Propositional Practice

(a). $()\exists x \in \mathbb{R})(x \notin \mathbb{Q})$
True. $\sqrt{2}$ is a real number and also not rational.

(b). $(\forall x \in \mathbb{Z})(((x \in \mathbb{N}) \vee (x < 0)) \wedge \neg((x \in \mathbb{N}) \wedge (x < 0)))$
True. We define the naturals to contain all integers which are not negative.

(c). $(\forall x \in \mathbb{N})(6 \mid x) \implies ((2 \mid x) \vee (3 \mid x))$
True. 6k = 2(3k) = 3(2k).

(d). All integers are rational.
True. Any interger can be represented as a/b, where a and b are coprime integers.

(e). Integers divisibe by 2 or 3 are also divisibe by 6.
False. Counterexample, 4.

(f). All natural number larger than 7 is a sum of two natural numbers.
True. Let a = x, b can be 0, they are both natural.

Proof Practice

(a).

$$\forall n \in \mathbb{N}, n = 2k + 1$$

$$n^2 + 1 = 4k^2 + 4k + 2 = 2(2K^2 + 2k + 1)$$

(b). if

$$x \geq y, (x + y - (x - y))/2 = y$$

if

$$x < y, (x + y - (y - x))/2 = x$$

(c). *Proof. Base Case:* $n = 1, \frac{n(n+1)}{2} = 1$,hold
Inductive Hypothesis: $n = k, \sum_{i=1}^{n} i = \frac{k(k+1)}{2}$
Inductive Step: $n = k + 1, \sum_{i=1}^{n} i = \frac{k(k+1)}{2} + k + 1 = \frac{(k+1)(k+2)}{2}$
we conclude. □

(d). *Proof.* Suppose $\Gamma \in \mathbb{P}(A)$, that is $\Gamma \subseteq A$
$\forall x \in \Gamma, x \in A$
since $A \subseteq B, x \in B, \Gamma \subseteq B$
so $\Gamma \in \mathbb{P}(B)$
we conclude.

□

Open Set Intersection

(a). Empty set is an open interval. Since any empty set can be writen as $\{x \in \mathbb{R} | x > a \wedge x < b\}$ where $a \geq b$.

(b). $(a, b) \cap (c, d)$ can be writen as $\{x \in \mathbb{R} | x > \max(a, c) \wedge x < \min(b, d)\}$.

(c). *Proof.* Prove by induction.
Base Case: n=1, I_1 is an open interval.
Inductive Hypothesis: n=k, $\cap_{i=1}^{k}$ is an open interval.
Inductive Step: n=k+1, $\cap_{i=1}^{k+1} = \cap_{i=1}^{k} \cap I_{k+1}$. Since *(b)* holds, so this is also an open interval. $\square$

(d). *Proof.* Prove by contradiction.
Assume it is an open interval.
$S = \{x\ in\mathbb{R} | x > a \wedge x < b\}$, $S = k$.
So $a < k < b$, where $a, k, b \in \mathbb{R}$, and there must $\exists k^1, a < k^1 < k$, lead S contains two numbers, contradict.
We conclude. $\square$

(e). Let $I_k = (-1/k, 1/k)$, 0 must be contained in set $\cap_{k=1}^{\infty}$ since $-1/k < 0 < 1/k$ always holds where $k \in \mathbb{N}$ and $k \geq 1$. If there exists another number x in the set, we can always find a $1/k \leq |x|$, make x is impossible in the set. So there must exist exactly one number 0 in the set, since *(d)* holds, therefore $\cap_{k=1}^{\infty}$ is not an open interval.

(f). Induction can only be used to prove that a proposition holds for every natural number. However, just because a proposition holds for every finite natural number does not mean that it holds as infinity.

Induction

(a). *Proof.* *Base Case:* $n = 3, 2^3 = 8 > 7$.
Inductive Hypothesis: $n = k, 2^k > 2k + 1$.
Inductive Step: $n = k+1, 2^{k+1} = 2^k \times 2 > 2(2k+1) = 4k+2$, since $2k > 1$, thus $4k + 2 > 2k + 3 = 2(k + 1) + 1$.
We conclude. $\square$

(b). *Proof.* *Base Case:* $n = 1$, holds.
Inductive Hypothesis: Assume $n = k$ holds.
Inductive Step: $n = k + 1$, $\frac{k(k+1)(2k+1)}{6} + k + 1 = \frac{(k+1)(k+2)(2k+3)}{6}$.
We conclude.

Proof. *Base Case:* $n = 1$, holds.
Inductive Hypothesis: Assume n=k holds.

Inductive Step: $n = k + 1$,

$$\begin{aligned}\frac{5}{4} \cdot 8^{k+1} + 3^{3(k+1)-1} &= 8 \cdot \frac{5}{4} \cdot 8^k + 27 \cdot 3^{3k-1} \\ &= 8(\frac{5}{4} \cdot 8^k + 3^{3k-1}) + 19 \cdot 3^{3k-1}\end{aligned}$$

We conclude. □

□

Make It Stronger

(a). when n=k+1, we can get $a_{n+1} \le 3^{2^{n+1}+1}$, can not resolve the extra plus 1 in the exponential.

(b). *Proof.* Prove $a_n \le 3^{2^n-1}$.
Inductive Step: $n = k + 1$,

$$\begin{aligned}a_{k+1} &= 3 \cdot a_k^2 \\ &\le 3 \cdot 3^{2^{n+1}-2} \\ &= 3^{2^{n+1}-1}\end{aligned}$$

Since $3^{2^n-1} < 3^{2^n}$, We conclude. □

A Coin Game

Proof. Proved by strong induction.
Base Case: n=1, holds.
Inductive Hypothesis: $\forall n \in (1, k]$, holds.
Inductive Step: $n = k + 1$, let $n = a + b$, where $1 \le a \le k, 1 \le b \le k$.

$$\begin{aligned}& a \times b + \frac{a(a-1)}{2} + \frac{b(b-1)}{2} \\ &= \frac{a^2 + b^2 + 2ab - (a+b)}{2} \\ &= \frac{(a+b)^2 - (a+b)}{2} \\ &= \frac{(a+b))a + b - 1}{2} \\ &= \frac{n(n-1)}{2}\end{aligned}$$

We conclude. $\square$

Preserving Set Operations

(a).

$$\begin{aligned}
&\forall x \in f^{-1}(A \cup B) \\
\implies &(f(x) \in A) \vee (f(x) \in B) \\
\implies &(x \in f^{-1}(A)) \vee (x \in f^{-1}(B)) \\
\implies &x \in (f^{-1}(A) \cup f^{-1}(B)) \\
\implies &f^{-1}(A \cup B) \subseteq f^{-1}(A) \cup f^{-1}(B)
\end{aligned}$$

$$\begin{aligned}
&\forall x \in f^{-1}(A) \cup f^{-1}(B) \\
\implies &(f(x) \in A) \cup (f(x) \in B) \\
\implies &f(x) \in (A \cup B) \\
\implies &x \in f^{-1}(A \cup B) \\
\implies &f^{-1}(A) \cup f^{-1}(B) \subseteq f^{-1}(A \cup B)
\end{aligned}$$

Thus, we conclude.

(b).

$$\begin{aligned}
&\forall x \in A \cup B, f(x) \in f(A \cup B) \\
\implies &(x \in A) \vee (x \in B) \\
\implies &(f(x) \in f(A)) \vee (f(x) \in f(B)) \\
\implies &f(x) \in (f(A) \cup f(B)) \\
\implies &f(A \cup B) \subseteq (f(A) \cup f(f(B))
\end{aligned}$$

$$\begin{aligned}
&\forall f(x) \in f(A) \cup f(B) \\
\implies &(f(x) \in f(A)) \vee (f(x) \in f(B)) \\
\implies &(x \in A) \vee (x \in B) \\
\implies &x \in (A \cup B) \\
\implies &f(x) \in f(A \cup B) \\
\implies &f(A) \cup f(B) \subseteq f(A \cup B)
\end{aligned}$$

Thus, we conclude.

Bijective Or Not

(a). (i). Bijective. If $f(a) \neq f(b)$, then $10^{-5}a \neq 10^{-5}b, a \neq b$, so two real numbers in range cannot be mapped to the same real number in domain, thus injective. $\forall f(x) \in \mathbb{R}, x = 10^5 \cdot f(x)$, so a $f(x)$ always have a mapped x, thus surjective.

(ii). Injective. Injective for the same reason as above. For surjective, $\exists f(x) \in \mathbb{R}$ has no mapped x. E.g. f(x) = 10^{-6}, x = $10^{-1} \notin \mathbb{Z} \cup \{\pi\}$.

(b). Injective but not surjective. For injective, let $f(a) = \{a\}, f(b) = \{b\}, \{a\} \neq \{b\}$, then $a \neq b$, thus injective. For surjective, some set in the range like $\{a, b\}$ or $\emptyset$ has no mapped x, thus not surjective.

(c). Yes if $X^{'}$ is not started with 0, since every number between 0 and 9 occurs precisely once in X, and thus precisely once in $X^{'}$ too, so no two digits in $X^{'}$ is the same. For surjective, there is two cases, if $X^{'}$ is not started with 0, since f(i) must be a digit of $X^{'}$ then $\forall f(i) \in (\mathbb{N} \cap [0, 9]), \exists i \in (\mathbb{N} \cap [0, 9])$, thus surjective. But when $X^{'}$ is started with 0, then only $\forall f(i) \in (\mathbb{N} \cap [1, 9])$ has a mapped i in the domain, that is $(\mathbb{N} \cap [0, 9])$, thus not surjective.