數學物理方程一期末報告

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這次報告我讀的主題主要是讀老師在講解 Nahari Manifold 的時候所補充的參考文獻 Michael Struwe 所撰寫的 Variational Methods，我主要讀完這本書的前兩章部分內容，1. The Direct Methods in the Calculus of Variations 和 2. Minimax Methods 的內容。

而在這篇報告中我將提到我覺得在讀完這些章節後帶給我的啟發。

首先，在微分方程上，我之前只有微分方程導論而已，所以對於許多符號定義，以及如每個定理中的條件所代表的涵義都不是十分清楚，像是 weak topology 以及弱收斂之類的詳細定義，以及每個不同符號所代表的空間像是(L^p 、$H^{m,p}$ 以及他們的對偶空間)所代表的意義，因此在這堂課以及讀這本書時，讓我第一次好好認真重新學，也比較能清楚釐清每個定理所適用的空間。

接下來在第一章時，介紹了許多不同種的方法來構造 Minimizer 以證明其存在性，主要有以下幾個：

1. classical lower semi-continuity
2. compensated compactness method
3. concentration compactness principle
4. Ekeland's Variational Principle
5. Minimization Problems Depending on Parameters

而這些方法大概求出解的方式，大概的流程是一開始先找出方程式的 weak solution 然後找出滿足甚麼條件下這個 weak solution 足夠 regular 而進而找出 classical solution，但是 regularity 的問題過於難以處理，所以我們暫時都只討論弱解的存在性的問題而已。

在 classical lower semi-continuity 的部分，他討論了在那些情況下我們的 functional 一定可以存在 minimizer.

1.1 Theorem. *Let M be a topological Hausdorff space, and suppose $E : M \to \mathbb{R} \cup +\infty$ satisfies the condition of bounded compactness:*

$$\text{(1.1)} \qquad \begin{array}{l} \textit{For any } \alpha \in \mathbb{R} \textit{ the set} \\ \qquad K_\alpha = \{u \in M \ ; \ E(u) \le \alpha\} \\ \textit{is compact (Heine-Borel property).} \end{array}$$

Then E is uniformly bounded from below on M and attains its infimum. The conclusion remains valid if instead of (1.1) we suppose that any sub-level set K_α is sequentially compact.

其中 lower semi-continuous 定義如下，

Note that if $E : M \to \mathbb{R}$ satisfies (1.1), then for any $\alpha \in \mathbb{R}$ the set

$$\{u \in M \; ; \; E(u) > \alpha\} = M \setminus K_\alpha$$

is open, that is, E is *lower semi-continuous*.

但因為在定理 1.1 的條件比較難以驗證，所以有以下的定理，可以告訴我們在種條件下 Minimizer 也存在(這個定理的條件比較好驗證。)

1.2 Theorem. *Suppose V is a reflexive Banach space with norm $\|\cdot\|$, and let $M \subset V$ be a weakly closed subset of V. Suppose $E : M \to \mathbb{R} \cup +\infty$ is coercive and (sequentially) weakly lower semi-continuous on M with respect to V, that is, suppose the following conditions are fullfilled:*
(1°) $E(u) \to \infty$ as $\|u\| \to \infty$, $u \in M$.
(2°) For any $u \in M$, any sequence (u_m) in M such that $u_m \rightharpoonup u$ weakly in V there holds:

$$E(u) \leq \liminf_{m \to \infty} E(u_m) \; .$$

Then E is bounded from below on M and attains its infimum in M.

然後由上面這個定理可以證明以下的定理：

Let Ω be a bounded domain in $\mathbb{R}^n$, and let S be a compact subset in $\mathbb{R}^N$. Also let $u_0 \in H^{1,2}(\Omega; \mathbb{R}^N)$ with $u_0(\Omega) \subset S$ be given. Define

$$H^{1,2}(\Omega; S) = \left\{u \in H^{1,2}(\Omega; \mathbb{R}^N) \; ; \; u(\Omega) \subset S \;\; \text{almost everywhere}\right\}$$

and let

$$M = \left\{u \in H^{1,2}(\Omega; S) \; ; \; u - u_0 \in H_0^{1,2}(\Omega; \mathbb{R}^N)\right\} \; .$$

Then, by Rellich's theorem, M is closed in the weak topology of $V = H^{1,2}(\Omega; \mathbb{R}^N)$. For $u = (u^1, \dots, u^N) \in H^{1,2}(\Omega; S)$ let

$$E(u) = \int_\Omega g_{ij}(u) \nabla u^i \nabla u^j \; dx \; ,$$

where $g = (g_{ij})_{1 \leq i,j \leq N}$ is a given positively definite symmetric matrix with coefficients $g_{ij}(u)$ depending continuously on $u \in S$, and where, by convention, we tacitly sum over repeated indices $1 \leq i, j \leq N$.

1.5 Theorem. *For any boundary data $u_0 \in H^{1,2}(\Omega; S)$ there exists an E-minimal extension $u \in M$.*

而這個定理的大致證明手段就是由 Rellich's theorem 讓我們知道一個序列是強收斂在 L^2 norm 下時，這時會有一個子序列會 converge 到一個函數 u almost everywhere，所以扣掉一個足扣小的定義域後我們函數的收斂會是 uniformly 地收斂，所以先對函數序列做極限來估算我們的 E(u)的值，再把我們的挖掉的定義域越弄越小，來得到我們的函數符合 Theorem 1.2 的第二點。

Remark：這個定理可以用來讓我們研究給定 boundary value 的 harmonic function 的存在性。

而對於我們的 functional 是以下形式的話，可以證明以下定理。

We consider variational integrals

$$E(u) = \int_\Omega F(x, u, \nabla u)\, dx \tag{1.5}$$

involving (vector-valued) functions $u : \Omega \subset \mathbb{R}^n \to \mathbb{R}^N$.

1.6 Theorem. *Let Ω be a domain in $\mathbb{R}^n$, and assume that $F : \Omega \times \mathbb{R}^N \times \mathbb{R}^{nN} \to \mathbb{R}$ is a Caratheodory function satisfying the conditions*
(1°) $F(x, u, p) \geq \phi(x)$ for almost every x, u, p, where $\phi \in L^1(\Omega)$.
(2°) $F(x, u, \cdot)$ is convex in p for almost every x, u.
Then, if u_m, $u \in H^{1,1}_{loc}(\Omega)$ and $u_m \to u$ in $L^1(\Omega')$, $\nabla u_m \to \nabla u$ weakly in $L^1(\Omega')$ for all bounded $\Omega' \subset\subset \Omega$, it follows that

$$E(u) \leq \liminf_{m \to \infty} E(u_m) \ ,$$

where E is given by (1.5).

當我們的 functional E 和我們的 space of admissible functions M 可能無法運用上面的定理時，我們可以嘗試在我們的 M 上做一些 constraint 或是調整我們的 functional 來讓我們使用我們前面用過的方法，最後再從解出來的解來驗證看看是不是符合原本條件的題目的解。

其中我覺得最有趣的方法應該是 Perron method，他可以用來幫助我們證明 Dirichlet Problem of Laplacian equation 在 bounded domain 上的解的存在性，因為我們可以藉由 Perron method 的方法來直接構造出一個 harmonic function 然後再建構 barrier functions 來驗證我們藉由 Perron method 所得到的 Perron solution 是符合我們 Dirichlet boundary condition，因此來證明我們的解的存在性。

Remark：在 Richard Schoen 和 Shing-Tung Yau 所撰寫的 Lectures on Differential Geometry 中甚至將一般的 Dirichlet Problem of Laplacian equation 我們是考慮在$\mathbb{R}^n$中的 bounded domain 上，但其實可以推廣至 compactified 的 Hadamard Manifold 上，而其中在證明解的存在性時，也是利用 perron method 來證明的，其中關鍵步驟需要利用 perron method 的建構方式來做出 Harmonic function on 整個 Hadamard Manifold. 在運用跟 perron method 很像的 barrier function 來驗證我們的 Perron solution 在我們定義的邊界上符合邊界條件。

所以現在來介紹課本中的 Perron method。首先，定義甚麼叫做

2.3 Weak sub- and super-solutions. Suppose Ω is a smooth, bounded domain in $\mathbb{R}^n$, and let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function with the property that $|g(x, u)| \leq C(R)$ for any $R > 0$ and all u such that $|u(x)| \leq R$ almost everywhere. Given $u_0 \in H^{1,2}_0(\Omega)$, we then consider the equation

$$-\Delta u = g(\cdot, u) \quad \text{in } \Omega \ , \tag{2.7}$$

$$u = u_0 \quad \text{on } \partial\Omega \ . \tag{2.8}$$

By definition $u \in H^{1,2}(\Omega)$ is a (weak) *sub-solution* to (2.7–2.8) if $u \leq u_0$ on $\partial\Omega$ and

$$\int_\Omega \nabla u \nabla \varphi \, dx - \int_\Omega g(\,\cdot\,, u)\varphi \, dx \leq 0 \qquad \text{for all } \varphi \in C_0^\infty(\Omega)\ ,\ \varphi \geq 0\ .$$

Similarly $u \in H^{1,2}(\Omega)$ is a (weak) *super-solution* to (2.7–2.8) if in the above the reverse inequalities hold.

而我們會有以下的定理：

2.4 Theorem. *Suppose $\underline{u} \in H^{1,2}(\Omega)$ is a sub-solution while $\overline{u} \in H^{1,2}(\Omega)$ is a super-solution to problem (2.7–2.8) and assume that with constants $\underline{c}, \overline{c} \in \mathbb{R}$ there holds $-\infty < \underline{c} \leq \underline{u} \leq \overline{u} \leq \overline{c} < \infty$, almost everywhere in Ω. Then there exists a weak solution $u \in H^{1,2}(\Omega)$ of (2.7–2.8), satisfying the condition $\underline{u} \leq u \leq \overline{u}$ almost everywhere in Ω.*

在看這個定理的證明之前，我們可以發現這個定理十分好用，因為有時候 sub-solution 和 super-solution 很好找，所以只要驗證一找出存在 sub-solution 和 super-solution 後，在驗證符合定理的條件馬上就能證明解的存在性。

Sketch the proof:

We rewrite (2.7) and (2.8) as the Euler-Lagrange equation of the functional E.

$$E(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 \, dx - \int_\Omega G(x,u)\, dx\ .$$

And restrict the space of admissible functions M to

$$M = \left\{ u \in H_0^{1,2}(\Omega)\ ;\ \underline{u} \leq u \leq \overline{u} \text{ almost everywhere} \right\}\ .$$

Then verify it satisfies the assumption of Theorem 1.2.

And using the Theorem 1.2 to get the minimizer $u \in M$. Then show that the minimizer u is just our weak solution.

To see that u weakly solves (2.7), for $\varphi \in C_0^\infty(\Omega)$ and $\varepsilon > 0$ let $v_\varepsilon = \min\{\overline{u}, \max\{\underline{u}, u + \varepsilon\varphi\}\} = u + \varepsilon\varphi - \varphi^\varepsilon + \varphi_\varepsilon \in M$ with

$$\varphi^\varepsilon = \max\{0, u + \varepsilon\varphi - \overline{u}\} \geq 0\ ,$$
$$\varphi_\varepsilon = \max\{0, \underline{u} - (u + \varepsilon\varphi)\} \geq 0\ .$$

Note that $\varphi_\varepsilon, \varphi^\varepsilon \in H_0^{1,2} \cap L^\infty(\Omega)$.

E is differentiable in direction $v_\varepsilon - u$. Since u minimizes E in M we have

$$0 \leq \langle (v_\varepsilon - u), DE(u)\rangle = \varepsilon\langle \varphi, DE(u)\rangle - \langle \varphi^\varepsilon, DE(u)\rangle + \langle \varphi_\varepsilon, DE(u)\rangle\ ,$$

so that

$$\langle \varphi, DE(u)\rangle \geq \frac{1}{\varepsilon}\left[\langle \varphi^\varepsilon, DE(u)\rangle - \langle \varphi_\varepsilon, DE(u)\rangle\right]\ .$$

Now, since $\overline{u}$ is a supersolution to (2.7), we have

$$\begin{aligned}\langle\varphi^{\varepsilon}, DE(u)\rangle &= \langle\varphi^{\varepsilon}, DE(\overline{u})\rangle + \langle\varphi^{\varepsilon}, DE(u) - DE(\overline{u})\rangle \\ &\geq \langle\varphi^{\varepsilon}, DE(u) - DE(\overline{u})\rangle \\ &= \int_{\Omega_{\varepsilon}} \{\nabla(u-\overline{u})\nabla(u+\varepsilon\varphi-\overline{u}) \\ &\qquad - \big(g(x,u)-g(x,\overline{u})\big)(u+\varepsilon\varphi-\overline{u})\}\, dx \\ &\geq \varepsilon\int_{\Omega_{\varepsilon}} \nabla(u-\overline{u})\nabla\varphi\, dx - \varepsilon\int_{\Omega_{\varepsilon}} \big|g(x,u)-g(x,\overline{u})\big|\, |\varphi|\, dx\ ,\end{aligned}$$

where $\Omega^{\varepsilon} = \big\{x\in\Omega\ ;\ u(x)+\varepsilon\varphi(x) \geq \overline{u}(x) > u(x)\big\}$. Note that $\mathcal{L}^n(\Omega^{\varepsilon}) \to 0$ as $\varepsilon \to 0$. Hence by absolute continuity of the Lebesgue integral we obtain that

$$\langle\varphi^{\varepsilon}, DE(u)\rangle \geq o(\varepsilon)\ ,$$

where $o(\varepsilon)/\varepsilon \to 0$ as $\varepsilon \to 0$. Similarly, we conclude that

$$\langle\varphi_{\varepsilon}, DE(u)\rangle \leq o(\varepsilon)\ ,$$

whence

$$\langle\varphi, DE(u)\rangle \geq 0$$

for all $\varphi \in C_0^{\infty}(\Omega)$. Reversing the sign of φ and since $C_0^{\infty}(\Omega)$ is dense in $H_0^{1,2}(\Omega)$ we finally see that $DE(u) = 0$, as claimed. □

第二部分主要是 Compensated Compactness，而主要的定理如下:

3.1 The compensated compactness lemma. *Let Ω be a domain in $\mathbb{R}^n$ and suppose that*
(1°) $u_m = (u_m^1, \ldots, u_m^N) \rightharpoonup u$ weakly in $L^2(\Omega;\mathbb{R}^N)$.
(2°) The set $\left\{\sum_{j,k} a_{jk}\frac{\partial u_m^j}{\partial x_k}\ ;\ m\in\mathbb{N}\right\}$ is relatively compact in $H_{loc}^{-1}(\Omega;\mathbb{R}^L)$ for a set of vectors $a_{jk}\in\mathbb{R}^L;\ 1\leq j\leq N,\ 1\leq k\leq n$. Let

$$\Lambda = \{\lambda\in\mathbb{R}^N\ ;\ \sum_{j,k} a_{jk}\lambda_j\xi_k = 0 \text{ for some } \xi\in\mathbb{R}^n\setminus\{0\}\}$$

and let Q be a (real) quadratic form such that $Q(\lambda)\geq 0$ for all $\lambda\in\Lambda$. Regarding $Q(u_m)\in L^1(\Omega)$ as Radon measures $Q(u_m)dx \in \left(C^0(\Omega')\right)^*$, we may assume that $\left(Q(u_m)\right)$ converges weak, locally.*
Then on any $\Omega' \subset\subset \Omega$ we have

$$\text{weak}^* - \lim_{m\to\infty} Q(u_m) \geq Q(u)$$

in the sense of measures. In particular, if $Q(\lambda)=0$ for all $\lambda\in\Lambda$, then

$$\text{weak}^* - \lim_{m\to\infty} Q(u_m) = Q(u)$$

locally, in the sense of measures.

而由上面那個可以推得底下的兩個重要定理

3.2 The Div-Curl Lemma. *Suppose $u_m \rightharpoonup u$, $v_m \rightharpoonup v$ weakly in $L^2(\Omega;\mathbb{R}^3)$ on a domain $\Omega \subset \mathbb{R}^3$ while the sequences $(\operatorname{div}\ u_m)$ and $(\operatorname{curl}\ v_m)$ are relatively compact in $H^{-1}(\Omega)$. Then for any $\varphi \in C_0^\infty(\Omega)$ we have*

$$\int_\Omega u_m \cdot v_m \varphi\, dx \to \int_\Omega u \cdot v \varphi\, dx$$

as $m \to \infty$.

3.4 Theorem. *Suppose $u_m \in H_0^{1,2}(\Omega)$ is a sequence of solutions to an elliptic equation*

$$\begin{aligned} -\Delta u_m &= f_m \qquad \text{in } \Omega \\ u_m &= 0 \qquad \text{on } \partial\Omega \end{aligned}$$

in a smooth and bounded domain Ω in $\mathbb{R}^n$. Suppose $u_m \rightharpoonup u$ weakly in $H_0^{1,2}(\Omega)$ while (f_m) is bounded in $L^1(\Omega)$. Then for a subsequence $m \to \infty$ we have $\nabla u_m \to \nabla u$ in $L^q(\Omega)$ for any $q < 2$, and $\nabla u_m \to \nabla u$ pointwise almost everywhere.

而其中定理 3.4 可以幫助我來解決以下狀況的問題。

Results like Theorem 3.4 are needed if one wants to solve nonlinear partial differential equations

$$-\Delta u = f(x, u, \nabla u) \tag{3.3}$$

with quadratic growth

$$|f(x,u,p)| \le c(1 + |p|^2)$$

而在第三部分是 concentration compactness principle，而這原理主要就是利用底下這個定理。

4.3 Concentration-Compactness Lemma I. *Suppose μ_m is a sequence of probability measures on $\mathbb{R}^n$: $\mu_m \ge 0$, $\int_{\mathbb{R}^n} d\mu_m = 1$. There is a subsequence (μ_m) such that one of the following three conditions holds:*
(1°) (Compactness) There exists a sequence $x_m \subset \mathbb{R}^n$ such that for any $\varepsilon > 0$ there is a radius $R > 0$ with the property that

$$\int_{B_R(x_m)} d\mu_m \ge 1 - \varepsilon$$

for all m.
(2°) (Vanishing) For all $R > 0$ there holds

$$\lim_{m\to\infty} \left(\sup_{x \in \mathbb{R}^n} \int_{B_R(x)} d\mu_m \right) = 0\ .$$

(3°) (Dichotomy) There exists a number λ, $0 < \lambda < 1$, such that for any $\varepsilon > 0$ there is a number $R > 0$ and a sequence (x_m) with the following property: Given $R' > R$ there are non-negative measures μ_m^1, μ_m^2 such that

$$\begin{aligned} &0 \le \mu_m^1 + \mu_m^2 \le \mu_m\ , \\ &\operatorname{supp}(\mu_m^1) \subset B_R(x_m),\ \operatorname{supp}(\mu_m^2) \subset \mathbb{R}^n \setminus B_{R'}(x_m)\ , \\ &\limsup_{m\to\infty} \left(\left| \lambda - \int_{\mathbb{R}^n} d\mu_m^1 \right| + \left| (1-\lambda) - \int_{\mathbb{R}^n} d\mu_m^2 \right| \right) \le \varepsilon\ . \end{aligned}$$

而由上面定理似乎看不出與微分方程之間的關係，而這定理一個常見的使用方式就是當我們考慮我們要 minimize 的 functional 如下，

$$J(f) = \int_{\mathbb{R}^n} F(x, f(x), \nabla f(x)) d\mathcal{L}^n(x) \ ,$$

且要滿足以下的限制式時

$$\int_{\mathbb{R}^n} |f(x)| d\mathcal{L}^n(x) = 1 \ .$$

我們運用上述定理的方式主要流程如下：

通常在這種問題都還會有額外假設就是我們的 functional 有著有限的 lower bound，所以我們主要的問題通常是我們的取的 sequence 是否是收斂到 minimizer。而我們要如何運用上面的定理，主要是考慮以下的測度。

$$\mu_j(E) = \int_E |f_j(x)| d\mathcal{L}^n(x)$$

而通常如果我們可以排除第二種和第三種的 case of concentration compactness principle，那可以得到 μ_j would converge weakly to some probability measure μ 而若我們又能在證明 $\mu << \mathcal{L}^n$, then $\mu = f\mathcal{L}^n$ and f_j 會弱收斂到一個L^1函數 f，那這 f 就會是我們要的 minimizer.

底下就是一個照此流程證出定理的例子，

4.9 Theorem. *Let $k \in \mathbb{N}$, $p > 1$, $kp < n$, $\frac{1}{q} = \frac{1}{p} - \frac{k}{n}$. Suppose (u_m) is a minimizing sequence for S in $D^{k,p} = D^{k,p}(\mathbb{R}^n)$ with $\|u_m\|_{L^q} = 1$. Then (u_m) up to translation and dilation is relatively compact in $D^{k,p}$.*

第四個部分則是 Ekeland's Variational Principle

這提供了一個幫助我們 construct the minimizing sequence 的方式，因為在前面的討論，我們都只有形式化的討論一個 minimizing sequence，並沒有寫下如何構造。而底下定理就是描述一個構造的方式：

5.1 Theorem. *Let M be a complete metric space with metric d, and let $E: M \to \mathbb{R} \cup +\infty$ be lower semi-continuous, bounded from below, and $\not\equiv \infty$.*
Then for any $\varepsilon, \delta > 0$, any $u \in M$ with

$$E(u) \le \inf_M E + \varepsilon,$$

there is an element $v \in M$ strictly minimizing the functional

$$E_v(w) \equiv E(w) + \frac{\varepsilon}{\delta} d(v, w) \ .$$

Moreover, we have

$$E(v) \le E(u), \quad d(u, v) \le \delta \ .$$

而上述定理證明流程大概如下，

1. 藉由 define a partial ordering on $M \times \mathbb{R}$

$$(v,\beta) \le (v',\beta') \Leftrightarrow (\beta' - \beta) + \alpha\, d(v,v') \le 0\ .$$

   且令 $S = \{(v,\beta) \in M \times \mathbb{R}\ ;\ E(v) \le \beta\}$

2. 而藉由:

   5.2 Lemma. *S contains a maximal element (v,β) with respect to the partial ordering $\le$ on $M \times \mathbb{R}$ such that $(u, E(u)) \le (v,\beta)$.*

   我們可以讓最大值就是 $(v, E(v)) \in S$

   the statement $(u,E(u)) \le (v,E(v))$ translates into the estimate

$$E(v) - E(u) + \alpha\, d(u,v) \le 0\ ;$$

   Then $E(v) \le E(u)$ and $d(u,v) \le \alpha^{-1}(E(u) - E(v)) \le \frac{\delta}{\varepsilon}\left(\inf_M E + \varepsilon - \inf_M E\right) = \delta$

   所以如果有個 w 滿足 $E_v(w) = E(w) + \alpha\, d(v,w) \le E(v) = E_v(v)$

   則會有 $(v, E(v)) \le (w, E(w))$ 但因為 $(v, E(v)) \in S$ 是最大值，所以 v=w，所以 v is a strict minimizer of E_v，則得證。

第五個部分是 Minimization Problems Depending on Parameters:

其中用到 Penalty method 是我覺得也很酷的想法，其設定如下:

7.1 Penalty method. Suppose that V is a Banach space and let E and G be non-negative functionals on V. We seek to minimize E on the set

$$M = \{u \in V\ ;\ G(u) = 0\}$$

of admissible functions.

而這個的作法不是硬去考慮符合限制的函數然後在嘗試求解，因為我們在這邊並沒有對於我們有任何 smooth 之類的條件，所以如果單純以上面講的方法嘗試求解應該是非常困難，所以我們將考慮以下的 Functional 序列，

$$E_\epsilon(u) = E(u) + \epsilon^{-1} G(u), \qquad u \in V.$$

而且同時我們我將我們原本的 admissible space 擴增到整個 V。

而這樣設計的好處是，當$\varepsilon \to 0$時，我們 minimizers of E_ϵ 會很接近 M，所以會 converge to a solution of minimization Problem.

(而我覺得這個做法的想法有點像是我們在微積分使用拉格朗日 multiplier 求極值的方法很像，我們考慮一個更 general 空間然後改變我們求極值的函數，來處理有一些 constraint 下的原函數的極值問題。) 而這也是上課用到 Ginzburg Landau model 的時候所講的解法。

最後快速說明一下第二章我所學到的。

第二章主要是在介紹我們要怎麼研究 saddle point，因為平常都是處理最大最小值，而 Minimax Method 可以幫助我們研究在有足夠好性質的 Functional 的 saddle point 的存在性。

一開始先討論了有限維空間空間下的情況

1.1 Theorem. *Suppose $E \in C^1(\mathbb{R}^n)$ is coercive and suppose that E possesses two distinct strict relative minima x_1 and x_2. Then E possesses a third critical point x_3 which is not a relative minimizer of E and hence distinct from x_1, x_2, characterized by the minimax principle*

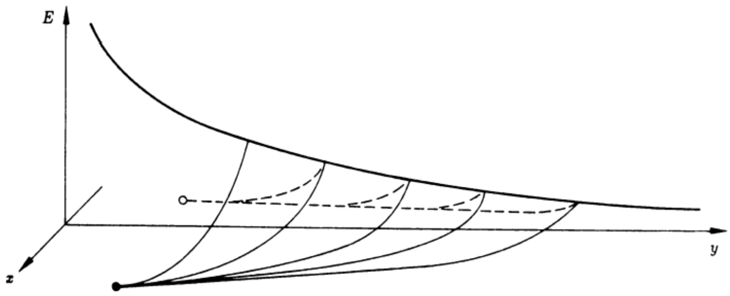
$$E(x_3) = \inf_{p \in P} \max_{x \in p} E(x) =: \beta \ ,$$

where

$$P = \{p \subset \mathbb{R}^n \ ; \ x_1, x_2 \in p, \ p \text{ is compact and connected}\}$$

is the class of "paths" connecting x_1 and x_2.

而這個定理很重要的一個條件是 P 是 compact 的因為正如老師上課舉例的圖，



我們知道這個圖中並沒有 saddle point。

所以我們接下來為了推廣至無限維空間，我們需要有一些條件。

所以有了底下兩種條件

(C) If S is a subset of V on which $|E|$ is bounded but on which $\|DE\|$ is not bounded away from zero, then there is a critical point in the closure of S.

Definition. *A sequence (u_m) in V is a Palais-Smale sequence for E if $|E(u_m)| \le c$, uniformly in m, while $\|DE(u_m)\| \to 0$ as $m \to \infty$.*

In terms of this definition our compactness condition may be phrased as follows.

(P.-S.) Any Palais-Smale sequence has a (strongly) convergent subsequence.

而可以發現第二個條件比第一個更嚴格，所以之後我們只需要驗證第二個條件

2.1 Proposition. *Suppose $E \in C^1(\mathbb{R}^n)$ and assume the function $\|DE\| + |E|: \mathbb{R}^n \to \mathbb{R}$ is coercive. Then (P.-S.) holds for E.*

2.2 Proposition. *Suppose that E has the following properties.*
(1°) Any Palais-Smale sequence for E is bounded in V.
(2°) For any $u \in V$ we can decompose

$$DE(u) = L + K(u) ,$$

where $L: V \to V^*$ is a fixed boundedly invertible linear map and the operator K maps bounded sets in V to relatively compact sets in V^*. Then E satisfies (P.-S.).

而最後底下給出的就是 Mountain Pass Lemma。他是用來證明 existence for saddle points.

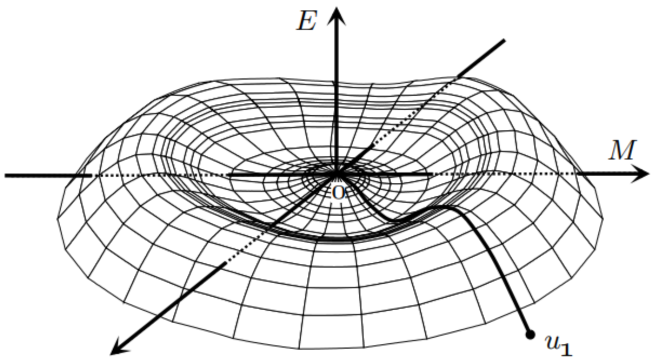
6.1 Theorem. *Suppose $E \in C^1(V)$ satisfies (P.-S.). Assume that*
(1°) $E(0) = 0$;
(2°) $\exists \rho > 0,\ \alpha > 0 :\ \|u\| = \rho \Rightarrow E(u) \geq \alpha;$
(3°) $\exists u_1 \in V :\ \|u_1\| \geq \rho$ and $E(u_1) < \alpha$.
Define

$$P = \{p \in C^0([0,1];V)\ ;\ p(0) = 0,\ p(1) = u_1\} .$$

Then

$$\beta = \inf_{p\in P} \sup_{u\in p} E(u) \geq \alpha$$

is a critical value.



這定理幾何圖象就是將我們 functional 的值當作高度，然後如果當從原點王外走走到比原本更低的位置時，其中原點與那個點中間被一個山所形成的環索圍住時，就像上圖一樣，那這樣一定會存在鞍點。

結語：在上完這一堂課以及讀了這一本書後，我覺得學到了很多分析的技巧，以及如何從已知的東西慢慢推廣成 general 的形式，以及如何將物理中對於系統的一些直覺性的猜測，如何把它嚴謹的運用到解其微分方程，讓我們更方便得到答案，也感謝老師這一整學期的教導，讓我從這堂課中更加了解到原來以前學的物理系統實際分析時所會遇到的狀況，有些時候可以靠物理直覺給出五們對於系統更深的理解，但其實也可以由方程式的角度，可以從已知的數學性質給出我們更好的預測，真的覺得是一件很酷的事情。

REF: Michael Struwe Variational Methods