STA452
Lecture Notes

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April 16, 2021

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1 Preface

In our course, you will not see very much data. Our job is to express the ideas that drive the logic (or the logic that drives the ideas). As a consequence, the most of the examples in elementary book appear as pure theory. But they are not defined, they are consequences of abstract mathematical ideas. For example, conditional density is a consequence of conditional expectation, which is an orthogonal projection in a vector space, a pure euclidean geometric idea.

1.1 Preliminary

Definition 1.1. A sequence of sets $A_n \to A$ iff $I(A_n) \to I(A)$.

Proposition 1.1. A limit exists when the limsup is equal to the liminf:

$$lim = \overline{lim} = \underline{lim} \tag{1.1}$$

Proof. For $w \in \Omega$,

$$\begin{aligned} \sup_{t\in T} I(A_t)(w) &= I(\cup_{t\in T} A_t)(w) && (1.2) \\ &= 1 \text{ or } 0 && (1.3) \\ \inf_{t\in T} I(A_t)(w) &= I(\cap_{t\in T} A_t)(w) && (1.4) \end{aligned}$$

Therefore,

$$\begin{aligned} \lim_{n\to\infty} I(A_n) &= \overline{\lim_{n\to\infty}} I(A_n) && (1.5) \\ &= \inf_{n=1}^{\infty} \sup_{k=n}^{\infty} I(A_k) && (1.6) \\ &= I\left(\cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k\right) && (1.7) \\ &= \underline{\lim_{n\to\infty}} I(A_n) && (1.8) \\ &= \sup_{n=1}^{\infty} \inf_{k=n}^{\infty} I(A_k) && (1.9) \\ &= I\left(\cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k\right) && (1.10) \end{aligned}$$

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Property 1.1. Therefore it is clear that

1. $$A_n \to A \iff A = \cap_{n=1}^{\infty} \cup_{k=n}^{\infty} A_k = \cup_{n=1}^{\infty} \cap_{k=n}^{\infty} A_k \tag{1.11}$$

2. $$A_n \uparrow \implies A_n \uparrow \cup_{n=1}^{\infty} A_n \tag{1.12}$$

3. $$A_n \downarrow \implies A_n \downarrow \cap_{n=1}^{\infty} A_n \tag{1.13}$$

2 Virtual Dice

2.1 The Law of Large Numbers

Consider a *mechanism/process/system*, W, which generates *outcomes*, w, in a sample space Ω:

$$W : w_1, w_2, \ldots, w_n, \ldots$$

The outcomes are often referred to as *trials* of the *process* W. w_n is called the nth trial, and the finite sequence $(w_1, w_2, \ldots, w_n)$ is the first n trials.
Consider any real-valued function $g : \Omega \to \mathbb{R}$ defined on the *sample space* Ω. Let $X = g(W)$ denote the *extended process* that applies the function g to the outcome w from W to produce the outcome $x = g(w)$. This new process has its own sequence of trial outcomes:

$$g(W) : g(w_1), g(w_2), \ldots, g(w_n), \ldots \tag{2.1}$$

$$\text{or} \quad X \quad : x_1, x_2, \ldots, x_n, \ldots \tag{2.2}$$

These transformed outcomes are all real values, with which we can do lots of easy arithmetic, while the abstract sample space Ω may not have this property.

Definition 2.1 (sample mean)**.** For each $n \in \mathbb{N}$, the *sample mean* over the first n trials is the *arithmetic average* of the function values over those n trials:

$$\widehat{E}_n X := \frac{g(w_1) + \ldots + g(w_n)}{n} = \bar{x}_n \tag{2.3}$$

Definition 2.2 (random variable)**.** A given process W is said to be a *random process / random variable* iff it satisfies the *empirical law of large numbers*, in that, for any real-valued $X = g(W)$, we have

1. *stability*: the sequence of *sample averages* $(\widehat{E}_n g(W), n \in \mathbb{N})$ converges;

2. *invariance*: the limit is independent of any particular realization $(w_n, n \in \mathbb{N})$.

Definition 2.3 (expected value)**.** For each real-valued $X = g(W)$, we obtain a *expected value* in the above limit:

$$EX := \lim_{n\to\infty} \widehat{E}_n g(W) = \lim_{n\to\infty} \hat{x}_n \tag{2.4}$$

Definition 2.4 (indicator function)**.** The indicator function of a subset A of a set X is a function $I_A : X \to \{0, 1\}$ defined as

$$I_A(x) := \begin{cases} 1 & x \in A \\ 0 & x \notin A \end{cases} \tag{2.5}$$

Definition 2.5 (probability)**.** Now *probability* itself is a special case of an *expected value*: for any *indicator function* $g = I_A$ with $A \subset \Omega$ we will get the usual sequence of averages, but now to be referred to as *empirical relative frequencies*. These averages give the proportion of times that A occurs in the first n trials.

$$\widehat{P}_n(W \in A) := \widehat{E}_n I_A(W) = \frac{I_A(w_1) + \ldots + I_A(w_n)}{n} \, \forall n \in \mathbb{N} \tag{2.6}$$

As $n \to \infty$, the above equation gives the *long-run frequency*, or *probability*:

$$P_W(A) = P(W \in A) := \lim_{n\to\infty} \widehat{E}_n I_A(W) = \lim_{n\to\infty} \widehat{P}_n(W \in A) \tag{2.7}$$

Notation 2.1. Given a random variable W and a *probability distribution* P_W, we can use the following notation:

$$W \sim P_W \quad \text{on} \quad \Omega$$

to be read as "W is distributed as P_W on Ω" or "W is distributed as P_W".

2.2 Some examples: "virtual dice"

Definition 2.6. For any specific $n \in \mathbb{N}$, the random variable X is said to have a *(finite discrete) uniform distribution* on the sample space $\Omega = \{1, \ldots, n\}$ (denoted $X \sim unif\{1, \ldots, n\}$)
iff

$$P(X = k) = \frac{1}{n}, \quad , k = 1, \ldots, n \tag{2.8}$$

Example 2.1. A ten-sided die: $Y \sim unif\{0, \ldots, 9\}$ Let Y be a 2-stage procedure:
Divide the ten digits $\Omega = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 0\}$ into two batches

$$A = \{1, 2, 3, 4, 5\} \quad \& \quad B = \{6, 7, 8, 9, 0\} \tag{2.9}$$

and then toss a standard six-sided die twice. On the first toss, if the die shows $1, 2$ or 3 then we go to A, if the die shows $4, 5$ or 6 then we go to B. Thus each batch is selected half the time. On the second toss, ignoring the digit 6, and if the die shows k we take the kth digit in the batch and report the result. It should be clear then we will arrive at each of the ten digits with identical frequency $\frac{1}{10}$.

2.2.1 A higher level of virtuality: 'continuous dice' and random 'real' numbers

Let U denote the hypothetical possibility of generating an infinite decimal expansion of a number between 0 and 1, by performing the physical algorithm outlined in Example (2.1) an infinite number of times. So an outcome u for U entails an infinite number of repetitions $Y_i, i = 1, 2, \ldots$ of the finite procedure Y:

$$U = \sum_{i=1}^{\infty} \frac{Y_i}{10^i} = 0.Y_1 Y_2 Y_3 \ldots \tag{2.10}$$

Example 2.2. If we generate U explicitly to four places $.y_1y_2y_3y_4$, then there are $10,000$ equally likely possibilities, and our 'actual' U is known to be somewhere between $.y_1y_2y_3y_4$ and 0.0001 higher. In other words, the outcome is in one particular of $10,000$ equally likely subintervals of $[0, 1]$:

$$P(.y_1y_2y_3y_4 \leq U \leq .y_1y_2y_3y_4 + 0.0001) = 1/10,000 \quad \forall y_1, y_2, y_3, y_4 \in \Omega \tag{2.11}$$

Thus we can deduce that

$$P(0 \leq U \leq .a_1a_2a_3a_4) = a_1a_2a_3a_4/10,000 = .a_1a_2a_3a_4 \forall a_1, a_2, a_3, a_4 \in \Omega \tag{2.12}$$

More generally, if u is an n-place finite decimal in the interval $[0, 1]$ for any $n \in \mathbb{Z}$, then $P(U \leq u) = u$, and for any pair of n-place finite decimals $a, b \in [0, 1]$ with $a \leq b$, we will have the *uniformity condition*

$$P(a \leq U \leq b) = b - a \tag{2.13}$$

Corollary 2.1. The probability of U obtaining any specific value u is zero.

$$P(U = u) = P(u \leq U \leq u) = u - u = 0 \tag{2.14}$$

Definition 2.7 (uniform distribution)**.** The random variable U is said to have a *(continuous) uniform distribution* on the unit interval $[0, 1]$ (denoted $U \sim unif[0, 1]$)
iff

$$P(U \leq u) = u \quad \forall 0 \leq u \leq 1 \tag{2.15}$$

Remark 2.1. This is a mathematical statement, which is different from physical existence as in Definition (2.6).

Corollary 2.2. If $X \sim unif[a,b]$ and $U \sim unif[0,1]$, then

$$x = a + (b-a) \cdot u \tag{2.16}$$

Example 2.3. Let $V = 1 - U$, then

$$\begin{aligned} P(V \le u) &= P(1-U \le u) = P(U \ge 1-u) && (2.17)\\ &= 1 - P(U \le 1-u) && (2.18)\\ &= 1 - (1-u) = u = P(U \le u) && (2.19) \end{aligned}$$

As random variables, U and V behave exactly the same way. They have the same *stochastic behavior*. Accordingly, they are said to be *equal-in-distribution*: $V \stackrel{d}{=} U$.

2.2.2 Equality-in-distribution

Definition 2.8 (equality-in-distribution)**.** Two random variables W_1, W_2 on the same sample space Ω are said to be *identically distributed / stochastically identical* (denoted $W_1 \stackrel{d}{=} W_2$)
iff

$$Eg(W_1) = Eg(W_2) \quad \forall g : \Omega \to \mathbb{R} \tag{2.20}$$

iff

$$P(W_1 \in A) = P(W_2 \in A) \quad \forall A \subset \Omega \tag{2.21}$$

Proposition 2.1 (invariance 1)**.** For any function $\phi : \Omega \to \chi$

$$W_1 \stackrel{d}{=} W_2 \implies \phi(W_1) \stackrel{d}{=} \phi(W_2) \tag{2.22}$$

Proof.

$$Eh(\phi(W_1)) = Eh(\phi(W_2)) \quad \forall h : \chi \to \mathbb{R} \tag{2.23}$$

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Proposition 2.2 (invariance 2)**.**

$$W_1 \stackrel{d}{=} W_2 \iff g(W_1) \stackrel{d}{=} g(W_2) \quad \forall g : \Omega \to \mathbb{R} \tag{2.24}$$

2.3 Nature makes them, so can you

2.3.1 Exponential distribution

Let $Z = -\ln U$ with $U \sim unif[0,1]$. Then it is straightforward to compute that, for any non-negative $0 \le s \le t \le \infty$:

$$P(s \le Z \le t) = e^{-s} - e^{-t} \tag{2.25}$$

Proof.

$$\begin{aligned} s \le Z \le t &\iff s \le -\ln U \le t && (2.26)\\ &\iff -t \le \ln U \le -s && (2.27)\\ &\iff e^{-t} \le U \le e^{-s} && (2.28) \end{aligned}$$

Therefore

$$P(s \le Z \le t) = P(e^{-t} \le U \le e^{-s}) \tag{2.29}$$
$$= e^{-s} - e^{-t} \tag{2.30}$$

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Definition 2.9 (standard exponential distribution)**.** The random variable Z is said to have a *standard exponential distribution* on $[0, \infty)$ (denoted $Z \sim \exp(1)$)
iff

$$P(Z \le z) = 1 - e^{-z} \quad \forall z \ge 0 \tag{2.31}$$

Definition 2.10 (scaled exponential distribution)**.** The random variable X is said to have a *scaled exponential distribution*, with *scale parameter* $\theta > 0$ on $[0, \infty)$ (denoted $X \sim \exp(\theta)$)
iff

$$X \overset{d}{=} \theta Z, \quad \text{where } Z \sim \exp(1) \tag{2.32}$$

2.3.2 Consider the generalization

Consider any strictly monotone and C^1 function, g on the interval $[0, 1]$, and let $X \overset{d}{=} g(U)$, where $U \sim unif[0, 1]$. Then

$$P(s < X \le t) = \begin{cases} g^{-1}(t) - g^{-1}(s), & g \uparrow\uparrow \\ g^{-1}(s) - g^{-1}(t), & g \downarrow\downarrow \end{cases} \tag{2.33}$$

Corollary 2.3. Suppose $F : \mathbb{R} \to [0, 1]\, x \mapsto P(X \le x)$. Then F is certainly *non-decreasing*, and for any $s \le t$,

$$P(s < X \le t) = F(t) - F(s) \tag{2.34}$$

Definition 2.11 (distribution function)**.** For any real-valued random variable, X, the *distribution function* of X is given by

$$F(x) \overset{or}{=} F_X(x) := P(X \le x) \quad \forall x \in \mathbb{R} \tag{2.35}$$

Remark 2.2. Let $f(x) = F'(x)$, then we immediately have

$$P(s < X \le t) = F(t) - F(s) = \int_s^t f(x)dx \quad \forall s, t \tag{2.36}$$

At each $x \in g[0, 1]$,

$$\lim_{s\uparrow x,\, t\downarrow x} \frac{P(s < X \le t)}{t - s} = \lim_{s\uparrow x,\, t\downarrow x} \frac{F(t) - F(s)}{t - s} = f(x) \tag{2.37}$$

Remark 2.3. $f(x)$ can be interpreted as "amount of probability per unit length at the point x".

Definition 2.12 (probability density function)**.** A real-valued random variable X is said to be *absolutely continuous* (wrt length measure)
iff

$$\exists f : \mathbb{R} \to [0, \infty), P(s < X \le t) = \int_s^t f(x)\, dx \quad \forall s \le t \tag{2.38}$$

in which case, the function f (not necessarily unique) is referred to as the *probability density function* of X.

Remark 2.4. For any abs. cont. X,

$$P(X = x) = \int_x^x f(x)\,dx = 0 \quad \forall x \tag{2.39}$$

so there is no discrete contribution to the distribution at any $x \in \mathbb{R}$. Thus,

$$P(s \le X \le t) = P(s < X < t) = P(s < X \le t) = P(s \le X < t)$$

Proposition 2.3. $F : [a, b] \to [0, 1]$ is C^1, iff

$$F(x) = \int_a^x f(s)\,ds \quad \text{with } f = F' > 0 \text{ cont. on } [a, b]$$

Proposition 2.4. If $g = F^{-1}$ and $g \in C^1$, then $F(X) \stackrel{d}{=} U$

Proof.

$$\begin{aligned} P(F(X) \le u) &= P(X \le g(u)) && (2.40)\\ &= P(X \le g(u)) && (2.41)\\ &= F(g(u)) && (2.42)\\ &= u && (2.43)\\ &= P(U \le u) && (2.44)\end{aligned}$$

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Definition 2.13 (quantile)**.** For any $0 \le p \le 1$, the value $x_p = g(p) = F^{-1}(p)$ is called the $100 \times p$th *quantile* (or *percentile*) of X. The function g is called the *quantile function.*

$$P(X \le x_p) = p \tag{2.45}$$

2.4 Expected Value

Property 2.1 (finite additivity of probability)**.** If two sets A and B are mutually disjoint, then

$$I(A + B) = I(A) + I(B) \tag{2.46}$$

Therefore

$$\begin{aligned} P(A + B) &= EI(A + B)(W) = E(I(A)(W) + I(B)(W)) && (2.47)\\ &= EI(A)(W) + EI(B)(W) && (2.48)\\ &= P(A) + P(B) && (2.49)\end{aligned}$$

We can prove by induction that

$$P\left(\sum_{i=1}^{n} A_i\right) = \sum_{i=1}^{n} P(A_i) \tag{2.50}$$

Property 2.2 (E is normed on constant random variables)**.** E is *normed* on $g(W) = c$ where $c \in \mathbb{R}$

$$Ec = c \quad \forall c \in \mathbb{R} \tag{2.51}$$

Property 2.3. The indicator function of the whole sample space Ω is 1

$$I_\Omega(W) = 1 \implies P(\Omega) = E1 = 1 \tag{2.52}$$

Property 2.4 (non-negativity of probability)**.**

$$0 \le I(A) \le 1 \implies 0 \le P(A) = EI(A) \le 1 \tag{2.53}$$

2.4.1 Expected Value for an Arbitrary Finite Discrete Distribution

Definition 2.14 (finite scheme)**.** For any finite discrete distribution, we can write a *finite scheme*

$$W \sim \begin{pmatrix} w_1 & \dots & w_N \\ p_1 & \dots & p_N \end{pmatrix} \tag{2.54}$$

to symbolize the *probability mass function*

$$P(W = w_i) = p_i, \quad i \in \{1, 2, \dots, N\}$$

where $\sum_{i=1}^{N} p_i = 1$.

Corollary 2.4. For any real-valued function $g(W), W \in \Omega$, the expected value is

$$Eg(W) = \sum_{i=1}^{N} g(w_i)P(W = w_i) = \sum_{i=1}^{N} g(w_i)p_i \tag{2.55}$$

Proof. $g(W)$ can be explicitly represented as a finite linear combination of simple indicator functions

$$g(W) = \sum_{i=1}^{N} g(w_i)I(W = w_i)$$

So that applying E to both sides gives us the result. ■

Remark 2.5. For $U \sim unif\{1, \dots, n\}$, we have $EU = \frac{1+\dots+n}{n}$ and generally $EU^k = \frac{1+\dots+n^k}{n}$. Note that $EU^k - E(U-1)^k = n^{k-1}$ for $k \in \mathbb{N}$. This provides an iterative basis for the computation of EU^k:

$$EU^2 - E(U-1)^2 = n \implies EU = \frac{n+1}{n} \tag{2.56}$$

$$EU^3 - E(U-1)^3 = n^2 \implies EU^2 = \frac{2n+1}{3}EU \tag{2.57}$$

$$EU^4 - E(U-1)^4 = n^3 \implies EU^3 = n(EU)^2 \tag{2.58}$$

$$\tag{2.59}$$

2.4.2 Full generality: lebesgue-stieltjes

Suppose we are given a distribution function $F(x) = P(X \leq x), x \in \mathbb{R}$, for a real-valued random variable $X = g(W)$, with $W \sim P$ on sample space Ω. Then consider some discrete approximation to X, for example,

$$X_n = \sum_{i=-n}^{n} \frac{i-1}{\sqrt{n}} I\left(\frac{i-1}{\sqrt{n}} < X < \frac{i}{\sqrt{n}}\right) \tag{2.60}$$

For this particular approximation,

$$|X - X_n| \leq \frac{1}{\sqrt{n}} + |X|I(|X| > \sqrt{n}) \tag{2.61}$$

Thus $X_n \to X$ as $n \to \infty$. Then any continuous real-valued function $h(X_n) \to h(X)$ as $n \to \infty$. If $h(X)$ is bounded, then

$$Eh(X) = \lim_{n\to\infty} \sum_{i=-n}^{n} h(\frac{i-1}{\sqrt{n}}) \left(F(\frac{i}{\sqrt{n}}) - F(\frac{i-1}{\sqrt{n}})\right) \tag{2.62}$$

which is called the *lebesgue-stieltjes integral* of the function $h(x)$. It may be denoted

$$Eh(X) := \int_{-\infty}^{\infty} h(x)\, dF(x) \tag{2.63}$$

2.4.3 Examples

Definition 2.15 (bernoulli trial)**.** The random variable Z is said to be a *bernoulli trial* (denoted $Z \sim bern(p), 0 \leq p \leq 1$) iff

$$Z \sim \begin{pmatrix} 0 & 1 \\ q & p \end{pmatrix}$$

2.4.4 Expected Value for Continuous Functions

2.4.5 Expected Value for C^1 functions

Consider the special case where function g is strictly monotone and C^1.

Proposition 2.5. If $X = g(U), g : [0,1] \to [a,b]$ is strictly monotone and C^1, then for any continuous function $h : \mathbb{R} \to \mathbb{R}$,

$$Eh(X) = \int_a^b h(x) f(x)\, dx \tag{2.64}$$

where

$$F = \begin{cases} g^{-1} & , g \uparrow\uparrow \\ 1 - g^{-1} & , g \downarrow\downarrow \end{cases} \quad \text{and} \quad f(x) = F'(x)$$

Proof. For any $0 \leq t < 1$, when $g \uparrow\uparrow$ and C^1, we have

$$\int_0^t h(g(u))\, du = ??? \tag{2.65}$$

∎

complete it later

2.5 Exponential Distribution

notes

2.6 Gamma Distribution

Property 2.5. As follows.

- $\Gamma(\alpha) = (\alpha - 1)!$ for positive integer α
- $\Gamma(p+1) = p\Gamma(p)$
- $\Gamma(1/2) = \pi^{1/2}$
- If $Z \sim Gamma(p, 1)$, then $EZ^s = \Gamma(p+s)/\Gamma(p) \quad \forall s \in \mathbb{R}$
- $Gamma(v/2, 1/2) \stackrel{d}{=} \chi^2_{(v)}$
- $Gamma(1, \lambda) \stackrel{d}{=} Exp(\lambda)$
- If $X_i \sim Gamma(\alpha_i, \beta)$ for $i = 1, 2, \ldots, N$, then $\sum_{i=1}^N X_i \sim Gamma\left(\sum_{i=1}^N \alpha_i, \beta\right)$
- If $X \sim Gamma(\alpha, \theta), Y \sim Gamma(\beta, \theta)$ are independently distributed, then $X/(X+Y) \sim Beta(\alpha, \beta)$ is independent of $X + Y$.
- If $X \sim Gamma(\alpha, \beta)$, then $cX \sim Gamma\left(\alpha, \frac{\beta}{c}\right)$
- If $X_i \sim Gamma(\alpha_i, 1)$ are independently distributed, then the vector $(X_1/S, \ldots, X_n/S)$, where $S = X_1 + \ldots + X_n$ follows a Dirichlet distribution with parameters $\alpha_1, \ldots, \alpha_n$.

2.7 Continuity Revisited

2.7.1 Sequential Continuity of Probability

Definition 2.16 (σ-additivity)**.** P is said to be σ*-additive / countably additive* iff for any mutually disjoint sequence of events A_n $(n \in \mathbb{N})$

$$P(\sum_1^\infty A_n) = \sum_1^\infty P(A_n) \tag{2.66}$$

Remark 2.6. Equation (2.66) is equivalent to the following pair of equations:

$$\text{finite-additivity: } P(\sum_1^n A_i) = \sum_1^n P(A_n) \tag{2.67}$$

$$\text{continuity: } A_n \to A \implies P(A_n) \to P(A) \tag{2.68}$$

Proposition 2.6. If $A_n \uparrow A$ or $A_n \downarrow A$, then

$$P(A_n) \to P(A)$$

Proof. if If $A_n \uparrow A$ then we have that

$$A = \cup_{n=1}^\infty A_n = \sum_{n=1}^\infty (A_n - A_{n-1})$$

where, for convenience, we have $A_0 = \emptyset$.
Then

$$P(A) = \sum_{n=1}^\infty (P(A_n) - P(A_{n-1}))x \tag{2.69}$$

$$= \lim_{n\to\infty} \sum_{i=1}^\infty (P(A_i) - P(A_{i-1})) \tag{2.70}$$

$$= \lim_{n\to\infty} P(A_n) \tag{2.71}$$

On the other hand, $A_n \downarrow A$ is equivalent to $A_n^c \uparrow A^c$. ■

Corollary 2.5 (sequential continuity)**.**

$$A_n \to A \implies P(A_n) \to P(A) \tag{2.72}$$

Proof. Suppose $A_n \to A$, then

$$\cup_{n=1}^\infty \cap_{k\geq n} A_k = A = \cap_{n=1}^\infty \cup_{k\geq n} A_k \tag{2.73}$$

$$\cap_{k\geq n} A_k \leq A_n, A \leq \cup_{k\geq n} A_k \tag{2.74}$$

$$P(\cap_{k\geq n} A_k) \leq P(A_n), P(A) \leq P(\cup_{k\geq n} A_k) \tag{2.75}$$

$$|P(A_n) - P(A)| \leq P(\cup_{k\geq n} A_k) - P(\cap_{k\geq n} A_k) \tag{2.76}$$

$$\to P(A) - P(A) \tag{2.77}$$

$$= 0 \tag{2.78}$$

Therefore,

$$|P(A_n) - P(A)| \to 0 \tag{2.79}$$

$$P(A_n) \to P(A) \tag{2.80}$$

■

2.7.2 Right Continuity of Cumulative Distribution Function

For any $x_n \downarrow x$, simply let $A_n = (-\infty, x_n]$ and $A = (-\infty, x]$.
Then $A_n \downarrow A$, so

$$F(x_n) = P(X \in A_n) \downarrow P(X \in A) = F(x)$$

Denoting the right-limit of F at x by $F(x+) := \lim_{y \downarrow x} F(y)$, and the left-limit $F(x-) := \lim_{y \uparrow x} F(y)$, we get the property of *right-continuity* for CDF

$$F(x+) = F(x) \quad \forall x \in \mathbb{R} \tag{2.81}$$

Remark 2.7. Any distribution function $F(x)$ can actually be discontinuous at no more than a countable number of points, which corresponds to all the jumps on the discrete part of the distribution.

Definition 2.17 (probability mass function)**.** For any real-valued random variable X, the *probability mass function* of X is given by

$$p(x) = p_X(x) = P(X = x) \quad \forall x \in \mathbb{R}$$

Proposition 2.7. Probability mass function

$$p(x) = F(x) - F(x-) \quad \forall x \in \mathbb{R} \tag{2.82}$$

Proof. For any $x_n \uparrow x$, simply let $A_n = (-\infty, x_n]$ and $A = (-\infty, x)$.
Then $A_n \uparrow A$, so

$$F(x-) := \lim_{n \to \infty} P(X \in A_n) = P(X \in A) = P(X < x)$$

Therefore

$$p(x) = P(X \leq x) - P(X < x) = F(x) - F(x-)$$

■

Remark 2.8. The points of continuity C_F of any distribution function correspond perfectly to the points where pmf is zero.

$$C_F = \{x \in \mathbb{R} | F(x-) = F(x+)\} \tag{2.83}$$
$$= \{x \in \mathbb{R} | F(x-) = F(x)\} \tag{2.84}$$
$$= \{x \in \mathbb{R} | p(x) = 0\} = p^{-1}(0) \tag{2.85}$$

The complementary region being the discrete part of the distribution

$$D_F = \{x \in \mathbb{R} | p(x) > 0\} = p^{-1}(0)^c \tag{2.86}$$

Proposition 2.8. D_F is at most countable.

$$\# D_F \leq \# \mathbb{N}$$

Proof. Note that

$$\{x \in \mathbb{R} | p(x) > 0\} = \cup_{n=1}^{\infty} \{x \in \mathbb{R} | p(x) > 1/n\}$$

It is clear that for every $n \in \mathbb{N}$, $\{x \in \mathbb{R} | p(x) > 1/n\}$ has less than n point in it. Otherwise

$$\exists A_n = \{a_1, \ldots, a_n\} \subset \{x \in \mathbb{R} | p(x) > 1/n\} \text{ with } P(A_n) > 1$$

which is a contradiction.
Since a countable union of countable sets is still countable, we have D_F is at most countable. ■

2.8 Back to the Uniform

Definition 2.18 (p-adic series)**.** For any $p \in \mathbb{N}$ with $p \geq 2$, any real number $U \in [0, 1)$ can be written as a base p expansion in the form

$$U = \sum_{i=1}^{\infty} Z_i p^{-1}$$

where $Z_i \in \{0, 1, 2, \ldots, p-1\}$.

Notation 2.2. Let $\dot{p}^{\infty}$ denote the collection of all the infinite p-sequences which do not end in $p-1$ repeated forever.

Lemma 2.1 (p-adic coding of the unit interval)**.** $u = \sum_{i=1}^{\infty} z_i p^{-i}$ defines a correspondence $\Phi : \dot{p}^{\infty} \mapsto [0, 1)$.

Lemma 2.2 (p-adic partitioning)**.** If $u = \sum_{i=1}^{\infty} z_i p^{-i}$ with $\mathbf{z} \in \dot{p}^{\infty}$ then

$$z_1 = b_1, \ldots, z_n = b_n \iff \sum_{i=1}^{n} z_i p^{-i} \leq u < \sum_{i=1}^{n} z_i p^{-i} + p^{-n}$$

Theorem 2.1 (digital coding of the uniform)**.** For $U = \sum_{i=1}^{\infty} z_i p^{-i}$ with $p \geq 2$ and $\mathbf{Z} \in p^{\infty}$,

$$U \sim unif[0, 1] \iff Z_i \overset{i.i.d.}{\sim} unif\{0, \ldots, p-1\}$$

Proof. Omitted here because it is very long and nuanced. ■

Remark 2.9. It is regarded as the *Fundamental Theorem of Applied Probability.*

2.9 Back to the Uniform II

2.9.1 Percentiles

For any given $X \sim F$ and any $0 < p < 1$

Definition 2.19 (percentile/quantile)**.** A p-th *percentile/quantile* of X is any value, $\theta = \theta_p$, such that $F(\theta-) \leq p \leq F(\theta)$

Remark 2.10. Not necessarily unique, could be a closed interval on the real line.

Definition 2.20 (lower and upper quantile functions)**.** We define the *lower quantile function* of the distribution function, F, to be the real-valued function $g : (0, 1) \to \mathbb{R}$ with

$$g(u) = \inf F^{-1}[u, 1] \tag{2.87}$$

and the *upper quantile function* to be $h : (0, 1) \to \mathbb{R}$ with

$$h(u) = \sup F^{-1}[0, u] \tag{2.88}$$

Remark 2.11. Both of these functions are non-decreasing. But even when $F(x)$ is strictly increasing, either of $g(u)$ or $h(u)$ may actually be constant over various intervals.

Proposition 2.9. For every $0 < u < 1$ and $x \in \mathbb{R}$ we have both

$$u \leq F(g(u)) \quad \text{and} \quad g(F(x)) \leq x \tag{2.89}$$

Proof. (1) By the definition of infimum, we may choose $x_n \downarrow g(u)$ with $F(x_n) \geq u \, \forall n$.
Since F is right-continuous, then $F(x_n) \downarrow F(g(u))$. So $\lim F(x_n) = F(g(u)) \geq u$.
(2)

$$x \in F^{-1}(F(x)) \subset F^{-1}[F(x), 1]$$

Since $g(F(x)) = \inf F^{-1}[F(x), 1]$, then $g(F(x)) \leq x$. ■

Proposition 2.10. For every $0 < u < 1$ and $x \in \mathbb{R}$ we have both

$$u \leq F(h(u)) \quad \text{and} \quad x \leq h(F(x)) \tag{2.90}$$

Proof. (1) (2) prove

$$x \in F^{-1}(F(x)) \subset F^{-1}[0, F(x)]$$

Since $h(F(x)) = \sup F^{-1}[0, F(x)]$, then $x \leq h(F(x))$. ■

Corollary 2.6.

$$g(u) \leq x \iff u \leq F(x) \tag{2.91}$$

Proof.

$$g(u) \leq x \overset{F}{\Longrightarrow} u \leq F(g(u)) \leq F(x) \overset{g}{\Longrightarrow} g(u) \leq gF(x) \leq x$$

■

Corollary 2.7.

$$F(g(p)-) \leq p \leq F(g(p)) \tag{2.92}$$

Proof. From $g(u) \leq x \iff u \leq F(x)$, we can conclude $x < g(u) \iff F(x) < u$. Then $F(x) < p \quad \forall x < g(p)$, so $F(g(p)-) \leq p$. ■

Remark 2.12. Indeed, the set of all pth percentiles is the simple compact interval $[g(p), h(p)]$.

Corollary 2.8. $F(x-) \leq pF(x) \iff g(p) \leq x \leq h(p)$ prove

Corollary 2.9.

$$FgF = F \quad \text{and} \quad gFg = g$$

Proof. Since $u \leq Fg(u)$, then $F(x) \leq F(g(F(x)))$.
Since $g(F(x)) \leq x$ and F is non-decreasing, then $F(g(F(x))) \leq F(x)$
Therefore, $F(x) \leq F(g(F(x))) \leq F(x) \implies FgF = F$.
Similarly, Since $u \leq F(g(u))$ and g is non-decreasing, then $g(u) \leq g(F(g(u)))$.
Since $g(F(x)) \leq x$, then $g(F(g(u))) \leq g(u)$.
Therefore, $g(u) \leq g(F(g(u))) \leq g(u) \implies gFg = g$. ■

Corollary 2.10. $g(u)$ is left-continuous.

$$u_n \uparrow u \implies g(u_n) \uparrow g(u) \tag{2.93}$$

Proof. $u_n \uparrow u \implies g(u_n) \uparrow c \leq g(u)$ for some upper limit c.
But $g(u_n) \leq c \quad \forall n$. Then $u_n \leq Fg(u_n) \leq F(c) \quad \forall n$
Then $u \leq F(c)$, then $g(u) \leq g(F(c)) \leq c$. Therefore $c \leq g(u) \leq c \implies g(u) = c$, then $g(u_n) \uparrow g(u)$. ■

Corollary 2.11. F is continuous iff $u = Fg(u) \quad \forall u$.

Proof. (⇒) Obvious, since $F(g(u)) = F(g(u)-) \leq u \leq F(g(u)) \implies u = Fg(u)$
(⇐) If $u = Fg(u)$ for every $0 < u < 1$, then we only need to show that F is left-continuous.
$x_n \uparrow x \implies F(x_n) \uparrow p = Fg(p) \leq F(x)$ for some p.
So if $g(p) = x$, we are done.
If $g(p) < x$, then $g(p) < x_n$ for n sufficient large.
So $p = Fg(p) \leq F(x_n) \leq p$ for n sufficient large, so $F(x_n) = p$ and ∎ ???

Proposition 2.11 (the quantile transform)**.**

$$U \sim unif[0,1] \implies g(U) \stackrel{d}{=} X \tag{2.94}$$

Proof. We know from Corollary (2.6) that $g(U) \leq x \iff U \leq F(x)$.
Therefore,

$$\begin{aligned} P(g(U) \leq x) &= P(U \leq F(x)) && (2.95) \\ &= F(x) && \text{(by the property of uniform distribution)} \\ &= P(X \leq x) && (2.96) \end{aligned}$$

∎

Corollary 2.12.

$$gF(X) \stackrel{d}{=} X \tag{2.97}$$

Proof. $gF(X) \stackrel{d}{=} gFg(U) \stackrel{d}{=} g(U) \stackrel{d}{=} X$. ∎

Corollary 2.13.

$$P(F(X) \leq F(x)) = P(X \leq x) \quad \forall x \in \mathbb{R} \tag{2.98}$$

Proof. Let $x \in \mathbb{R}$
$F(X) \leq F(x) \implies gF(X) \leq \underbrace{g(F(x))}_{X} \leq x \implies F(X) \leq F(x)$
Therefore, $F(X) \leq F(x)$ iff $X \leq x$
Then $P(F(X) \leq F(x)) = P(X \leq x)$. ∎

Proposition 2.12 (probability integral transform)**.** F is continuous iff $F(X) \stackrel{d}{=} U$

Proof. (⇒): Assume F is continuous. From Proposition (2.11), we know $g(U) \stackrel{d}{=} X$, so $F(X) \stackrel{d}{=} Fg(U)$. But since F is continuous, $Fg(U) = U$. Therefore, $F(X) \stackrel{d}{=} U$. (⇐): Assume $F(X) \stackrel{d}{=} U$.
$X = x$ implies $F(X) = F(x)$. This means $F(X) = F(x)$ may have a higher probability than $X = x$. Therefore,

$$P(X = x) \leq P(F(X) = F(x)) = P(U = F(x)) = 0$$

Hence $P(X = x) = 0$ for all $x \in \mathbb{R}$ so F is continuous. ∎

Proposition 2.13. Both g and F are continuous iff $g = h = F^{-1}$ on $(0,1)$.

Proof. (⇒): Assume g and F are continuous.
Then from Corollary 2.11, $u = Fg(u)$. Also we can easily conclude that g is onto. ∎ prove

Property 2.6. Given any $f : \mathbb{R} \to (0,\infty) \in C$ s.t. $\int_{-\infty}^{\infty} f(x)\,dx = 1$, the function defined by $F(x) = \int_{-\infty}^{x} f(s)\,ds, x \in \bar{\mathbb{R}} := \mathbb{R} \cup \{\pm\infty\} = [-\infty,\infty]$ determines a homeomorphism $F : \bar{\mathbb{R}} \stackrel{\cong}{\to} [0,1]$ with quantile function $g = h = F^{-1}$.

2.9.2 Medians

Definition 2.21. A *median* for a random variable X is any $\theta = \theta_{1/2}$ s.t. $F(\theta-) \leq \frac{1}{2} \leq F(\theta)$ (denoted $\theta = median(X)$).

Remark 2.13. A median is simply a 50th percentile.

Proposition 2.14. Assuming $E|X| < \infty$ (the mean of X exists):

$$\theta = median(X) \iff E|X-\theta| = \inf_{t\in\mathbb{R}} E|X-t| \tag{2.99}$$

Remark 2.14. A median of a r.v. X is the closest constant to X in L_1 metric, a specific way of measuring the distance between two random objects:

$$d_1(x,y) = E|x-y|$$

Proposition 2.15. Assuming $E|X| < \infty$ (the mean of X exists):

$$\mu = EX \iff \sqrt{E(X-\mu)^2} = \inf_{t\in\mathbb{R}} \sqrt{E(X-t)^2} \tag{2.100}$$

Remark 2.15. A mean of a r.v. X is the closest constant to X in L_2 metric:

$$d_2(x,y) = \sqrt{E(x-y)^2}$$

3 Reduction to an Axiomatic System

3.1 The Kolmogorov Axioms

Definition 3.1 (probability space)**.** A *probability space (distribution)* is a triple of objects (Ω, L, E)

1. Ω: any set, called the *sample space*
2. L: any vector space of real-valued functions on Ω that contains the constants, and is closed under taking absolute values ($X \in L \implies |X| \in L$), the elements of which are referred to as *random variables*
3. $E : L \to \mathbb{R}$, any functional that is
   - *normed*: $Ec = c$
   - *non-negative*: $X \geq 0 \implies EX \geq 0$
   - *linear*: $E\sum_1^n a_iX_i = \sum_1^n a_iEX_i$
   - *continuous*: $0 \leq X_n \uparrow X \implies 0 \leq EX_n \uparrow EX$

   referred to as an *expectation operator*, while its value EX at any $X \in L$ is called the *expected value* of that X.

Property 3.1 (continuity)**.** A useful variant of E's *continuous* property is stated as:
If $Z_n \geq 0, n = 1, 2, \ldots$, then

$$E\sum_{i=1}^{\infty} Z_n = \sum_{i=1}^{\infty} EZ_n \tag{3.1}$$

3.1.1 Reducing the Reduction

As understood, probability is a very special case of expected values. Thus we can reduce the definition of a probability space as follows

Definition 3.2 (probability space)**.** A *probability space (distribution)* is a triple of objects $(\Omega, \mathcal{F}, P)$

1. Ω: any set, called the *sample space*
2. $\mathcal{F}$: any σ-algebra of subsets of Ω, which is a non-empty collection closed under countable unions and complements. The elements of $\mathcal{F}$ are referred to as *events*
3. $E : \mathcal{F} \to \mathbb{R}$, any functional that is
   - *normed*: $Ec = c$
   - *non-negative*: $X \geq 0 \implies EX \geq 0$
   - *σ-additive*: $P(\sum_1^\infty A_i) = \sum_1^\infty P(A_i)$

   referred to as *probability measure*, while its value $P(A)$ at any $A \in \mathcal{F}$, is called the *probability* of that A.

Remark 3.1. σ-algebra is identical to σ-field.

Proposition 3.1 (nullity)**.**

$$P(\emptyset) = 0$$

Proof. First we show that $\Omega \in \mathcal{F}$.
If $F \neq \emptyset$, then $\exists A \in \mathcal{F}$ s.t. $A^c \in \mathcal{F}$.
So let $A_1 = A, A_n = A^c \ \forall n \geq 2$.
Then

$$\begin{aligned} \cup_1^\infty A_n &= A \cup A^c \cup A^c \cup \dots & (3.2)\\ &= A \cup A^c & (3.3)\\ &= \Omega \in \mathcal{F} & (3.4) \end{aligned}$$

Define the sequence of mutually disjoint events

$$A_1 = \Omega \quad \& \quad A_n = \emptyset, \quad n \geq 2$$

Then we have $\Omega = \sum_{n=1}^\infty A_n$ and thus

$$1 = 1 + \lim_{n\to\infty} nP(\emptyset)$$

which forces the result. ■

Proposition 3.2 (finite-additivity)**.**

$$P(A + B) = P(A) + P(B)$$

Corollary 3.1 (complementarity)**.**

$$P(A^c) = 1 - P(A)$$

Corollary 3.2 (negative additivity)**.**

$$P(A - B) = P(A) - P(A \cap B)$$

Proof. Since $A = AB + AB^c = AB + (A - B)$, then $P(A) = P(AB) + P(A - B)$, hence the result. ■

Corollary 3.3 (monotonicity)**.**

$$A \subset B \implies P(A) \leq P(B)$$

Proof. Since $B = (B - A) \cup A$, then $P(B) - P(A) = P(B - A) \geq 0$, hence the result. ∎

Proposition 3.3. Assuming *normed, non-negative* and *σ-additive*. If $A_n \uparrow A$ or $A_n \downarrow A$, then

$$P(A_n) \to P(A)$$

3.1.2 Recovering the Expectation Operator

The space of bernoulli trials (indicator functions) is

$$\mathcal{J} = \{I_A | A \in \mathcal{F}\}$$

On this collection, we have to define the expectation operator to be $E : \mathcal{J} \to \mathbb{R}$

$$E(I_A) = P(A) \quad \forall A \in \mathcal{F}$$

Starting from $\mathcal{J}$, we can create a vector space that contains finite linear combinations of indicator functions. They are all the finite discrete random variables

$$\mathcal{S} = \left\{ S | S = \sum_{i=1}^{m} a_i I(A_i), a_i \in \mathbb{R}, A_i \in \mathcal{F}, i = 1, \ldots, m, m \in \mathbb{N} \right\}$$

In this case, E is required to be linear, so we define it as $E : \mathcal{S} \to \mathbb{R}$

$$E(S) = \sum_{i=1}^{m} a_i P(A_i) \quad \forall S = \sum_{i=1}^{m} a_i I(A_i)$$

Lemma 3.1 (invariance at zero)**.**

$$\sum_{i=1}^{m} a_i I(A_i) = 0 \implies \sum_{i=1}^{m} a_i P(A_i) = 0 \tag{3.5}$$

Corollary 3.4 (invariance)**.**

$$\sum_{i=1}^{m} a_i I(A_i) = \sum_{j=1}^{m} b_j I(B_j) \implies \sum_{i=1}^{m} a_i P(A_i) = \sum_{j=1}^{m} b_j P(B_j) \tag{3.6}$$

Corollary 3.5 (linearity)**.**

$$E : \mathcal{S} \overset{linear}{\to} \mathbb{R}$$

by

$$ES = \sum_{i=1}^{m} a_i P(A_i)$$

for any

$$S = \sum_{i=1}^{m} I(A_i)$$

3.1.3 Classical Random Variables

Consider the event $(X \leq x)$ for any $x \in \mathbb{R}$. This is a subset of the original sample space Ω:

$$(X \leq x) = \{w \in \Omega | X(w) \leq x\} = X^{-1}(-\infty, x]$$

Definition 3.3 (classical real-valued random variables)**.** The classical real-valued random variables, X, wrt to a given distribution, $(\Omega, \mathcal{F}, P)$, consist in the collection, $\mathcal{R} = \langle \mathcal{F} \rangle$ (generated by $\mathcal{F}$):

$$\mathcal{R} = \langle \mathcal{F} \rangle = \{X : \Omega \to \mathbb{R} | (X \leq x) \in \mathcal{F} \, \forall x \in \mathbb{R} \tag{3.7}$$

Proposition 3.4 ($\mathcal{R}$ is closed wrt countable maxima and minima)**.** For any given sequence $X_n, n \in \mathbb{N}$ in $\mathcal{R}$, provided they are real-valued, both $\inf_{n=1}^{\infty} X_n \in \mathcal{R}$ and $\sup_{n=1}^{\infty} X_n \in \mathcal{R}$

Corollary 3.6 ($\mathcal{R}$ is sequentially closed)**.** For a given sequence $X_n, n \in \mathbb{N}$ in $\mathcal{R}$,

$$X_n \to X \implies X \in \mathcal{R}$$

Proposition 3.5 (fundamental representation)**.** For any $Z \geq 0$ in $\mathcal{R}$, there are $S_n \geq 0$ in $\mathcal{S}$ s.t.

$$0 \leq S_n \uparrow Z$$

Proposition 3.6. $\mathcal{R}$ is an algebra (a vector space with multiplication), closed wrt sequential limits and absolute values.

3.1.4 Expectation Operator

Theorem 3.1 (monotone convergence theorem (MCT))**.**

$$0 \leq Z_n \uparrow Z \implies 0 \leq EZ_n \uparrow EZ \tag{3.8}$$

Corollary 3.7 (dominated convergence theorem (DCT))**.** Assume $X_n, Y \in \mathcal{R}$,

$$X_n \to X \, w. |X_n| \leq Y, EY < \infty \implies E|X_n - X| \implies 0 \tag{3.9}$$

Property 3.2 (decomposition)**.** $X \in \mathcal{R}$ can be decomposed into its positive and negative parts:

$$X^+ = \max(X, 0)$$

$$X^- = -\min(X, 0)$$

$$X = X^+ - X^-$$

$$EX = \begin{cases} EX^+ - EX^-, & EX^+ < \infty or EX^- < \infty \\ undefined, & EX^+ = EX^- = \infty \end{cases} \tag{3.10}$$

3.2 Recovering the Physics - Kolmogorov Synthesis

Theorem 3.2 (empirical law of large numbers (ELLN))**.** Suppose $X_i, i \in \{1, 2, ..., n\}$ i.i.d. and the sample mean $\overline{X_n} = \frac{X_1 + ... + X_n}{n}$

$$EX = \lim_{n \to \infty} \overline{X_n} \tag{3.11}$$

Theorem 3.3 (strong law of large numbers (SLLN))**.** Suppose $X_i, i \in \{1, 2, ..., n\}$ i.i.d. and the sample mean $\overline{X_n} = \frac{X_1 + ... + X_n}{n}$

$$P\left(\overline{X_n} \to EX\right) = 1 \tag{3.12}$$

4 Geometry of Data

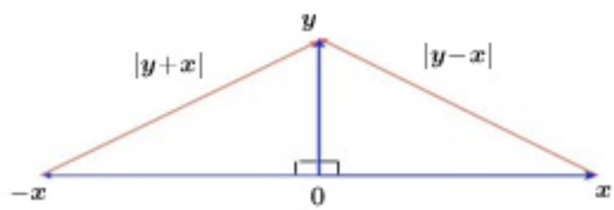
4.1 The Natural Geometry of $\mathbb{R}^n$

On $\mathbb{R}^n$ we define three standard geometric devices

1. **inner product** $\mathbf{x} \cdot \mathbf{y} = \mathbf{x}'\mathbf{y} = \sum_{i=1}^n x_i y_i$
2. **length (norm)** $|\mathbf{x}| = \sqrt{\mathbf{x} \cdot \mathbf{x}}$
3. **distance (metric)** $d(\mathbf{x}, \mathbf{y}) = |\mathbf{y} - \mathbf{x}|$

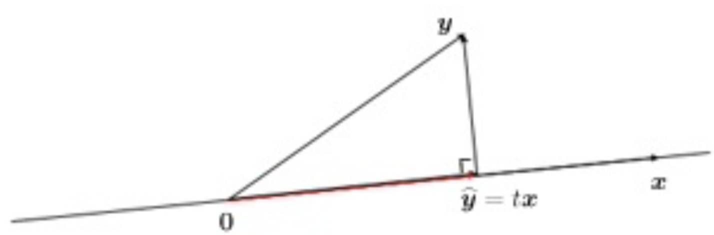
Definition 4.1 (orthogonality)**.** For non-zero vectors $\mathbf{x}$ and $\mathbf{y}$,

$$\mathbf{x} \perp \mathbf{y} \iff |\mathbf{y} - \mathbf{x}| = |\mathbf{y} + \mathbf{x}| \iff \mathbf{x} \cdot \mathbf{y} = 0 \tag{4.1}$$



Definition 4.2 (orthogonal projection)**.** The *orthogonal projection* $\hat{\mathbf{y}}$ of $\mathbf{y}$ on any non-zero $\mathbf{x}$ is a scalar multiple of $\mathbf{x}$ and the *residual vector* $\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to $\mathbf{x}$:

$$\begin{cases} \hat{\mathbf{y}} = t\mathbf{x} & \text{for some } t \in \mathbb{R} \\ \mathbf{y} - \hat{\mathbf{y}} \perp \mathbf{x} \end{cases} \tag{4.2}$$



Remark 4.1. As follows.

1. Then $\mathbf{x} \cdot (\mathbf{y} - t\mathbf{x}) = 0$ and $t = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}|^2}$.
2. Let θ be the angle described by $\mathbf{x}$ and $\mathbf{y}$. Then $\cos\theta = sign(\mathbf{x} \cdot \mathbf{y}) \frac{|\hat{\mathbf{y}}|}{|\mathbf{y}|} = \frac{\mathbf{x} \cdot \mathbf{y}}{|\mathbf{x}||\mathbf{y}|}$

Property 4.1 (Pythagorean Theorem)**.**

$$|\mathbf{y}|^2 = |\hat{\mathbf{y}} + (\mathbf{y} - \hat{\mathbf{y}})|^2 = |\hat{\mathbf{y}}|^2 + |\mathbf{y} - \hat{\mathbf{y}}|^2 \tag{4.3}$$

Property 4.2 (Cauchy-Schwarz inequalities)**.** As follows.

1. $|\hat{\mathbf{y}}| \leq |\mathbf{y}|$ w.eq. iff $\mathbf{y} = \hat{\mathbf{y}}$
2. $|\mathbf{x} \cdot \mathbf{y}| \leq |\mathbf{x}||\mathbf{y}|$ w.eq. iff $\mathbf{y} = t\mathbf{x}$ for $t \in \mathbb{R}$
3. $|\cos\theta(\mathbf{x}, \mathbf{y})| \leq 1$ w.eq. iff $\mathbf{y} = t\mathbf{x}$ for $t \in \mathbb{R}$

Property 4.3 (Triangle Inequality)**.**

$$|\mathbf{x} + \mathbf{y}| \leq |\mathbf{x}| + |\mathbf{y}| \quad \text{w.eq. iff } \mathbf{y} = t\mathbf{x} \text{ for } t > 0 \tag{4.4}$$

4.2 The Natural Geometry of Random Variables

Definition 4.3 (L2 space)**.** By definition, L is the vector space of real-valued random variables. We move to the particular sub-space of L where the natural geometry of $\mathbb{R}^n$ has been reinvested in the random variables.

$$L2 = \{X \in L | EX^2 < \infty\} \tag{4.5}$$

In L2 we define

1. **inner product** $\langle X, Y \rangle = EXY$
2. **length (norm)** $||X|| = \sqrt{\langle X, X \rangle} = \sqrt{EX^2}$
3. **distance (metric)** $d(X, Y) = ||Y - X|| = \sqrt{E(Y - X)^2}$

4.2.1 Markov & Chebyshev

Theorem 4.1 (Markov's Inequality)**.** For any $Z \geq 0, t \geq 0$ and non-decreasing $g : [0, \infty) \to [0, \infty)$, we have

$$P(Z \geq t) \leq \frac{Eg(Z)}{g(t)} \quad \forall t \text{ s.t. } g(t) > 0 \tag{4.6}$$

Proof.

$$g(t)I(Z \geq t) \leq g(Z) \tag{4.7}$$

$$g(t)P(Z \geq t) \leq Eg(Z) \qquad \text{(applying } E \text{ to both sides)}$$

$$P(Z \geq t) \leq \frac{Eg(Z)}{g(t)} \tag{4.8}$$

■

Consider an arbitrary random variable X in $L2$ with $\mu = EX$ and $\sigma^2 = E(X - \mu)^2$ and let $Z = |X - \mu|$ and $g(t) = \epsilon^2$ to get

Corollary 4.1 (Chebyshev I)**.** For any $X \in L_2, \epsilon > 0$, we have

$$P(|X - \mu| \geq \epsilon) \leq \frac{\sigma^2}{\epsilon^2} \tag{4.9}$$

Or take $Z = \frac{|X-\mu|}{\sigma}$ and $g(t) = k^2$ to find

Corollary 4.2 (Chebyshev II)**.** For any $X \in L_2, k > 0$, we have

$$P\left(\frac{|X - \mu|}{\sigma} > k\right) \leq \frac{1}{k^2} \tag{4.10}$$

Corollary 4.3 (Markov's Equality)**.**

$$E|X| = 0 \iff X \overset{wP1}{=} 0 \tag{4.11}$$

4.2.2 The Geometry of L_2

Definition 4.4 (orthogonality).

$$X \perp Y \iff ||Y - X|| = ||Y + X|| \iff \langle X, Y\rangle = EXY = 0 \tag{4.12}$$

Definition 4.5 (orthogonal projection).

$$\begin{cases} \hat{Y} = tX \quad \text{for some } t \in \mathbb{R} \\ Y - \hat{Y} \perp X \end{cases} \tag{4.13}$$

Remark 4.2.

1. Then $E(Y - tX)X = 0$ and $t = \frac{EXY}{EX^2}$ provided $EX^2 > 0$.
2. $\cos\theta(X,Y) = sign(EXY)\frac{||\hat{Y}||}{||Y||} = \frac{EXY}{\sqrt{EX^2EY^2}}$

Property 4.4 (Pythagorean Theorem).

$$||Y||^2 = ||\hat{Y} + (Y - \hat{Y})||^2 = ||\hat{Y}||^2 + ||Y - \hat{Y}||^2 \tag{4.14}$$

Property 4.5 (Cauchy-Schwarz inequalities). As follows.

1. $||\hat{Y}|| \leq ||Y||$ w.eq. iff $Y \overset{wP1}{=} \hat{Y}$
2. $(EXY)^2 \leq EX^2Y^2$ w.eq. iff $Y \overset{wP1}{=} tX$ for $t \in \mathbb{R}$
3. $|\cos\theta(X,Y)| \leq 1$ w.eq. iff $Y = tX$ for $t \in \mathbb{R}$

Property 4.6 (Triangle Inequality).

$$||X + Y|| \leq ||X|| + ||Y|| \quad \text{w.eq. iff } Y \overset{wP1}{=} tX \text{ for } t > 0 \tag{4.15}$$

4.3 Covariance & Correlation

Definition 4.6 (centred random variables). We define *centred* version of random variable X as

$$\dot{X} = X - EX$$

The expected value of any centred random variables is zero:

$$E\dot{X} = 0$$

Remark 4.3.

- Variance is a quadratic operation instead of a linear one
- It is the inner product of the two centred variables:

$$cov(X,Y) := E\dot{X}\dot{Y} = E(X - EX)(Y - EY) \tag{4.16}$$

Definition 4.7 (correlation coefficient). We define the *correlation coefficient* of X and Y to be the *cosine of the angle* between the centred X and Y

$$\rho(X,Y) := \cos\theta(\dot{X}, \dot{Y}) = \frac{cov(X,Y)}{\sigma(X)\sigma(Y)} \tag{4.17}$$

Property 4.7 (Cauchy-Schwarz inequalities). As follows.

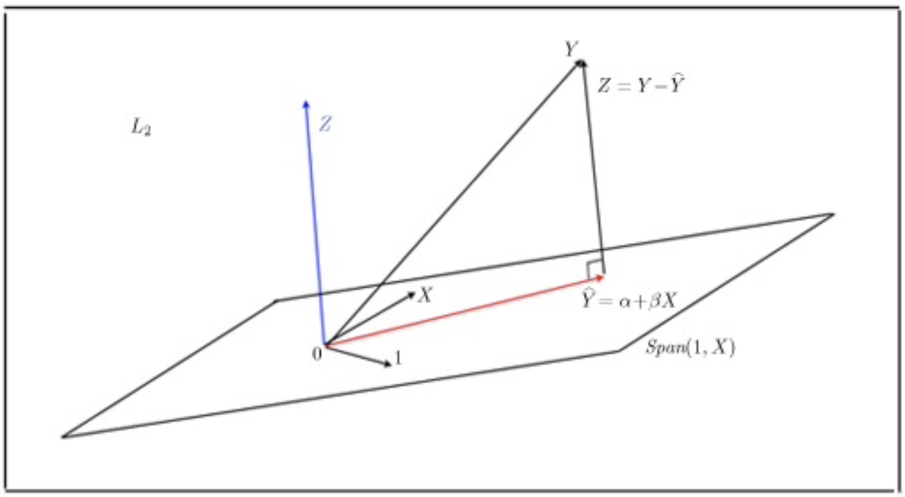
1. $|cov(X,Y)| \leq \sqrt{varXvarY}$ w.eq. iff $Y \overset{wP1}{=} \alpha + \beta X$
2. $|\rho(X,Y)| \leq 1$ w.eq. iff $Y \overset{wP1}{=} \alpha + \beta X$

4.4 Simple Linear Model

Proposition 4.1. Given any two real-valued r.v.s X and Y, in L_2 there exist unique scalars α and β and a unique r.v. Z s.t.

$$Y = \alpha + \beta X + Z \quad \text{w. } EZ = 0 = \rho(Z, X) \tag{4.18}$$

Remark 4.4. We call $Z = Y - \alpha - \beta X$ a *residual* variable.



Corollary 4.4.

$$||Y - \alpha - \beta X|| \leq ||Y - s - tX|| \quad \text{w. eq. iff } s = \alpha, t = \beta \tag{4.19}$$

Property 4.8 (Pythagorean expression)**.**

$$||Y||^2 = ||\alpha + \beta X||^2 + ||Z||^2 \tag{4.20}$$

$$\implies varY = var(\alpha + \beta X) + varZ \tag{4.21}$$

$$= \underbrace{\beta^2 varX}_{cov(X,Y) = \beta varX} \quad + varZ \tag{4.22}$$

$$= \rho(X, Y)^2 varY + varZ \tag{4.23}$$

$$\implies ||Z||^2 = varZ = (1 - \rho(X, Y)^2) varY \tag{4.24}$$

$$\implies \frac{||Y - \alpha - \beta X||}{||Y - EY||} = \frac{\sigma(Z)}{\sigma(Y)} = \sqrt{1 - \rho(X, Y)^2} \leq 1 \tag{4.25}$$

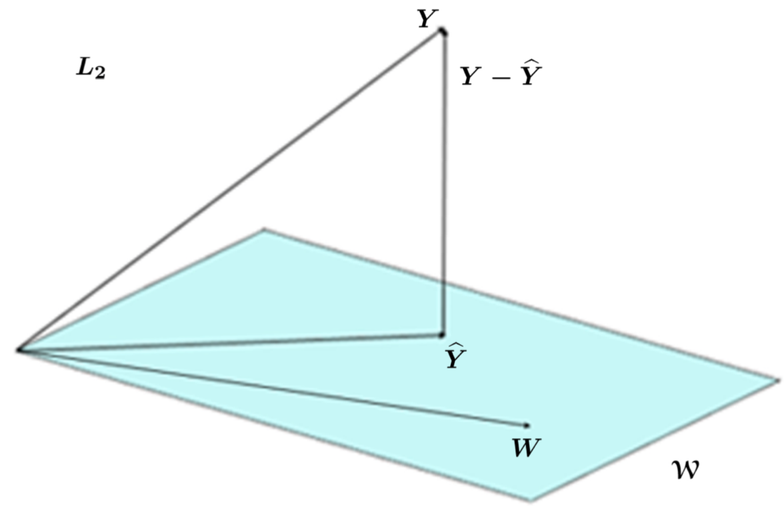
Remark 4.5. (Equation 4.25) tells us the ratio of two physical distances: from Y to $\hat{Y} = \alpha + \beta$ and from Y to its own mean value. It will be strictly less than 1 if $\rho(X, Y)$ is not trivial, which means $\hat{Y}$, a linear transformation of a random variable, can easily do better than a single real value.

4.5 General Linear Model

A L_2 *prediction space* is any closed vector subspace $\mathcal{W} \subseteq L_2$, of r.v.s that contains the constants:

$$1 \in \mathcal{W} \overset{closed}{\subseteq} L_2$$

The best predictor of Y in $\mathcal{W}$ is the unique element $\hat{Y}$ that is closest to Y.



The *orthogonal complement* of $\mathcal{W}$ is the vector subspace $\mathcal{W}^{\perp}$, of all r.v.s that are orthogonal to everything in $\mathcal{W}$ itself:

$$\mathcal{W}^{\perp} = \{V \in L_2 | EVW = 0 \quad \forall W \in \mathcal{W} \tag{4.26}$$

It is clear that $\mathcal{W} \cap \mathcal{W}^{\perp} = \{0\}$.

Definition 4.8 (orthogonal projection)**.** The *orthogonal projection* of Y on $\mathcal{W}$, denoted $\hat{Y} = op(Y|\mathcal{W})$, is the random variable $\hat{Y}$ s.t.

$$\hat{Y} \in \mathcal{W} \quad \text{and} \quad Y - \hat{Y} \in \mathcal{W}^{\perp}$$

Proposition 4.2 (orthogonal projection and minimum distance)**.**

$$\hat{Y} = op(Y|\mathcal{W}) \iff ||Y - \hat{Y}|| = \inf_{W \in \mathcal{W}} ||Y - W|| \tag{4.27}$$

Proposition 4.3 (general linear model)**.** For any $Y \in L_2$ and $1 \in \mathcal{W} \subseteq L_2$, there are unique $\hat{Y} \in \mathcal{W}$ and Z s.t.

$$Y = \hat{Y} + Z \quad w. \quad EZ = 0 = \rho(Z, W) \quad \forall W \in \mathcal{W}$$

Proposition 4.4 (maximum correlated estimator)**.** $Y = \hat{Y} + Z$ with $\hat{Y} \in \mathcal{W}, Z \in \mathcal{W}^{\perp}$ iff

$$\rho(\hat{Y}, Y) = \sup_{W \in \mathcal{W}} \rho(W, Y) \quad w. \quad EZ = 0 = \rho(Z, \hat{Y})$$