Week 1 Report

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Title: Optimal Quadratization (α value) for Reachability Analysis of the first Example System

1 Introduction to the Problem

From previous week, we study the ODE $x' = -x + ax^3$ with the following quadratization:

$$\begin{cases} x' = -x + axy \\ y' = -2y + 2ay^2 \end{cases}$$

where we introduced a new variable $w_0 = y = x^2$ to quadratize the system. Then, we studied the reachability analysis of the system using the Carleman linearization.

Now we want to take a step further and study the optimal quadratization for the system which also produce the least error bound in the reachability analysis. For optimal quadratization, we have several options: $w_0 = y = x^2$ or $w_0 = y = \alpha x^2$ which are equivalent. If we take $w_0 = y = \alpha x^2$, then we have the following computation:

$$\begin{aligned} y' &= 2\alpha x x' \\ &= 2\alpha x(-x + ax^3) \\ &= -2\alpha x^2 + 2\alpha a x^4 \end{aligned}$$

Since we have $y = \alpha x^2 \Rightarrow x^2 = \frac{1}{\alpha}y$, therefore we have:

$$-2\alpha x^2 + 2\alpha a x^4 = -2y + \frac{2a}{\alpha}y^2$$

Then we have the following system:

$$\begin{cases} x' = -x + \frac{a}{\alpha}xy \\ y' = -2y + \frac{2a}{\alpha}y^2 \end{cases}$$

If we denote the new parameter $k = \frac{a}{\alpha}$, then the system goes back to the original format. Therefore, we can only study the reachability analysis with b.

$$\begin{cases} x' = -x+ = kxy \\ y' = -2y + 2ky^2 \end{cases}$$

2 Computing the Error Bound and Reachability

From previous analysis, we have the $F_1 \in \mathbb{R}^{n \times n}$ matrix for linear part:

$$F_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

And for the nonlinear part we have $F_2 \in \mathbb{R}^{n \times n^2}$ matrix:

$$F_2 = \begin{bmatrix} 0 & 2k = \frac{2a}{\alpha} & 0 & 0 \\ 0 & 0 & 0 & k = \frac{a}{\alpha} \end{bmatrix}.$$

For matrix F_1 we have the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Then we got $\Re(\lambda_1) = -1$ (the real part of λ_1). We cite the [paper] for reachabiliy analysis, we have following part:

Definition 1. System is said to be weakly nonlinear if the ratio

$$R := \frac{\|X_0\| \|F_2\|}{|\Re(\lambda_1)|}$$

satisfies $R < 1$.

Definition 2. System (1) is said to be dissipative if $\Re(\lambda_1) < 0$ (i.e., the real part of all eigenvalues is negative).

The conditions $\Re(\lambda_1) < 0$ and $R < 1$ ensure arbitrary-time convergence.

Theorem 1 ([30, Corollary 1]) Assuming that (1) is weakly nonlinear and dissipative, the error bound associated with the linearized problem (2) truncated at order N satisfies

$$\|\eta_1(t)\| \le \varepsilon(t) := \|X_0\| R^N \left(1 - e^{\Re(\lambda_1)t}\right)^N,$$

with R as defined in (5). This error bound holds for all $t \ge 0$.

From the result of [norm test], we know for $\|X_0\|$ is $l_2 - norm$ and for $\|F_2\|$ is $l_\infty - norm$. Then we compute the error bound for our example system:

$$\|F_2\| = \max_{1 \le i,j \le n} |a_{ij}| = 2b$$

We denote $X_0 = [x_0, y_0] = [x_0, \alpha x_0^2]$ where x_0 is the initial value of x and $y_0 = \alpha x_0^2$. Then we have:

$$\|X_0\| = \|[x_0, \alpha x_0^2]\| = \sqrt{x_0^2 + \alpha^2 x_0^4} = x_0\sqrt{1 + \alpha^2 x_0^2}$$

Then we can compute the value of R:

$$R := \frac{\|X_0\| \|F_2\|}{|\Re(\lambda_1)|}$$

$$= x_0\sqrt{1 + \alpha^2 x_0^2} \cdot 2k$$

$$= x_0\sqrt{1 + \alpha^2 x_0^2} \cdot 2\frac{a}{\alpha}$$

$$< 1$$

In order to simplify the computation, we assume the value $x_0, \alpha, a > 0$, then we have:

$$x_0\sqrt{1 + \alpha^2 x_0^2} \cdot 2\frac{a}{\alpha} < 1$$

$$1 + \alpha^2 x_0^2 < \frac{\alpha^2}{4a^2 x_0^2}$$

$$4a^2 x_0^2 < \alpha^2(1 - 4a^2 x_0^4)$$

We have the inequality:

$$\alpha > \frac{2ax_0}{\sqrt{1 - 4a^2 x_0^4}}$$

Then we have the boundary condition that $\alpha > \frac{2ax_0}{\sqrt{1-4a^2 x_0^4}}$ and $ax_0^2 < \frac{1}{2}$, we can compute the error bound:

$$\|\eta_1(t)\| \le \varepsilon(t) := \|X_0\| R^N \left(1 - e^{\Re(\lambda_1)t}\right)^N$$

$$= \|X_0\|^{N+1}(2\frac{a}{\alpha})^N \left(1 - e^{-t}\right)^N$$

$$= x_0^{N+1}(1 + \alpha^2 x_0^2)^{\frac{N+1}{2}} \frac{(2a)^N}{\alpha^N} \left(1 - e^{-t}\right)^N$$

Since α is our parameter, then we only need to analyze the function $(1 + \alpha^2 x_0^2)^{\frac{N+1}{2}} \frac{1}{\alpha^N}$

3 Current Problem

So currently, I am stuck on how to analyze the function $(1 + \alpha^2 x_0^2)^{\frac{N+1}{2}} \frac{1}{\alpha^N}$. The behavior of the function depends on N and x_0, for example:
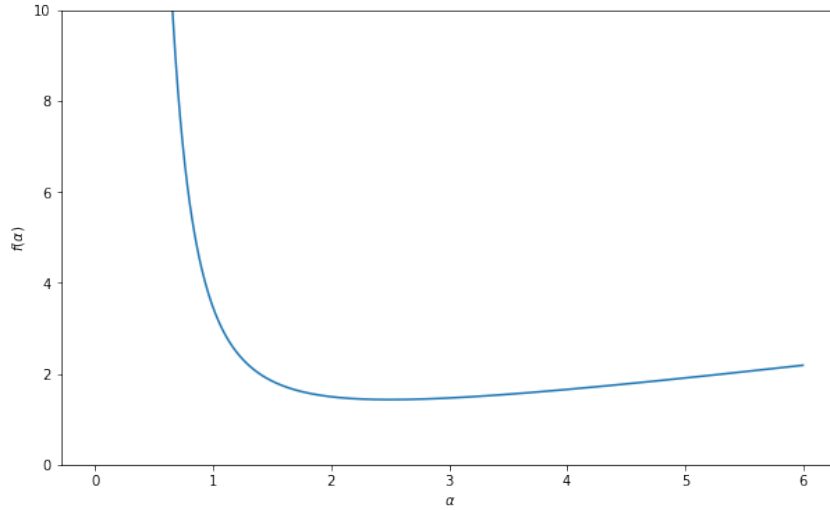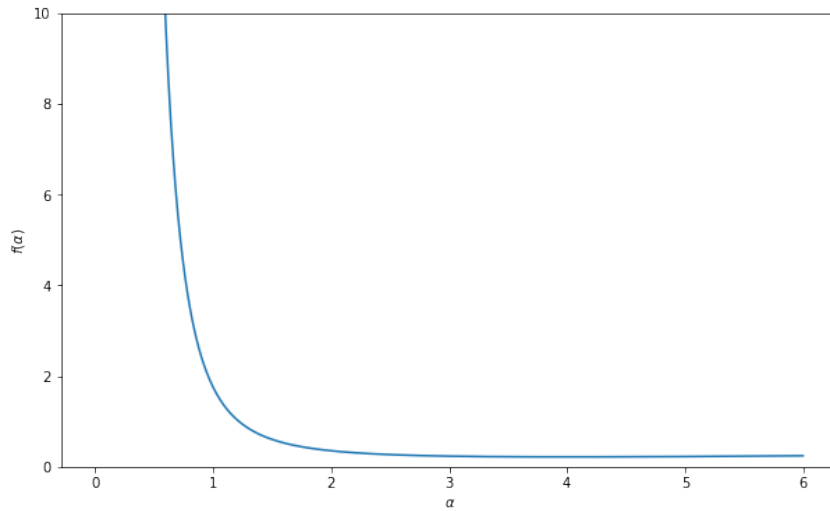

Figure 1: With $N = 4$ and $x_0 = 0.8$

Figure 2: With $N = 4$ and $x_0 = 0.5$

So, in the above graph, the first one has a local minimum but not the second one. Also, I have to consider the boundary condition that $\alpha > \frac{2ax_0}{\sqrt{1-4a^2x_0^4}}$ and $ax_0^2 < \frac{1}{2}$. I don't know how to use a great mathematical way to analyze this problem. The graph is from this [optimi test].

I try to take the derivative of $f(\alpha) = (1 + \alpha^2 x_0^2)^{\frac{N+1}{2}} \frac{1}{\alpha^N}$, then I got

$$f'(\alpha) = \frac{(n+1)x_0^2(1 + \alpha^2 x_0^2)^{\frac{N-1}{2}} - N\alpha^{N-1}(1 + \alpha^2 x_0^2)^{\frac{N+1}{2}}}{\alpha^{2N}}$$

Make $f'(\alpha) = 0$, then we have

$$Nx_0^2\alpha^{N+1} + N\alpha^{N-1} - (N+1)x_0^2 = 0$$

So I am wondering, is there any method to fix it, or should I fix N and x_0?