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Week 3 Report

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Title: Numerical Analysis base on different N for error function of the first system

1 Analytic Expression for Error Function

From our previous result, we try to analyze the error function $f(\alpha) = (1+\alpha^2 x_0^2)^{\frac{N+1}{2}} \frac{1}{\alpha^N}$ by computing the derivative. However, this method is not efficient and the analytic expression of the equation does not correspond to the numerical solution. However, for function:

$$f(\alpha) = \frac{(1+\alpha^2 x_0^2)^{\frac{N+1}{2}}}{\alpha^N}$$

We first take the logarithm of the error equation, we have:

$$\log f(\alpha) = \frac{N+1}{2} \log(1+\alpha^2 x_0^2) - N \log(\alpha)$$

We take the derivative of the logarithm of the error equation, we have:

$$\begin{aligned} \frac{d \log f(\alpha)}{d\alpha} &= \frac{N+1}{2} \frac{2\alpha x_0^2}{1+\alpha^2 x_0^2} - \frac{N}{\alpha} \\ &= \frac{(N+1)x_0^2 \alpha}{1+\alpha^2 x_0^2} - \frac{N}{\alpha} \end{aligned}$$

Since log is increasing function, therefore, in order to find the local minimum of the error equation, we need to find the point where the derivative of the logarithm of the error equation is zero, i.e.

$$\frac{d \log f(\alpha)}{d\alpha} = \frac{(N+1)x_0^2 \alpha}{1+\alpha^2 x_0^2} - \frac{N}{\alpha} = 0$$

Then we have:

$$\begin{aligned} (N+1)x_0^2\alpha^2 &= N + N\alpha^2 x_0^2 \\ x_0^2 \alpha^2 &= N \\ \alpha &= \sqrt{\frac{N}{x_0^2}} \end{aligned}$$

Here we find the Analytic expression for the local minimum of the error equation, we have:

$$\alpha = \sqrt{\frac{N}{x_0^2}} = \frac{\sqrt{N}}{x_0}$$

Then, we find the optimal α for quadratization of the first system only depends on the order of Carleman linearization (N) and initial condition x_0. Then we need to check whether the optimal solution satisfies the **boundary condition**. We need to compare the analytic expression with the boundary condition of $\alpha > \frac{2ax_0}{\sqrt{1-4a^2x_0^4}}$ and $ax_0^2 < \frac{1}{2}$. We have:

$$\begin{aligned} \alpha = \sqrt{\frac{N}{x_0^2}} &> \frac{2ax_0}{\sqrt{1-4a^2x^4}} \\ \frac{N}{x_0^2} &> \frac{4a^2x_0^4}{1-4a^2x_0^4} \\ N - 4Na^2x_0^4 &> 4a^2x_0^4 \\ N &> (4N+4)a^2x_0^4 \end{aligned}$$

Therefore, we have the condition that:

$$ax_0^2 < \frac{1}{2}\sqrt{\frac{N}{N+1}} < \frac{1}{2}$$

So we can see that for all the cases, the local minimum exists since the condition is always satisfied.

2 Testing for Analytic Expression with Numerical Method

The numerical testing and analysis file:

1. [numerical_optimi.ipynb]
2. [quadra_reachability.jl]

So, in [numerical_optimi.ipynb], I computed the local minimum of the error function and plot them as follows

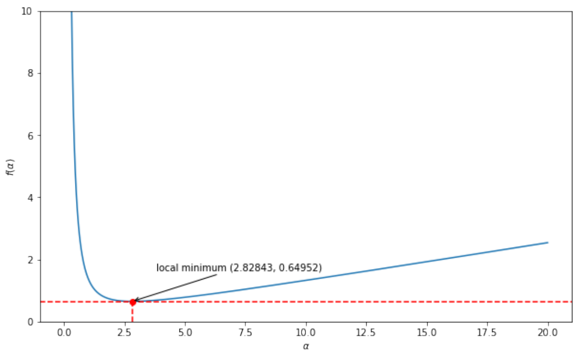

Figure 1: Error Function with $N = 2$, $x_0 = 0.5$

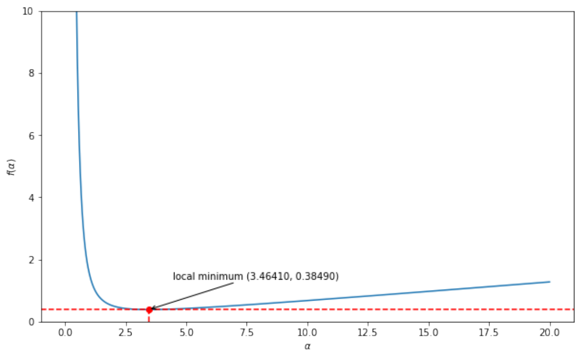

Figure 2: Error Function with $N = 3$, $x_0 = 0.5$

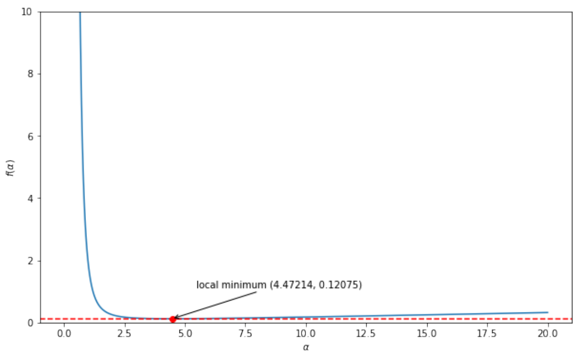

Figure 3: Error Function with $N = 5$, $x_0 = 0.5$

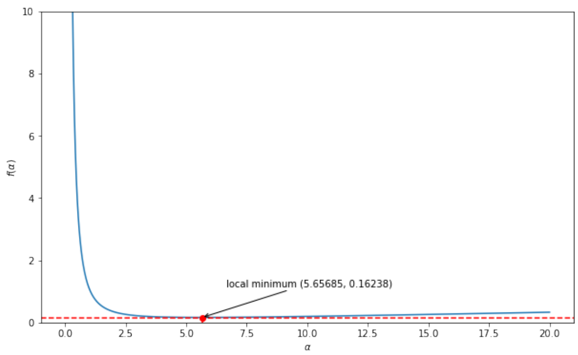

Figure 4: Error Function with $N = 2$, $x_0 = 0.25$

From our plot, we can see that the result works well for the first system. Then we need to verify this result and plot to see whether is true for our example in Julia code. In the file [quadra_reachability.jl] we verify with different N and x_0 seperately. We choose the following cases

1. $N = 2$, $x_0 = 0.5$, with optimal choice $\alpha = \frac{\sqrt{N}}{x_0} = 2\sqrt{2}$
2. $N = 3$, $x_0 = 0.5$, with optimal choice $\alpha = \frac{\sqrt{N}}{x_0} = 2\sqrt{3}$
3. $N = 2$, $x_0 = 0.25$, with optimal choice $\alpha = \frac{\sqrt{N}}{x_0} = 4\sqrt{2}$

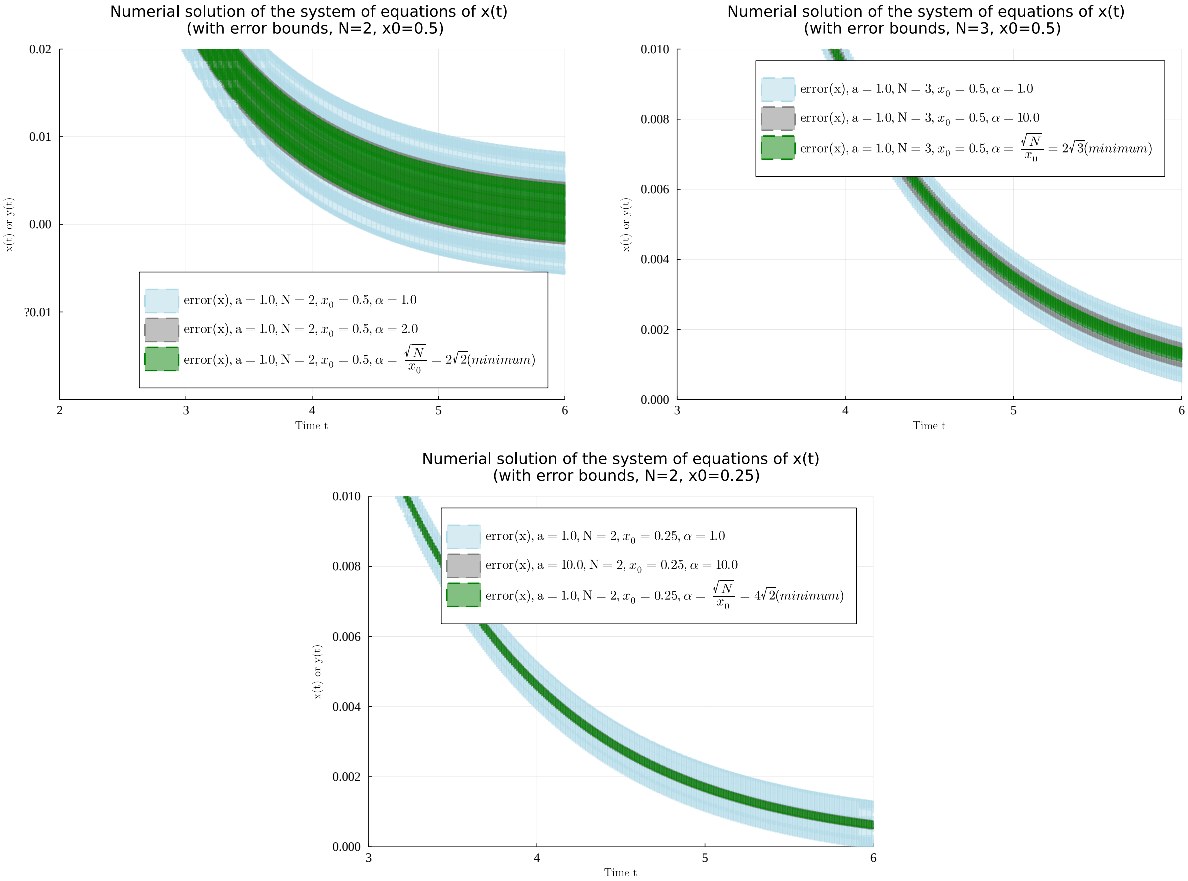

Figure 5: Polts with different N and x_0

We can clearly see from the graph that the green line, which is our optimal choice from the analytic solution, has the smallest error bound. This verifies our conclusion. Then we can conclude for our first model that the optimal quadratization is $y = x^2 + \frac{\sqrt{N}}{x_0}$. Therefore, we have the following system that

$$\begin{cases} x' = -x + \frac{a x_0}{\sqrt{N}} xy \\ y' = -2y + \frac{2 a x_0}{\sqrt{N}} y^2 \end{cases}$$