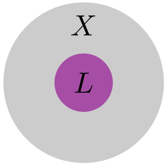
Design and Analysis of Algorithm
Basics of Complexity Theory

Outline

Decision Problem

Decision Problem: recognition of a set of strings $L \subseteq X$

- X: a set of strings
- x: a string in X (each string corresponds to an instance)
- L: language (a subset of X satisfying some property)



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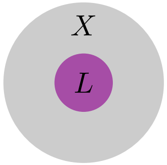

Task: Decide membership — if $x \in L$

Example

- $X = \mathbb{N}$
- L are Primes $= \{2, 3, 5, 7, 11, 13, \ldots\}$
- decide if x is a prime.

Motivation for Complexity Theory

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1. Precisely model algorithms
    - What is computation?
    - What is computable?
2. Precisely define what does it means for efficient.

Outline

Turing Machine

1936, London Mathematical Society: On computable numbers, with an application to the Entscheidungs Problem.

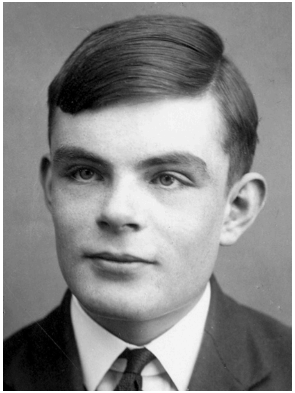
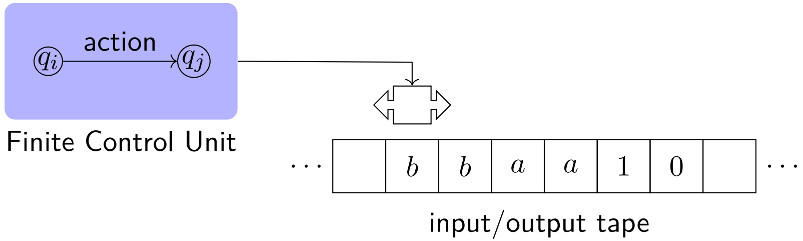


Figure: Alan Turing

Turing Machine

Turing machine: automatic machine that has a tape (divided into infinite cells), a control unit and a read/write head.



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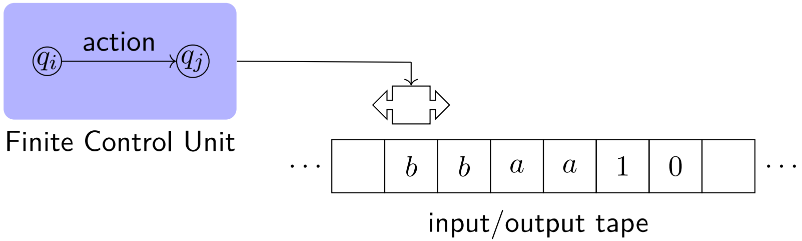

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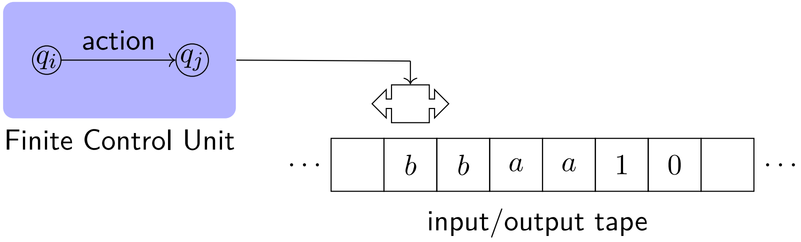
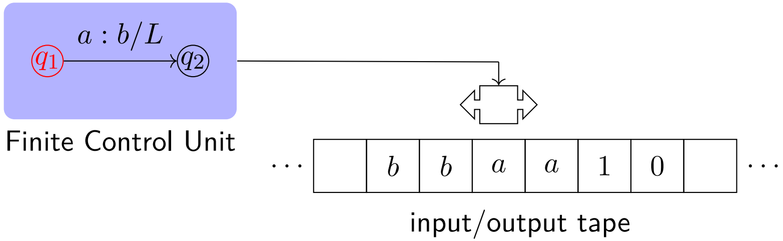
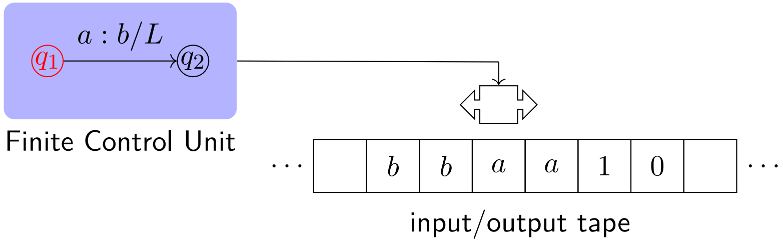


- At the beginning, the tape contains the input in several cells. Other places are empty.
- During computation, the control unit monitor current state and the head value, can do the following operations:
  1. wipe off old value and write new values
  2. change the current state
  3. move head left or right

An Example


$a : b/L$
q_1
q_2
Finite Control Unit
$\cdots$
b
b
a
a
1
0
$\cdots$
input/output tape

An Example



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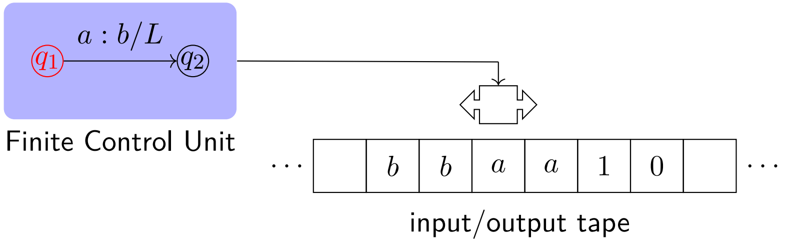

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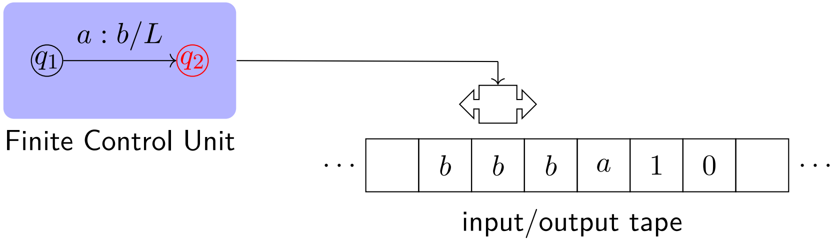

Intuition of Turing Machine

Mimic how human being solve a problem

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Mimic how human being solve a problem

- TM has a finite number of states (memory)
- TM is provided a tape, which contains infinite cells (paper)
- a symbol can be scanned from a cell or printed to a cell (reading and writing)

Formal Definition

Definition 1 (Turing Machine)

TM consists $(Q, \Sigma, \Gamma, \delta, q_0, q_{\text{acc}}, q_{\text{rej}})$

- Q: a finite set of states
- Σ: input alphabets
- Γ: working alphabets (including $\perp$, $\Sigma \subseteq \Gamma$)
- q_0: the initial state of Q;
- $q_{\text{acc}}, q_{\text{rej}}$: accept and reject state of Q
- δ: transition function

$$\delta : (Q \backslash \{q_{\text{acc}}, q_{\text{rej}}\}) \times \Gamma \to Q \times \Gamma \times \{L, R\}$$

Running Time of TM

Definition 2

We denote the running time of TM by $t_M(n)$, which is the maximum steps that TM runs on all inputs of length n

Polynomial Time

$$\bigcup_{k\in\mathbb{N}} \mathsf{TIME}(n^k)$$

The Extended Church-Turing Thesis

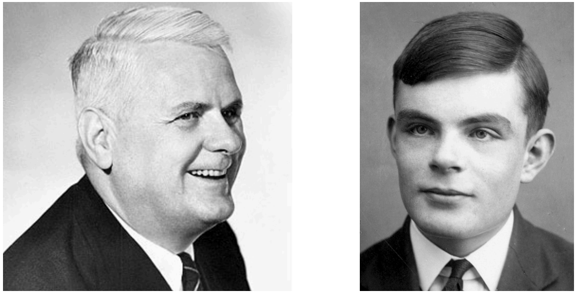
Figure: Alonzo Church & Alan Turing

Everyone's intuition of **Efficient** Algorithms $=$ **Polynomial-Time** deterministic TMs

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Non-determinism doesn't give TM any power to recognize more languages.

- Any NDTM can be simulated by a TM (with potentially exponential time overhead) by trying all branches of the NDTM machine "in parallel" by using BFS.

Notes

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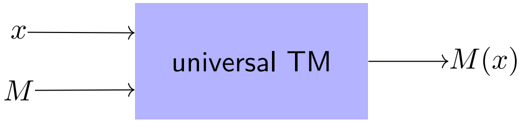
Why TMs are so powerful?

- TM has a working tape (好记性不如烂笔头)
- TM itself can be treated as data! TM can take another TM as its input.

Universal TM



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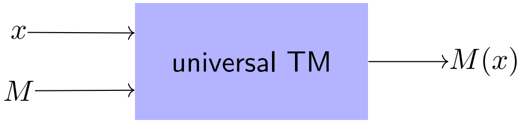

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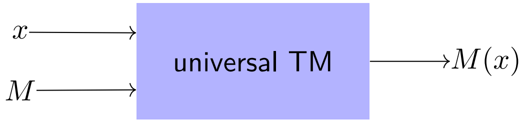

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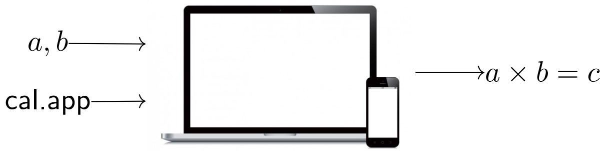

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Next, we introduce two important sets of problems, characterized by time complexity by DTM and NDTM:

$\mathcal{P}$ and $\mathcal{NP}$

$\mathcal{P}$ Complexity

Definition 3 ($\mathcal{P}$ Language)

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Example of $\mathcal{P}$ Languages

- $L = \{$even integers$\}$, M just need to check if the last bit is 0.
- $L = \mathsf{PRIME}$, M is the AKS primality test algorithm.

$\mathcal{NP}$ Complexity

Definition 4 ($\mathcal{NP}$ Languages - Conventional)

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Warning

$\mathcal{NP}$ means non-deterministic poly-time, not non-poly-time!

Modern Definition

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Equivalence between traditional and modern definitions

- Even though M is a deterministic machine, its second argument w captures the nondeterminism in the definition.

Examples of $\mathcal{NP}$ Language - Composites

$L =$ COMPOSITE

- instance x is an integer
- a witness w for $x \in L$ is a non-trivial factor of x
- M just need to check if w divides x, which could be done in polynomial time.

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$\mathcal{P}$ is closed under complement, i.e., $L \in \mathcal{P} \Rightarrow \overline{L} \in \mathcal{P}$.

- A deterministic TM can simply flip the accept/reject output at the end, and this does not change the running time.

Examples of $\mathcal{NP}$ Language - SAT and 3-SAT

SAT: Given a CNF formula Φ, check if it has a satisfying truth assignment.

3-SAT: SAT where each clause contains exactly 3 literals

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Example of 3-SAT

- instance $\Phi = (\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (\overline{x_1} \vee x_2 \vee x_4)$
- witness: $x_1 = 1$, $x_2 = 1$, $x_3 = 0$, $x_4 = 0$

Examples of $\mathcal{NP}$ Language - Hamilton Path

Hamilton Graph: Given an undirected graph $G = (V, E)$, does there exists a simple path that visits every node?

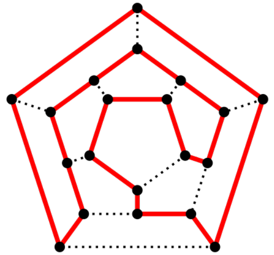

Figure: Hamiltonian Graph (a path traverses through each verticals exactly once)

witness: a path

M check if the path contains each node in V exactly once

$\mathcal{P}$ vs. $\mathcal{NP}$

As per definition, $\mathcal{P} \subseteq \mathcal{NP}$. Because $L \in \mathcal{P} \Rightarrow L \in \mathcal{NP}$:

- $M'(x, w)$ can always sets $w = \perp$ and decide whether $x \in L$ using M.
- Alternatively, “short” M can be viewed as a witness for $x \in L$. Think about why the description of M is short?

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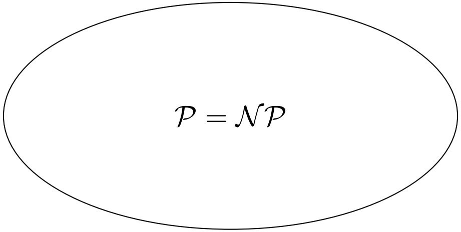
1971: Cook, Edmonds, Levin, Yablonski, Gödel

Perhaps the most prominent question in TCS:

$$\mathcal{P} =? \mathcal{NP}$$

$\mathcal{P} = \mathcal{NP}$



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In principle, every aspect of life could be efficiently and globally optimized $\cdots$

$\cdots$ life as we know it would be different!

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Let $f : \{0,1\}^n \to \{0,1\}^m$. To efficiently find a pre-image x of y, the idea is to determine x bit-by-bit. $f(x_1 || \cdots || x_n) = y$.

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Define a collection of languages $L_i = \{(y,z) | \exists w$ s.t. $y = f(z||w)\}$, where $z \in \{0,1\}^i$, $w \in \{0,1\}^{n-i}$

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Algorithm 4: Invert(y)

```
1: $z = \epsilon$;
2: for $i \leftarrow 1$ to $n$ do
3:     if $(y, z||0) \in L_i$ then $z = z||0$;
4:     else $z = z||1$;
5: end
6: return $z$
```

The Reverse Direction

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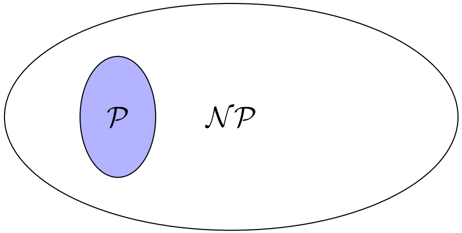
Warning

OWFs do not exist *does not imply* $\mathcal{P} = \mathcal{NP}$

Consensus



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Exhaustive search: try all 2^n truth assignments.

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- Because $\mathcal{NP}$ contains many languages that are not believed to be in $\mathcal{P}$, such as: 3-SAT and Hamiltonian graph.

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Conjecture: $\underbrace{\text{No poly-time algorithm}}_{\text{intractable}}$ for 3-SAT

Outline

Motivation of Reduction

$\mathcal{NP}$ is the set of many problems.

How to figure out the relations among them?

A central approach is finding reductions

Polynomial Time Reducibility

Language L' is *poly-time reducible* or *reduces* to language L, written as $L' \leq_p L$, if there is a determinstic poly-time function $\mathcal{R} : L' \to L$ so that:

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We should pay attention to:

- the direction of $\mathcal{R}$
- the time complexity of $\mathcal{R}$

$\mathcal{NP}$-Hard

Definition 6 ($\mathcal{NP}$-Hard)

L is said to be $\mathcal{NP}$-hard if for every $\mathcal{NP}$-language L', there is a deterministic poly-time algorithm (a reduction) $\mathcal{R}$:

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Fact: languages in $\mathcal{NP}$-hard may not fall in $\mathcal{NP}$.

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Definition Intuition: $\mathcal{NP}$-complete represents the set of hardest problems in $\mathcal{NP}$.

- We can solve all problems in $\mathcal{NP}$ if we find an efficient algorithm for any problems in $\mathcal{NP}$-complete.

The Meaning of $\mathcal{NP}$-complete

Theorem 8

Suppose $Y \in \mathcal{NP}$*-complete, then* $Y \in \mathcal{P} \iff \mathcal{P} = \mathcal{NP}$.

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- This theorem essentially states that if $\mathcal{P} \cap \mathcal{NPC}$ is non-empty iff $\mathcal{P} = \mathcal{NP}$.

$\mathcal{P}$ vs. $\mathcal{NP}$ revisited

Overwhelming consensus (still): $\mathcal{P} \neq \mathcal{NP}$.

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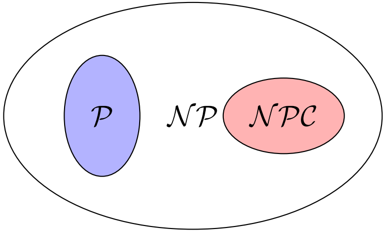


Figure: $\mathcal{P} \neq \mathcal{NP}$

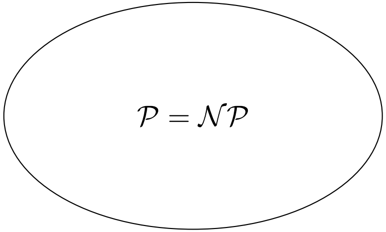


Figure: $\mathcal{P} = \mathcal{NP}$

$\mathcal{P}$ vs. $\mathcal{NP}$ revisited

Overwhelming consensus (still): $\mathcal{P} \neq \mathcal{NP}$.

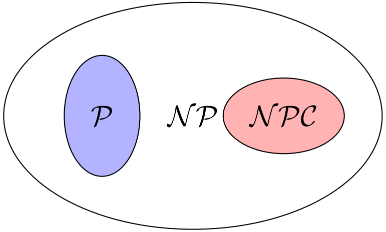


Figure: $\mathcal{P} \neq \mathcal{NP}$

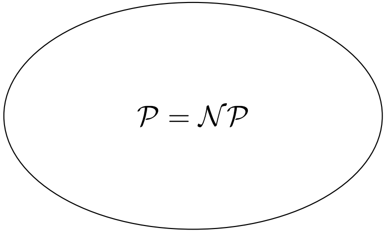
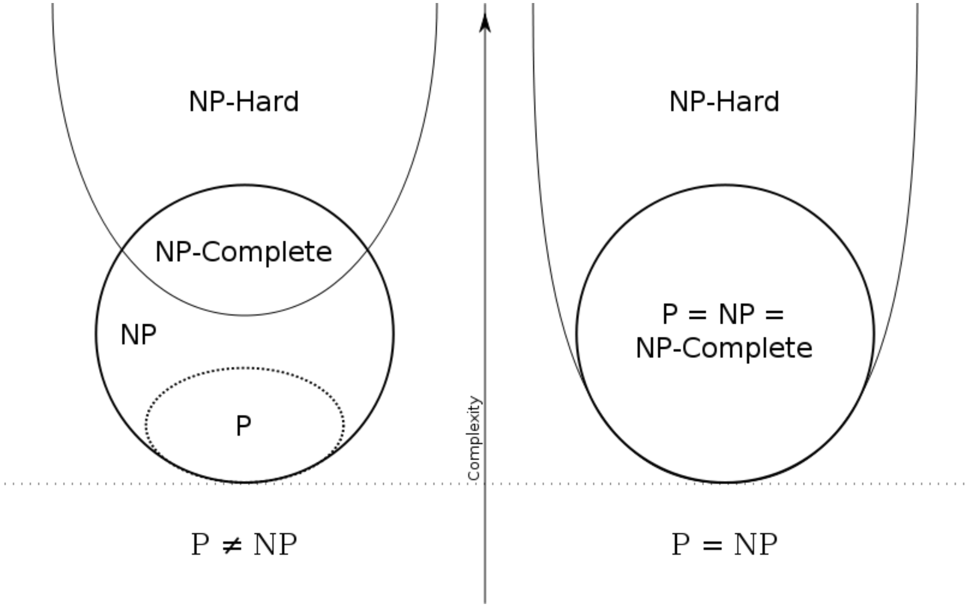


Figure: $\mathcal{P} = \mathcal{NP}$

Why we believe $\mathcal{P} \neq \mathcal{NP}$? Because some problems appear significantly harder.


NP-Hard
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NP
P
Complexity
NP-Hard
P = NP =
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Cook-Levin Theorem

Do there exist "natural" $\mathcal{NP}$-complete problems?

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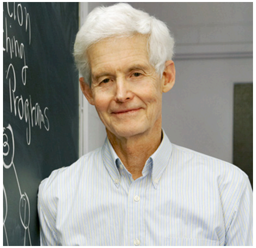
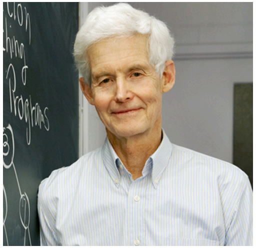


Figure: Stephen Cook & Leonid Levin

Cook-Levin Theorem

Do there exist "natural" $\mathcal{NP}$-complete problems?



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Cook-Levin theorem proves that SAT $\in \mathcal{NPC}$. Via Cook-Levin reduction, we can reduce any problem in $\mathcal{NP}$ to SAT.

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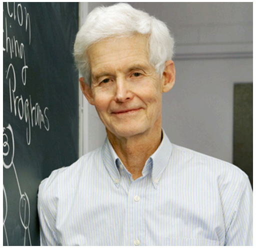

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Cook-Levin theorem proves that SAT $\in \mathcal{NPC}$. Via Cook-Levin reduction, we can reduce any problem in $\mathcal{NP}$ to SAT.

- Other examples of $\mathcal{NP}$-complete including graph 3-colorability, graph Hamiltonicity, and so on.

Outline

Motivation for co-$\mathcal{NP}$

Asymmetry of $\mathcal{NP}$: We need short certificates only for yes instances.

$$x \in L \iff \exists w \in \{0,1\}^{\mathsf{poly}(|x|)} \text{ s.t. } M(x,w) = 1$$
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- Can prove a CNF formula is satisfiable by specifying an assignment.
- How could we prove that a formula is not satisfiable?

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HAM-CYCLE vs. NO-HAM-CYCLE

- Can prove a graph is Hamiltonian by specifying a path.
- How could we prove that a graph is not Hamiltonian?

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- $L = \{4, 6, 8, 9, 10, 12, 14, 15, \dots\}$, $\overline{L} = \{2, 3, 5, 7, 11, 13, 17, 23, 29, \dots\}$

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Complements of decision problems in $\mathcal{NP}$.

Examples: UN-SAT, NO-HAM-CYCLE, and PRIMES.

$\mathcal{NP}$ vs. co-$\mathcal{NP}$

Fundamental open question: Does $\mathcal{NP} =$ co-$\mathcal{NP}$?

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If $\mathcal{NP} \neq \text{co-}\mathcal{NP}$, then $\mathcal{P} \neq \mathcal{NP}$.

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Proof idea.

- $\mathcal{P}$ is closed under complementation.
- If $\mathcal{P} = \mathcal{NP}$, then $\mathcal{NP}$ is closed under complementation. In other words, $\mathcal{NP} = \text{co-}\mathcal{NP}$.
- This is the contrapositive of the theorem.

$\mathcal{NP} \cap$ **co-**$\mathcal{NP}$

Good characterization: [Edmonds 1965] $\mathcal{NP} \cap$ co-$\mathcal{NP}$

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If problem $X \in \mathcal{NP} \cap$ co-$\mathcal{NP}$, then:

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- for no instance, there is a succinct disqualifier

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Provides conceptual leverage for reasoning about a problem.

Outline

Motivation of Randomized Algorithm

TM models deterministic algorithms.

TM does not seem to capture one aspect of reality — the ability to make random choices during computation

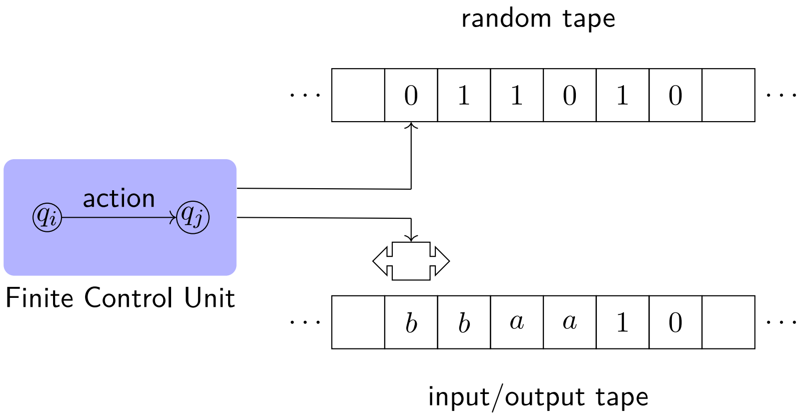
- Most programming languages provide a built-in RNG.

It makes sense to consider algorithms that can toss a coin, a.k.a. use a source of random bits. Such algorithms have been implicitly studied for a long time.

- estimate facts about a large sample by taking a small sample
- simulate real-world systems that are themselves probabilistic, such as nuclear fission and the stock market
- differential equations

Probabilistic Turing Machine

Probabilistic Polynomial-time TM models probabilistic algorithm.



random tape
$\cdots$ 0 1 1 0 1 0 $\cdots$
q_i
action
q_j
Finite Control Unit
$\cdots$ b b a a 1 0 $\cdots$
input/output tape

PTM vs. NDTM

NDTM is a TM with two transition functions. PTM is syntactically similar.

The difference is in how we interpret the working of TM.

- In a PTM, each transition is taken with probability $1/2$, a computation that runs for time t gives rise 2^t branches in the graph of all computations, each of which is taken with probability $1/2^t$. $\Pr[M(x) = 1]$ is simply the *fraction* of branches that end with M outputting a 1.
- In a NDTM, $M(x) = 1$ iff there exists a branch that outputs 1

On a conceptual level, PTM and NDTM are very different

- PTM like TM and unlike NDTM, is intended to model realistic computation devices.

Outline

Bounded-Error Probabilistic Polynomial Time

Definition 11 ($\mathcal{BPP}$ Complexity)

$L \in \mathcal{BPP}$ iff there exists a probabilistic polynomial time TM M such that:

$$\begin{aligned} \forall x \in L: \quad & \Pr[M(x) = 1] \geq \tfrac{1}{2} + \epsilon \\ \forall x \notin L: \quad & \Pr[M(x) = 1] \leq \tfrac{1}{2} - \epsilon \end{aligned}$$

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Key properties

- Bounded error: M must be noticeably more likely to be correct than incorrect, i.e., the error is bounded away from the random guess $1/2$ by a noticable gap.
- Error reduction: By repetition and then taking majority vote, the error can be reduced to 2^{-n} with only polynomial overhead.
- Practical meaning: $\mathcal{BPP}$ captures efficient randomized algorithms used in practice (e.g., primality testing before AKS).

Intuition: M is more likely correct than wrong.

Error Reduction

In practice, an error probability $1/2 - \epsilon$ (the gap is ϵ) might not be acceptable.

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However, the choice of gap ϵ could be arbitrary and the set $\mathcal{BPP}$ will be unchanged.

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- It does not even have to be constant: ϵ could be as small as n^{-c}, where c is any positive constant, and n is the length of input.

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The method is running M indepedently several times, then taking the majority vote. Chernoff Bound gurantees the probability that the majority of the runs are wrong drops off exponentially.

Chernoff Bound (Lower Tail): Let $\{X_i\}$ be independent 0-1 random variables such that $\Pr[X_i = b] \geq 1/2 + \epsilon$, $X = \sum_{i=1}^{m} X_i$.

$$\Pr[X \neq b] \leq \frac{1}{e^{\epsilon^2 m/2}}$$

Setting $m \geq n^{2c} + 1$ ensures the error probability is negligible in n.

Why Does Soundness Amplification Not Work?

In the ZKP literature, the error we care about is soundness error:

$$\epsilon = \Pr[\langle P, V\rangle(x) : x \notin L]$$

We can reduce ϵ to ϵ^n by:

- Run the protocol n times in sequential/parallel
- Accept iff all of the repetitions accept

Such method does not work for $\mathcal{BPP}$, cause in ZKP the completeness error is typically 0, while in $\mathcal{BPP}$ is error is two-sided.

- The decision algorithm does not know the input instance belongs to YES or NO. It has to apply the same strategy.

Probabilistic Polynomial Time

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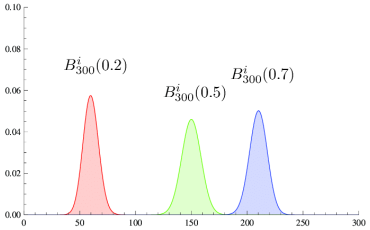
Key properties

- Unbounded error: The acceptance probability may exceed $1/2$ by an exponentially tiny amount.
- No error reduction: amplification: The gap away from $1/2$ may be too small to reduce efficiently.
- Extremely powerful: $\mathcal{PP}$ contains many hard classes.

Intuition: M is just barely more likely correct than wrong.

Why Does Majority Vote Work?

Recall that $X = X_1 + \cdots + X_m$ is a Binomial distribution. When ϵ is noticable, after polynomial times of repetition, the two distributions induced by $x \in L$ and $x \notin L$ are largely detached. Otherwise, they are mingled together $\rightsquigarrow$ hard to find a split line



This also explain for $\mathcal{PP}$, the majority vote does not work if ϵ is negligibly close to 0. For example:

$$x \in L \Rightarrow \Pr[M(x) = 1] \geq \frac{1}{2} + \frac{1}{2^n}$$
$$x \notin L \Rightarrow \Pr[M(x) = 1] \leq \frac{1}{2} - \frac{1}{2^n}$$

Relationship and Power

Containment

$$\mathcal{BPP} \subseteq \mathcal{PP}$$

- Any bounded-error algorithm is also an unbounded-error one.
- The containment is believed to be strict.

The difference between n^{-c} vs 2^{-n} looks small numerically, but it is enormous computationally:

- $\mathcal{BPP}$ requires a noticable probability advantage and models realistic randomized computation
- $\mathcal{PP}$ allows an infinitesimal advantage over guessing, making it far more powerful and closer to counting complexity than to efficient computation.

About $\mathcal{BPP}$

$\mathcal{BPP}$ is one of the largest practical class of problems, since problems in $\mathcal{BPP}$ have efficient probabilistic algorithms that can be run quickly on real modern machines.

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For a long time, one of the most famous problems that was known to be in $\mathcal{BPP}$ but not known to be in $\mathcal{P}$ was the PRIME.

About $\mathcal{BPP}$

$\mathcal{BPP}$ is one of the largest practical class of problems, since problems in $\mathcal{BPP}$ have efficient probabilistic algorithms that can be run quickly on real modern machines.

- Clearly, $\mathcal{BPP} \supseteq \mathcal{P}$, since a deterministic machine is a special case of a probabilistic machine.
- Many problems were known to be in $\mathcal{BPP}$ but not known to be in $\mathcal{P}$. The number of such problems is decreasing, and it is conjectured that $\mathcal{P} = \mathcal{BPP}$.

For a long time, one of the most famous problems that was known to be in $\mathcal{BPP}$ but not known to be in $\mathcal{P}$ was the PRIME.

[AKS04] gave a deterministic polynomial-time algorithm for PRIME, thus showing that it is in $\mathcal{P}$.

Gödel Prize and Fulkerson Prize

One-sided and Zero-sided Error

$\mathcal{ZPP}$: probabilistic polynomial-time TM always returns correct YES or NO answer, or halts with low probability, a.k.a. running time is polynomial in expectation for every input

two-sided error

$$\begin{tikzcd}
 & \mathcal{BPP} \arrow[dl] \arrow[d] \arrow[dr] & \\
\mathcal{RP} & \mathcal{ZPP} & \text{co-}\mathcal{RP}
\end{tikzcd}$$

$\mathcal{RP}$:
$x \in L \Rightarrow \Pr[M(x) = 1] \geq 2/3$
$x \notin L \Rightarrow M(x) = 0$

$\mathcal{ZPP}$:
alway correct
no error

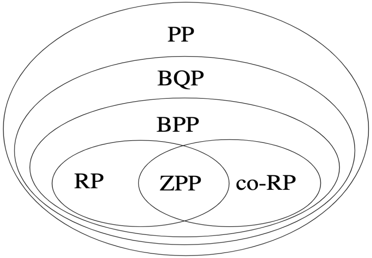
co-$\mathcal{RP}$:
$x \in L \Rightarrow M(x) = 1$
$x \notin L \Rightarrow \Pr[M(x) = 1] \leq 1/3$

- $\mathcal{BPP}$: Monte Carlo algorithms (probabilistic) likely to be correct in strict polynomial running time
- $\mathcal{ZPP}$: Las Vegas algorithms (probabilistic) are always correct in expected polynomial running time

$\mathcal{BPP}$ in Relation to Other Probabilistic Complexity Classes

$\mathcal{BQP}$ (bounded-error quantum polynomial time): the class of decision problems solvable by a quantum TM in polynomial time with bounded error

- It is the quantum analogue of $\mathcal{BPP}$



Limits of $\mathcal{BPP}$

Consensus: $\mathcal{P} \subseteq \underline{\mathcal{ZPP} = \mathcal{RP} \cap \text{co-}\mathcal{RP}} \subseteq \mathcal{BPP} \subseteq \mathcal{NP}$

$\mathcal{P} \subseteq \mathcal{BPP}$

- An important example of a problem in $\mathcal{BPP}$ still not known to be in $\mathcal{P}$ is polynomial identity testing — determining whether a polynomial is identically equal to the zero polynomial, when you have access to the value of the polynomial for any given input, but not to the coefficients.

$\mathcal{BPP} \subseteq \mathcal{NP}$

- Adleman's theorem: $\mathcal{BPP} \subseteq P/\text{poly}$ (polynomial-size Boolean circuits)
- Karp-Levin theorem: $\mathcal{NP} \subseteq P/\text{poly} \Rightarrow \text{PH} = \sum_2^P$

Thus, $\mathcal{NP} \subseteq \mathcal{BPP}$ will imply collapse of PH, which is unlikely to be true. In other words, $\nexists$ bounded-error probabilistic algorithms for $\mathcal{NPC}$ problems.

Outline

PSPACE

$\mathcal{P}$: Decision problems solvable by DTM in polynomial time.

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PSPACE Example

- Binary counter. Count from 0 to $2^n - 1$ in binary.
- Algorithm. Use n bit odometer.

Observation $\mathcal{P} \subseteq$ PSPACE

- This is because poly-time algorithm can consume only polynomial space.

Relation Between NP and PSPACE

Claim: 3-SAT $\in$ PSPACE.

Proof of claim

- Enumerate all 2^n possible truth assignments using counter.
- Check each assignment to see if it satisfies all clauses.

Theorem: $\mathcal{NP} \subseteq$ PSPACE

- Consider arbitrary problem $Y \in \mathcal{NP}$.
- Since $Y \leq_p$ 3-SAT, there exists algorithm that solves Y in poly-time plus polynomial number of calls to 3-SAT black box.
- Thus, Y can be solved by DTM in poly-space.

It is easy to verify that co-$\mathcal{NP} \subseteq$ PSPACE.

The Merit of 3-SAT

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But for 3-SAT, given an instance x, we know the exact length of its witness.

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Decision vs. Search

So far, we only consider decision problems, i.e., membership problem

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A: For $\mathcal{NP}$ languages. Yes!

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$$L = \{(\phi, r) : \exists w \text{ s.t. } \phi(w) = 1 \wedge w < r\}$$

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Conclusion: decision 3-SAT $\approx_{\text{hard}}$ search 3-SAT

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- Given a black-box for FACTOR one can use binary search to find a prime factor of m.

Outline

Languages Defined by Cryptographic Problems

Example based on DDH w.r.t. $(\mathbb{G}, p, g_1, g_2)$

- $X = \mathbb{G} \times \mathbb{G}$;
- $L_{g_1,g_2} = \{(g_1^w, g_2^w)\}$ where $w \in \mathbb{Z}_p$;

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Remark

Cryptographic problems assume average-case complexity (for $x \xleftarrow{\mathsf{R}} X$), while problems in computer science consider worst-case complexity.

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Remark

Proving the existence of hard-on-average problems in $\mathcal{NP}$ using the $\mathcal{P} \neq \mathcal{NP}$ assumption is a major open problem.

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worst-case $=$ average-case

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Leading candidate: Lattice problems (enjoy both)

Reference I

Manindra Agrawal, Neeraj Kayal, and Nitin Saxena.
Primes is in p.
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