Tensor Calculus Worksheet!?

Cheng

December 19, 2024

$$
\begin{array}{rr}
\textit{Cartesian} & \textit{Spherical} \\
x(r,\varphi,\theta) = & r(x,y,z) = \\
y(r,\varphi,\theta) = & \varphi(x,y,z) = \\
z(r,\varphi,\theta) = & \theta(x,y,z) =
\end{array}
$$

basis

$$\mathbf{r} = x\hat{x} + y\hat{y} + z\hat{z} = x(r,\varphi,\theta)\ \hat{x} + y(r,\varphi,\theta)\ \hat{y} + z(r,\varphi,\theta)\ \hat{z}$$

$$
\begin{array}{ll}
\dfrac{\partial\vec{r}}{\partial x} = \vec{x} = & \dfrac{\partial\vec{r}}{\partial r} = \vec{r} = \\
\dfrac{\partial\vec{r}}{\partial y} = \vec{y} = & \dfrac{\partial\vec{r}}{\partial \varphi} = \vec{\varphi} = \\
\dfrac{\partial\vec{r}}{\partial z} = \vec{z} = & \dfrac{\partial\vec{r}}{\partial \theta} = \vec{\theta} =
\end{array}
$$

$$\mathbf{r} = r(x,y,z)\ \vec{r} + \varphi(x,y,z)\ \vec{\varphi} + \theta(x,y,z)\ \vec{\theta} = r\vec{r} + \varphi\vec{\varphi} + \theta\vec{\theta}$$

$$
\begin{array}{ll}
\dfrac{\partial\vec{r}}{\partial x} = \vec{x} = & \dfrac{\partial\vec{r}}{\partial r} = \vec{r} = \\
\dfrac{\partial\vec{r}}{\partial y} = \vec{y} = & \dfrac{\partial\vec{r}}{\partial \varphi} = \vec{\varphi} = \\
\dfrac{\partial\vec{r}}{\partial z} = \vec{z} = & \dfrac{\partial\vec{r}}{\partial \theta} = \vec{\theta} =
\end{array}
$$

Notice:

$$\frac{\partial\mathbf{r}}{\partial r} = \vec{r} = \partial_r x\ \hat{x} + \partial_r y\ \hat{y} + \partial_r z\ \hat{z}$$

$$J_{xyz\to r\varphi\theta} = J^{(xyz)}_{(r\varphi\theta)} = \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = \begin{bmatrix} \partial_r x & \partial_\varphi x & \partial_\theta x \\ \partial_r y & \partial_\varphi y & \partial_\theta y \\ \partial_r z & \partial_\varphi z & \partial_\theta z \end{bmatrix}$$

$$\begin{tikzcd}
\vec{r}, \mathbf{Z_1} & \vec{\varphi}, \mathbf{Z_2} & \vec{\theta}, \mathbf{Z_3} \\
\partial_r x \arrow[dd, dash] & \partial_\varphi x \arrow[dd, dash] & \partial_\theta x \arrow[dd, dash] \\
\partial_r y & \partial_\varphi y & \partial_\theta y \\
\partial_r z & \partial_\varphi z & \partial_\theta z
\end{tikzcd}$$

$$J_{r\varphi\theta\to xyz} = J^{(r\varphi\theta)}_{(xyz)} = \frac{\partial(r,\varphi,\theta)}{\partial(x,y,z)} = \begin{bmatrix} \partial_x r & \partial_y r & \partial_z r \\ \partial_x \varphi & \partial_y \varphi & \partial_z \varphi \\ \partial_x \theta & \partial_y \theta & \partial_z \theta \end{bmatrix}$$

$$\begin{tikzcd}
\vec{x} & \vec{y} & \vec{z} & \\
\partial_x r \arrow[r, dash] \arrow[dd] & \partial_y r \arrow[r, dash] \arrow[dd] & \partial_z r \arrow[dd] & \mathbf{Z^1} \\
\partial_x \varphi \arrow[r, dash] & \partial_y \varphi \arrow[r, dash] & \partial_z \varphi & \mathbf{Z^2} \\
\partial_x \theta \arrow[r, dash] & \partial_y \theta \arrow[r, dash] & \partial_z \theta & \mathbf{Z^3}
\end{tikzcd}$$

$$\frac{\partial(r,\varphi,\theta)}{\partial(x,y,z)} \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} [\mathbf{Z^1}] \\ \\ [\mathbf{Z^2}] \\ \\ [\mathbf{Z^3}] \end{pmatrix} \begin{pmatrix} [\mathbf{Z_1}] & [\mathbf{Z_2}] & [\mathbf{Z_3}] \end{pmatrix} = \begin{pmatrix} \mathbf{Z^1}\cdot\mathbf{Z_1} & \mathbf{Z^1}\cdot\mathbf{Z_2} & \mathbf{Z^1}\cdot\mathbf{Z_3} \\ \mathbf{Z^2}\cdot\mathbf{Z_1} & \mathbf{Z^2}\cdot\mathbf{Z_2} & \mathbf{Z^2}\cdot\mathbf{Z_3} \\ \mathbf{Z^3}\cdot\mathbf{Z_1} & \mathbf{Z^3}\cdot\mathbf{Z_2} & \mathbf{Z^3}\cdot\mathbf{Z_3} \end{pmatrix}$$

$$\mathbf{Z^i}\cdot\mathbf{Z_j} = \delta_{ij}$$

$$\frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}^T \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = \begin{pmatrix} [\mathbf{Z_1}] \\ \\ [\mathbf{Z_2}] \\ \\ [\mathbf{Z_3}] \end{pmatrix} \begin{pmatrix} [\mathbf{Z_1}] & [\mathbf{Z_2}] & [\mathbf{Z_3}] \end{pmatrix} = \begin{pmatrix} \mathbf{Z_1}\cdot\mathbf{Z_1} & \mathbf{Z_1}\cdot\mathbf{Z_2} & \mathbf{Z_1}\cdot\mathbf{Z_3} \\ \mathbf{Z_2}\cdot\mathbf{Z_1} & \mathbf{Z_2}\cdot\mathbf{Z_2} & \mathbf{Z_2}\cdot\mathbf{Z_3} \\ \mathbf{Z_3}\cdot\mathbf{Z_1} & \mathbf{Z_3}\cdot\mathbf{Z_2} & \mathbf{Z_3}\cdot\mathbf{Z_3} \end{pmatrix} = [Z_{ij}]$$

$$\frac{\partial(r,\varphi,\theta)}{\partial(x,y,z)} \frac{\partial(r,\varphi,\theta)}{\partial(x,y,z)}^T = \begin{pmatrix} [\mathbf{Z^1}] \\ \\ [\mathbf{Z^2}] \\ \\ [\mathbf{Z^3}] \end{pmatrix} \begin{pmatrix} [\mathbf{Z^1}] & [\mathbf{Z^2}] & [\mathbf{Z^3}] \end{pmatrix} = [Z^{ij}]$$

$$[Z^{ij}][Z_{ij}] = \frac{\partial(r,\varphi,\theta)}{\partial(x,y,z)} \frac{\partial(r,\varphi,\theta)}{\partial(x,y,z)}^T \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}^T \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)} = I$$

$$\therefore Z^{ij} Z_{jk} = \delta_{ik}$$

$$[Z^{ij}] \begin{pmatrix} [\mathbf{Z_1}] \\ [\mathbf{Z_2}] \\ [\mathbf{Z_3}] \end{pmatrix} \stackrel{?}{=} \begin{pmatrix} [\mathbf{Z_1}] \\ [\mathbf{Z_2}] \\ [\mathbf{Z_3}] \end{pmatrix}$$

$$[Z^{ij}] \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}^T = \frac{\partial(r,\varphi,\theta)}{\partial(x,y,z)} \frac{\partial(r,\varphi,\theta)}{\partial(x,y,z)}^T \frac{\partial(x,y,z)}{\partial(r,\varphi,\theta)}^T = \frac{\partial(r,\varphi,\theta)}{\partial(x,y,z)}$$

$$\therefore Z^{ij} \mathbf{Z_j} = \mathbf{Z^i}$$

Bases:

$$\begin{array}{ll} \textit{covariant} & \textit{contravariant} \\ \mathbf{Z_1} = & \mathbf{Z^1} = \\ \mathbf{Z_2} = & \mathbf{Z^2} = \\ \mathbf{Z_3} = & \mathbf{Z^3} = \end{array}$$

$$\mathbf{Z^i} \cdot \mathbf{Z_j} = \delta^i_j$$

$$\begin{array}{ll} \textit{covariant} & \textit{contravariant} \\ \mathbf{V} = V^i \mathbf{Z_i} & = V_i \mathbf{Z^i} \end{array}$$

$$\begin{aligned} \mathbf{V} \cdot \mathbf{V} &= covariant \cdot covariant = Z_{ij} V^i V^j \\ &= covariant \cdot contravariant = {\delta_i}^j V^i V_j = V^j V_j \\ &= contravariant \cdot contravariant = Z^{ij} V_i V_j \end{aligned}$$