Design and Analysis of Algorithm
Divide and Conquer (I)

Outline

Divide-and-Conquer Paradigm

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   - when the subproblems are so small, they are solved outright
3. Combine: compose solutions to subproblems into overall solution
   - coordinated by the algorithm's core recursive structure

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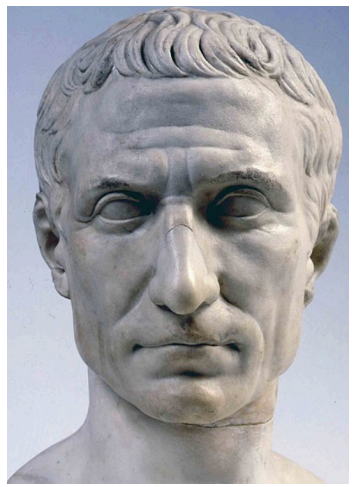
particularly applicable in parallel computing environment (will be more efficient)

Origin of Divide-and-Conquer: Western

Divide and rule (Latin: divide et impera), or divide and conquer, in politics and sociology is gaining and maintaining power by breaking up larger concentrations of power into pieces that individually have less power than the one implementing the strategy.


- The maxim divide et impera has been attributed to Philip II of Macedon. It was utilised by the Roman ruler Julius Caesar and the French emperor Napoleon.

Origin of Divide-and-Conquer: Eastern

故用兵之法，十则围之，五则攻之，倍则战之，敌则能分之，...

—《孙子兵法》

Figure: 秦王扫六合时，虎视何雄哉

General Divide-and-Conquer Algorithm

Algorithm 1: Divide-and-Conquer(P)

1: **if** $|P| \leq s^*$ **then** Solve(P); // direct solve
2: **else** divide P into $P_1, P_2, \ldots, P_k$; // divide
3: **for** $i \leftarrow 1$ *to* k **do**
4: $\quad y_i \leftarrow$ Divide-and-Conquer(P_i) // solve subproblems
5: **end**
6: **return** Merge($y_1, y_2, \ldots, y_k$) // combine answers

Complexity of Divide-and-Conquer

Recurrence relation:

$$\begin{cases} T(n) = T(|P_1|) + T(|P_2|) + \cdots + T(|P_k|) + f(n) \\ T(s^*) = C \end{cases}$$

- $P_1, P_2, \ldots, P_k$ are subproblems after dividing
- $f(n)$ is the complexity of dividing subproblems and combining answers of subproblems to answer to the original problem
- C is the complexity of the smallest subproblem of size s^*

Next, we introduce two canonical types of recurrence relations.

Case 1: Subproblems Reduce Size by a Constant

$$T(n) = \sum_{i=1}^{k} a_i T(n-i) + f(n)$$

Solving method

1. Iteration (direct iteration or simplify-then-iteration)
2. Recursive tree

Example. Hanoi tower: $T(n) = 2T(n-1) + 1$

Case 2: Subproblems Reduce Size Linearly

$$T(n) = aT\left(\frac{n}{b}\right) + f(n), h(n) = n^{\log_b a}$$

Solving method: recursion tree, master theorem

$$T(n) = \begin{cases} \Theta(h(n)) & \text{if } f(n) = o(h(n)) \\ \Theta(h(n)\log n) & \text{if } f(n) = \Theta(h(n)) \\ \Theta(f(n)) & \text{if } f(n) = \omega(h(n)) \\ & \wedge \exists\ r < 1 \text{ s.t. } af(n/b) < rf(n) \end{cases}$$

Example 1. Binary search: $W(n) = W(n/2) + 1$

Example 2. Merge sort: $W(n) = 2W(n/2) + (n-1)$

In this section, we illustrate the main idea of divide-and-conquer by several introductory examples.

Hanoi Tower

Algorithm 2: Hanoi(A, C, n) // n disk from A to C

Input: $A(n), B(0), C(0)$
Output: $A(0), B(0), C(n)$

1: **if** $n = 1$ **then** move (A, C); //one disk from A to C
2: **else**
3: Hanoi$(A, B, n-1)$;
4: move (A, C);
5: Hanoi$(B, C, n-1)$
6: **end**

Complexity of Hanoi Tower

1. Reduce the original problem to two subproblem of size $n-1$
2. Continue to reduce until the size of subproblem is 1
3. From input size 1 to $n-1$, combine the answers until the size go back to n.

Let $T(n)$ be the complexity of moving n disks: the minimum number of moves required

$$\left.\begin{array}{c} T(n) = 2T(n-1)+1 \\ T(1) = 1 \end{array}\right\} \Rightarrow T(n) = 2^n - 1$$

There is no worst-case, best-case, average-case distinctions for this problem, since the input only depend on the input size.

Binary Search

Algorithm 3: BinarySearch(A, l, r, x)

Input: sorted $A[l, r]$ in ascending order, target x
Output: j // if $x \in T$, j is the index, else $j = 0$

```
1:  if l = r then                                   //the smallest subproblem
2:      if x = A[l] then return l;
3:      else return 0;
4:  end
5:  m ← ⌊(l + r)/2⌋          // m is the middle position;
6:  if x ≤ A[m] then                                //compare to median
7:      BinarySearch(A, l, m, x)
8:  end
9:  else
10:     BinarySearch(A, m + 1, r, x)
11: end
```

Complexity of Binary Search

1. Reduce the original problem to a subproblem with half size by comparing x with the median:
   - if $x \leq A[m]$, then $A[l, r] := A[l, m]$, else $A[l, r] := A[m + 1, r]$
2. Repeatedly search T until its size becomes 1, i.e. $l = r$
   - At this point, directly compare x and $A[l]$, return l if equal and "0" otherwise.

Worst-case complexity of binary search

$$h(n) = 1,\ f(n) = \Theta(h(n)) \Rightarrow \text{master theorem (case 2)}$$

$$\left.\begin{array}{c} W(n) = W(\lceil n/2 \rceil) + 1 \\ W(1) = 1 \end{array}\right\} \Rightarrow W(n) = \Theta(\log n)$$

Example of MergeSort

Algorithm 4: MergeSort(A, n)

Input: unsorted $A[n]$

Output: sorted $A[n]$ in ascending order

1: $l \leftarrow 1$, $r \leftarrow n$;
2: **if** $l < r$ **then**
3: $\quad m \leftarrow \lfloor (l + r)/2 \rfloor$ //partition by half;
4: $\quad$ MergeSort(A, l, m) // subproblem 1;
5: $\quad$ MergeSort$(A, m + 1, r)$ // subproblem 2;
6: $\quad$ Merge$(A[l, m], A[m + 1, r])$ // merge sorted sub-array
7: **end**

How to implement Merge *recursively?*

Recursive Merge Algorithm

Algorithm 5: Merge$(A[1,k], B[1,l])$

1: **if** $k = 0$ **then return** $B[1,l]$;
2: **if** $l = 0$ **then return** $A[1,k]$;
3: **if** $A[1] \leq B[1]$ **then return** $A[1] \circ$ Merge$(A[2,k], B[1,l])$;
4: **else return** $B[1] \circ$ Merge$(A[1,k], B[2,l])$;

- The Merge procedure does a constant amount of work per recursive call, for a total running time of $O(k+l)$.

Complexity of MergeSort

1. Partition the original problem to 2 subproblem of size $n/2$
2. Continue the partition step until the size of subproblem is 1
3. From input size 1 to $n/2$, merge two neighbored sorted sub-array.
   - The size of sub-array doubles after each merge, until reach the original size.

Assume $n = 2^k$, the worse-case complexity of MergeSort is:

$$h(n) = n,\ f(n) = \Theta(h(n)) \Rightarrow \text{master theorem (case 2)}$$

$$\left.\begin{array}{c} W(n) = 2W(n/2) + n - 1 \\ W(1) = 0 \end{array}\right\} \Rightarrow W(n) = \Theta(n \log n)$$

Recap of MergeSort

The dividing of subproblems is already done thanks to data structure of array, all the real work need to be done is merging.

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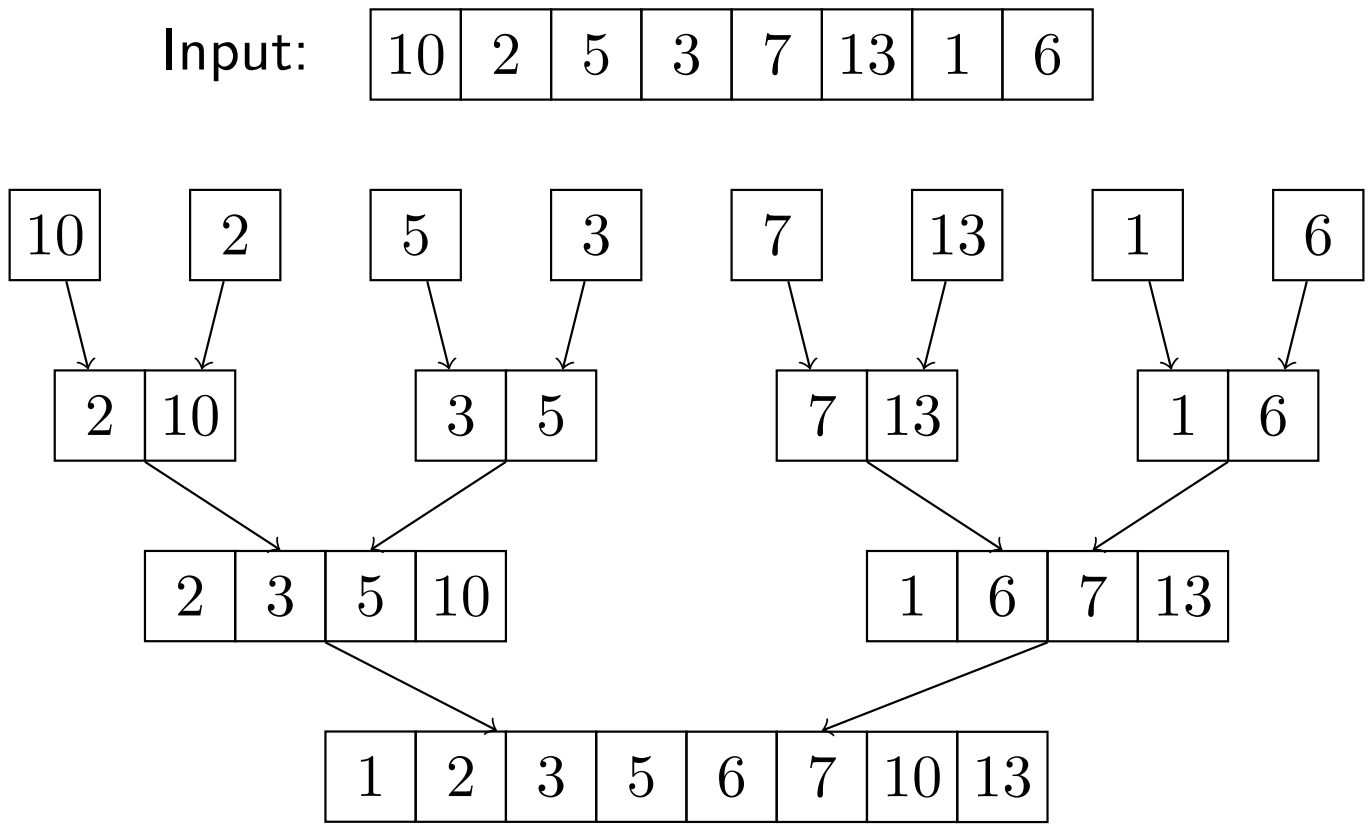


Recap

We exemplify the features of divide-and-conquer algorithm:

- Divide the original problem to independent subproblems with smaller size
  - the subproblem and the original problem are of the same type
  - when the subproblems are sufficiently small, they can be solved outright
- The algorithm can be solved recursively or iteratively
- Complexity analysis: solving recurrence relation

Outline

Basic Idea

1. Choose the first element x as pivot, partition A into two sub-array:
   - low sub-array A_L: elements less than x
   - high sub-array A_R: elements greater than x
   - x is at the right position

2. recursively sort A_L and A_R, until the size of sub-array is 1

Pseudocode of QuickSort

Algorithm 6: QuickSort(A, l, r)

Input: $A[l \dots r]$

Output: sorted A in ascending order

1: **if** $l = r$ **then return**; //reach the smallest case
2: **if** $l < r$ **then**
3: $k \leftarrow \text{Partition}(A, l, r)$;
4: $A[l] \leftrightarrow A[k]$;
5: $\text{QuickSort}(A, l, k-1)$;
6: $\text{QuickSort}(A, k+1, r)$;
7: **end**

Pseudocode of Partition

Algorithm 7: Partition(A, l, r)

1: $x \leftarrow A[l]$ //set the first element as pivot;
2: $i \leftarrow l$, $j \leftarrow r + 1$ //initialize left/right pointer;
3: **while** *true* **do**
4: **repeat** $j \leftarrow j - 1$ **until** $A[j] \leq x$; //less than x
5: **repeat** $i \leftarrow i + 1$ **until** $A[i] > x$; //greater than x
6: **if** $i < j$ **then** $A[i] \leftrightarrow A[j]$;
7: **else return** j; //cross happen, find the position
8: **end**

Demo of Partition

27	99	0	8	13	64	86	16	7	10	88	25	90
	i										j	
27	25	0	8	13	64	86	16	7	10	88	99	90
					i				j			
27	25	0	8	13	10	86	16	7	64	88	99	90
						i		j				
27	25	0	8	13	10	7	16	86	64	88	99	90
							i	j				
16	25	0	8	13	10	7	27	86	64	88	99	90

Complexity Analysis

Worst-case:

$$\left.\begin{array}{c} W(n) = W(n-1) + n - 1 \\ W(1) = 0 \end{array}\right\} \Rightarrow W(n) = n(n-1)/2$$

Best-case:

$$\left.\begin{array}{c} T(n) = 2T(n/2) + n - 1 \\ T(1) = 0 \end{array}\right\} \Rightarrow T(n) = \Theta(n \log n)$$

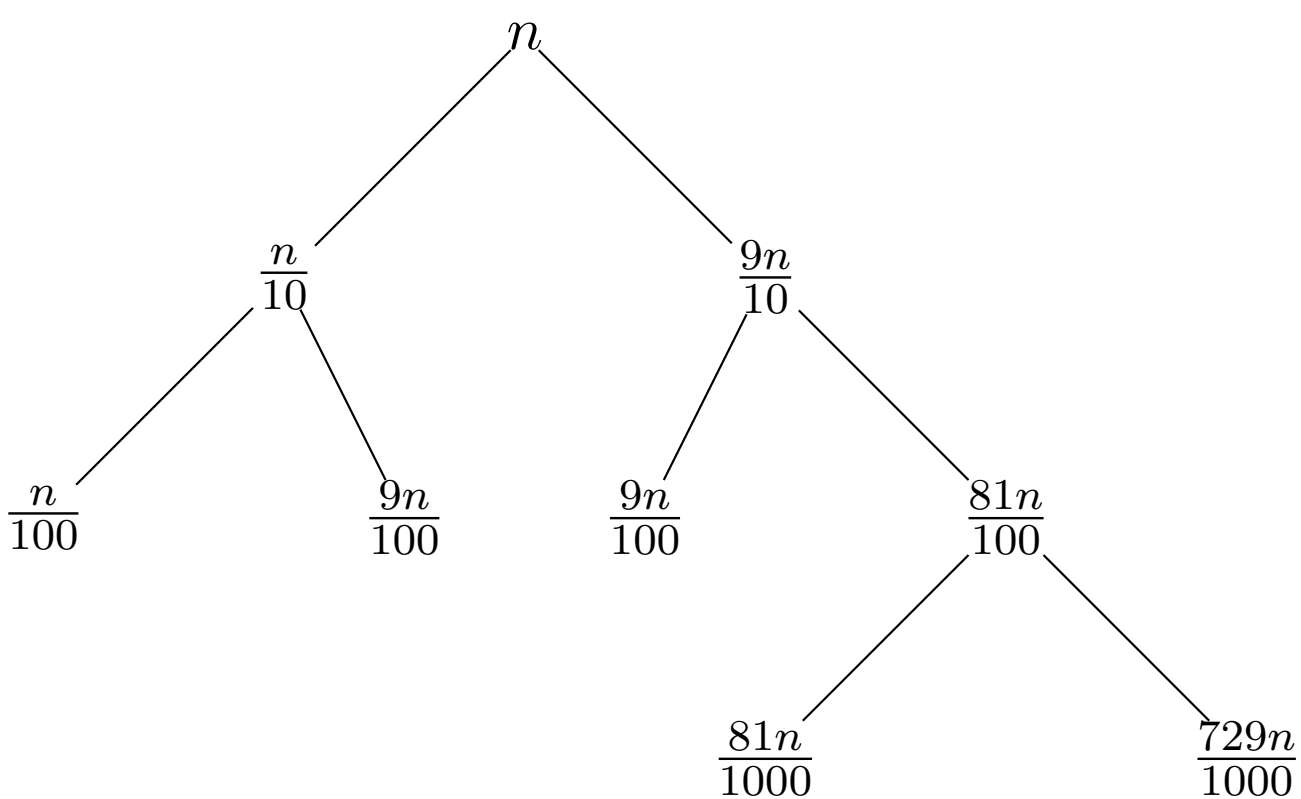
Complexity of Constant Partition

Constant Partition. The ratio of subproblems vs. original problem is a fixed constant, such as $1 : 9$.

$$\begin{cases} T(n) = T(n/10) + T(9n/10) + n \\ T(1) = 0 \end{cases}$$

Solving via recursion tree $\Rightarrow T(n) = \Theta(n \log n)$

Recursion Tree



$\cdots$

$O(n)$

$$T(n) = \Theta(n \log n)$$

Average-Case Complexity

Suppose the first element finally appear at position $1, 2, \ldots, n$ with equal probability, i.e., $1/n$. We analyze the size of resulting subproblems

- appear at position 1: $T(0)$, $T(n-1)$
- appear at position 2: $T(1)$, $T(n-2)$
- $\ldots$
- appear at position $n-1$: $T(n-2)$, $T(1)$
- appear at position n: $T(n-1)$, $T(0)$

The cost of all subproblems: $2(T(1) + T(2) + \cdots + T(n-1))$

The cost of partition: $n-1$ compares

Average-Case Complexity

$$T(n) = \frac{1}{n}\sum_{k=1}^{n-1}(T(k) + T(n-k)) + n - 1$$

$$T(n) = \frac{2}{n}\sum_{k=1}^{n-1} T(k) + n - 1$$

$$T(1) = 0$$

We simplify the recurrence relation via subtraction $\Rightarrow$

$$T(n) = \Theta(n \log n)$$

See pp. 47 on Lec 3 if you cannot remember it.

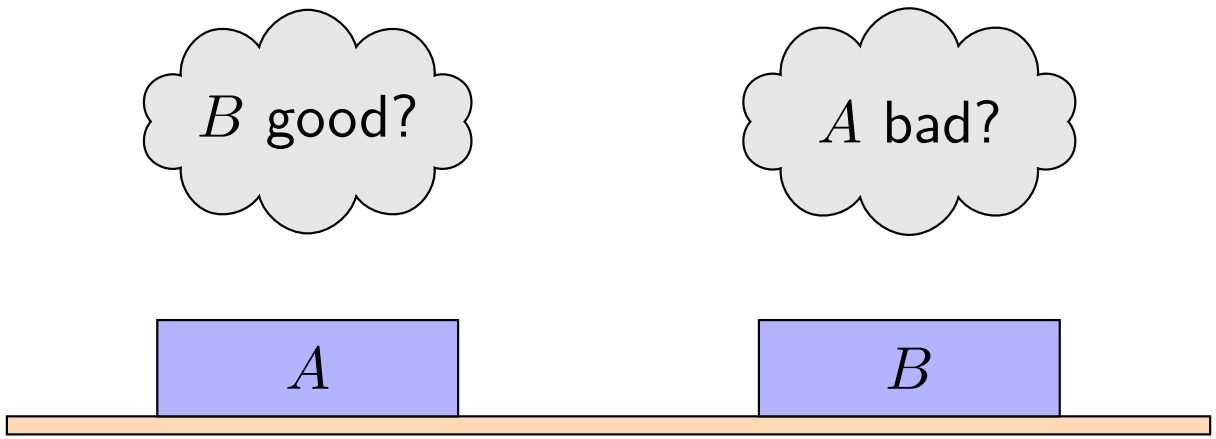
Outline

Chip Test

Chip factory only admits basic test method.

Basic test method. Put two chips A and B on the testbed, begin the mutual test

- the test report is "good" or "bad"



Assumption. The report from good chip is always correct, but the report from bad chip is non-deterministic (probably wrong)

Analysis of Test Report

A's report	B's report	Conclusion
B is good	A is good	A, B are both good or bad
B is good	A is bad	at least one is bad
B is bad	A is good	at least one is bad
B is bad	A is bad	at least one is bad

Problem of Chip Test

Input. n chips, $\#(\text{good}) - \#(\text{bad}) \geq 1$

Question. Devise a test method to choose one good chip from n chips

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Starting point. Given a chip A, how to check if A is good or bad

Method. Using other $n-1$ chip to test A.

- Idea: utilize the odevity of n

Case 1: n is Odd

Example. $n = 7$, #(good chips) ≥ 4.

- A is good $\Leftrightarrow$ at least 3 among 6 reports "good"
- A is bad $\Leftrightarrow$ at least 4 among 6 reports "bad"

Generalize to n is odd, #(good chips) $\geq (n+1)/2$.

- A is good: $\Leftrightarrow$ at least $(n-1)/2$ reports "good"
- A is bad: $\Leftrightarrow$ at least $(n+1)/2$ reports "bad"

Key observation. The test result is of if and only if flavor. Thus, it constitutes a necessary and sufficient condition.

Criteria: in $n-1$ reports

- at least one half reports "good" $\Rightarrow$ A is good
- more than one half reports "bad" $\Rightarrow$ A is bad

Case 2: n is Even

Example. $n = 8$, #(good chips) ≥ 5.

- A is good $\Leftrightarrow$ at least 4 from 7 report "good"
- A is bad $\Leftrightarrow$ at least 5 from 7 report "bad"

Generalize to n is even, #(good chips) $\geq n/2 + 1$.

- A is good $\Leftrightarrow$ at least $n/2$ report "good"
- A is bad $\Leftrightarrow$ at least $n/2 + 1$ report "bad"

Key observation. The test result is also of if and only if flavor. Thus, it constitutes a necessary and sufficient condition.

Criteria: in $n - 1$ reports

- at least one half reports "good" $\Rightarrow$ A is good
- more than one half reports "bad" $\Rightarrow$ A is bad

Brute Force Algorithm

Test method. Randomly pick a chip, apply the aforementioned test. If it is good, then the test is over. Else, discard it and randomly pick another chip from the rest, until get a good chip.

- correctness: #(good chips) is always more than half.

Time complexity

- 1-st round: random one is bad, at most $n-1$ time tests
- 2-rd round: random one is bad, at most $n-2$ time tests
- ...
- i-th round: random one is bad, at most $n-i$ time tests

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The overall complexity in the worst-case is $\Theta(n^2)$

Optimizations

Nice discovery by [2020 江锴杰]: in $i > 1$ round, we can randomly discard one chip then test $\rightsquigarrow$ requiring at most $n - 1 - 2i$ time tests

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However, the above tricks do not change the worst-case complexity.

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Assume n is even, divide n chips into two groups and begin mutual test; the rest chips form a subproblem and begin the next round test

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- "good, good" $\rightsquigarrow$ pick a random one into the next round
- other cases $\rightsquigarrow$ discard them all

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- other cases $\rightsquigarrow$ discard them all

The end condition of recursion: $n \leq 3$

- 3 chips: one test suffices (think why? retain the same property as the original problem)
  1. "good, good": randomly pick one and output it
  2. "good, bad": output the rest one
  3. "bad, bad": output the rest one
- 1 or 2 chips: both are good, no more test is needed

Correctness of Divide-and-Conquer Algorithm

Claim. When n is even, after one round of test, in the rest chips, #(good chips) - #(bad chips) ≥ 1

Proof. Consider the following three cases:

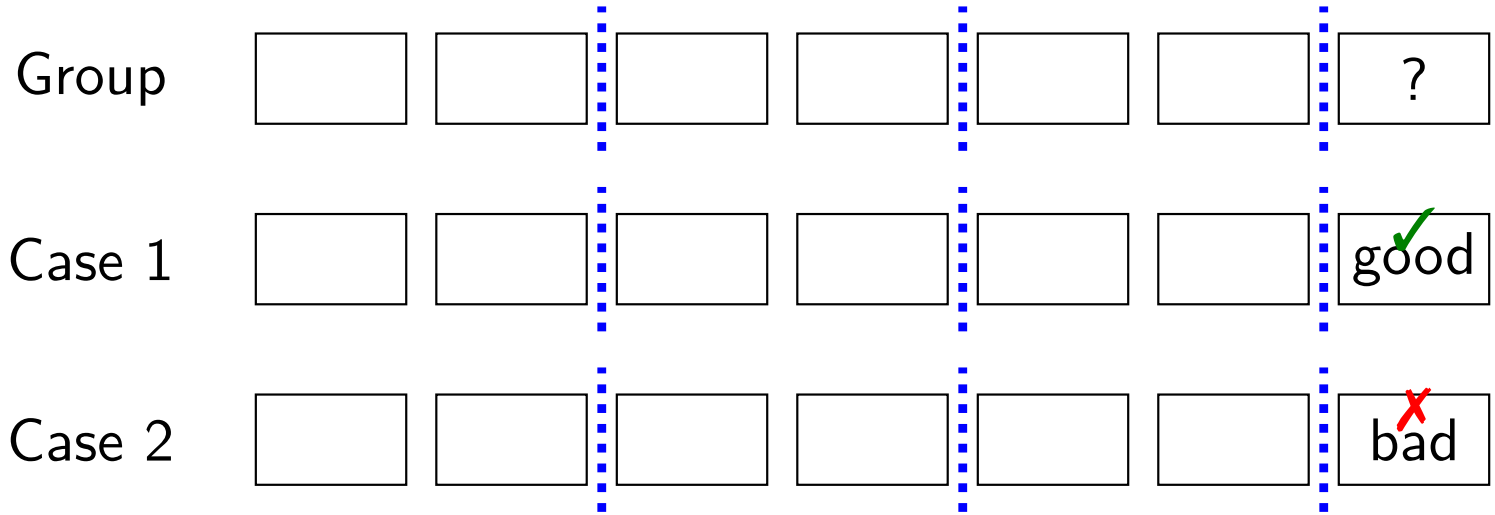
1. Both are good (i groups) $\rightsquigarrow$ keep a random one
2. One is good, one is bad (j groups) $\rightsquigarrow$ discard them all
3. Both are bad (k groups) $\rightsquigarrow$ keep a random one or discard them all

After one round test, #(good chips) $= i$, #(bad chips) $\leq k$

$$\left.\begin{cases} 2i + 2j + 2k = n & \#\text{(chips) before test} \\ 2i + j > 2k + j & \#\text{(good chips)} > \#\text{(bad chips)} \end{cases}\right\} \Rightarrow i > k$$

Adjust when n is Odd

When n is odd, there would one chip left without group member.



Adjustment. When n is odd, add one-round direct test for the ungrouped chip

- if it is good, the algorithm is over
- else, discard it and enter into the next round (since $n-1$ chips satisfying original property)

Pseudocode

Algorithm 8: ChipTest(n)

1: **if** $n = 3$ **then** //smallest case
2: randomly pick 2 chips;
3: **if** *both are good* **then return** a random one;
4: **else return** the rest one;
5: **end**
6: **if** n =*2 or 1* **then return** a random one;
7: ---
8: divide into $\lfloor n/2 \rfloor$ groups; // adjust when n is odd
9: **for** $i = 1$ *to* $\lfloor n/2 \rfloor$ **do**
10: **if** *both are good* **then** keep a random one;
11: **else** discard both of them;
12: **end**
13: $n \leftarrow$ #(rest chips);
14: ChipTest(n);

Complexity Analysis

For input size n, after each round test, the number of chips reduces at least by half

- #(test) (include addition adjustment when n is odd): $\Theta(n)$

Recurrence relation

$$\left.\begin{array}{c} W(n) = W(n/2) + \Theta(n) \\ W(3) = 1, W(2) = W(1) = 0 \end{array}\right\} \Rightarrow W(n) = \Theta(n)$$

Summary of Divide-and-Conquer chip test algorithm

- Adjustment $\rightsquigarrow$ guarantee the subproblem is of the same type as the original problem
- branching factor $a = 1$ & dividing-merging cost $f(n) = \Theta(n)$ $\rightsquigarrow$ ensure remarkable efficiency improvement over brute-force algorithm

Outline

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General Selection Problem

Selection. Given n elements from a totally ordered universe S, find k-th smallest.

- Minimum: $k = 1 \rightsquigarrow$ min element
- Maximum: $k = n \rightsquigarrow$ max element
- Median: $k = \lfloor (n+1)/2 \rfloor$
  - n is odd, the median is unique, $k = (n+1)/2$
  - n is even, the median has two choices: $n/2$ and $n/2+1$, typically we choose $k = n/2$

Known results

- $O(n)$ compares for min or max.
- Naive algorithms for general selection: $O(n \log n)$ compares by sorting, and $O(n \log k)$ compares with a binary heap.

Applications. order statistics; select the "top k"; bottleneck paths

Q. Can we accomplish general selection with $O(n)$ compares?

A. Yes! Selection is easier than sorting.

About Median

Median of the list of numbers is its 50th percentile: half the numbers are larger than it, and half are smaller.

Example. The median of $[45, 1, 10, 30, 25]$ is 25.

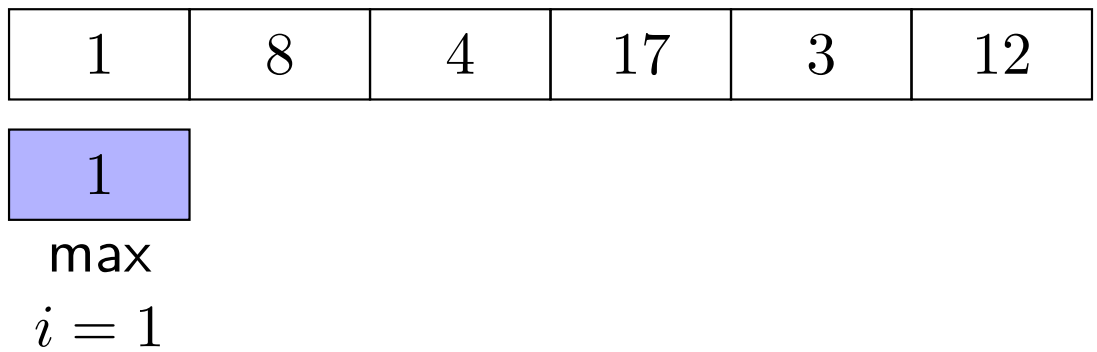
Meaning of median. Summarize a set of numbers by a single, typical value.

- The *mean* or *average* is also very commonly used for this purpose.
- But, median is in a sense more typical
  - always one of the data values, unlike the mean
  - less sensitive to outliers.

Counterexample. The median of hundreds 1's is 1, as is mean. However, if just one of these numbers gets corrupted to 10000, the mean shoots above 100, while the median is unaffected with large probability.

Selecting Max

Algorithm. Sequential compare



Output. $max = 17$, $i = 4$

Worst-case complexity. $W(n) = n - 1$

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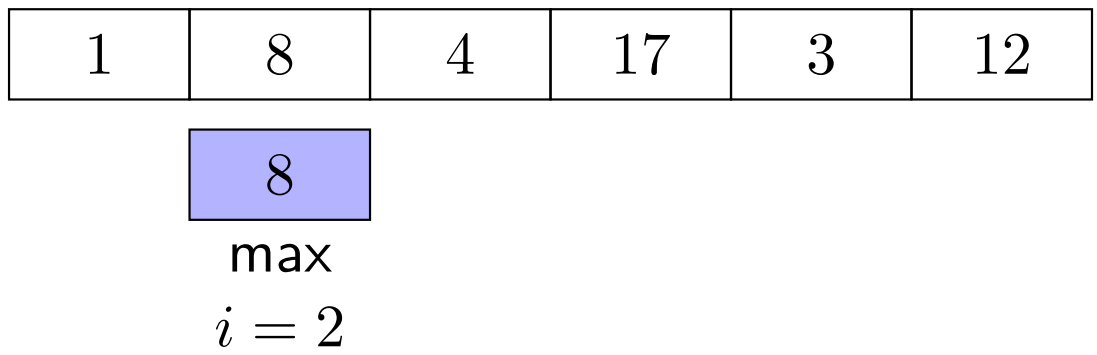

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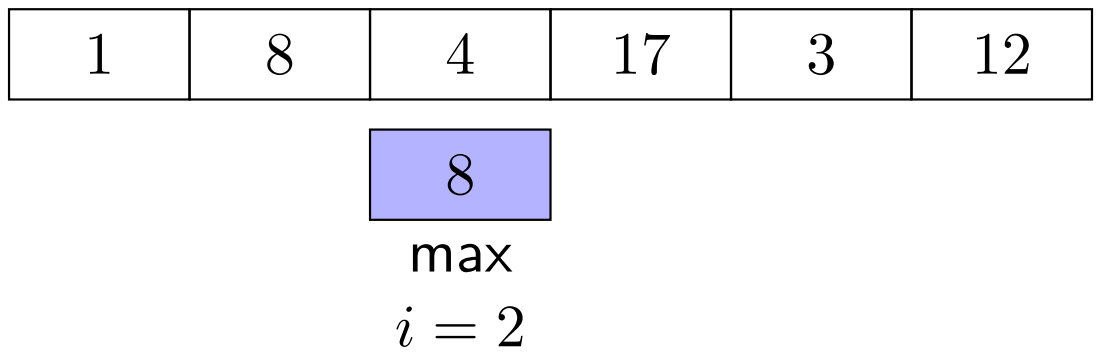
Selecting Max

Algorithm. Sequential compare



Output. $max = 17$, $i = 4$

Worst-case complexity. $W(n) = n - 1$

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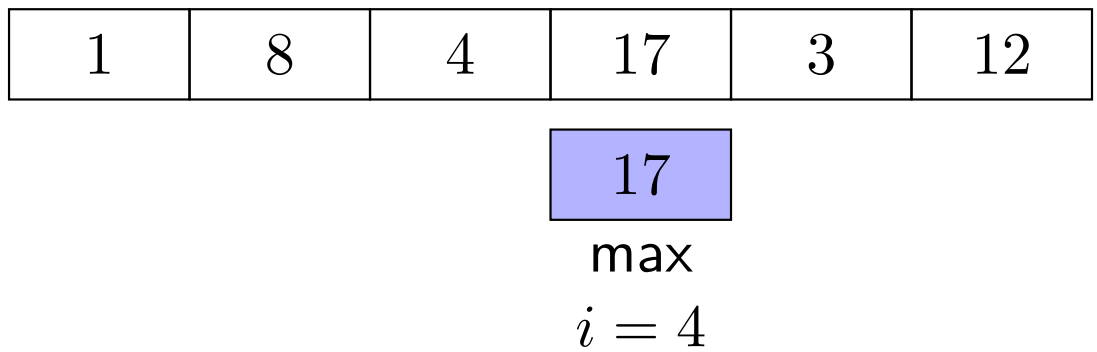

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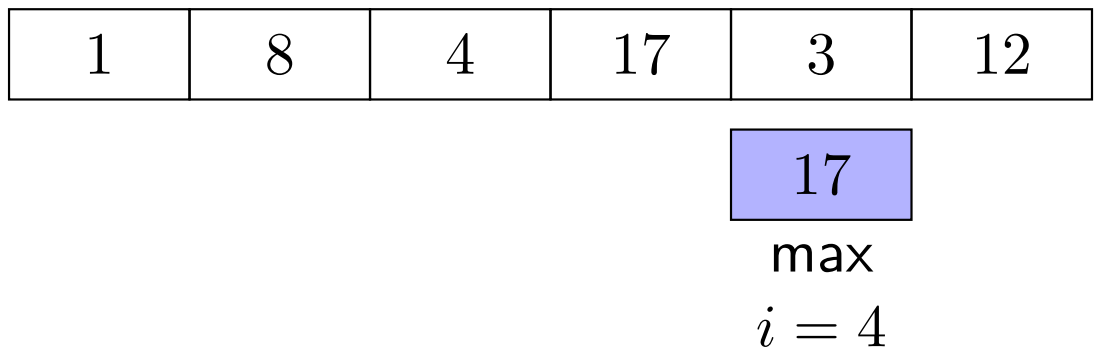

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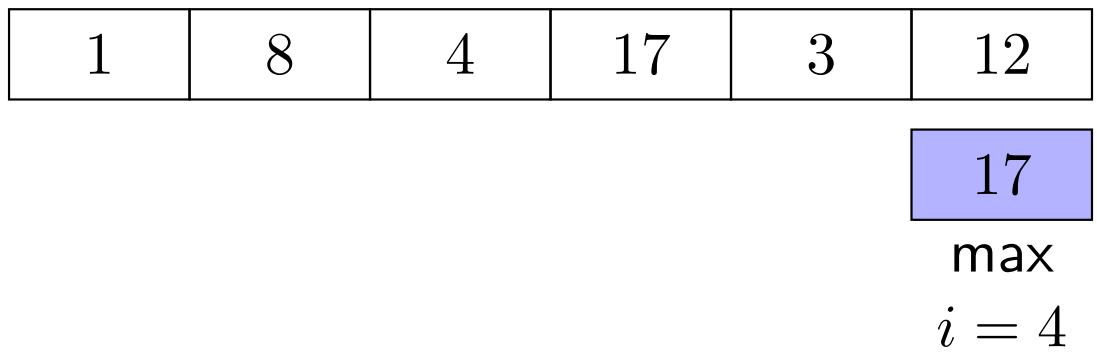

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Worst-case complexity. $W(n) = n - 1$

Pseudocode

Algorithm 9: SelectMax(A, n)

Input: $A[n]$
Output: max, j

1: $max \leftarrow A[1]$;
2: $j \leftarrow 1$;
3: **for** $i \leftarrow 2$ *to* n **do**
4: **if** $max < A[i]$ **then**
5: $max \leftarrow A[i]$;
6: $j \leftarrow i$;
7: **end**
8: **end**
9: **return** max, j

Selecting Max and Min

Naive Algorithm

1. Sequential compare, first choose max and remove it
2. Then choose min in the left list, using the same algorithm but retain smaller element after each compare.

Selecting Max and Min

Naive Algorithm

1. Sequential compare, first choose max and remove it
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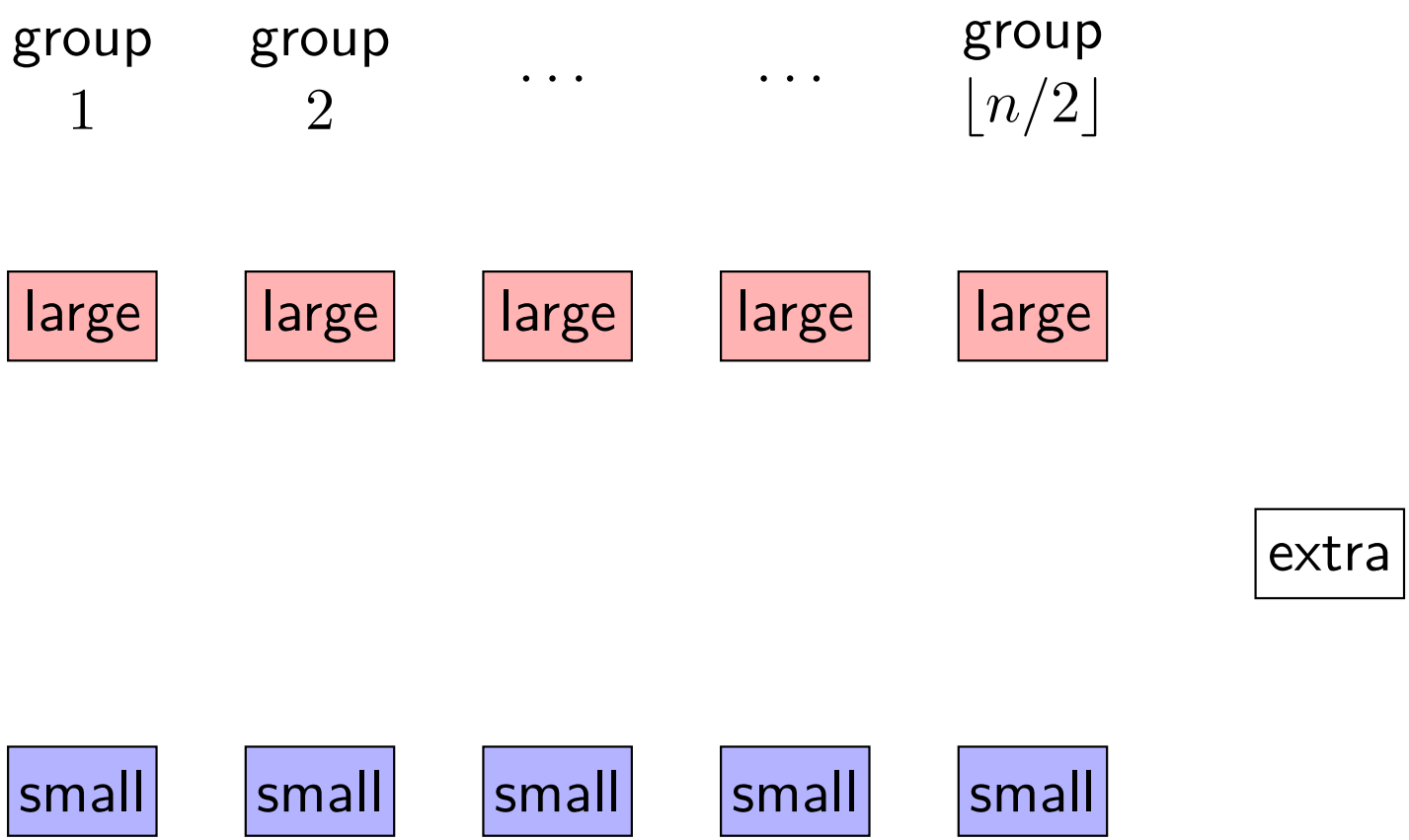
Worst-case time complexity

$$W(n) = n - 1 + n - 2 = 2n - 3$$

Grouping Algorithm

Idea. Split list into higher list and lower list.



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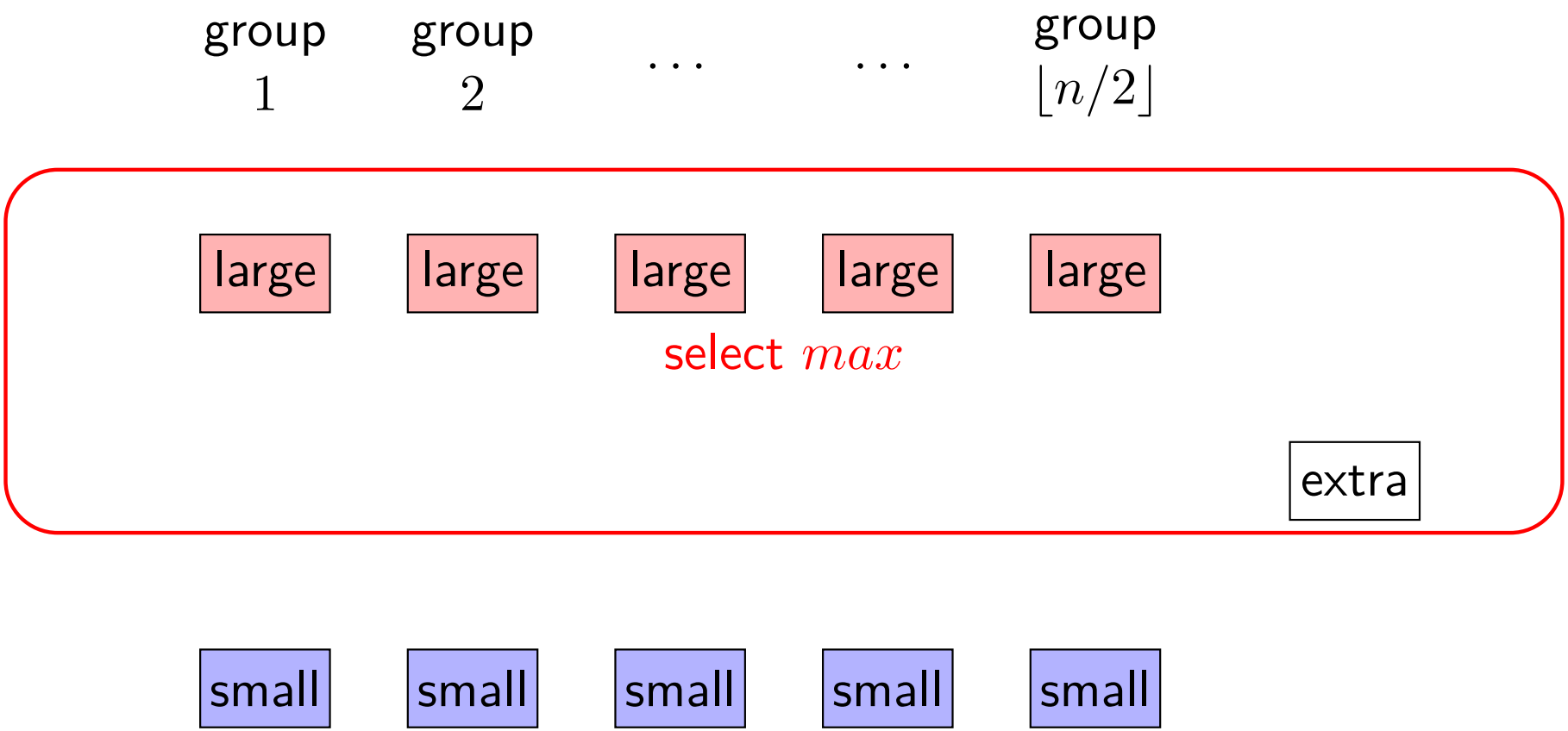

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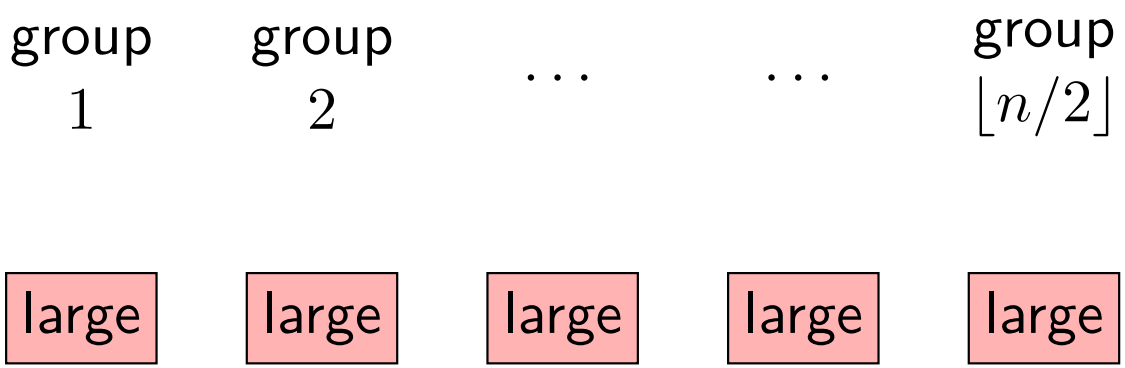

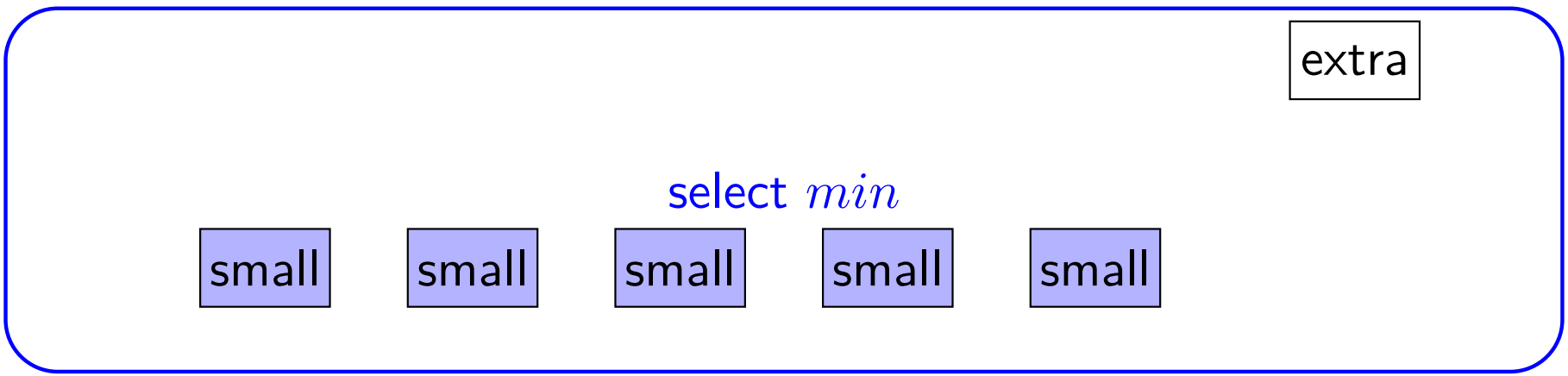

Pseudocode of Grouping Method for Selecting Max and Min

Algorithm 10: SelectMaxMin(A, n)

Input: unsorted $A[n]$

Output: max, min

1: divide n elements into $\lfloor n/2 \rfloor$ groups;

2: compare two elements in each group, obtain $\lfloor n/2 \rfloor$ smaller and $\lfloor n/2 \rfloor$ larger;

3: select max in $\lfloor n/2 \rfloor$ larger elements and the extra element;

4: select min in $\lfloor n/2 \rfloor$ smaller elements and the extra element;

Pseudocode of Grouping Method for Selecting Max and Min

Algorithm 11: SelectMaxMin(A, n)

Input: unsorted $A[n]$

Output: max, min

1: divide n elements into $\lfloor n/2 \rfloor$ groups;

2: compare two elements in each group, obtain $\lfloor n/2 \rfloor$ smaller and $\lfloor n/2 \rfloor$ larger;

3: select max in $\lfloor n/2 \rfloor$ larger elements and the extra element;

4: select min in $\lfloor n/2 \rfloor$ smaller elements and the extra element;

Summing it up, $W(n) \approx 3\lfloor n/2 \rfloor$

- Group inside compare: $\lfloor n/2 \rfloor$
- When n is even: select max: $n/2 - 1$, select min: $n/2 - 1$
- When n is odd: select max: $(n-1)/2 + 1 - 1$, select min: $(n-1)/2 + 1 - 1$

Divide-and-Conquer Strategy

Grouping algorithm outperforms naive algorithm cause group inside compare costs $\lfloor n/2 \rfloor$ compares, but saves $\approx (n/2) \times 2$ compares.

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Can we design SelectMaxMin via divide-and-conquer?

---

1. Divide A into left halve A_1 and right halve A_2
2. Recursively select max_1 and min_1 in A_1
3. Recursively select max_2 and min_2 in A_2
4. $max \leftarrow \max\{max_1, max_2\}$
5. $min \leftarrow \min\{min_1, min_2\}$

Divide-and-Conquer Strategy

Grouping algorithm outperforms naive algorithm cause group inside compare costs $\lfloor n/2 \rfloor$ compares, but saves $\approx (n/2) \times 2$ compares.

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5. $min \leftarrow \min\{min_1, min_2\}$

[2020 游泓慧] This recursive algorithm can be made iteratively like MergeSort.

Worse-Case Complexity

Assume $n = 2^k$, the recurrence relation of $W(n)$ is as below:

$$\begin{cases} W(n) = 2W(n/2) + 2 \\ W(2) = 1 \end{cases}$$

Solving for the exact value via substitute-then-iterate method:

$$\begin{aligned} W(2^k) =& 2W(2^{k-1}) + 2 \\ =& 2(2W(2^{k-2}) + 2) + 2 \\ =& 2^2 W(2^{k-2}) + 2^2 + 2 \\ =& 2^i W(2^{k-i}) + 2^i + \cdots + 2 \end{aligned}$$

The right side reach the initial value when $i = k - 1$, the summation is:

$$2^{k-1} + (2^{k-1} + \cdots + 2^2 + 2) = 3 \cdot 2^{k-1} - 2 = 3n/2 - 2$$

Summary

Select Max. Sequentially compare, requires at most $n-1$ compares

---

Select Max and Min. (worst-case)

- Naive algorithm: $2n-3$
- Grouping algorithm: $3\lfloor n/2 \rfloor$
- Divide-and-Conquer: $3n/2-2$

It can be proved that grouping algorithm and divide-and-conquer algorithm are optimal for SelectMaxMin, achieving the lower bound

Outline

Selecting the Second Largest

Input. $A[n]$

Output. The second largest max'

Selecting the Second Largest

Input. $A[n]$

Output. The second largest max'

Naive algorithm: sequential compare

1. select max from $A[n]$ via sequential compare
2. select max' from $A[n] \backslash max$, which is exactly the second largest

Time complexity: $W(n) = (n-1) + (n-2) = 2n - 3$

Optimized Method

Observation. The sufficient and necessary condition to be the second largest: only beaten by the largest

To determine the second largest element, we must know the largest element first.

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Idea. Trade space for time

- Record the elements that are beaten by the largest element in a set L along the way of selecting the largest
- Selecting the largest element among the elements in L.

Tournament Algorithm for Second Largest

1. Divide elements into groups of size 2
2. In each group, two elements compare, the larger one goes to the next level, and (only) records the beaten element in its list.
3. Repeat the above steps until there is only one element left, a.k.a. max
4. Select the largest element form the list of max, a.k.a. max'

The name comes from single-elimination tournament: players play in two-sided matches, and the winner is promoted to the next level up. The hierarchy continues until the final match determines the ultimate winner. The tournament determines the best player, but the player who was beaten in the final match may not be the second best - he may be inferior to other players the winner bested.

Pseudocode of SelectSecond

Algorithm 12: SelectSecond(A, n)

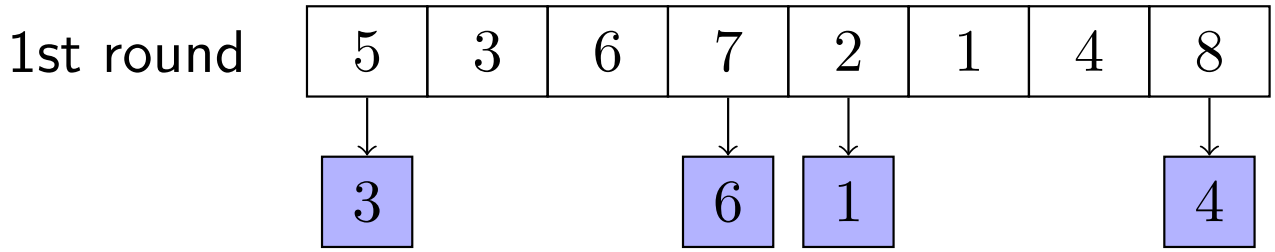
Input: $A[n]$

Output: Second largest element max'

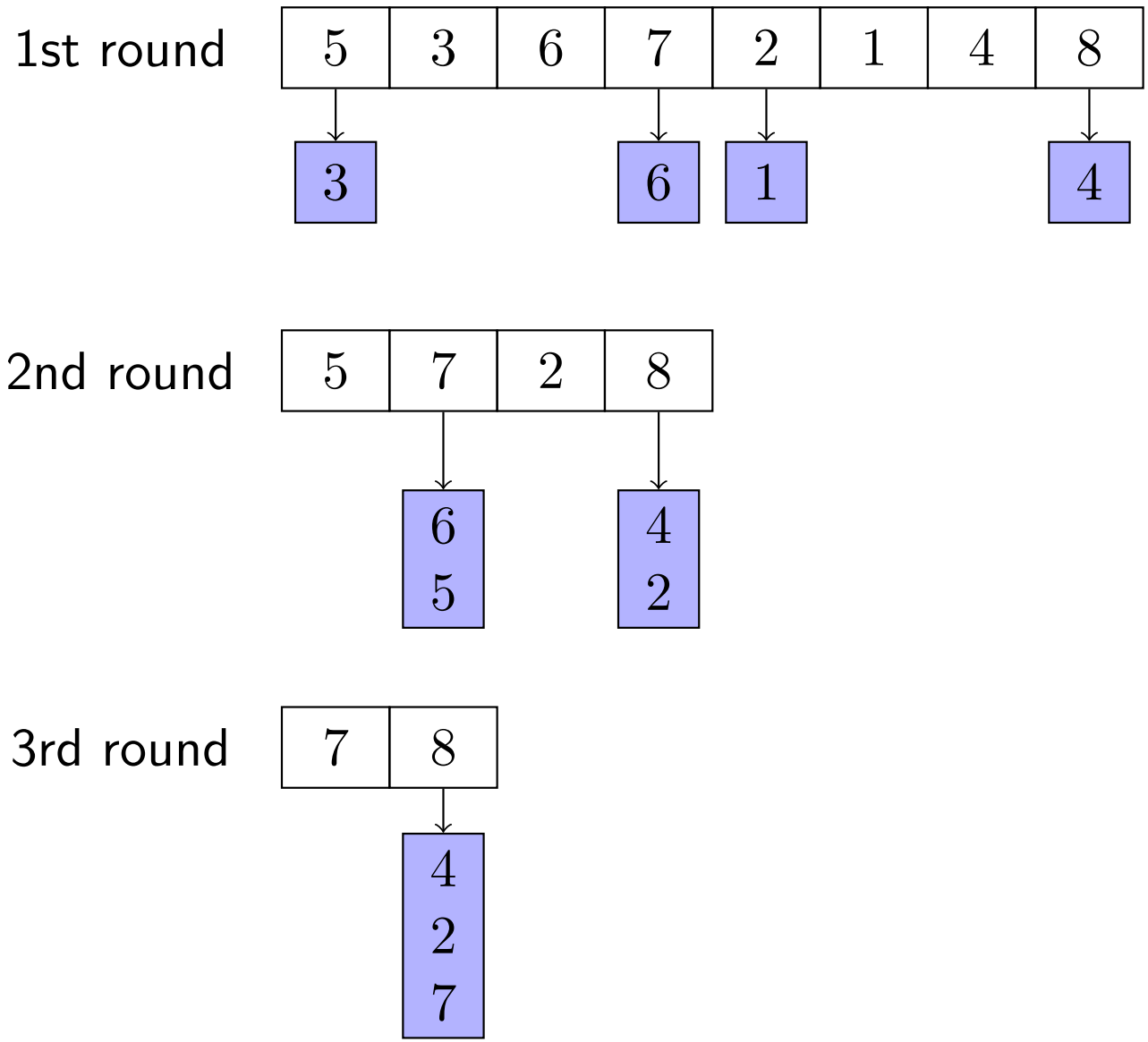
1: $k \leftarrow n$ //number of elements;
2: divide k elements into $\lfloor k/2 \rfloor$ groups;
3: In each group, two elements compare to select larger;
4: record the loser into the list of winner;
5: **if** k *is odd* **then** $k \leftarrow 1 + \lfloor k/2 \rfloor$;
6: **else** $k \leftarrow k/2$;
7: **if** $k > 1$ **then** goto 2;
8: $max \leftarrow$ ultimate winner;
9: $max' \leftarrow max$ in the list of ultimate winner

- line 2-4: one round match
- line 5-6: compute the number of winners — $\lceil k/2 \rceil$
- line 7: next round match

Demo of SelectSecond



Demo of SelectSecond



Complexity Analysis (1/3)

Proposition 1. Assume there are n elements, at most $\lceil n/2^i \rceil$ elements are left after i-th round match.

Proof. Carry mathematical induction over i:

- Induction basis $i = 1$: divide into $\lfloor n/2 \rfloor$ groups, kick off $\lfloor n/2 \rfloor$ elements, the number of elements prompted to the next level is
$$n - \lfloor n/2 \rfloor = \lceil n/2 \rceil$$
- Induction step: $P(i) \Rightarrow P(i+1)$. Assume the number of elements after i-th match is at most $\lceil n/2^i \rceil$, then after $i+1$-th match, the number of elements is

$$\text{continuous rounding property} \Rightarrow \lceil \lceil n/2^i \rceil / 2 \rceil = \lceil n/2^{i+1} \rceil$$

Complexity Analysis (2/3)

Proposition 2. max compares with $\lceil \log n \rceil$ elements

Proof. Assume max is selected after k round match. According to Proposition 1, $\lceil n/2^k \rceil = 1$.

- if $n = 2^d$ for some $d \in \mathbb{Z}$, then:

$$\log n = \lceil \log n \rceil$$
$$k = d = \lceil \log n \rceil$$

- else $2^d < n < 2^{d+1}$ for some $d \in \mathbb{Z}$, then

$$d < \log n < d + 1$$
$$k = d + 1 = \lceil \log n \rceil$$

Complexity Analysis (3/3)

Phase 1: number of elements is n

- number of total compares $= n - 1 \Leftarrow n - 1$ elements are eliminated (one compare kicks off one element)

Phase 2: number of elements is $\lceil \log n \rceil$, which is exactly the size of winner's list according to Proposition 2

- number of compares $= \lceil \log n \rceil - 1 \Leftarrow$ sequential compare or tournament algorithm ($\lceil \log n \rceil - 1$ elements are eliminated)

The overall time complexity:

$$\begin{aligned} W(n) &= n - 1 + \lceil \log n \rceil - 1 \\ &= n + \lceil \log n \rceil - 2 \end{aligned}$$

Summary of Selecting Second Max

Select the second max

- Naive algorithm (invoking SelectMax twice) $2n - 3$
- Tournament Algorithm: $n + \lceil \log n \rceil - 2$
  - main trick: trade space for efficiency

Outline

General Selection Problem

Problem. Select k-th smallest

Input. list $A[n]$, integer $k \in [n]$

Output. the k-th smallest

General selection has a broad range of applications

- Example 1. $A = \{3, 4, 8, 2, 5, 9, 10\}$, $k = 4$, solution $= 5$
- Example 2. Statistical data set S, $|S| = n$, select the median, $k = \lceil n/2 \rceil$

Naive Algorithms

Algorithm 1

- invoke algorithm SelectMin k times
- time complexity is: $O(kn)$

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Algorithm 2

- sort then select the k-smallest number
- time complexity is: $O(n \log n)$

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- time complexity is: $O(kn)$

Algorithm 2

- sort then select the k-smallest number
- time complexity is: $O(n \log n)$

Ideally we expect linear complexity

- This is hopeful cause sorting does far more than we really need — we do not care about the relative ordering of the rest of them.

QuickSelect

[Hoa71] Algorithm 65: Find

QuickSelect

[Hoa71] Algorithm 65: Find

QuickSelect uses the same overall approach as QuickSort — choosing one element m^* as pivot to partition S so that m^* is in place, smaller elements in left subarray S_1 and larger elements in right subarray S_2

1. If $k \leq |S_1|$, then select the k-smallest in S_1
2. If $k = |S_1| + 1$, then m^* is the k-smallest
3. If $k > |S_1| + 1$, then select the $k - |S_1| - 1$-smallest in S_2

QuickSelect

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Instead of recursing into both sides as in QuickSort, QuickSelect only recurses into one side — the side with the element it is searching for.

- this reduces the average-case complexity from $O(n \log n)$ to $O(n)$.
- the best-case complexity is $O(n)$, while the worst-case complexity is $O(n^2)$.

How to Improve the Worst-Case Complexity?

Like QuickSort, the complexity of QuickSelect is determined by quality of the pivot

- $\leadsto$ efficient in practice and has good average-case performance, but has poor worst-case performance.

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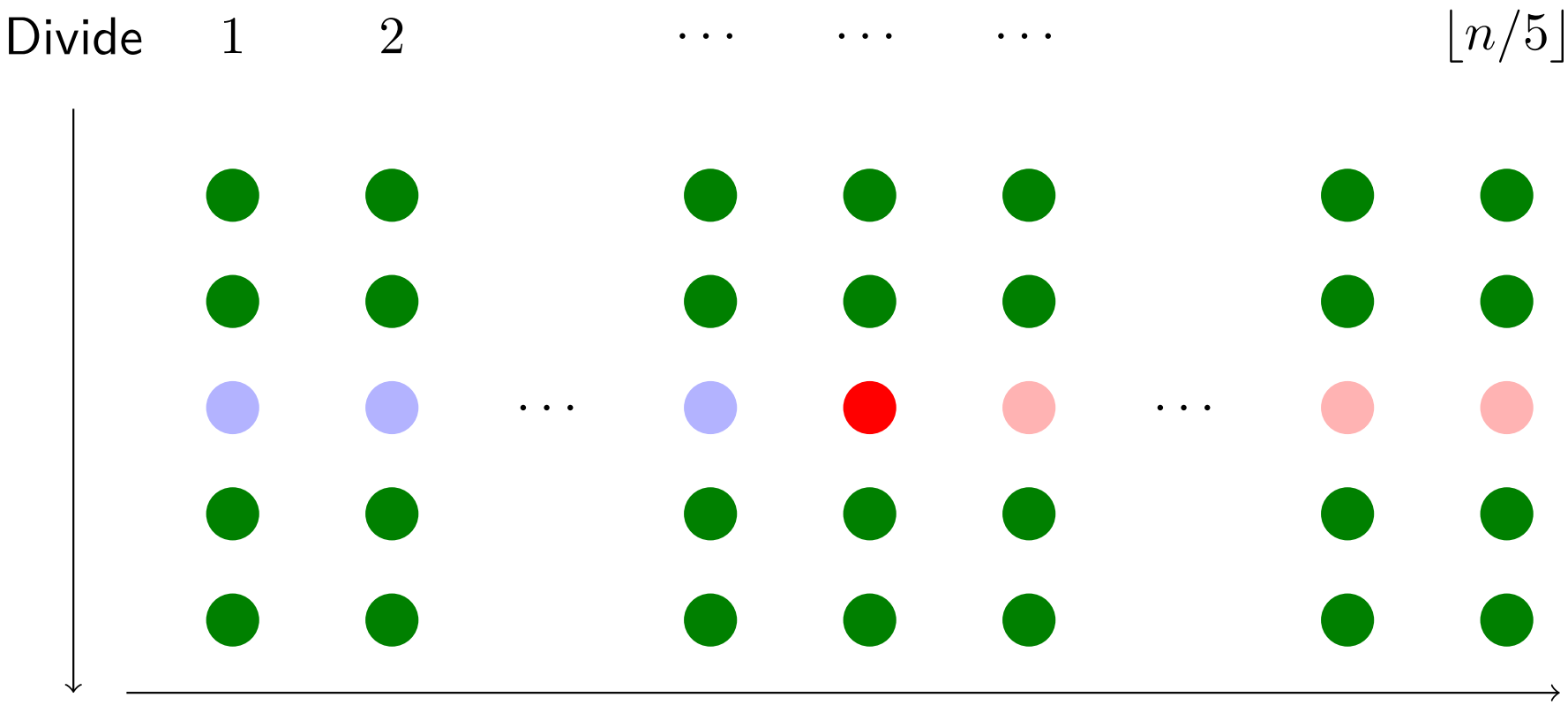
How to choose good pivot m^* to attain good average-case and worst-case performance?

Ideal case: select the exact median, but this means we have to solve a problem of the same size first

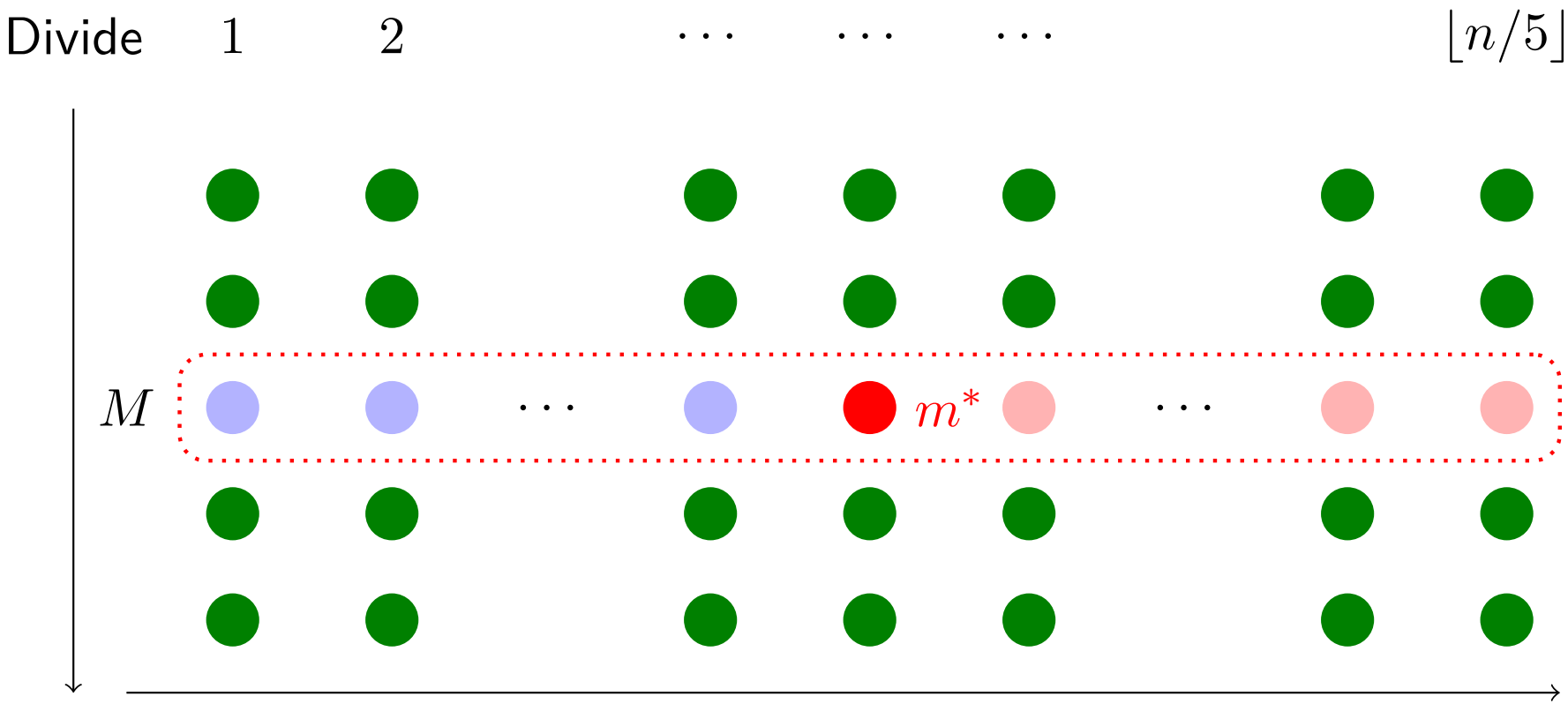
Real case: use quasi-median instead — the median of medians

Select the Median of Medians



1. sort each column in descending order

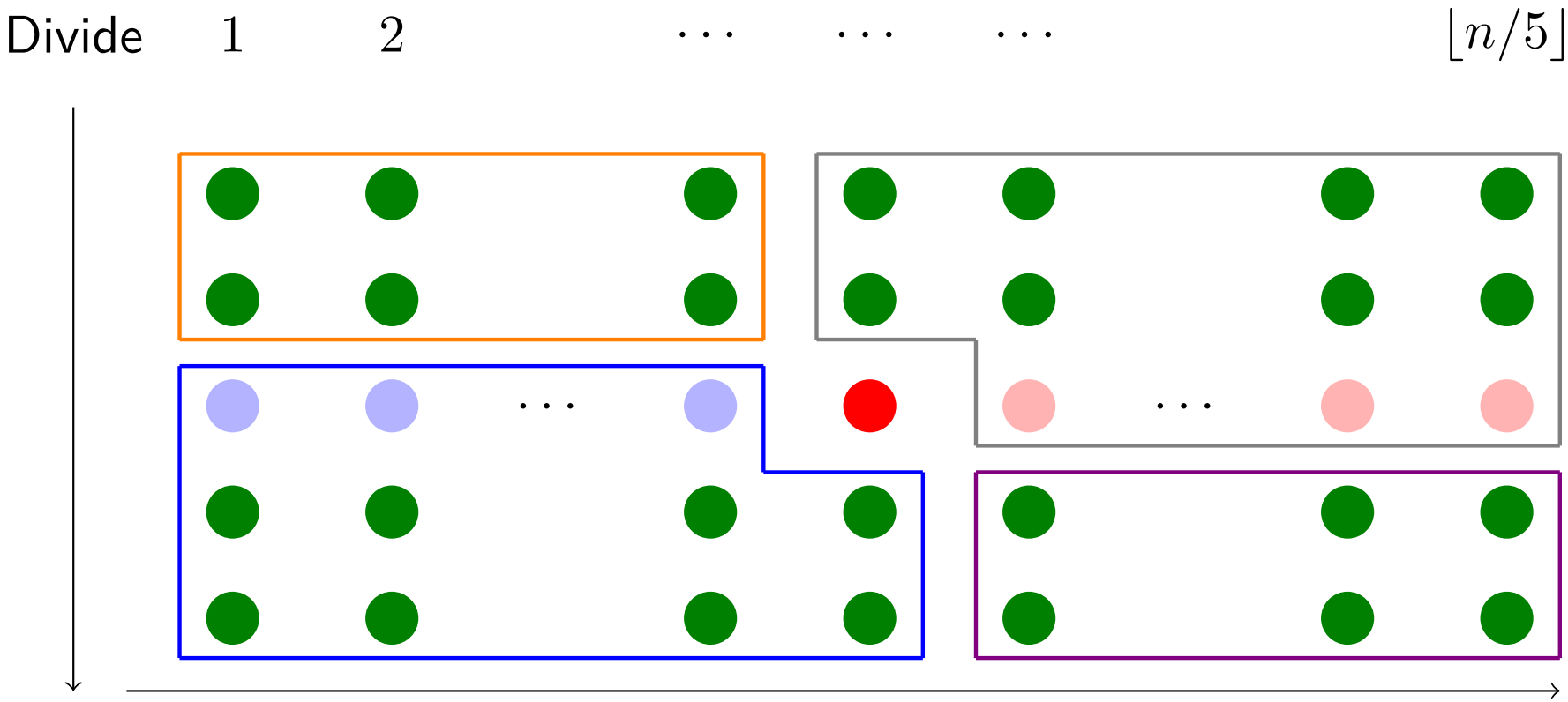
Select the Median of Medians



1. sort each column in descending order
2. select the median of medians — m^*

Partition

4. re-organize the columns to place the column contains m^* in the middle, columns whose median less than m^* to the left, and columns whose median larger than m^* to the right



- A zone: require compares with m^*
- B zone: larger than m^*; C zone: smaller than m^*
- D zone: require compares with m^*

Demo: $n = 15$, $k = 6$

8	2	3	5	7	6	11	14	1	9	13	10	4	12	15

	8	14	15	
	7	11	13	
M	5	9	12	$m^* = 9$
	3	6	10	
	2	1	4	

	8	14	15	
A	7	11	13	B
	5	9	12	
C	3	6	10	D
	2	1	4	

Divide to Subproblems

	8	14	15	
	7	11	13	
S_1	5	9	12	S_2
	3	6	10	
	2	1	4	

subproblem: $\{8, 7, 5, 3, 2, 6, 1, 4\}$

size of subproblem $= 8$, $k = 6$

Pseudocode of QuickSelect

Algorithm 13: QuickSelect$(A[n], k)$

1: divide elements in A into groups of size 5, there are totally $m = \lceil n/5 \rceil$ groups;
2: sort each group and place the medians into M;
3: $m^* \leftarrow$ QuickSelect$(M, \lceil |M|/2 \rceil)$ //split S into A, B, C, D;
4: For elements in A and D, record the ones smaller than m^* into S_1, the ones larger than m^* into S_2;
5: $S_1 \leftarrow S_1 \cup C$, $S_2 \leftarrow S_2 \cup B$;
6: **if** $k = |S_1| + 1$ **then** output m^*;
7: **else if** $k \leq |S_1|$ **then**
8: QuickSelect(S_1, k);
9: **else** QuickSelect$(S_2, k - |S_1| - 1)$;

- line 4-5: split
- line 7-9: recursively solve subproblems

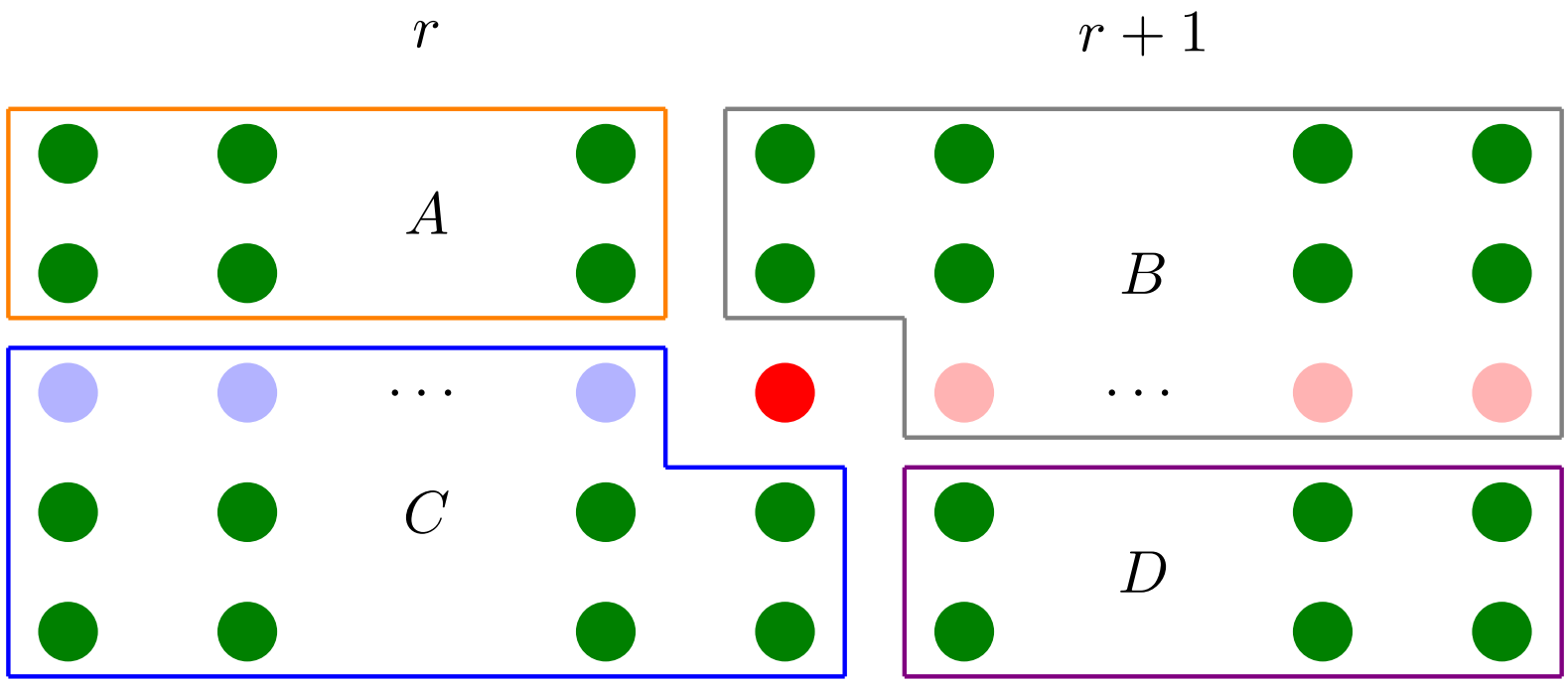
Complexity Analysis

Each round of QuickSelect algorithm consists of two recursive calls of QuickSelect

1. select median from median M as pivot for dividing
2. the real subproblem

The overall complexity of the algorithm is determined by the quality of pivot

Worst-Case Complexity



unbalanced divide $\rightsquigarrow$ bad complexity

We consider an extreme case: elements in A zone and D zone go to the same side.

- $n = 5(2r+1)$, $|A| = |D| = 2r$
- the size of subproblems is at most: $2r + 2r + 3r + 2 = 7r + 2$

Estimation of the Size of Subproblems

Assume $n = 5(2r + 1)$, $|A| = |D| = 2r$

$$r = \frac{n/5 - 1}{2} = \frac{n}{10} - \frac{1}{2}$$

The size of subproblem after dividing is at most:

$$\begin{aligned} 7r + 2 &= 7\left(\frac{n}{10} - \frac{1}{2}\right) + 2 \\ &= \frac{7n}{10} - \frac{3}{2} < \frac{7n}{10} \end{aligned}$$

Recurrence Relation for Worst-Case Complexity

Worse-case complexity $W(n)$

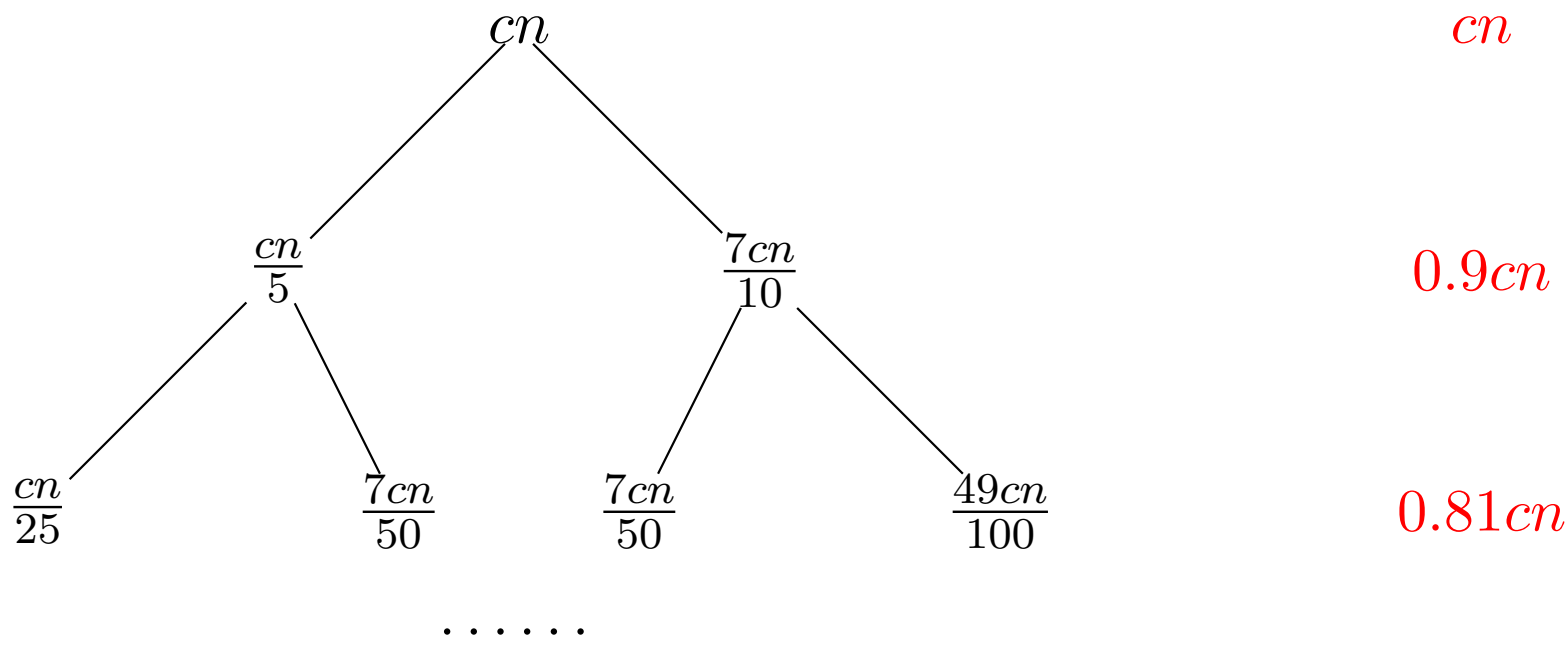
- line 2: $\Theta(n)$ //select median among each 5 elements (constant time), form M
- line 3: $W(n/5)$ //select median m^* of M
- line 4: $\Theta(n)$ //divide S using m^* (only need to compare A and D)
- line 8-9: $W(7n/10)$ //recursive call to the subproblem

The recurrence relation is:

$$W(n) \leq W(n/5) + W(7n/10) + \Theta(n)$$

Solving via Recurrence Tree

$$W(n) \leq W(n/5) + W(7n/10) + \Theta(n)$$



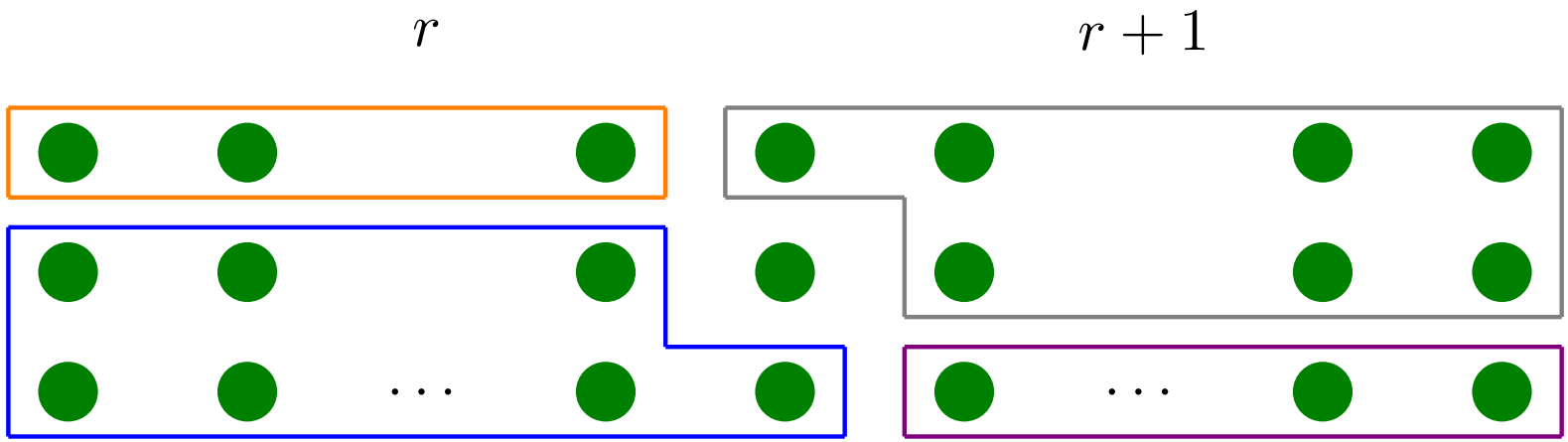
- the depth of tree is $\Theta(\log n) \Rightarrow$ the number of leaf nodes is $\Theta(n)$; the cost of solving smallest problem is constant $\Rightarrow$ the cost of all smallest problem is $\Theta(n)$

$$W(n) \leq cn(1 + 0.9 + 0.9^2 + \dots) + \Theta(n) = \Theta(n)$$

Discussion

Why we have to divide the elements into groups of size 5? Is it your lucky number? Can we choose the group size as 3 or 7?

Case Study: $t = 3$



$n = 3(2r + 1)$, $r = (n/3 - 1)/2 = n/6 - 1/2$

The subproblem size is at most: $4r + 1 = 4n/6 - 1$

Recurrence relation of worst-case complexity is:

$$W(n) = W(n/3) + W(4n/6) + cn$$

Solving by recurrence tree $\Rightarrow W(n) = \Theta(n \log n)$

About the Group Size

The group size affects the overall complexity. Let t be the group size.

1. The cost of selecting m^* is related to $|M| = n/t$. The larger is t, the smaller is $|M|$.
2. The size of subproblem after dividing is related to t. The larger is t, the larger is $|S_i|$.

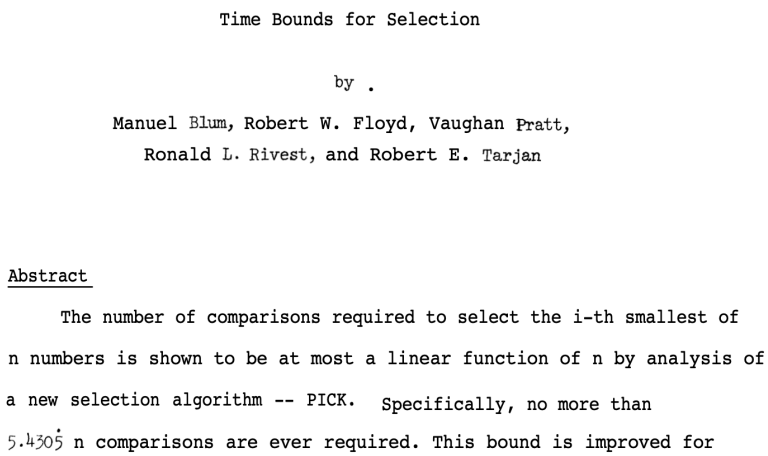
We have to hit the sweet balance.

Crux. When $|M| + |S_i| < n$, then the total cost on the inner nodes of recurrence tree forms geometric series with common ratio less than 1. $W(n)$ is $\Theta(n)$ only in this case.

The Median of Medians

Step 1-3 of QuickSelect constitutes approximate median (the median of medians) selection algorithm.

[BFP+73] Blum-Floyd-Pratt-Rivest-Tarjan 1973: There exists a compare-based selection algorithm whose $W(n) = O(n)$.



Time Bounds for Selection

by .

Manuel Blum, Robert W. Floyd, Vaughan Pratt,
Ronald L. Rivest, and Robert E. Tarjan

Abstract

The number of comparisons required to select the i-th smallest of n numbers is shown to be at most a linear function of n by analysis of a new selection algorithm -- PICK. Specifically, no more than $5.4305\dot{5}$ n comparisons are ever required. This bound is improved for

It was orginally studied alone under the name of PICK, which is frequently used to supply a good pivot for an exact selection algorithm, most commonly QuickSelect.

More About The Median of Medians

Theory.

- Optimized version of BFPRT: $\leq 5.4305n$ compares.
- Best known upper bound [Dor-Zwick 1995]: $\leq 2.95n$.
- Best known lower bound [Dor-Zwick 1999]: $\geq (2+\epsilon)n$.

Practice.

- Constant and overhead (currently) too large to be useful.

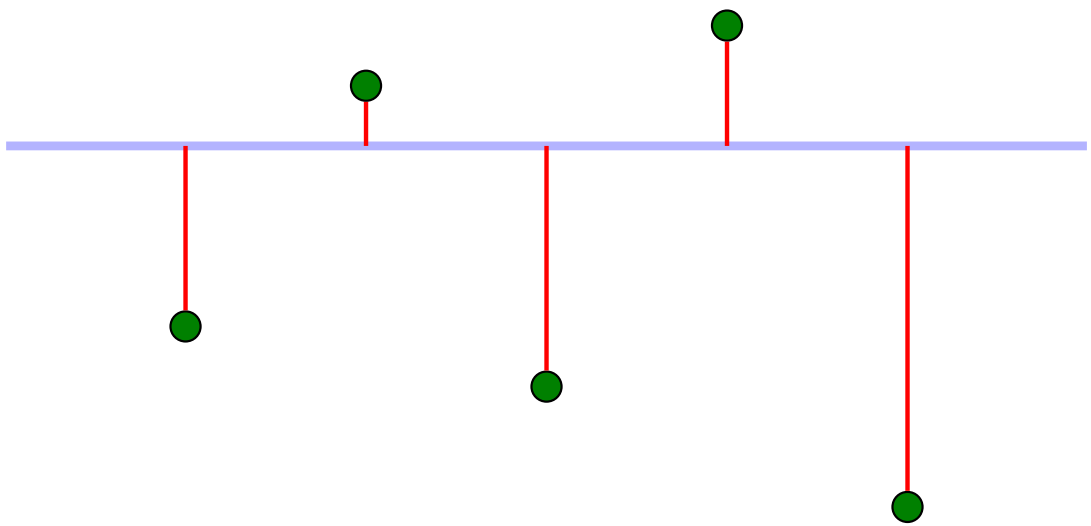
Musser [Mus97] (known for his work in generic programming $\rightsquigarrow$ C++ Standard Template Library), with the purpose of providing generic algorithms for the C++ STL, introduced IntroSelect (short for "introspective selection"): a hybrid of original random version QuickSelect and Median of Medians

- optimistically starting out with QuickSelect and only switching to a worst-case linear-time selection algorithm if it recurses too many times without making sufficient progress

Application of Selecting Median: Optimal Pipeline Design

Problem. Assume there are n oil wells, the task is building a pipeline system to connect n oil wells. The pipeline system consists of a horizontal main pipeline, each oil well connects to the main pipeline via a vertical pipeline.

Optimization Goal. How to choose the position of main pipeline to minimize the total length of vertical pipelines?



Optimal Solution: the Median of Y Coordinates

The main pipeline is horizontal $\Rightarrow$ optimal solution is independent of the distribution of X coordinates

y_n

$\vdots$

$y_{n/2+1}$

$y_{n/2}$ y

y_2

y_1

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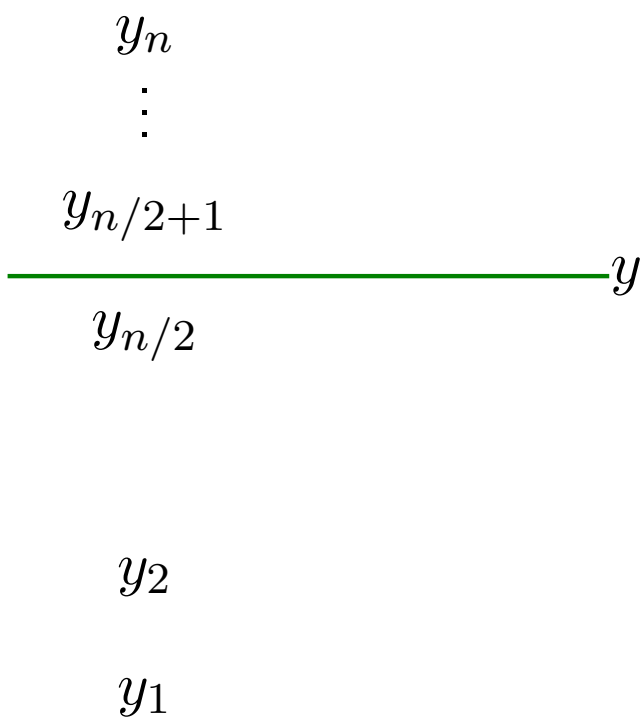

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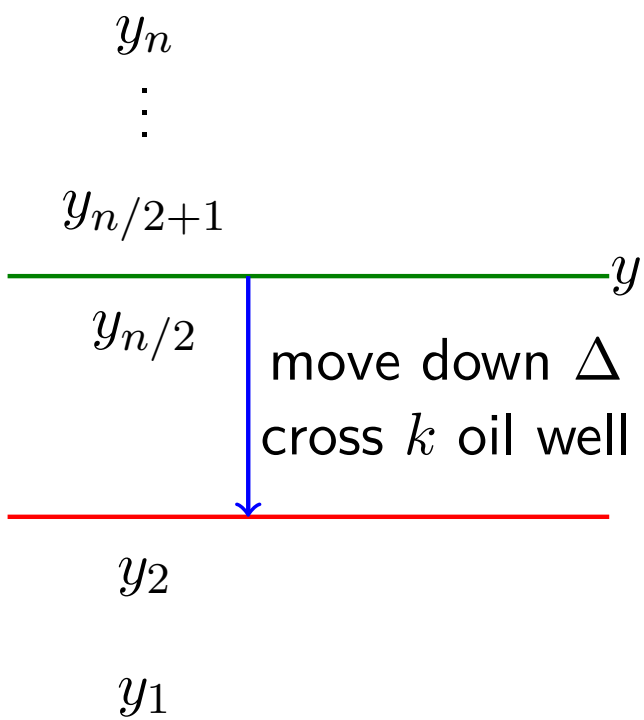

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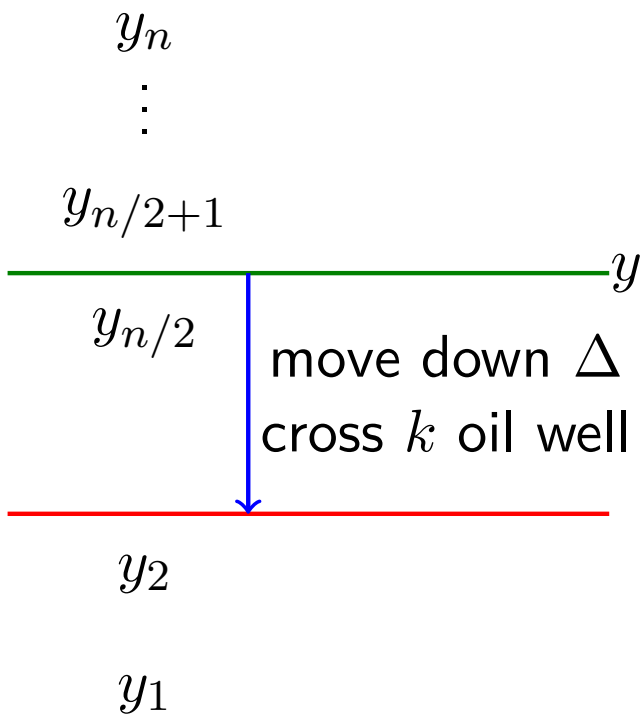

each vertical pipeline increases Δ

each vertical pipeline increases/decreases at most Δ

each vertical pipeline decreases length Δ

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The same analysis also applies to the case of moving up, with the same effect.

Summary of Selection Problem

Selecting max or min

- Naive sequential compare: $W(n) = n - 1$

Selecting max and min

- grouping algorithm: $W(n) = 3\lfloor n/2 \rfloor$
- divide-and-conquer: $W(n) = 3n/2 - 2$

Selecting the second largest: the tournament algorithm

$$W(n) = n + \lceil \log n \rceil - 2$$

General selecting problem: divide-and-conquer algorithm

$$W(n) = \Theta(n)(\approx 44n)$$

Outline

Finding the Closest Pair of Points

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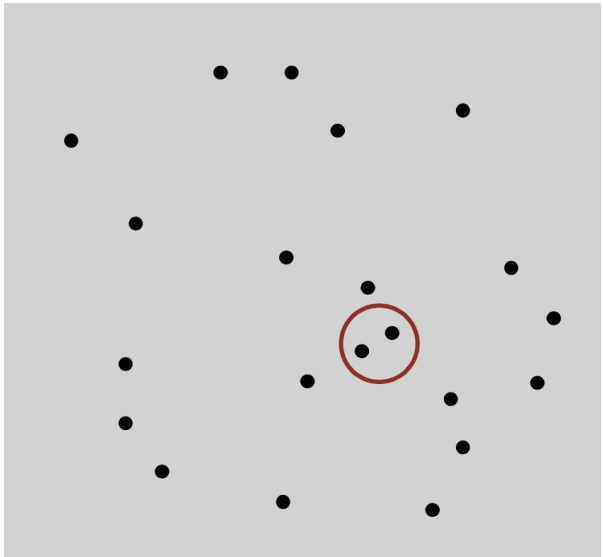

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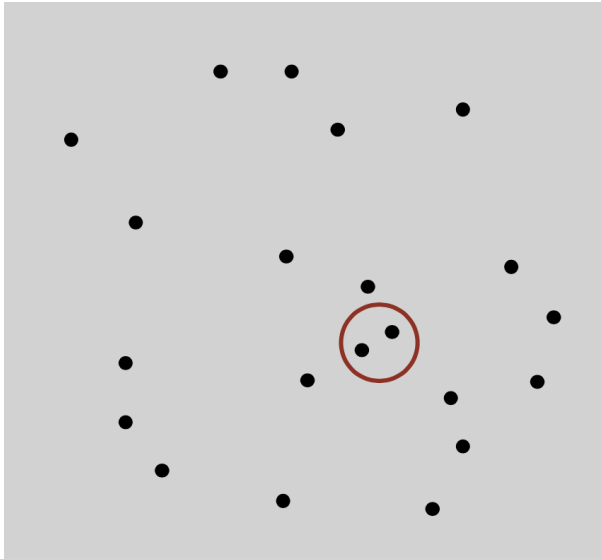

Fundamental geometric primitive.

- Graphics, computer vision, geographic information systems, molecular modeling, air traffic control.
- Special case of $\underbrace{\text{nearest neighbor, Euclidean MST, Voronoi}}_{\text{fast closest pair inspired fast algorithms for these problems}}$

Attempts

1d version. $O(n \log n)$ algorithm if n points are on a line (note: inputs are not given in order)

1. sort n points according to x coordinate
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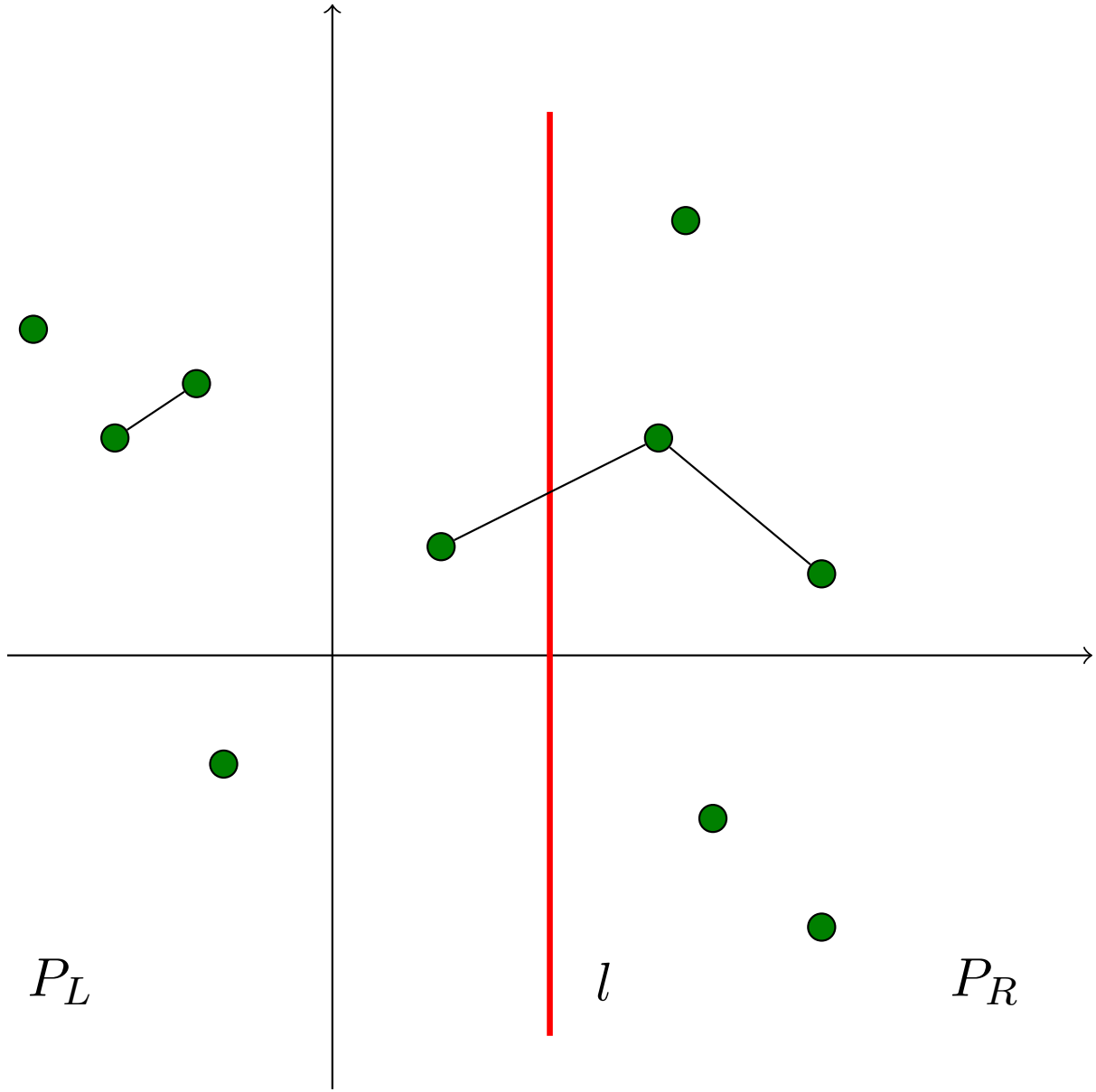
$\Theta(n^2)$ complexity seems inevitable. Can we do better?

Divide-and-Conquer

Idea. Partition P into P_L and P_R of roughly the same size

- Divide: draw vertical line l so that $n/2$ points on each side: P_L and P_R
- Conquer: find closest pair of points in each side recursively
- Combine: find closest pair with one point in each side.
- Return the closest one among 3 solutions.

Demo of Partition: $n = 10$



Pseudocode of MinDistance

Algorithm 14: MinDistance(P, X, Y)

Input: Points set P, coordinates set X and Y

Output: the closest pair of points and distance

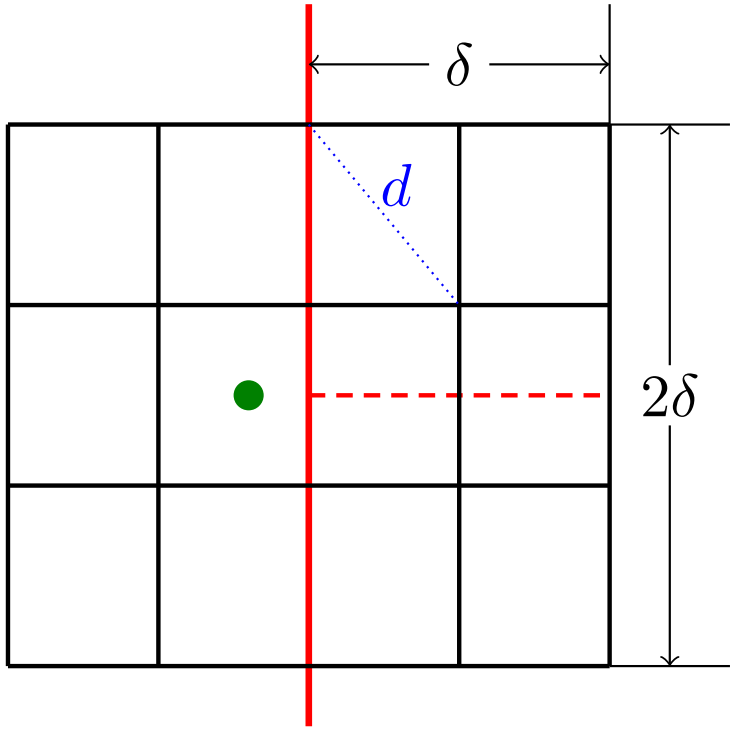
1: **if** $|P| < 3$ **then** direct compute;
2: sorted X and Y;
3: draw midline l to partition P into P_L and P_R;
4: $\delta_L \leftarrow$ MinDistance(P_L, X_L, Y_L);
5: $\delta_R \leftarrow$ MinDistance(P_R, X_R, Y_R);
6: $\delta = \min(\delta_L, \delta_R)$; // δ_L, δ_R are solutions to subproblems ;
7: check nodes within certain distance to l;
8: **if** *distance is smaller than* δ **then** update δ as this value;

The combine step seems again require $\Theta(n^2)$.

Step 7 requires delicated desgin and analysis.

How to find closest pair with one point in each side?

Observation 1. Only need to consider points within δ of line l.



$$\begin{aligned} d &= \sqrt{(\delta/2)^2 + (2\delta/3)^2} \\ &= \sqrt{\delta^2/4 + 4\delta^2/9} \\ &= \sqrt{25\delta^2/36} = 5\delta/6 \end{aligned}$$

Observation 2. In each rectangles on the right side: # point ≤ 1

- each point at most is required to compare with 6 points in the opposite side (because there are at most 1 point in 1 cell)
- checking one point requires constant time $\rightsquigarrow$ compare $\Theta(n)$ points requires $\Theta(n)$ time

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Why not sort two δ-strips separately instead? Cause fix one point at one side, it could be complicated to locate "the 6 neighbors" on the opposite side.

Complexity Analysis

step	operation	time complexity
1	smallest problem	$O(1)$
2	sort X and Y	$\Theta(n \log n)$
3	partition	$O(n)$
4-5	subproblems	$2W(n/2)$
6	$\delta = \min\{\delta_L, \delta_R\}$	$O(1)$
7	cross midline treatment	$\Theta(n)$

Why sorting X and Y is necessary?

- sort X: partition P to P_L and P_R
- sort Y: deal with the strip

$$\begin{cases} W(n) = 2W(n/2) + \Theta(n \log n) \\ W(n) = O(1), n \leq 3 \end{cases}$$

Applying master theorem (case 2), we have:

$$\text{recursion tree} \Rightarrow W(n) = \Theta(n \log^2 n)$$

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Improved approach.

1. Preprocessing. sort X and Y before recursion
2. split sorted X and Y when partitioning, obtaining sorted X_L, Y_L for P_L, and sorted X_R, Y_R for P_R
   - splitting X is simple: split by the median
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sorting points from scratch each time $\rightsquigarrow$ sorting once and then splitting

Details of Sorting and Splitting

Data structure. two lists $X[n], Y[n]$, each element is a label i, sorted in ascending order of x, y coordinates once

Splitting of X: simple, but additional trick is needed to facilitate splitting of Y. Let n be the current size of problem

- generate an indication map H of size n, $H[i] = 0$ indicates point i in the left, $H[i] = 1$ indicates point in the right.
- time complexity is $\Theta(n)$

Splitting of Y: sequentially scanning $Y[n]$

- if $H[Y[i]] = 0$, classify $Y[i]$ to the left side, else classify it to the right side, yielding sorted Y_L and Y_R.
- time complexity is $\Theta(n)$

Demo of Splitting in Recursion

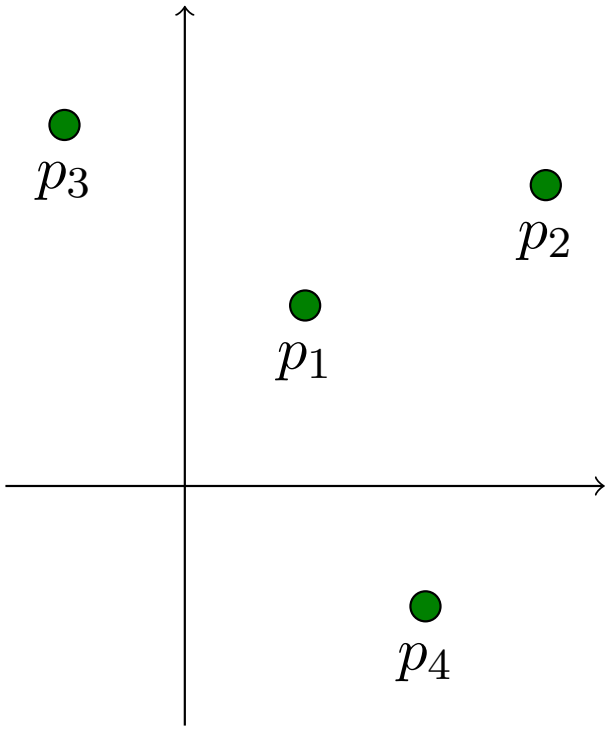

Table: Input

P	1	2	3	4
x	0.5	2	-2	1
y	2	3	4	-1

Table: Preprocessing: sort

X	3	1	4	2
Y	4	1	2	3

Table: Splitting

X_L	3	1
Y_L	1	3

Table: Splitting

X_R	4	2
Y_R	4	2

Improved Divide-and-Conquer Algorithm

$T(n)$ is the overall complexity, $\Theta(n \log n)$ is the complexity of global preprocessing, $T'(n)$ is the complexity of main recursive algorithm,

$$\begin{cases} T(n) = T'(n) + \Theta(n \log n) \\ T'(n) = 2T'(n/2) + \Theta(n) \\ T'(n) = O(1) \quad n \leq 3 \end{cases}$$

$$\text{master theorem (case 2)} \Rightarrow T'(n) = \Theta(n \log n)$$

Putting all the above together, $T(n) = \Theta(n \log n)$

Lower bound. In quadratic decision tree model (compute the Euclidean distance then compare), any algorithm for closest pair (even in 1D) requires $\Theta(n \log n)$ quadratic tests.

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