Lecture 15-16 Risk

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Understanding Risk

What does *risk* encompass?

- Potential **loss**
- Market **volatility**
- **Correlation** with other assets
- The **worst-case scenarios**
- **Sensitivity** to various factors

Various measures of risk aim to provide insights into these aspects.

Return, Price, PnL, and Beyond

Risk analysis spans different asset classes.

- The principles apply to various types of assets.

Focus of Analysis:

- **Price** - Relevant for bonds and options
- **Return** - Key for equities, futures, and currencies

In some cases, analyzing **profit and loss (PnL)** is more pertinent.

Price and Return Analysis of iShares 7-10 Year Treasury Bond

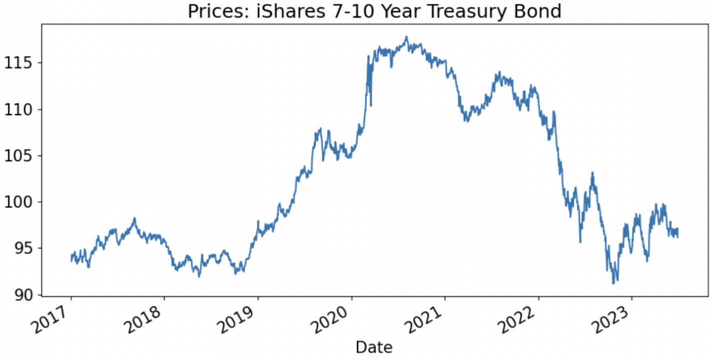


Figure: Price Trends of iShares 7-10 Year Treasury Bond (2017-2023)

Price and Return Analysis of iShares 7-10 Year Treasury Bond

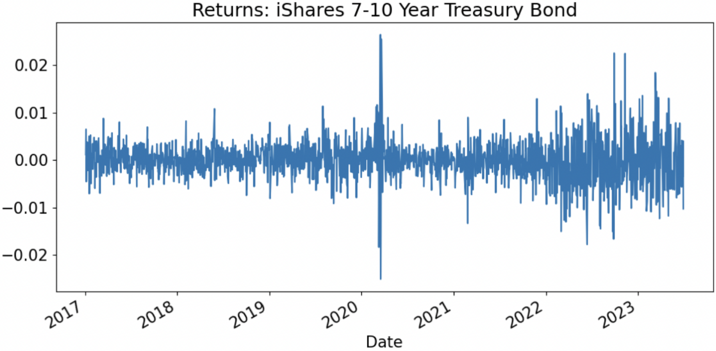


Figure: Daily Returns of iShares 7-10 Year Treasury Bond (2017-2023)

Tools for Risk Analysis

Tools Overview:

- Risk assessment for equities and other assets often employs more statistical methods and fewer mathematical models compared to fixed income and options pricing.

For Bonds and Options:

- Risk is often evaluated by examining the **mathematical sensitivity** of prices.

For Equities and Other Assets:

- We analyze the **statistical sensitivity** of returns, which can be thought of as scaled prices.

Modeling Approach to Risk

Key Aspects of the Modeling Approach:

- **Stationarity:**
  - Often necessary for meaningful analysis.
  - Drives the choice between prices, returns, or profits.
- **Independence and Identical Distribution (iid):**
  - Critical for statistical methods.
  - Achieving iid can be complex.
- **Normal Distribution:**
  - Not commonly required but useful for comparisons.
  - Sometimes used for simplification.

Risk Assessment versus Pricing

Distinct Approaches:

- Risk analysis does not rely on alternative **probability measures**.
- **Volatility** is understood as actual market variation, not derived from options pricing.
- Primarily uses **discrete-time** models, rather than continuous-time stochastic calculus.

Initial Considerations in Risk Analysis

Risk Analysis Question:

- Is a 10-year Treasury Note considered risky?

Focus of Risk Assessment:

- Primarily evaluates risk over a single period.
- Useful for practical risk management and forecasting.
- For example, metrics like Delta and Duration focus on short-term risk.

Extended Analysis:

- Sometimes necessary to consider multi-period risk and cumulative returns.

Moments and Data for Risk Analysis

Data Example:

- We use ETF data from various asset classes.
- Daily data spanning from 2017 to the present.

Advantages of Using ETF Data:

- Focuses on traded securities, avoiding issues with index data.
- Prevents problems related to rolling futures or FX carry trades.
- Be aware of potential differences due to fund expenses and tracking errors (e.g., in oil ETFs).

Data Example

ticker	quoteType	shortName	volume	totalAssets	longBusinessSummary
SPY	ETF	SPDR S&P 500	3695818	4.227204e+11	The Trust seeks to achieve its investment obje...
VEA	ETF	Vanguard FTSE Developed Markets	293687	1.680537e+11	The fund employs an indexing investment approa...
UPRO	ETF	ProShares UltraPro S&P 500	433011	2.758412e+09	The fund invests in financial instruments that...
GLD	ETF	SPDR Gold Trust	577948	5.666016e+10	The Trust holds gold bars and from time to tim...
USO	ETF	United States Oil Fund	142872	1.516177e+09	USO invests primarily in futures contracts for...
CL=F	FUTURE	Crude Oil Sep 23	88194	NaN	NaN
FXE	ETF	Invesco CurrencyShares Euro Cur	1019	2.520237e+08	NaN
BTC-USD	CRYPTOCURRENCY	Bitcoin USD	13960983552	NaN	NaN
HYG	ETF	iShares iBoxx $ High Yield Corp	1387739	1.456480e+10	The underlying index is a rules-based index co...
IEF	ETF	iShares 7-10 Year Treasury Bond	773587	2.911525e+10	The underlying index measures the performance ...
TIP	ETF	iShares TIPS Bond ETF	73739	2.148979e+10	The index tracks the performance of inflation-...
SHV	ETF	iShares Short Treasury Bond ETF	315152	2.004322e+10	The fund will invest at least 80% of its asset...

Figure: ETF Data Across Various Asset Classes

The Mean: First Moment of a Distribution

Definition:

- The mean (**first moment**) of an unspecified distribution is given by:

$$\mu = E[r]$$

Sample Estimator:

$$\hat{\mu} = \frac{1}{N} \sum_{t=1}^{N} r_t$$

Note: This is often denoted as $\bar{r}$, but we use $\hat{\mu}$ for consistency with other estimators.

Importance of Mean in Risk Analysis

Relevance to Risk:

- For risk assessment, we primarily use **de-meaned** data to understand deviations from the mean.
- Accurate models of the mean are crucial for forecasting returns in directional investing.

Key Considerations:

- An incorrect mean can distort risk analysis.
- Short-term risk evaluation is less affected by the mean, focusing more on volatility.

Mean Data for ETFs

Analyzing Mean Data:

- Examining mean values for both **price** and **return** helps in understanding the baseline performance of different assets.

Price and Return Data for Various ETFs:

ETF	Price	Return
SPY	318.28	13.92%
VEA	39.63	7.94%
UPRO	33.03	36.52%
GLD	148.31	8.24%
USO	73.73	2.87%
CL1	63.34	-40.90%
HYG	70.86	3.06%
IEF	102.58	0.65%
TIP	104.04	2.31%
SHV	105.01	1.31%

Table: Mean Prices and Returns for Various ETFs

Variance and Volatility

Definition of Variance:

- The variance is the **second centered moment** of a distribution:

$$\sigma^2 = E\left[(r-\mu)^2\right]$$

Sample Estimator:

$$\hat{\sigma}^2 = \frac{1}{N-1}\sum_{t=1}^{N}(r_t-\hat{\mu})^2$$

Why $N-1$?

- This adjustment accounts for the degrees of freedom lost by estimating the mean.

Standard Deviation and Volatility

Definition:

- The standard deviation is the **square root** of the variance.
- Often referred to as **volatility**, indicating the variation in asset returns.

Technical Points:

- **Non-negativity:** Standard deviation is always non-negative, ensuring realistic modeling.
- **Centering:** Variance is the second centered moment, subtracting the mean before squaring.
- Alternative view:

$$\sigma^2 = E\left[r^2\right] - (E[r])^2$$

Volatility and Variance for ETFs

ETF Volatility and Variance Data:

ETF	Volatility	Variance
SPY	19.51%	3.81%
VEA	18.20%	3.31%
UPRO	58.89%	34.68%
GLD	13.79%	1.90%
USO	40.96%	16.78%
CL1	139.29%	194.03%
FXE	7.38%	0.54%
BTC	74.95%	56.17%
HYG	9.38%	0.88%
IEF	6.75%	0.46%
TIP	6.22%	0.39%
SHV	0.26%	0.00%

Table: Volatility and Variance for Various ETFs

Higher Moments: Skewness and Kurtosis

Skewness:

- Skewness is the **third moment** (centered and scaled).

$$\varsigma = \frac{1}{\sigma^3} E\left[(r-\mu)^3\right]$$

- Sample estimator:

$$\hat{\varsigma} = \frac{1}{\hat{\sigma}^3} \frac{1}{N-1} \sum_{t=1}^{N} (r_t - \hat{\mu})^3$$

- Can be positive or negative and is not typically a percentage.
- Negative skewness indicates a distribution with a long left tail.

Kurtosis:

- Kurtosis is the **fourth moment** (centered and scaled).

$$\kappa = \frac{1}{\sigma^4} E\left[(r-\mu)^4\right]$$

- Sample estimator:

$$\hat{\kappa} = \frac{1}{\hat{\sigma}^4} \frac{1}{N-1} \sum_{t=1}^{N} (r_t - \hat{\mu})^4$$

- Non-negative and typically expressed as **excess kurtosis**, $\kappa - 3$.
- This compares to the normal distribution, which has a kurtosis of 3.

Higher Moments Data for ETFs

Higher Moments:

ETF	Mean	Volatility	Skewness	Kurtosis
SPY	13.92%	19.51%	-0.57	12.55
VEA	7.94%	18.20%	-1.04	16.41
UPRO	36.52%	58.89%	-0.56	14.21
GLD	8.24%	13.79%	-0.22	3.39
USO	2.87%	40.96%	-1.36	16.50
CL1	-40.90%	139.29%	-27.75	931.85
FXE	0.27%	7.38%	0.06	1.03
BTC	80.42%	74.95%	-0.03	5.69
HYG	3.06%	9.38%	0.12	24.73
IEF	0.65%	6.75%	0.25	4.38
TIP	2.31%	6.22%	0.44	16.76
SHV	1.31%	0.26%	0.76	5.18

Table: Mean, Volatility, Skewness, and Kurtosis for Various ETFs

Annualizing Moments

Annualizing:

- Use frequency of data per year, τ, to annualize.

Moment	**Annualize**	**Sign**	**Quote**
Mean	τ	+ / -	%
Volatility	$\sqrt{\tau}$	+	%
Skewness	1	+ / -	Number
Kurtosis	1	+	Number

Table: Annualizing Moments

For example:

- **Daily data**: $\tau = 252$ (trading days per year).
- **Monthly data**: $\tau = 12$.

Distribution and Quantiles

Moments vs. Quantiles:

- Moments use the entire distribution, offering high statistical power.
- Quantiles estimate specific points on the distribution without assuming a specific distribution form.

Quantiles:

$$q_\pi = F^{-1}(\pi)$$

Sample Estimator:

- Sort sample data in ascending order: $\{r_{(1)}, \ldots, r_{(N)}\}$.
- Quantile estimate:

$$\hat{q}_\pi = r_{(\pi N)}$$

Median and Quantiles

Median:

- For $\pi = 0.5$, the quantile is the median:

$$\hat{q}_{0.5} = r_{\text{median}}$$

Technical Note:

- If πN is not an integer, interpolate linearly between the nearest points:

$$\hat{q}_{\pi} = \frac{r_{(\pi N-)} + r_{(\pi N+)}}{2}$$

Normal Distribution and Returns

Key Point:

- **None** of the risk measures discussed rely on the assumption of a normal distribution.
- Future estimates may depend on assuming normality and understanding how well it fits returns data.

Question:

- How well does a normal distribution approximate actual returns?

Returns vs Normal Distribution

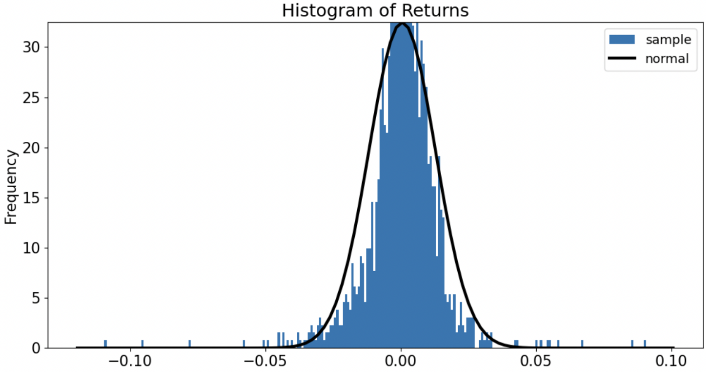


Figure: Histogram of Returns Compared to a Normal Distribution

Considering Outliers

Initial Observation:

- The histogram suggests a reasonable fit for normality.

Outlier Impact:

- A normal distribution implies extreme outliers (5 or 10 standard deviations) are extremely rare.
- In reality, many returns exhibit these extreme outliers.

Outlier Analysis

Outlier Analysis:

ETF	z Min	z Max	Normal Prob Min	Normal Prob Max
SPY	-8.95	7.33	1.81×10^{-19}	1.18×10^{-13}
VEA	-9.78	7.73	6.83×10^{-23}	5.33×10^{-15}
UPRO	-9.46	7.50	1.55×10^{-21}	3.21×10^{-14}
GLD	-6.22	5.55	2.54×10^{-10}	1.45×10^{-8}
USO	-9.81	6.45	4.88×10^{-23}	5.44×10^{-11}
CL1	-34.85	4.31	2.08×10^{-266}	8.14×10^{-6}
FXE	-4.42	4.56	4.85×10^{-6}	2.51×10^{-6}
BTC	-7.94	5.28	1.01×10^{-15}	6.45×10^{-8}
HYG	-9.32	11.05	5.87×10^{-21}	0.00×10^{0}
IEF	-5.90	6.21	1.77×10^{-9}	2.68×10^{-10}
TIP	-7.33	11.34	1.11×10^{-13}	0.00×10^{0}
SHV	-5.23	6.79	8.41×10^{-8}	5.60×10^{-12}

Table: Outliers: Min and Max z-scores with Normal Distribution Probabilities

Astronomical Probabilities

Key Insight:

- The extremely low probabilities of outliers highlight that returns are not normally distributed.
- While a normal distribution can be a good rough approximation, it fails for the extreme events crucial in risk management.

Impact of Time Frequency on Normality

Normality and Data Frequency:

- The approximation of normality can vary with the time frequency of the data.
- Higher frequency data (daily or intra-daily) may show stronger deviations.
- Over longer intervals, such deviations might average out.

Monthly Returns Example:

ETF	z Min	z Max	Normal Prob Min	Normal Prob Max
SPY	-2.80	2.38	2.54×10^{-3}	8.56×10^{-3}
VEA	-3.25	2.82	5.75×10^{-4}	2.37×10^{-3}
UPRO	-3.38	2.28	3.68×10^{-4}	1.12×10^{-2}
GLD	-2.06	2.69	1.99×10^{-2}	3.59×10^{-3}
USO	-4.49	2.80	3.58×10^{-6}	2.54×10^{-3}
CL1	-3.71	5.80	1.03×10^{-4}	3.33×10^{-9}
FXE	-2.36	2.65	9.21×10^{-3}	4.04×10^{-3}
BTC	-1.80	2.61	3.62×10^{-2}	4.58×10^{-3}
HYG	-4.19	2.63	1.38×10^{-5}	4.21×10^{-3}
IEF	-2.52	2.06	5.84×10^{-3}	1.99×10^{-2}
TIP	-4.38	2.62	6.00×10^{-6}	4.46×10^{-3}
SHV	-1.47	3.07	7.04×10^{-2}	1.08×10^{-3}

Table: Outliers in Monthly Returns with Normal Distribution Probabilities

Maximum Drawdown

Definition:

- The Maximum Drawdown (MDD) represents the largest cumulative loss from peak to trough during a given period.
- It's a critical metric for performance evaluation, indicating potential worst-case losses.

Technical Note:

- MDD is often used in backtesting to gauge the risk profile of a strategy.
- Unlike moment-based statistics, MDD is path-dependent and less precise for future forecasts.

Maximum Drawdown Example

Visual Example:

- Consider the price chart below.
- The MDD is the largest drop from a peak to the lowest point in the sample period.

Practical Insight:

- MDD provides a tangible measure of risk but may be difficult to discern directly from a price graph.

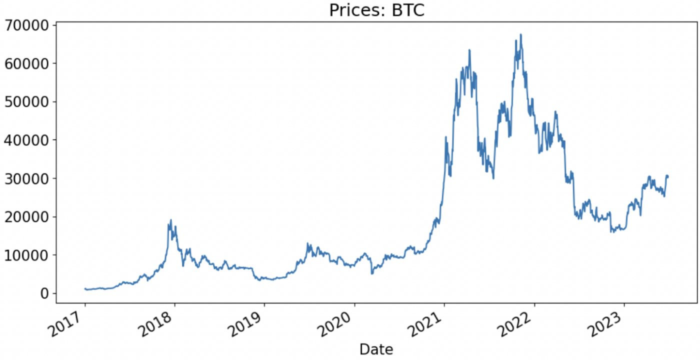
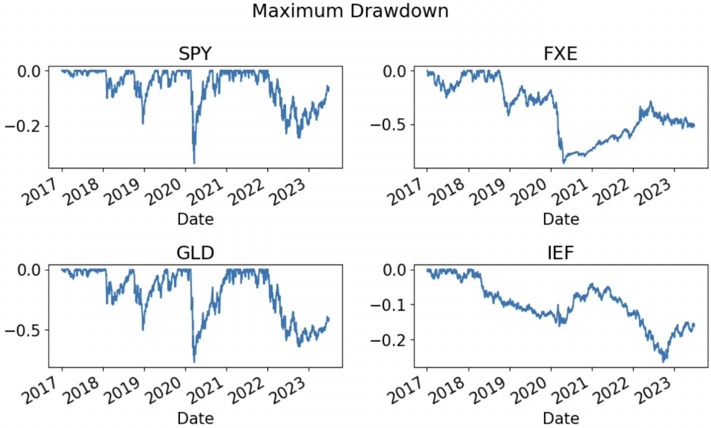


Figure: Example of Maximum Drawdown on BTC

Interpreting Maximum Drawdown (MDD):

- The MDD chart displays how far the strategy is below its historical maximum for any point in time.
- A value of 0 indicates the strategy is at a new peak.



Multivariable Risk Analysis

Overview:

- Previous risk measures were **univariate**, focusing on single return distributions.
- We will now explore **multivariable** measures necessary for portfolio risk analysis.

Notation:

- Returns on assets i and j are denoted as r_i and r_j respectively.
- Here, the superscripts are identifiers, not exponents.

Covariance: Understanding Asset Relationships

Definition:

$$\sigma_{i,j} = E\left[(r_{i,t} - \mu_i)(r_{j,t} - \mu_j)\right]$$

- Covariance measures how two assets move together.
- A variable's covariance with itself is its variance.

Sample Estimator:

$$\hat{\sigma}_{i,j} = \frac{1}{N}\sum_{t=1}^{N}(r_{i,t} - \hat{\mu}_i)(r_{j,t} - \hat{\mu}_j)$$

Covariance Matrix: A Multivariable Approach

Covariance Matrix:

- For K assets, the covariance is represented in a matrix form:

$$\Sigma_{i,j} = \mathbf{\Sigma} = E\left[(\boldsymbol{r} - \boldsymbol{\mu})(\boldsymbol{r} - \boldsymbol{\mu})'\right]$$

- The diagonal elements represent variances, and off-diagonal elements represent covariances.

Sample Estimator:

$$\hat{\Sigma} = (\boldsymbol{R} - \hat{\boldsymbol{\mu}})(\boldsymbol{R} - \hat{\boldsymbol{\mu}})' \left(\frac{1}{N - K}\right)$$

- $\hat{\boldsymbol{\mu}}$ is the vector of sample means.
- $\boldsymbol{R}$ is the matrix of returns.

Technical Properties of the Covariance Matrix

Key Properties:

- The covariance matrix Σ is symmetric with $K(K+1)/2$ unique elements.
- It is also positive (semi-)definite:

$$w'\Sigma w \geq 0 \quad \text{for any vector } w$$

- This implies that any linear combination of assets will have a non-negative variance.

Correlation: Scaling Covariance

Definition:

- Correlation rescales covariance to lie between -1 and 1, facilitating easier interpretation:

$$\rho_{i,j} = \frac{\sigma_{i,j}}{\sigma_i \sigma_j}$$

- It retains the sign of the covariance and indicates the strength of the relationship.

Correlation Matrix:

- Similar to the covariance matrix, the correlation matrix is positive semi-definite.

Correlation Matrix Analysis

Correlation Matrix for Various Assets:

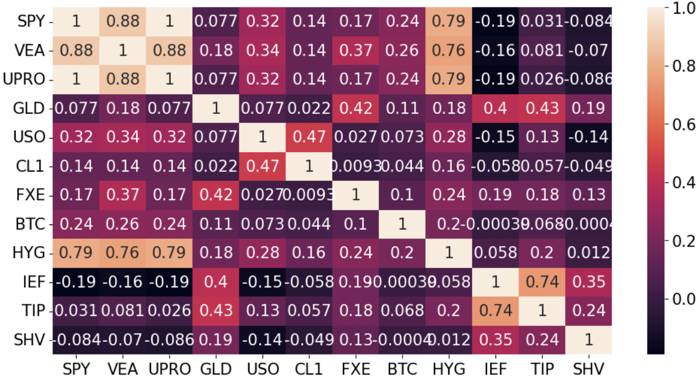


Figure: Correlation Matrix for Various Assets (2017-2023)

Understanding Beta: Linear Decomposition

Beta Definition:

- Beta measures the sensitivity of asset i to asset j.
- Linear regression model:

$$r_{i,t} = \alpha + \beta r_{j,t} + \epsilon_t$$

OLS Estimator:

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta} \end{bmatrix} = (\boldsymbol{R}_j' \boldsymbol{R}_j)^{-1} \boldsymbol{R}_j' \boldsymbol{r}_i$$

- $\boldsymbol{R}_j$ is an $N \times 2$ matrix with a vector of ones and sample returns of $r_{j,t}$.

Scaled Correlation and Beta

Beta as Scaled Correlation:

- Beta can also be defined through scaled correlation:

$$\beta = \frac{\sigma_i}{\sigma_j} \rho_{i,j}$$

- It uses the standard deviations and correlation of the assets.

Comparison:

- Covariance, correlation, and beta are three perspectives on asset relationships.

Data Comparison:

Asset	Correlation with SPY	Covariance with SPY	Beta
SPY	100.00%	3.81%	1.00
VEA	88.36%	3.14%	0.82
UPRO	99.92%	11.48%	3.02
GLD	7.69%	0.21%	0.05
USO	32.19%	2.57%	0.68
CL1	14.01%	3.81%	1.00
FXE	17.04%	0.25%	0.06
BTC	24.24%	3.54%	0.93
HYG	79.38%	1.45%	0.38
IEF	-18.82%	-0.25%	-0.07
TIP	3.14%	0.04%	0.01
SHV	-8.38%	-0.00%	-0.00

Table: Comparison of Correlation, Covariance, and Beta with SPY

Multivariate Beta

Multivariate Regression:

- Extends the single-variable beta to multiple variables:

$$r_{i,t} = \alpha + \beta_j r_{j,t} + \beta_k r_{k,t} + \epsilon_t$$

OLS Estimator:

$$\begin{bmatrix} \hat{\alpha} \\ \hat{\beta}_j \\ \hat{\beta}_k \end{bmatrix} = (\boldsymbol{R}'\boldsymbol{R})^{-1}\boldsymbol{R}'\boldsymbol{r}_i$$

- $\boldsymbol{R}$ is the matrix with columns of ones, $r_{j,t}$, and $r_{k,t}$.

Comparison of Multivariate and Univariate Beta

Comparison Table:

Asset	Multivariate (SPY, VEA)	Univariate
SPY	1.00	1.00
VEA	0.00	0.82
UPRO	3.02	3.02
GLD	-0.28	0.05
USO	0.17	0.68
CL1	0.56	1.00
FXE	-0.27	0.06
BTC	0.26	0.93
HYG	0.26	0.38
IEF	-0.07	-0.07
TIP	-0.06	0.01
SHV	-0.00	-0.00

Table: Comparison of Multivariate and Univariate Beta

Coherent Risk Measures

Defining Risk Measures:

- A risk measure ϱ estimates the potential dollar amount of losses, or capital required to protect against these losses.

Different Risk Measures:

- Examples include volatility, variance, and quantiles.
- Negative $\varrho(\Pi_j)$ implies capital can be withdrawn.

Risk Capital

Risk Capital:

- Refers to the capital needed to cover potential losses.
- Risk measures help allocate a capital budget for different portfolios.

Properties of a Coherent Risk Measure

Coherence Criteria:

- **Translation Invariance:**

$$\varrho(\alpha + \Pi_j) = \varrho(\Pi_j) - \alpha$$

- **Positive Homogeneity:**

$$\varrho(\lambda \Pi_j) = \lambda \varrho(\Pi_j) \quad \text{for } \lambda > 0$$

- **Monotonicity:**

$$\text{If } \Pi_j > \Pi_k, \text{ then } \varrho(\Pi_j) < \varrho(\Pi_k)$$

- **Subadditivity:**

$$\varrho(\Pi_j + \Pi_k) \leq \varrho(\Pi_j) + \varrho(\Pi_k)$$

Convex Risk Measure:

- Combines subadditivity and positive homogeneity into one condition:

$$\varrho(\lambda\Pi_j + (1-\lambda)\Pi_k) \leq \lambda\varrho(\Pi_j) + (1-\lambda)\varrho(\Pi_k)$$

- $\lambda \in [0, 1]$

Which Measures Are Coherent?

- Standard deviation
- Variance
- 5th quantile of the loss distribution
- Maximum drawdown

Data for Evaluation:

Asset	Std Dev	Variance	Quantile 0.05	MDD
SPY	1.23%	0.02%	-1.85%	-33.72%
UPRO	3.71%	0.14%	-5.65%	-76.82%
Portfolio	4.94%	0.24%	-7.50%	-87.79%
Sum of Parts	4.94%	0.24%	-7.50%	-110.53%

Table: Comparison of Risk Measures

Dynamic Risk Measures

Static vs. Conditional Risk:

- Static measures assume iid returns.
- Conditional measures account for time-varying risk, such as volatility.

Conditional Risk Measures:

- Techniques include:
  - Rolling window
  - Stochastic volatility models (GARCH)
  - Simulation

Value at Risk (VaR)

Definition:

- **Value-at-Risk (VaR)** is a quantile of the distribution.
- For a τ-day period and q quantile, VaR of an asset or portfolio is defined as $\Pi^{\mathrm{VaR}_{q,\tau}}$ where:

$$\Pr\left(\Pi_{t,t+\tau} \leq \Pi_t^{\mathrm{VaR}_{q,\tau}}\right) = q$$

- Π represents the dollar amount of the profit and loss:

$$\Pi = \Pi_{t+\tau} - \Pi_t$$

- Π is the value of the portfolio.

Interpretation:

- There is a probability of q that over τ days, the portfolio PnL will be less than $\Pi^{\mathrm{VaR}_{q,\tau}}$.
- Alternatively:

$$\Pr\left(L_{t,t+\tau} > L^{\mathrm{VaR}_{q,\tau}}\right) = q$$

VaR for Returns

Definition:

- Discuss the VaR in terms of return $r^{\mathrm{VaR}_{q,\tau}}$:

$$\Pr\left(r_{t,t+\tau} \leq r_t^{\mathrm{VaR}_{q,\tau}}\right) = q$$

- Relationship between PnL and return VaR:

$$\Pi_t^{\mathrm{VaR}_{q,\tau}} = r_t^{\mathrm{VaR}_{q,\tau}} \cdot \Pi_t$$

Profits vs. Losses:

- $\Pi^{\mathrm{VaR}_{q,\tau}}$ and $r^{\mathrm{VaR}_{q,\tau}}$ are negative numbers for interesting cases.
- VaR can be defined in terms of the left tail of profit or the right tail of losses.
- Focus on returns and PnL; often use absolute values for clarity.

VaR Distribution

Distribution of PnL:

- If PnL has a cumulative density function (cdf) that is continuous and strictly increasing, then:

$$\Pi^{\mathrm{VaR}_{q,\tau}} = \Phi_{\Pi}^{-1}(q)$$

- Φ_{Π} denotes the cdf of returns, and Φ_{Π}^{-1} is its inverse.

For Returns:

$$r^{\mathrm{VaR}_{q,\tau}} = \Phi_{r}^{-1}(q)$$

Technical Note:

- If the returns' distribution lacks such a cdf, a general definition is required:

$$\Pi^{\mathrm{VaR}_{q,\tau}} = \sup \{\Pi \mid \Phi_{\Pi}(\Pi) \leq q\}$$

- Similarly for returns, $r^{\mathrm{VaR}_{q,\tau}}$.

Normal Distribution and VaR

Using Normal Distribution:

- VaR is often modeled using a normal distribution.
- Even if returns are not normally distributed, normal VaR formulas are common for quoting and comparing VaR.

Assumption:

- Assume returns are normally distributed:

$$r_{t,t+\tau} \sim \mathcal{N}\left(\mu_\tau, \sigma_\tau^2\right)$$

- Mean and variance depend on the return horizon τ.

Normal Formula for VaR

Formula:

- For normally distributed returns:

$$r^{\mathrm{VaR}_{q,\tau}} = \mu_\tau + \mathtt{z}_q \sigma_\tau$$

- $\mathtt{z}_q$ is the z-score for the quantile q:

$$\Phi_{\mathtt{z}}(\mathtt{z}_q) = q$$

- Using:

$$\Pi_t^{\mathrm{VaR}_{q,\tau}} = r_t^{\mathrm{VaR}_{q,\tau}} \cdot \Pi_t$$

Lognormal Distribution and VaR

Lognormal Assumption:

- Assume prices are lognormal, making log returns normal with one-period mean μ and variance σ^2.
- For τ-period horizon:

$$\mathtt{r}_{t,t+\tau} \sim \mathcal{N}\left(\tau\mu, \tau\sigma^2\right)$$

Under Assumptions:

- Assuming log returns are iid and $\mu = 0$:

$$\mathtt{r}^{\mathrm{VaR}_{q,\tau}} = \sqrt{\tau}\,\mathtt{r}^{\mathrm{VaR}_{q,1}}$$

Implication:

- VaR scales with the square root of the horizon.
- Commonly used as a convention despite real-world deviations.

VaR for Log Returns

Log Returns Formula:

- For log returns with normal distribution, the VaR for profits is:

$$\Pi^{\mathrm{VaR}_{q,\tau}} = \Pi_t \left[\exp\left\{ \tau\mu + \mathsf{z}_q \sqrt{\sigma^2} \right\} - 1 \right]$$

Technical Note:

- Different formula for log returns; PnL may no longer be normally distributed.

Conditional Value-at-Risk (CVaR)

Definition:

- **Conditional VaR (CVaR)** or **Expected Shortfall (ES)** is the expected value conditional on being less than the VaR threshold.

$$\Pi^{\mathrm{CVaR}_{q,\tau}} = E\left[\Pi_{t,t+\tau} \,\middle|\, \Pi_{t,t+\tau} < \Pi^{\mathrm{VaR}_{q,\tau}}\right]$$

- Similarly for returns:

$$r^{\mathrm{CVaR}_{q,\tau}} = E\left[r_{t,t+\tau} \,\middle|\, r_{t,t+\tau} < r^{\mathrm{VaR}_{q,\tau}}\right]$$

Relationship:

$$\Pi_t^{\mathrm{CVaR}_{q,\tau}} = r_t^{\mathrm{CVaR}_{q,\tau}} \cdot \Pi_t$$

Normal Distribution for CVaR

Normal Distribution:

- Assume returns are normally distributed, CVaR is the conditional expectation:

$$r_t^{\mathrm{CVaR}_{q,\tau}} = \mu_{\tau,t} - \frac{\phi_{\mathtt{z}}(\mathtt{z}_q)}{q}\sigma_{\tau,t}$$

Technical Point:

- Derived from the truncated normal distribution, conditional expectation of a normal distribution.
- For any threshold a:

$$E\left[\mathtt{z} \,|\, \mathtt{z} < a\right] = -\frac{\phi_{\mathtt{z}}(a)}{\Phi_{\mathtt{z}}(a)}$$

Sufficient Statistics of a Normal Distribution

Key Insight:

- VaR and CVaR formulas for normal distribution:

$$r^{\mathrm{VaR}_{q,\tau}} = \mu_{\tau,t} + \mathtt{z}_q \sigma_{\tau,t}$$

- CVaR:

$$r^{\mathrm{CVaR}_{q,\tau}} = \mu_{\tau,t} + \left(- \frac{\phi_{\mathtt{z}(\mathtt{z}_q)}}{q} \right) \sigma_{\tau,t}$$

Implication:

- Normal distribution characterized by mean and variance (sufficient statistics).
- Any statistic can be rewritten as a function of μ and σ.

Is VaR Coherent?

- VaR is not a coherent risk measure.
- It generally fails to meet the subadditivity criterion.

Payoff Outcomes:

- Consider the payoffs for Π_i and Π_j across 5 potential states of the world:

$$\Pi_i \in \begin{bmatrix} -2 \\ -1 \\ 0 \\ 1 \\ 2 \end{bmatrix}, \quad \Pi_j \in \begin{bmatrix} -1 \\ -2 \\ 0 \\ 1 \\ 2 \end{bmatrix}$$

- The states are ordered, such that in state s, both securities deliver the payoff in row s of their respective arrays.
- Portfolio payoff is the sum: $\Pi_p = \Pi_i + \Pi_j$.

VaR Calculation:

$$\Pi_i^{\mathrm{VaR}.2}, \quad \Pi_j^{\mathrm{VaR}.2}, \quad \Pi_i^{\mathrm{VaR}.2} + \Pi_j^{\mathrm{VaR}.2}, \quad \Pi_p^{\mathrm{VaR}.2}$$

Example: Returns Not Subadditive

Example Setup:

- Consider 100 potential outcomes (”states”).
- Two assets, r_i and r_j.
- In almost every state, the return is:

$$r_{s,i} = r_{s,j} = \frac{s - 50}{100}$$

- Exception:

$$r_{i,(5)} = -0.45, \quad r_{j,(5)} = -0.46$$
$$r_{i,(4)} = -0.46, \quad r_{j,(4)} = -0.45$$

VaR Calculation:

- For individual returns:

$$r_i^{\text{VaR}.05} = -0.45 = r_j^{\text{VaR}.05}$$

- For the equal-weighted portfolio:

$$r_p^{\text{VaR}.05} = -0.455$$

Issue with VaR

Sensitivity to Ordering:

- VaR is sensitive to the ordering around the specified quantile.
- If two series do not share the timing of returns around the q threshold, discrepancies may occur.
- Example: Worst and second-worst dates swapped in two series can change VaR.

Data Example

Real-World Example:

- Similar issues observed with SPY and UPRO across 2017-2023.
- VaR differences highlight the sensitivity to the distribution of returns.

Normal VaR and Coherence

Normal VaR:

- Is **Normal VaR** coherent?
- It often fails the subadditivity property, hence not coherent.

CVaR Coherence:

- Is CVaR coherent?
- For a normal distribution?
- For the examples above?

Conditional Moments

Dynamic VaR:

- VaR should be dynamic across time, using information up to time t.
- Conditional VaR utilizes this dynamic modeling.

Importance of Conditional Measures:

- Conditional risk statistics provide a broader view than unconditional measures.

Ignoring the Mean

Why Ignore the Mean?

- Typically, calculations focus on the second moment, not the centered second moment.
- Ignoring the mean simplifies the estimation.

Model for the Mean:

- Including the mean is valid if necessary, adjusting for degrees of freedom.
- Consider residuals after demeaning in calculations.

Volatility Estimates

Notation:

- σ_t^2 is estimated using data up to $t-1$ to estimate next period's variance.

Expanding Volatility:

$$\sigma_t^2 = \frac{1}{t-1}\sum_{i=1}^{t-1} r_i^2$$

Rolling Volatility:

$$\sigma_t^2 = \frac{1}{m}\sum_{i=1}^{m} r_{t-i}^2$$

EWMA for Volatility

Exponential Weighted Moving Average (EWMA):

- EWMA assigns more weight to recent data with a decay parameter λ (typically $\lambda \in (.9, 1)$).

$$\sigma_t^2 = \sum_{i=1}^{N-1} \lambda^{(i-1)} r_{t-i}^2$$

IGARCH and GARCH

IGARCH:

- Integrated GARCH:

$$\sigma_t^2 = \lambda\sigma_{t-1}^2 + (1-\lambda)r_{t-1}^2$$

- In the limit, similar to EWMA.

GARCH:

- Generalized Autoregressive Conditional Heteroskedasticity:

$$\sigma_t^2 = \alpha_0 + \gamma_1\sigma_{t-1}^2 + \alpha_1 r_{t-1}^2$$

- Unlike EWMA, parameters do not need to sum to 1 and can include lags.

Evaluating VaR

Hit Test:

- Common method to backtest VaR.
- Checks historical daily VaR against actual performance.

Consideration:

- Future market conditions may differ from past environments.
- Many VaR models appeared adequate before the 2007-2008 crisis.

Simulation Methods for VaR

Overview:

- Historical Simulation
- Monte Carlo Simulation
- Other Approaches

Historical Simulation

Empirical CDF:

- Historical simulation is widely used for calculating VaR.
- It does not assume normally distributed losses.
- This method is unconditional and assumes independent observations.
- The cdf is constructed from the histogram of past losses.
- VaR is calculated as the empirical quantile.

CVaR Calculation

Conditional Value-at-Risk (CVaR):

- CVaR is obtained using order statistics to build the cdf:

$$r^{\mathrm{CVaR}_{q,1}} = \frac{1}{qN}\sum_{i=1}^{qN} r_{(i)}$$

- This represents the sample mean of the left tail of the empirical cdf.

Advantages of Historical Simulation

Key Advantages:

- Nonparametric estimation of cdf based on historical data.
- Flexibility in not assuming a predefined probability distribution.
- Easy implementation and adaptability to actual data shapes.

Disadvantages of Historical Simulation

Statistical Power:

- VaR estimation requires accurate tail distribution.
- Small sample sizes result in large standard errors for order statistics.
- For extreme conditions, large sample sizes are necessary to obtain precise estimates.

Dynamic Assumptions:

- Assumes returns are iid, which is often not true.
- Large samples may include irrelevant historical data, while small samples lack precision.

Monte Carlo Simulation

Overview:

- Generates data based on a statistical model.
- Each simulated observation is used to construct portfolio loss.
- An empirical cdf is built from these simulated losses to calculate VaR.

Applications:

- Historical approach skips data generation by using past observations.
- Useful for complex dynamics and nonlinear valuations where keeping cdf tractable is difficult.

Simulating Bonds and Options

Applications:

- Useful for portfolios with nonlinear value dependencies.
- Example: Simulate stock prices and use Black-Scholes for options to obtain losses.
- Simulate interest rates and calculate bond portfolio losses as a nonlinear function of these rates.

Other Approaches for VaR

Alternative Distributions:

- **Student's t Distribution:**
  - Provides fatter tails than normal distribution.
  - Less useful for quoting and more computationally intensive.
- **Extreme Value Theory:**
  - Focuses on modeling the tail of the distribution.
  - Uses mathematical results to estimate extreme values.
- **Quantile Regression:**
  - Estimates quantiles using conditioning information.
  - Offers a more direct approach to estimate extreme risks.

Historic Simulation: Empirical CDF

Introduction:

- Historical simulation is straightforward and commonly used for VaR.
- It relies on past data without assuming a specific distribution.
- Produces an empirical cdf from past losses to estimate VaR.

Calculating VaR:

- VaR is calculated as an empirical quantile of the historical loss distribution.

Calculating CVaR with Empirical CDF

Procedure:

- Use order statistics to determine the cdf:

$$r^{\mathrm{CVaR}_{q,1}} = \frac{1}{qN} \sum_{i=1}^{qN} r_{(i)}$$

- Represents the sample mean of the left tail.

Advantages of Historical Simulation

Benefits:

- Estimates cdf nonparametrically, reflecting actual probabilities.
- Flexibility in estimating cdf shapes based on historical data.
- Simple implementation without pre-assumed distribution.

Disadvantages of Historical Simulation

Statistical Limitations:

- Requires accurate estimation of tail distribution.
- Small samples lead to high standard errors.
- For precise tail estimates, large samples are necessary.

Dynamic Issues:

- Assumes returns are iid, which is often incorrect.
- Large samples may include outdated data, while small samples lack precision.

Monte Carlo Simulation

Overview:

- Generates data based on statistical models.
- Constructs portfolio losses for each simulated observation.
- Builds an empirical cdf from simulated losses for VaR calculation.

Applications:

- Historical approach uses past data as simulated observations.
- Ideal for scenarios with complex dynamics and nonlinear valuations.

Monte Carlo for Bonds and Options

Use Cases:

- Applicable for portfolios with nonlinear valuation dependencies.
- Simulate stock prices and use models like Black-Scholes for options.
- Simulate interest rates to determine bond portfolio losses.

Alternative Distributions for VaR

Other Approaches:

- **Student's t Distribution:**
  - Offers fatter tails than a normal distribution.
  - Involves more computation, less useful for quoting.
- **Extreme Value Theory:**
  - Models the extreme values of the distribution.
  - Focuses on the tail rather than the entire distribution.
- **Quantile Regression:**
  - Estimates quantiles using conditioning information.
  - Provides a more direct estimation of extreme risks.

Thank You! Questions or Comments? Contact:

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