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Week 4 Report

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Title:

1 Building the system for the second example

We have the second system of the following form:

$$x' = -x^3$$

Normally, we have the following quadratization, where $y = x^2$:

$$y' = -2xx' = -2x^4 = -2y^2$$

Then we have the following system:

$$\begin{cases} y' = -2y^2 \\ x' = -xy \end{cases}$$

However, we can find that **there is no linear part** in the system. Therefore, we try to add linear terms like ax or a. After computation, we find that ax can't produce eigenvalues for matrix F_1. Therefore, we try with the term a. We can change $y = x^2 - a \Rightarrow a = x^2 - y$, then we have the following system:

$$x' = -x^3 = -x(y+a) = -xy - ax$$

$$\begin{aligned} y' &= 2xx' = 2x(-x^3) - 2x^4 \\ &= -2(y+a)^2 = -2y^2 - 4ay - 2a^2 \\ &= -2y^2 - 4ay - 2a(x^2 - y) \\ &= -2y^2 - 4ay - 2ax^2 + 2ay \\ &= -2y^2 - 2ax^2 - 2ay \end{aligned}$$

Then we can finalize the system as following:

$$\begin{cases} y' = -2y^2 \ + 2ax^2 + 2ay \\ x' = -xy + ax \end{cases}$$

Then we have the $F_1 \in \mathbb{R}^{n \times n}$ matrix for linear part:

$$F_1 = \begin{array}{c c} & \begin{array}{cc} x & y \end{array} \\ \begin{array}{c} x' \\ y' \end{array} & \begin{pmatrix} -a & 0 \\ 0 & -2a \end{pmatrix} \end{array} \tag{1}$$

And for the nonlinear part we have $F_2 \in \mathbb{R}^{n \times n^2}$ matrix:

$$F_2 = \begin{array}{c c} & \begin{array}{cccc} x^2 & xy & yx & y^2 \end{array} \\ \begin{array}{c} x' \\ y' \end{array} & \begin{pmatrix} 0 & -1 & 0 & 0 \\ -2a & 0 & 0 & -2 \end{pmatrix} \end{array} \tag{2}$$

For matrix F_1 we have the eigenvalues $\lambda_1 = -a$ and $\lambda_2 = -2a$. Then we got $\Re(\lambda_1) = -1$ (the real part of λ_1). We cite the [paper]for reachabiliy analysis, we have following part:

Definition 1. System is said to be weakly nonlinear if the ratio

$$R := \frac{\|X_0\| \|F_2\|}{|\Re(\lambda_1)|}$$

satisfies $R < 1$.

Definition 2. System (1) is said to be dissipative if $\Re(\lambda_1) < 0$ (i.e., the real part of all eigenvalues is negative).

The conditions $\Re(\lambda_1) < 0$ and $R < 1$ ensure arbitrary-time convergence.

Theorem 1 ([30, Corollary 1]) Assuming that (1) is weakly nonlinear and dissipative, the error bound associated with the linearized problem (2) truncated at order N satisfies

$$\|\eta_1(t)\| \leq \varepsilon(t) := \|X_0\| R^N \left(1 - e^{\Re(\lambda_1)t}\right)^N,$$

with R as defined in (5). This error bound holds for all $t \geq 0$.

2 Computation of the SVD decomposition of F_2

From above, we have the nonlinear system matrix that

$$F_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ -2a & 0 & 0 & -2 \end{bmatrix}.$$

with $a > 0$. In the [paper], it mentioned that we apply the $l - \infty$ norm in the computation of $\|F_2\|$. However, when I look into the code of the function _error_bound_specabs_R(x0c, F1, F2; check=true), I have the following algorithms in **Julia**.

```
# See Definition (2.2) in [2]. These bounds use the spectral norm (p = 2)
function _error_bound_specabs_R(x₀, F₁, F₂; check=true)
    nx₀ = norm(x₀, 2)
    nF₂ = opnorm(F₂, 2)

    # compute eigenvalues and sort them by increasing real part
    λ = eigvals(F₁, sortby=real)
    λ₁ = last(λ)
    Re_λ₁ = real(λ₁)
    if check
        @assert Re_λ₁ <= 0 "expected Re(λ₁) ≤ 0, got $Re_λ₁"
    end
    R = nx₀ * nF₂ / abs(Re_λ₁)
    return (R, Re_λ₁)
end
```

Figure 1: Code for function _error_bound_specabs_R(x0c, F1, F2; check=true)

Therefore, we can see that in the computation of $\|F_2\|$, we apply the $l - 2$ norm. If we check the $l - 2$ norm in **Julia**, we have the code as follows [link]:

```
function opnorm(A::AbstractMatrix, p::Real=2)
    F = factorize(Hermitian(A) |> Matrix)
    lambda = maximum(svdvals(F.U))
    return lambda^(1/p)
end
```

Therefore, in order to compute $\|F_2\|_2$, we need to apply **singular value decomposition (SVD)** to F_2. We denote $A = F_2$ here and then compute $A^T A$ and AA^T. We have:

$$A^T A = \begin{bmatrix} 1 & 0 \\ 0 & 4a^2 \end{bmatrix}.$$

$$AA^T = \begin{bmatrix} 4a^2 & 0 & 0 & -4a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -4a & 0 & 0 & 4 \end{bmatrix}.$$

For matrix $A^T A$, it's easy to find the eigenvalue $\lambda_1 = 1$ and $\lambda_2 = 4a^2$ with the eigenvector of $v_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$.

Then we try to compute the eigenvalue and eigenvector of AA^T, we first compute its **characteristic polynomial**, we have

$$\begin{aligned} \mathcal{X}_{AA^T} &= \begin{vmatrix} 4a^2 - x & 0 & 0 & 4a \\ 0 & 1-x & 0 & 0 \\ 0 & 0 & -x & 0 \\ 4a & 0 & 0 & 4-x \end{vmatrix} \\ &= (1-x) \begin{vmatrix} 4a^2 - x & 0 & 4a \\ 0 & -x & 0 \\ 4a & 0 & 4-x \end{vmatrix} \\ &= x^2(x-1)(x-(4+4a^2)) \end{aligned}$$

Therefore, we have the eigenvalues that $\begin{bmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{bmatrix} = \begin{bmatrix} 4a^2+4 \\ 0 \\ 1 \\ 0 \end{bmatrix}$. Then we got the eigenvectors that $\begin{bmatrix} u_1, u_2, u_3, u_4 \end{bmatrix} = \begin{bmatrix} \sqrt{\frac{a^2}{a^2+1}} & \frac{1}{a}\sqrt{\frac{a^2}{a^2+1}} & 0 & 4a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{a}\sqrt{\frac{a^2}{a^2+1}} & \sqrt{\frac{a^2}{a^2+1}} & 0 & 4 \end{bmatrix}$. Then we can compute the singular value, we have $\sigma_i = \sqrt{\lambda_i}$ formula to compute the singular value that $\sigma_1 = \sqrt{4+4a^2}$ and $\sigma_2 = 1$. Finally, we have the SVD of F_2 that:

$$F_2 = \begin{bmatrix} \sqrt{\frac{a^2}{a^2+1}} & \frac{1}{a}\sqrt{\frac{a^2}{a^2+1}} & 0 & 4a \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -\frac{1}{a}\sqrt{\frac{a^2}{a^2+1}} & \sqrt{\frac{a^2}{a^2+1}} & 0 & 4 \end{bmatrix} \begin{bmatrix} \sqrt{4+4a^2} & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

Therefore, since $l-2$ norm is the maximum of the absolute value of the singular value, we have $\|F_2\|_2 = \sqrt{4+4a^2}$.

3 Computation of the error bound function

We have the weakly nonlinear condition that

$$\begin{aligned} R &:= \frac{\|X_0\|\|F_2\|}{|\Re(\lambda_1)|} \\ &= \frac{2\sqrt{a^2+1}\sqrt{x_0^4 + (1-2a)x_0^2 + a^2}}{a} \\ &< 1 \end{aligned}$$

Then for the error bound function we have

$$\varepsilon(t) := \frac{\|X_0\|^{N+1}\|F_2\|^N}{a^N} \left(1 - e^{-at}\right)^N$$