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Week 0 Report

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Title: Familiarization with the QBee package and quadratization of the system

1 Useful informations

1. **QBee package**: [link]
2. **QBee documentation**: [link]
3. **Examples**: [link]
4. **Optimal monomial quadratization for ODE systems**: [link]

2 Introduction to Quadratization Problem

For a nonlinear ODE system, on the right hand, we have variables with degree ≥ 2. For example, the following system is nonlinear:

$$x' = -x + x^3 + y^2$$

However, we find that for this system, there exists x^3. Quadratization is to introduce new variables in order to decrease the degree of the system to at most 2. Notice that xy is in degree 2 but xy^2 is in degree 3.

3 Quadratization with QBee

Here we use the QBee package to quadratize the system, for example, the following system:

$$x' = -x + ax^3$$

```
a = parameters("a")
x = functions("x")

eq = - x + a * x ** 3
system = [(x, eq)]

quadr_system = polynomialize_and_quadratize(system,
                                            new_vars_name ="w")

==================================================
Quadratization result
==================================================
Number of introduced variables: 1
Nodes traversed: 3
Introduced variables:
w{0} =x**2
```

From the result, we can introduce a new variable $w_0 = y = x^2$ to quadratize the system. Then for the original equation, we have:

$$\begin{aligned} x' = x' &= -x + ax^3 \\ &= -x + axy \end{aligned}$$

However, since we also introduce a new variable $w_0 = y = x^2$, we can also need to compute the derivative of w_0 in order to find the ODE of it.

$$
\begin{aligned}
y' &= 2xx' = 2x(-x + ax^3) \\
&= -2x^2 + 2ax^4 \\
&= -2y + 2ay^2
\end{aligned}
$$

Then we can combine the two equations to get the quadratized system:

$$
\begin{cases}
y' = -2y + 2ay^2 \\
x' = -x + axy
\end{cases}
$$

Here, on the right hand side, we can see that all the variables are in degree at most 2.

4 Other Examples

For more examples and detailed code, please refer to the [examples]. Here I only show the quadratization of the following system:

We have the third system of following form:

$$x'' = kx + ax^3$$

(we set $x_0 = x$ and $x_1 = x'$). Then we have the system of the ODEs:

$$
\begin{cases}
x_0' = x_1 \\
x_1' = kx_0 + ax_0^3
\end{cases}
$$

```
k, a =parameters("k, a")
x0, x1 =functions("x0, x1")

eq1 =-k * x0 - a * x0**3
system =[
    (x1, eq1),
    (x0, x1),
]

quadr_system =polynomialize_and_quadratize(system, input_der_orders={x0: 2})

==================================================
Quadratization result
==================================================
Number of introduced variables: 4
Nodes traversed: 83
Introduced variables:
w_{0} =x0**3
w_{1} =x0**3 # Actually, I don't know why it is the same as w_{0}
w_{2} =x0*x0'
w_{3} = x0**2
```

From the program we introduce 3 variables:

$$
\begin{cases}
w_0 = x_0^3 \\
w_1 = x_0 x_0' \\
w_2 = x_0^2
\end{cases}
$$

Then we got the following:

$$\begin{cases} x_0' = x_1 \\ x_1' = kx_0 + aw_0 \end{cases}$$

Now we need to dealing with w_0, w_1 and w_2, we have the following computations:

$$\begin{aligned} w_0' &= 3x_0^2 x_0' \\ &= 3w_2 x_1 \end{aligned}$$

$$\begin{aligned} w_1' &= x_0 x_0'' + x_0' x_0' \\ &= x_1^2 + x_0 x_1' = x_1^2 + x_0(kx_0 + ax_0^3) \\ &= x_1^2 + kx_0^2 + akx_0^4 \\ &= x_1^2 + kw_2 + akw_2^2 \end{aligned}$$

$$w_2' = 2x_0 x_0' = 2w_1$$

Therefore, we have the following system:

$$\begin{cases} x_0' = x_1 \\ x_1' = kx_0 + aw_0 \\ w_0' = 3w_2 x_1 \\ w_1' = x_1^2 + kw_2 + akw_2^2 \\ w_2' = 2w_1 \end{cases}$$