MAT237 Multivariable Calculus
Lecture Notes

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1 Taylor's Theorem

1.1 Review of Taylor's Theorem in 1 Dimenson

Definition of Taylor polynomials Assume $I \subset \mathbb{R}$ is an open interval and that $f : I \to \mathbb{R}$ is a function of class C^k on I.

For a point $a \in I$, the kth order Taylor polynomial of f at a is the unique polynomial of order at most k, denoted$P_{a,k}(h)$ such that

$$\begin{aligned} f(a) &= P_{a,k}(0) \\ f'(a) &= P'_{a,k}(0) \\ &\vdots \\ f^{(k)}(a) &= P^{(k)}_{a,k}(0) \end{aligned}$$

$$\begin{aligned} P_{a,k}(h) &= f(a) + hf'(a) + \frac{h^2}{2} f"(a) + ... + \frac{h^k}{k!} f^{(k)}(a) \\ &= \sum_{j=0}^{k} \frac{h^j}{j!} f^{(j)}(a) \end{aligned}$$

Remark Taylor's Theorem guarantees that $P_{a,k}(h)$ is a very good approximation of $f(a+h)$, and that the quality of the approximation increases as k increases.

Taylor's Theorem in 1D Assume $I \subset \mathbb{R}$ is an open interval and that $f : I \to \mathbb{R}$ is a function of class C^k on I. For $a \in I$ and $h \in \mathbb{R}$ such that $a+h \in I$, let $P_{a,k}(h)$ denote the kth-order Taylor polynomial at a and define the remainder

$$R_{a,k}(h) := f(a+h) - P_{a,k}(h)$$

Then

$$\lim_{h \to 0} \frac{R_{a,k}(h)}{h^k} = 0$$

1.2 Taylor's Theorem in higher dimensions

Assume $S \subset \mathbb{R}^n$ is an open set and that $f : S \to \mathbb{R}$ is a function of class C^k on S. For a point $a \in S$, the kth order Taylor polynomial of f at a is the unique polynomial of order at most k, denoted$P_{a,k}(\mathbf{h})$ such that

$$f(\mathbf{a}) = P_{\mathbf{a},k}(\mathbf{0})$$
$$\partial^\alpha f(\mathbf{a}) = \partial^\alpha P_{\mathbf{a},k}(\mathbf{0})$$

for all partial derivatives of order up to k.

Taylor's Theorem in nD Assume $S \subset \mathbb{R}^n$ is an open interval and that $f : S \to \mathbb{R}$ is a function of class C^k on I. For $a \in S$ and $h \in \mathbb{R}^n$ such that $a + h \in S$, let $P_{a,k}(h)$ denote the kth-order Taylor polynomial at a and define the remainder

$$R_{a,k}(h) := f(a+h) - P_{a,k}(h)$$

Then

$$\lim_{h \to 0} \frac{R_{a,k}(h)}{|h|^k} = 0$$

A Taylor polynomial formula for k = 2

$$P_{\mathbf{a},2}(\mathbf{h}) = f(\mathbf{a}) + \nabla f(\mathbf{a}) \cdot \mathbf{h} + \frac{1}{2}(H(\mathbf{a})\mathbf{h}) \cdot \mathbf{h}$$

where we remember that $\mathbf{h} = \mathbf{x} - \mathbf{a}$ if we want the result in terms of x, y.

2 Critical Points

Definition A symmetric $n \times n$ matrix A is

1. **positive definite** if $\mathbf{x}^T A\mathbf{x} > 0$ for all $x \in \mathbb{R}^n \setminus \{\mathbf{0}\}$
2. **nonnegative definite** if $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $x \in \mathbb{R}^n$

In addition, we say that A is

1. **negative definite** if -A is positive definite
2. **nonpositive definite** if -A is nonnegative definite

A matrix A is **indefinite** if none of the above holds. Equivalently, A is indefinite if there exist $\mathbf{x},\ \mathbf{y} \in \mathbb{R}$ such that $\mathbf{x}^T A\mathbf{x} < 0 < \mathbf{y}^T A\mathbf{y}$

Theorem 1 Assume that A is a symmetric matrix. Then

1. A is positive definite $\iff$ all its eigenvalues are positive
   $\iff \exists\lambda_1 > 0$ such that $\mathbf{x}^T A\mathbf{x} \geq \lambda_1|\mathbf{x}|^2$ for all $\mathbf{x} \in \mathbb{R}^n$
2. A is nonnegative definite $\iff$ all its eigenvalues are nonnegative
3. A is indefinite $\iff$ A has both positive and negative eigenvalues

Remark If A is a symmetric matrix then
The smallest eigenvalue of A $= \min_{\{\mathbf{u}\in\mathbb{R}^n:|\mathbf{u}|=1\}} \mathbf{u}^T A\mathbf{u}$

Theorem 2 For the matrix $A = \begin{pmatrix} \alpha & \beta \\ \beta & \gamma \end{pmatrix}$,

1. if $detA < 0$, then A is indefinite
2. if $detA > 0$, then
   if $\alpha > 0$ then A is positive definite
   if $\alpha < 0$ then A is negative definite
3. if $detA = 0$ then at least one eigenvalue equals zero.

Definition A critical point $\mathbf{a}$ of C^2 function $\mathbf{f}$ is <u>degenerate</u> if $\det(D_{\mathbf{H}}(\mathbf{a})) = 0$

Theorem 3 - first derivative test If $\mathbf{f} : S \in \mathbb{R}^n \to \mathbb{R}$ is differentiable, then every local extremum is a critical point.

Theorem 4 - second derivative test

1. If $f : S \to \mathbb{R}$ is C^2 and $\mathbf{a}$ is a local minimum point for f, then $\mathbf{a}$ is a critical point of f and $H(\mathbf{a})$ is nonnegative definite.
2. If $\mathbf{a}$ is a critical point and $H(\mathbf{a})$ is positive definite, then $\mathbf{a}$ is a local minimum point.

Corollary Assume that f is C^2 and $\nabla f(\mathbf{a}) = \mathbf{0}$

1. If H(a) is positive definite, then a is a local min;
2. If H(a) is negative definite, then a is a local max;
3. If H(a) is indefinite, then a is a saddle point;
4. If none of the above hold, then we cannot determine the character of the critical point without further thought.

E.Knight's approach to critical points. In solving a question of $f : \mathbb{R}^2 \to \mathbb{R}$ we could use the following "quick check" approach:

1. Calculate the gradient of F, equating it to zero to find the critical points
2. Calculate the Hessian of F, find the correspnding matrices for each critical points, where the Hessian is defined as

$$H(f) = \begin{bmatrix} \partial_{xx} f & \partial_{xy} f = \partial_{yx} f \\ \partial_{xy} f = \partial_{yx} f & \partial_{yy} f \end{bmatrix}$$

3. Calculate the determinant of the hessian, and there are the following cases to consider
   (a) $\det H < 0$, then $sig(H) = (1, 1)$ and the point is a saddle point
   (b) $\det H > 0$, then
      i. $tr(H) < 0 \implies sig(H) = (2, 0)$ and the point is a local minimum
      ii. $tr(H) > 0 \implies sig(H) = (0, 2)$ and the point is a local maximum
   (c) $\det H = 0$, then the test is inconclusive. We have to do this case by starring at it.

3 Lagrange Multipliers

3.1 Two constraints

Assume that f, g_1 and g_2 are functions $\mathbb{R}^n \to \mathbb{R}$ of class C^1. Assume also that $\{\nabla g_1(\mathbf{x}), \nabla g_2(\mathbf{x})\}$ are linearly independent at all $\mathbf{x}$ where $g_1(\mathbf{x}) = g_2(\mathbf{x}) = 0$

Then if x is any solution to the optimization problem, there exists $\lambda_1, \lambda_2 \in \mathbb{R}$ such that the following system of equations is satisfies by $\mathbf{x}, \lambda_1$ and λ_2:

$$\begin{cases} \nabla f(\mathbf{x}) + \lambda_1 \nabla g_1(\mathbf{x}) + \lambda_2 \nabla g_2(\mathbf{x}) & = \mathbf{0} \\ g_1(\mathbf{x}) & = 0 \\ g_2(\mathbf{x}) & = 0 \end{cases}$$

3.2 Inequality Constraints

Consider the problem

$$\begin{cases} \text{minimize/maximize } f(\mathbf{x}) \\ \text{subject to the constraint: } g(\mathbf{x}) \leq 0 \end{cases}$$

where we assume that g is C^2, say, and that $\nabla g(\mathbf{x} \neq 0$ on the set $\{\mathbf{x} \in \mathbb{R}^n : g(\mathbf{x}) = 0\}$
We can reduce this to problems we already know how to solve. EVT guarantees that the problem has a solution.

Case 1 The max or min occurs in the set $\{x \in \mathbb{R}^n : g(\mathbf{x}) < 0$
Then it is a critical point, which we know how to find and classify

Case 2 The max or min occurs in the set $\{x \in \mathbb{R}^n : g(\mathbf{x}) = 0$
Then we can find it by the Lagrange Multiplier technique.
Finally we can choose the smallest/largest value of f and the point where that value is attained from among all the candidates found in steps 1 and 2 above.

4 The Implicit Function Theorem

Assume that S is an open subset of $\mathbb{R}^{n+k}$ and that $F : S \to \mathbb{R}^k$ is a function of class C^1. Assume also that $(\mathbf{a}, \mathbf{b})$ is a point in S such that **F(a, b)=0** and $\det D_{\mathbf{y}}\mathbf{F}(\mathbf{a}, \mathbf{b}) \neq 0$
1. Then there exists $r_0, r_1 > 0$ such that for every $\mathbf{x} \in \mathbb{R}^n$ such that $|\mathbf{x} - \mathbf{a}| < r_0$, there exists a unique $\mathbf{y} \in \mathbb{R}^k$ such that $|\mathbf{y} - \mathbf{b}| < r_1$

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{0}(1)$$

In other words, equation (1) implicitly defines a function $\mathbf{y} = \mathbf{f}(\mathbf{x})$ for $x \in \mathbb{R}^n$ near $\mathbf{a}$, with $\mathbf{y} = \mathbf{f}(\mathbf{x})$ close to $\mathbf{b}$. Note in particular that $\mathbf{b} = \mathbf{f}(\mathbf{a})$.

2. Moreover, the function $\mathbf{f}: B(r_0, \mathbf{a}) \to B(r_1, \mathbf{b}) \subset \mathbb{R}^k$ from part (1) above is of class C^1, and its derivatives may be determined by differentiating the identity

$$\mathbf{F(x,f(x)) = 0}$$

and solving to find the partial derivatives of $\mathbf{f}$.

Remark

$$D\mathbf{f}(\mathbf{a}) = -[D_{\mathbf{y}}\mathbf{F}(\mathbf{a},\ \mathbf{b})]^{-1} D_{\mathbf{x}}\mathbf{F}(\mathbf{a},\ \mathbf{b})$$

5 The Inverse Function Theorem

Let U and V be open sets in $\mathbb{R}^n$, and assume that $\mathbf{f}: U \to V$ is a mapping of class C^1.
Assume that $\mathbf{a} \in U$ is a point such that $D\mathbf{f}(\mathbf{a})$ is invertible.
and let $\mathbf{b} := \mathbf{f}(\mathbf{a})$. Then there exist open sets $M \subset U$ and $N \subset V$ such that

1. $\mathbf{a} \in M$ and $\mathbf{b} \in N$
2. $\mathbf{f}$ is one-to-one from M onto N (hence invertible), and
3. the inverse function $f^{-1} : N \to M$ is of class C^1

Moreover, if $x \in M$ and $y = \mathbf{f}(\mathbf{x}) \in N$, then

$$D(\mathbf{f}^{-1})(\mathbf{y}) = [D\mathbf{f}(\mathbf{x})]^{-1}$$

In particular,

$$D(\mathbf{f}^{-1})(\mathbf{b}) = [D\mathbf{f}(\mathbf{a})]^{-1}$$

6 Some Important Coordinate Systems

6.1 Polar Coordinates in $\mathbb{R}^2$

$$\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \end{pmatrix} = \mathbf{f}(r, \theta)$$

For $\mathbf{f}$ to be a bijection between open sets, we have to restrict its domain and range. A common choice is to specify that $\mathbf{f}$ is a function $U \to V$ where

$$U := \{(r, \theta) : r > 0, |\theta| < \pi\}, V := \mathbb{R}^2 \backslash \{(x, 0) : x \leq 0\}$$

(Note that there is a half of the x-axis missing)

6.2 Spherical Coordinates in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta\sin\varphi \\ r\sin\theta\sin\varphi \\ r\cos\varphi \end{pmatrix} = \mathbf{f}(r,\theta,\varphi)$$

If we want $\mathbf{f}$ to be a bijection between open sets U and V, it is necessary to restrict the domain and range in some appropriate way.

6.3 Cylindrical Coordinates in $\mathbb{R}^3$

$$\begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} r\cos\theta \\ r\sin\theta \\ z \end{pmatrix} = \mathbf{f}(r,\theta,z)$$

7 k-Dimensional Manifolds in $\mathbb{R}^n$

7.1 The General Case

Fix $k < n$. For a k-dimensional manifold M in $\mathbb{R}^n$, we say that M has "degrees of freedom" k. There are 3 natural ways to represent M (be careful with the dimensions!!!):

1. As a graph:

$$\mathbf{f} : U \subset \mathbb{R}^k \to \mathbb{R}^{n-k}$$

where U is open.

$$S = \{(\mathbf{x}, \mathbf{f}(\mathbf{x})) \in \mathbb{R}^n : \mathbf{x} = \mathbf{f}(\mathbf{x}), \forall \mathbf{x} \in U\}$$

2. As a level set:

$$\mathbf{F} : U \in \mathbb{R}^n \to \mathbb{R}^{n-k}$$

where U is open.

$$S = \{\mathbf{x} \in U : \mathbf{F}(\mathbf{x}) = \mathbf{c}\}$$

for some $\mathbf{c} \in \mathbb{R}^{n-k}$.
This is also called the *"zero locus"* of $\mathbf{F}$ when $\mathbf{c} = \mathbf{0}$

Remark 1 The regularity conditions that guarantees that S is smooth is that

1. $\nabla F_1(\mathbf{x}), ..., \nabla F_{n-k}(\mathbf{x})$ are linearly independent at each $\mathbf{x} \in S$. Or equivalently,
2. the matrix $D\mathbf{F}(\mathbf{x})$ has rank $n-k$ at every $\mathbf{x} \in S$.

Remark 2 It can happen that the above conditions are satisfied but S is not smooth. Example: The square of a smooth F, c = 0

3. Parametrically

$$\mathbf{f} : U \subset \mathbb{R}^k \to \mathbb{R}^n$$

where U is open.

$$S = \{\mathbf{f}(\mathbf{u}) : \mathbf{u} \in U\}$$

Remark The regularity conditions that guarantees that S is smooth is that

1. $\partial_{u_1}\mathbf{f}(\mathbf{u}), ..., \partial_{u_k}\mathbf{f}(\mathbf{u})$ are linearly independent at each $\mathbf{u} \in U$. Or equivalently,
2. the matrix $D\mathbf{f}(\mathbf{u})$ has rank k at every $\mathbf{u} \in U$.

Notes We can prove that if the above conditions are satisfied, then S is smooth. Construct $\mathbf{F} : \mathbb{R}^{2k} \to \mathbb{R}^k$, then use IFT (the proof is hard but worthwhile to think about since the general case implies every specific case).

7.2 The Specific Cases

Theorem 1 - When is a curve regular? Assume that $\mathbf{F} : \mathbb{R}^2 \to \mathbb{R}$ is C^1, and let

$$S := \{\mathbf{x} \in \mathbb{R}^2 : F(\mathbf{x}) = 0\}$$

If $\mathbf{a} \in S$ and $\nabla F(\mathbf{a}) \neq 0$, then there exists some $r > 0$ such that $B(r, \mathbf{a}) \cap S$ is a C^1 graph.
(Prove directly using IFT)

Theorem 2 - When is the parametrization regular? Assume that $\mathbf{f} : (a, b) \to \mathbb{R}^2$ is C^1, and let

$$S := \{\mathbf{f}(t) : t \in (a, b)\}$$

If $\mathbf{f}'(c) \neq 0$ for some $c \in (a, b)$, then there exists some $r > 0$ such that $\{\mathbf{f}(t) : |t - c| < r\}$ is a C^1 graph.

Remark It says only that the parametrization is regular near $t = c$, it does not say that S is regular near $\mathbf{f}$(c). What it means is that when increasing/decreasing t, we have no control over the path of $f(t)$.

Theorem 3- When is a surface regular? conditions: $\mathbf{a} \in S$ and $\nabla F(\mathbf{a}) \neq 0$

Theorem 4 - When is the parametrization regular? conditions: $D\mathbf{f}(\mathbf{c})$ has rank 2 at some c

7.3 Remarks on smoothness of a parametric smooth curve

Definition. If $I \subseteq \mathbb{R}$ is an interval, a $\mathcal{C}^1$ map $\gamma : I \to \mathbb{R}^2$ is said to be

1. A *regular curve* if $\gamma'(t) \neq 0, \forall t \in I$
2. A *simple curve* if γ is injective on the interior of I.

Hence if γ is regular, then there is a neighbourhood if each point whose image looks like a graph of a $\mathcal{C}^1$-function. Simplicity guarantees that no funny overlaps can happen, and this is what is need for the curve to be smooth. **As a conclusion, we say a parametric curve is smooth if and only if it is *regular* and *simple*.** Equivalently we could also convert such parametrization into a zero locus of a F to use the good old method of gradient directly.

8 Integration

8.1 Zero content

Zero content in 1-D A set $S \subset \mathbb{R}$ is said to have zero content if

$$\forall \epsilon > 0, \exists \text{ intervals } I_1, ..., I_n \ s.t. \ S \subseteq \bigcup_{i=1}^{n} I_i \wedge \sum_{i=1}^{n} Len(I_i) < \epsilon$$

Multidimensional zero content. A set $S \subset \mathbb{R}^n$ is said to have zero content if

$$\forall \epsilon > 0, \exists \text{ boxes } B_1, ..., B_n \ s.t. \ S \subseteq \bigcup_{i=1}^{n} B_i \wedge \sum_{i=1}^{n} Area(B_i) < \epsilon$$

Consequence of zero content. If a set Z has zero content, then

$$\forall \epsilon > 0, \exists \text{ boxes } B_1, ..., B_n \ s.t. \ S \subseteq \bigcup_{i=1}^{n} B_i^{int} \wedge \sum_{i=1}^{n} Area(B_i) < \epsilon$$

Notice the extra *int*.

Proposition on zero content

1. If $Z \subset \mathbb{R}^2$ has zero content and $U \subset Z$, then U has zero content.
2. If $Z_1, ..., Z_k$ have zero content, then so does $\bigcup_1^k Z_j$
3. $\mathbf{f} : (a_0, b_0) \to \mathbb{R}^2$ is of class C_1, then $\mathbf{f}([a, b])$ has zero content whenever $a_0 < a < b < b_0$

8.2 Theorems of 1-D Integral Calculus

Lemma: Refined partitions give better approximations Let P be some partition over an interval and let P' be a refinement of P, then

$$LS_{P'}f \geq LS_P f \wedge US_{P'} \leq US_P f$$

Where LS and US stands for lower sum and upper sum respectively.

Lemma: Lower sum is always less then or equal to upper sum If P and Q are any partitions of $[a, b]$, then $LS_P f \leq US_Q f$. The essence of this proof is to consider the common refinement of these two partitions.

Lemma. $\epsilon - \delta$ definition of integrability If f is a bounded function on $[a, b]$, the following conditions are equivalent:

1. f is integrable on $[a, b]$
2. $\forall \epsilon > 0, \exists P$ of $[a, b]$ such that $US_P f - LS_P f < \epsilon$

Theorem: Integration is "Linear"

1. Suppose $a < b < c$. If f is integrable on $[a, b]$ and on $[b, c]$, then f is integrable on $[a, c]$, further more

$$\int_a^c f(x)dx = \int_a^b f(x)dx + \int_b^c f(x)dx$$

2. If f and g are integrable on $[a, b]$, then so is $f + g$, further more

$$\int_a^b [f(x) + g(x)]dx = \int_a^b f(x)dx + \int_a^b g(x)dx$$

Theorem. Suppose f is integrable on $[a, b]$.

1. If $c \in \mathbb{R}$, the cf is integrable on $[a, b]$, and $\int_a^b cf(x) = c \int_a^b f(x)dx$
2. Of $[c, d] \subset [a, b]$, then f is integrable on $[c, d]$.
3. If g is integrable on $[a, b]$ and $f(x) \leq g(x), \forall x \in [a, b]$, then $\int_a^b f(x)dx \leq \int_a^b g(x)dx$
4. $|f|$ is integrable on $[a, b]$, and $\left|\int_a^b f(x)dx\right| \leq \int_a^b |f(x)|dx$

Theorem: Bounded + monotone $\implies$ integrable If f is bounded and monotone on $[a, b]$, then f is integrable on $[a, b]$. The proof of this uses the $\epsilon - \delta$ definition of integrability

Theorem: Continuous $\implies$ integrable If f is continuous on $[a, b]$, then f is integrable on $[a, b]$. Note that continuous is a sufficient but not necessary condition of integrability

Theorem: discontinuous at only finite pts $\implies$ integrable If f is bounded on $[a, b]$ and continuous at all except finitely many points in $[a, b]$, then f is integrable on $[a, b]$. A easy example of this would be any $\mathbb{R}$ function that has a hole in it.

Theorem: Discontinuous at only zero content $\implies$ integrable If f is bounded on $[a, b]$ and the set of points in $[a, b]$ at which f is discontinuous has zero content, then f is integrable on $[a, b]$.

Proposition. Suppose f and g are integrable on $[a, b]$ and $f(x) = g(x)$ for all except finitely many points $x \in [a, b]$. Then $\int_a^b f(x)dx = \int_a^b g(x)dx$.

The Fundamental Theorem Of Calculus

1. Let f be an integrable function on $[a, b]$. For $x \in [a, b]$, let $F(x) = \int_a^x f(t)dt$. Then F is continuous on $[a, b]$; more-over, $F'(x)$ exists and equals $f(x)$ at every x at which f is continuous,
2. Let F be a continuous function on $[a, b]$ that is differentiable except perhaps at finitely many points in $[a, b]$, and let f be a function on $[a, b]$ that agrees with F' at all points where the latter is defined. If f is integrable on $[a, b]$, then $\int_a^b f(t)dt = F(b) - F(a)$

Proposition. Suppose f is integrable on $[a, b]$. Given $\epsilon > 0, \exists \delta > 0$ such that if $P = \{x_0, ..., x_J\}$ is any partition of $[a, b]$ satisfying

$$max\{x_j - x_{j-1}|1 \leq j \leq J\} < \delta$$

the sums $LS_P f$ and $US_P f$ differ from $\int_a^b f(x)dx$ by at most ϵ.

8.3 Generalized Integral Calculus

Theorems of double integrals

1. If f_1 and f_2 are integrable on the bounded set S and $c_1, c_2 \in \mathbb{R}$, then $c_1 f_1 + c_2 f_2$ is integrable on S, and

$$\iint_S [c_1 f_1 + c_2 f_2] dA = c_1 \iint_S f_1 dA + c_2 \iint_S f_2 dA$$

2. Let S_1 and S_2 be bounded sets with no points in common (intersection $= \emptyset$), and let f be a bounded function. If f is integrable on S_1 and on S_2, then f is integrable on $S_1 \cup S_2$, in which case

$$\iint_{S_1 \cup S_2} f dA = \iint_{S_1} f dA + \iint_{S_2} f dA$$

3. If f and g are integrable on S and $f(\mathbf{x}) \leq g(\mathbf{x})$ for $\mathbf{x} \in S$, then $\iint_S f dA \leq \iint_S g dA$

4. If f is integrable on S, then so is $|f|$, and

$$\left| \iint_S f dA \right| \leq \iint_S |f| dA$$

Theorem. Suppose f is a bounded function on the rectangle R. If the set of points in R at which f is discontinuous has zero content, then f is integrable on R.

Discontinuity of characteristic function The function χ_S is discontinuous at $\mathbf{x}$ if and only if $\mathbf{x}$ is in the boundary of S.

Theorem. Let S be a measurable subset of $\mathbb{R}^2$. Suppose $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded and the set of points in S at which f is discontinuous has zero content. Then f is integrable on S.

Remark on this theorem: The only points where $f_{\chi S}$ can be discontinuous are those points in the closure of S where either f or χ_S is discontinuous. Both of these cases are discontuinity on a set of zero content. And we can definitely fix S inside of a rectangle, then by the previously stated theorem (The theorem directly above), such function is integrable.

Proposition: Integration on a set of zero content evaluates to zero. Suppose $Z \subset \mathbb{R}^2$ has zero content. If $f : \mathbb{R}^2 \to \mathbb{R}$ is bounded, then f is integrable on Z and $\int_Z f dA = 0$

Corollary

1. Suppose that f is integrable on the set $S \subset \mathbb{R}^2$. If $g(\mathbf{x}) = f(\mathbf{x})$ except for $\mathbf{x}$ in a set of zero content, then g is integrable on S and $\int_S g dA = \int_S f dA$
2. Suppose that f is integrable on S and on T, and $S \cap T$ has zero content. Then f is integrable on $S \cup T$, and $\int_{S \cup T} f dA = \int_S f d + \int_T f dA$

Fubini's Theorem Let $R = \{(x, y) : a \le x \le b, c \le y \le d\}$, and let f be an integrable function on R. Suppose that, for each $y \in [c, d]$, the function f_y defined by $f_y(x) = f(x, y)$ is integrable on $[a, b]$, and the function $g(y) = \int_a^b f(x, y)dx$ is integrable on $[c, d]$. Then

$$\iint_R f dA = \int_c^d \left[\int_a^b f(x, y)dx \right] dy$$

Likewise, if $f^x(y) = f(x, y)$ is integrable on $[c, d]$ for each $x \in [a, b]$, and $h(x) = \int_c^d f(x, y)dy$ is integrable on $[a, b]$, then

$$\iint_R f dA = \int_a^b \left[\int_c^d f(x, y)dy \right] dx$$

8.4 Change of Variables

Change of Variable formula 1D If g is a one-to-one function of class C^1 on the interval $[a, b]$, then for any continuous function f,

$$\int_{[a,b]} f(g(u))|g'(u)|du = \int_{g([a,b])} f(x)dx$$

In practice it is often more convenient to have all the g's on one side of the equation. If we set $I = g([a,b])$, we have $[a,b] = g^{-1}(I)$, and

$$\int_I f(x) fx = \int_{g^{-1}(I)} f(g(u))|g'(u)|du$$

goal: find the analogous formula for multiple integrals. The questions is: How does the volume of a tiny piece of n-space change when one applies the transformation G?

Theorem - Change of Variable for linear mappings Let A be an invertible $n \times n$ matrix, and let $\mathbf{G}(\mathbf{u}) = A\mathbf{u}$ be the corresponding linear transformation of $\mathbb{R}^n$. Suppose S is a measurable region in $\mathbb{R}^n$ and f is an integrable function on S. Then $G^{-1}(S) = \{A^{-1}\mathbf{x} : \mathbf{x} \in S\}$ is measurable and $f \circ \mathbf{G}$ is integrable on $\mathbf{G}^{-1}(S)$, and

$$\int \dots \int_S f(\mathbf{x}) d^n\mathbf{x} = |detA| \int \dots \int_{\mathbf{G}^{-1}(S)} f(A\mathbf{u}) d^n\mathbf{u}$$

Theorem - Change of Variable for general functions Given open sets U and V in $\mathbb{R}^n$, let $\mathbf{G} : U \to V$ be a one-to-one transformation of class C^1 whose derivative $D\mathbf{G}(\mathbf{u})$ is invertible for all $\mathbf{u} \in U$. Suppose that $T \subset U$ and $S \subset V$ are measurable sets such that $\mathbf{G}(T) = S$. If f is an integrable function on S, then $f \circ \mathbf{G}$ is integrable on T, and

$$\int \dots \int_S f(\mathbf{x}) d^n\mathbf{x} = \int \dots \int_T f(\mathbf{G}(\mathbf{u}))|\det D\mathbf{G}(\mathbf{u})| d^n\mathbf{u}$$

Some important determinants

1. polar coordinates: factor $= r$
2. cylindrical coordinates: factor $= |\det(Dg)| = r$
3. spherical coordinates: factor $= r^2 \sin\varphi$

8.5 Functions Defined by Integrals

Question

$$F(\mathbf{x}) = \int \dots \int_S f(\mathbf{x}, \mathbf{y}) d^n\mathbf{y}$$

What condition on f guarantee that F behaves well?

Theorem 1 - Continuity of F Suppose S and T are compact subsets of $\mathbb{R}^n$ and $\mathbb{R}^m$, respectively, and S is measurable. If $f(\mathbf{x},\mathbf{y})$ is continuous on the set $T \times S = \{(\mathbf{x},\mathbf{y}) : \mathbf{x} \in T, \mathbf{y} \in S\}$, then the function F defined by

$$F(\mathbf{x}) = \int \dots \int_S f(\mathbf{x},\mathbf{y}) d^n\mathbf{y}$$

is continuous on T.

Theorem 2 - Differentiability of F Suppose $S \subset \mathbb{R}^n$ is compact and measurable, and $T \subset \mathbb{R}^m$ is open. If f is a continuous function on $T \times S$ that is of class C^1 as a function of $\mathbf{x} \in T$ for each $\mathbf{y} \in S$, then the function F defined by

$$F(\mathbf{x}) = \int \dots \int_S f(\mathbf{x},\mathbf{y}) d^n\mathbf{y}$$

is of class C^1 on T, and

$$\frac{\partial F}{\partial x_j}(\mathbf{x}) = \int \dots \int_S \frac{\partial f}{\partial x_j}(\mathbf{x},\mathbf{y}) d^n\mathbf{y} (\mathbf{x} \in T)$$

Remark Situation often occur in which the variable $\mathbf{x}$occurs in the limits of integration as well as the integrand. For simplicity we consider the case where x and y are scalar variables:

$$F(x) = \int_a^{\varphi(x)} f(x,y) dy (*)$$

We suppose that f is continuous in x and y and of class C^1 in x for each y, and that φ is of class C^1. If f does not depend on x, the derivative of F can be computed by the fundamental theorem of calculus together with the chain rule:

$$\frac{d}{dx} \int_a^{\varphi(x)} f(y) f y = f(\varphi(x)) \varphi'(x)$$

For the more general case (*), we can differentiate F by combining this result with Theorem 2: Differentiate with respect to each x in (*) while treating the others as constants, and add the results

$$F'(x) = f(\varphi(x))\varphi'(x) + \int_a^{\varphi(x)} \frac{\partial f}{\partial x}(x,y) dy$$

8.6 Improper Multiple Integrals

There are many situations where one needs to integrate functions over infinite intervals(e.g. half-space or the whole space) or functions that are unbounded near some point in the region of integration. Suppose, for example, that f is a continuous function on $\mathbb{R}^2$ and we wish to define $\iint_{\mathbb{R}^2} f dA$. The obvious idea is to set

$$\iint_{\mathbb{R}^2} f dA = \lim_{r \to \infty} \iint_{S_r} f dA$$

where the S_r's are a family of measurable sets that fill out $\mathbb{R}^2$ as $r \to \infty$. For example, we could take S_r to be

1. the disc of radius r about the origin
2. the square of side length r centered at the origin
3. the rectangle of side lengths r and r^2 centered at the origin
4. the disc of radius r centered at $(15, -37)$
5. ...

No rationale for choosing one over another and no guarantee that different families S_r will yield the same limit.

Proposition For $p > 0$, define f_p on $\mathbb{R}^n \setminus \{\mathbf{0}\}$ by $f_p(\mathbf{x}) = |\mathbf{x}|^{-p}$. The integral of f_p over a ball $\{\mathbf{x} : |\mathbf{x}| < a\}$ is finite if and only if $p < n$; the integral of f_p over the complement of a ball, $\{\mathbf{x} : |\mathbf{x}| > a\}$, is finite if and only if $p > n$.

Theorem Suppose $f : \mathbb{R}^n \to \mathbb{R}$ is bounded and integrable on any bounded set and satisfies $|f(\mathbf{x})| \leq C \cdot \frac{1}{||\mathbf{x}||^p}$ whenever $||\mathbf{x}|| > R$, for some constants $C, R > 0$.
If $p > n$ then $\int \dots \int_{\mathbb{R}^n} f d^n\mathbf{x}$ exists.

proof: (n=2)Suppose there exist a $p > 2$ such that the hypothesis of the theorem is satisfied. Consider $B(\mathbf{0}, R)$.
Then

$$\int \dots \int_{\mathbb{R}^n} f d^n\mathbf{x} = \int \dots \int_{B(\mathbf{0},R)} f d^n\mathbf{x} + \int \dots \int_{\mathbb{R}^2 \setminus B(\mathbf{0},R)} f d^n\mathbf{x}$$

(if it exists)
It's sufficient to show both (1) and (2) exist for all $R > 0$.
By the hypothesis of the theorem (that f is integrable on any bounded set), (1) exists.
Consider (2)
Convert to polar coordinates:

$$\begin{aligned}
\int \cdots \int_{\mathbb{R}^2 \setminus B(\mathbf{0},R)} f d^n\mathbf{x} &= \int_0^{2\pi} \int_0^R frdrd\theta \\
&\leq \int_0^{2\pi} \int_0^R |f|rdrd\theta \\
&= 2\pi \int_0^R |f|rdrd\theta \\
&\leq 2\pi \int_0^R C\frac{1}{||\mathbf{x}||^p} rdr \\
&= 2\pi \int_0^R C\frac{1}{r^p} rdr && (||\mathbf{x}|| = r) \\
&= 2\pi C \int_0^R r^{1-p} dr \\
&= 2\pi C \frac{1}{2-p} R^{2-p}
\end{aligned}$$

As $R \to \infty$,

$$\lim_{R\to\infty} 2\pi C \frac{1}{2-p} R^{2-p}$$

exists iff

$$2 - p < 0 \iff p > 2$$

■

Proposition

$$\int_{-\infty}^{\infty} e^{-x^2} dx = \sqrt{\pi}$$

9 Line and Surface Integrals; Vector Analysis

Let $\mathbf{F}$ be an $\mathbb{R}^n$-valued function defined on some subset of $\mathbb{R}^n$. We have encountered such things in previous chapters, where we generally thought of them as representing transformations from one region of $\mathbb{R}^n$ to another or coordinate systems on regions of $\mathbb{R}^n$. In this chapter, however, we think of such an $\mathbf{F}$ as a function that assigns to each point $\mathbf{x}$in its domain a vector $\mathbf{F}(\mathbf{x})$, represented pictorially as an arrow based at $\mathbf{x}$, and we therefore call it a **vector field**.

9.1 Arc Length and Line Integrals

line integrals: integrals over curves
based on the idea of cutting up the curve into many tiny pieces, forming appropriate Riemann sums, and passing to the limit

Differentials on Curve; Arc Length Suppose C is a smooth curve in $\mathbb{R}^n$. We consider two nearby points $\mathbf{x}$and $\mathbf{x} + d\mathbf{x}$ on the curve; here

$$d\mathbf{x} = (dx_1, \ldots, dx_n)$$

is the vector difference between the two points, and we imagine it as being infinitely small. We may, however, be more interested in the distance between the two points, traditionally denoted by ds, which is

$$ds = |d\mathbf{x}| = \sqrt{dx_1^2 + \ldots + d^2x_n}(*)$$

To give these differentials a precise meaning that can be used for calculations, the best procedure is to parametrize the curve. Thus, we assume that C is given by parametric equations $\mathbf{x} = \mathbf{g}(t), a \leq t \leq b$, where g is of class C^1 and $g'(t) \neq \mathbf{0}$. Then the neighboring points $\mathbf{x}$and $\mathbf{x} + d\mathbf{x}$ are given by $\mathbf{g}(t)$ and $\mathbf{g}(t + dt)$, so

$$d\mathbf{x} = \mathbf{g}(t + dt) - \mathbf{g}(t) = \mathbf{g}'(t)dt = (\frac{dx_1}{dt}, \ldots, \frac{dx_n}{dt})dt$$

Moreover,

$$|d\mathbf{x}| = |\mathbf{g}'(t)|dt = \sqrt{(\frac{dx_1}{dt})^2 + \ldots + (\frac{dx_n}{dt})^2}dt$$

which is just what one gets by formally multiplying and dividing the expression on the right of (*) by dt
What happens if we sum up all the infinitesimal increments $d\mathbf{x}$ or ds - that is, if we integrate the differentials $d\mathbf{x}$ or $ds = |d\mathbf{x}|$ over the curve? Integration of the vector increments $d\mathbf{x}$ just gives the total vector increment, that is, the vector difference between the initial and final points on the curve:

$$\int_C d\mathbf{x} = \int_a^b \mathbf{g}'(t)dt = \mathbf{g}(b) - \mathbf{g}(a)$$

On the other hand, ds is the straight-line distance between two infinitesimally close points $\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$ on the curve, and since smooth curves are indistinguishable from their linear approximations on the infinitesimal level, ds is the *arc length* of the bit of curve between $d\mathbf{x}$ and $\mathbf{x} + d\mathbf{x}$. Adding these up gives the total arc length of the curve

$$\text{Arc length} = \int_C ds = \int_a^b |\mathbf{g}'(t)| dt$$

This is the **definition of arc length for a smooth curve**

Notes The arc length and the vector difference between the two endpoints of the curve should not depend on the particular parametrization we use. The issue here is that a parametrization $\mathbf{x} = \mathbf{g}(t)$ determines an **orientation** for the curve C, that is, a determination of which direction along the curve is "forward" (the direction in which the point $\mathbf{g}(t)$ moves as t increases) and which direction is "backward".

Definition The function $\mathbf{g} : [a, b] \to \mathbb{R}^n$ is called **piecewise smooth** if

1. it is continuous
2. its derivative exists and is continuous except perhaps at finitely many points t_j, at which the one-sided limits $\lim_{t \to t_j \pm} \mathbf{g}'(t)$ exist.

Line Integrals of Scalar Functions If f is a continuous function whose domain includes a smooth (or piecewise smooth) curve C in $\mathbb{R}^n$, we can integrate f over the curve, taking the differential in the integral to be the element of arc length ds. Thus, if C is parametrized by $\mathbf{x} = \mathbf{g}(t), a \leq t \leq b$, we define

$$\int_C f ds = \int_a^b f(\mathbf{g}(t)) |\mathbf{g}'(t)| dt$$

This is independent of the parametrization and the orientation. ¡¡¡¡¡¡¡ HEAD

Line Integrals of Vector Fields We can define the integral of an $\mathbb{R}^m$-valued function over a curve in $\mathbb{R}^n$, simply by integrating each component separately; that is, if $\mathbf{F} = (F_1, \ldots, F_m)$, then $\int_C \mathbf{F} ds = (\int_C F_1 ds, \ldots, \int_C F_m ds)$

Definition of line integral If C is a smooth (or piece wise smooth) curve in $\mathbb{R}^n$ and $\mathbf{F}$ is a continuous vector field defined on some neighborhood of C in $\mathbb{R}^n$, the **line integral** of $\mathbf{F}$ over C is

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_C (F_1 dx_1 + F_2 dx_2 + \ldots + F_n dx_n)$$

That is, if C is described parametrically by $\mathbf{x} = \mathbf{g}(t), a \leq t \leq b$, then

$$\int_C \mathbf{F} \cdot d\mathbf{x} = \int_a^b \mathbf{F}(\mathbf{g}(t)) \cdot \mathbf{g}'(t) dt$$

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9.2 Green's Theorem

Piece-wise smooth. Let $R \subset \mathbb{R}^2$ be a region such that $R = \overline{R^{int}}$. Assume that R is bounded. We say that R is piece-wise defined smooth if the boundaries satisfy:

$$\partial f = \bigcup_{i=1}^{n} C_i \text{ with } C_i \text{ being smooth, } \forall i$$

Simple-Closed curve. A Jordan Simple-Closed curve is a curve in $\mathbb{R}^2$ that is closed and non-self-overlapping. Such curve divides the region of $\mathbb{R}^2$ into to portions: those that are inside this curve, and those that are outside.

Statement of the Green's Theorem. Let C be a positively oriented, piece-wise smooth, simple closed curve in a plane, and let D be the region bounded by C. If $L(\cdot)$ and $M(\cdot)$ are functions of (x, y) defined on an open region containing D and have continuous partial derivatives there, then

$$\oint_C (L dx + M dy) = \iint_D \left(\frac{\partial M}{\partial x} - \frac{\partial L}{\partial y} \right) dx dy$$

where the path of integration along C is anticlockwise.

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