Morse Theory

The goals of this reports.
First, We study the real value function on M, and define the Morse function. Then prove the existence of Morse function on any smooth real manifold.
Second, we study the critical point of Morse function and we can determine the homotopy type of M by the local behavior of the Morse function near the critical point.
Third, We use the morse theory to Riemannion geometry and focus on the space of curve on the manifold. Define the topology on the space of curve on manifold. Then prove the Morse index theorem.
Last, We want to prove the Bott Periodicity theorem. In this part we will review some Algebraic topology result and property of $U(n)$.

1.Definition and existence of morse function

Let $f: M \to \mathbb{R}$ f be a smooth function, at each p we can induce $df_p: T_pM \longrightarrow T_{f(p)}\mathbb{R}$

Def 1.1: $P \in M$ is a critical point of f if $df_p \equiv 0$. Which means in local coordinate $(x^1, \cdots, x^n)$ $\frac{\partial f}{\partial x^1}(p) = \frac{\partial f}{\partial x^2}(p) = \cdots = \frac{\partial f}{\partial x^n}(p) = 0$.

Def 1.2: $H_f(p)$ can be defined as $H_f(p): M_p \times M_p \longrightarrow \mathbb{R}$ $\quad X, Y \in \mathfrak{X}(M)$ and

$(X(p), Y(p)) \longmapsto X(Yf)(p)$

$H_f(p)$ only depend on $X(p)$, $Y(p)$ and $H_f(p)$ is bilinear and $H_f(p) = \frac{\partial^2 f}{\partial x_i \partial x_j}$ in local coordinate.

(pf) From $([\tilde{X}, \tilde{Y}])f(p) = \tilde{X}(\tilde{Y}f)(p) - \tilde{Y}(\tilde{X}f)(p) = df([\tilde{X}, \tilde{Y}])_p = 0$ $\because df = 0$ p is critical point,

$\Rightarrow X(\tilde{Y}f)(p) = Y(\tilde{X}f)(p)$ $\because X(\tilde{Y}f)(p)$ only depend on $X(p)$ and $Y(\tilde{X}f)(p)$ depend on $Y(p)$

$\Rightarrow X(\tilde{Y}f)(p)$ only depend on $X(p)$ and $Y(p)$ and such definition is bilinear.

In local chart $(x^1, x^2, x^3 \cdots x^n)$ $V = \sum a_i \frac{\partial}{\partial x^i}|_p$ $W = \sum b^j \frac{\partial}{\partial x^j}|_p$ Then we take extension $\tilde{W} = \sum b_j \frac{\partial}{\partial x^j}$ with b_j constant, $H_f(p)(V, W) = \sum a_i \frac{\partial}{\partial x^i}\left(\sum b_j \frac{\partial f}{\partial x^j}\right)(p)$

$$= \sum_{i,j} a_i b_j \frac{\partial^2 f}{\partial x_i \partial x^j}(p)$$

Then we can see that $H_f(p)$ can be reprensented by $\left(\frac{\partial^2 f}{\partial x^i \partial x^j}(p)\right)$.

Def 1.3: The index of bilinear form H is the maximal dimension of subspace of V on which H is negative definite

Def 1.4: A map $f: M \longrightarrow \mathbb{R}$ is a Morse function if all critical points of f are nondegenerate. (If $H_f(p)$ at each critical point are non-singular).

Next, we want to know how many morse function on M.

We first prove that the existence of morse function.

lemma 1.1: Let $M \subset \mathbb{R}^n$ be a submanifold. For almost any $P \in \mathbb{R}^n$, the function $f_p: M \to \mathbb{R}$ $x \to \|x - p\|^2$ is a morse function.

pf: $df_{p,x}(V) = 2(x-P, V)$. When x is critical point means T_xM normal to $x-P$

let $(u_1, u_2 \cdots, u_d)$ be local coordinate near x.

$\frac{\partial f}{\partial u_i} = 2(x-P) \cdot \frac{\partial x}{\partial u_i}$ $\quad \frac{\partial^2 f}{\partial u_i \partial u_j} = 2\left(\frac{\partial x}{\partial u_i} \cdot \frac{\partial x}{\partial u_j} + (x-P) \cdot \frac{\partial^2 x}{\partial u_i \partial u_j}\right)$

The point x is nondegenerate critical point $\Leftrightarrow$ $x-P \perp T_xM$ and $\frac{\partial^2 f}{\partial u_i \partial u_j}$ has rank d.

To show f_p is Morse function for almost all P in M,

$\Leftrightarrow$ P doesn't satisfy the condition are critical values of a C^∞ map and apply Sard's thm.

$N = \{(x, v) \in U \times \mathbb{R}^n \mid v \perp T_xU\} \subset \times \mathbb{R}^n$.

define a map $E: N \longrightarrow \mathbb{R}^n$

$(x, v) \longmapsto x + v$

claim: N is submanifold of $U\times\mathbb{R}^n$ and $P=X+V\in\mathbb{R}^n$ is critical value of E iff $\frac{\partial^2 f}{\partial u_i \partial u_j}=2\left(\frac{\partial X}{\partial u_i}\cdot\frac{\partial X}{\partial u_j}-V\cdot\frac{\partial^2 X}{\partial u_i\partial u_j}\right)$ is non-invertible.

(pf) $\because$ U is submanifold of $\mathbb{R}^n$ so we have chart to send $\mathbb{R}^n$ on to the open subset of $\mathbb{R}^d$. And the tangent map of chart send $\mathbb{R}^n$ onto the basis of tangent space of U. Then by orthonormal complement, we can have $V_1,\cdots,V_{n-d}$ as the orthonormal basis of $(T_xU)^\perp$.

$$\Rightarrow (u_1,\cdots u_d,t_1,\cdots,t_{n-d})\longrightarrow \left(X(u_1,\cdots,u_d),\sum_{i=1}^{n-d}t_iV_i(u_1,\cdots,u_d)\right)$$

We can see that N be a submanifold of $U\times\mathbb{R}^n$.

$$\frac{\partial E}{\partial u_i}=\frac{\partial X}{\partial u_i}+\sum_{k=1}^{n-d}t_k\frac{\partial V_k}{\partial u_i}\qquad \frac{\partial E}{\partial t_i}=V_i$$

$$dE=\begin{pmatrix}\frac{\partial X}{\partial u_i}\cdot\frac{\partial X}{\partial u_j}+\sum_{k=1}^{n-d}t_k\frac{\partial V_k}{\partial u_j}\cdot\frac{\partial X}{\partial u_j} & \sum_{k=1}^{n-d}t_k\frac{\partial V_k}{\partial u_i}\cdot V_l\\ 0 & Id\end{pmatrix}\qquad \because \frac{\partial X}{\partial u_i}\perp V_l$$

$\Rightarrow$ $\because$ $V_k\cdot\frac{\partial X}{\partial u_j}=0$ $\quad \frac{\partial}{\partial u_i}\left(V_k\cdot\frac{\partial X}{\partial u_j}\right)=0=\frac{\partial V_k}{\partial u_i}\cdot\frac{\partial X}{\partial u_j}+V_k\cdot\frac{\partial^2 X}{\partial u_i\partial u_j}$

$\Rightarrow$ $P=X+V\in\mathbb{R}^n$ is critical value iff $\frac{\partial X}{\partial u_i}\cdot\frac{\partial X}{\partial u_j}-\sum_{k=1}^{n-d}t_kV_k\cdot\frac{\partial^2 X}{\partial u_i\partial u_j}$ is singular

$\Leftrightarrow \frac{\partial X}{\partial u_i}\cdot\frac{\partial X}{\partial u_j}+(X-P)\cdot\frac{\partial^2 X}{\partial u_i\partial u_j}$ is singular.

$\square$

Therefore the set of all f_p (with p vary in $\mathbb{R}^n$) which are not Morse function correspond to the subset of critical values of E which by Sard's thm. has measure zero in $\mathbb{R}^n$. $\Rightarrow$ for almost all p, f_p is Morse function. $\square$

Remark1: By strong Whitney embeding thm, we know that any smooth m-dim (real) manifold can be smoothly embeded in the real $2m$-space $\mathbb{R}^{2m}$, So for the general real smooth manifold, we have a lot of morse function on M.

Remark2.: In the proof, we have used the sard's thm. The statement of Sard thm is $f:N\to M$ f is a C^k function and M, N are differentiable manifold with dim m and n, respectively. Let X be the set that df has rank less than m, then $Im(X)$ is measure zero in M.

And the following is some results without proof.

Result 1.: M be manifold, $f:M\to\mathbb{R}$ smooth. Let $k\in\mathbb{N}$, Then on any compact set of M. f can be approximated by Morse functions in C^k-norm.

2. If M is compact, the set of Morse function is dense open subset in $C^\infty(M)$.

2.Morse theory：analyzing the critical points of morse function to determine the homotopy type of manifold

Because we want to study the critical point P, so we want to choose an good coordinate near P.

Morse lemma: If P is non-degenerate critical point of f and index of f at p is λ, then $\exists$ coordinate $(y^1, \cdots, y^n)$ in nbd U of P with $y^i(p)=0$ and $f = f(p) - (y^1)^2 - \cdots - (y^\lambda)^2 + (y^{\lambda+1})^2 + \cdots + (y^n)^2$ on whole U.

sublemma: f is smooth function in a convex nbd V of 0 in $\mathbb{R}^n$, with $f(0)=0$.

Then $f(x_1, \cdots, x_n) = \sum_{i=1}^n x_i g_i(x_1, \cdots, x_n)$

(pf) $f(x_1, \cdots, x_n) = \int_0^1 \frac{d f(x_1 t, \cdots, x_n t)}{dt} dt = \int_0^1 \sum_{i=1}^n \frac{\partial f}{\partial x_i}(tx_1, \cdots, tx_n) x_i \, dt$

$\Rightarrow$ define $g_i(x_1, \cdots, x_n) = \int_0^1 \frac{\partial f}{\partial x_i}(tx_1, \cdots, tx_n)\, dt$. □

(proof of Morse lemma) By linear alg $\Rightarrow$ any expression of f, λ must be the index of f at p.

We replace $f \to f - f(p)$ then we have $f(0)=0$, and assume P be 0 of $\mathbb{R}^n$.

$\Rightarrow f(x_1, \cdots x_n) = \sum_{i=1}^n x_i g_i(x_1, \cdots x_n)$ by sublemma. (let $(x_1, \cdots, x_n)$ be nbd of $p=0$.)

$\left(\because \frac{\partial f}{\partial x_i}(0) = g_i(0) = 0\right)$ Use sublemma again $\Rightarrow g_i(x_1, \cdots, x_n) = \sum_{j=1}^n x_j h_{ij}(x_1, \cdots, x_n)$

$h_{ij}(0) = \frac{\partial g_i(0)}{\partial x_j} = \int_0^1 \frac{\partial^2 f}{\partial x_i \partial x_j}(tx_1, \cdots, tx_n) t \, dt \Big|_0 = \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}(0)$

$\Rightarrow f(x_1, \cdots, x_n) = \sum_{i,j=1}^n x_i x_j h_{ij}(x_1, \cdots, x_n)$. and $\bar{h}_{ij}(0) = \frac{1}{2} \frac{\partial^2 f}{\partial x_i \partial x_j}$

let $\bar{h}_{ij} = \frac{1}{2}(h_{ij} + h_{ji}) \Rightarrow f(x_1, \cdots, x_n) = \sum_{i,j=1}^n x_i x_j \bar{h}_{ij}(x_1, \cdots x_n)$ (with $\bar{h}_{ij} = \bar{h}_{ji}$)

Then we prove by induction. Suppose $\exists (u_1, \cdots, u_n)$ in a nbd U_1 of 0 st

$f = \pm (u_1)^2 \pm \cdots \pm (u_{k-1})^2 + \sum_{i,j \geq k} H_{ij}(u_1, \cdots, u_n)$ throughout U_1, H_{ij} are symmetric, and nonsingular

WLOG we can use cooridnate change to let $H_{kk}(0) \neq 0$.

let $L(u_1, \cdots u_n) = \sqrt{|H_{kk}(u_1, u_2, \cdots, u_n)|}$ Then in the smaller $U_2 \subseteq U_1$ of 0.

We can introduce new coordinate $(v_1, \cdots, v_n)$ by

$v_i = u_i$ for $i \neq k$ $\quad v_k = L \cdot [u_k + \frac{1}{H_{kk}} \sum_{i>} u_i H_{ik}]$

By IFT $(v_1, \cdots, v_n)$ can serve as a coordinate with U_3 be nbd of 0. $U_3 \subseteq U_2$.

$\Rightarrow f = \sum_{i \leq k} \pm v_i^2 + \sum_{i,j > k} v_i v_j H_{ij}'$ throughout U_3.

Then by induction we can complete the proof. □

Cor 2.1. Non degenerate critical points are isolated.

(pf) Use Morse lemma, f can locally written as $x_1^2 + \cdots x_{n-\lambda}^2 - y_1^2 - \cdots - y_\lambda^2$ then the critical point of f, we can let ϕ be the chart then $f \circ \phi$ has critical point only at 0. $\Rightarrow f$ has only one critical point p in the nbd of p. #

Then we begin to study the relation between critical points and homotopy type.

Def 2.1: $M^a = \{p \in M \mid f(p) \le a\}$ which $f: M \to \mathbb{R}$ is smooth.

Def 2.2: A subspace A of X is called deformation retract of X if $\exists F: X \times I \to X$ st $\forall x \in X$ $a \in A$ (1) $F(x,0) = x$ (2) $F(x,1) \in A$ (3) $F(a,1) = a$.

Thm 2.1. If $f: M \to \mathbb{R}$ is smooth on M, $a \le b$ and $f^{-1}[a,b]$ is compact and contains no critical points of f, then M^a is diffeomorphic to M^b. In fact, M^a is a deformation retract of M^b.

(pf) The idea is pull M^a down M^b along the orthogonal trajectories of hypersurface of f = constant.

$\langle X, \nabla f \rangle = X(f)$ for any vector field X. ∇f vanish only at critical point of f, and for a curve $C: \mathbb{R} \to M$ $\langle \frac{dc}{dt}, \nabla f \rangle = \frac{dc}{dt}(f) = \frac{d(c \circ f)}{dt}$.

Then we can define $\rho: M \to \mathbb{R}$ $\rho(x) = \frac{1}{\langle \nabla f, \nabla f \rangle_x}$ thorough all $f^{-1}[a,b]$, and vanished outside a cpt nbd of $f^{-1}[a,b]$.

Then define $X_q = \rho(q) \nabla f_q$. Claim X_q generate the 1-parameter group of diffeomorphism φ_t.

lemma: smooth vector field on M vanishes outside of a cpt set $K \subset M$. generate a unique 1-parameter group of diffeomorphism of M.

(pf): Let φ be the 1-parameter group of diffeomorphism generated by X, Then for fixed $q \in M$ the curve $t \to \varphi_t(q)$ satisfity $\frac{d\varphi_t(q)}{dt} = X_{\varphi_t(q)}$.

($\because$ for any smooth f, $X_q(f) = \lim_{h \to 0} \frac{f(\varphi_h(q)) - f(q)}{h}$ which X is said to generate φ.)

$\Rightarrow \frac{d\varphi_t(q)}{dt}(f) = \lim_{h \to 0} \frac{f(\varphi_{t+h}(q)) - f(\varphi_t(q))}{h} = X_{\varphi_t(q)}(f)$. #

By ODE uniqueness, for any $q \in M$ $\exists$ nbd U and $\varepsilon > 0$ st $\frac{d\varphi_t(q)}{dt} = X_{\varphi_t(q)}$, $\varphi_0(q) = q$. which is unique for $q \in U$ and $|t| < \varepsilon$.

The K is cpt, mean $\exists$ finite number such U cover K.

Let $\varepsilon_0 > 0$ be the smallest number of all ε. ($\because$ finite number of ε have minimal)

We set $\varphi_t(q) = q$ if $q \notin K$. it's follow that differential equation have unique solution for $|t| < \varepsilon_0$. $\forall q \in M$, and by def $\varphi_{t+s} = \varphi_t \circ \varphi_s$ if $|t|, |s|, |t+s| < \varepsilon_0$.

Then the only thing we need to define φ_t when $|t| > \varepsilon_0$. $\because \varepsilon_0 > 0$ then any $t = k(\frac{\varepsilon_0}{2}) + r$ $\exists k \in \mathbb{Z}$ and $|r| < \frac{\varepsilon_0}{2}$

Then we define $\varphi_t = \underbrace{\varphi_{\frac{\varepsilon_0}{2}} \circ \cdots \circ \varphi_{\frac{\varepsilon_0}{2}}}_{k\text{-}\mathrm{th}} \circ \varphi_r$ if $k < 0$ then $\frac{\varepsilon_0}{2} \to -\frac{\varepsilon_0}{2}$.

They φ_t is well-defined, smooth and satisfity $\varphi_{t+s} = \varphi_t \circ \varphi_s$. $\square$

Then with the above lemma, we can see that X can generated 1-parameter group of diffeomorphism φ_t.

for each $q \in M$, $g_q(t) = f(\varphi_t(q))$ If $\varphi_t(q) \in f^{-1}[a,b]$,

Then $\frac{d g_q(t)}{dt} = \frac{d f(\varphi_t(q))}{dt} = \langle \frac{d\varphi_t(q)}{dt}, \nabla f \rangle = \langle X, \nabla f \rangle = +1$.

$\therefore \frac{d g_q(t)}{dt} = 1 \Rightarrow f(\varphi_t(q)) = t + f(q) \Rightarrow \varphi_{b-a}: M \to M$ is a diffeomorphism map $M^a \to M^b$.

check M^a is deformation retract of M^b

define $r_t(q) = \begin{cases} q & \text{if } f(q) \le a \\ \varphi_{t(a-f(q))}(q) & \text{if } a \le f(q) \le b \end{cases}$ $\quad r_0$ is identity and r_1 is retraction from M^b to M^a. $\square$

Thm 2.2: Let $f: M \to \mathbb{R}$ be smooth function. Let p be the nondegenerate critical point of f with index λ. If $f(p) = c$ suppose for some $\varepsilon > 0$. $f^{-1}[c-\varepsilon, c+\varepsilon]$ is compact. It contain no critical point other than p. Then for all small enough $\varepsilon > 0$, $M^{c+\varepsilon}$ has homotopy type of $M^{c-\varepsilon}$ with a λ cell attached.

The idea of the proof

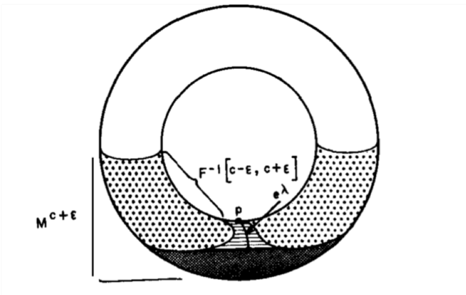

Diagram 4.

(pf) By morse lemma $f = c - (x^1)^2 - \cdots - (x^\lambda)^2 + (x^{\lambda+1})^2 + \cdots + (x^n)^2$.

where $(x^1, \cdots, x^n)$ be a local nbd U of p st $x^i(p) = 0$ $\forall i \in 1, 2, 3 \cdots n$.

choose $\varepsilon > 0$ which is small enough that $Im(U)$ under the diffeomorphism imbedding $(x^1, \cdots, x^n): U \to \mathbb{R}^n$ contain the closed ball $\{(x^1, \cdots, x^n) \mid \sum (x^i)^2 \le 2\varepsilon\}$

and $f^{-1}[c-\varepsilon, c+\varepsilon]$ is cpt and contain no critical point other than p.

let g be smooth st 1. $g(0) > \varepsilon$ 2. $g(r) = 0$ for $r \ge 2\varepsilon$ 3. $-1 < g'(r) \le 0$ $\forall r$.

define $F := f - g((x^1)^2 + \cdots + (x^\lambda)^2 + 2((x^{\lambda+1})^2 + \cdots (x^n)^2))$.

Denote $\alpha = (x^1)^2 + \cdots + (x^\lambda)^2$, $\beta = (x^{\lambda+1})^2 + \cdots + (x^n)^2$.

Then $f = c - \alpha + \beta$ $\quad F = c - \alpha + \beta - g(\alpha + 2\beta)$

By construction of $g \ge 0$ $\forall r \ge 0$ $g(r) = 0$ when $r > 2\varepsilon$. So find that

$F \le f$ when $\alpha + 2\beta \le 2\varepsilon$

$F = f$ when $\alpha + 2\beta > 2\varepsilon$.

Claim 1. $F^{-1}(-\infty, c+\varepsilon]$ coincide with $M^{c+\varepsilon}$.

(pf) when $\alpha + 2\beta > 2\varepsilon$ $F = f$

$\alpha + 2\beta \le 2\varepsilon$ $F \le f = c - \alpha + \beta \le c + \frac{\alpha}{2} + \beta \le c + \varepsilon$

region of $\alpha + 2\beta \le 2\varepsilon \subseteq F^{-1}(-\infty, c+\varepsilon]$ and $M^{c+\varepsilon}$

Then $F^{-1}(-\infty, c+\varepsilon] \cap \{\alpha+2\beta > 2\varepsilon\} = f^{-1}(-\infty, c+\varepsilon] \cap \{\alpha+2\beta > 2\varepsilon\}$

$\Rightarrow F^{-1}(-\infty, c+\varepsilon] \subseteq M^{c+\varepsilon}$

converse, when $x \in M^{c+\varepsilon} \Rightarrow f(x) \leq c+\varepsilon$ when x in $\alpha+2\beta > 2\varepsilon$ then $F(x) = f(x) < c+\varepsilon$

when x in $\alpha+2\beta \leq 2\varepsilon \Rightarrow F(x) \leq f(x) \leq c+\varepsilon$

$\Rightarrow x \in F^{-1}(-\infty, c+\varepsilon]$ #

Claim 2. The critical points of F in U are the same as those of f in U.

(pf) $\frac{\partial F}{\partial \alpha} = -1 - g'(\alpha+2\beta) < 0$ $(\because -1 < g'(x) \leq 0)$

$\frac{\partial F}{\partial \beta} = 1 - 2g'(\alpha+2\beta) > 1$

$dF = \frac{\partial F}{\partial \alpha} d\alpha + \frac{\partial F}{\partial \beta} d\beta \Rightarrow dF = 0$ in region $\alpha + 2\beta \leq 2\varepsilon$ iff $d\alpha$, $d\beta$ are 0.

$\Rightarrow F$ has no critical points in U other than the origin. □

Claim 3. $F^{-1}(-\infty, c-\varepsilon]$ is a deformation retract $M^{c+\varepsilon}$.

(pf) $F^{-1}[c-\varepsilon, c+\varepsilon] \subset f^{-1}[c-\varepsilon, c+\varepsilon]$ $\left(\because f \geq F \text{ all the time} \Rightarrow F^{-1}[c-\varepsilon, \infty) \subseteq f^{-1}[c-\varepsilon, \infty) \text{ and claim 1. } F^{-1}(-\infty, c+\varepsilon] = f^{-1}(-\infty, c+\varepsilon]\right)$

$\because f^{-1}[c-\varepsilon, c+\varepsilon]$ is cpt and F is continuous $\Rightarrow F^{-1}[c-\varepsilon, c+\varepsilon]$ is closed

$\Rightarrow F^{-1}[c-\varepsilon, c+\varepsilon]$ is closed in cpt set $f^{-1}[c-\varepsilon, c+\varepsilon]$.

$\Rightarrow F^{-1}[c-\varepsilon, c+\varepsilon]$ is cpt.

$F(p) = c - g(0) < c - \varepsilon$ $(\because g(0) > \varepsilon) \Rightarrow F^{-1}[c-\varepsilon, c+\varepsilon]$ don't have any critical point and cpt.

$\Rightarrow$ From the Thm 2.1 we have $F^{-1}(-\infty, c-\varepsilon]$ is deformation retract of $F^{-1}(-\infty, c+\varepsilon]$

which is means $F^{-1}(-\infty, c-\varepsilon]$ is deformation retract of $M^{c+\varepsilon}$ ($\because$ claim 1).

Then denote $F^{-1}(-\infty, c-\varepsilon]$ by $M^{c-\varepsilon} \cup H$ which is closure of $F^{-1}(-\infty, c-\varepsilon] \setminus M^{c-\varepsilon}$.

Consider $e^{\lambda} := \{q \in M \mid \alpha(q) \leq \varepsilon, \beta(q) = 0\}$

note that $e^{\lambda} \subseteq H$ $\left(\because q \in e^{\lambda},\ \frac{\partial F}{\partial \alpha} < 0 \Rightarrow F(q) \leq F(p) < c-\varepsilon.\right.$ $\left.\text{But } f(q) = c - \alpha(q) \geq c-\varepsilon \Rightarrow q \subset F^{-1}(-\infty, c-\varepsilon] \setminus M^{c-\varepsilon} \subset H.\right)$

Claim 4. $M^{c-\varepsilon} \cup e^{\lambda}$ is a deformation retract of $M^{c-\varepsilon} \cup H$.

(pf) Case 1.

When $\alpha \leq \varepsilon$ define $r_t(x^1, \cdots, x^n) = (x^1, \cdots, x^{\lambda}, t x^{\lambda+1}, \cdots, t x^n)$

r_t map $F^{-1}(-\infty, c-\varepsilon]$ into itself since $\frac{\partial F}{\partial \beta} > 0$ ($\because$ as $t < 1$ then $\beta \downarrow \Rightarrow F \downarrow$)

r_1 is identity and r_0 map $F^{-1}(-\infty, c-\varepsilon]$ to e^{λ}.

Case 2.

When $\varepsilon < \alpha \leq \beta + \varepsilon$ define $r_t(x^1, \cdots, x^n) = (x^1, \cdots, x^{\lambda}, s_t x^{\lambda+1}, \cdots, s_t x^n)$

$s_t = t + (1-t)\sqrt{\frac{\alpha-\varepsilon}{\beta}}$

when $t=1$ r_1 is identity, when $t=0$ $s_0 = \sqrt{\frac{\alpha-\varepsilon}{\beta}}$

$\Rightarrow f(x^1, \cdots, x^{\lambda}, s_0 x^{\lambda+1}, \cdots, s_0 x^n) = c - \alpha + \frac{\alpha-\varepsilon}{\beta} \cdot \beta = c - \varepsilon$

$\Rightarrow$ such r_0 map the region $\varepsilon \leq \alpha \leq \beta+\varepsilon$ to $f^{-1}(c-\varepsilon)$

And check as $\alpha \to \varepsilon$ $\beta \to 0 \Rightarrow s_t \longrightarrow t$ coincide with case 1 ($\alpha = \varepsilon$)

($\because$ given any $\beta > 0$ we let $\alpha \to \varepsilon \Rightarrow s_0 = \sqrt{\frac{\alpha-\varepsilon}{\beta}} \to 0$)

case 3.

When $\beta+\varepsilon<\alpha \Leftrightarrow c-\alpha+\beta\le c-\varepsilon \Leftrightarrow$ when in $M^{c-\varepsilon}$

Then we define $r_t=$ identity. And when $\alpha\to\beta+\varepsilon$ $s_t\to t+1-t=1$

Then we have r_t coincide with case 2 when $\alpha\to\beta+\varepsilon$.

Then finally we get the desired r_t $\square$

$\Rightarrow \because M^{c-\varepsilon}\cup e^\lambda$ is a deformation retract of $M^{c-\varepsilon}\cup H=F^{-1}[-\infty,c-\varepsilon]$

And from claim 3 we know $F^{-1}[-\infty,c-\varepsilon]$ is a deformation retract of $M^{c+\varepsilon}$ $\Rightarrow M^{c-\varepsilon}\cup e^\lambda$ homotopy to $M^{c+\varepsilon}$ $\square$

Remark: from the local behavior of Morse function f can determine the homotopy type of M.

The application of the above theorem 2.1 and theorem 2.2.

(Reeb thm): If M is cpt manifold and f is differentiable function on M with only two critical points, and both of them are non-degenerate, then M is homeomorphic to sphere.

(pf) From thm 2.1 and Morse lemma, the two critical points must be maximum and minimum. Let $f(p)=0$ and $f(q)=1$ p,q are minimum point and maximum point, If ε is small enough then $M^\varepsilon=f^{-1}[0,\varepsilon]$ and $f^{-1}[1-\varepsilon,1]$ are closed n-cell from Morse lemma. And M^ε homeomorphic to $M^{1-\varepsilon}$ By the thm 2.1.

$\Rightarrow M$ is the union of two closed n-cells. $M^{1-\varepsilon}$ and $f^{-1}[1-\varepsilon,1]$ matched along their common boundary, which is easy to see that M homeomorphic to S^n.

3.Morse theory on Riemannian geometry and the space of curve on manifold. The proof of Morse index theorem.

Next, we turn to the use of Morse theory in Riemannian geometry, its application to space of curve on manifold.

Def 3.1: $\Omega(p,q)$ be the set of continuous piecewise smooth curve in M from p to q. $\gamma\in\Omega(p,q)$ $\gamma(0)=p$ $\gamma(1)=q$ $\gamma:[0,1]\to M$ which is piecewise smooth.

We want to define the metric on $\Omega(p,q)$. Let $\gamma,\gamma_0\in\Omega(p,q)$ and $L(t),L_0(t)$ be the arclength fuction correpond to γ and γ_0.

$L(t)=\int_0^t\|\frac{d\gamma}{dt}\|dt$ $L_0(t)=\int_0^t\|\frac{d\gamma_0}{dt}\|dt$. Denote $\rho(p,q)$ be the distance of p,q on M. $p,q\in M$.

Then define $d(\gamma,\gamma_0)=\max\limits_{0\le t\le 1}\{\rho(\gamma(t),\gamma_0(t))+[\int_0^1(\frac{dL}{dt}-\frac{dL_0}{dt})^2dt]^{\frac{1}{2}}\}$

It is easy to check d is a well define metric on $\Omega(p,q)$.

Def 3.2: $E(\gamma) = \int_0^1 (\frac{dL}{dt})^2 dt$ which is the energe function.

Which is continuous from $\Omega(P,q)$ to $\mathbb{R}$.

Let T be tangent vector of $\gamma \Rightarrow E(\gamma) = \int_0^1 ||T||^2 dt$

$|L(\gamma)|^2 = (\int_a^b ||T|| dt)^2 \leq (\int_a^b ||T||^2 dt)(\int_a^b 1 dt) = E(\gamma)(b-a)$ from schwartz inequality.

(with equality iff $||T||$ is constant)

Def 3.3: $\Omega^C(P,q) = \{\gamma \in \Omega(P,q) \mid E(\gamma) \leq C\}$

Def $\Omega(t_0, \cdots, t_k)$ be the set of path γ in $\Omega(P,q)$ and $\gamma[t_i, t_{i+1}]$ is geodesic.

Let $\Omega^C(t_0, \cdots, t_k) = \Omega(t_0, \cdots, t_k) \cap \Omega^C$

The broken geodesic γ in $\Omega^C(t_0, \cdots, t_k)$ is uniquely determined by $\{\gamma(t_i)\}$

$\Rightarrow \Omega^C(t_0, \cdots, t_k) \to M \times M \cdots \times M$ (k-1 time) $\gamma \to (\gamma(t_1), \cdots, \gamma(t_{k-1}))$ is inj.

The image of the above map will be submanifold of $M \times M \cdots \times M$ with boundary.

The interior is $\{\gamma \in \Omega(t_0, \cdots, t_k) \mid E(\gamma) < C\}$

Therefore we can endow $\Omega^C(t_0, t_1, \cdots, t_k)$ with structure of manifold and $E|\Omega^C(t_0, \cdots, t_k)$ be smooth function.

Then we constract deformation retract of $\Omega^C(P,q)$ onto $\Omega^C(t_0, t_1 \cdots, t_k)$

Let $\gamma \in \Omega^C(P,q)$ let σ_s^i be the unique geodesic from $\gamma(t_i)$ to $\gamma(t_i + s(t_{i+1} - t_i))$

Define $r_s(t) = \begin{cases} \sigma_s^i(t) & t_i \leq t \leq t_i + s(t_{i+1} - t_i) \\ \gamma(t) & t_i + s(t_{i+1} - t_i) \leq t \leq t_{i+1} \end{cases}$

Denote $\gamma_s^i = \gamma|[t_i, t_i + s(t_{i+1} - t_i)]$

$E(\gamma_s^i) s(t_{i+1} - t_i) \geq |L(\gamma_s^i)|^2 \geq |L(\sigma_s^i)|^2 = E(\sigma_s^i) s(t_{i+1} - t_i)$

$\Rightarrow E(\gamma_s^i) \geq E(\sigma_s^i)$ so $\forall s$ $r_s \in \Omega^C(P,q) \Rightarrow$ we have a deformation retract of $\Omega^C(P,q)$ onto $\Omega^C(t_0, t_1, \cdots, t_k)$

Before we go further, we discuss the critical pome of E.

Let $\{C_s\}$ be one parameter family of curve.

$\frac{1}{2} \frac{d}{ds} E(C_s)|_{s=0} = -\int_0^1 \langle W, \nabla_T T \rangle dt - \sum_i \Delta_{t_i} \langle W, T \rangle$ $(W = \frac{\partial}{\partial s}(C_s(t))$ $T = \frac{\partial}{\partial t}(C_s(t)))$

If C_0 is broken geodesic $\Rightarrow \langle W, \nabla_T T \rangle = 0$

If and only if the C_0 is an unbroken geodesic then $\sum_i \Delta_{t_i} \langle W, T \rangle = 0$

Similarly, we can discuss the second variation of E.
Let $v \& w$ be the variation parameter of $\gamma \in \Omega(P,q)$ which is geodesic.
Let $V \& W$ be the vector field induced by $v \& w$ variation.
$\frac{\partial^2 E}{\partial v \partial w} = \int_0^1 \langle \nabla_T V, \nabla_T W\rangle + \langle R(V,T)W,T\rangle \doteq I(V,W)$

When we consider $E|\Omega^c(t_0,\cdots,t_i)$ then γ is varied through broken geodesic.
Then $V \& W$ be broken Jacobi field. Then the index of a critical point of $E|\Omega^c(t_0,\cdots,t_i)$ is the index of $I(\ ,\ )$ on broken Jacobi field.

Def 3.4: $X_0(\gamma)$ be the set of piecewise smooth vector field vanish on $p \& q$.
Morse Index theorem: The subspace $X_0(\gamma)$ on I is negative definite is finite dimension, and its dim is equal to the number of conjugate point to p on $\gamma([0,1])$ (counting the multiplicity).
The null space is zero unless p conjugate q, the dim of null space is order of conjugate points.

The proof of Morse Index Theorem.
Let $V \& W$ in $X_0(\gamma)$, $I(V,W) = \sum_i \Delta t_i \langle \nabla_T V, W\rangle - \int_0^1 \langle \nabla_T^2 V, W\rangle + \langle R(T,V)T,W\rangle$
If $\forall i$ $V|[t_i,t_{i+1}]$ is a Jacobi field. $\Rightarrow I(V,W) = \sum \Delta t_i \langle \nabla_T V, W\rangle$.
We can pick $0=t_0<t_1<\cdots<t_n=t\le 1$ st $\gamma|[t_i,t_{i+1}]$ has no pair of conjugate point.
($\because$ By sard theorem we know that for almost $q\in M$ p is not conjugate to q along any geodesic. And for any $p\in M$ $\exists$ normal nbd, st we can find t_1 such that $\gamma|[0,t_1]$ has no pair of conjugate points. ($\because \gamma[0,1]$ is cpt $\exists$ finite $t_1, t_2 \cdots t_n$ st $\gamma|[t_i,t_{i+1}]$ has no pair of conjugate points.)
Let $X' = \{V\in X_0(\gamma) \mid V(t_i)=0\}$ and $X(t_0,\cdots,t_k)$ be the subspace of those vector fields in $X_0(\gamma)$ which is Jacobi field on each $[t_i,t_{i+1}]$.
But Jacobi field can be unique determined by the value of end points.
$\Rightarrow X(t_0,t_1\cdots,t_k) \to M_{\gamma(t_1)} \oplus M_{\gamma(t_2)} \cdots \oplus M_{\gamma(t_{k-1})}$
And this identification is just the tangent map at γ to the injection,
$\Omega^c(t_0,t_1\cdots,t_k) \to M \times M \times \cdots \times M$ ($k-1$ times).
And $X_0(\gamma) = X' \oplus X(t_0,\cdots,t_k)$ for $V\in X_0(\gamma)$ the element of $X(t_0,\cdots,t_k)$ is determined by $V(t_1),\cdots,V(t_{k-1})$.
lemma 1. $I|_{X'}$ is positive definite.
(pf) Let I_i be the index form on $\gamma[t_i,t_{i+1}]$. Let $W\in X'$.
$I_i(W,W) \geq I_i(J,J)$ where $J_{t_i} = J_{t_{i+1}} = 0$ J is Jacobi
(which is by the index lemma in our lecture).

But as our assumption their is no conjugate pair in $\gamma[t_i, t_{i+1}]$
$\Rightarrow J = 0$ on $[t_i, t_{i+1}]$ $\Rightarrow I_i(W,W) \geq 0$ and $I_i(W,W) = 0$ iff $W = J = 0$.
$\Rightarrow I(W,W) = \sum_i I_i(W,W) \geq 0$ $\Rightarrow$ It $I(W,W) = 0$ iff $W = 0$ on whole γ.

lemma 2. X' and $X(t_0, \dots, t_k)$ are perpendicular with respect to I,
(pf) Let $W \in X(t_0, \dots, t_k)$, $W \in X'$. The lemma is immediate from the fact that W vanishes at the jumps of $\nabla_T V$, for I when V is broken Jacobi field.

Use lemmas $\Rightarrow$ $\text{Index}(I) = \text{Index}(I|X(t_0, t_1, \dots, t_k))$

Then we continue the proof, By use of the identification of $X(t_0, \dots, t_{k-1}, t)$ for varible t. $M_{\gamma(t_1)} \oplus M_{\gamma(t_2)} \cdots \oplus M_{\gamma(t_{k-1})} = D$ is fixed vector space.
Then $I|X(t_0, \dots, t_{k-1}, t)$ as 1-parameter family of bilinear forms I_t on D.
Let $v \in M_{\gamma(t_{k-1})}$ and $J_{t,v}$ be the Jacobi field on $[t_{k-1}, t]$ st $J_{t,v}(t_{k-1}) = v$
$J_{t,v}(t) = 0$, Clearly $I_t \cdot J_{t,v}$ varies conti with t.
Moreover, for $t' > t$, let $\bar{J}_{t,v}$ be the field on $[t_{k-1}, t']$ which agree with $J_{t,v}$ on $[t_{k-1}, t]$ and 0 on $[t, t']$,
Then by First Index lemma. $I(\bar{J}_{t,v}, \bar{J}_{t,v}) > I(J_{t',v}, J_{t',v})$
$\Rightarrow$ For fixed v in D. $t' > t$ imply $I_{t'}(V,V) < I_t(V,V)$.
Then the index of $I_{t'}$, $t' > t$ at lease equal to index of I_t plus the nullity. (When $t'-t$ is small enough $\Rightarrow$ null space of I_t become negative definitely $\Rightarrow$ Index $I_{t'}$ = Index I_t + Nullity of I_t) ($\because$ I_t depends on t continuously.
Because I_t depend on time continously on t, For t sufficiently small, the index of I_t is zero by basic index lemma.
Then by the counting argument we have done that.
In particular, we know $V \in X_0(\gamma)$ is in null space of I iff V is Jacobi field. So the null space is zero unless p, q are conjugate. $\square$

By the morse index thm, we can calculate the multiplicity by the following thm.

Thm 3.1: Let $\gamma: \mathbb{R} \to M$, be geodesic in a locally symmetric manifold ($\nabla R = 0$) Let $V = \frac{d\gamma}{dt}(0)$, $\gamma(0) = p$, Det $K_V: T_pM \to T_pM$ $K_V(W) = R(V,W)V$.
Let $\lambda_1, \lambda_2, \dots \lambda_n$ be eigenvalue of K_V. The conjugate points to p along γ are the points $\gamma(\frac{\pi k}{\sqrt{\lambda_i}})$ $k \in \mathbb{N}$. and i be positive eigenvalue.
Then the multipilicity of $\gamma(t)$ as a conjugate point is equal to the number of λ_i st t is multiple of $\frac{\pi}{\sqrt{\lambda_i}}$.

(pf) Choose orthonormal basis $U_1, \dots U_n$ for M_p and satisify $K_V(U_i) = \lambda_i U_i$ and extend V, U_i along γ by parrellel translation.
$\because \nabla R = 0 \Rightarrow \nabla R(V, U_i, V, W, \gamma') = \gamma' R(V, U_i, V, W) - R(\nabla_{\gamma'} V, U_i, V, W) - R(V, \nabla_{\gamma'} U_i, V, W)$
$- R(V, U_i, \nabla_{\gamma'} V, W) - R(V, U_i, V, \nabla_{\gamma'} W)$

$= \gamma' R(V,U_i,V,W) - \langle R(V,U_i)V, \nabla_{\gamma'} W\rangle$

$= \langle \nabla_{\gamma'} R(V,U_i)V, W\rangle \Rightarrow \nabla_{\gamma'} R(V,U_i)V = 0$

Then $\Rightarrow K_V(U_i) = \lambda_i U_i$ is true along γ.

So Any vector field along γ, $W(t) = w_1(t)U_1(t) + \cdots w_n(t)U_n(t)$.

Consider the Jacobbi equation $\frac{D^2W}{dt^2} + K_V(W) = 0$ $\quad \sum \frac{D^2w}{dt^2} U_i + \sum \lambda_i w_i U_i = 0$

Because U_i are everywhere linearly indep $\Rightarrow \frac{d^2 w_i}{dt^2} + \lambda_i w_i = 0$

If $\lambda_i > 0$ $w_i(t) = c_i \sin(\sqrt{\lambda_i} t) \Rightarrow$ The zero of $w_i(t)$ are the multiple of $t = \frac{\pi}{\sqrt{\lambda_i}}$

If $\lambda_i = 0$, then $w_i(t) = c_i t$

If $\lambda_i < 0$, then $w_i(t) = c_i \sinh(\sqrt{\lambda_i} t)$ $\Big]$ vanish only at $t=0$.

#

Denote $E^{-1}[0,d]$ by Ω^d

Thm 3.2 If the space of minimal geodesic from p to q is topological manifold, and if every non-minimal geodesic from p to q has index at least λ_0. Then relative homotopy gp $\pi_i(\Omega, \Omega^d)$ is zero for $0 \le i < \lambda_0$.

The proof of Thm 3.2 base on the following lemma.

Lemma 1. Let K be cpt set of $\mathbb{R}^n$, and U be nbd of K. Let $f: U \to \mathbb{R}$ be a smooth function st all critical pt of f in K have index $\ge \lambda_0$. If $g: U \to \mathbb{R}$ smooth function st $\left|\frac{\partial g}{\partial x_i} - \frac{\partial f}{\partial x_i}\right| < \varepsilon$ and $\left|\frac{\partial^2 g}{\partial x_i \partial x_j} - \frac{\partial^2 f}{\partial x_i \partial x_j}\right| \le \varepsilon$ $\forall i,j$ uniformly throughout K for some small ε. Then all critical point of g have index $\ge \lambda$.

(pf) $K_g = \sum_i \left|\frac{\partial g}{\partial x_i}\right| > 0$ let $e_g^1(x) \le \cdots \le e_g^n(x)$ be eigenvalues of the matrix that $\left(\frac{\partial^2 g}{\partial x_i \partial x_j}\right)$ $\because$ the index $\ge \lambda \Leftrightarrow e_g^\lambda(x)$ is negative. $\Rightarrow m_g(x) > 0$

Consider $m_g(x) = \max\{K_g(x), -(e_g)^{\lambda_0}\} > 0$ and $m_f(x) = \max\{K_f(x), -(e_f)^{\lambda_0}\}$

($\because K_f(x) = 0$ but $-e_f^{\lambda_0}(x) > 0$) $\Rightarrow m_f(x) > 0$ $\forall x \in K$ Let δ be minimum of m_f.

Pick ε is small enough st $|K_g(x) - K_f(x)| < \delta$ and $|e_g^\circ(x) - e_f^\circ(x)| < \delta$.

Then, m_g is always positive, then for x is critical point $K_g(x) = 0 \Rightarrow -(e_g)^{\lambda_0} > 0$ $\Rightarrow e_g^{\lambda_0} < 0 \Rightarrow$ Every critical point of g has index $\ge \lambda_0$. #

Lemma 2. Let $f: M \to \mathbb{R}$ be smooth function with minimum 0. st $M^c = f^{-1}[0,c]$ is compact. If M^0 is a manifold, and critical point of $M \setminus M^0$ has index at least λ_0, then $\pi_r(M, M^0) = 0$ for $0 \le r < \lambda_0$.

(pf) M^0 is a retract of some nbd of $U \subset M$. And restrict U st. each pt of U is joint to the corresponding point of M^0 by unique minimal geodesic. Thus U can be deformed into M^0 within M.

Let I^r be unit cube of dim $r < \lambda_0$. $h: (I^r, S^r) \to (M, M^0)$

The following we are going to show h is homotopic to h' where $f(I^r) \subset M^0$.

First, choose g approxmated f on M^c, where c is the maximum of f on $h(I^r)$. By lemma 1, we can choose g s.t it has no degenerate critical and each critial point has index at least λ.

And g satisify the following properties.

1. $|g(x)-f(x)|<\delta \quad \forall x \in M^{c+2\delta}$
2. The critical points of g in $f^{-1}[\delta, c+2\delta]$ have index larger than λ_0 (This can be done, because $f^{-1}[\delta, c+2\delta]$ is compact so we can have finitely coverings, so we can construct such g.)

Let 3δ be minimum of f on $M \setminus U$, we can see $g^{-1}[2\delta, c+\delta] \subseteq f^{-1}[\delta, c+2\delta]$ and all critical points with index $\geq \lambda_0$, $h(I^r) \subseteq M^c \subseteq g^{-1}(-\infty, c+\delta]$.

$g^{-1}(-\infty, c+\delta]$ has the same homotopy type as $g^{-1}(-\infty, 2\delta]$ with some λ-cells. $(\lambda \geq \lambda_0)$

$\Rightarrow h:(I^r, S^r) \to (M^c, M^0) \subset (g^{-1}(-\infty, c+\delta], M^0)$

Since $r \leq \lambda$ then h is homopotic to some h' that maps into $(g^{-1}(-\infty, 2\delta], M^0)$ This is true because all critical point of g have index $\geq \lambda$. However, $g^{-1}(-\infty, 2\delta]$ is contained in U and U can be deformed into M^0, so have $\pi_r(M, M^0) = 0$

Then go back to the proof of thm 3.2.

(pf) $\pi_r(\Omega, \Omega^d) = \pi_r(\mathrm{Int}\,\Omega^c(t_0, t_1, \cdots, t_k), \Omega)$

($\because \mathrm{Int}\,\Omega^c$ contain smooth manifold $\mathrm{Int}\,\Omega^c(t_0, \cdots, t_k)$ as deformation retract. $\Omega^d \subseteq \mathrm{Int}\,\Omega^c(t_0, t_1 \cdots, t_k)$)

$E: \Omega \to \mathbb{R}$ restrict on $\mathrm{Int}\,\Omega^c(t_0, \cdots t_k)$. It almost satisify the assumption of lemma 2. But In this region we have the $d \leq E(\omega) < c$, so we let $F: [d, c) \to [0, \infty)$ be a diffeomorphism.

$F \circ E : \mathrm{Int}\,\Omega^c(\,_0, \cdots, t_k) \to \mathbb{R}$ Then $F \circ E$ satisify the assumption of Lemma 2.

$\Rightarrow \pi_r(\mathrm{Int}\,\Omega^c(t_0, \cdots, t_k), \Omega^d) \cong \pi_r(\mathrm{Int}\,\Omega^c, \Omega^d) = 0$ if $r < \lambda_0$ #.

Cor of thm 3.2: If the space of minimal geodesic is a topological manifold, and if every non-minimal geodesic has index at least λ_0, then $\pi_i(\Omega^d)$ is isomorphic to $\pi_{i+1}(M)$, $\forall i \leq \lambda_0 - 2$.

(pf) $\pi_i(\Omega^d)$ isomophic to $\pi_i(\Omega)$ for $i \leq \lambda_0 - 1$. $\because \pi_i(\Omega, \Omega^d) = 0$

But we know that p, q is path connected.

Denote $\Omega(p, p)$ be all conti map α st $\alpha(0) = \alpha(1) = p$

$\Omega(p, q)$ be all conti map β st $\beta(0) = p \quad \beta(1) = q$

Fix α path from p to q.

$F: \Omega(p,p) \to \Omega(p,q)$ $(\gamma\alpha)$ be the concatenation
$\gamma \mapsto [\gamma\alpha]$ $(\gamma\alpha)(t) = \begin{cases} \gamma(2t) & t \in [0,\frac{1}{2}] \\ \alpha(2t-1) & t \in [\frac{1}{2},1] \end{cases}$

$G: \Omega(p,q) \to \Omega(p,p)$
$\beta \mapsto (\beta\alpha^{-1})$

$F \circ G: \Omega(p,q) \to \Omega(p,q)$ $G \circ F: \Omega(p,p) \to \Omega(p,q)$
$\beta \quad \beta\alpha^{-1}\alpha$ $\gamma \to \gamma\alpha\alpha^{-1}$

$\because \alpha\alpha^{-1}$ and $\alpha^{-1}\alpha$ are homotopic to constant map.
$\Rightarrow \Omega(p,q)$ homotopic to $\Omega(p,p) = \Omega$.

But $\pi_i(\Omega) = \{\gamma : (I^i, \partial I^i) \to (\Omega, \gamma_0)\}/\sim$ consider γ_0 = contant map on p
$\pi_{i+1}(M) = \{\beta : (I^{i+1}, \partial I^{i+1}) \to (M,p)\}/\sim$ $\gamma_0[I] = p$

$\gamma : (I^i \times I) \to M$ $\Rightarrow \gamma|_{\partial I^i \times I \cup I^i \times \partial I} = p$
$\because \partial I^i \times I \cup I^i \times \partial I = \partial I^{i+1}$

From the result of algebraic topology we know $\pi_i(\Omega) \simeq \pi_{i+1}(M)$.
Then $\pi_i(\Omega) \simeq \pi_i(\Omega^d) \simeq \pi_{i+1}(M)$ #

4.Bott Periodicity Theorem of unitary group

Bott Periodicity Theorem.

Def 4.1: The unitary group $U(n)$ is $S\in U(n)$ $S:\mathbb{C}^n\to\mathbb{C}^n$ linear transformation $SS^{+}=I$.

Def 4.2: exponential of A which is $n\times n$ $\mathbb{C}$ matrix. $\exp A = I + A + \frac{1}{2!}A^2 + \frac{1}{3!}A^3 \cdots$

The following property are easily verified

1. $\exp(A^{+}) = \exp(A)^{+}$ $\quad \exp(TAT^{-1}) = T\exp(A)T^{-1}$
2. If A, B commute $\Rightarrow \exp(A+B) = \exp(A)\exp(B)$ $\quad (\exp A)(\exp -A) = I$
3. The exp map in the nbd of 0 is diffeomorphism from $n\times n$ matrix space onto the nbd of I.
4. $\exp A$ unitary $\Leftrightarrow A + A^{+} = 0$. ($A$ is skew-Hermitian iff $A + A^{+} = 0$.)
5. $U(n)$ is a smooth submanifold of space of $n\times n$ matrices.
6. $T_I U(n)$ can be identified with the space of $n\times n$ skew-Hermitian matrices.

Then, we want to define the Riemannian metric on $U(n)$. First, consider the inner product on $\mathfrak{g}$ as $\langle A,B\rangle := \sum_{i,j} Re(A_{ij}\bar{B}_{ij})$. This inner product is positive definite on $\mathfrak{g}$ and we extend this inner product by $\langle A,B\rangle_g = \langle dL_{g^{-1}}A, dL_{g^{-1}}B\rangle_e$. Then this inner product determines a unique left invariant Riemannian metric on $U(n)$.

Next, we can check that the left invariant metric is also right invariant.

Def: The adjoint action; $S\in U(n)$ $Ad_S(X) := SXS^{-1} = (L_S R_S^{-1})X$. Then $Ad_S: TU(n)_I \to TU(n)_I$ is an automorphism of Lie algebra of $U(n)$.

$$\begin{aligned}
\langle Ad_S A, Ad_S B\rangle &= Re\,(tr(Ad_S A)(Ad_S B)) \\
&= Re\,(tr(SAS^{-1}(SBS^{-1}))) \\
&= Re\,(tr(SAS^{-1}(S^{-1})^{+}B^{+}S^{+})) \\
&= Re\,(tr(SAB^{+}S^{+})) \quad (\because S\in U(n)) \\
&= Re\,(tr(AB^{+})) = \langle A,B\rangle
\end{aligned}$$

Then, the above left invariant Riemannian metric is also right invariant.

Since the geodesics on G passing e are one parameter subgp of G.

From linear alegebra, given $A\in\mathfrak{g}$ we know that there exist $U\in U(n)$ st UAU^{-1} is diagonal form.

$$UA^{-1}U^{-1} = \begin{pmatrix} ia_1 & & & \\ & ia_2 & & \\ & & \ddots & \\ & & & ia_n \end{pmatrix} \quad a_i \text{ are real.}$$

And $S\in U(n)$, then $\exists U\in U(n)$ st

$$USU^{-1} = \begin{pmatrix} e^{ia_1} & & \\ & \ddots & \\ & & e^{ia_n} \end{pmatrix} \quad \text{for } a_i \text{ are real.}$$

Then $\exp: \mathfrak{g}\to U(n)$ is surjective.

When we consider the special unitary group $SU(n)$ which is defined as the subgp of $U(n)$ with the determinant equal to 1,

Then let $A' \in g'$ which is the Lie algebra of $SU(n)$. $\exists U \in U(n)$ st $UA'U^{-1}$ is diagonal.

Then $\det(\exp A') = \det(U \exp A' U^{-1}) = \det(\exp(UA'U^{-1})) = e^{\text{trace } A'}$.

Lie algebra of $SU(n)$ which is the set of all matrices which satisfy $A + A^t = 0$ and trace $A = 0$.

Then we want to use Morse theory to the topology of $U(n)$ and $SU(n)$, we have to study the geodesic from I to $-I$. ($\because$ In the above Morse theory we study the geodesic from p to q $p, q \in M$, so here we consider $p = I$ $q = -I$.)

So we consider all $A \in g = T_I U(n)$ st $\exp(A) = -I$.

If A is not diagonal form, $\exists U \in U(n)$ st UAU^{-1} is diagonal.

$\exp(TAT^{-1}) = T\exp(A)T^{-1} = T(-I)T^{-1} = -I$, so WLOG we can replace A by UAU^{-1} which is diagonal.

$\Rightarrow \exp(A) = \begin{pmatrix} e^{ia_1} & & 0 \\ & \ddots & \\ 0 & & e^{ia_n} \end{pmatrix}$ $\Rightarrow \exp(A) = -I \Leftrightarrow ia_m = ik_m\pi$, k_m are odd integer, $\forall m \in \{1, 2, 3 \dots n\}$

$\Leftrightarrow A = \begin{pmatrix} ik_1\pi & & 0 \\ & \ddots & \\ 0 & & ik_n\pi \end{pmatrix}$

The length of the above geodesic $t \to \exp(tA)$ from $t=0$ to $t=1$

$= |A| = \sqrt{\text{Re}(\text{trace}(AA^t))} = \sqrt{tr(AA^t)} \Rightarrow l = \pi\sqrt{k_1^2 + \dots + k_n^2}$ $k_1, \dots k_n$ are odd integers.

If A determine the minimal geodesic, then $k_m = \pm 1$ $\forall m \in \{1, 2, \dots n\}$. $\Rightarrow l = \pi\sqrt{n}$.

Next, consider A as linear transformation $\mathbb{C}^n \to \mathbb{C}^n$. Then A is determined by the Eigen$(i\pi)$ and Eigen$(-i\pi)$, but if Eigen$(i\pi)$ is determined then Eigen$(-i\pi)$ is determined.

$\Rightarrow$ Eigen$(i\pi)$ can determine A. But A determine the minimal geodesic from I to $-I$.

$\Rightarrow$ The space of all minimal geodesic in $U(n)$ from I to $-I$ can be identified by the space of all sub-vector-space of $\mathbb{C}^n$.

Similarly, we can consider the case of $SU(n)$ and set $n = 2m$ for $m \in \mathbb{N}$.

Then we have the addition condition $k_1 + k_2 + k_3 \dots + k_{2m} = 0$ (Lie algebra of $SU(n)$ is skew-Hermitian matrix with trace zero.)

$\Rightarrow$ Eigen$(i\pi)$ must be m-dim subspace of $\mathbb{C}^n$

Then we get the following lemma,

Lemma 4.1: The space of minimal geodesic space from I to $-I$ in $SU(2m)$ is homeomorphic to the complex Grassmann manifold $G_m(\mathbb{C}^{2m})$.

In order to use the cor of thm3.2, we have to compute the index of non-minimal geodesics.

Lemma 4.2 ; Every non-minimal geodesic from I to $-I$ in $SU(2m)$ has index $\geq 2m+2$.

(pf) By above dicussion about the $\mathfrak{g}'$, we know $A\in\mathfrak{g}'$ then A has eigenvalue $ik_1\pi, \cdots, ik_n\pi$ with $k_1\cdots k_n$ are odd integers and $\sum_{i=1}^{n} k_i=0$.

The index of non-minimal geodesic is the numbers of points on $\gamma(t)$ conjugate with $\gamma(0)=I$ (counting multiplicity) (By Morse Index theorem).

From last semester lecture we have $R(V,W)V=\frac{1}{4}[[V,W],V]$ ($\because SU(2m)$ has bi-invariant metric.) And $SU(2m)$ is locally symmetric manifold.

Then WLOG $A\in\mathfrak{g}'$, let $A=\begin{pmatrix} ik_1\pi & & 0 \\ & \ddots & \\ 0 & & ik_n\pi \end{pmatrix}$ with $k_1\geq k_2\geq\cdots\geq k_n$.

Let $W=(W_{j\ell})$

$[A,W]=i\pi(k_j-k_\ell)W_{j\ell}$

$[A,[A,W]]=-\pi^2(k_j-k_\ell)^2W_{j\ell}$

$R(A,W)A=\frac{\pi^2}{4}(k_j-k_\ell)^2W_{j\ell}$

Then the eigenvalue of $K_V(W):=R(V,W)V$ with $W\in\mathfrak{g}'$.

(1) For each $j<\ell$, $E_{j\ell}$ with 1 and $E_{\ell j}=-1$ and else are zero. Then such eigenvector corresponds to eigenvalue $\frac{\pi^2}{4}(k_j-k_\ell)^2$

(2) For each $j<\ell$, $E_{j\ell}$ with i and $E_{\ell j}=i$ and else are zero. Then such eigenvector corresponds to eigenvalue $\frac{\pi^2}{4}(k_j-k_\ell)^2$

(3) Each diagonal matrix in $\mathfrak{g}'$ correspond to eigenvalue 0.

We find the eigenbasis of $\mathfrak{g}'$, and the non-zero eigenvalue $\frac{\pi^2}{4}(k_j^2-k_\ell^2)$ are to be counted twice.

Then consider geodesic $\gamma(t)=e^{tA}$. Then by the thm3.1, each eigenvalue $e=\frac{\pi^2}{4}(k_j-k_\ell)^2$ given rise to a seires of conjugate points along γ correspond to $t=\frac{\pi}{\sqrt{e}}, \frac{2\pi}{\sqrt{e}}\cdots=\frac{2}{k_j-k_\ell}, \frac{4}{k_j-k_\ell}, \cdots$

Then the number of the conjugate point correspond to e is $\frac{1}{\frac{2}{k_j-k_\ell}}-1=\frac{k_j-k_\ell}{2}-1$

But each eigenvalue e should be counted twice

$\Rightarrow$ Index $=\sum_{k_j>k_\ell}\left(\frac{k_j-k_\ell}{2}-1\right)\times 2=\sum_{k_j>k_\ell}(k_j-k_\ell-2)$

The easy check when γ is minimal $\Rightarrow$ all $k_i=\pm1 \Rightarrow \sum_{k_j>k_\ell}(+1-(-1)-2)=0$.

When γ is non-minimal case, let $n=2m$,

Case 1. At least $m+1$ negative k_i. Then must have one of positive $k_j\geq 3$

($\because$ If all positive $k_j=1 \Rightarrow 0=\sum_{j=1}^{n}k_j\geq m-1-(m+1)\to\leftarrow$)

Then $\lambda\geq\sum_{1}^{m+1}(3-k_i-2)\geq\sum_{1}^{m+1}(3-2-(-1))=2(m+1)$

Case 2. At least $m+1$ positive k_i. Then must have one of negative $k_j\leq -3$

$\Rightarrow \lambda\geq\sum_{1}^{m+1}(k_i-(-3)-2)\geq\sum_{1}^{m+1}1+3-2=2(m+1)$

case 3. m positive and m negative k_i. But not all $|k_i|=1$ ($\because$ Our assumption is non-minimal.) $\Rightarrow \exists\, k_i, k_j,\ k_i \geq 3,\ k_j \leq -3$ $\left(\because \sum_{i=1}^{2m} k_i = 0\right.$ so there is one $k_i \geq 3$ and one $k_j \leq -3\left.\right)$

$$\lambda \geq \sum_{1}^{m-1}(3-(-1)-2) + \sum_{1}^{m+1}(1-(-3)-2) + (3-(-3)-2) = 4m \geq 2(m+1)$$

$\Rightarrow$ If γ is an non-minimal geodesic then the index of $\gamma \geq 2(m+1)$. $\square$

Finally, we are going to prove the last goal "Bott Periodicity Theorem."

Theorem 4.1 The inclusion map $G_m(\mathbb{C}^{2m}) \to \Omega(SU(2m), I, -I)$ induce isomorphism of homotopy groups in dimension $\leq 2m$, Hence, $\pi_i G_m(\mathbb{C}^{2m}) \simeq \pi_{i+1} SU(2m)\ \forall\, i \leq 2m$

(pf) From Lemma 4.1 $\Rightarrow \Omega(SU(2m), I, -I)$ is a topological manifold. ($\because$ It is isomorphic to $G_m(\mathbb{C}^{2m})$.) From Lemma 4.2, every non-minimal geodesic has index at least $\lambda_0 = 2m+2$. Then, it satisifies the assumption of Cor of thm 3.2 $\Rightarrow \pi_i G_m(\mathbb{C}^{2m}) = \pi_i(\Omega^{\pi^2 n}) \cong \pi_{i+1}(SU(2m))$ for $i \leq \lambda_0 - 2 = 2m$. $\square$

Then we want to establish the relation between homotopy group of $U(m)$ and homotopy group of $SU(m)$

Lemma 4.3 The group $\pi_i G_m(\mathbb{C}^{2m})$ is isomorphic to $\pi_{i-1} U(m)$ for $i \leq 2m$.
Moreover, $\pi_{i-1} U(m) \simeq \pi_{i-1} U(m+k)$ for $i \leq 2m$ $k \in \mathbb{N}$.
And $\pi_j U(m) \simeq \pi_j SU(m)$ for $j \neq 1$.

(pf) Pick the fibration $U(m) \to U(m+1) \to S^{2m+1}$ and $U(m) \to U(2m) \to U(2m)/U(m)$.
Use snake lemma we get long exact sequence.

$$\cdots \to \pi_i S^{2m+1} \to \pi_{i-1} U(m) \to \pi_{i-1} U(m+1) \to \pi_{i-1} S^{2m+1} \to \cdots$$

$\because \pi_i S^{2m+1} = 0$ when $0 \leq i \leq 2m$ $\Rightarrow 0 \to \pi_{i-1} U(m) \to \pi_{i-1} U(m+1) \to 0$

Use second fibration we get the following long exact sequence.

$$\cdots \to \pi_i(U(2m)/U(m)) \to \pi_{i-1} U(m) \to \pi_{i-1} U(2m) \to \pi_{i-1}(U(2m)/U(m)) \to \cdots$$

$\because \pi_{i-1} U(m) \simeq \pi_{i-1} U(2m) \Rightarrow \pi_i(U(2m)/U(m)) = 0$ for $i \leq 2m$.

Because the complex Grassmann manifold $G_m(\mathbb{C}^{2m})$ can be identified with $U(2m)/U(m) \times U(m)$ $\Rightarrow$ we have the fibration $U(m) \to U(2m)/U(m) \to G_m(\mathbb{C}^{2m})$

Then $\pi_i(U(2m)/U(m)) \to \pi_i G_m(\mathbb{C}^{2m}) \to \pi_{i-1} U(m) \to \pi_{i-1}(U(2m)/U(m))$

$\because \pi_i(U(2m)/U(m)) = 0 \Rightarrow \pi_i G_m(\mathbb{C}^{2m}) \simeq \pi_{i-1} U(m)\ \forall\, i \leq 2m$.

Then pick fibration $SU(m) \to U(m) \to S^1$

similarly we get $\pi_i SU(m) \simeq \pi_i U(m)$ for $i \neq 1$. $\square$

Remark: We can denote $\pi_i U$ as $\pi_i U(n)$ for any $n \geq \frac{i}{2}$

Then we can see that $\pi_{i-1} U = \pi_{i-1} U(m) \simeq \pi_i G_m(\mathbb{C}^{2m}) \simeq \pi_{i+1} SU(2m) \simeq \pi_{i+1} U$

This is the Bott Periodicity Theorem: For $i \geq 1$ $\pi_{i-1} U \simeq \pi_{i+1} U$ #

$\pi_0(U) \cong \pi_0(U(1))$ $\because$ $U(1) \simeq S^1$ $\ni$ $\pi_0(U(1)) = 0$ ($\because S^1$ is path connected.)

$\pi_1(U(1)) = \pi_1(S^1) = \mathbb{Z}$

$\ni \pi_i U \cong \mathbb{Z}$ when i is odd

0 when i is even.

Reference: 1. Morse Theory, J. Milnor

2. Comparison Theorems in Riemannian Geometry, Jeff Cheeger, David G. Ebin.