NANYANG TECHNOLOGICAL UNIVERSITY SINGAPORE

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Bell's Theorem and Quantum Variational Methods

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INTRODUCTION

In 1935, Einstein, Podolsky and Rosen (EPR) argued that the description of physical reality provided by quantum mechanics was incomplete. They attempted to identify elements of reality that were not included in quantum mechanics. They theorised that it is possible to predict with certainty, the value that the property will have, immediately before measurement. In 1965, an experimental test was proposed by John Bell, which invalidated EPR's argument. The CHSH inequality can be used in the proof of Bell's Theorem, which states that certain consequences of entanglement in quantum mechanics cannot be reproduced by local hidden variables. The inequality is a constraint on the statistics of "coincidences" in a Bell test, which is necessarily true if there exists underlying local hidden variables. However, the constraint can be infringed by quantum mechanics, particularly, entanglement.

Violation of CHSH Inequality with 2 Qubits

There are 2 possible CHSH inequalities:

$|\langle CHSH1\rangle| = |\langle AB\rangle| + |\langle Ab\rangle| - |\langle aB\rangle| + |\langle ab\rangle| \leq 2$

$|\langle CHSH2\rangle| = |\langle AB\rangle| - |\langle Ab\rangle| + |\langle aB\rangle| + |\langle ab\rangle| \leq 2$

where $\{A,\ a\}$ and $\{B,\ b\}$ are sets of orthogonal bases measured by parties Alice and Bob respectively.

The maximum expectation of CHSH operator for a *quantum system* is $2\sqrt{2}$ (Tsirelson's bound). However, the violation of Bell inequality is a sufficient criterion for certifying entanglement but not a necessary one.

The 4 circuits below will give the expectation values of AB, Ab, aB and ab, for calculating the CHSH values.

State creation ***Bases Selection*** ***Measurement***

Fig 1.1: Circuit 1 for $\langle AB\rangle$

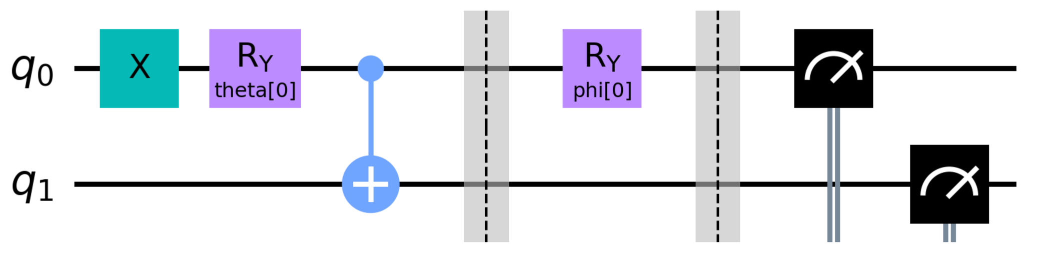


Fig 1.2: Circuit 2 for $\langle Ab\rangle$

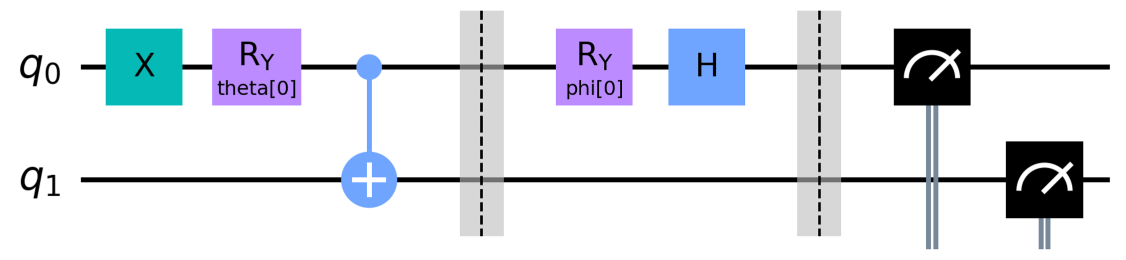


Fig 1.3: Circuit 3 for $\langle aB\rangle$

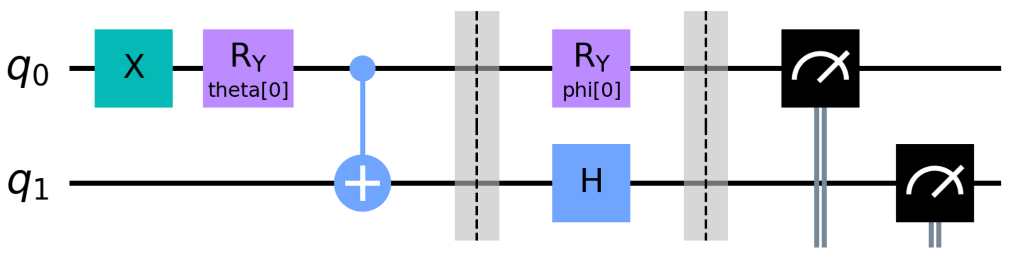
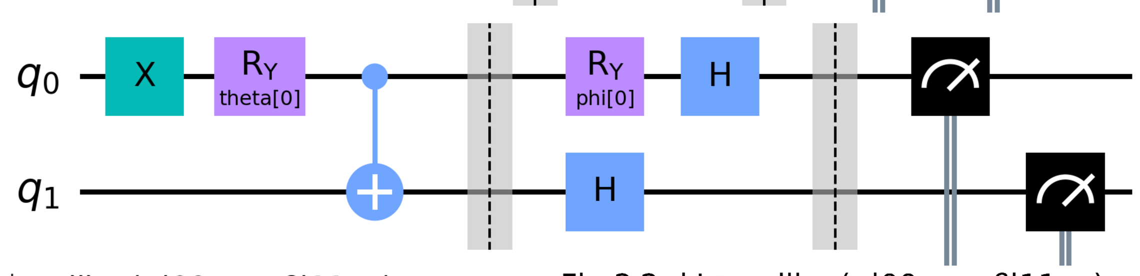


Fig 1.4: Circuit 4 for $\langle ab\rangle$



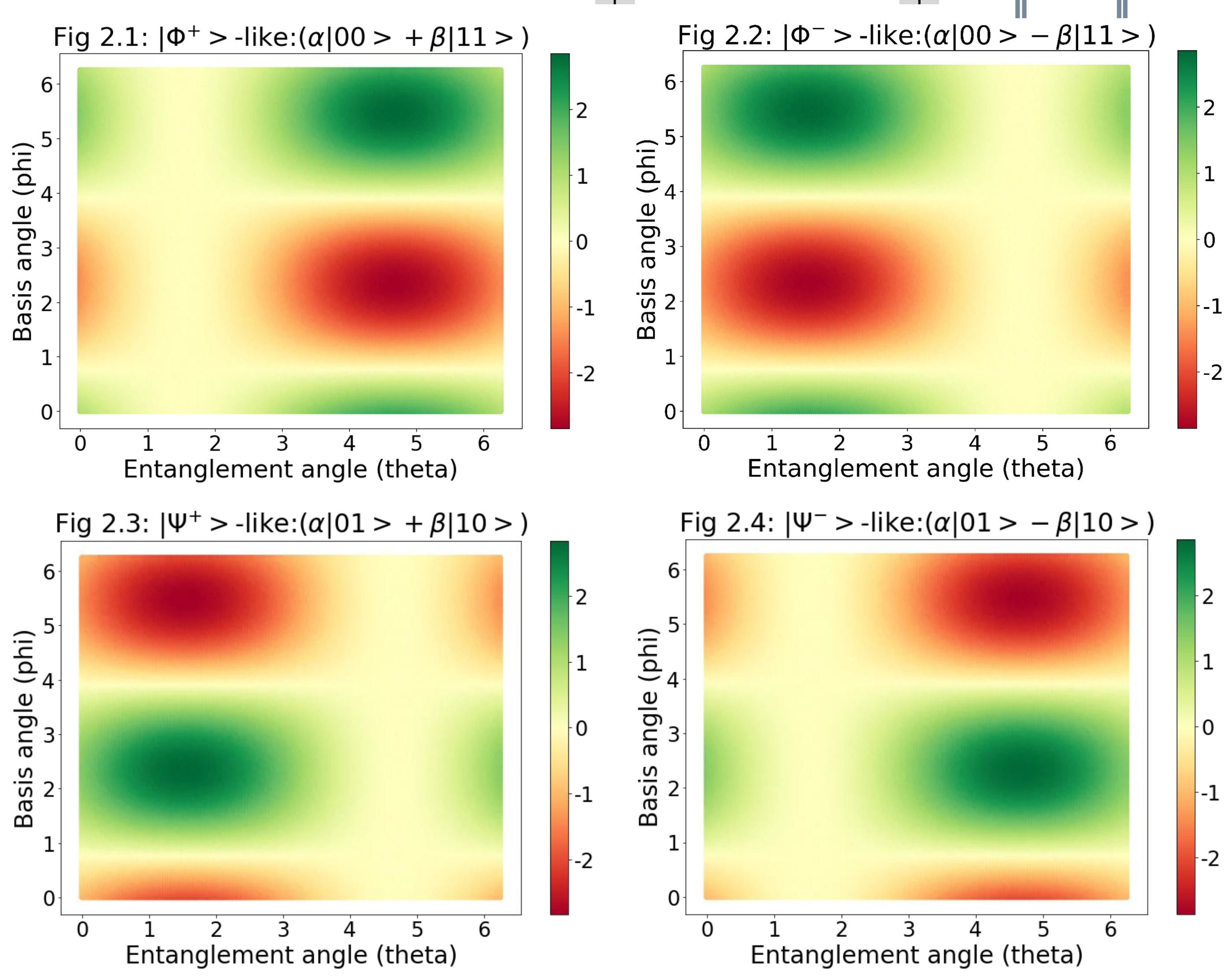


Fig 2: Simulated Results for $\langle CHSH1\rangle$ with Different Bell States

Operator Generalisation for Multiple Qubits

The Hamiltonian / CHSH operator can be generalised for any n-qubit GHZ state. When n is odd, n-qubit GHZ states is the eigenstate, with eigenvalue $2\sqrt{2}$, of the operator:

$$\hat{H}_1 = \sqrt{2}\left(\sigma_X \otimes \sigma_X \otimes \ldots \otimes \sigma_X^{nth} + \sigma_Z \otimes \sigma_Z \otimes \ldots \otimes \sigma_Z^{(n-1)th} \otimes \underline{\mathbb{I}}\right)$$

When n is even, the operator takes the form:

$$\hat{H}_1 = \sqrt{2}\left(\sigma_X \otimes \sigma_X \otimes \ldots \otimes \sigma_X^{nth} + \sigma_Z \otimes \sigma_Z \otimes \ldots \otimes \sigma_Z^{(n-1)th} \otimes \underline{\sigma_Z}\right)$$

Variational Quantum Eigensolver on Multiple Qubits

VQE allows us to significantly reduce the number of measurements for 2 qubits.

Table 1: CHSH Operators for 2 Qubits

Hamiltonian/CHSH Operator	λ	$\vec{v}$
$\hat{H}_1 = \sqrt{2}\,(\sigma_X \otimes \sigma_X + \sigma_Z \otimes \sigma_Z)$	$-2\sqrt{2}$	$\lvert\Psi^-\rangle = \frac{1}{\sqrt{2}}(\lvert 10\rangle - \lvert 01\rangle)$
$\hat{H}_2 = \sqrt{2}\,(\sigma_X \otimes \sigma_X - \sigma_Z \otimes \sigma_Z)$	$-2\sqrt{2}$	$\lvert\Phi^-\rangle = \frac{1}{\sqrt{2}}(\lvert 11\rangle - \lvert 00\rangle)$
$\hat{H}_3 = \sqrt{2}\,(-\sigma_X \otimes \sigma_X + \sigma_Z \otimes \sigma_Z)$	$-2\sqrt{2}$	$\lvert\Psi^+\rangle = \frac{1}{\sqrt{2}}(\lvert 01\rangle + \lvert 10\rangle)$
$\hat{H}_4 = \sqrt{2}\,(-\sigma_X \otimes \sigma_X - \sigma_Z \otimes \sigma_Z)$	$-2\sqrt{2}$	$\lvert\Phi^+\rangle = \frac{1}{\sqrt{2}}(\lvert 00\rangle + \lvert 11\rangle)$

Table 2: CHSH Operators for 3 Qubits

Hamiltonian/CHSH Operator
$\hat{H}_1 = \sqrt{2}\,(\sigma_X \otimes \sigma_X \otimes \sigma_X + \sigma_Z \otimes \sigma_Z \otimes \mathbb{I})$
$\hat{H}_2 = \sqrt{2}\,(\sigma_X \otimes \sigma_X \otimes \sigma_X - \sigma_Z \otimes \sigma_Z \otimes \mathbb{I})$
$\hat{H}_3 = \sqrt{2}\,(-\sigma_X \otimes \sigma_X \otimes \sigma_X + \sigma_Z \otimes \sigma_Z \otimes \mathbb{I})$
$\hat{H}_4 = \sqrt{2}\,(-\sigma_X \otimes \sigma_X \otimes \sigma_X - \sigma_Z \otimes \sigma_Z \otimes \mathbb{I})$

The VQE must run with at least 2 CHSH operators ($\hat{H}_1 \& \hat{H}_2$). The output with the lowest eigenvalue is the CHSH value we need. The 2 non-biseparable classes of 3-qubit states are $|GHZ\rangle = \frac{1}{\sqrt{2}}(|000\rangle + |111\rangle)$ and $|W\rangle = \frac{1}{\sqrt{3}}(|001\rangle + |010\rangle + |100\rangle)$.

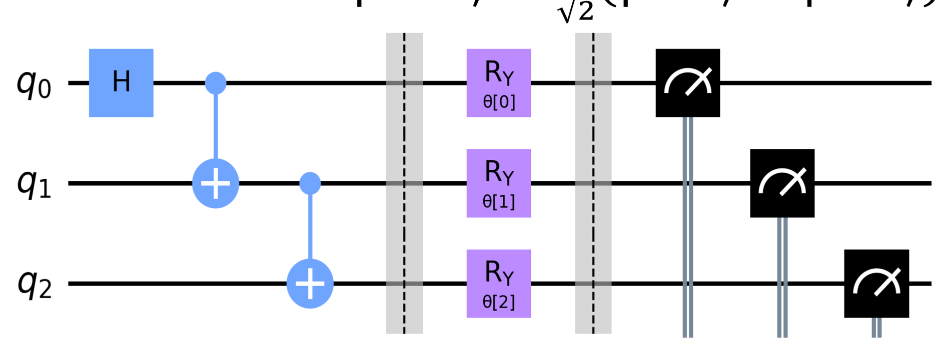


Fig 3.1: Variational Quantum Circuit (GHZ-state)

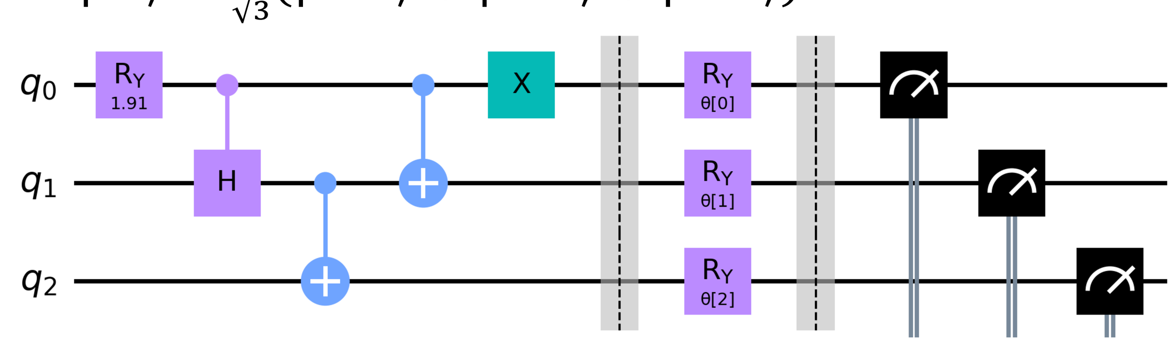


Fig 3.2: Variational Quantum Circuit (W-state)

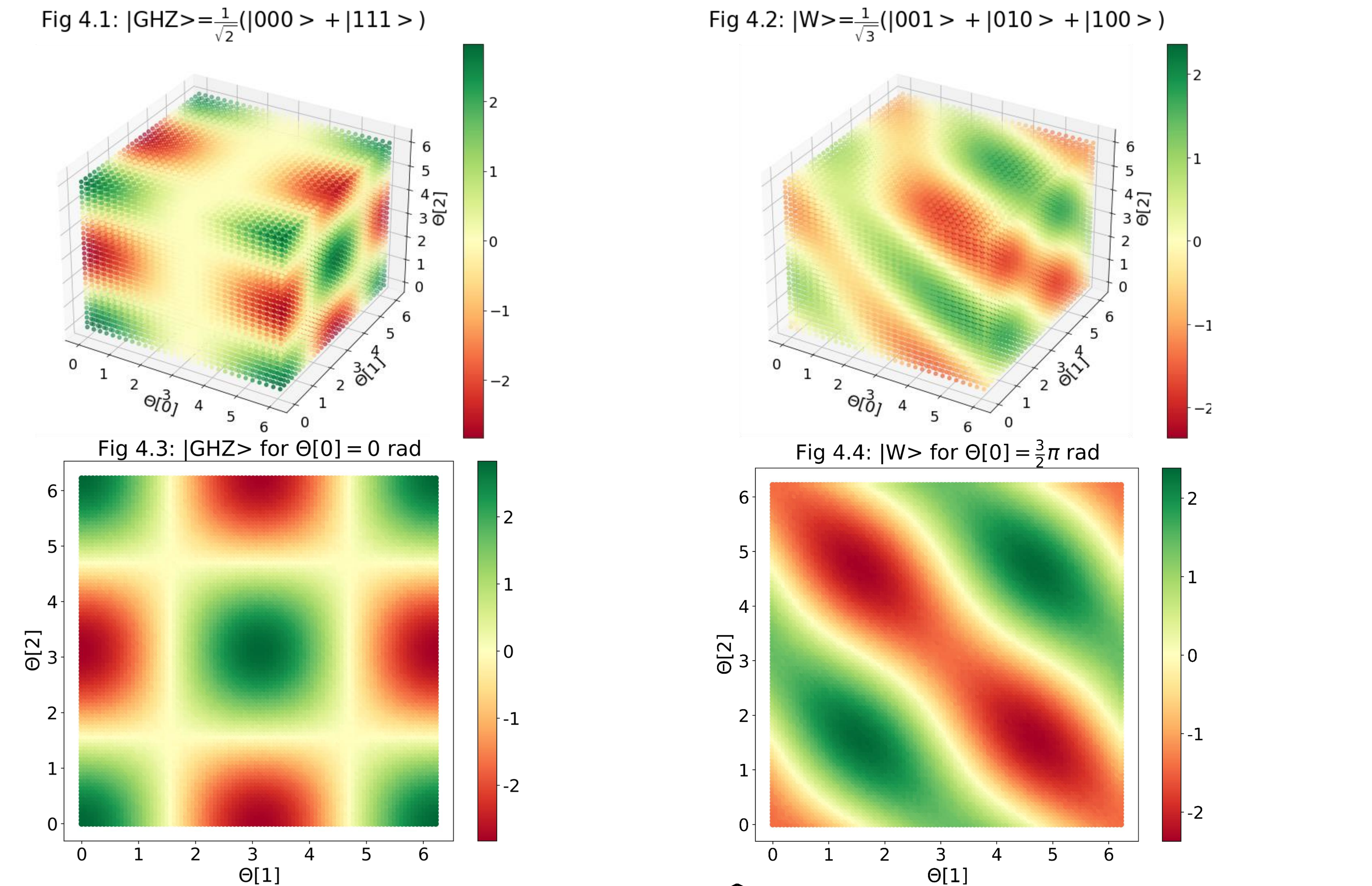


Fig 4: Simulated Results for CHSH Eigenvalue using $\hat{H}_1$ for 3-qubit $|GHZ\rangle$ and $|W\rangle$ States

Note the minimum eigenvalue obtained for the W-state is only ≈ -2.357, as it is not an eigenstate of any of the above 4 Hamiltonians. It is difficult, if not impossible, to construct the Hamiltonian for which the W-state is an eigenstate of.

Limitations of VQE Method on 3 or More Qubits

VQE is unable to distinguish between bi-separable and genuinely n-qubit entangled pure states. The output eigenvalue is a measure of how strongly entangled any pair of qubits are in the ansatz, and will be > 2 as long as there is sufficient entanglement between any pair. For example, both $|GHZ\rangle$, a genuinely entangled pure state, and $|\psi\rangle = |\Phi^+\rangle \otimes |0\rangle$, a 3-qubit bi-separable state, will output eigenvalue of $2\sqrt{2}$.

Future Work

In future, such experiments shall be conducted on real quantum devices which are far from the ideal simulations in this project. Gate fidelity and noise factors are some challenges to overcome as we try to reproduce ideal results.

References

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