① Use potential method:

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$$\alpha(T) = \frac{T.num}{T.size}$$

$$\hat{c}_i = c_i + \phi(D_i) - \phi(D_{i-1})$$

potential function
$$\phi(T) = \begin{cases} 2 \times T.num - T.size & \text{if } \alpha(T) \geq \frac{1}{2} \\ \frac{T.size}{2} - T.num & \text{if } \alpha(T) < \frac{1}{2} \end{cases}$$

insert:

Case 1: $\alpha_{i-1} \geq \frac{1}{2}$, $\alpha_i \geq \frac{1}{2}$

Case 1.1: insert don't cause expansion: $\begin{cases} c_i = 1 \\ num_i = num_{i-1} + 1 \\ size_i = size_{i-1} \end{cases}$

$$\begin{aligned} \hat{c}_i &= c_i + \phi(D_i) - \phi(D_{i-1}) \\ &= 1 + (2num_i - size_i) - (2num_{i-1} - size_{i-1}) \\ &= 1 + \cancel{2num_{i-1}} + 2 - \cancel{size_{i-1}} - \cancel{2num_{i-1}} + \cancel{size_{i-1}} \\ &= 3 \end{aligned}$$

Case 1.2: insert cause expansion: $\begin{cases} c_i = num_i \\ num_i = num_{i-1} + 1 \\ size_{i-1} = num_{i-1} \\ size_i = 2\,size_{i-1} \end{cases}$

$$\begin{aligned} \hat{c}_i &= c_i + \phi(D_i) - \phi(D_{i-1}) \\ &= num_i + (2num_i - size_i) - (2num_{i-1} - size_{i-1}) \\ &= \cancel{num_{i-1}} + 1 + \cancel{2num_{i-1}} + 2 - \cancel{2num_{i-1}} - \cancel{2num_{i-1}} + \cancel{num_{i-1}} \\ &= 3 \end{aligned}$$

Case 2: $\alpha_{i-1} < \frac{1}{2}$, $\alpha_i < \frac{1}{2}$

$\begin{cases} c_i = 1 \\ num_i = num_{i-1} + 1 \\ size_i = size_{i-1} \end{cases}$

$$\begin{aligned} \hat{c}_i &= c_i + \phi(D_i) - \phi(D_{i-1}) \\ &= 1 + \left(\frac{size_i}{2} - num_i\right) - \left(\frac{size_{i-1}}{2} - num_{i-1}\right) \\ &= 1 + \cancel{\frac{size_{i-1}}{2}} - \cancel{num_{i-1}} - 1 - \cancel{\frac{size_{i-1}}{2}} + \cancel{num_{i-1}} = 0 \end{aligned}$$

Case 3: $\alpha_{i-1} < \frac{1}{2}$, $\alpha_i \geq \frac{1}{2}$

$\begin{cases} c_i = 1 \\ num_i = num_{i-1} + 1 \\ size_i = size_{i-1} \end{cases}$

$$\begin{aligned} \hat{c}_i &= c_i + \phi(D_i) - \phi(D_{i-1}) \\ &= 1 + (2num_i - size_i) - \left(\frac{size_{i-1}}{2} - num_{i-1}\right) \\ &= 1 + 2num_{i-1} + 2 - size_{i-1} - \frac{size_{i-1}}{2} + num_{i-1} \\ &= 3 - \frac{3}{2} size_{i-1} + 3\,num_{i-1} \\ &\overset{(num = \alpha \cdot size)}{=} 3 - \frac{3}{2} size_{i-1} + 3\alpha_{i-1} \cdot size_{i-1} < 3 \quad (\because \alpha_{i-1} < \tfrac{1}{2}) \end{aligned}$$

delete:

Case 1: $\alpha_{i-1} < \frac{1}{2}$, $\alpha_i < \frac{1}{2}$

Case 1.1: delete don't cause contraction: $\begin{cases} c_i = 1 \\ num_i = num_{i-1} - 1 \\ size_i = size_{i-1} \end{cases}$

$$1 + \left(\frac{size_i}{2} - num_i\right) - \left(\frac{size_{i-1}}{2} - num_{i-1}\right)$$
$$= 1 + \cancel{\frac{size_{i-1}}{2}} - \cancel{num_{i-1}} + 1 - \cancel{\frac{size_{i-1}}{2}} + \cancel{num_{i-1}}$$
$$= 2$$

Case 1.2: delete cause contraction: $\begin{cases} c_i = num_i + 1 \\ num_i = num_{i-1} - 1 \\ size_i = size_{i-1} \times \frac{1}{2} \\ num_{i-1} = size_{i-1} \times \frac{1}{4} \end{cases}$

$$num_i + 1 + \left(\frac{size_i}{2} - num_i\right) - \left(\frac{size_{i-1}}{2} - num_{i-1}\right)$$
$$= num_{i-1} + \frac{size_{i-1}}{4} - \cancel{num_{i-1}} + 1 - \frac{size_{i-1}}{2} + \cancel{num_{i-1}}$$
$$= 1 - \frac{size_{i-1}}{4} + num_{i-1} = 1 - num_{i-1} + num_{i-1}$$
$$= 1$$

Case 2: $\alpha_{i-1} \geq \frac{1}{2}$, $\alpha_i \geq \frac{1}{2}$

$\begin{cases} c_i = 1 \\ num_i = num_{i-1} - 1 \\ size_i = size_{i-1} \end{cases}$

$$1 + (2num_i - size_i) - (2num_{i-1} - size_{i-1})$$
$$= 1 + \cancel{2num_{i-1}} - 2 - \cancel{size_{i-1}} - \cancel{2num_{i-1}} + \cancel{size_{i-1}} = -1$$

Case 3: $\alpha_{i-1} \geq \frac{1}{2}$, $\alpha_i < \frac{1}{2}$

$\begin{cases} c_i = 1 \\ num_i = num_{i-1} - 1 \\ size_i = size_{i-1} \end{cases}$

$$1 + \left(\frac{size_i}{2} - num_i\right) - (2num_{i-1} - size_{i-1})$$
$$= 1 + \frac{size_{i-1}}{2} - num_{i-1} + 1 - 2num_{i-1} + size_{i-1}$$
$$= 2 - 3num_{i-1} + \frac{3}{2} size_{i-1}$$
$$\overset{(num = \alpha \cdot size)}{=} 2 - 3\alpha_{i-1} \cdot size_{i-1} + \frac{3}{2} size_{i-1}$$
$$\underset{(\alpha_{i-1} \geq \frac{1}{2})}{\leq} 2$$

All cases above are constant, so dynamic table (single) insertion and deletion are constant.

②

$$A_k(j) = \begin{cases} j+1 & , k=0 \\ A_{k-1}^{(j+1)}(j) & , k=1 \end{cases}$$

$$\begin{cases} A_{k-1}^{(0)}(j) = j \\ A_{k-1}^{(i)}(j) = A_{k-1}(A_{k-1}^{(i-1)}(j)) & \text{for } i \geq 1 \end{cases}$$

if $k=0$, $j=0$ $\quad A_0(0) = 1$

$k=0$, $j=1$ $\quad A_0(1) = 2$

$k=0$, $j=2$ $\quad A_0(2) = 3$

$\vdots$

$\Rightarrow A_0(j) = j+1$

$A_1(j) = 2j+1$

$k=1$, $j=0$ $\quad A_1(0) = A_0^1(0) = A_0(A_0^0(0)) = A_0(0) = 1$

$k=1$, $j=1$ $\quad A_1(1) = A_0^2(1) = A_0(A_0(A_0^0(1))) = A_0(A_0(1)) = A_0(2) = 3$

$k=1$, $j=2$ $\quad A_1(2) = A_0^3(2) = A_0(A_0(A_0(A_0^0(2)))) = A_0(A_0(A_0(2))) = A_0(A_0(3)) = A_0(4) = 5$

$\vdots$

$k=2$, $j=0$ $\quad A_2(0) = A_1^1(0) = A_1(A_1^0(0)) = A_1(0) = 1$

$k=2$, $j=1$ $\quad A_2(1) = A_1^2(1) = A_1(A_1(A_1^0(1))) = A_1(A_1(1)) = A_1(3) = 7$

$k=2$, $j=2$ $\quad A_2(2) = A_1^3(2) = A_1(A_1(A_1(A_1^0(2)))) = A_1(A_1(A_1(2))) = A_1(A_1(5)) = A_1(11) = 23$

$\vdots$

$\Rightarrow A_2(j) = 2^{(j+1)} \cdot (j+1) - 1$

if $k=3,\ j=1$ $\quad A_3(1) = A_2^{2}(1) = A_2(A_2(A_2^{0}(1))) = A_2(A_2(1))$

$= A_2(7) = 2^{8}\cdot 8 - 1 = 2047$

$k=4,\ j=1$ $\quad A_4(1) = A_3^{2}(1) = A_3(A_3(A_3^{0}(1))) = A_3(A_3(1))$

$= A_3(2047)$

$= A_2^{2048}(2047) \gg A_2(2047)$

$= 2^{2048}\cdot 2048 - 1$

$> 2^{2048}$

$= (2^{4})^{512}$

$= 16^{512} \gg 10^{80}$

$$\boxed{\begin{aligned} A_1(1) &= 3 \\ A_2(1) &= 7 \\ A_3(1) &= 2047 \\ A_4(1) &\gg 10^{80} \end{aligned}}$$