Dimensions of continued fraction sets

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Based on work in 'Intermediate dimensions of infinitely generated attractors' (with Jonathan M. Fraser), arXiv preprint (2021), `https://arxiv.org/abs/2104.15133`

Continued fractions

- Recall that every irrational number in $(0,1)$ has a unique continued fraction expansion $\cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{\ddots}}}$ for $b_i \in \mathbb{N}$.
- For a non-empty proper subset $I \subseteq \mathbb{N}$, define the real continued fraction set

$$F_I := \left\{ z \in (0,1)\backslash\mathbb{Q} : z = \cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{\ddots}}}, b_n \in I \text{ for all } n \in \mathbb{N} \right\}.$$

- More generally, for $I \subseteq \{\, m+ni : m \in \mathbb{N}, n \in \mathbb{Z} \,\} = \mathbb{N} \times \mathbb{Z}i$, define the complex continued fraction set

$$F_I := \left\{ z \in \mathbb{C} : z = \cfrac{1}{b_1+\cfrac{1}{b_2+\cfrac{1}{\ddots}}}, b_n \in I \text{ for all } n \in \mathbb{N} \right\}.$$

Conformal maps

- These sets are limit sets of infinite conformal iterated function systems.
- Conformal maps locally preserve angles.
- If $V \subseteq \mathbb{R}^d$ is open then $f \colon V \to \mathbb{R}^d$ is conformal if for all $x \in V$ the differential $f'|_x$ exists, is non-zero, is Hölder continuous in x, and is a similarity map: $||f'|_x(y)|| = ||f'|_x|| \cdot ||y||$ for all $y \in \mathbb{R}^d$.
- In one dimension, they are simply functions with non-vanishing Hölder continuous derivative.

  In two dimensions, they are holomorphic functions with non-vanishing complex derivative on their domain.

  In dimension three and higher they have a very restricted form: by a theorem of Liouville (1850) they are Möbius transformations, so can be composed from translations, similarities, orthogonal transformations, and inversions.

Infinite conformal iterated function systems (Mauldin-Urbański, 1996)

- Let $X \subset V \subset \mathbb{R}^d$, X closed with non-empty interior, V open, bounded and connected. Consider a countable number of $C^{1+\epsilon}$ maps S_i from V into an open subset of V, mapping X into X. Assume the maps are uniformly contracting: there exists $\rho \in (0,1)$ such that for all $i \in I$ we have $||S_i'|| < \rho$, where $||\cdot||$ is the supremum norm over V.
- This forms a conformal iterated function systems (CIFS) if the following additional properties are satisfied:
- (Conformality) Each S_i is conformal on V.
- (OSC) The set X has non-empty interior $U := \mathrm{Int}_{\mathbb{R}^d} X$, and $S_i(U) \subset U$ for all $i \in I$ and $S_i(U) \cap S_j(U) = \varnothing$ for all $i,j \in I$ with $i \neq j$.
- (Cone condition) $\inf_{x \in X} \inf_{r \in (0,1)} \mathcal{L}^d(B(x,r) \cap \mathrm{Int}_{\mathbb{R}^d} X)/r^d > 0$.
- (Bounded distortion property) There exists $K > 0$ such that for all $x, y \in X$ and $w_1, \ldots, w_n \in I$ we have $||(S_{w_1} \circ \cdots \circ S_{w_n})'|_y|| \leq K ||(S_{w_1} \circ \cdots \circ S_{w_n})'|_x||$.

Limit sets

- To each CIFS is an associated limit set/attractor: the largest set (by inclusion) $F \subseteq X$ that satisfies the invariance

$$F = \bigcup_{i \in I} S_i(F).$$

  It is non-empty but not generally compact.
- Working in $\mathbb{C} \simeq \mathbb{R}^2$, with $X = D = \overline{B}(1/2, 1/2)$ and $V := B(1/2, 3/4)$, Mauldin and Urbański (1996) verified that $\{ S_b(z) := 1/(b+z) : b \in I \}$ is a CIFS (if $1 \notin I$) with limit set F_I. Conformality holds since the maps are holomorphic. Bounded distortion property holds by the Koebe distortion theorem.

Hausdorff dimension

- The topological pressure function is defined by

$$P(t) \coloneqq \lim_{n\to\infty} \frac{1}{n} \log \sum_{w_1,\ldots,w_n\in I} ||(S_{w_1} \circ \cdots \circ S_{w_n})'||^t.$$

- Mauldin and Urbański (1996) showed that for any CIFS, $P(t)$ exists in $\mathbb{R} \cup \{\infty\}$ and is decreasing on $[0,\infty)$, and is strictly decreasing and continuous on $(\inf\{\, t \geq 0 : P(t) < \infty \,\}, \infty)$.

Theorem (Mauldin-Urbański, 1996)

For any CIFS, $\dim_{\mathrm{H}} F = \inf\{\, t \geq 0 : P(t) < 0 \,\}$.

There may not exist $t \geq 0$ such that $P(t) = 0$.

Hausdorff dimension - numerical approximation

There has been much work trying to find good numerical approximations for the Hausdorff dimension.
Mauldin-Gardner (1983): $1 < \dim_{\mathrm{H}} F_{\mathbb{N}\times\mathbb{Z}i} < 2$.
Mauldin-Urbański (1996): $1.24 < \dim_{\mathrm{H}} F_{\mathbb{N}\times\mathbb{Z}i} < 1.9$.
Falk-Nussbaum (2018): $1.85574 < \dim_{\mathrm{H}} F_{\mathbb{N}\times\mathbb{Z}i} < 1.85589$.

Box dimension

Theorem (Mauldin-Urbański, 1999)

If F is the limit set of a CIFS, for any $x \in X$,

$$\overline{\dim}_{\mathrm{B}} F = \max\{\dim_{\mathrm{H}} F, \overline{\dim}_{\mathrm{B}}\{\, S_i(x) : i \in I \,\}\}.$$

(When there are finitely many contractions, the pressure function is finite, strictly decreasing, continuous, and has a unique zero, which is both the Hausdorff and box dimension of the self-conformal set.)

For any $I \subseteq \mathbb{N} \times \mathbb{Z}i$,

$$\overline{\dim}_{\mathrm{B}} F_I = \max\{\dim_{\mathrm{H}} F_I, \overline{\dim}_{\mathrm{B}}\{\, 1/b : b \in I \,\}\}.$$

Alternative definitions of dimensions

- The Hausdorff dimension of any $F \subseteq \mathbb{R}$ can be defined without using Hausdorff measure by

$$\dim_H F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists a finite or countable cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \sum_i |U_i|^s \leq \epsilon \,\}$$

- $$\overline{\dim}_B F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } |U_i| = \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

- Falconer, Fraser and Kempton (2020) noted that the Hausdorff and box dimensions "may be regarded as two extreme cases of the same definition, one with no restriction on the size of covering sets, and the other requiring them all to have equal diameters..."

Intermediate dimensions

- and defined the upper θ-intermediate dimension of F for $\theta \in (0,1)$ by

$$\overline{\dim}_\theta F = \inf\{\, s \geq 0 : \text{for all } \epsilon > 0 \text{ there exists } \delta_0 \in (0,1] \text{ such that for all } \delta \in (0,\delta_0) \text{ there exists a cover } \{U_1, U_2, \ldots\} \text{ of } F \text{ such that } \delta^{1/\theta} \leq |U_i| \leq \delta \text{ for all } i, \text{ and } \sum_i |U_i|^s \leq \epsilon \,\}.$$

- As expected, $\dim_{\mathrm{H}} F \leq \overline{\dim}_\theta F \leq \overline{\dim}_{\mathrm{B}} F$ for all $\theta \in (0,1)$. There is also a lower version of the intermediate dimensions.
- The intermediate dimensions satisfy many of the properties that would be expected of most reasonable notions of dimensions. For any set F, the function $\theta \mapsto \overline{\dim}_\theta F$ is increasing in $\theta \in [0,1]$ and continuous for $\theta \in (0,1]$.
- The intermediate dimensions are an example of dimension interpolation: where one considers two notions of dimension dim and Dim for which $\dim F \leq \mathrm{Dim} F$ for all 'reasonable' sets F and finds a geometrically natural family of dimensions that lie between $\dim F$ and $\mathrm{Dim} F$ for all such sets and shares some similarities with both. Example: Assouad spectrum (Fraser-Yu, 2018).

Main result - intermediate dimensions

Theorem (B.-Fraser, 2021)

If F is the limit set of a CIFS, for all $\theta \in [0,1]$, for any $x \in X$,

$$\begin{aligned}\overline{\dim}_\theta F &= \max\{\dim_{\mathrm{H}} F, \overline{\dim}_\theta\{\, S_i(x) : i \in I \,\}\} \\ &= \max\{\dim_{\mathrm{H}} F, \overline{\dim}_\theta\{\, \text{fixed points in } X \text{ of the } S_i \text{ for } i \in I \,\}\}.\end{aligned}$$

Lower bound is trivial. One step in the proof of the upper bound is an induction argument, using efficient covers at larger scales to construct efficient covers at smaller scales.

In particular

$$\overline{\dim}_\theta F_I = \max\{\dim_{\mathrm{H}} F_I, \overline{\dim}_\theta\{\, 1/b : b \in I \,\}\}.$$

Example

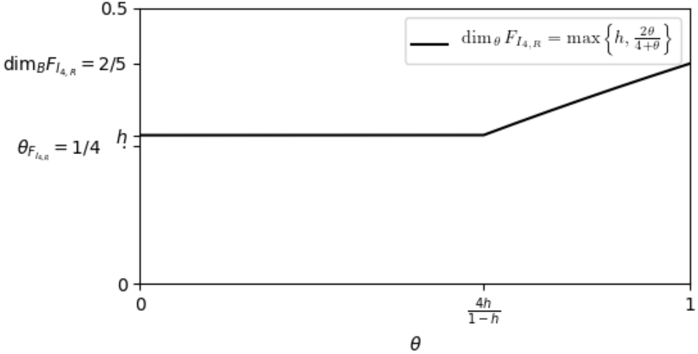

Figure: Graph of $\dim_\theta F_{I_{4,R}}$ for some large $R >> 0$ where $I_{4,R} := \{\, m^4 + n^4 i : n, m \in \mathbb{N} \,\} \backslash B(0, R)$. Note that the graph of the intermediate dimensions has a phase transition, and is neither convex on the whole domain nor concave on the whole domain.

Continuity at $\theta = 0$

It can be shown that $\dim_\theta\{\,1/b : b \in \mathbb{N} \times \mathbb{Z}i\,\} = \frac{2\theta}{1+\theta}$, so the intermediate dimensions of F_I are continuous at $\theta = 0$.

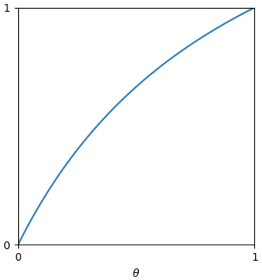

Figure: Intermediate dimensions of $\{\,1/b : b \in \mathbb{N} \times \mathbb{Z}i\,\}$

The intermediate dimensions of limit sets of general infinite CIFS may not be continuous at $\theta = 0$, for example when the fixed points are at $\{\,1/(\log n) : n \in \mathbb{N}\,\}$ and the contraction ratios are small enough for the Hausdorff dimension to be less than 1.

Fractional Brownian motion

- For $\alpha \in (0,1)$, index-α fractional Brownian motion is a stochastic process $B_\alpha \colon \mathbb{R}^d \to \mathbb{R}^d$ such that the increments $B_\alpha(x) - B_\alpha(y)$ are stationary and normally distributed with mean 0 and variance $||x-y||^\alpha$ for all $x, y \in \mathbb{R}^d$ (the increments are not independent unless $\alpha = 1/2$, which is usual Brownian motion).
- It follows from the continuity of the intermediate dimensions and a result of Burrell (2020) that
- for $I \subset \mathbb{N}$ and B_α fractional Brownian motion on $\mathbb{R}$, almost surely $\overline{\dim}_B B_\alpha(F_I) = 1$ if and only if $\alpha \leq \dim_{\mathrm{H}} F_I$.
- for $I \subseteq \mathbb{N} \times \mathbb{Z}i$ and B_α fractional Brownian motion on $\mathbb{C} \simeq \mathbb{R}^2$, $\overline{\dim}_B B_\alpha(F_I) = 2$ if and only if $\alpha \leq (\dim_{\mathrm{H}} F_I)/2$.
- This is an example of how the intermediate dimensions can be used to give non-trivial information about the box dimensions of sets.

Application: Hölder images

- If $f\colon F \to \mathbb{R}^d$ is an α-Hölder map for some $\alpha \in (0,1]$, meaning that there exists $c > 0$ such that $||f(x) - f(y)|| \leq c||x - y||^\alpha$ for all $x, y \in F$, then

$$\overline{\dim}_\theta f(F) \leq \alpha^{-1} \overline{\dim}_\theta F$$

- If $f\colon F \to G$ is α-Hölder with $f(F) \supseteq G$ then

$$\alpha \leq \frac{\overline{\dim}_\theta F}{\overline{\dim}_\theta G}.$$

- Applying this with the continued fraction sets $F = F_I$ and $G = F_J$ for appropriate $I, J \subset \mathbb{N}$ shows $\theta \in (0,1)$ can give better information than the Hausdorff or box dimensions.

Hölder images

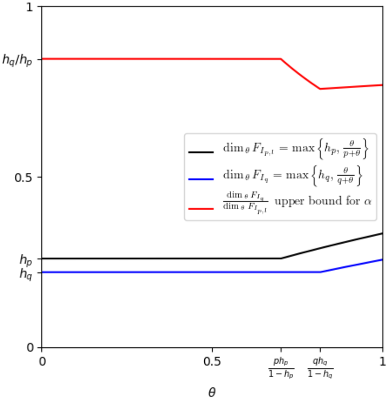

Figure: The black curve is the graph of the intermediate dimensions of the continued fraction set F_I, where $I = \{n^2, (n+1)^2, (n+2)^2, \ldots\}$ for some fixed large $n \in \mathbb{N}$. The red curve is the upper bound for α.

Thank you for listening!

Questions?