Intermediate dimensions and infinite conformal iterated function systems

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Based on work in:

- 'Generalised intermediate dimensions', arXiv preprint (2020), https://arxiv.org/abs/2011.08613
- 'Intermediate dimensions of infinitely generated attractors' (with Jonathan M. Fraser), arXiv preprint (2021), https://arxiv.org/abs/2104.15133

Fractals and dimension

- There is no universal definition of a fractal, but typical features include fine structure at arbitrarily small scales, and some sort of self-similarity.
- There are many different notions of dimension which attempt to quantify the 'thickness' of sets at small scales.

Box dimension

- Throughout, $F \subset \mathbb{R}^d$ will be non-empty and bounded.
- The (upper) box/Minkowski dimension is defined by

$$\overline{\dim}_B F := \limsup_{\delta \to 0^+} \frac{\log N_\delta(F)}{-\log \delta}$$

  where $N_\delta(F)$ is the smallest number of balls of radius δ needed to cover F.
- Intuitively, a disc has box dimension 2 because the number of discs of size r needed to cover it scales approximately like r^{-2} as $r \to 0^+$.

Hausdorff dimension

- The diameter of a set $U \subseteq \mathbb{R}^d$ is $|U| = \sup\{\, ||x - y|| : x, y \in U \,\}$. For $\delta > 0$, a δ-cover of F is a set of subsets $\{U_i\}$ of $\mathbb{R}^d$ such that $0 \leq |U_i| \leq \delta$ for each i, that cover F, i.e. $F \subseteq \bigcup_i U_i$.
- For $s \geq 0$ define

$$H_\delta^s(F) = \inf \left\{ \sum_{i=1}^{\infty} |U_i|^s \,\middle|\, \{U_i\} \text{ is a (finite or) countable } \delta\text{-cover of } F \right\}.$$

- As δ decreases, the class of δ-covers of F is reduced, so the infimum increases, so converges to a limit

$$H_\delta^s(F) \to H^s(F) \in [0, \infty] \text{ as } \delta \to 0^+,$$

  called the *s*-dimensional Hausdorff measure of F, whose restriction to the Borel sets is a measure.
- It is straightforward to see that there is a unique $s \geq 0$, called the Hausdorff dimension of F, denoted $\dim_{\mathrm{H}} F$, such that if $0 \leq t < s$ then $H^t(F) = \infty$ and if $t > s$ then $H^t(F) = 0$.
- Intuitively, disc has Hausdorff dimension 2 because it has positive and finite area.

Assoud dimension

- The Assouad dimension is the largest notion of dimension that can reasonably be defined using covers. It captures the scaling behaviour of the thickest parts of the set.
- $$\begin{aligned}\dim_{\mathrm{A}} F = \inf \{\, \alpha : \ &\text{there exists } C > 0 \text{ such that for all } x \in F \text{ and} \\ &0 < r < R, \text{ we have } N_r(B(x,R) \cap F) \leq C(R/r)^{\alpha} \,\}.\end{aligned}$$
- Originally introduced in the context of embedding theory.

Interpolating between dimensions

- We always have $\dim_{\mathrm{H}} F \leq \overline{\dim}_B F \leq \dim_{\mathrm{A}} F$, and we will see that these can be strict. Since different notions of dimension can take different values for the same set, how to define dimension is a non-trivial question.
- How/when/why do different dimensions of the same set differ?
- One way to improve our understanding is to use dimension interpolation: if dim and Dim are notions of dimensions for which $\dim F \leq \mathrm{Dim} F$ for all 'reasonable' sets F, then we want to find a geometrically natural family of dimensions that lie between $\dim F$ and $\mathrm{Dim} F$ for all such sets.
- The family of dimensions should have some similarities with both dim and Dim, should satisfy properties that would be expected of all notions of dimension, and should lead to an interesting theory.
- This can lead to a better understanding of the similarities and differences between dim and Dim.
- Example: the Assouad spectrum (Fraser and Yu, 2018) and ϕ-Assouad dimensions (Fraser-Yu (2018), García-Hare-Mendivil (2019)) interpolate between the box and Assouad dimensions.

Alternative definitions

- Equivalent definitions:
- $$\overline{\dim}_B F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$
- $$\dim_H F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \epsilon \,\}$$

Intermediate dimensions - definition

- Falconer, Fraser and Kempton (2020) noted that these "may be regarded as two extreme cases of the same definition, one with no restriction on the size of covering sets, and the other requiring them all to have equal diameters" and defined the upper θ-intermediate dimension of F for $\theta \in (0,1)$ by

$$\begin{aligned}\overline{\dim}_\theta F = \inf\{\, s \geq 0 : &\text{ for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all} \\ &\delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that} \\ &\delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.\end{aligned}$$

- As expected, $\dim_{\mathrm{H}} F \leq \overline{\dim}_\theta F \leq \overline{\dim}_{\mathrm{B}} F$ for all $\theta \in (0,1)$.
- There is also a lower version of the box and intermediate dimensions.

Properties

- For each fixed θ, the dimension cannot increase under Lipschitz maps. The dimension can increase by a multiple of at most α^{-1} under α-Hölder maps. The upper intermediate dimensions are finitely stable: the dimension of a finite union of sets is the maximum of the dimension of the individual sets. There are bounds for the dimensions of product sets in terms of the dimensions of the marginals, etc.
- Monotonicity: the function $\theta \mapsto \overline{\dim}_\theta F$ is increasing in $\theta \in [0,1]$.
- Continuity: FFK (2020) showed that $\theta \mapsto \overline{\dim}_\theta F$ is continuous for $\theta \in (0,1]$. More than this, they gave a quantitative continuity result which can be improved to the following:

Proposition (B., 2020)

If $0 < \theta \leq \phi \leq 1$ then

$$\overline{\dim}_\theta F \leq \overline{\dim}_\phi F \leq \overline{\dim}_\theta F + \frac{(\phi - \theta)\overline{\dim}_\theta F(\dim_A F - \overline{\dim}_\theta F)}{(\phi - \theta)\overline{\dim}_\theta F + \theta \dim_A F}.$$

Lower bound

- Of particular interest is the case $\phi = 1$, which upon rearranging gives the general lower bound

$$\overline{\dim}_\theta F \geq \frac{\theta \dim_{\mathrm{A}} F \overline{\dim}_{\mathrm{B}} F}{\dim_{\mathrm{A}} F - (1-\theta)\overline{\dim}_{\mathrm{B}} F}.$$

- For some interesting fractals, for example some Bedford-McMullen carpets, for some θ this gives the best lower bound that is currently known.
- We have the following mutual dependency between the box and intermediate dimensions: $\overline{\dim}_{\mathrm{B}} F > 0$ if and only if $\overline{\dim}_\theta F > 0$ for all (equivalently, any) $\theta \in (0,1]$.
  This means that in order to check that the box dimension of a set is 0, it suffices to check the *a priori* weaker condition that the θ-intermediate dimension of the set is 0 at any (small) $\theta \in (0,1]$; this may be of independent interest outside the realm of dimension interpolation.

Examples - polynomial sequences

- For $p \in (0, \infty)$ define $F_p := \{0\} \cup \{\, n^{-p} : n \in \mathbb{N} \,\}$. These sets satisfy $\dim_{\mathrm{H}} F_p = 0$, $\dim_{\mathrm{B}} F_p = \frac{1}{p+1}$, $\dim_{\mathrm{A}} F_p = 1$.
- FFK showed that $\dim_\theta F_p = \frac{\theta}{p+\theta}$ for all $\theta \in [0,1]$. Note that the box, intermediate and Assouad dimensions of countable sets can be positive.

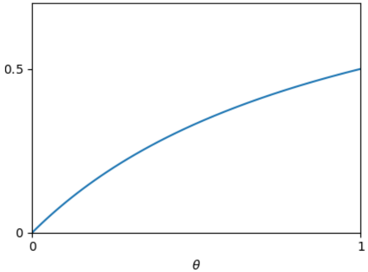

Figure: Intermediate dimensions of $\{\, 1/b : b \in \mathbb{N} \,\}$

- These examples show that the quantitative continuity result and general lower bound above are sharp.

Discontinuity at $\theta = 0$

- Define $F_{log} := \{0\} \cup \{\, 1/(\log n) : n \in \mathbb{N}\,\}$.
- Straightforward to show $\dim_{\mathrm{B}} F_{log} = 1$.
- Since 1 is also the ambient spatial dimension, by a result of FFK, $\dim_\theta F_{log} = 1$ for all $\theta \in (0,1]$.
- But F_{log} is countable so has Hausdorff dimension 0.
- In particular, the intermediate dimensions of F_{log} are discontinuous at $\theta = 0$ and do not interpolate all the way between the Hausdorff and box dimensions.
- In fact there are many compact sets whose intermediate dimensions are discontinuous at $\theta = 0$.

Recovering the interpolation

Theorem (B., 2020)

If $F \subset \mathbb{R}^d$ is non-empty and compact then for all $s \in [\dim_H F, \overline{\dim}_B F]$ there exists a function $\Phi_s : (0,1) \to (0,1)$ that is increasing and satisfies $\Phi_s(\delta)/\delta \to 0$ as $\delta \to 0^+$ such that if we define the new dimension

$$\overline{\dim}^{\Phi_s} F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \Phi_s(\delta) \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}$$

then $\overline{\dim}^{\Phi_s} F = s$.

- It can be shown that $\overline{\dim}^{\Phi_s}$ is a 'reasonable' notion of dimension with many of the same properties as $\overline{\dim}_\theta$, and these dimensions will distinguish between sets which the usual notions of dimension will not distinguish between.
- The compact assumption cannot be removed in general: consider $\mathbb{Q} \cap [0,1]$.
- The choice of Φ_s is not unique, and depends on F.

Iterated function systems

- An iterated function system (IFS) on a compact set $X \subseteq \mathbb{R}^d$ is a finite set of contractions $S_i \colon X \to X$ for i in some finite indexing set I (so there exists $\rho < 1$ such that for all i and $x, y \in X$ we have $||S_i(x) - S_i(y)|| \leq \rho ||x - y||$).
- We write I^n for the set of words of length n on the alphabet I, and I^* for the set of infinite words. We write $w|_n$ for the word comprised of the first n letters of w. Define $S_{w_1 \cdots w_n} := S_{w_1} \circ \cdots \circ S_{w_n}$.
- For $w \in I^*$, $|S_{w|_n}(X)| \leq \rho^n |X|$ for all n, so $\cap_{n=1}^{\infty} S_{w|_n}(X)$ is a single point. Thus it makes sense to define the limit set/attractor of the IFS as

$$F := \bigcup_{w \in I^*} \bigcap_{n=1}^{\infty} S_{w|_n}(X).$$

- F is non-empty and compact, and satisfies $F = \cup_{i \in I} S_i(F)$. It is the largest of many sets, and the only non-empty compact set, that satisfies this invariance property (Hutchinson (1981), using the Banach contraction mapping theorem).

Finite IFSs

- Defining $S(E) := \cup_{i \in I} S_i(E)$ for $E \subseteq X$, the limit set F satisfies $F = \cap_{n=1}^{\infty} S^n(X)$.
- Moreover, F is the closure of the set of fixed points of all finite compositions of the S_i.

Self-similar sets

- If each of the contractions S_i is a similarity (so there exists $c_i \in [0,1)$ such that $||S_i(x) - S_i(y)|| = c_i||x - y||$ for all $x, y \in X$), then F is a self-similar set and has equal Hausdorff and box dimensions.
- If the IFS satisfies the open set condition (OSC), meaning that there exists a non-empty bounded open set U such that $U \supseteq \cup_{i \in I} S_i(U)$ with the union disjoint, then the dimension of F is the unique $h \geq 0$ such that

$$\sum_{i \in I} c_i^h = 1$$

(Moran (1946), Hutchinson (1981)).
Note this is also the unique zero of the topological pressure function

$$P(t) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{w_1 \cdots w_n \in I^n} (c_{w_1} \cdots c_{w_n})^t.$$

Infinite iterated function systems (Mauldin-Urbański, 1996)

- In an infinite iterated function system there are a countable number of maps $S_i \colon X \to X$ which are uniformly contractive (so there exists $\rho < 1$ such that for all i and $x, y \in X$ we have $||S_i(x) - S_i(y)|| \leq \rho ||x - y||$).
- One can again define the limit set

$$F := \bigcup_{w \in I^*} \bigcap_{n=1}^{\infty} S_{w|_n}(X).$$

- F again satisfies $F = \cup_{i \in I} S_i(F)$, and is the largest of many sets which do.
- F is non-empty but may not be closed.
- The two alternative characterisations of the limit set in the case of finite IFSs may now give sets which strictly contain the limit set.

Conformal maps

- Conformal maps locally preserve angles.
- If $V \subseteq \mathbb{R}^d$ is open then $f\colon V \to \mathbb{R}^d$ is conformal if for all $x \in V$ the differential $f'|_x$ exists, is non-zero, is Hölder continuous in x, and is a similarity map: $||f'|_x(y)|| = ||f'|_x|| \cdot ||y||$ for all $y \in \mathbb{R}^d$.
- In one dimension, they are simply functions with non-vanishing Hölder continuous derivative.
  In two dimensions, they are holomorphic functions with non-vanishing complex derivative on their domain.
  In dimension three and higher they have a very restricted form: by a theorem of Liouville (1850) they are Möbius transformations, so can be composed from homotheties, isometries, reflections in hyperplanes, and inversions in spheres.

Conformal iterated function systems

- Assume that the IFS satisfies the following properties:
- (Conformality) There exists an open, bounded, connected subset $V \subset \mathbb{R}^d$ such that $X \subset V$ and such that for each $i \in I$, S_i extends to a conformal map from V to an open subset of V. Moreover, there exists $\rho \in (0,1)$ such that for all $i \in I$ we have $||S_i'|| < \rho$, where $||\cdot||$ is the supremum norm over V.
- (OSC) Letting $U := \mathrm{Int}_{\mathbb{R}^d} X$, we have $S_i(U) \subset U$ for all $i \in I$ and $S_i(U) \cap S_j(U) = \varnothing$ for all $i, j \in I$ with $i \neq j$.
- (Cone condition) $\inf_{x \in X} \inf_{r \in (0,1)} \mathcal{L}^d(B(x,r) \cap \mathrm{Int}_{\mathbb{R}^d} X)/r^d > 0$.
- (Bounded distortion property) There exists $K > 0$ such that for all $x, y \in X$ and any finite word w we have $||S_w'|_y|| \leq K ||S_w'|_x||$.

Pressure functions for infinite conformal IFSs (Mauldin-Urbański, 1996)

- Define the pressure function

$$P(t) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{w \in I^n} ||S'_w||^t$$

- If $\theta_S := \inf\{ t \geq 0 : P(t) < \infty \}$ then P is strictly decreasing and continuous on (θ_S, ∞).
- $\dim_{\mathrm{H}} F = \inf\{ t \geq 0 : P(t) \leq 0 \}$, but there may not exist $t \geq 0$ such that $P(t) = 0$.
- In particular, if each S_i is a similarity with contraction ratio c_i then $\dim_{\mathrm{H}} F = \inf\{ t \geq 0 : \sum_{i \in I} c_i^t \leq 1 \}$.

Box and intermediate dimensions

Again assume conformality and the other assumptions as above.

Theorem (Mauldin-Urbański, 1999)

For any $x \in X$,

$$\overline{\dim}_{\mathrm{B}} F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}}\{\, S_i(x) : i \in I \,\}\}.$$

When there are finitely many contractions, the pressure function is finite, strictly decreasing, continuous, and has a unique zero, which is both the Hausdorff and box dimension of the self-conformal set.

Theorem (B.-Fraser, 2021)

For all $\theta \in [0,1]$, for any $x \in X$,

$$\begin{aligned}\overline{\dim}_{\theta} F &= \max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\theta}\{\, S_i(x) : i \in I \,\}\} \\ &= \max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\theta}\{\, \text{fixed points in } X \text{ of the } S_i \text{ for } i \in I \,\}\}.\end{aligned}$$

Continued fractions

- Each irrational number in $(0,1)$ has a unique continued fraction expansion $\frac{1}{b_1+\frac{1}{b_2+\frac{1}{\ddots}}}$ for $b_i \in \mathbb{N}$.
- For a non-empty subset $I \subseteq \mathbb{N}$, define

$$F_I := \left\{ z \in (0,1)\backslash\mathbb{Q} : z = \frac{1}{b_1+\frac{1}{b_2+\frac{1}{\ddots}}}, b_n \in I \text{ for all } n \in \mathbb{N} \right\}.$$

- Working in $\mathbb{R}$, with $X = [0,1]$ and $V := (-1/8, 9/8)$, $\{ S_b(x) := 1/(b+x) : b \in I \}$ is an infinite conformal iterated function system (if $1 \notin I$) with limit set F_I.
- Therefore a corollary of our theorem is that $\overline{\dim}_\theta F_I = \max\{\dim_H F_I, \overline{\dim}_\theta\{ 1/b : b \in I \}\}$ for all $\theta \in [0,1]$.

Continued fractions - example

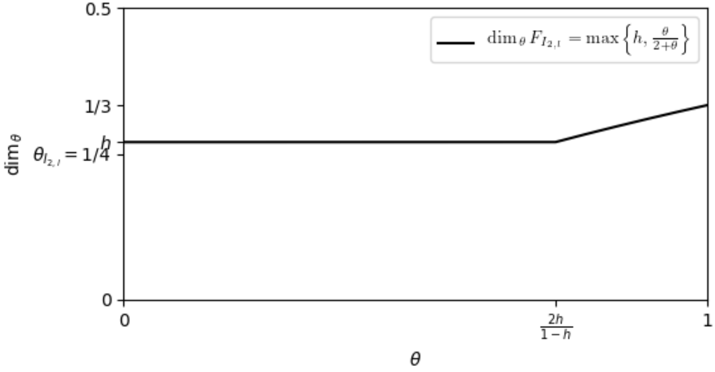

Figure: Graph of the intermediate dimensions of the continued fraction set $\overline{\dim}_\theta F_I = \max\left\{\dim_H F_I, \frac{\theta}{2+\theta}\right\}$, where $I = \{n^2, (n+1)^2, (n+2)^2, \ldots\}$ for some fixed large $n \in \mathbb{N}$. Note that the graph of the intermediate dimensions has a phase transition, and is neither convex on the whole domain nor concave on the whole domain.

Fractional Brownian motion

- In particular, since $\overline{\dim}_\theta\{\,1/b : b \in \mathbb{N}\,\} = \frac{\theta}{1+\theta}$, the intermediate dimensions of F_I are continuous at $\theta = 0$.
- The intermediate dimensions of limit sets of general infinite CIFS may not be continuous at $\theta = 0$, for example when the fixed points are at $\{\,1/(\log n) : n \in \mathbb{N}\,\}$ and the contraction ratios are small enough for the Hausdorff dimension to be less than 1.
- For $\alpha \in (0, 1)$, index-α fractional Brownian motion is a stochastic process $B_\alpha\colon \mathbb{R} \to \mathbb{R}$ such that the increments $B_\alpha(x) - B_\alpha(y)$ are normally distributed with mean 0 and variance $|x - y|^\alpha$ for all $x, y \in \mathbb{R}$ (so $\alpha = 1/2$ gives usual Brownian motion).
- It follows from the continuity of the intermediate dimensions and a result of Burrell (2020) that almost surely, $\overline{\dim}_B B_\alpha(F_I) = 1$ if and only if $\alpha \leq \dim_{\mathrm{H}} F_I$.
- This is another example of how the intermediate dimensions can be used to give non-trivial information about the box dimensions of sets.

Thank you for listening!

Questions?