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Week 5 - 6 Report

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Title: Conclusion with the second system of Example

1 Analysis of the Error Bound

From **Weekly Report 4**, we have already computed the weakly nonlinear condition that

$$\begin{aligned} R &:= \frac{\|X_0\|\|F_2\|}{|\Re(\lambda_1)|} \\ &= \frac{2\sqrt{a^2+1}\sqrt{x_0^4+(1-2a)x_0^2+a^2}}{a} \\ &< 1 \end{aligned}$$

Then for the error-bound function we have

$$\begin{aligned} \varepsilon(t) &:= \frac{\|X_0\|^{N+1}\|F_2\|^N}{a^N}\left(1-e^{-at}\right)^N \\ &= \frac{2^N(x_0^4+(1-2a)x_0^2+a^2)^{\frac{N+1}{2}}(a^2+1)^{\frac{N}{2}}}{a^N}\left(1-e^{-at}\right)^N \end{aligned}$$

Then we try to simplify the error-bound function. First $(1-e^{-at})^N$ increase extremely slow and the maximum of $(1-e^{-at})$ is 1. This part has a relatively small effect on the value of error. Then, we remove all the constant value, we got the error bound

$$\begin{aligned} \varepsilon(t) &:= \frac{(x_0^4+(1-2a)x_0^2+a^2)^{\frac{N+1}{2}}(a^2+1)^{\frac{N}{2}}}{a^N} \\ &= (x_0^4+(1-2a)x_0^2+a^2)^{\frac{N+1}{2}}(1+\frac{1}{a^2})^{\frac{N}{2}} \end{aligned}$$

The function can be transferred into the form of a quartic equation with one unknown a and coefficient containing x_0 and N.

However, **after we test the error bound in Julia**, we find that the weakly nonlinear condition can not be satisfied.

$$\begin{aligned} R &:= \frac{\|X_0\|\|F_2\|}{|\Re(\lambda_1)|} \\ &= \frac{2\sqrt{a^2+1}\sqrt{x_0^4+(1-2a)x_0^2+a^2}}{a} \\ &= 2\sqrt{1+\frac{1}{a^2}}\sqrt{(x_0^2-a)^2+x_0^2} \end{aligned}$$

From our optimization result, we find that $R \geq 2$ no matter what value chose for a and x_0. We can also see this in our plot

Then, we can deduce that after quadratization for the ODE $x' = -x^3$, the boundary condition (R) can not be satisfied therefore we are not able to apply the carleman linearization to the system.

2 Computation for the case with quadratizaiton operator ax

We denote the quadratization method $y = x^2 - ax \Rightarrow ax = x^2 - y$, then we have the following system:

$$x' = -x^3 = -x(ax+y) = -ax^2 + xy = -a(ax+y) + xy = -a^2x - ay + xy$$

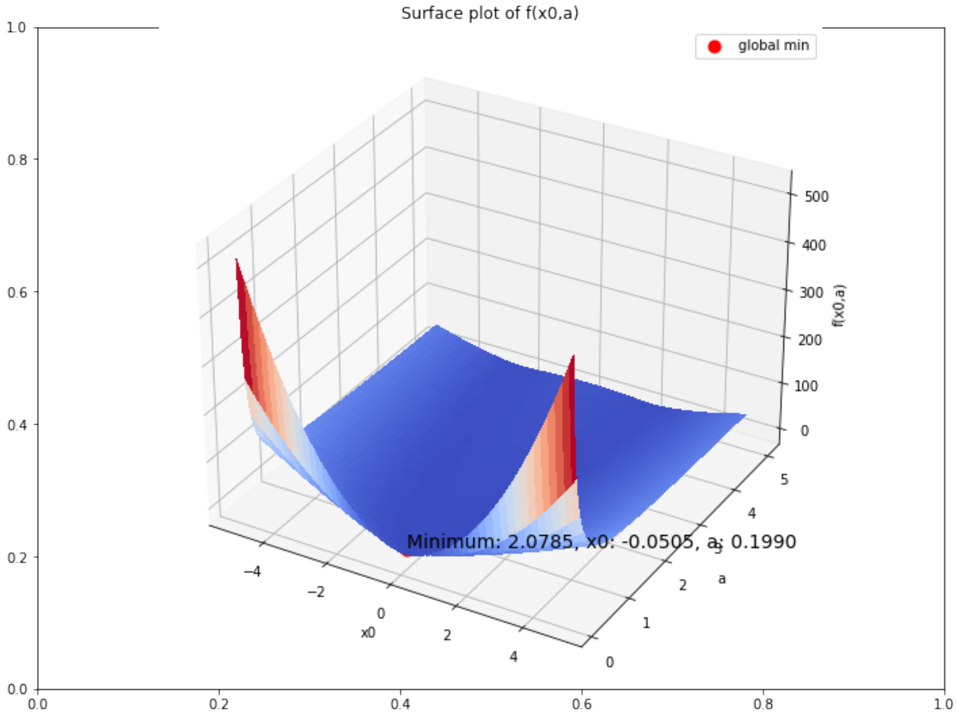

Figure 1: Plot for $R = 2\sqrt{1+\frac{1}{a^2}}\sqrt{(x_0^2-a)^2+x_0^2}$, we can see that the minimum of R is larger than 2

$$\begin{aligned} y' &= 2xx' = -2x^4 \\ &= -2(a^2x^2 + y^2 + 2axy) \quad \text{(Here we still not have linear part)} \\ &= -2(a^2y + a^3x + y^2 + 2axy) \end{aligned}$$

Then we can finalize the system as following:

$$\begin{cases} y' = -2(a^2y + a^3x + y^2 + 2axy) \\ x' = a^2x - ay + xy \end{cases}$$

Then we have the $F_1 \in \mathbb{R}^{n\times n}$ matrix for linear part:

$$F_1 = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x' \\ y' \end{array} & \begin{pmatrix} a^2 & -a \\ a^3 & -2a^2 \end{pmatrix} \end{array} \tag{1}$$

However, we can still see that the first and second rows of the matrix F_1 are colinear, then we can't compute a negative eigenvalue. This is due to equation y', we don't have the linear part, which we need to substitute x^2 into $ax + y$. This transformation is the same as the transformation method in the equation of x' and causes linearity.

3 Analysis of the failure of carleman linearization of the equation $x' = -x^3$

Since after we quadratic a system, the highest degree of the ODE is 2. Also, the operator of the quadratization is $[c, x^1, \cdots, x^n]$ where n is the highest degree of the variable that we introduced for quadratization. In this example, we tried all possibilities with a, ax, x^2 which we can find a method for carleman linearization.