MAT237 Multivariable Calculus
Lecture Notes

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1 Taylor's Theorem

1.1 Review of Taylor's Theorem in 1 Dimenson

Definition of Taylor polynomials Assume $I \subset \mathbb{R}$ is an open interval and that $f : I \to \mathbb{R}$ is a function of class C^k on I.

For a point $a \in I$, the kth order Taylor polynomial of f at a is the unique polynomial of order at most k, denoted$P_{a,k}(h)$ such that

$$\begin{aligned} f(a) &= P_{a,k}(0) \\ f'(a) &= P'_{a,k}(0) \\ &\vdots \\ f^{(k)}(a) &= P^{(k)}_{a,k}(0) \end{aligned}$$

$$\begin{aligned} P_{a,k}(h) &= f(a) + hf'(a) + \frac{h^2}{2} f\text{"}(a) + ... + \frac{h^k}{k!} f^{(k)}(a) \\ &= \sum_{j=0}^{h^j} f^{(j)}(a) \end{aligned}$$

Remark Taylor's Theorem guarantees that $P_{a,k}(h)$ is a very good approximation of $f(a+h)$, and that the quality of the approximation increases as k increases.

Taylor's Theorem in 1D Assume $I \subset \mathbb{R}$ is an open interval and that $f : I \to \mathbb{R}$ is a function of class C^k on I. For $a \in I$ and $h \in \mathbb{R}$ such that $a+h \in I$, let $P_{a,k}(h)$ denote the kth-order Taylor polynomial at a and define the remainder

$$R_{a,k}(h) := f(a+h) - P_{a,k}(h)$$

Then

$$\lim_{h \to 0} \frac{R_{a,k}(h)}{h^k} = 0$$

1.2 Taylor's Theorem in higher dimensions

Assume $S \subset \mathbb{R}^n$ is an open set and that $f : S \to \mathbb{R}$ is a function of class C^k on S. For a point $a \in S$, the kth order Taylor polynomial of f at a is the unique polynomial of order at most k, denoted$P_{a,k}(\mathbf{h})$ such that

$$f(\mathbf{a}) = P_{\mathbf{a},k}(\mathbf{0})$$
$$\partial^\alpha f(\mathbf{a}) = \partial^\alpha P_{\mathbf{a},k}(\mathbf{0})$$

for all partial derivatives of order up to k.

Taylor's Theorem in nD Assume $S \subset \mathbb{R}^n$ is an open interval and that $f : S \to \mathbb{R}$ is a function of class C^k on I. For $a \in S$ and $h \in \mathbb{R}^n$ such that $a + h \in S$, let $P_{a,k}(h)$ denote the kth-order Taylor polynomial at a and define the remainder

$$R_{a,k}(h) := f(a+h) - P_{a,k}(h)$$

Then

$$\lim_{h\to 0} \frac{R_{a,k}(h)}{|h|^k} = 0$$

A Taylor polynomial formula for k = 2

$$P_{\mathbf{a},2}(\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2}(H(\mathbf{a})\mathbf{h}) \cdot \mathbf{h}$$

2 Critical Points

Definition A symmetric $n \times n$ matrix A is

1. **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $x \in \mathbb{R}^n \backslash \{\mathbf{0}\}$
2. **nonnegative definite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $x \in \mathbb{R}^n$

In addition, we say that A is

1. **negative definite** if -A is positive definite
2. **nonpositive definite** if -A is nonnegative definite

A matrix A is **indefinite** if none of the above holds. Equivalently, A is indefinite if there exist $\mathbf{x}, \mathbf{y} \in \mathbb{R}$ such that $\mathbf{x}^T A\mathbf{x} < 0 < \mathbf{y}^T A\mathbf{y}$

Theorem 1 Assume that A is a symmetric matrix. Then

1. A is positive definite $\iff$ all its eigenvalues are positive
$\iff$ $\exists\lambda_1 > 0$ such that $\mathbf{x}^T A\mathbf{x} \geq \lambda_1|\mathbf{x}|^2$ for all $\mathbf{x} \in \mathbb{R}^n$

2. A is nonnegative definite $\iff$ all its eigenvalues are nonnegative

3. A is indefinite $\iff$ A has both positive and negative eigenvalues

Remark If A is a symmetric matrix then
The smallest eigenvalue of A $= \min_{\{\mathbf{u}\in\mathbb{R}^n:|\mathbf{u}|=1\}} \mathbf{u}^T A\mathbf{u}$

Theorem 2 For the matrix $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$,

1. if $detA < 0$, then A is indefinite

2. if $detA > 0$, then

   if $\alpha > 0$ then A is positive definite

   if $\alpha < 0$ then A is negative definite

3. if $detA = 0$ then at least one eigenvalue equals zero.

Definition A critical point $\mathbf{a}$ of C^2 function $\mathbf{f}$ is <u>degenerate</u> if $\det(D_{\mathbf{H}}(\mathbf{a})) = 0$

Theorem 3 - first derivative test If $\mathbf{f} : S \in \mathbb{R}^n \to \mathbb{R}$ is differentiable, then every local extremum is a critical point.

Theorem 4 - second derivative test

1. If $f : S \to \mathbb{R}$ is C^2 and $\mathbf{a}$ is a local minimum point for f, then $\mathbf{a}$ is a critical point of f and $H(\mathbf{a})$ is nonnegative definite.

2. If $\mathbf{a}$ is a critical point and $H(\mathbf{a})$ is positive definite, then $\mathbf{a}$ is a local minimum point.

Corollary Assume that f is C^2 and $\nabla f(\mathbf{a}) = \mathbf{0}$

1. If H(a) is positive definite, then a is a local min;
2. If H(a) is negative definite, then a is a local max;
3. If H(a) is indefinite, then a is a saddle point;
4. If none of the above hold, then we cannot determine the character of the critical point without further thought.

E.Knight's approach to critical points. In solving a question of $f : \mathbb{R}^2 \to \mathbb{R}$ we could use the following "quick check" approach:

1. Calculate the gradient of F, equating it to zero to find the critical points
2. Calculate the Hessian of F, find the correspnding matrices for each critical points, where the Hessian is defined as

$$H(f) = \begin{bmatrix} \partial_{xx} f & \partial_{xy} f = \partial_{yx} f \\ \partial_{xy} f = \partial_{yx} f & \partial_{yy} f \end{bmatrix}$$

3. Calculate the determinant of the hessian, and there are the following cases to consider
   (a) det$H < 0$, then $sig(H) = (1, 1)$ and the point is a saddle point
   (b) det$H > 0$, then
      i. $tr(H) < 0 \implies sig(H) = (2, 0)$ and the point is a local minimum
      ii. $tr(H) > 0 \implies sig(H) = (0, 2)$ and the point is a local maximum
   (c) det$H = 0$, then the test is inconclusive. We have to do this case by starring at it.

3 The Implicit Function Theorem

Assume that S is an open subset of $\mathbb{R}^{n+k}$ and that $F : S \to \mathbb{R}^k$ is a function of class C^1. Assume also that $(\mathbf{a}, \mathbf{b})$ is a point in S such that **F(a, b)=0** and det$D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b}) \neq 0$

1. Then there exists $r_0, r_1 > 0$ such that for every $\mathbf{x} \in \mathbb{R}^n$ such that $|\mathbf{x} - \mathbf{a}| < r_0$, there exists a unique $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{b}| < r_1$

$$\mathbf{F(x,\ y) = 0}(1)$$

In other words, equation (1) implicitly defines a function $\mathbf{y = f(x)}$ for $x \in \mathbb{R}^n$ near $\mathbf{a}$, with $\mathbf{y = f(x)}$ close to $\mathbf{b}$. Note in particular that $\mathbf{b = f(a)}$.
2. Moreover, the function $\mathbf{f} : B(r_0, \mathbf{a}) \to B(r_1, \mathbf{b}) \subset \mathbb{R}^k$ from part (1) above is of class C^1, and its derivatives may be determined by differentiating the identity

$$\mathbf{F(x,f(x)) = 0}$$

and solving to find the partial derivatives of $\mathbf{f}$.

Remark

$$D\mathbf{f(a)} = -[D_{\mathbf{y}}\mathbf{F(a,\ b)}]^{-1} D_{\mathbf{x}}\mathbf{F(a,\ b)}$$

4 The Inverse Function Theorem

Let U and V be open sets in $\mathbb{R}^n$, and assume that $\mathbf{f} : U \to V$ is a mapping of class C^1.
Assume that $\mathbf{a} \in U$ is a point such that $D\mathbf{f(a)}$ is invertible.
and let $\mathbf{b} := \mathbf{f(a)}$. Then there exist open sets $M \subset U$ and $N \subset V$ such that

1. $\mathbf{a} \in M$ and $\mathbf{b} \in N$
2. $\mathbf{f}$ is one-to-one from M onto N (hence invertible), and
3. the inverse function $f^{-1} : N \to M$ is of class C^1

Moreover, if $x \in M$ and $y = \mathbf{f(x)} \in N$, then

$$D(\mathbf{f}^{-1})(\mathbf{y}) = [D\mathbf{f(x)}]^{-1}$$

In particular,

$$D(\mathbf{f}^{-1})(\mathbf{b}) = [D\mathbf{f(a)}]^{-1}$$

5 Some Important Coordinate Systems

5.1 Polar Coordinates in $\mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \mathbf{f}(r, \theta)$$

For **f** to be a bijection between open sets, we have to restrict its domain and range. A common choice is to specify that **f** is a function $U \to V$ where

$$U := \{(r,\theta) : r > 0, |\theta| < \pi\}, V := \mathbb{R}^2 \backslash \{(x,0) : x \leq 0\}$$

(Note that there is a half of the x-axis missing)

5.2 Spherical Coordinates in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta\sin\varphi \\ r\sin\theta\sin\varphi \\ r\cos\varphi \end{pmatrix} = \mathbf{f}(r,\theta,\varphi)$$

If we want **f** to be a bijection between open sets U and V, it is necessary to restrict the domain and range in some appropriate way.

5.3 Cylindrical Coordinates in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} = \mathbf{f}(r,\theta,z)$$

6 k-Dimensional Manifolds in $\mathbb{R}^n$

6.1 The General Case

Fix $k < n$. For a k-dimensional manifold M in $\mathbb{R}^n$, we say that M has "degrees of freedom" k. There are 3 natural ways to represent M (be careful with the dimensions!!!):

1. As a graph:

$$\mathbf{f} : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$$

where U is open.

$$S = \{(\mathbf{x}, \mathbf{f}(\mathbf{x})) \in \mathbb{R}^n : \mathbf{x} = \mathbf{f}(\mathbf{x}), \forall \mathbf{x} \in U\}$$

2. As a level set:

$$\mathbf{F} : U \in \mathbb{R}^n \to \mathbb{R}^{n-k}$$

where U is open.

$$S = \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = c\}$$

for some $c \in \mathbb{R}$.
This is also called the *"zero locus"* of **F** when $c = 0$

Remark The regularity conditions that guarantees that S is smooth is that

1. $\nabla F_1(\mathbf{x}), ..., \nabla F_{n-k}(\mathbf{x})$ are linearly independent at each $\mathbf{x} \in S$. Or equivalently,
2. the matrix $D\mathbf{F}(\mathbf{x})$ has rank $n-k$ at every $\mathbf{x} \in S$.

3. Parametrically

$$\mathbf{f} : U \subset \mathbb{R}^k \to \mathbb{R}^n$$

where U is open.

$$S = \{\mathbf{f}(\mathbf{u}) : \mathbf{u} \in U\}$$

Remark The regularity conditions that guarantees that S is smooth is that

1. $\partial_{u_1}\mathbf{f}(\mathbf{u}), ..., \partial_{u_k}\mathbf{f}(\mathbf{u})$ are linearly independent at each $\mathbf{u} \in U$. Or equivalently,
2. the matrix $D\mathbf{f}(\mathbf{u})$ has rank k at every $\mathbf{u} \in U$.

Notes We can prove that if the above conditions are satisfied, then S is smooth. Construct $\mathbf{F} : \mathbb{R}^{2k} \to \mathbb{R}^k$, then use IFT (the proof is hard but worthwhile to think about since the general case implies every specific case).

6.2 The Specific Cases

Theorem 1 - When is a curve regular? Assume that $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}$ is C^1, and let

$$S := \{\mathbf{x} \in \mathbb{R}^2 : F(\mathbf{x}) = 0\}$$

If $\mathbf{a} \in S$ and $\nabla F(\mathbf{a}) \neq 0$, then there exists some $r > 0$ such that $B(r, \mathbf{a}) \cap S$ is a C^1 graph.
(Prove directly using IFT)

Theorem 2 - When is the parametrization regular? Assume that $\mathbf{f} : (a, b) \to \mathbb{R}^2$ is C^1, and let

$$S := \{\mathbf{f}(t) : t \in (a, b)\}$$

If $\mathbf{f}'(c) \neq 0$ for some $c \in (a, b)$, then there exists some $r > 0$ such that $\{\mathbf{f}(t) : |t - c| < r\}$ is a C^1 graph.

Remark It says only that the parametrization is regular near $t = c$, it does not say that S is regular near $\mathbf{f}$(c). What it means is that when increasing/decreasing t, we have no control over the path of $f(t)$.

Theorem 3- When is a surface regular? conditions: $\mathbf{a} \in S$ and $\nabla F(\mathbf{a}) \neq 0$

Theorem 4 - When is the parametrization regular? conditions: $D\mathbf{f}(\mathbf{c})$ has rank 2 at some c

7 Zero content

Zero content in 1-D A set $S \subset \mathbb{R}$ is said to have zero content if

$$\forall \epsilon > 0, \exists \text{ intervals } I_1, ..., I_n \ s.t. \ S \subseteq \bigcup_{i=1}^{n} I_i \wedge \sum_{i=1}^{n} Len(I_i) < \epsilon$$

Multidimensional zero content. A set $S \subset \mathbb{R}^n$ is said to have zero content if

$$\forall \epsilon > 0, \exists \text{ boxes } B_1, ..., B_n \ s.t. \ S \subseteq \bigcup_{i=1}^{n} B_i \wedge \sum_{i=1}^{n} Area(B_i) < \epsilon$$

Consequence of zero content. If a set Z has zero content, then

$$\forall \epsilon > 0, \exists \text{ boxes } B_1, ..., B_n \ s.t. \ S \subseteq \bigcup_{i=1}^{n} B_i^{int} \wedge \sum_{i=1}^{n} Area(B_i) < \epsilon$$

Notice the extra *int*.

8 Theorems of 1-D Integral Calculus

Lemma: Refined partitions give better approximations Let P be some partition over an interval and let P' be a refinement of P, then

$$LS_{P'}f \geq LS_P f \wedge US_{P'} \leq US_P f$$

Where LS and US stands for lower sum and upper sum respectively.

Lemma: Lower sum is always less then or equal to upper sum If P and Q are any partitions of $[a,b]$, then $LS_P f \leq US_Q f$. The essence of this proof is to consider the common refinement of these two partitions.

Lemma. $\epsilon - \delta$ definition of integrability If f is a bounded function on $[a,b]$, the following conditions are equivalent:

1. f is integrable on $[a,b]$
2. $\forall \epsilon > 0, \exists P$ of $[a,b]$ such that $US_P f - LS_P f < \epsilon$

Theorem: Integration is "Linear"

1. Suppose $a < b < c$. If f is integrable on $[a,b]$ and on $[b,c]$, then f is integrable on $[a,c]$, further more
$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$
2. If f and g are integrable on $[a,b]$, then so is $f+g$, further more
$$\int_a^b [f(x)+g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Theorem. Suppose f is integrable on $[a,b]$.

1. If $c \in \mathbb{R}$, the cf is integrable on $[a,b]$, and $\int_a^b cf(x) = c\int_a^b f(x)dx$
2. Of $[c,d] \subset [a,b]$, then f is integrable on $[c,d]$.
3. If g is integrable on $[a,b]$ and $f(x) \leq g(x), \forall x \in [a,b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
4. $|f|$ is integrable on $[a,b]$, and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

Theorem: Bounded + monotone $\implies$ integrable If f is bounded and monotone on $[a,b]$, then f is integrable on $[a,b]$. The proof of this uses the $\epsilon - \delta$ definition of integrability

Theorem: Continuous $\implies$ integrable If f is continuous on $[a,b]$, then f is integrable on $[a,b]$. Note that continuous is a sufficient but not necessary condition of integrability

Theorem: discontinuous at only finite pts $\implies$ integrable If f is bounded on $[a, b]$ and continuous at all except finitely many points in $[a, b]$, then f is integrable on $[a, b]$. A easy example of this would be any $\mathbb{R}$ function that has a hole in it.

Theorem: Discontinuous at only zero content $\implies$ integrable If f is bounded on $[a, b]$ and the set of points in $[a, b]$ at which f is discontinuous has zero content, then f is integrable on $[a, b]$.

Proposition. Suppose f and g are integrable on $[a, b]$ and $f(x) = g(x)$ for all except finitely many points $x \in [a, b]$. Then $\int_a^b f(x)dx = \int_a^b g(x)dx$.

The Fundamental Theorem Of Calculus

1. Let f be an integrable function on $[a, b]$. For $x \in [a, b]$, let $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a, b]$; more-over, $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous,
2. Let F be a continuous function on $[a, b]$ that is differentiable except perhaps at finitely many points in $[a, b]$, and let f be a function on $[a, b]$ that agrees with F' at all points where the latter is defined. If f is integrable on $[a, b]$, then $\int_a^b f(t)dt = F(b) - F(a)$

Proposition. Suppose f is integrable on $[a, b]$. Given $\epsilon > 0, \exists \delta > 0$ such that if $P = \{x_0, ..., x_J\}$ is any partition of $[a, b]$ satisfying

$$max\{x_j - x_{j-1} | 1 \leq j \leq J\} < \delta$$

the sums $LS_P f$ and $US_P f$ differ from $\int_a^b f(x)dx$ by at most ϵ.

9 Generalized Integral Calculus

Theorems of double integrals

1. If f_1 and f_2 are integrable on the bounded set S and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2$ is integrable on S, and

$$\iint_S [c_1 f_1 + c_2 f_2] dA = c_1 \iint_S f_1 dA + c_2 \iint_S f_2 dA$$

2. Let S_1 and S_2 be bounded sets with no points in common (intersection $= \emptyset$), and let f be a bounded function. If f is integrable on S_1 and on S_2, then f is integrable on $S_1 \cup S_2$, in which case

$$\iint_{S_1 \cup S_2} f dA = \iint_{S_1} f dA + \iint_{S_2} f dA$$

3. If f and g are integrable on S and $f(\mathbf{x}) \leq g(\mathbf{x})$ for $\mathbf{x} \in S$, then $\iint_S f dA \leq \iint_S g dA$

4. If f is integrable on S, then so is $|f|$, and

$$\left| \iint_S f dA \right| \leq \iint_S |f| dA$$

Theorem. Suppose f is a bounded function on the rectangle R. If the set of points in R at which f is discontinuous has zero content, then f is integrable on R.

Proposition: on zero content

1. If $Z \subset \mathbb{R}^2$ has zero content and $U \subset Z$, then U has zero content.
2. If $Z_1, ..., Z_k$ have zero content, then so does $\bigcup_1^k Z_j$
3. $\mathbf{f} : (a_0, b_0) \to \mathbb{R}^2$ is of class C_1, then $\mathbf{f}([a, b])$ has zero content whenever $a_0 < a < b < b_0$

Discontinuity of characteristic function The function χ_S is discontinuous at $\mathbf{x}$ if and only if $\mathbf{x}$ is in the boundary of S.

Theorem. Let S be a measurable subset of $\mathbb{R}^2$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded and the set of points in S at which f is discontinuous has zero content. Then f is integrable on S.

Remark on this theorem: The only points where $f_{\chi S}$ can be discontinuous are those points in the closure of S where either f or χ_S is discontinuous. Both of these cases are discontuinity on a set of zero content. And we can definitely fix S inside of a rectangle, then by the previously stated theorem (The theorem directly above), such function is integrable.

Proposition: Integration on a set of zero content evaluates to zero. Suppose $Z \subset \mathbb{R}^2$ has zero content. If $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded, then f is integrable on Z and $\int_Z f dA = 0$

Corollary

1. Suppose that f is integrable on the set $S \subset \mathbb{R}^2$. If $g(\mathbf{x}) = f(\mathbf{x})$ except for $\mathbf{x}$ in a set of zero content, then g is integrable on S and $\int_S g dA = \int_S f dA$
2. Suppose that f is integrable on S and on T, and $S \cap T$ has zero content. Then f is integrable on $S \cup T$, and $\int_{S \cup T} f dA = \int_S f d + \int_T f dA$