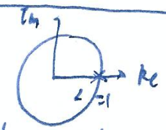
Math SE Calc. Prob.

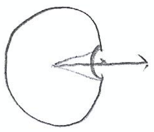
$$\int_{|z|=1} \frac{z^2+3z+2i}{(z+4)(z-1)}\,dz$$



$z=4$

pole on circle !

(src: pole on a Contour. Problem with Integration)



$$\oint \frac{z^2+3z+2i}{(z+4)(z-1)}\,dz = 0$$

$$\int_{\epsilon}^{2\pi-\epsilon} \frac{z^2+3z+2i}{(z+4)(z-1)}\,dz$$

$$+\int_{-\epsilon}^{\epsilon} \frac{z^2+3z+2i}{(z+4)(z-1)}\,dz$$

clockwise, so $\boxed{\text{neg.}}$

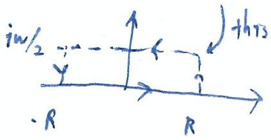
$$\text{Res}\,(z=1) = \frac{1+3+2i}{5} = \frac{4+2i}{5}$$

$$\therefore \int_{|z|=1} \frac{z^2+3z+2i}{(z+4)(z-1)}\,dz = \frac{4+2i}{5}\pi i$$

$$\int_{-\infty}^{\infty} e^{-t^2} e^{-itw}\,dt$$

(src: can we use contour integration to compute the Fourier transform of Gaussian?)

idea: $\left(t+\frac{iw}{2}\right)^2 = t^2 - \frac{w^2}{4} + itw$



this is along the $y = \frac{iw}{2}$ line

$R \to \infty$

$$\oint e^{-z^2}\,dz = 0 \qquad ; z = x+y$$

along sides: $\uparrow$ $\displaystyle\int_{R}^{R+iw/2} e^{-(x^2+y^2+2xy)}\,dz$

(x const., y changes)

$$\therefore e^{-R^2}\int_0^{iw/2} e^{-\frac{w^2}{4}-iRw}\,dw$$

$\to 0$

similarly for $\downarrow$

$$\leftarrow \quad -\int_{-R+iw/2}^{R+iw/2} e^{-(x^2+y^2+2xy)}\,dz$$

x changes, y const.

$$\to -e^{+w^2/4}\int_{-\infty}^{\infty} e^{-x^2-iwx}\,dx$$

$$\to \int_{-R}^{R} e^{-(x^2+y^2+2xy)}\,dz$$

$$\to \int_{-\infty}^{\infty} e^{-x^2}\,dx$$

$$\therefore \int_{-\infty}^{\infty} e^{-x^2-iwx}\,dx = \sqrt{\pi}\,e^{-w^2/4}$$

$e^{\frac{1}{z}} = ?$

idea: e^z Taylor series is valid everywhere

$\frac{1}{z}$ is ... except at zero

so $e^{\frac{1}{z}}$ is valid except at zero

plug $\frac{1}{z}$ into e^z

$$e^{\frac{1}{z}} = \sum \frac{(1/z)^n}{n!} \quad \#$$

$\frac{1}{\sin z}$? Laurent series ! (src: Calculate Laurent series for $1/\sin(z)$)

$$\# \quad \left(\sum_{k=0}^{\infty} a_k z^k\right)\left(\sum_{k=0}^{\infty} b_k z^k\right) = \sum_{k=0}^{\infty} c_k z^k$$

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}$$

$$\left(\frac{1}{\sin z}\right)(\sin z) = 1 \quad ?$$

But $\frac{1}{\sin z}$ cannot be a Taylor series ! (only powers ≥ 0)

so we use

$$\frac{z}{\sin z}\,\frac{1}{z} = \frac{1}{\sin z}$$

no singularity

and

$$\left(\frac{z}{\sin z}\right)\left(\frac{\sin z}{z}\right) = 1$$

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots = \sum_{\text{odd } n} \frac{z^n}{n!}(-1)^{n-1}$$

$$\frac{\sin z}{z} = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \dots$$

$$= \sum_{n \text{ odd}} \frac{z^{n-1}}{n!}(-1)^{n-1} \quad (n \geq 1)$$

$$\left(\frac{\sin z}{z}\right)\left(\frac{z}{\sin z}\right) = 1$$

$$c_k = \sum_{j=0}^{k} a_j b_{k-j}$$

$$b_k = \frac{1}{a_0}\left\{c_k - \sum_{j=1}^{k} a_j b_{k-j}\right\}$$

$$c_k = \begin{cases} 1 & k = 0 \\ 0 & \text{otherwise} \end{cases}$$

$$a_j = \frac{(-1)^{j-1}}{j!} \quad (j \text{ odd})$$

j/k	0	1	2	3	4	5	6
c_k	1	0	0	0	0	0	0
a_k	1	0	$-1/3!$	0	$1/5!$	0	$-1/7!$
b_k	1	0	$+1/3!$	0	$+7/360$	0	$+\frac{31}{15120}$

(b_0) $c_0 = a_0 b_0$

(b_1) $c_1 = 0 = a_0 b_1 + a_1 b_0$ (indices add up to k)

(b_2) $c_2 = a_0 b_2 + a_1 b_1 + a_2 b_2$

p.s. try pointing your fingers

6 5 4 3 2 1
1 2 3 4 5 6

$b_k z^{k-1}$

$$\therefore \frac{1}{\sin z} = \frac{1}{z} + \frac{1}{3!} z + \frac{7}{360} z^3 + \dots$$

16/1

Laurent Series Expansion w/o Geom. Series

$\frac{1}{z^6+1}$ let $\alpha^6+1=0$, $\alpha^6=-1$

$z = \alpha + w$?

$= \alpha + \alpha w$?

$z^6 = \alpha^6(1+w)^6$

$z^6+1 = \alpha^6[(1+w)^6-1] = 1-(1+w)^6$

$\therefore \frac{1}{z^6+1} = \frac{1}{1-(1+w)^6}$

singular at $w=0$

$\frac{1}{z^6-\alpha^6}$ is there something we can transform onto?...

idea: we want to expand in terms of $z-\alpha$.

$$\frac{1}{z^6+1} = \frac{1}{(\underbrace{z-\alpha+\alpha}_{?\ w \text{ or } \alpha w?})^6+1}$$

$$= \frac{1}{1-(1+w)^6}$$

1
1 1
1 2 1 $(\ )^2$
1 3 3 1
1 4 6 4 1 $(\ )^4$
1 5 10 10 5 1
1 6 15 20 15 6 1 $(\ )^6$

$$= -\frac{1}{6w+15w^2+20w^3+15w^4+6w^5+w^6}$$

$$= -\frac{1}{w}\,\frac{1}{6+15w+20w^2+15w^3+6w^4+w^5}$$

$$= \sum_{k=-1}^{\infty} b_k w^k$$

how low must k go?
check: as $w \to 0$, what $w^n \times f(w)$ would give a const.? $n=1$!

Math stack exchange Prob.

$-1 = (6+15w+20w^2+15w^3+6w^4+w^5)\sum_{k=-1}^{\infty} b_k w^{k+1}$

$\therefore k=-1$, $6b_{-1} = -1$, $b_{-1} = 6$

↑ lowest w_0

w_0	w_1	w_2	w_3	w_4	w_5
$6b_{-1}$	$6b_0$	$6b_1$	$6b_2$	$6b_3$	$6b_4$
	$15b_{-1}$	$15b_0$	$15b_1$	$15b_2$	$15b_3$
		$20b_{-1}$	$20b_0$	$20b_1$	$20b_2$
			$15b_{-1}$	$15b_0$	$15b_1$
				$6b_{-1}$	$6b_0$
					b_{-1}
-1	0	0	0	0	0

$b_{-1} = -\frac{1}{6}$ $6b_0 + 15b_{-1} = 0$

$36b_0 - 15 = 0$

$b_0 = \frac{15}{36}$

$b_0 = \frac{5}{12}$

$6b_1 + 16b_0 + 20b_{-1} = 0$

$6b_1 + \frac{5}{2}(5) - \frac{10}{3} = 0$

$b_1 = -\frac{5}{9}$

$6b_2 = -(\frac{55}{18}$ $b_2 = \frac{55}{108}$

$b_3 = -\frac{8}{27}$

for $n \geq 4$,

$\cancel{6}b_n = -(15b_{n-1}+20b_{n-2}+15b_{n-3}+6b_{n-4}+b_{n-5})\frac{1}{6}$

$b_n = \frac{5}{2}b_{n-1}+\frac{10}{3}b_{n-2}+\frac{5}{2}b_{n-3}+b_{n-4}+\frac{1}{6}b_{n-5}$

$= -\frac{1}{6}\frac{1}{w}+\frac{5}{12}-\frac{5}{9}w+\frac{55}{108}w^2+\frac{8}{27}w^3+\dots$

Physics Forums:

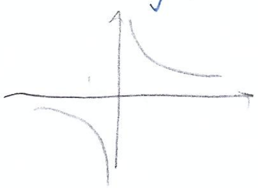
Laurent Series of Sqrt(z)

→ branch pt., so cannot!

Why $z^{\wedge(-1/2)}$ cannot be expanded in Laurent series with center $z=0$?

→ non isolated singularities?

understanding Cauchy Principle value:



consider $\int_{-1}^{1} \frac{1}{x}\, dx$

$$\int_{-1}^{1} \frac{1}{x} dx \overset{?}{=} \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{1}{x} + \int_{\varepsilon}^{1} \frac{1}{x} dx$$

$= 0$ as same area, opp. sign

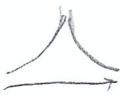
$$\int_{-1}^{1} \frac{1}{x} dx \overset{?}{=} \lim_{\varepsilon \to 0} \int_{-1}^{-\varepsilon} \frac{1}{x} + \int_{2\varepsilon}^{1} \frac{1}{x} dx$$

$$= \ln|x| \Big|_{-1}^{-\varepsilon} + \ln|x| \Big|_{2\varepsilon}^{1}$$

$$= (\ln \varepsilon - \ln 1) + \ln 1 - \ln 2\varepsilon$$

$$= \ln \frac{1}{2} \; !$$

$$\int_{-1}^{1} \frac{dx}{x^2} = \left[-x\right]_{-1}^{1} = -2 \; ?$$



consider: $\int_{-1}^{1} \frac{1}{(x-i\epsilon)^2} dx$

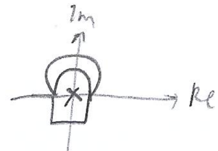
$$= \frac{-1}{x-i\epsilon} \Big|_{-1}^{1} = -\frac{1}{1-i\epsilon} - \frac{1}{1+i\epsilon}$$

$$= \frac{-2}{1+\epsilon^2}$$

$$\lim_{\epsilon \to 0} \int_{-1}^{1} \frac{1}{(x-i\epsilon)^2} dx \neq \int_{-1}^{1} \lim_{\epsilon \to 0} \frac{1}{(x-i\epsilon)^2} dx \; !$$

$$-2 \neq \infty$$

in fact,



any complex path around the pole gives -2!

"path-independence", except through origin

show that:

- - - - -

$$\int_{-1}^{-\epsilon} \frac{dx}{x^2} + \int_{\epsilon}^{1} \frac{dx}{x^2} = -\frac{1}{x}\Big|_{-1}^{-\epsilon} + \left(-\frac{1}{x}\right)\Big|_{+\epsilon}^{1}$$

$$= -2 + \frac{2}{\epsilon}$$

$$\int_{\pi}^{0} \frac{i\,d\theta}{\rho e^{i\theta}} = \frac{i}{-i} \rho^{-1} e^{-i\theta} \Big|_{\pi}^{0}$$

$$= -\frac{1}{\epsilon}\left[1-(-1)\right]$$

$$= -\frac{2}{\epsilon}$$

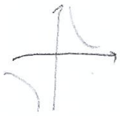
so divergence principal and circle cancel out!

(see: How to "fix" $\int_{-1}^{1} \frac{dx}{x^2}$ with complex numbers)

space is dark green

PSE: what makes the Cauchy Principal value the "correct" value for a integral

Consider the $\frac{1}{x}$ potential



$V = \frac{1}{x}$; $V(1) - V(-1) = 2$

$\nabla V = F = -\frac{1}{x^2}$

$W = \int -F dx = \int_{-1}^{1} \frac{1}{x^2} dx = ?$ — PV ∞ / contour integral / naive evaluation, 2 !

PSE: Integration along real axis with singularities

(Green's functions

understanding Kramers Kronig Relation :

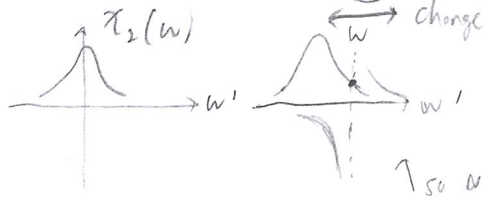
① $\chi(\omega) = \frac{1}{i\pi} P \int_{-\infty}^{\infty} \frac{\chi(\omega')}{\omega'-\omega} d\omega'$

- $\chi = \chi_1(\omega) + i\chi_2(\omega)$
- $\chi \to 0$ when $|\omega| \to \infty$

② $\chi_1(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi_2(\omega')}{\omega'-\omega} d\omega'$

$\chi_2(\omega) = -\frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{\chi_1(\omega')}{\omega'-\omega} d\omega'$

① what's $\frac{\chi_2(\omega')}{\omega'-\omega}$??



↑ so a areas cancel

ex: $\chi_2(\omega) = \frac{1}{1+\omega^2}$

$\chi_1(\omega) = \frac{1}{\pi} P \int_{-\infty}^{\infty} \frac{1}{1+\omega'^2} \frac{1}{\omega'-\omega} d\omega'$

$\pi\chi_1(\omega) = \int_{-\infty}^{\omega-\epsilon} \frac{1}{(1+\omega'^2)} \frac{1}{\omega'-\omega} d\omega'$

$+ \int_{\omega+\epsilon}^{+\infty} \frac{1}{1+\omega'^2} \frac{1}{\omega'-\omega} d\omega'$

→ Sokhotski-Plemelj theorem

$\lim_{\varepsilon \to 0^+} \frac{1}{x \pm i\varepsilon} = \mp i\pi\delta(x) + P\left(\frac{1}{x}\right)$

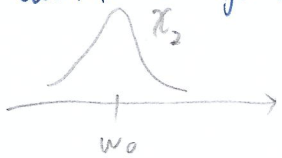
$\updownarrow$

$\lim_{\varepsilon \to 0^+} \int_a^b \frac{f(x)}{x \pm i\varepsilon} dx = \mp i\pi f(0) + P\int \frac{f(x)}{x} dx$

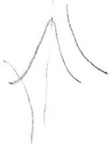
pole at origin! $a < 0 < b$

→ see Numerical Evaluation of truncated Kramers Kronig transforms, (Exact Evaluation)

Qualitatively, sign : χ_1 from χ_2



$\Rightarrow \chi_1(\omega < \omega_0) > 0$



$\Rightarrow \chi_1(\omega_0) = 0$

$\Rightarrow \chi_1(\omega > \omega_0) < 0$

Problems from Web

Contour Integration, from YT

Type 1 : trigonometric

Feynman trick $\int_0^{2\pi} \frac{d\theta}{a+b\cos\theta}$, $\int_0^{2\pi} \frac{d\theta}{a+b\sin\theta}$

$\int_0^{2\pi} \frac{d\theta}{(a+b\cos\theta)^2}$, $\int_0^{\pi} \frac{d\theta}{(a+b\cos\theta)^2}$

Type 2 : Algebraic func., Improper Integrals

$\int_{-\infty}^{\infty} \frac{dx}{(x^2+1)^2}$ $\int_{-\infty}^{\infty} \frac{x^2}{(x^2+a^2)^3}$

$\int_{-\infty}^{\infty} \frac{x^4}{(a+bx^2)^4} dx$ (consider $\int_{-\infty}^{\infty} \frac{1}{a+bx^2} dx$ then Feynman)

$\text{Res}(f, \overset{a+i}{\downarrow}{i}) =$; poles of order "3"

Type 3 : trigo + algebra

$\int_0^{\infty} \frac{\cos mx}{a^2+x^2} dx$ $\int_0^{\infty} \frac{x\sin mx}{a^2+x^2} dx$

$\int_{-\infty}^{\infty} \frac{a\cos x + x\sin x}{x^2+a^2} dx$

$\int_{-\infty}^{\infty} \frac{\cos x}{(x^2+a^2)(x^2+b^2)} dx$

- even & odd, extend to $\int_{-\infty}^{\infty}$
- poles on $\mathbb{R}$
- take real part to isolate corresponding trigo func.
  Im

Type 4 :

$\int_0^{\infty} \frac{\sin x}{x} = \frac{\pi}{2}$ $\int_0^{\infty} \frac{1-\cos x}{x^2} = \frac{\pi}{2}$

$\int_0^{\infty} \frac{\cos ax - \cos bx}{x^2} dx = \frac{\pi}{2}(b-a)$

$\int_{-\infty}^{\infty} \frac{((x^2+a^2)\sin mx)}{x(x^2+b^2)} dx$

Type 5: Multi valued

$\int_0^{\infty} \frac{x^{\alpha-1}}{1+x} = \frac{\pi}{\sin\alpha\pi}$

$\int_0^{\infty} \frac{x^{\alpha-1}}{1-x} dx = -\pi\cot\alpha\pi$