Ecole Polytechnique
Personal Webpage: link
Mobile: +33 7 67 19 79 16

Week 2 Report

Email : yubo.cai@polytechnique.edu Authur: Yubo Cai Supervisor: Gleb Pogudin

Title: Optimal Quadratization (α value) for Reachability Analysis of the first Example System

1 Introduction to the Problem

From previous week, we study the ODE $x' = -x + ax^3$ with the following quadratization:

$$\begin{cases} x' = -x + axy \\ y' = -2y + 2ay^2 \end{cases}$$

where we introduced a new variable $w_0 = y = x^2$ to quadratize the system. Then, we studied the reachability analysis of the system using the Carleman linearization.

Now we want to take a step further and study the optimal quadratization for the system which also produce the least error bound in the reachability analysis. For optimal quadratization, we have several options: $w_0 = y = x^2$ or $w_0 = y = \alpha x^2$ which are equivalent. If we take $w_0 = y = \alpha x^2$, then we have the following computation:

$$\begin{aligned} y' &= 2\alpha x x' \\ &= 2\alpha x(-x + ax^3) \\ &= -2\alpha x^2 + 2\alpha a x^4 \end{aligned}$$

Since we have $y = \alpha x^2 \Rightarrow x^2 = \frac{1}{\alpha} y$, therefore we have:

$$-2\alpha x^2 + 2\alpha a x^4 = -2y + \frac{2a}{\alpha} y^2$$

Then we have the following system:

$$\begin{cases} x' = -x + \frac{a}{\alpha} xy \\ y' = -2y + \frac{2a}{\alpha} y^2 \end{cases}$$

If we denote the new parameter $k = \frac{a}{\alpha}$, then the system goes back to the original format. Therefore, we can only study the reachability analysis with b.

$$\begin{cases} x' = -x + = kxy \\ y' = -2y + 2ky^2 \end{cases}$$

2 Computing the Error Bound and Reachability

From previous analysis, we have the $F_1 \in \mathbb{R}^{n\times n}$ matrix for linear part:

$$F_1 = \begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}.$$

And for the nonlinear part we have $F_2 \in \mathbb{R}^{n\times n^2}$ matrix:

$$F_2 = \begin{bmatrix} 0 & 2k = \frac{2a}{\alpha} & 0 & 0 \\ 0 & 0 & 0 & k = \frac{a}{\alpha} \end{bmatrix}.$$

For matrix F_1 we have the eigenvalues $\lambda_1 = -1$ and $\lambda_2 = -2$. Then we got $\Re(\lambda_1) = -1$ (the real part of λ_1). We cite the [paper]for reachabiliy analysis, we have following part:

Definition 1. System is said to be weakly nonlinear if the ratio

$$R := \frac{\|X_0\|\|F_2\|}{|\Re(\lambda_1)|}$$

satisfies $R < 1$.

Definition 2. System (1) is said to be dissipative if $\Re(\lambda_1) < 0$ (i.e., the real part of all eigenvalues is negative).

The conditions $\Re(\lambda_1) < 0$ and $R < 1$ ensure arbitrary-time convergence.

Theorem 1 ([30, Corollary 1]) Assuming that (1) is weakly nonlinear and dissipative, the error bound associated with the linearized problem (2) truncated at order N satisfies

$$\|\eta_1(t)\| \leq \varepsilon(t) := \|X_0\| R^N \left(1 - e^{\Re(\lambda_1)t}\right)^N,$$

with R as defined in (5). This error bound holds for all $t \geq 0$.

From the result of [norm test], we know for $\|X_0\|$ is $l_2 - norm$ and for $\|F_2\|$ is $l_\infty - norm$. Then we compute the error bound for our example system:

$$\|F_2\| = \max_{1\leq i,j\leq n} |a_{ij}| = 2b$$

We denote $X_0 = [x_0, y_0] = [x_0, \alpha x_0^2]$ where x_0 is the initial value of x and $y_0 = \alpha x_0^2$. Then we have:

$$\|X_0\| = \|[x_0, \alpha x_0^2]\| = \sqrt{x_0^2 + \alpha^2 x_0^4} = x_0\sqrt{1 + \alpha^2 x_0^2}$$

Then we can compute the value of R:

$$\begin{aligned} R &:= \frac{\|X_0\|\|F_2\|}{|\Re(\lambda_1)|} \\ &= x_0\sqrt{1 + \alpha^2 x_0^2} \cdot 2k \\ &= x_0\sqrt{1 + \alpha^2 x_0^2} \cdot 2\frac{a}{\alpha} \\ &< 1 \end{aligned}$$

In order to simplify the computation, we assume the value $x_0, \alpha, a > 0$, then we have:

$$\begin{aligned} x_0\sqrt{1 + \alpha^2 x_0^2} \cdot 2\frac{a}{\alpha} &< 1 \\ 1 + \alpha^2 x_0^2 &< \frac{\alpha^2}{4a^2 x_0^2} \\ 4a^2 x_0^2 &< \alpha^2(1 - 4a^2 x_0^4) \end{aligned}$$

We have the inequality:

$$\alpha > \frac{2ax_0}{\sqrt{1 - 4a^2 x_0^4}}$$

Then we have the boundary condition that $\alpha > \frac{2ax_0}{\sqrt{1-4a^2x_0^4}}$ and $ax_0^2 < \frac{1}{2}$, we can compute the error bound:

$$\begin{aligned} \|\eta_1(t)\| \leq \varepsilon(t) &:= \|X_0\| R^N \left(1 - e^{\Re(\lambda_1)t}\right)^N \\ &= \|X_0\|^{N+1} (2\frac{a}{\alpha})^N \left(1 - e^{-t}\right)^N \\ &= x_0^{N+1} (1 + \alpha^2 x_0^2)^{\frac{N+1}{2}} \frac{(2a)^N}{\alpha^N} \left(1 - e^{-t}\right)^N \end{aligned}$$

Since α is our parameter, then we only need to analyze the function $(1 + \alpha^2 x_0^2)^{\frac{N+1}{2}} \frac{1}{\alpha^N}$

3 Current Problem

So currently, I am stuck on how to analyze the function $(1+\alpha^2 x_0^2)^{\frac{N+1}{2}}\frac{1}{\alpha^N}$. The behavior of the function depends on N and x_0, for example:

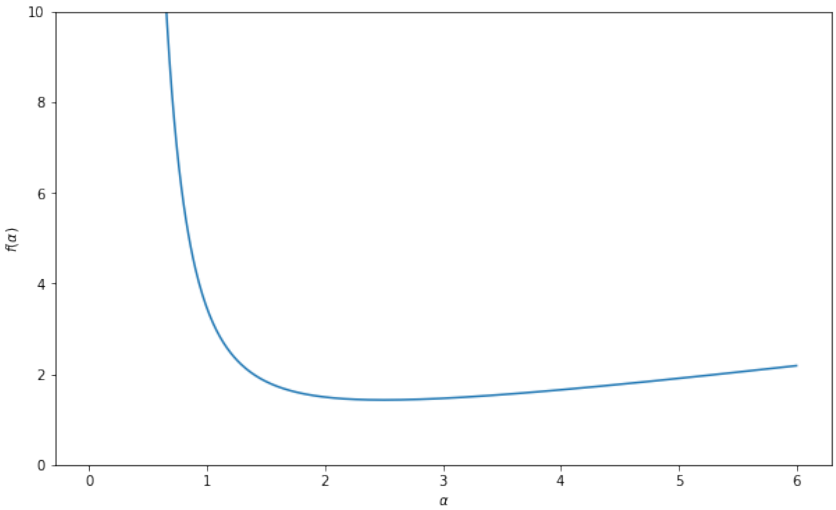

Figure 1: With $N=4$ and $x_0=0.8$

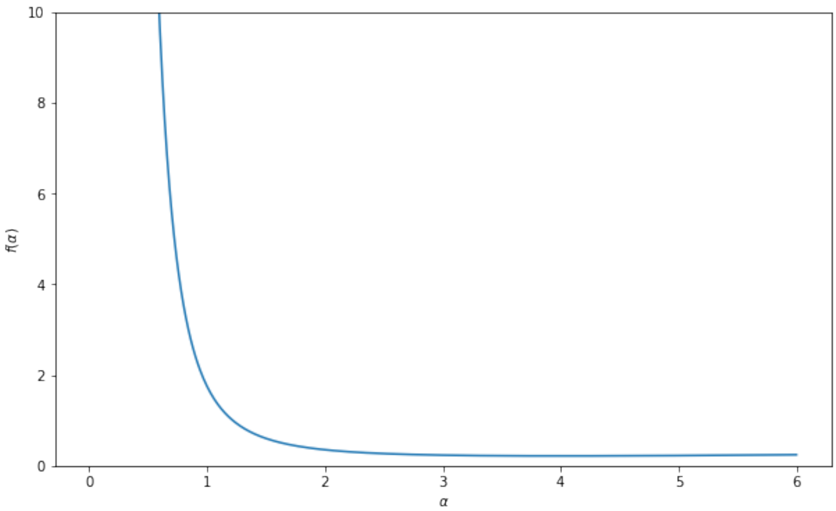

Figure 2: With $N=4$ and $x_0=0.5$

So, in the above graph, the first one has a local minimum but not the second one. Also, I have to consider the boundary condition that $\alpha > \frac{2ax_0}{\sqrt{1-4a^2x_0^4}}$ and $ax_0^2 < \frac{1}{2}$. I don't know how to use a great mathematical way to analyze this problem. The graph is from this [optimi test].

I try to take the derivative of $f(\alpha) = (1+\alpha^2 x_0^2)^{\frac{N+1}{2}}\frac{1}{\alpha^N}$, then I got

$$f'(\alpha) = \frac{(N+1)x_0^2\alpha(1+\alpha^2x_0^2)^{\frac{N-1}{2}} - N\alpha^{N-1}(1+\alpha^2x_0^2)^{\frac{N+1}{2}}}{\alpha^{2N}}$$

Make $f'(\alpha)=0$, then we have

$$(N+1)X_0^2\alpha = N\alpha^{N-1}(1+\alpha^2x_0^2)$$

Then we have

$$N x_0^2 \alpha^{N+1} + N\alpha^{N-1} - (N+1) X_0^2 \alpha = 0$$
$$\alpha(N x_0^2 \alpha^{N} + N\alpha^{N-2} - (N+1) X_0^2) = 0$$

Since $\alpha \neq 0$, therefore, we got

$$N x_0^2 \alpha^{N} + N\alpha^{N-2} - (N+1) X_0^2 = 0$$