

**Solution:**

(a) Let  $x$  and  $y$  be arbitrary strings.

Assume for any string  $w$  where  $|w| < |x|$  that  $\text{digsum}(w \bullet y) = \text{digsum}(w) + \text{digsum}(y)$ .

There are two cases to consider:

- If  $x = \epsilon$ , then

$$\begin{aligned}
 \text{digsum}(x \bullet y) &= \text{digsum}(\epsilon \bullet y) && \text{because } x = \epsilon \\
 &= \text{digsum}(y) && \text{by Lemma 1} \\
 &= 0 + \text{digsum}(y) \\
 &= \text{digsum}(\epsilon) + \text{digsum}(y) && \text{because } \text{digsum}(\epsilon) = 0 \\
 &= \text{digsum}(x) + \text{digsum}(y) && \text{because } x = \epsilon
 \end{aligned}$$

- Otherwise,  $x = aw$  for some symbol  $a \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  and some string  $w$ .

$$\begin{aligned}
 \text{digsum}(x \bullet y) &= \text{digsum}(aw \bullet y) && \text{because } x = aw \\
 &= \text{digsum}(a \cdot (w \bullet y)) && \text{by definition of } \bullet \\
 &= a + \text{digsum}(w \bullet y) && \text{by definition } \text{digsum}(ax) = a + \text{digsum}(x) \\
 &= a + \text{digsum}(w) + \text{digsum}(y) && \text{by the induction hypothesis} \\
 &= \text{digsum}(aw) + \text{digsum}(y) && \text{by definition } \text{digsum}(ax) = a + \text{digsum}(x) \\
 &= \text{digsum}(x) + \text{digsum}(y) && \text{because } x = aw
 \end{aligned}$$

Also, since  $\text{digsum}(y \bullet x) = \text{digsum}(y) + \text{digsum}(x) = \text{digsum}(x) + \text{digsum}(y)$ , we conclude that,

$$\text{digsum}(x \bullet y) = \text{digsum}(y \bullet x) = \text{digsum}(x) + \text{digsum}(y).$$

(b) Let  $x$  be an arbitrary string.

Assume for any string  $w$  where  $|w| < |x|$  that  $\text{digsum}(w^R) = \text{digsum}(w)$ .

There are two cases to consider:

- If  $x = \epsilon$ , then

$$\begin{aligned}
 \text{digsum}(x^R) &= \text{digsum}(\epsilon) && \text{because } x = \epsilon \text{ and } \epsilon^R = \epsilon \\
 &= \text{digsum}(x) && \text{because } x = \epsilon
 \end{aligned}$$

- Otherwise,  $x = aw$  for some symbol  $a \in \{1, 2, 3, 4, 5, 6, 7, 8, 9\}$  and some string  $w$ .

$$\begin{aligned}
 \text{digsum}(x^R) &= \text{digsum}((aw)^R) && \text{because } x = aw \\
 &= \text{digsum}(w^R a) && \text{by definition of } {}^R \\
 &= \text{digsum}(w^R \bullet a) && \text{because } a \text{ can be seen as a string of just one symbol} \\
 &= \text{digsum}(w^R) + \text{digsum}(a) && \text{by conclusion of (a)} \\
 &= \text{digsum}(w) + \text{digsum}(a) && \text{by the induction hypothesis} \\
 &= \text{digsum}(a) + \text{digsum}(w) \\
 &= \text{digsum}(a \bullet w) && \text{by conclusion of (a)} \\
 &= \text{digsum}(x) && \text{because } x = aw
 \end{aligned}$$

In both cases, we conclude that  $\text{digsum}(x^R) = \text{digsum}(x)$ . ■

**Solution:**

(a) By definition ①  $a \in L_{odd}$  for  $a \in \{1, 3, 5, 7, 9\}$ , the string 374 obviously fails to match since it contains 3 symbols.

By definition ②  $ax \in L_{odd}$  for  $a \in \{0, 2, 4, 6, 8\}$  and  $x \in L_{odd}$ , the string 374 obviously fails to match since its first symbol is 3.

By definition ③  $axb \in L_{odd}$  for  $a, b \in \{1, 3, 5, 7, 9\}$  and  $x \in L_{odd}$ , the string 374 can be written as  $3 \cdot 7 \cdot 4$ ,  $3 \in \{1, 3, 5, 7, 9\}$  but  $4 \notin \{1, 3, 5, 7, 9\}$ . 374 fails to match definition ③, therefore,  $374 \notin L_{odd}$ .

(b)  $x$  is an arbitrary string that  $x \in L_{odd}$ .

Assume for any string  $w$  where  $|w| < |x|$  that  $w \in L_{odd}$  and  $\text{digsum}(w)$  is odd.

There are 3 cases to consider:

- If  $x$  is in the form of " $a$ ", by definition ①  $a \in L_{odd}$  for  $a \in \{1, 3, 5, 7, 9\}$ ,  $x$  is either 1 or 3 or 5 or 7 or 9,  $\text{digsum}(x)$  is also either 1 or 3 or 5 or 7 or 9, so  $\text{digsum}(x)$  is odd.
- If  $x$  is in the form of " $aw$ ", by definition ②  $aw \in L_{odd}$  for  $a \in \{0, 2, 4, 6, 8\}$  and  $w \in L_{odd}$ , we have:

$$\begin{aligned} \text{digsum}(x) &= \text{digsum}(aw) && \text{because } x = aw \\ &= a + \text{digsum}(w) && \text{by definition } \text{digsum}(ax) = a + \text{digsum}(x) \\ &\quad a \text{ is even} && a \in \{0, 2, 4, 6, 8\} \\ &\quad \text{digsum}(w) \text{ is odd} && \text{by the induction hypothesis} \\ &\quad a + \text{digsum}(w) \text{ is odd} \end{aligned}$$

- If  $x$  is in the form of " $awb$ ", by definition ③  $awb \in L_{odd}$  for  $a, b \in \{1, 3, 5, 7, 9\}$  and  $w \in L_{odd}$ , we have:

$$\begin{aligned} \text{digsum}(x) &= \text{digsum}(awb) && \text{because } x = awb \\ &= \text{digsum}(a \bullet (wb)) && \text{by definition of } \bullet \\ &= a + \text{digsum}(wb) && \text{by definition } \text{digsum}(ax) = a + \text{digsum}(x) \\ &= a + \text{digsum}(w \bullet b) && \text{by definition of } \bullet \\ &= a + b + \text{digsum}(w) && \text{by definition } \text{digsum}(ax) = a + \text{digsum}(x) \\ &\quad a, b \text{ are odd} && a, b \in \{1, 3, 5, 7, 9\} \\ &\quad \text{digsum}(w) \text{ is odd} && \text{by the induction hypothesis} \\ &\quad a + b + \text{digsum}(w) \text{ is odd} \end{aligned}$$

In conclusion, for any  $x \in L_{odd}$ ,  $\text{digsum}(x)$  is odd. ■

**Solution:** Assume  $w$  is an arbitrary string over  $\{1, 0\}$ ,  $L_{bad}$  is defined as follows:

- $w \in L_{bad}$  for  $|w| < 3$
- for  $|w| \geq 3$  and  $w = ax$ ,  $w \in L_{bad}$  for  $a \in \{1, 0\}$  if  $x \in L_{bad}$  and  $ax_1x_2$  is not 000 or 111

The idea is the same as Problem(2). First we define base case, that if the string is only consisted of less than 3 digits, then it belongs to  $L_{bad}$  since it can not contain 000 or 111. Second, if the string has 3 or more digits, we can express the string as the concatenation of its leading digit and the remaining string. The remaining string is a substring of the string to be examined, if it not belong to  $L_{bad}$ , then the superstring must have 000 or 111 and not belong to  $L_{bad}$  either. If it belongs to  $L_{bad}$ , we guarantee that from  $x_3$  to the last digit  $x_n$ , there is no 000 or 111, but we must take a step further to examine whether  $ax_1x_2$  is 000 or 111, if it is not 000 or 111, then the superstring belongs to  $L_{odd}$ . ■