CS/ECE 374 B Fall 2019

Homework 3 Problem 1

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Solution:

Denote the language in each case as L.

(a) The language is regular.

We can denote this language with a regular expression as:

$$L = \sum_{n=0}^{9}(0+1+\ldots+9+\#)^* n\#^n n(0+1+\ldots+9+\#)^*$$

Where $n \in \{0, \ldots, 9\}$.

- Let x be an arbitrary string that $x \in L$, then x must contain the substring $c\#^c c$, where $c \in \{0, \ldots, 9\}$.
- For any string $y = a \cdot c\#^c c \cdot b$ that contains the substring $c\#^c c$, where $c \in \{0, \ldots, 9\}$, the prefix a and suffix b of y could be any string over $\Sigma = \{0, \ldots, 9, \#\}$, and $(0+1+\ldots+9+\#)^*$denotes the set of all strings over Σ, so $a, b \in (0+1+\ldots+9+\#)^*$. Therefore, $y \in L$.

In conclusion, every string in L contains the substring $c\#^c c$ and any string that has the substring $c\#^c c$ is in L, so we prove that L can be expressed as the regular expression $\sum_{n=0}^{9}(0+1+\ldots+9+\#)^* n\#^n n(0+1+\ldots+9+\#)^*$. Therefore, L is regular.

(b) The language is not regular.

Let $F = \{< n+1 > \#^1 | n \in \mathbb{N}\}$.

Let x and y be arbitrary elements of F.

Then $x =< i+1 > \#^1$, $y =< j+1 > \#^1$, for some natural numbers $i \neq j$.

Let $z = \#^j$.

Then $xz =< i+1 > \#^{j+1} \notin L$, since $i+1 \neq j+1$.

And $yz =< i+1 > \#^{j+1} \in L$.

Thus, F is a fooling set of L.

Because F is infinite, L can not be regular.

(c) The language is regular.

We can denote this language with a regular expression as:

$$L = \sum_{n=1}^{15600}(a+b+\ldots+z)^* \cdot f(n) \cdot (a+b+\ldots+z)^* \cdot f(n) \cdot (a+b+\ldots+z)^*$$

Where $n \in \{0, 1, \ldots, 15600\}$, and $f(n)$ is a bijective function that maps n to its corresponding 3-character string:

$$f(1) = aaa, f(2) = aab, \ldots, f(15600) = zzz$$

There are P_3^{26} = 15600 permutations of a 3-character string, so $n \in \{0, 1, \ldots, 15600\}$, and

$f(n) \in \{aaa, aab, ...zzz\}$.

Similar to (a),

- Let X be an arbitrary string that $X \in L$, then X must contain the substring $Z = f(n) \in \{aaa, aab, ...zzz\}$, where $n \in \{1, 2, ..., 15600\}$, that at least appears twice.
- For any string $Y = A \cdot Z \cdot B \cdot Z \cdot C$ that contains the substring $Z = f(n) \in \{aaa, aab, ...zzz\}$, where $n \in \{1, 2, ..., 15600\}$, the prefix A, suffix C, and substring B of Y could be any string over $\Sigma = \{a, ..., z\}$, and $(a + b + ... + z)^*$denotes the set of all strings over Σ, so $A, B, C \in (a + b + ... + z)^*$. Therefore, $Y \in L$.

In conclusion, every string in L contains the substring $Z = f(n) \in \{aaa, aab, ...zzz\}$, where $n \in \{1, 2, ..., 15600\}$, that appears at least twice, and any string that has the substring Z appears at least twice is in L, so we prove that L can be expressed as the regular expression $\sum_{n=1}^{15600}(a+b+...+z)^* \cdot f(n) \cdot (a+b+...+z)^* \cdot f(n) \cdot (a+b+...+z)^*$. Therefore, L is regular. ■

Solution:

Function $f : \Sigma_1 \to \Sigma_2^*$ maps an arbitrary symbol a in Σ_1 to a string w in Σ_2^*, such that,

$$f(a) = w,\ f^{-1}(w) = a, \text{ where } a \in \Sigma_1, w \in \Sigma_2^*$$

$$f(\epsilon) = \epsilon, \text{ and } f(ax) = f(a) \cdot f(x) = w \cdot f(x), \text{ where } a \in \Sigma_1, x \in \Sigma_1^*$$

Suppose the given DFA M which accepts L can be expressed as $M = (\Sigma_1, Q_1, \delta_1, s_1, A_1)$, we can construct the corresponding NFA N that accepts $f(L)$ as $N = (\Sigma_2, Q_2, \delta_2, s_2, A_2)$, where,

Σ_2 contains symbols in $f(a), a \in \Sigma_1$

$Q_2 = (Q_1 \times a \times p) \cup \{s_2\}, a \in \Sigma_1$ and p is a prefix of $f(a)$

$A_2 = \{(s, a, w) | s \in A_1\}$

s_2 is an explicit state in Q_2

$\delta_2(s_2, \epsilon) = \{(s_1, a, \epsilon) | a \in \Sigma_1\} \in A_2$

$$\delta_2((s, a, w), m) = \begin{cases} (s, a, w \cdot m) & if\, w \cdot m \notin \{x | x = f(b), b \in \Sigma_1\} \\ \{(\delta_1(s, b), n, \epsilon) | n \in \Sigma_1\} & if\, w \cdot m \in \{x | x = f(b), b \in \Sigma_1\} \end{cases}$$

Furthermore,

- for every symbol $a \in \Sigma_1$, there exists a string $w \in \Sigma_2^*$, which is a's homomorphism in Σ_2^*, following $f : \Sigma_1 \to \Sigma_2^*$ that $f(a) = w$.
- for every string x over Σ_1 such that $x \in L$ that is accepted by M, its corresponding homomorphism $f(x) \in f(L)$ is also accepted by N: $x \in L, \delta_1^*(s_1, x) \in A_1 \implies f(x) \in f(L), \delta_2^*(s_2, f(x)) \in A_2$.

Suppose w is an arbitrary string over Σ_1, for every string x shorter than w over $\Sigma_1, f(x)$ is accepted by N,

$$x \in L, \delta_1^*(s_1, x) \in A_1 \implies f(x) \in f(L), \delta_2^*(s_2, f(x)) \in A_2$$

There are two cases to consider,

- Suppose $w = \epsilon$, by definition $f(w) = f(\epsilon) = \epsilon$, therefore,

$$\delta_2^*(s_2, f(w)) = \delta_2^*(s_2, f(\epsilon))$$

$$= \delta_2^*(s_2, \epsilon)$$

$$= \{(s_1, a, \epsilon) | a \in \Sigma_1\} \in A_2$$

So N accepts $f(\epsilon)$.

- Suppose $w = x \cdot a$ for some symbol $a \in \Sigma_1$ and some string $x \in \Sigma_1^*$. Assume $f(x) = y$ and $f(a) = z$. Let,

$$\delta_2^*(s_2, f(x)) = \{(s_n, b, \epsilon) | b \in \Sigma_1\}$$

$$\delta_2^*(s_2, f(w)) = \delta_2^*(s_2, f(x \cdot a))$$

$$= \delta_2^*(s_2, f(x) \cdot z)$$

$$= \{\delta_2^*((s_n, b, \epsilon), z) | b \in \Sigma_1\}$$

Suppose z can be written as $z[1]z[2]...z[n]$, or $z[1:n]$, where $z[1], z[2], z[n]$ are symbols in z, we can write,

$$\delta_2^*(s_2, f(w)) = \{\delta_2^*((s_n, b, \epsilon), z) | b \in \Sigma_1\}$$

$$= \{\delta_2^*((s_n, b, z[1]), z[2:n]) | b \in \Sigma_1\}$$

$$= \{\delta_2^*((s_n, b, z[1:2]), z[3:n]) | b \in \Sigma_1\}$$

$$\ldots$$

$$= \{\delta_2^*((s_n, b, z[1:n-1]), z[n]) | b \in \Sigma_1\}$$

by definition and assumption of δ_2, δ_2^*, we can write,

$$= \{(\delta_1(s_n, a), d, \epsilon) | a, d \in \Sigma_1, f(a) = z\}$$

$$= \{((\delta_1(\delta_1^*(s_1, x), a), d, \epsilon) | a, d \in \Sigma_1, f(a) = z\}$$

$$= \{(\delta_1^*(s_1, xa), d, \epsilon) | a, d \in \Sigma_1, f(a) = z\}$$

Since $\delta_1^*(s_1, xa) \in A_1 \implies (\delta_1^*(s_1, xa), d, \epsilon) \in A_2$, $w = xa$, therefore, $\delta_2^*(s_2, f(w)) = \{(\delta_1^*(s_1, xa), d, \epsilon) | a, d \in \Sigma_1, f(a) = z\} \in A_2$. We conclude that N accepts $f(w)$ if and only if M accepts w.

In coclusion, N is the NFA we construct that accepts $f(L)$. ■