

# MAT237 Multivariable Calculus

## Lecture Notes

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# 1 Taylor's Theorem

## 1.1 Review of Taylor's Theorem in 1 Dimension

**Definition of Taylor polynomials** Assume  $I \subset \mathbb{R}$  is an open interval and that  $f : I \rightarrow \mathbb{R}$  is a function of class  $C^k$  on  $I$ .

For a point  $a \in I$ , the  $k$ th order Taylor polynomial of  $f$  at  $a$  is the unique polynomial of order at most  $k$ , denoted  $P_{a,k}(h)$  such that

$$\begin{aligned} f(a) &= P_{a,k}(0) \\ f'(a) &= P'_{a,k}(0) \\ &\vdots \\ f^{(k)}(a) &= P^{(k)}_{a,k}(0) \end{aligned}$$

$$\begin{aligned} P_{a,k}(h) &= f(a) + hf'(a) + \frac{h^2}{2}f''(a) + \dots + \frac{h^k}{k!}f^{(k)}(a) \\ &= \sum_{j=0}^k \frac{h^j}{j!}f^{(j)}(a) \end{aligned}$$

**Remark** Taylor's Theorem guarantees that  $P_{a,k}(h)$  is a very good approximation of  $f(a+h)$ , and that the quality of the approximation increases as  $k$  increases.

**Taylor's Theorem in 1D** Assume  $I \subset \mathbb{R}$  is an open interval and that  $f : I \rightarrow \mathbb{R}$  is a function of class  $C^k$  on  $I$ . For  $a \in I$  and  $h \in \mathbb{R}$  such that  $a+h \in I$ , let  $P_{a,k}(h)$  denote the  $k$ th-order Taylor polynomial at  $a$  and define the remainder

$$R_{a,k}(h) := f(a+h) - P_{a,k}(h)$$

Then

$$\lim_{h \rightarrow 0} \frac{R_{a,k}(h)}{h^k} = 0$$

## 1.2 Taylor's Theorem in higher dimensions

Assume  $S \subset \mathbb{R}^n$  is an open set and that  $f : S \rightarrow \mathbb{R}$  is a function of class  $C^k$  on  $S$ . For a point  $a \in S$ , the  $k$ th order Taylor polynomial of  $f$  at  $a$  is the

unique polynomial of order at most  $k$ , denoted  $P_{a,k}(\mathbf{h})$  such that

$$\begin{aligned} f(\mathbf{a}) &= P_{\mathbf{a},k}(\mathbf{0}) \\ \partial^\alpha f(\mathbf{a}) &= \partial^\alpha P_{\mathbf{a},k}(\mathbf{0}) \end{aligned}$$

for all partial derivatives of order up to  $k$ .

**Taylor's Theorem in nD** Assume  $S \subset \mathbb{R}^n$  is an open interval and that  $f : S \rightarrow \mathbb{R}$  is a function of class  $C^k$  on  $I$ . For  $a \in S$  and  $h \in \mathbb{R}^n$  such that  $a + h \in S$ , let  $P_{a,k}(h)$  denote the  $k$ th-order Taylor polynomial at  $a$  and define the remainder

$$R_{a,k}(h) := f(a + h) - P_{a,k}(h)$$

Then

$$\lim_{h \rightarrow 0} \frac{R_{a,k}(h)}{|h|^k} = 0$$

**A Taylor polynomial formula for  $k = 2$**

$$P_{\mathbf{a},2}(\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2} (H(\mathbf{a})\mathbf{h}) \cdot \mathbf{h}$$

where we remember that  $\mathbf{h} = \mathbf{x} - \mathbf{a}$  if we want the result in terms of  $x, y$ .

## 2 Critical Points

**Definition** A symmetric  $n \times n$  matrix  $A$  is

1. **positive definite** if  $\mathbf{x}^T A \mathbf{x} > 0$  for all  $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$
2. **nonnegative definite** if  $\mathbf{x}^T A \mathbf{x} \geq 0$  for all  $x \in \mathbb{R}^n$

In addition, we say that  $A$  is

1. **negative definite** if  $-A$  is positive definite
2. **nonpositive definite** if  $-A$  is nonnegative definite

A matrix  $A$  is **indefinite** if none of the above holds. Equivalently,  $A$  is indefinite if there exist  $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$  such that  $\mathbf{x}^T A \mathbf{x} < 0 < \mathbf{y}^T A \mathbf{y}$

**Theorem 1** Assume that  $A$  is a symmetric matrix. Then

1.  $A$  is positive definite  $\iff$  all its eigenvalues are positive  
 $\iff \exists \lambda_1 > 0$  such that  $\mathbf{x}^T A \mathbf{x} \geq \lambda_1 |\mathbf{x}|^2$  for all  $\mathbf{x} \in \mathbb{R}^n$
2.  $A$  is nonnegative definite  $\iff$  all its eigenvalues are nonnegative
3.  $A$  is indefinite  $\iff A$  has both positive and negative eigenvalues

**Remark** If  $A$  is a symmetric matrix then

The smallest eigenvalue of  $A = \min_{\{\mathbf{u} \in \mathbb{R}^n : |\mathbf{u}|=1\}} \mathbf{u}^T A \mathbf{u}$

**Theorem 2** For the matrix  $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$ ,

1. if  $\det A < 0$ , then  $A$  is indefinite
2. if  $\det A > 0$ , then
  - if  $\alpha > 0$  then  $A$  is positive definite
  - if  $\alpha < 0$  then  $A$  is negative definite
3. if  $\det A = 0$  then at least one eigenvalue equals zero.

**Definition** A critical point  $\mathbf{a}$  of  $C^2$  function  $\mathbf{f}$  is degenerate if  $\det(D_{\mathbf{H}}(\mathbf{a})) = 0$

**Theorem 3 - first derivative test** If  $\mathbf{f} : S \subset \mathbb{R}^n \rightarrow \mathbb{R}$  is differentiable, then every local extremum is a critical point.

**Theorem 4 - second derivative test**

1. If  $f : S \rightarrow \mathbb{R}$  is  $C^2$  and  $\mathbf{a}$  is a local minimum point for  $f$ , then  $\mathbf{a}$  is a critical point of  $f$  and  $H(\mathbf{a})$  is nonnegative definite.
2. If  $\mathbf{a}$  is a critical point and  $H(\mathbf{a})$  is positive definite, then  $\mathbf{a}$  is a local minimum point.

**Corollary** Assume that  $f$  is  $C^2$  and  $\nabla f(\mathbf{a}) = \mathbf{0}$

1. If  $H(\mathbf{a})$  is positive definite, then  $\mathbf{a}$  is a local min;
2. If  $H(\mathbf{a})$  is negative definite, then  $\mathbf{a}$  is a local max;
3. If  $H(\mathbf{a})$  is indefinite, then  $\mathbf{a}$  is a saddle point;
4. If none of the above hold, then we cannot determine the character of the critical point without further thought.

**E.Knight's approach to critical points.** In solving a question of  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  we could use the following “quick check” approach:

1. Calculate the gradient of  $F$ , equating it to zero to find the critical points
2. Calculate the Hessian of  $F$ , find the corresponding matrices for each critical points, where the Hessian is defined as

$$H(f) = \begin{bmatrix} \partial_{xx}f & \partial_{xy}f = \partial_{yx}f \\ \partial_{xy}f = \partial_{yx}f & \partial_{yy}f \end{bmatrix}$$

3. Calculate the determinant of the hessian, and there are the following cases to consider
  - (a)  $\det H < 0$ , then  $\text{sig}(H) = (1, 1)$  and the point is a saddle point
  - (b)  $\det H > 0$ , then
    - i.  $\text{tr}(H) < 0 \implies \text{sig}(H) = (2, 0)$  and the point is a local minimum
    - ii.  $\text{tr}(H) > 0 \implies \text{sig}(H) = (0, 2)$  and the point is a local maximum
  - (c)  $\det H = 0$ , then the test is inconclusive. We have to do this case by starring at it.

### 3 The Implicit Function Theorem

Assume that  $S$  is an open subset of  $\mathbb{R}^{n+k}$  and that  $F : S \rightarrow \mathbb{R}^k$  is a function of class  $C^1$ . Assume also that  $(\mathbf{a}, \mathbf{b})$  is a point in  $S$  such that  $\mathbf{F}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$  and  $\det D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b}) \neq 0$

1. Then there exists  $r_0, r_1 > 0$  such that for every  $\mathbf{x} \in \mathbb{R}^n$  such that  $|\mathbf{x} - \mathbf{a}| < r_0$ , there exists a unique  $\mathbf{y} \in \mathbb{R}^k$  such that  $|\mathbf{y} - \mathbf{b}| < r_1$

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}(1)$$

In other words, equation (1) implicitly defines a function  $\mathbf{y} = \mathbf{f}(\mathbf{x})$  for  $x \in \mathbb{R}^n$  near  $\mathbf{a}$ , with  $\mathbf{y} = \mathbf{f}(\mathbf{x})$  close to  $\mathbf{b}$ . Note in particular that  $\mathbf{b} = \mathbf{f}(\mathbf{a})$ .

2. Moreover, the function  $\mathbf{f} : B(r_0, \mathbf{a}) \rightarrow B(r_1, \mathbf{b}) \subset \mathbb{R}^k$  from part (1) above is of class  $C^1$ , and its derivatives may be determined by differentiating the identity

$$\mathbf{F}(\mathbf{x}, \mathbf{f}(\mathbf{x})) = \mathbf{0}$$

and solving to find the partial derivatives of  $\mathbf{f}$ .

**Remark**

$$D\mathbf{f}(\mathbf{a}) = -[D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b})]^{-1}D_{\mathbf{x}}\mathbf{F}(\mathbf{a}, \mathbf{b})$$

## 4 The Inverse Function Theorem

Let  $U$  and  $V$  be open sets in  $\mathbb{R}^n$ , and assume that  $\mathbf{f} : U \rightarrow V$  is a mapping of class  $C^1$ .

Assume that  $\mathbf{a} \in U$  is a point such that  $D\mathbf{f}(\mathbf{a})$  is invertible.

and let  $\mathbf{b} := \mathbf{f}(\mathbf{a})$ . Then there exist open sets  $M \subset U$  and  $N \subset V$  such that

1.  $\mathbf{a} \in M$  and  $\mathbf{b} \in N$
2.  $\mathbf{f}$  is one-to-one from  $M$  onto  $N$  (hence invertible), and
3. the inverse function  $\mathbf{f}^{-1} : N \rightarrow M$  is of class  $C^1$

Moreover, if  $x \in M$  and  $y = \mathbf{f}(\mathbf{x}) \in N$ , then

$$D(\mathbf{f}^{-1})(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1}$$

In particular,

$$D(\mathbf{f}^{-1})(\mathbf{b}) = [D\mathbf{f}(\mathbf{a})]^{-1}$$

## 5 Some Important Coordinate Systems

### 5.1 Polar Coordinates in $\mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \end{pmatrix} = \mathbf{f}(r, \theta)$$

For  $\mathbf{f}$  to be a bijection between open sets, we have to restrict its domain and range. A common choice is to specify that  $\mathbf{f}$  is a function  $U \rightarrow V$  where

$$U := \{(r, \theta) : r > 0, |\theta| < \pi\}, V := \mathbb{R}^2 \setminus \{(x, 0) : x \leq 0\}$$

(Note that there is a half of the x-axis missing)

### 5.2 Spherical Coordinates in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \theta \sin \varphi \\ r \sin \theta \sin \varphi \\ r \cos \varphi \end{pmatrix} = \mathbf{f}(r, \theta, \varphi)$$

If we want  $\mathbf{f}$  to be a bijection between open sets  $U$  and  $V$ , it is necessary to restrict the domain and range in some appropriate way.

### 5.3 Cylindrical Coordinates in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r \cos \theta \\ r \sin \theta \\ z \end{pmatrix} = \mathbf{f}(r, \theta, z)$$

## 6 *k*-Dimensional Manifolds in $\mathbb{R}^n$

### 6.1 The General Case

Fix  $k < n$ . For a  $k$ -dimensional manifold  $M$  in  $\mathbb{R}^n$ , we say that  $M$  has "degrees of freedom"  $k$ . There are 3 natural ways to represent  $M$  (be careful with the dimensions!!! ):

#### 1. As a graph:

$$\mathbf{f} : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^{n-k}$$

where  $U$  is open.

$$S = \{(\mathbf{x}, \mathbf{f}(\mathbf{x})) \in \mathbb{R}^n : \mathbf{x} \in U, \forall \mathbf{x} \in U\}$$

#### 2. As a level set:

$$\mathbf{F} : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n-k}$$

where  $U$  is open.

$$S = \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = c\}$$

for some  $c \in \mathbb{R}$ .

This is also called the "zero locus" of  $\mathbf{F}$  when  $c = 0$

**Remark** The regularity conditions that guarantees that  $S$  is smooth is that

1.  $\nabla F_1(\mathbf{x}), \dots, \nabla F_{n-k}(\mathbf{x})$  are linearly independent at each  $\mathbf{x} \in S$ . Or equivalently,
2. the matrix  $D\mathbf{F}(\mathbf{x})$  has rank  $n - k$  at every  $\mathbf{x} \in S$ .

### 3. Parametrically

$$\mathbf{f} : U \subset \mathbb{R}^k \rightarrow \mathbb{R}^n$$

where  $U$  is open.

$$S = \{\mathbf{f}(\mathbf{u}) : \mathbf{u} \in U\}$$

**Remark** The regularity conditions that guarantees that  $S$  is smooth is that

1.  $\partial_{u_1}\mathbf{f}(\mathbf{u}), \dots, \partial_{u_k}\mathbf{f}(\mathbf{u})$  are linearly independent at each  $\mathbf{u} \in U$ . Or equivalently,
2. the matrix  $D\mathbf{f}(\mathbf{u})$  has rank  $k$  at every  $\mathbf{u} \in U$ .

**Notes** We can prove that if the above conditions are satisfied, then  $S$  is smooth. Construct  $\mathbf{F} : \mathbb{R}^{2k} \rightarrow \mathbb{R}^k$ , then use IFT (the proof is hard but worthwhile to think about since the general case implies every specific case).

## 6.2 The Specific Cases

**Theorem 1 - When is a curve regular?** Assume that  $\mathbf{F} : \mathbb{R}^2 \rightarrow \mathbb{R}$  is  $C^1$ , and let

$$S := \{\mathbf{x} \in \mathbb{R}^2 : F(\mathbf{x}) = 0\}$$

If  $\mathbf{a} \in S$  and  $\nabla F(\mathbf{a}) \neq 0$ , then there exists some  $r > 0$  such that  $B(r, \mathbf{a}) \cap S$  is a  $C^1$  graph.

(Prove directly using IFT)

**Theorem 2 - When is the parametrization regular?** Assume that  $\mathbf{f} : (a, b) \rightarrow \mathbb{R}^2$  is  $C^1$ , and let

$$S := \{\mathbf{f}(t) : t \in (a, b)\}$$

If  $\mathbf{f}'(c) \neq 0$  for some  $c \in (a, b)$ , then there exists some  $r > 0$  such that  $\{\mathbf{f}(t) : |t - c| < r\}$  is a  $C^1$  graph.



**Remark** It says only that the parametrization is regular near  $t = c$ , it does not say that  $S$  is regular near  $\mathbf{f}(c)$ . What it means is that when increasing/decreasing  $t$ , we have no control over the path of  $f(t)$ .

**Theorem 3- When is a surface regular?** conditions:  $\mathbf{a} \in S$  and  $\nabla F(\mathbf{a}) \neq 0$

**Theorem 4 - When is the parametrization regular?** conditions:  $Df(c)$  has rank 2 at some  $c$

### 6.3 Remarks on smoothness of a parametric smooth curve

**Definition.** If  $I \subseteq \mathbb{R}$  is an interval, a  $\mathcal{C}^1$  map  $\gamma : I \rightarrow \mathbb{R}^2$  is said to be

1. A *regular curve* if  $\gamma'(t) \neq 0, \forall t \in I$
2. A *simple curve* if  $\gamma$  is injective on the interior of  $I$ .

Hence if  $\gamma$  is regular, then there is a neighbourhood of each point whose image looks like a graph of a  $\mathcal{C}^1$ -function. Simplicity guarantees that no funny overlaps can happen, and this is what is needed for the curve to be smooth. Equivalently we could also convert such parametrization into a zero locus of a  $F$  to use the good old method of gradient directly.

## 7 Zero content

**Zero content in 1-D** A set  $S \subset \mathbb{R}$  is said to have zero content if

$$\forall \epsilon > 0, \exists \text{ intervals } I_1, \dots, I_n \text{ s.t. } S \subseteq \bigcup_{i=1}^n I_i \wedge \sum_{i=1}^n \text{Len}(I_i) < \epsilon$$

**Multidimensional zero content.** A set  $S \subset \mathbb{R}^n$  is said to have zero content if

$$\forall \epsilon > 0, \exists \text{ boxes } B_1, \dots, B_n \text{ s.t. } S \subseteq \bigcup_{i=1}^n B_i \wedge \sum_{i=1}^n \text{Area}(B_i) < \epsilon$$

**Consequence of zero content.** If a set  $Z$  has zero content, then

$$\forall \epsilon > 0, \exists \text{ boxes } B_1, \dots, B_n \text{ s.t. } S \subseteq \bigcup_{i=1}^n B_i^{int} \wedge \sum_{i=1}^n \text{Area}(B_i) < \epsilon$$

Notice the extra *int*.

## 8 Theorems of 1-D Integral Calculus

**Lemma: Refined partitions give better approximations** Let  $P$  be some partition over an interval and let  $P'$  be a refinement of  $P$ , then

$$LS_{P'}f \geq LS_Pf \wedge US_{P'} \leq US_Pf$$

Where LS and US stands for lower sum and upper sum respectively.

**Lemma: Lower sum is always less then or equal to upper sum** If  $P$  and  $Q$  are any partitions of  $[a, b]$ , then  $LS_Pf \leq US_Qf$ . The essence of this proof is to consider the common refinement of these two partitions.

**Lemma.  $\epsilon - \delta$  definition of integrability** If  $f$  is a bounded function on  $[a, b]$ , the following conditions are equivalent:

1.  $f$  is integrable on  $[a, b]$
2.  $\forall \epsilon > 0, \exists P$  of  $[a, b]$  such that  $US_Pf - LS_Pf < \epsilon$

**Theorem: Integration is “Linear”**

1. Suppose  $a < b < c$ . If  $f$  is integrable on  $[a, b]$  and on  $[b, c]$ , then  $f$  is integrable on  $[a, c]$ , further more

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

2. If  $f$  and  $g$  are integrable on  $[a, b]$ , then so is  $f + g$ , further more

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

**Theorem.** Suppose  $f$  is integrable on  $[a, b]$ .

1. If  $c \in \mathbb{R}$ , the  $cf$  is integrable on  $[a, b]$ , and  $\int_a^b cf(x) = c \int_a^b f(x)dx$
2. Of  $[c, d] \subset [a, b]$ , then  $f$  is integrable on  $[c, d]$ .
3. If  $g$  is integrable on  $[a, b]$  and  $f(x) \leq g(x), \forall x \in [a, b]$ , then  $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
4.  $|f|$  is integrable on  $[a, b]$ , and  $\left| \int_a^b f(x)dx \right| \leq \int_a^b |f(x)|dx$

**Theorem: Bounded + monotone  $\implies$  integrable** If  $f$  is bounded and monotone on  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ . The proof of this uses the  $\epsilon - \delta$  definition of integrability

**Theorem: Continuous  $\implies$  integrable** If  $f$  is continuous on  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ . Note that continuous is a sufficient but not necessary condition of integrability

**Theorem: discontinuous at only finite pts  $\implies$  integrable** If  $f$  is bounded on  $[a, b]$  and continuous at all except finitely many points in  $[a, b]$ , then  $f$  is integrable on  $[a, b]$ . A easy example of this would be any  $\mathbb{R}$  function that has a hole in it.

**Theorem: Discontinuous at only zero content  $\implies$  integrable** If  $f$  is bounded on  $[a, b]$  and the set of points in  $[a, b]$  at which  $f$  is discontinuous has zero content, then  $f$  is integrable on  $[a, b]$ .

**Proposition.** Suppose  $f$  and  $g$  are integrable on  $[a, b]$  and  $f(x) = g(x)$  for all except finitely many points  $x \in [a, b]$ . Then  $\int_a^b f(x)dx = \int_a^b g(x)dx$ .

### The Fundamental Theorem Of Calculus

1. Let  $f$  be an integrable function on  $[a, b]$ . For  $x \in [a, b]$ , let  $F(x) = \int_a^x f(t)dt$ . Then  $F$  is continuous on  $[a, b]$ ; more-over,  $F'(x)$  exists and equals  $f(x)$  at every  $x$  at which  $f$  is continuous,
2. Let  $F$  be a continuous function on  $[a, b]$  that is differentiable except perhaps at finitely many points in  $[a, b]$ , and let  $f$  be a function on  $[a, b]$  that agrees with  $F'$  at all points where the latter is defined. If  $f$  is integrable on  $[a, b]$ , then  $\int_a^b f(t)dt = F(b) - F(a)$

**Proposition.** Suppose  $f$  is integrable on  $[a, b]$ . Given  $\epsilon > 0, \exists \delta > 0$  such that if  $P = \{x_0, \dots, x_J\}$  is any partition of  $[a, b]$  satisfying

$$\max\{x_j - x_{j-1} | 1 \leq j \leq J\} < \delta$$

the sums  $LS_P f$  and  $US_P f$  differ from  $\int_a^b f(x)dx$  by at most  $\epsilon$ .

## 9 Generalized Integral Calculus

### Theorems of double integrals

1. If  $f_1$  and  $f_2$  are integrable on the bounded set  $S$  and  $c_1, c_2 \in \mathbb{R}$ , then  $c_1 f_1 + c_2 f_2$  is integrable on  $S$ , and

$$\iint_S [c_1 f_1 + c_2 f_2] dA = c_1 \iint_S f_1 dA + c_2 \iint_S f_2 dA$$

2. Let  $S_1$  and  $S_2$  be bounded sets with no points in common (intersection  $= \emptyset$ ), and let  $f$  be a bounded function. If  $f$  is integrable on  $S_1$  and on  $S_2$ , then  $f$  is integrable on  $S_1 \cup S_2$ , in which case

$$\iint_{S_1 \cup S_2} f dA = \iint_{S_1} f dA + \iint_{S_2} f dA$$

3. If  $f$  and  $g$  are integrable on  $S$  and  $f(\mathbf{x}) \leq g(\mathbf{x})$  for  $\mathbf{x} \in S$ , then  $\iint_S f dA \leq \iint_S g dA$
4. If  $f$  is integrable on  $S$ , then so is  $|f|$ , and

$$\left| \iint_S f dA \right| \leq \iint_S |f| dA$$

**Theorem.** Suppose  $f$  is a bounded function on the rectangle  $R$ . If the set of points in  $R$  at which  $f$  is discontinuous has zero content, then  $f$  is integrable on  $R$ .

### Proposition: on zero content

1. If  $Z \subset \mathbb{R}^2$  has zero content and  $U \subset Z$ , then  $U$  has zero content.
2. If  $Z_1, \dots, Z_k$  have zero content, then so does  $\bigcup_1^k Z_j$
3.  $\mathbf{f} : (a_0, b_0) \rightarrow \mathbb{R}^2$  is of class  $C_1$ , then  $\mathbf{f}([a, b])$  has zero content whenever  $a_0 < a < b < b_0$

**Discontinuity of characteristic function** The function  $\chi_S$  is discontinuous at  $\mathbf{x}$  if and only if  $\mathbf{x}$  is in the boundary of  $S$ .

**Theorem.** Let  $S$  be a measurable subset of  $\mathbb{R}^2$ . Suppose  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is bounded and the set of points in  $S$  at which  $f$  is discontinuous has zero content. Then  $f$  is integrable on  $S$ .

**Remark on this theorem:** The only points where  $f_{\chi_S}$  can be discontinuous are those points in the closure of  $S$  where either  $f$  or  $\chi_S$  is discontinuous. Both of these cases are discontinuity on a set of zero content. And we can definitely fix  $S$  inside of a rectangle, then by the previously stated theorem (The theorem directly above), such function is integrable.

**Proposition: Integration on a set of zero content evaluates to zero.** Suppose  $Z \subset \mathbb{R}^2$  has zero content. If  $f : \mathbb{R}^2 \rightarrow \mathbb{R}$  is bounded, then  $f$  is integrable on  $Z$  and  $\int_Z f dA = 0$

**Corollary**

1. Suppose that  $f$  is integrable on the set  $S \subset \mathbb{R}^2$ . If  $g(\mathbf{x}) = f(\mathbf{x})$  except for  $\mathbf{x}$  in a set of zero content, then  $g$  is integrable on  $S$  and  $\int_S g dA = \int_S f dA$
2. Suppose that  $f$  is integrable on  $S$  and on  $T$ , and  $S \cap T$  has zero content. Then  $f$  is integrable on  $S \cup T$ , and  $\int_{S \cup T} f dA = \int_S f dA + \int_T f dA$