Design and Analysis of Algorithms
Greedy Algorithms

Outline

Motivation

A game like chess can be won only by *thinking ahead*

- a player who is foucsed entirely on immediate advanatges is easy to defeat.

But in many other games, such as Scrabble

- it's fine to make whichever move seems best at the moment and not worrying too much about future consquences.

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- it's fine to make whichever move seems best at the moment and not worrying too much about future consquences.

The sort of myopic behavior is easy and convinient, making it an attractive algorithmic strategy

Greedy Algorithm

Greedy algorithm works: proof of correctness

- Interval scheduling: induction on step
- Optimal loading: induction on input size
- Scheduling to minimum lateness: exchange argument

Greedy algorithm does not work: find a counter-example

- Coin changing problem

Outline

Interval Scheduling

Input. $S = \{1, 2, \ldots, n\}$ is a set of n jobs, job i starts at s_i and finishes at f_i.

- Two jobs i and j are compatible if they don't overlaps: $s_i \geq f_j$ or $s_j \geq f_i$

Goal: find maximum subset of mutually compatible jobs.

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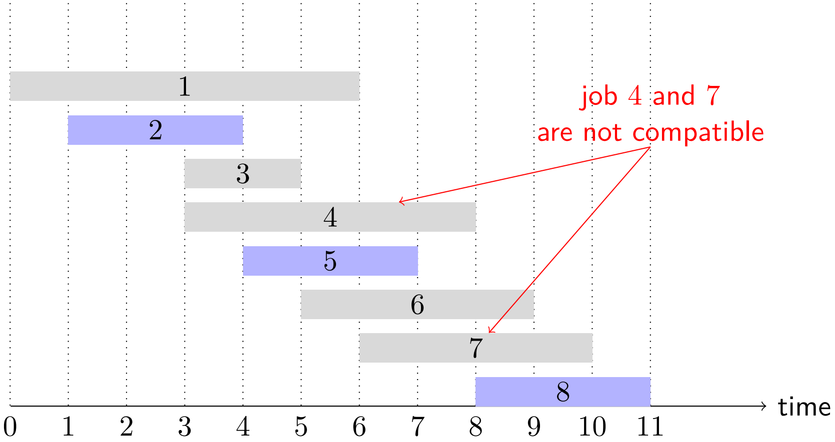
Instance

i	1	2	3	4	5	6	7	8
s_i	0	1	3	3	4	5	6	8
f_i	6	4	5	8	7	9	10	11

Solution. $\{2, 5, 8\}$

Example



1
2
3
4
5
6
7
8
job 4 and 7
are not compatible
time
0 1 2 3 4 5 6 7 8 9 10 11

Interval Scheduling: Greedy Algorithm

Greedy template

- Consider jobs in some natural order, then take each job provided it's compatible with the ones already taken.
- Selection strategy is short-sighted $\rightsquigarrow$ the order might not be optimal

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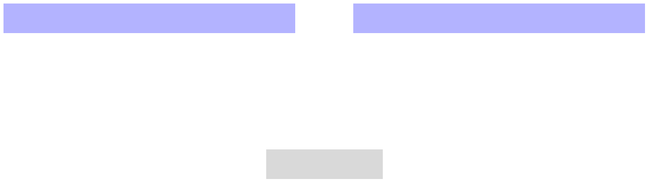
Candidate selection strategies

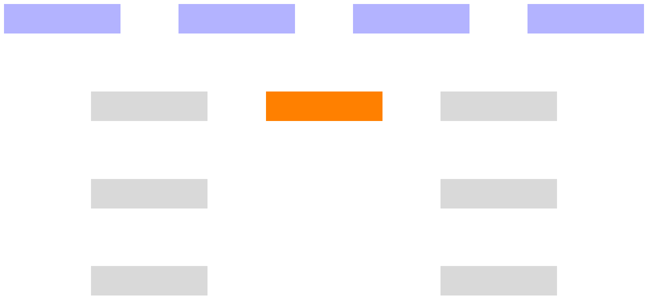
- [Earliest start time] Consider jobs in ascending order of s_i
- [Earliest finish time] Consider jobs in ascending order of f_i
- [Shortest interval] Consider jobs in ascending order of $f_i - s_i$
- [Fewest conflicts] For each job j, count the number of conflicting jobs c_j. Schedule in ascending order of c_j.

Counterexample for Earliest Start Time

Counterexample for Shortest Interval



Counterexample for Fewest Conflicts



Greedy Algorithm: Earliest-Finish-Time-First

Algorithm 1: GreedySelect$(S, s_i, f_i, i \in [n])$

Output: maximum compatible subset $A \subseteq S$

Sort jobs by finish time so that $f_1 \leq \cdots \leq f_n$;
$n \leftarrow |S|$;
$A \leftarrow \emptyset$;
for $i \leq 1$ *to* n **do**
  if *job* i *is compatible with* A **then** $A \leftarrow A \cup \{i\}$;
end
return A;

Q. How to decide job i is compatible with A?

A. Keep track of job j^* that was last added to A. Job i is compatible with A iff $s_i \geq f_{j^*}$ holds.

Demo of Earliest Finish Time First

Input. $S = \{1, 2, \ldots, 8\}$

i	1	2	3	4	5	6	7	8
s_i	0	1	3	3	4	5	6	8
f_i	6	4	5	8	7	9	10	11

Solution. $A = \{2, 4, 8\}$

Complexity. overall $\Theta(n \log n)$

- Sorting by finish time: $\Theta(n \log n)$
- Compare to check compatible: $O(n)$

Lemma. Earliest-finish-time-first algorithm always give the correct solution.

How to prove it?

Mathematic Induction for Greedy Algorithm

Proof template for greedy algorithm

1. Describe the correctness as a proposition about natural number n, which claims greedy algorithm yields correct solution.
    - Here, n could be the algorithm steps or input size.
2. Prove the proposition is true for all natural number.
    - Induction basis: from the smallest instance
    - Induction steps: type 1 or type 2 induction

Proposition for Earliest-Finish-Time-First

Let S be the job set of size n, s_i and f_i are the start time and finish time, A be a maximum compatible subset of S.

Proposition. When algorithm GreedySelect carries on the k-th step, it choose k jobs $(i_1 = 1, i_2, \ldots, i_k)$, which is exactly the first k jobs of A.

According the above proposition, $\forall k$, the first k-step choice is exactly the first k-jobs of some maximum compatible subset A, and will yield A in at most n steps.

Mathematic Induction: Induction Basis

Let $S = \{1, 2, \ldots, n\}$ be the sorted job set: $f_1 \leq \cdots \leq f_n$

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Let $S = \{1, 2, \ldots, n\}$ be the sorted job set: $f_1 \leq \cdots \leq f_n$

Induction basis. $k = 1$, prove A includes job 1

For an arbitrary maximum compatible subset A, sort jobs in A in ascending order according to the finish time.

If the first job in A is j and $j \neq 1$, then replace job j with job 1, yielding A':

$$A' = (A - \{j\}) \cup \{1\}$$

- 1 won't appear in $(A - \{j\}) \Rightarrow |A| = |A'|$
- $f_1 \leq f_j \Rightarrow$ replacement does not affect compatibility $\Rightarrow A'$ is also one of the maximum compatible subset of A and includes job 1.

Mathematic Induction: Induction Step (1/2)

Assume Proposition is true for k, prove it is also true for $k+1$

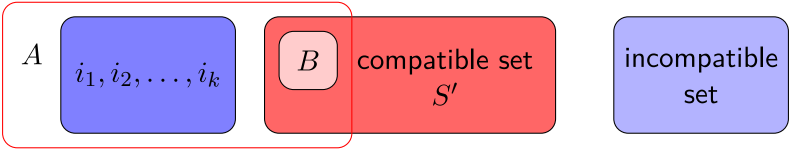
- $(k+1)$-step choice job i_{k+1} and $(i_1, \ldots, i_k)$ forms the first $k+1$ jobs of some A for S.

Proof. After k steps, algorithm chooses $i_1 = 1, i_2, \ldots, i_k$.

Premise $\Rightarrow$ $\exists$ a maximum compatible A that contains $i_1, i_2, \ldots, i_k$.

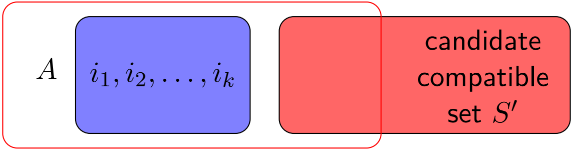
- Let B the set of other elements in A (already sorted and not empty), and S' be the set of compatible elements w.r.t. $\{i_1, i_2, \ldots, i_k\}$.

$$A = \{i_1, i_2, \ldots, i_k\} \cup B$$
$$S' = \{i \mid i \in S, s_i \geq f_k\}$$



Mathematic Induction: Induction Step (2/2)

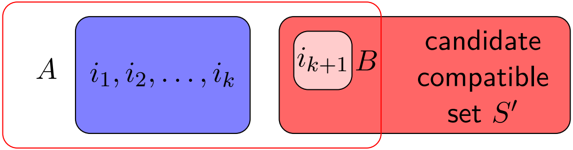
Consider two cases according to if job i_{k+1} is the 1st job in B.



Mathematic Induction: Induction Step (2/2)

Consider two cases according to if job i_{k+1} is the 1st job in B.

- If i_{k+1} happens to the first job in B, then the desired result immediately follows, $(k+1)$-step choice still yields the partial solution of A.



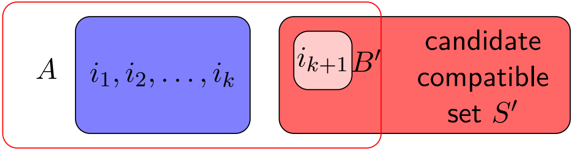
Mathematic Induction: Induction Step (2/2)

Consider two cases according to if job i_{k+1} is the 1st job in B.

- If i_{k+1} happens to the first job in B, then the desired result immediately follows, $(k+1)$-step choice still yields the partial solution of A.
- If i_{k+1} is not the first job in B, then we must have $i_{k+1} \notin B$
  - the strategy choice of the greedy algorithm $\Rightarrow$ the finish time of i_{k+1} must be earlier than the first job in B
  - At this point, we can replace the first job in B with job i_{k+1}, yielding B'. Obviously, $|B'| = |B|$.

$$\{i_1, i_2, \ldots, i_k\} \cup B' = A'$$

Note that $|A| = |A'| \Rightarrow A'$ is still a maximum compatible set of S. This proves the induction step.



Outline

1. Introduction of Greedy Algorithm
2. Interval Scheduling
3. Optimal Loading
4. Scheduling to Minimizing Lateness
5. Fractional Knapsack Problem
6. Greedy Algorithm Does Not Work (not teach in class)

Optimal Loading Problem

Problem. Given n containers with weight w_i and a boat with maximum weight capacity W (no volume limit).

Goal. A loading plan that maximizes the number of containers on the ship.

Analysis. This problem is a special case of 0-1 knapsack problem.

- item: container
- boat: knapsack
- all $v_i = 1$

Modeling

Let $(x_1, x_2, \ldots, x_n)$ be the solution vector, $x_i \in \{0, 1\}$.

- $x_i = 1$ iff i-th container is on the boat

Goal function:

$$\max \sum_{i=1}^{n} x_i$$

Constraint:

$$\sum_{i=1}^{n} w_i x_i \leq W, x_i = \{0, 1\}, i \in [n]$$

Algorithm Design

Greedy strategy. lightest first

Algorithm steps

- sorting container according to weight in ascending order, to ensure $w_1 \leq w_2 \leq \cdots \leq w_n$
- loading the container from the smallest label, and stop until loading next container will exceed the limit

Proof of Correctness (Induction on Input Size)

Lemma. $\forall$ input size n, the algorithm yields the correct solution.

Let $S = \{1, 2, \ldots, n\}$ be the set of containers that has been sorted in ascending order, and $w_1 \leq w_2 \leq \cdots \leq w_n$.

- Induction basis. Prove when the input size $n = 1$ (there is only one container), the greedy algorithm will yield the correct solution. Obviously hold.
- Induction steps. Prove if the greedy algorithm yield optimal solution for input size n, it will also yield optimal solution for input size $n + 1$.

Analysis of Greedy Algorithm: Interpretation

$S = \{1, 2, \ldots, n+1\}$, $w_1 \le \cdots \le w_{n+1}$

$\downarrow$

if $W < w_1$, return $S = \{\perp\}$

$\downarrow$

else remove container 1, let $W' = W - w_1$
input size is n: $S' = \{2, 3, \ldots, n+1\}$

$\downarrow$

optimal solution I'
for (S', W')

$\downarrow$

$I \leftarrow \{I'\} \cup \{1\}$

$\downarrow$

prove I is the optimal solution for (S, W)

Correctness Proof (1/2)

Premise of induction: greedy strategy will yield optimal solution for input size n, consider input size $n+1$

$$S = \{1, 2, \ldots, n+1\}, w_1 \leq w_2 \leq \cdots \leq w_{n+1}$$

Premise of induction $\Rightarrow$ for input size n

$$S' = \{2, \ldots, n+1\}, W' = W - w_1$$

Greedy strategy yields optimal solution I' for (S', W').

Let $I = I' \cup \{1\}$.

Correctness Proof (2/2)

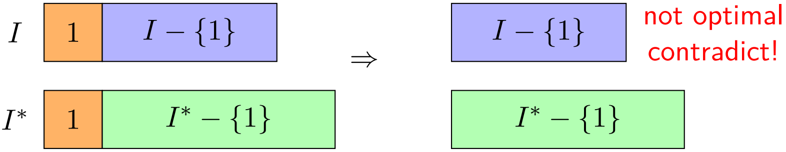
Claim. I is the optimal solution for (S, W).

Proof by contradiction. If not, suppose there exists an optimal solution I^* for (S, W) and $|I^*| > |I|$.

- Assume w.l.o.g. $1 \in I^*$, since otherwise we can replace 1 with the first container in I^*, also yield the optimal solution.
- $I^* - \{1\}$ forms a solution for (S', W') and

$$|I^* - \{1\}| > |I - \{1\}| = |I'|$$

The existence of I^* contradicts to the premise that I' is the optimal solution for (S', W').



Summary

0-1 knapscak is an $\mathcal{NP}$-hard problem

- optimal loading is a variant of 0-1 knapscak problem, and can be solevd using greedy algorithm efficiently

Correctness proof. Induction on input size

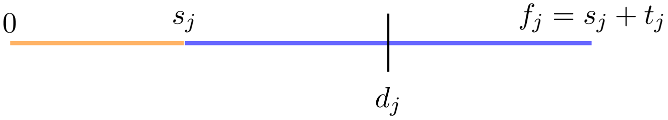
Outline

Scheduling to Minimizing Lateness

Minimizing lateness problem (最小延迟调度)

- A job set A, single resource processes one job at a time, all jobs come in at time 0
- Job j requires t_j units of processing time and is due at time d_j (ddl). Clearly, $t_j \leq d_j$.
- If job j starts at time s_j, it finishes at time $f_j = s_j + t_j$.
- Scheduling: $S : A \to \mathbb{N}$, $S(j) = s_j$ is the start time of job j.
- Lateness: Lateness function computes the lateness of job:

$$L(j) = \ell_j = \max\{0, f_j - d_j\} = \max\{0, s_j + t_j - d_j\}$$



Goal. Schedule all jobs to minimize max lateness

$$\min\{\max_{j \in A} \ell_j\} = \min\{\max_{j \in A}\{\max\{0, s_j + t_j - d_j\}\}\}$$

Constraint. No overlap

$$\forall i, j \in A, i \neq j$$
$$s_i + t_i \leq s_j \lor s_j + t_j \leq s_i$$

Example 1

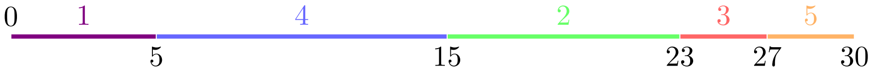
A	1	2	3	4	5
S	0	5	13	17	27
T	5	8	4	10	3
D	10	12	15	11	20
L	0	1	2	16	10

Table: Sequential scheduling

Example 2

A	1	4	2	3	5
S	0	5	15	23	27
T	5	10	8	4	3
D	10	11	12	15	20
L	0	4	11	12	10

Table: Earliest-deadline first



Minimizing Lateness: Greedy Algorithms

Greedy template. Schedule jobs according to some natural order.

- [Shortest processing time first] Schedule jobs in ascending order of processing time t_j.

A	1	2
T	1	10
D	100	10

  - $\ell_1 = 0$, $\ell_2 = 11 - 10 = 1$
  - $\ell_2 = 0$, $\ell_1 = 0$ (better)

- [Smallest slack] Schedule jobs in ascending order of slack $d_j - t_j$.

A	1	2
T	1	10
D	2	10

  - $\ell_2 = 10 - 10 = 0$, $\ell_1 = 11 - 2 = 9$
  - $\ell_1 = 0$, $\ell_2 = 10 + 1 - 10 = 1$ (better)

Minimizing Lateness: Earliest Deadline First

Algorithm 2: Schedule(A, T, D)

sort n jobs in A so that $d_1 \le d_2 \le \cdots \le d_n$;
$t \leftarrow 0$ //from time 0;
for $j = 1$ *to* n **do**
    assign job j to interval $[t, t + t_j]$;
    $s_j \leftarrow t$;
    $f_j \leftarrow t + t_j$;
    $t \leftarrow t + t_j$
end
return *intervals* $[s_1, f_1], \ldots, [s_n, f_n]$

Main idea

- earliest deadline first
- assign jobs one after another, no idle time

Correctness Proof: Exchange Argument

Proof sketch

- Analyze the difference between optimal solution and algorithm solution (e.g. different order)
- Design a transform operation (e.g. swap), thus we can gradually convert an optimal solution to algorithm solution in finite steps.
- The transformation does not affect optimality of solution, since every step preserving optimality.

In this case, two properties of greedy algorithm solution:

- No idle time: every time there is a job being processed
- No inversion. We say (i, j) forms an inversion if $d_i > d_j$ but $s_i < s_j$

Key Lemma about Algorithm Solution

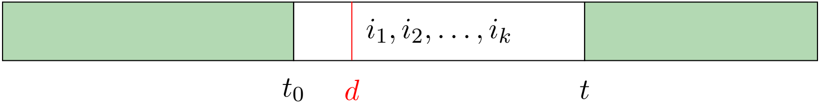
Lemma. All schedulings with no inversion and idle time have the same minimal max lateness time.

Proof. No inversion $\Rightarrow$ tasks are sorted in ascending order of d_i.

It is possible that several jobs has the same deadline. Jobs $i_1, i_2, \ldots, i_k$ with the same deadline d are assigned arbitrarily. (green parts are identical)

- The start time is t_0, the finish time for one of these jobs is t, among this jobs, the maximal lateness is $\max\{0, t-d\} \Leftarrow$ irrelevant to the order of $i_1, i_2, \ldots, i_k$.

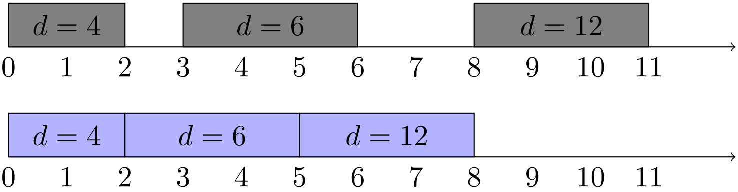
$$t = t_0 + (t_{i_1} + t_{i_2} + \cdots + t_{i_k})$$



Corollary. All possible algorithm solutions have the same minimal max lateness time.

Examine the Optimal Solution

Observation. There always exists an optimal schedule with no idle time.



Algorithm solution: the earliest-deadline-first schedule has no idle time.

- We have eliminate one difference between optimal solution and algorithm solution.
- There is another one: inversion

Minimizing Lateness: Inversions

Inversion. Given a schedule S, an inversion is a pair of jobs i and j such that $d_i < d_j$ but j scheduled before i, i.e., $s_j < s_i$.

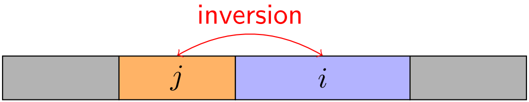


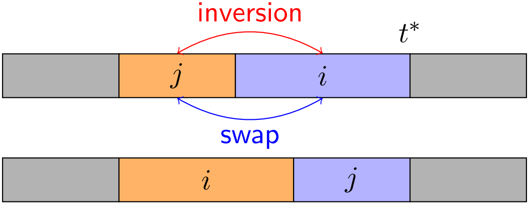
Figure: As before, jobs are numbered so that $d_1 \leq d_2 \leq \cdots \leq d_n$

Algorithm solution: the earliest-deadline-first schedule has no inversions.

Fact. If a schedule (with no idle time) has an inversion, it has at least one pair of inverted jobs scheduled consecutively. (according to definition)

Minimizing Lateness: Inversions

Claim. Swapping two adjacent, inverted jobs reduces the number of inversions by one and does not increase the max lateness.



Proof. Let ℓ be the lateness before the swap, ℓ' be it afterwards.

- $i \leftrightarrow j$ does not affect the latest time of other jobs: $\ell'_k = \ell_k$ for all $k \neq i, j$
- $\ell'_i \leq \ell_i$ (because job i has been moved forwards)
- $\ell'_j = \max\{0, t^* - d_j\}$ (definition), i and j are inverted $\Rightarrow$ $d_i < d_j$, thus $\ell'_j \leq \max\{0, t^* - d_i\} = \ell_i$

$$\Rightarrow \max\{\ell_i, \ell_j\} \geq \max\{\ell'_i, \ell'_j\}$$

Putting All the Above Together

Theorem. The earliest-deadline-first schedule S is optimal.

Proof. Define S^* to be an optimal schedule that has the fewest number of inversions, and let's see what happens.

- Can always assume S^* has no idle time.
- If S^* has no inversions, then key lemma $S \sim S^*$, stop here.
- If S^* has an inversion, let $i \leftrightarrow j$ be an adjacent inversion. Swapping i and j:
  - does not increase the max lateness
  - strictly decreases the number of inversions
- Continue the above process until there is no inversion, we can also conclude that $S \sim S^*$.

Max number of inversion is $n(n-1)/2$ (completely inverted), thus the transformation will stop in finite steps.

Summary of Greedy Analysis Trick

Analysis. Find the difference between optimal solution and algorithm solution.

Exchange argument. Gradually transform an optimal solution to the one found by the greedy algorithm.

- at most require finite steps (seems unnecessary)
- each step of transformation does not hurt its quality

Outline

Fractional Knapsack Problem

Input. Given n items with weight vector $(w_1, \ldots, w_n)$ and value vector $(v_1, \ldots, v_n)$, and weight limit $W > 0$.

Goal. Find $x = (p_1, \ldots, p_n) \in [0, 1]^n$ (choose some fractions of n items) to satisfy:

- Optimized goal: maximizes $\sum_{i=1}^{n} p_i v_i$
- Constraint: $\sum_{i=1}^{n} p_i w_i \leq W$

The difference is that now the items are infinitely divisible.

Greedy Algorithm

Greedy strategy. greatest value-per-weight ratio first

Algorithm

- Sort n items according to the decending order of value-per-weight ratio $\alpha_i = v_i/w_i$.
- iteratively picks the item with the greatest value-per-weight ratio
- if, at some step, the knapsack cannot fit the entire last item with current greatest value-per-weight ratio items, we will take a fraction of it to fill the knapsack.

Correctness Proof (1/3)

Lemma. $\forall$ input size n, the algorithm yields the optimal solution.

Proof idea. Mathematical reduction on input size.

Induction basis. When $n = 1$, the greedy algorithm is obviously the optimal solution.

Induction step. Suppose the algorithm is optimal for $n = k$, then it is also optimal for $n = k + 1$.

- Let p_1 be the algorithm's output for the first item, $I' = (p_2, \ldots, p_{k+1})$ be the output on instance $(w_2, \ldots, w_{k+1})$, $(v_2, \ldots, v_{k+1})$, and $W - p_1 w_1$.
- According to the induction premise, I' is the optimal solution of the above sub-instance of size $n = k$. Let $I = p_1 \cup I'$.

Claim. Then, we claim I is the optimal solution for $n = k + 1$.

Correctness Proof (2/3)

Proof by contradiction. If not, suppose there exists a more optimal solution I^* with maximal value V^*.

Prove the first element p_1^* of I^* must be equal to p_1 of I.

1. $p_1^* = p_1$: we have nothing to prove.
2. $p_1^* > p_1$ is impossible, because the greedy strategy guarantees that p_1 of I is as large as possible.
3. If $p_1^* < p_1$, we can always increase it to p_1 by decreasing total weight of its remaining k items by $\Delta = (p_1 - p_1^*)w_1$. Note that such adjustment makes sense since the total weight of the remaining k items must be larger than Δ. Otherwise, we must have $V^* < V$, which is not possible by premise. We then consider two sub-cases after adjustment:
   - The total value is unchanged. This is only possible when there exists at least one more item j such that $\alpha_j = \alpha_1$.
   - The total value is higher. In this case, we must have there is no j such that $\alpha_j = \alpha_1$. However, this case will never occur since it goes against the assumed optimality of I^*.

Correctness Proof (3/3)

We conclude that either $p_1^* = p_1$ or we can adjust it to this case without compromising optimality.

$I^* - \{p_1\}$ forms a solution for $W - p_1^* w_1 = W - p_1 w_1$ with items $(2, \ldots, n+1)$ with total value $V^* - \alpha_1 p_1 > V - \alpha_1 p_1$
$\rightsquigarrow$ contradicts the optimality of I'

This proves I is the optimal solution for input size $n = k + 1$.

Outline

What if Greedy Algorithm Does not Work

Input analysis

- Determine the range of input that greedy strategy works.

Error analysis

- Greedy algorithm is the approximation algorithm of the problem: estimate the distance between greedy solution and optimal solution (the upper bound over all inputs)

Coin Changing Problem

Coin changing. Given n currency denominations

- $v_1 = 1, v_2, \ldots, v_n$, $v_1 < v_2 < \cdots < v_n$,
- weight $w_1, w_2, ..., w_n$.

Goal. Devise a method to pay amount y using coins with lightest weight.

Example. $v_1 = 1$, $v_2 = 5$, $v_3 = 14$, $v_4 = 18$, $w_i = 1$, $i \in [n]$, $y = 28$. In this case, the problem is equivalent to using fewest number of coins.

Solution.: $x_3 = 2$, $x_1 = x_2 = x_4 = 0$, total weight is 2.

Modeling

Let x_i be the number of coin i, $i \in [n]$

Goal function.

$$\min \left\{ \sum_{i=1}^{n} w_i x_i \right\}$$

Constraint.

$$\sum_{i=1}^{n} v_i x_i = y, x_i \in \mathbb{N}, i \in [n]$$

Next, we consider a special case: $w_i = 1$ for all $i \in [n]$.

Dynamic Programming

$F_k(y)$: the lightest weight using first k types of coins to pay amount y

The iteration equation

$$\begin{cases} F_k(y) = \min_{0 \le x_k \le \lfloor \frac{y}{v_k} \rfloor} \{F_{k-1}(y - v_k x_k) + 1 \cdot x_k\} \\ F_1(y) = \frac{y}{v_1} = y \end{cases}$$

- Dynamic programming requires the domination of the first coin is 1 to ensure the constraint can always be met.
- Dynamic programming always give the optimal solution.

Greedy Algorithm

Strategy. Smallest w_i/v_i coin first. Since all $w_i = 1$, this means largest domination coin first and $v_1 = 1$.

$$\frac{1}{v_1} > \frac{1}{v_2} > \cdots > \boxed{\frac{1}{v_n}}$$

$G_k(y)$: greedy solution of using first k types coins to pay y

$$\begin{cases} G_k(y) = \left\lfloor \frac{y}{v_k} \right\rfloor + G_{k-1}(y \bmod v_k), k > 1 \\ G_1(y) = \frac{y}{v_1} = y \end{cases}$$

Thinking. Why we require all $w_i = 1$? Otherwise, we cannot guarantee $v_1 = 1$ appears at first place in line with greedy algorithm's input order. Looking ahead, we will use dynamic programming as a reference.

$n = 1, 2$: Greedy Strategy Yield Optimal Solution

$n = 1$: only one type of coin and we must have $v_1 = 1$.

- In this case, $F_1(y) = G_1(y) = w_1 y$

$n = 2$: for dynamic programming algorithm, the larger is x_2, the better is the solution

$$F_2(y) = \min_{0 \leq x_2 \leq \lfloor y/v_2 \rfloor} \{F_1(y - v_2 x_2) + x_2\}$$

Goal: prove $F_2(y) = G_2(y)$

Technique: decide the monotonicity of function $F_1(y - v_2 x_2) + x_2$ about x_2

$$\begin{aligned} & [F_1(y - v_2(x_2 + \delta)) + (x_2 + \delta)] - [F_1(y - v_2 x_2) + x_2] \\ = & [(y - v_2 x_2 - v_2 \delta) + x_2 + \delta] - [(y - v_2 x_2) + x_2] \\ = & - v_2 \delta + \delta = \delta(1 - v_2) < 0 \end{aligned}$$

This proves the greedy that choice is optimal for $n = 2$.

Criteria

Theorem. Let n_0 be an integer. Suppose $\forall k \leq n_0$, $G_k(y) = F_k(y)$ for all $y \in \mathbb{N}$. Let (p, δ) be the tuple such that $v_{k+1} = pv_k - \delta$, where $0 \leq \delta < v_k$, $v_k < v_{k+1}$, $p \in \mathbb{Z}^+$.
The following propositions are equivalent:

1. $G_{k+1}(y) = F_{k+1}(y)$ for all $y \in \mathbb{Z}^+$;
2. $G_{k+1}(pv_k) = F_{k+1}(pv_k)$ (can be used to give counterexample)
3. $1 + G_k(\delta) \leq p$ (can be used to decide if the first statement holds)

The uniqueness of (p, δ):

- Since $v_{k+1} > v_k$, v_{k+1} can be uniquely expressed as $p'v_k + \eta$, where $0 \leq \eta < v_k$.
- $p'v_k + \eta = (p' + 1)v_k - (v_k - \eta)$. Set $p' + 1 = p$, $v_k - \eta = \delta$. The uniqueness of (p', η) implies the uniqueness of (p, δ).

Some Remarks

By the equivalence of (1) and (3), we can decide if greedy algorithm gives the optimal solution for $k \geq 3$.

Verifying the truth of statement (3) requiring $O(k)$ complexity.

Statement (2) is a special case of proposition (1) when $y = pv_k$.

Statement (1) is true $\Rightarrow$ Statement (2) is true
Statement (2) is false $\Rightarrow$ Statement (1) is false

The amount $y = pv_k$ provide a counterexample for the correctness of greedy algorithm.

Demo: $n = 3$

$$v_{k+1} = pv_k - \delta, 0 \le \delta < v_k, p \in \mathbb{Z}^+$$

$$\boxed{\text{proposition (3)} : 1 + G_k(\delta) \le p}$$

Example. $v_1 = 1$, $v_2 = 5$, $v_3 = 14$, $v_4 = 18$.

$\forall y$: $G_1(y) = F_1(y)$, $G_2(y) = F_2(y)$

Decide if $G_3(y) = F_3(y)$

To utilize proposition (3), we first compute tuple (p, δ):
$v_3 = pv_2 - \delta \Rightarrow p = 3, \delta = 1$

$$1 + G_2(\delta) = 1 + 1 = 2 \le 3 = p$$

Conclusion: proposition (3) is true thus greedy algorithm still works for $n = 3$.

Demo: $n = 4$

Example. $v_1 = 1$, $v_2 = 5$, $v_3 = 14$, $v_4 = 18$.

$\forall y$ we have: $G_1(y) = F_1(y)$, $G_2(y) = F_2(y)$, $G_3(y) = F_3(y)$

Decide if $G_4(y) = F_4(y)$

To utilize proposition (3), we first compute tuple (p, δ):
$v_4 = pv_3 - \delta \Rightarrow p = 2, \delta = 10$

$$1 + G_3(\delta) = 1 + 2 > p = 2$$

Conclusion: proposition (3) is false thus greedy algorithm does not work for $n = 3$.

Counterexample is give by proposition (2), i.e. $n = 4$,
$y = pv3 = 28$

Optimal solution $x_3 = 2$ vs. Greedy solution $(x_4 = 1, x_2 = 2)$