i-1}) + V_N^2 > 0 \qquad \forall V \neq 0. \end{aligned}$$

   Therefore L is symmetric positive definite.

4. One verifies that $L = 2(Id - M^J)$. Since $0 \notin Sp(L)$, then $0 \notin Sp(Id - M^J)$ and $1 \notin Sp(M^J)$. One verifies similarily that $-1 \notin Sp(M^J)$. Then the eigenvalues of M^J satisfy

$$Sp(M^J) \subset B(0,1) \backslash \{-1, 1\}$$

   and $\rho(M^J) < 1$.