Dimensions of infinitely generated attractors

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Based on work in 'Intermediate dimensions of infinitely generated attractors' (with Jonathan M. Fraser), arXiv preprint (2021), https://arxiv.org/abs/2104.15133

Finite iterated function systems

- Recall that many fractals are generated using iterated function systems. An IFS is a finite set of contractions $\{S_i \colon X \to X\}_{i \in I}$ where $X \subset \mathbb{R}^d$ is compact.
- Hutchinson (1981) showed there is a unique non-empty compact attractor satisfying
$$F = \cup_{i \in I} S_i(F).$$
There are many non-compact sets satisfying this relation, all contained in F.
- F is the closure of the set of fixed points of all finite compositions of the S_i. Defining $S(E) := \cup_{i \in I} S_i(E)$ for $E \subseteq X$, the limit set F satisfies $F = \cap_{n=1}^{\infty} S^n(X)$.

Self-similar sets

- If each of the contractions S_i is a similarity with contraction ratio c_i then F is a self-similar set, for example the Sierpinski gasket.
- The Hausdorff and box dimensions are equal, and the open set condition (OSC) is satisfied, meaning that there exists a non-empty bounded open set U such that $U \supseteq \cup_{i \in I} S_i(U)$ with the union disjoint, then the dimension is the unique $h \geq 0$ such that
$$\sum_{i \in I} c_i^h = 1$$
(Moran (1946), Hutchinson (1981)) e.g. $\log 3 / \log 2$ for the Sierpinski gasket.
- Note that the dimension is also the unique zero of the topological pressure function
$$P(t) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{w_1 \cdots w_n \in I^n} (c_{w_1} \cdots c_{w_n})^t.$$
- Equality of Hausdorff and box dimensions indicates that self-similar sets are very homogeneous in space. But can get inhomogeneity if either the maps are less nice (e.g. affine), or if there are infinitely many maps.

Infinite iterated function systems (Mauldin-Urbański, 1996)

- In an infinite iterated function system (IIFS) there are a countable number of maps $\{S_i\colon X \to X\}_{i \in I}$ which are uniformly contractive (so there exists $\rho < 1$ such that for all i and $x, y \in X$ we have $||S_i(x) - S_i(y)|| \leq \rho||x - y||$).
- Notation: we write I^n for the set of words of length n on the alphabet I, I^* for the set of finite words, and $I^{\mathbb{N}}$ for the set of infinite words. We write $w|_n$ for the word comprised of the first n letters of w. Define $S_{w_1 \cdots w_n} := S_{w_1} \circ \cdots \circ S_{w_n}$ and $M_n := \{ S_w : w \in I^n \}$.
- For $w \in I^{\mathbb{N}}$, $|S_{w|_n}(X)| \leq \rho^n |X|$ for all n, so $\cap_{n=1}^{\infty} S_{w|_n}(X)$ is a single point. Thus it makes sense to define the limit set/attractor of the IIFS as

$$F := \bigcup_{w \in I^{\mathbb{N}}} \bigcap_{n=1}^{\infty} S_{w|_n}(X).$$

  When I is finite, F is the same as the attractor in the sense of Hutchinson.
- F again satisfies $F = \cup_{i \in I} S_i(F)$, and is the largest (by inclusion) of many sets which do, and is non-empty but may not be closed. The two alternative characterisations of the limit set in the case of finite IFSs may now give sets which strictly contain the limit set.

Topological pressure functions

- For $w \in I^n$ define

$$R_w = R_{S_w} := \sup_{x,y \in X, x \neq y} \frac{||S_w(x) - S_w(y)||}{||x - y||},$$

the smallest possible Lipschitz constant for S_w. These are clearly submultiplicative; for $v, w \in I^*$ we have $R_{vw} \leq R_v R_w$, so the sequence $\left(\log \sum_{\sigma \in M_n} R_\sigma^t\right)_{n \in \mathbb{N}}$ is subadditive.
- Therefore by Fekete's subadditivity lemma, the limit exists if we define the topological pressure function $P \colon (0, \infty) \to [-\infty, \infty]$ by

$$P(t) := \lim_{n \to \infty} \frac{1}{n} \log \sum_{\sigma \in M_n} R_\sigma^t.$$

- Define

$$\theta_S := \inf\{\, t > 0 : P(t) < \infty \,\} \in [0, \infty];$$

$$h := \inf\{\, t > 0 : P(t) < 0 \,\} \in [0, \infty].$$

It is straightforward to see that $P(t)$ is strictly decreasing for $t \in (\theta_S, \infty)$.

Hausdorff dimension - upper bound

The Hausdorff dimension of any $F \subseteq \mathbb{R}$ can be defined without using Hausdorff measure by

$$\dim_H F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \epsilon \,\}$$

Proposition

If F is the limit set of an infinite IFS then $\dim_{\mathrm{H}} F \leq h$.

Proof sketch

$\{\, \sigma(X) : \sigma \in M_n \,\}$ forms a cover of F with the maximum diameter tending to 0 as $n \to \infty$ by the uniform contractivity. If $s > h$ then $P(s) < 0$ so

$$\sum_{\sigma \in M_n} |\sigma(X)|^s \leq |X|^s \sum_{\sigma \in M_n} R_\sigma^s \leq |X|^s e^{nP(s)/2} \xrightarrow[n \to \infty]{} 0,$$

completing the proof.

Box dimension

- The (upper) box/Minkowski dimension of a non-empty, bounded $F \subset \mathbb{R}^d$ is defined by
$$\overline{\dim}_B F := \limsup_{\delta \to 0^+} \frac{\log N_\delta(F)}{-\log \delta}$$
where $N_\delta(F)$ is the smallest number of balls of radius δ needed to cover F.
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$$\overline{\dim}_B F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$
- Falconer, Fraser and Kempton (2020) noted that the Hausdorff and box dimensions "may be regarded as two extreme cases of the same definition, one with no restriction on the size of covering sets, and the other requiring them all to have equal diameters..."

Intermediate dimensions

- and defined the upper θ-intermediate dimension of F for $\theta \in (0,1)$ by

$$\overline{\dim}_\theta F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

- As expected, $\dim_{\mathrm{H}} F \leq \overline{\dim}_\theta F \leq \overline{\dim}_{\mathrm{B}} F$ for all $\theta \in (0,1)$. There is also a lower version of the intermediate dimensions.
- The intermediate dimensions satisfy many of the properties that would be expected of most reasonable notions of dimensions. For any set F, the function $\theta \mapsto \overline{\dim}_\theta F$ is increasing in $\theta \in [0,1]$ and continuous for $\theta \in (0,1]$ but not necessarily at $\theta = 0$.
- The intermediate dimensions are an example of dimension interpolation: where one considers two notions of dimension dim and Dim for which $\dim F \leq \mathrm{Dim} F$ for all 'reasonable' sets F and finds a geometrically natural family of dimensions that lie between $\dim F$ and $\mathrm{Dim} F$ for all such sets and shares some similarities with both. Example: Assouad spectrum (Fraser-Yu, 2018).

Upper bounds

Theorem (B.-Fraser, 2021)

For all $\theta \in [0,1]$ we have

$$\overline{\dim}_\theta F \leq \max\{h, \lim_{n\to\infty} \inf\{\overline{\dim}_\theta P_n : P_n \subseteq X \text{ and } P_n \cap S_w(X) \neq \varnothing \, \forall w \in I^n\}\}.$$

The particular case $\theta = 1$ shows this for $\overline{\dim}_\theta = \overline{\dim}_{\mathrm{B}}$.

Proof idea: consider $\delta \in \left(\frac{1}{n+1}, \frac{1}{n}\right]$ and induct on n.

Conformal maps

- Conformal maps locally preserve angles.
- If $V \subseteq \mathbb{R}^d$ is open then $f\colon V \to \mathbb{R}^d$ is conformal if for all $x \in V$ the differential $f'|_x$ exists, is non-zero, is Hölder continuous in x, and is a similarity map: $||f'|_x(y)|| = ||f'|_x|| \cdot ||y||$ for all $y \in \mathbb{R}^d$.
- In one dimension, they are simply functions with non-vanishing Hölder continuous derivative.
  In two dimensions, they are holomorphic functions with non-vanishing complex derivative on their domain.
  In dimension three and higher they have a very restricted form: by a theorem of Liouville (1850) they are Möbius transformations, so can be composed from homotheties, isometries, reflections in hyperplanes, and inversions in spheres.

Conformal iterated function systems (Mauldin-Urbański, 1996)

- Assume that the IFS satisfies the following properties:
- (Conformality) There exists an open, bounded, connected subset $V \subset \mathbb{R}^d$ such that $X \subset V$ and such that for each $i \in I$, S_i extends to a conformal map from V to an open subset of V. Moreover, there exists $\rho \in (0,1)$ such that for all $i \in I$ we have $||S_i'|| < \rho$, where $||\cdot||$ is the supremum norm over V.
- (OSC) The set X has non-empty interior $U := \mathrm{Int}_{\mathbb{R}^d} X$, and $S_i(U) \subset U$ for all $i \in I$ and $S_i(U) \cap S_j(U) = \varnothing$ for all $i, j \in I$ with $i \neq j$.
- (Cone condition) $\inf_{x \in X} \inf_{r \in (0,1)} \mathcal{L}^d(B(x,r) \cap \mathrm{Int}_{\mathbb{R}^d} X)/r^d > 0$.
- (Bounded distortion property) There exists $K > 0$ such that for all $x, y \in X$ and any finite word w we have $||S_w'|_y|| \leq K ||S_w'|_x||$.

Hausdorff dimension for infinite conformal IFSs

Theorem (Mauldin-Urbański, 1996)

Assuming conformality and the other conditions, $\dim_{\mathrm{H}} F = h$.

- We have already seen that $\dim_{\mathrm{H}} F \leq h$ holds more generally, but the proof that $\dim_{\mathrm{H}} F = h$ crucially relies on conformality etc.
- Note that there may not exist $t \geq 0$ such that $P(t) = 0$.
- In particular, if each S_i is a similarity with contraction ratio c_i then $\dim_{\mathrm{H}} F = \inf\{\, t \geq 0 : \sum_{i \in I} c_i^t \leq 1 \,\}$.

Box dimension

Again assume conformality and the other assumptions as above.

Theorem (Mauldin-Urbański, 1999)

For any $x \in X$,

$$\overline{\dim}_{\mathrm{B}} F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}}\{\, S_i(x) : i \in I \,\}\}.$$

When there are finitely many contractions, the pressure function is finite, strictly decreasing, continuous, and has a unique zero, which is both the Hausdorff and box dimension of the self-conformal set.

Box dimension proof outline

Proposition (Mauldin-Urbański, 1999)

For all $x, y \in X$ and $n \in \mathbb{N}$, $\overline{\dim}_{\mathrm{B}}\{\, S_w(x) : w \in I^n \,\} = \overline{\dim}_{\mathrm{B}}\{\, S_w(y) : w \in I^n \,\}$.

Lemma (Mauldin-Urbański, 1999)

For all $x \in X$ and $n \in \mathbb{N}$, $\overline{\dim}_{\mathrm{B}}\{\, S_w(x) : w \in I^n \,\} = \overline{\dim}_{\mathrm{B}}\{\, S_w(x) : w \in I \,\}$.

The proofs of the Prop and Lemma are geometric arguments using the conformality etc. The proof of the lemma uses the fact that $\overline{\dim}_{\mathrm{B}}\{\, S_w(x) : w \in I \,\} \geq \theta_S$ which is not always true for $\overline{\dim}_{\theta}$. The proof is then an induction on n.
These together with an upper bound like the general upper bound we have seen prove the result.

Main result - intermediate dimensions for a CIFS

Theorem (B.-Fraser, 2021)

Assuming conformality and the other conditions, for all $\theta \in [0,1]$, if P is any subset of X which intersects $S_i(X)$ in exactly one point for each $i \in I$ (for example the iterates of a given point, or the set of fixed points of the contractions), then

$$\overline{\dim}_\theta F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_\theta P\}.$$

Intermediate dimensions proof outline

Lemma

If P and Q are both subsets of X which intersect $S_w(X)$ in exactly one point for each $w \in I^n$ then $\overline{\dim}_\theta P = \overline{\dim}_\theta Q$ for each $\theta \in [0,1]$.

Lemma

If for each $n \in \mathbb{N}$, $P_n \subseteq X$ intersects $S_w(X)$ in exactly one point for each $w \in I^n$, then for all $\theta \in [0,1]$, either $\overline{\dim}_\theta P_n \leq \theta_S \leq h$ for all $n \in \mathbb{N}$ or $\overline{\dim}_\theta P_n = \overline{\dim}_\theta P_1$ for all $n \in \mathbb{N}$.

(note difference with Mauldin and Urbański's Lemma. Also an induction on n). These together with the general upper bound, and the fact that $\dim_H = h$ by conformality etc, prove the result.

Continued fractions

- Each irrational number in $(0,1)$ has a unique continued fraction expansion $\frac{1}{b_1+\frac{1}{b_2+\frac{1}{\ddots}}}$ for $b_i \in \mathbb{N}$.
- For a non-empty subset $I \subseteq \mathbb{N}$, define

$$F_I := \left\{ z \in (0,1)\backslash\mathbb{Q} : z = \frac{1}{b_1 + \frac{1}{b_2+\frac{1}{\ddots}}}, b_n \in I \text{ for all } n \in \mathbb{N} \right\}.$$

- Working in $\mathbb{R}$, with $X = [0,1]$ and $V := (-1/8, 9/8)$, $\{ S_b(x) := 1/(b+x) : b \in I \}$ is an infinite conformal iterated function system (if $1 \notin I$) with limit set F_I.
- Therefore a corollary of our theorem is that $\overline{\dim}_\theta F_I = \max\{\dim_{\mathrm{H}} F_I, \overline{\dim}_\theta\{ 1/b : b \in I \}\}$ for all $\theta \in [0,1]$.

Continuity at $\theta = 0$

In particular, since $\overline{\dim}_\theta \{ 1/b : b \in \mathbb{N} \} = \frac{\theta}{1+\theta}$ by a result of Falconer, Fraser and Kempton (2020), the intermediate dimensions of F_I are continuous at $\theta = 0$.

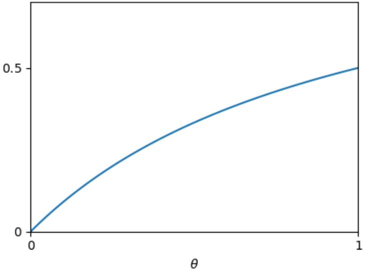

Figure: Intermediate dimensions of $\{ 1/b : b \in \mathbb{N} \}$

The intermediate dimensions of limit sets of general infinite CIFS may not be continuous at $\theta = 0$, for example when the fixed points are at $\{ 1/(\log n) : n \in \mathbb{N} \}$ and the contraction ratios are small enough for the Hausdorff dimension to be less than 1.

Fractional Brownian motion

- For $\alpha \in (0,1)$, index-α fractional Brownian motion is a stochastic process $B_\alpha \colon \mathbb{R} \to \mathbb{R}$ such that the increments $B_\alpha(x) - B_\alpha(y)$ are normally distributed with mean 0 and variance $|x-y|^\alpha$ for all $x, y \in \mathbb{R}$ (so $\alpha = 1/2$ gives usual Brownian motion).
- It follows from the continuity of the intermediate dimensions and a result of Burrell (2020) that almost surely, $\overline{\dim}_B B_\alpha(F_I) = 1$ if and only if $\alpha \leq \dim_{\mathrm{H}} F_I$.
- This is an example of how the intermediate dimensions can be used to give non-trivial information about the box dimensions of sets.

Hölder maps

- If $f \colon F \to \mathbb{R}^d$ is an α-Hölder map for some $\alpha \in (0,1]$, meaning that there exists $c > 0$ such that $||f(x) - f(y)|| \leq c||x - y||^\alpha$ for all $x, y \in F$, then
$$\overline{\dim}_\theta f(F) \leq \alpha^{-1} \overline{\dim}_\theta F$$
- If $f \colon F \to G$ is α-Hölder with $f(F) \supseteq G$ then
$$\alpha \leq \frac{\overline{\dim}_\theta F}{\overline{\dim}_\theta G}.$$
- Applying this with the continued fraction sets $F = F_I$ and $G = F_J$ for appropriate $I, J \subset \mathbb{N}$ shows $\theta \in (0,1)$ can give better information than the Hausdorff or box dimensions.

Hölder maps

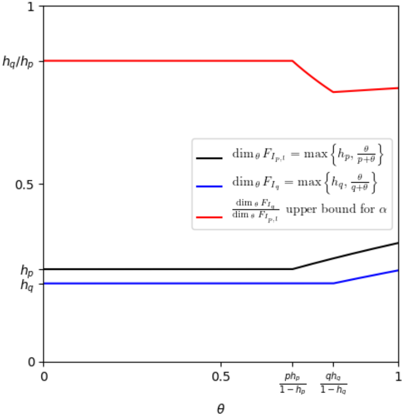

Figure: The black curve is the graph of the intermediate dimensions of the continued fraction set $\overline{\dim}_\theta F_I = \max\left\{\dim_H F_I, \frac{\theta}{2+\theta}\right\}$, where $I = \{n^2, (n+1)^2, (n+2)^2, \ldots\}$ for some fixed large $n \in \mathbb{N}$. Note that the graph of the intermediate dimensions has a phase transition, and is neither convex on the whole domain nor concave on the whole domain.

Generic attractors

- Falconer (1988) showed that if one fixes a finite set of matrices then for Lebesgue-almost every choice of translations, the Hausdorff and box dimensions of the associated self-affine set satisfies the affinity dimension formula depending only on the matrices.
- Fix an infinite IFS of contractions $S_i \colon [0,2]^d \to [0,1]^d$. Assume the contraction ratios accumulate only at 0. Let $V := [0,1)^d$ and for $t = (t_1, t_2, \dots) \in V^{\mathbb{N}}$ let F_t denote the attractor of the IIFS $\{S_i + t_i\}_{i \in \mathbb{N}}$ with $S_i + t_i$ defined on $[0,2]^d$ by $(S_i + t_i)(x) = S_i(x) + t_i$.

Theorem (B.-Fraser, 2021)

For a 'generic' set of translates $t \in V^{\mathbb{N}}$, the limit set F_t is somewhere dense in $[0,2]^d$, so in particular for all $\theta \in (0,1]$

$$\dim_\theta F_t = \dim_{\mathrm{B}} F_t = d,$$

where 'generic' can mean any of the following three things:

Meanings of 'generic'

- For almost every $t \in V^{\mathbb{N}}$ with respect to the natural probability measure on $V^{\mathbb{N}}$ formed by taking the infinite product of the restriction of the Lebesgue measure to V. *Not* true for the Hausdorff dimension which Käenmäki and Reeve (2014) showed satisfies an analogue of Falconer's affinity dimension formula for almost every set of translates of affine maps;
- For a prevalent set of $t \in V^{\mathbb{N}}$: if we equip $V^{\mathbb{N}}$ with a topological group structure by taking the infinite product of the group V under addition mod 1 with the product topology, then the complement of this set of t is a *Haar-null* set, i.e. a Borel set B for which there exists a Borel probability measure ν on $V^{\mathbb{N}}$ such that $\nu(gBh) = 0$ for all $g, h \in V^{\mathbb{N}}$;
- For a residual/co-meagre (topologically generic) set of $t \in V^{\mathbb{N}}$ if we endow $V^{\mathbb{N}}$ with the Hilbert cube metric

$$d(t,s) = \left(\sum_{i=1}^{\infty} \frac{||t_i - s_i||^2}{i^2} \right)^{1/2}$$

which generates the product topology on $V^{\mathbb{N}}$.

Thank you for listening!

Questions?