Visual Complex Analysis

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LOG(M)

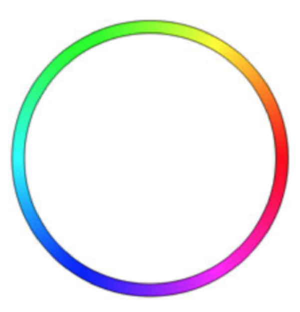
Introduction

Question

Can we find a geometric relationship between the roots of a complex polynomial and the roots of its derivative?

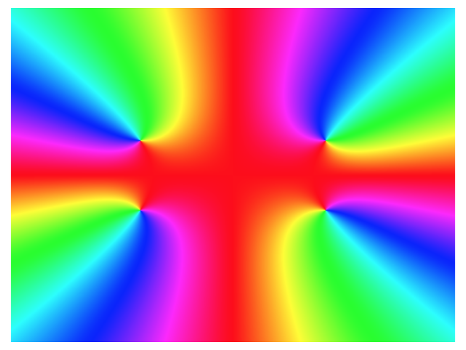
Visualization Tool

For a complex number $z = re^{i\theta}$, the *phase* of z is $e^{i\theta}$, which is a number on the unit circle. For a complex function $f : \mathbb{C} \to \mathbb{C}$, we can associate a color to the phase of $f(z)$ and plot the colors on the domain to construct a *phase plot* for f.



Phase Color Wheel [1]

Identity Map



Quartic Polynomial

It can be shown that two analytic functions have identical phase plots if and only if they are positive scalar multiples of each other. We recognize zeros and poles of analytic functions as places where all colors come together. The argument principle tells us that the order of a zero or a pole is equal to the number of times the colors cycle around the point; colors cycle counterclockwise around zeros and clockwise around poles. We now present a result about *monochromatic curves*, i.e. curves of constant phase.

Theorem 1. *Let f be an analytic function such that $f(z_0) \neq 0$. Then f' has a zero of order n at $z = z_0$ if and only if the phase plot of f has $n+1$ monochromatic curves passing through $z = z_0$.*

From these results, we can identify the zeros of a function and of its derivative simply by looking at its phase plot. This is demonstrated in the first two images below; in each case, the crossing cyan phase lines at the origin tell us that f' attains a zero at the origin, by Theorem 1.

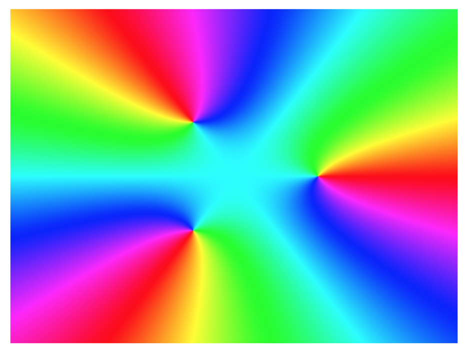
Motivation

Some known results provide background and motivation for our topic.

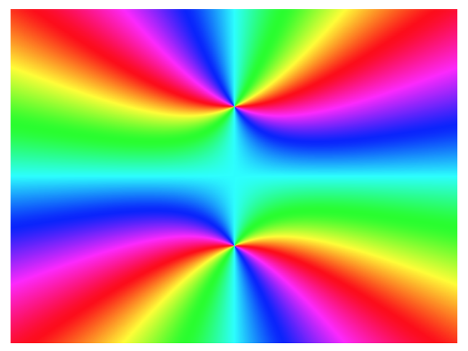
Theorem 2 (Gauss-Lucas)**.** *Given a non-constant complex polynomial p, the zeros of the derivative p' lie in the convex hull of the roots of p.*

Theorem 3 (Marden [2])**.** *Let p be a cubic complex polynomial with non-colinear zeros z_1, z_2, z_3. The triangle formed by z_1, z_2, z_3 admits a unique inscribed ellipse that is tangent to the sides at their midpoints. The foci of this ellipse are the zeros of the derivative p'.*

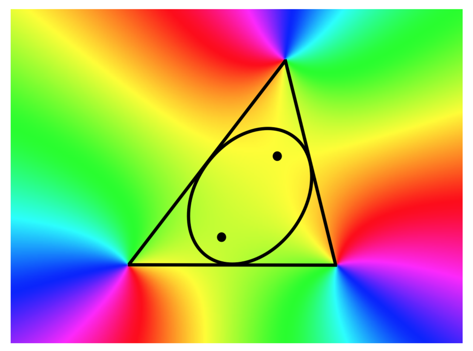
The Gauss-Lucas theorem gives us the initial inspiration to find a stronger relationship between the zeros of a polynomial and the zeros of its derivative. Marden's theorem then presents a geometric relationship in the case of third degree polynomials (see the third picture below), motivating us to find a similar relationship for higher degrees.



$f(z) = z^3 - 1$



$f(z) = -(z^2+1)^2$



Marden's Theorem

Current Progress

Affine Mapping

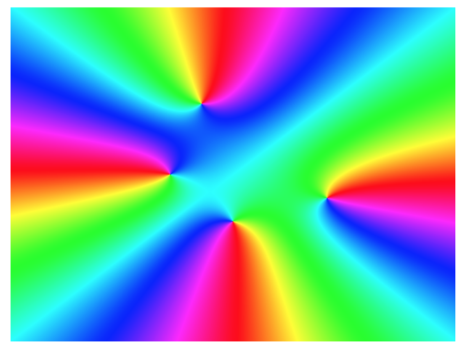
Our first result reduces the number of polynomials we have to consider by allowing us to move them while preserving the relevant geometry.

Theorem 4. *Let $p(z)$ be a monic polynomial with roots $z_1, z_2, \ldots, z_n$, and $T : \mathbb{C} \to \mathbb{C}$ be a transformation that is some composition of scalings, rotations and translations. Let $P_T(z)$ denote the monic polynomial with roots $T(z_1), T(z_2), \ldots, T(z_n)$. Then $w \in \mathbb{C}$ is a root of p' if and only if $T(w)$ is a root of $p_T(z)$.*

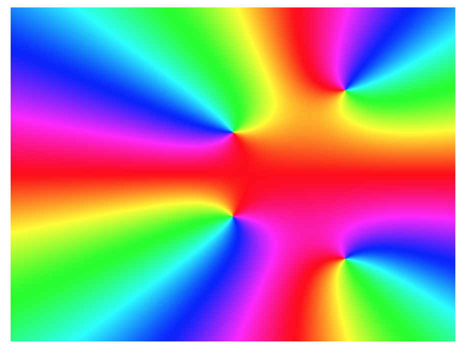
We can define an equivalence relation $\sim$: two complex coefficient polynomials p and q are equivalent if and only if p can be transformed to q through a composition of scalings, rotations and translations. Letting P be the set of all complex coefficient polynomials of finite degree, $P/\sim$ is our subject of interest.

A main focus of ours has been on quartic polynomials. For any quartic polynomial with four complex roots forming a symmetric trapezoid, by Theorem 4 we can always transform it to the form

$$p(z) = (z^2+1)(z-u)(z-\bar{u}). \tag{1}$$



$z^4 + (1.5i)z^2 - (1.5 - 1.5i)z - 2.5$



$(z^2+1)(z^2-4z+8)$

Constant Phase Lines and Root Flow

With p as above in equation (1), we have two interesting observations:

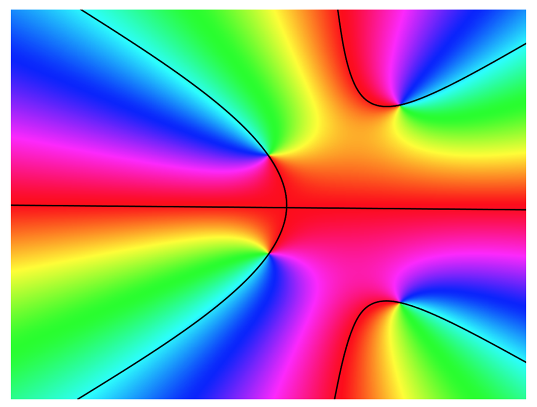
- $p(z)$ is always a quartic polynomial with real coefficients.
- Changing the constant term of $p(z)$ leaves the roots of $p'(z)$ unchanged.

The first observation allows us to focus on real coefficient quartic polynomials, and the second observation enables us further to consider special real coefficient quartic polynomials, as illustrated in the next panel under Classes of Polynomials.

Exploring the structure of the lines of constant phase allows us to understand the paths of root flow as we change the constant term of p. In particular, the lines where p assumes real values (i.e. the phase plot is red or cyan) are of interest. When changing the constant term of p by a real number, the real phase lines are the trajectories of the roots of p. This leads to the next result:

Theorem 5. *For a quartic polynomial $p(z)$ with real coefficients, where $z = x + iy$, the real phase lines are given by*

$$24x + p'''(0)y^3 = 6p'(x)y. \tag{2}$$



$(z^2+1)(z^2-4z+8)$ with real phase lines

Future Direction

Classes of Polynomials

While investigating the roots of the derivative of any fourth degree polynomial with real coefficients, we first observed that by changing the constant term, there is at least one constant term such that the polynomial has a real root of order at least two. Second, using affine mapping, any such polynomial with a double root can be mapped to one of the following types:

Type 1: $f(z) = (z - r_1)^2(z - r_2)(z - r_3)$, where $r_1, r_2, r_3 \in \mathbb{R}$, possibly equal

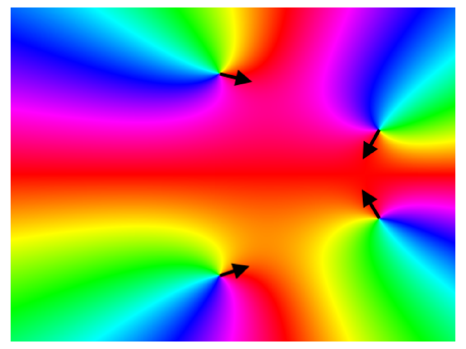
Type 2: $f(z) = (z - r)^2(z - i)(z + i)$, where $r \in \mathbb{R}$

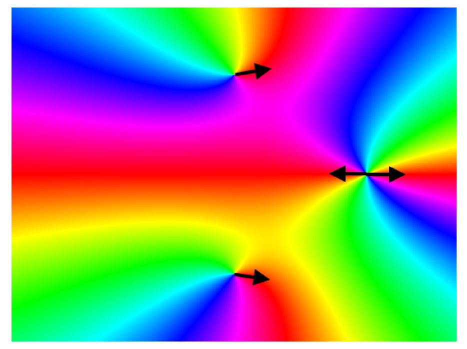
Example and Observations

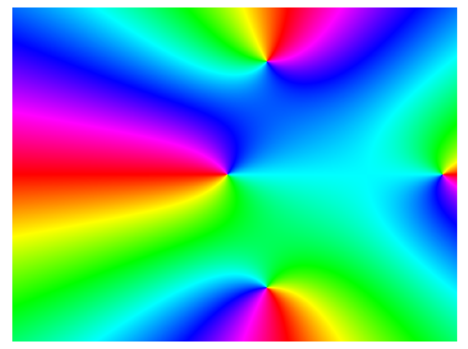
Here, we take one polynomial of the second type to investigate:

$$f(z) = (z-1)^2(z-i)(z+i) = z^4 - 2z^3 + 2z^2 - 2z + 1$$

While we change the constant term of $f(z)$, the following graphs show how the roots of the polynomial change:



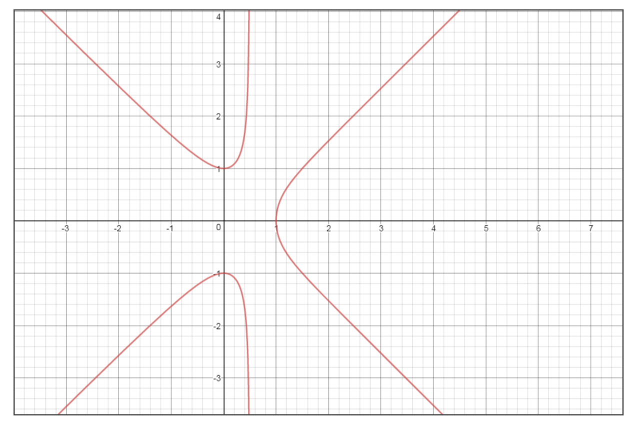




In the second picture, there is a double root at $z = 1$. Also note that manipulating the constant term doesn't affect the derivative, so the roots of the derivative are the same for all these graphs. Based on these graphs, we have made some observations for this specific polynomial. Write the conjugate roots as $a + bi, a - bi$, where a, b depend on the constant term c. Then, as we change the constant term of c of $f(z)$,

1. $|b|$ is minimized when $a = 0, b = 1$, and the curve $\mathrm{Im}(z) = 1$ is tangent to the curve $\mathrm{Im}(f(z)) = 0$ at this point.
2. $\mathrm{Re}(a) < \frac{1}{2}$ for all c. $\mathrm{Re}(z) = \frac{1}{2}$ is an asymptote for the conjugate roots.
3. The area of the convex hull of the roots is minimized when there is a double root.

The graph below shows possible values of the conjugate roots $a \pm bi$ parameterized by the constant term c.



Future investigations will be focused on the relationship between the four roots of a fourth degree polynomial with real coefficients and the three roots of its derivative, using the strategy 'flowing' the roots by manipulating the constant term.

References

[1] Elias Wegert, *Visual Complex Functions*, Springer Basel, 2012.

[2] Dan Kalman, *An Elementary Proof of Marden's Theorem*, Amer. Math. Monthly 115:4 2008, 330-338.

[3] Illinois Geometry Lab, *IGL Poster Template*, University of Illinois at Urbana-Champaign Department of Mathematics, 2017.

[4] Jim Fowler, *Interactive Phase Plot Visualizer*, https://people.math.osu.edu/fowler.291/phase/index.html