

Design and Analysis of Algorithm

Basics of Complexity Theory

- 1 Decision Problem
- 2 Deterministic Computation
- 3 Several Important Complexity Classes
 - \mathcal{P} vs. \mathcal{NP}
 - \mathcal{NP} -complete
 - $\text{co-}\mathcal{NP}$
- 4 Randomized Computation
 - \mathcal{BPP}
 - PSPACE
- 5 Decision vs. Search
- 6 Impact on Cryptography

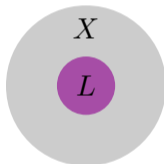
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Decision Problem

Decision Problem: recognition of a set of strings $L \subseteq X$

- X : a set of strings
- x : a string in X (each string corresponds to an instance)
- L : language (a subset of X satisfying some property)

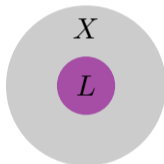


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Example

- $X = \mathbb{N}$
- L are Primes = $\{2, 3, 5, 7, 11, 13, \dots\}$
- decide if x is a prime.

Motivation for Complexity Theory

We always want to know if a given problem can be *efficiently* solved by an algorithm.

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We always want to know if a given problem can be *efficiently* solved by an algorithm.

- ① Precisely model algorithms
 - What is computation?
 - What is computable?
- ② Precisely define what does it means for efficient.

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Turing Machine

1936, London Mathematical Society: On computable numbers, with an application to the Entscheidungs Problem.

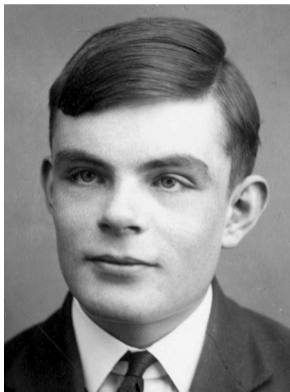
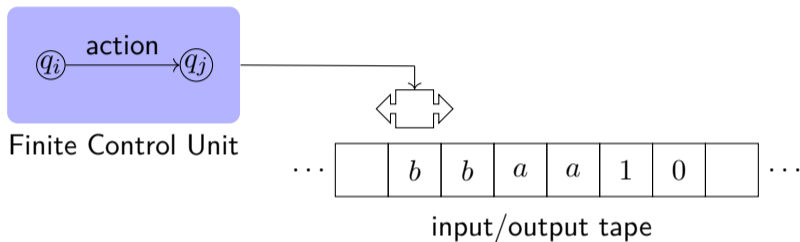


Figure: Alan Turing

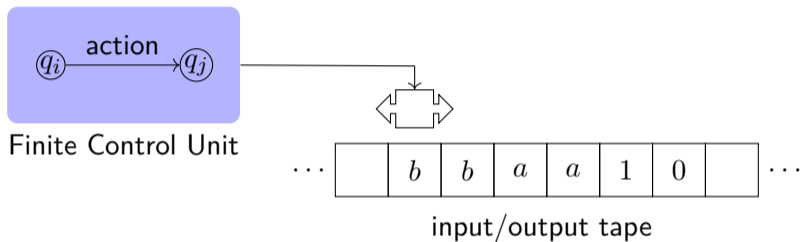
Turing Machine

Turing machine: automatic machine that has a tape (divided into infinite cells), a control unit and a read/write head.



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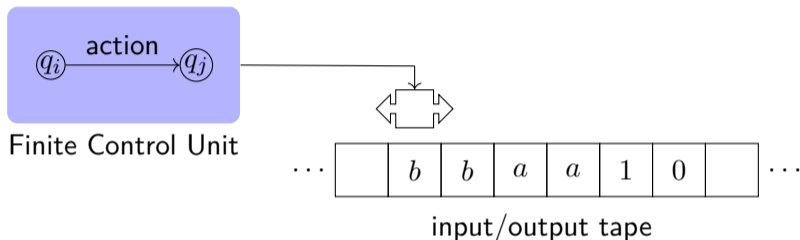
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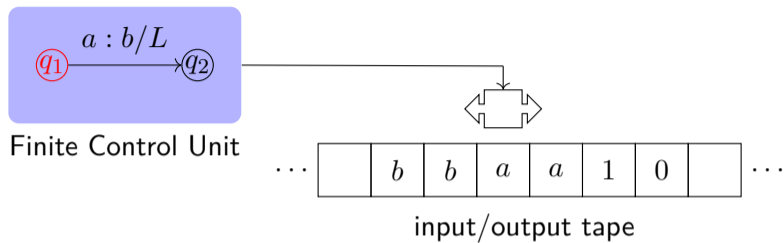
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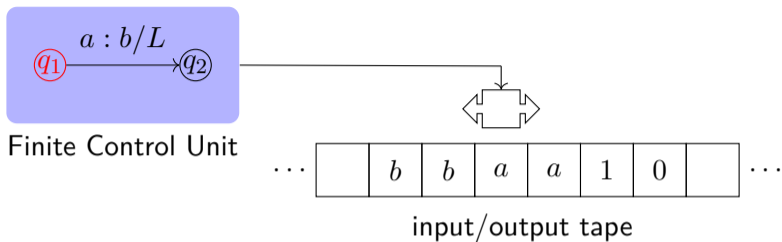


- At the beginning, the tape contains the input in several cells. Other places are empty.
- During computation, the control unit monitor current state and the head value, can do the following operations:
 - 1 wipe off old value and write new values
 - 2 change the current state
 - 3 move head left or right

An Example

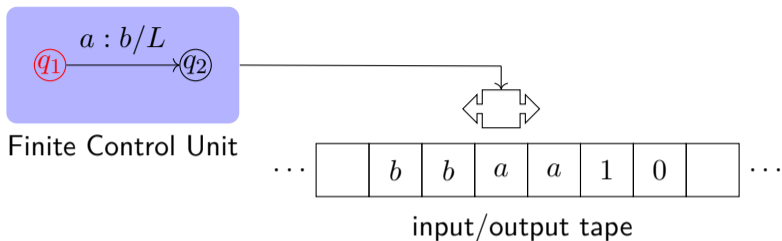


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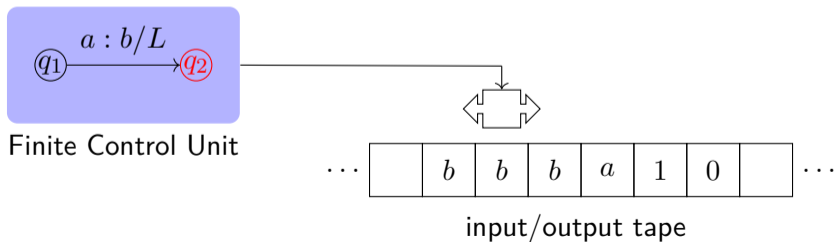


- $a \rightarrow b$; move left; current state $q_1 \rightarrow q_2$

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Mimic how human being solve a problem

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- TM has a finite number of states (memory)
- TM is provided a tape, which contains infinite cells (paper)
- a symbol can be scanned from a cell or printed to a cell (reading and writing)

Definition 1 (Turing Machine)

TM consists $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$

- Q : a finite set of states
- Σ : input alphabets
- Γ : working alphabets (including \perp , $\Sigma \subseteq \Gamma$)
- q_0 : the initial state of Q ;
- $q_{\text{acc}}, q_{\text{rej}}$: accept and reject state of Q
- δ : transition function

$$\delta : (Q \setminus \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma \rightarrow Q \times \Gamma \times \{L, R\}$$

Running Time of TM

Definition 2

We denote the running time of TM by $t_M(n)$, which is the maximum steps that TM runs on all inputs of length n

Polynomial Time

$$\bigcup_{k \in \mathbb{N}} \text{TIME}(n^k)$$

The Extended Church-Turing Thesis

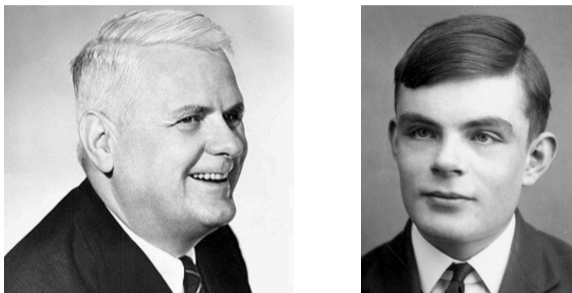


Figure: Alonzo Church & Alan Turing

Everyone's intuition of **Efficient** Algorithms = **Polynomial-Time** deterministic TMs

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Non-determinism doesn't give TM any power to recognize more languages.

- Any NDTM can be simulated by a TM (with potentially exponential time overhead) by trying all branches of the NDTM machine “in parallel” by using BFS.

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- TM has a working tape (好记性不如烂笔头)
- TM itself can be treated as data! TM can take another TM as its input.

Universal TM



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Next, we introduce two important sets of problems, characterized by time complexity by DTM and NDTM:

$$\mathcal{P} \text{ and } \mathcal{NP}$$

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Example of \mathcal{P} Languages

- $L = \{\text{even integers}\}$, M just need to check if the last bit is 0.
- $L = \text{PRIME}$, M is the AKS primality test algorithm.

Definition 4 (\mathcal{NP} Languages - Conventional)

$L \in \mathcal{NP}$ if there exists a non-deterministic poly-time TM M :

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Warning

\mathcal{NP} means non-deterministic poly-time, not **non-poly-time**!

Modern Definition

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$L \in \mathcal{NP}$ if there exists a deterministic poly-time TM M :

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Equivalence between traditional and modern definitions

- Even though M is a deterministic machine, its second argument w captures the nondeterminism in the definition.

Examples of \mathcal{NP} Language - Composites

$L = \text{COMPOSITE}$

- instance x is an integer
- a witness w for $x \in L$ is a non-trivial factor of x
- M just need to check if w divides x , which could be done in polynomial time.

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\mathcal{P} is closed under complement, i.e., $L \in \mathcal{P} \Rightarrow \bar{L} \in \mathcal{P}$.

- A deterministic TM can simply flip the accept/reject output at the end, and this does not change the running time.

Examples of \mathcal{NP} Language - SAT and 3-SAT

SAT: Given a CNF formula Φ , check if it has a satisfying truth assignment.

3-SAT: SAT where each clause contains exactly 3 literals

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Example of 3-SAT

- instance $\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$
- witness: $x_1 = 1, x_2 = 1, x_3 = 0, x_4 = 0$

Examples of \mathcal{NP} Language - Hamilton Path

Hamilton Graph: Given an undirected graph $G = (V, E)$, does there exist a simple path that visits every node?

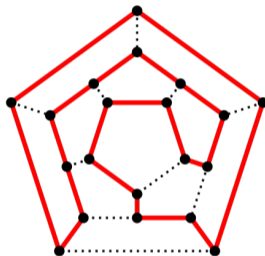


Figure: Hamiltonian Graph (a path traverses through each vertex exactly once)

witness: a path

M check if the path contains each node in V exactly once

\mathcal{P} vs. \mathcal{NP}

As per definition, $\mathcal{P} \subseteq \mathcal{NP}$. Because $L \in \mathcal{P} \Rightarrow L \in \mathcal{NP}$:

- $M'(x, w)$ can always sets $w = \perp$ and decide whether $x \in L$ using M .
- Alternatively, “short” M can be viewed as a witness for $x \in L$. Think about why the description of M is short?

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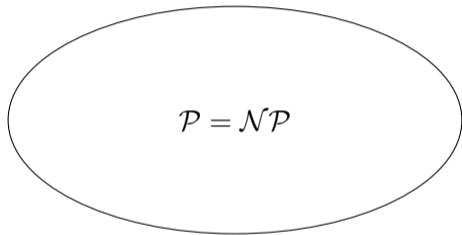
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1971: Cook, Edmonds, Levin, Yablonski, Gödel

Perhaps the most prominent question in TCS:

$$\mathcal{P} \stackrel{?}{=} \mathcal{NP}$$

$$\mathcal{P} = \mathcal{NP}$$



If $\mathcal{P} = \mathcal{NP}$

The foundation of modern cryptography collapse!



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In principle, every aspect of life could be efficiently and globally optimized ...
... life as we know it would be different!

The Consequence of $\mathcal{P} = \mathcal{NP}$

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Define a collection of languages $L_i = \{(y, z) | \exists w \text{ s.t. } y = f(z || w)\}$, where $z \in \{0, 1\}^i$, $w \in \{0, 1\}^{n-i}$

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Algorithm 4: Invert(y)

```
1:  $z = \epsilon$ ;  
2: for  $i \leftarrow 1$  to  $n$  do  
3:   if  $(y, z || 0) \in L_i$  then  $z = z || 0$ ;  
4:   else  $z = z || 1$ ;  
5: end  
6: return  $z$ 
```

The Reverse Direction

OWF exists $\Rightarrow \mathcal{P} \neq \mathcal{NP}$

- We have many candidates of OWFs, but they require assumptions.

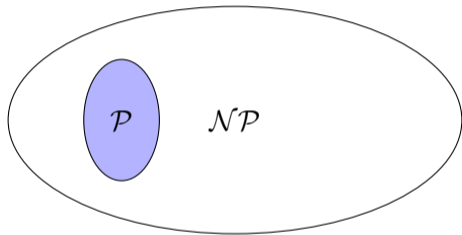
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Warning

OWFs do not exist *does not imply* $\mathcal{P} = \mathcal{NP}$



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Conjecture: $\underbrace{\text{No poly-time algorithm for 3-SAT}}_{\text{intractable}}$

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Motivation of Reduction

\mathcal{NP} is the set of many problems.

How to figure out the relations among them?

A central approach is finding reductions

Polynomial Time Reducibility

Language L' is *poly-time reducible* or *reduces* to language L , written as $L' \leq_p L$, if there is a deterministic poly-time function $\mathcal{R} : L' \rightarrow L$ so that:

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We should pay attention to:

- the direction of \mathcal{R}
- the time complexity of \mathcal{R}

Definition 6 (\mathcal{NP} -Hard)

L is said to be \mathcal{NP} -hard if for every \mathcal{NP} -language L' , there is a deterministic poly-time algorithm (a reduction) \mathcal{R} :

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Fact: languages in \mathcal{NP} -hard may **not** fall in \mathcal{NP} .

Definition 7 (\mathcal{NP} -Complete)

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Definition Intuition: \mathcal{NP} -complete represents the set of hardest problems in \mathcal{NP} .

- We can solve all problems in \mathcal{NP} if we find an efficient algorithm for any problems in \mathcal{NP} -complete.

The Meaning of \mathcal{NP} -complete

Theorem 8

Suppose $Y \in \mathcal{NP}$ -complete, then $Y \in \mathcal{P} \iff \mathcal{P} = \mathcal{NP}$.

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- This theorem essentially states that if $\mathcal{P} \cap \mathcal{NPC}$ is non-empty iff $\mathcal{P} = \mathcal{NP}$.

\mathcal{P} vs. \mathcal{NP} revisited

Overwhelming consensus (still): $\mathcal{P} \neq \mathcal{NP}$.

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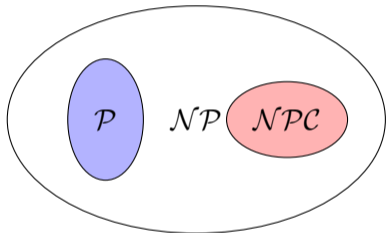


Figure: $\mathcal{P} \neq \mathcal{NP}$

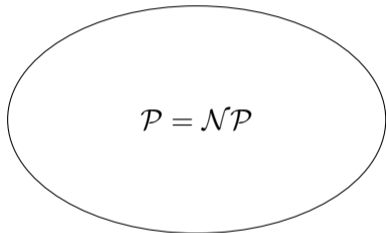


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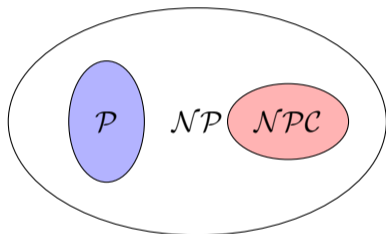


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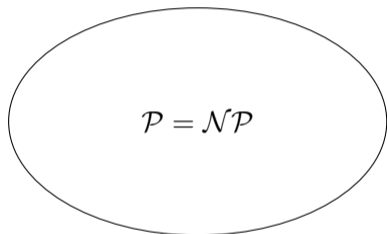
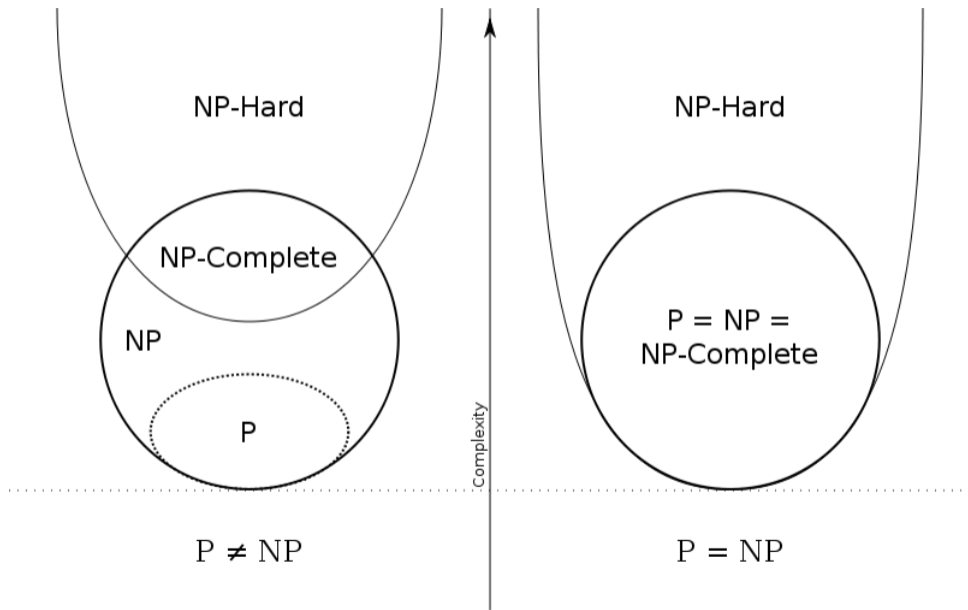


Figure: $\mathcal{P} = \mathcal{NP}$

Why we believe $\mathcal{P} \neq \mathcal{NP}$? Because some problems appear significantly harder.



Cook-Levin Theorem

Do there exist “natural” \mathcal{NP} -complete problems?

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Figure: Stephen Cook & Leonid Levin

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- Other examples of \mathcal{NP} -complete including graph 3-colorability, graph Hamiltonicity, and so on.

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Motivation for $\text{co-}\mathcal{NP}$

Asymmetry of \mathcal{NP} : We need short certificates only for yes instances.

$$x \in L \iff \exists w \in \{0, 1\}^{\text{poly}(|x|)} \text{ s.t. } M(x, w) = 1$$

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HAM-CYCLE vs. NO-HAM-CYCLE

- Can prove a graph is Hamiltonian by specifying a path.
- How could we prove that a graph is not Hamiltonian?

$\text{co-}\mathcal{NP}$

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- $L = \{4, 6, 8, 9, 10, 12, 14, 15, \dots\}$, $\overline{L} = \{2, 3, 5, 7, 11, 13, 17, 23, 29, \dots\}$

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Complements of decision problems in \mathcal{NP} .

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Examples: UN-SAT, NO-HAM-CYCLE, and PRIMES.

\mathcal{NP} vs. $\text{co-}\mathcal{NP}$

Fundamental open question: Does $\mathcal{NP} = \text{co-}\mathcal{NP}$?

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Theorem 10

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Proof idea.

- \mathcal{P} is closed under complementation.
- If $\mathcal{P} = \mathcal{NP}$, then \mathcal{NP} is closed under complementation. In other words, $\mathcal{NP} = \text{co-}\mathcal{NP}$.
- This is the contrapositive of the theorem.

$$\mathcal{NP} \cap \mathbf{co}\text{-}\mathcal{NP}$$

Good characterization: [Edmonds 1965] $\mathcal{NP} \cap \mathbf{co}\text{-}\mathcal{NP}$

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Provides conceptual leverage for reasoning about a problem.

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Motivation of Randomized Algorithm

TM models deterministic algorithms.

TM does not seem to capture one aspect of reality — the ability to make random choices during computation

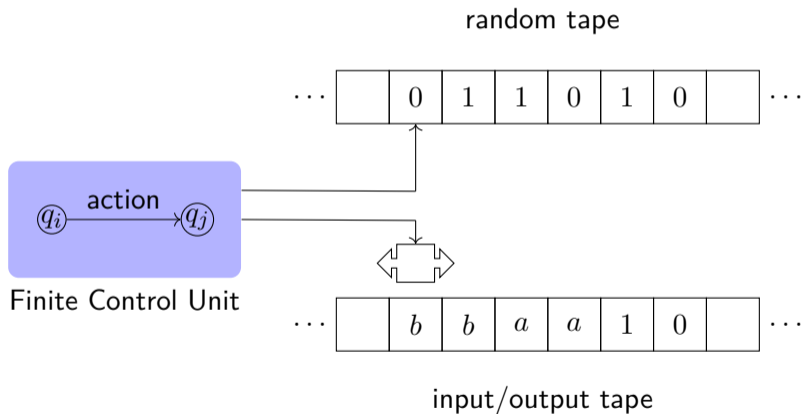
- Most programming languages provide a built-in RNG.

It makes sense to consider algorithms that can toss a coin, a.k.a. use a source of random bits. Such algorithms have been implicitly studied for a long time.

- estimate facts about a large sample by taking a small sample
- simulate real-world systems that are themselves probabilistic, such as nuclear fission and the stock market
- differential equations

Probabilistic Turing Machine

Probabilistic Polynomial-time TM models probabilistic algorithm.



PTM vs. NDTM

NDTM is a TM with two transition functions. PTM is syntactically similar.

The difference is in how we interpret the working of TM.

- In a PTM, each transition is taken with probability $1/2$, a computation that runs for time t gives rise 2^t branches in the graph of all computations, each of which is taken with probability $1/2^t$. $\Pr[M(x) = 1]$ is simply the *fraction* of branches that end with M outputting a 1.
- In a NDTM, $M(x) = 1$ iff there exists a branch that outputs 1

On a conceptual level, PTM and NDTM are very different

- PTM like TM and unlike NDTM, is intended to model realistic computation devices.

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Bounded-Error Probabilistic Polynomial Time

Definition 11 (\mathcal{BPP} Complexity)

$L \in \mathcal{BPP}$ iff there exists a probabilistic polynomial time TM M such that:

$$\forall x \in L : \Pr[M(x) = 1] \geq \frac{1}{2} + \epsilon$$

$$\forall x \notin L : \Pr[M(x) = 1] \leq \frac{1}{2} - \epsilon$$

Intuition: M is more likely correct than wrong.

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Key properties

- Bounded error: M must be noticeably more likely to be correct than incorrect, i.e., the error is bounded away from the random guess $1/2$ by a noticable gap.
- Error reduction: By repetition and then taking majority vote, the error can be reduced to 2^{-n} with only polynomial overhead.
- Practical meaning: \mathcal{BPP} captures efficient randomized algorithms used in practice (e.g., primality testing before AKS).

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Error Reduction

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However, the choice of gap ϵ could be arbitrary and the set \mathcal{BPP} will be unchanged.

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The method is running M independently several times, then taking the majority vote.

Chernoff Bound guarantees the probability that the majority of the runs are wrong drops off exponentially.

Chernoff Bound (Lower Tail): Let $\{X_i\}$ be independent 0-1 random variables such that $\Pr[X_i = 1] \geq 1/2 + \epsilon$, $X = \sum_{i=1}^m X_i$.

$$\Pr[X \neq b] \leq \frac{1}{e^{\epsilon^2 m/2}}$$

Setting $m \geq n^{2c} + 1$ ensures the error probability is negligible in n .

Why Does Soundness Amplification Not Work?

In the ZKP literature, the error we care about is soundness error:

$$\epsilon = \Pr[\langle P, V \rangle(x) : x \notin L]$$

We can reduce ϵ to ϵ^n by:

- Run the protocol n times in sequential/parallel
- Accept iff all of the repetitions accept

Such method does not work for \mathcal{BPP} , cause in ZKP the completeness error is typically 0, while in \mathcal{BPP} is error is two-sided.

- The decision algorithm does not know the input instance belongs to YES or NO. It has to apply the same strategy.

Probabilistic Polynomial Time

Definition 12 (\mathcal{PP} Complexity)

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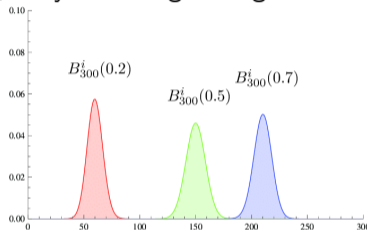
Key properties

- Unbounded error: The acceptance probability may exceed $1/2$ by an exponentially tiny amount.
- No error reduction: The gap away from $1/2$ may be too small to reduce efficiently.
- Extremely powerful: \mathcal{PP} contains many hard classes.

Intuition: M is just barely more likely correct than wrong.

Why Does Majority Vote Work?

Recall that $X = X_1 + \dots + X_m$ is a Binomial distribution. When ϵ is noticeable, after polynomial times of repetition, the two distributions induced by $x \in L$ and $x \notin L$ are largely detached. Otherwise, they are mingled together \leadsto hard to find a split line



This also explain for \mathcal{PP} , the majority vote does not work if ϵ is negligibly close to 0. For example:

$$x \in L \Rightarrow \Pr[M(x) = 1] \geq \frac{1}{2} + \frac{1}{2^n}$$

$$x \notin L \Rightarrow \Pr[M(x) = 1] \leq \frac{1}{2} - \frac{1}{2^n}$$

Containment

$$\mathcal{BPP} \subseteq \mathcal{PP}$$

- Any bounded-error algorithm is also an unbounded-error one.
 - The containment is believed to be strict.
-

The difference between n^{-c} vs 2^{-n} looks small numerically, but it is enormous computationally:

- \mathcal{BPP} requires a noticable probability advantage and models realistic randomized computation
- \mathcal{PP} allows an infinitesimal advantage over guessing, making it far more powerful and closer to counting complexity than to efficient computation.

About BPP

BPP is one of the largest practical class of problems, since problems in BPP have efficient probabilistic algorithms that can be run quickly on real modern machines.

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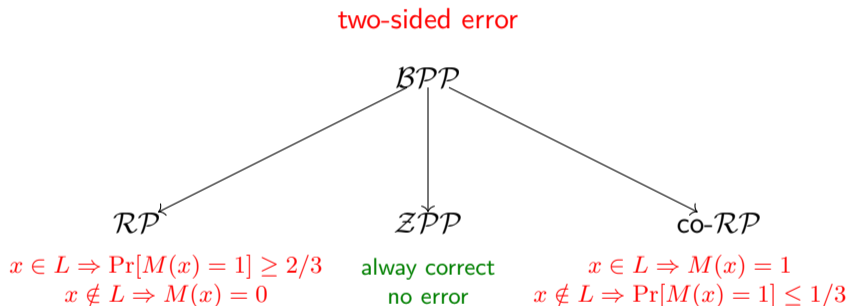
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[AKS04] gave a deterministic polynomial-time algorithm for PRIME, thus showing that it is in \mathcal{P} .

Gödel Prize and Fulkerson Prize

One-sided and Zero-sided Error

\mathcal{ZPP} : probabilistic polynomial-time TM always returns correct YES or NO answer, or halts with low probability, a.k.a. running time is polynomial **in expectation** for every input

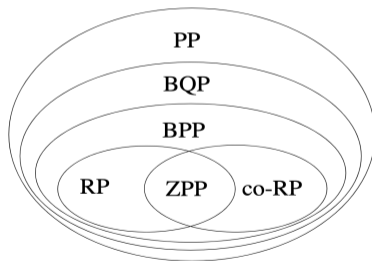


- \mathcal{BPP} : Monte Carlo algorithms (probabilistic) likely to be correct in strict polynomial running time
- \mathcal{ZPP} : Las Vegas algorithms (probabilistic) are always correct in expected polynomial running time

BPP in Relation to Other Probabilistic Complexity Classes

BQP (bounded-error quantum polynomial time): the class of decision problems solvable by a quantum TM in polynomial time with bounded error

- It is the quantum analogue of BPP



Limits of \mathcal{BPP}

Consensus: $\mathcal{P} \subseteq \underline{\mathcal{ZPP} = \mathcal{RP} \cap \text{co-}\mathcal{RP}} \subseteq \mathcal{BPP} \subseteq \mathcal{NP}$

$\mathcal{P} \subseteq \mathcal{BPP}$

- An important example of a problem in \mathcal{BPP} still not known to be in \mathcal{P} is **polynomial identity testing** — determining whether a polynomial is identically equal to the zero polynomial, when you have access to the value of the polynomial for any given input, but not to the coefficients.

$\mathcal{BPP} \subseteq \mathcal{NP}$

- Adleman's theorem: $\mathcal{BPP} \subseteq P/\text{poly}$ (polynomial-size Boolean circuits)
- Karp-Levin theorem: $\mathcal{NP} \subseteq P/\text{poly} \Rightarrow \text{PH} = \sum_2^P$

Thus, $\mathcal{NP} \subseteq \mathcal{BPP}$ will imply collapse of PH, which is unlikely to be true. In other words, \nexists bounded-error probabilistic algorithms for \mathcal{NPC} problems.

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PSPACE

\mathcal{P} : Decision problems solvable by DTM in polynomial time.

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PSPACE Example

- Binary counter. Count from 0 to $2^n - 1$ in binary.
- Algorithm. Use n bit odometer.

Observation $\mathcal{P} \subseteq \text{PSPACE}$

- This is because poly-time algorithm can consume only polynomial space.

Relation Between NP and PSPACE

Claim: $3\text{-SAT} \in \text{PSPACE}$.

Proof of claim

- Enumerate all 2^n possible truth assignments using counter.
- Check each assignment to see if it satisfies all clauses.

Theorem: $\mathcal{NP} \subseteq \text{PSPACE}$

- Consider arbitrary problem $Y \in \mathcal{NP}$.
- Since $Y \leq_p 3\text{-SAT}$, there exists algorithm that solves Y in poly-time plus polynomial number of calls to 3-SAT black box.
- Thus, Y can be solved by DTM in poly-space.

It is easy to verify that $\text{co-}\mathcal{NP} \subseteq \text{PSPACE}$.

The Merit of 3-SAT

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But for 3-SAT, given an instance x , we know the exact length of its witness.

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A: For \mathcal{NP} languages. Yes!

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$$L = \{(\phi, r) : \exists w \text{ s.t. } \phi(w) = 1 \wedge w < r\}$$

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Conclusion: decision 3-SAT \approx_{hard} search 3-SAT

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Languages Defined by Cryptographic Problems

Example based on DDH w.r.t. $(\mathbb{G}, p, g_1, g_2)$

- $X = \mathbb{G} \times \mathbb{G}$;
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Remark

Cryptographic problems assume average-case complexity (for $x \xleftarrow{R} X$), while problems in computer science consider worst-case complexity.

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Remark

Proving the existence of hard-on-average problems in \mathcal{NP} using the $\mathcal{P} \neq \mathcal{NP}$ assumption is a major open problem.

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Leading candidate: Lattice problems (enjoy both)

Reference I



Manindra Agrawal, Neeraj Kayal, and Nitin Saxena.

Primes is in P .

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