Harmonic function on Manifold with Negative Curvature

In this report, we consider the Manifold with negative sectional curvature $-b^2 \leq K_M \leq -a^2 < 0$, which is Hadamard manifold.
By Cartan-Hadamard theorem, we have M diffeomorphic to $\mathbb{R}^n$ and the geodesics on M have many good properties, so we can have some good estimation on gradient or laplacian and the green function exists.
Then we listed the goals of this report.
Firstly, we compactify M by adding $S(\infty)$, then $\overline{M} := M \cup S(\infty)$ is compact in the cone topology.
Secondly, we prove the existence and uniquiness of Dirichelet problem on $\overline{M}$ with the given value on $S(\infty)$.
In the bounded domain of $\mathbb{R}^n$, we have many useful property like Poisson kernel and Martin representation formula for positive harmonic function on M. So in the following section we try to prove that it also works on $\overline{M}$.
Third part of this report is to constract Poisson kernel and prove the uniqueness and existence of Poisson kernel.
Fourth part is using the concept of constructing Poisson kernel to define Martin Boundary. Then Using some result of measure theory and harmonic measure, then we can derive Martin Integral representation.
Lastly, we prove the very important Harnack inequalities.

1. Compactified of M

1.1 Preliminary

Assume that M is simply connected, complete, n-dimensional Riemannian manifold with metric g whose sectional curvature K_M satisifies

$$-b^2 \leq K_M \leq -a^2 < 0 \text{ where } a, b \text{ are positive constant.} \quad (1,1)$$

Some notations we may use in the whole reports.

1. AB means the unique geodesic joining A and B. ($A, B \in M$)
2. $\triangle ABC$ is the geodesic triangle, where $A, B, C \in M$
3. Angle of two vectors V_1, V_2 at p is defined by $\cos\theta = \dfrac{g(V_1, V_2)}{\sqrt{g(V_1,V_1)}\sqrt{g(V_2,V_2)}}$
4. $\angle A$ is the angle between AB and AC.
5. $H(r)$ means the simply connected form with constant curvature r.

Then we list some theorems we may use without proof.

1. Cartan-Hadamard theorem:

   M is simply connected, complete, n-dimensional Riemannian manifold with all sectional curvatures of any P on M is non-positive.

   Then $\exp_p: T_pM \longrightarrow M$ is diffeomorphism.

2. Rauch Comparison Theorem

   Suppose (1,1) is hold, and let $\rho(x)$ be the distance of x to a fixed point $P \in M$. Then we have following inequality.

   $$a\coth(a\rho)(g - d\rho \otimes d\rho) \leq D^2\rho \leq b\coth(b\rho)(g - d\rho \otimes d\rho)$$

3. Toponogor Comparison Theorem

Let (1.1) hold and Let $\triangle ABC$ be a geodesic triangle in M.

Suppose $\triangle A'B'C'$ is geodesic triangle in $H(-b^2)$ and $\triangle A''B''C''$ is geodesic triangle in $H(-a^2)$ with the following condition.

If $|A'B'|=|A''B''|=|AB|$, $|B'C'|=|B''C''|=|BC|$, $|C'A'|=|C''A''|=|CA|$

Then we have $\angle A' \leq \angle A \leq \angle A''$.

If $|A'B'|=|A''B''|=|AB|$, $|A'C'|=|A''C''|=|AC|$, $\angle A'=\angle A''=\angle A$.

Then we have $|B'C'| \geqq |BC| \geqq |B''C''|$.

4. Suppose $\triangle ABC$ is a geodesic triangle in the space form $H(-a^2)$ and if we set $\rho(A,B)=r$ $\rho(A,C)=\sigma$ $\rho(B,C)=s$ and $\angle A=\theta$.

Then $\cos\theta \sinh ar \sinh a\sigma = \cosh ar \cosh a\sigma - \cosh as$.

With 3. and 4., we can immediately get the following result.

Proposition 1.1.1 Let M be a simply connected, complete Riemannian manifold whose sectional curvature $-b^2 \leq K_M \leq -a^2 < 0$. Let $\triangle ABC$ be a geodesic triangle in M with $\rho(A,B)=r$ $\rho(A,C)=\sigma$ $\rho(B,C)=s$. and $\angle A=\theta$.

Then $\cos\theta \geqq \coth ar \coth a\sigma - \dfrac{\cosh as}{\sinh ar \sinh a\sigma}$

$\cos\theta \leq \coth br \coth b\sigma - \dfrac{\cosh bs}{\sinh br \sinh b\sigma}$

(pf) Use the notation in 3. $\angle A'=\theta'$ $\angle A''=\theta''$

We have $\theta' \le \theta \le \theta'' \Rightarrow \cos\theta' \ge \cos\theta \ge \cos\theta''$

由3. $\exists$ $\cos\theta' = \coth br \coth b\sigma - \frac{\cosh bs}{\sinh br \sinh b\sigma} \ge \cos\theta$

$$\cos\theta'' = \coth ar \coth a\sigma - \frac{\cosh as}{\sinh ar \sinh a\sigma} \le \cos\theta$$

□

Proposition 1.1.2 Let M have the same condition as Proposition 1.1. Let x_1、x_2、$O \in M$. Suppose that $\rho(O,x_1)=\rho(O,x_2)=r$. Denote the geodesic ray from O to x_i as γ_i for $i=1,2$. And Denote θ as the angle between γ_1 and γ_2 at O. Then for large enough r and suffiently small θ we have the following stimation of $\rho(x_1,x_2)$.

$$2r + \frac{2}{a}(\log\theta - 1) \le \rho(x_1,x_2) \le 2r + \frac{2}{b}(\log\theta + 1)$$

(pf) Let $\sigma = r$ and $\rho(x_1,x_2)=s$, From Proposition 1.1 we have,

$$\cos\theta \sinh^2 ar \ge \cosh^2 ar - \cosh as$$

$$\cos\theta \sinh^2 ar \ge 1 + \sinh^2 ar - \cosh as$$

$$\cosh as - 1 \ge (1-\cos\theta)\sinh^2 ar$$

$$2\sinh^2\frac{as}{2} \ge (1-\cos\theta)\sinh^2 ar \quad ①$$

Let c be the constant which $1 < c < \frac{e}{2}$ and satisify the following conditions.

$2c^2(1-\cos\theta) \le \theta^2$ (when θ is small enough) ②

$e^{ar} - e^{-ar} \ge 2ce^{ar-1}$ (when r is large enough) ③

Then substitute the right terms in ① with ②、③.

$$2\sinh^2\frac{as}{2} \ge \frac{1}{2}e^{2(ar-1)}\theta^2. \quad ④$$

$\ln$ ④ $\Rightarrow \ln 2 + 2\ln\left(\frac{(e^{\frac{as}{2}} - e^{\frac{-as}{2}})}{2}\right) \geq -\ln 2 + 2(ar-1) + 2\ln\theta$

$\Rightarrow 2\ln(e^{\frac{as}{2}} - e^{\frac{-as}{2}}) \geq 2(ar-1) + 2\ln\theta$

$\Rightarrow as \geq 2ar + 2(\ln\theta - 1)$

For any side of inequality, let $6 = r$, $\rho(x_1, x_2) = s$

$\cos\theta \sinh^2 br \leq \cosh^2 br - \cosh bs$

$(\cos\theta - 1)\sinh^2 br \leq 1 - \cosh bs = -2\sinh^2(\frac{bs}{2})$

$4\sinh^2(\frac{bs}{2}) \leq 2(1-\cos\theta)\sinh^2 br$

Let c be a constant with $c > 1$

$2c^2(1-\cos\theta) \geq \theta^2$

$e^{br} - e^{-br} \leq 2ce^{br}$

$4\sinh^2(\frac{bs}{2}) \leq \frac{\theta^2}{c^2} \cdot \frac{4c^2}{4} e^{2br}$

$4 \cdot \frac{1}{4}(e^{\frac{bs}{2}} - e^{\frac{-bs}{2}})^2 \leq \theta^2 e^{2br}$

$2\ln(e^{\frac{bs}{2}} - e^{\frac{-bs}{2}}) \leq 2\ln\theta + 2br$

$2(\frac{bs}{2} - 1) \leq 2\ln\theta + 2br$

$\Rightarrow bs \leq 2\ln\theta + 2(br+1)$

$\Rightarrow s \leq 2r + 2\left(\frac{\ln\theta + 1}{b}\right)$ □

1.2 Geometric boundary $S(\infty)$ and Compactification of M.

Define: Let γ_1、γ_2 be the two geodesic rays. Then two geodesic rays γ_1、γ_2 are said to be equivalent $\gamma_1 \sim \gamma_2$ if $\exists C > 0$, $\rho(\gamma_1(t), \gamma_2(t)) \leq C \quad \forall t \geq 0$.

Define: The sphere at infinity $S(\infty)$ is defined to be the space of all equivalence classes of geodesic rays.

$$S(\infty) := \text{The set of all geodesic rays}/\sim$$

We skip the proof of following lemma.

Lemma 1.2.1 For any geodesic ray σ and point $m \in M$, there is a geodesic ray γ starting at m and γ is equivalent to σ.

With the above lemma 1.2.1 and proposition 1.1.2, we can identify $S(\infty)$ with the set of all geodesic ray starting at a given fixed point O (or more percisely can be identified as the unit sphere of T_OM) in the following steps.

First, from lemma 1.2.1, any equivalent class of geodesic rays can be represented by a geodesic passing O.

Secondly, any two geodesic rays γ_1, γ_2 with $\gamma_1(0) = \gamma_2(0) = O$ is equivalent if and only if $\gamma_1 = \gamma_2$. Because of Proposition 1.1.2, we have $2t + \frac{2}{a}(\log\theta - 1) \leq \rho(\gamma_1(t), \gamma_2(t)) \leq C$, $\forall t \geq 0$ (θ is the angle between γ_1 and γ_2 at O.

But as $t \to \infty$, we must have $\theta = 0$.

Then we can identify $S(\infty)$ as the sphere of T_OM.

Now, we are going to compactify M.

Define: For any $O \in M$ $v \in T_OM \setminus \{0\}$, define the cone $C_O(v, \delta)$ by $C_O(v,\delta) := \{x \in M : \angle(v, T_{\overline{Ox}}) < \delta\}$ where $\angle(v, T_{\overline{Ox}})$ denote the angle between v and the tangent vector of geodesic OX at O.

And define $T_O(v, \delta, R) = C_O(v, \delta) \setminus B_O(R)$ which is called truncated cone. (Where $B_O(R)$ is the geodesic Ball centered at O with radius R).

Then, all truncated cone $\{T_O(v, \delta, R) : \delta > 0 \ R > 0\}$ and all geodesic Balls $\{B_q(r) : q \in M, r > 0\}$ form a local basis of "cone topology" of $\bar{M} = M \cup S(\infty)$. And this topology gives a compactification of M.

2. The Dirichelet problem for harmonic function on M with the Dirichelet boundary condition posed on $S(\infty)$.

Before we prove the existence and uniqueness of solution for Dirichelet problem on $M \cup S(\infty)$, we introduce some properties we may use in the following proof.

Define: A fuction $\varphi \in C^2(M)$ is said to be subharmic if $\Delta\varphi \geq 0$,
superharmonic if $\Delta\varphi \leq 0$.

Briefly introducing Perron method which is the way to construct the solution of Dirichelet problem $\Delta u = 0$ on Ω and $u = g$ on $\partial\Omega$.

Perron method constructs the solution in the following steps.

1. Consider S_g be the set of all subharmonic functions $v \in C^0(\bar{\Omega})$ with $v \leq g$ on $\partial\Omega$. Then consider $u(x) := \sup_{v \in S_g} v(x)$.

(Since $\mathrm{Min}_{\partial\Omega}\, g$ a constant function which belongs S_g. So S_g is not empty.)

Easy to see that $\min_{\partial\Omega} g \leq u(x) \leq \max_{\partial\Omega} g$. $u : \Omega \to \mathbb{R}$

2. Then use the trick of harmonic lifting of subharmonic function.

(Harmonic lifting of w which is a subharmonic function
$\bar{w}(x) := w(x)$ when $x \in \Omega \setminus B_a(r)$ $a \in \Omega$, $r \in \mathbb{R}^+$.
$= h(x)$ when $x \in B_a(r)$
(where $h(x)$ be the solution of $\Delta h = 0$ on $B_a(r)$ $h = w$ on $\partial B_a(r)$.))

We can prove that u is harmonic and belong to $C^2(\Omega)$.

3. The we can prove that u can be extended such that u belongs to $C^0(\bar{\Omega})$ and $u|_{\partial\Omega}=g$.

The key is to find barrior functions u_+ and u_- such that $u_- \le u \le u_+$ which u_- is subharmonic and u_+ is a superharmonic function. and u_+ & u_- have boundary value very close to g, then we find the sequence of $\{u_{+i}\}$ and $\{u_{-i}\}$ st $u_+|_{\partial\Omega} \to g$.
$u_-|_{\partial\Omega} \to g$.

Then we know the such u also satisfies the boundary condition

Remark: The above result is for the bounded domain.

However the case we may meet is on a unbounded domain M.

But we can use the similar way to construct u which is harmonic on M even though u may not satisfy the boundary condition.

Theorem 2.1. Let M be simply connected, complete, n-dimensional Riemannian manifold whose sectional curvature K_M satisifies $-b^2 \le K_M \le -a^2 < 0$ where a & b are positive constant.

Then, given any $\varphi \in C^0(S(\infty))$, there exist a unique harmonic function $u \in C^\infty(M) \cap C^0(\bar{M})$ such that $u|S(\infty) = \varphi$.

(pf) Fixed $O \in M$, identity $S(\infty)$ as $S_O(1)$ (unit sphere in T_OM).

WLOG, we let $\varphi \in C^\infty(S_O(1))$. Because any continuous function on $S_O(1)$ can be approximated by smooth function, and uniform convergence of harmonic functions on $S(\infty)$ implies uniform convergence of the sequence on $\bar{M}$ by Maximum principle.

From Cartan-Hadamard theorem, we know $\exp_O : T_OM \to M$ is

diffeomorphism, so let (r,θ) be the normal polar coordinate at O.
Then φ can be written as $\varphi(\theta)$. And extend φ to $M\setminus\{o\}$ by defining $\varphi(r,\theta)=\varphi(\theta)$ $\forall r>0$. (Note that this extention is smooth and bounded in $M\setminus\{o\}$)

Define: $osc_{B_x(1)}\varphi = \sup_{y\in B_x(1)} |\varphi(y)-\varphi(x)|$

Then the proof follows the following steps.

1. $osc_{B_x(1)}\varphi = O(e^{-a\rho(x)})$
2. Take the average $\overline{\varphi}$ of φ which satisifies $\Delta\overline{\varphi}=O(e^{-a\rho(x)})$
3. Consider $g(x)=e^{-\delta\rho(x)}$, and we can find $\alpha>0$ such that $\Delta(\overline{\varphi}+\alpha g)\le 0$ and $\Delta(\overline{\varphi}-\alpha g)\ge 0$, then use the classical Perron method to construct a harmonic function u.
4. Check u satisisfy the boundary condition and the uniqueness follows from maximum principle.

Then we are going to prove each step.

1. Let $y\in B_x(1)$, $|\varphi(y)-\varphi(x)|=|\varphi(\theta')-\varphi(\theta)|\le C|\theta'-\theta|$ $\left(\because S_o(1) \text{ is compact}\right)$

 where θ、θ' are the spherical coordinate of x、y.

 By Propisition 1.1.2, $2\rho(x)+\frac{2}{a}(\log|\theta-\theta'|-1)\le\rho_x(y)\le 1$

 where $\rho(x)=\rho(0,x)$, $\rho_x(y)=\rho(x,y)$.

 $\Rightarrow |\theta-\theta'|\le C_1 e^{-a\rho(x)}$, C_1 depend only on a.

 $\Rightarrow osc_{B_x(1)}\varphi\le Ce^{-a\rho(x)}$, C depend on a、φ. $\square$

2. Let $\chi\in C_0^\infty(\mathbb{R})$ with $0\le\chi\le 1$, $\chi(t)=0$ for $|t|\ge 1$, $\chi(t)=1$ for $|t|\le\frac{1}{2}$.

Define $\overline{\varphi}(x) = \dfrac{\int_M \chi(\rho_x^2(y))\varphi(y)\,dy}{\int_M \chi(\rho_x^2(y))\,dy}$

$$\Rightarrow |\overline{\varphi}(x) - \varphi(x)| = \frac{\int_M \chi(\rho_x^2(y))(\varphi(y) - \varphi(x))\,dy}{\int_M \chi(\rho_x^2(y))\,dy}$$

$$\leq \sup_{B_x(1)} |\varphi(y) - \varphi(x)| = osc_{B_x(1)}\varphi = O(e^{-a\rho(x)}) \text{ (By 1.)}$$

$$\Delta\overline{\varphi}(x_0) = \Delta(\overline{\varphi}(x) - \varphi(x_0))\big|_{x=x_0}$$

$$= \int_M \Delta\left(\frac{\chi(\rho_y^2(x))}{\int_M \chi(\rho_y^2(x))\,dy}\right)(\varphi(y) - \varphi(x_0))\,dy\Big|_{x=x_0}$$

But we know that $\Delta\left(\frac{u}{v}\right) = \nabla\cdot\left(\frac{\nabla u}{v} - \frac{u\nabla v}{v^2}\right)$

$$= \frac{\Delta u}{v} - \frac{\nabla u\cdot\nabla v}{v^2} - \frac{\nabla u\cdot\nabla v}{v^2} - \frac{u\Delta v}{v^2} + \frac{2u(\nabla v\cdot\nabla v)}{v^3}$$

Then we let $u = \chi(\rho_y^2(x))$, $v = \int_M \chi(\rho_y^2(x))\,dy$, $\rho = \rho_y(x)$.

$$\nabla u = \chi'(\rho^2)\,2\rho\nabla\rho, \quad \Delta u = 4\rho^2\chi''(\rho^2)|\nabla\rho|^2 + 2\chi'(\rho^2)|\nabla\rho|^2 + 2\rho\chi'(\rho^2)\Delta\rho.$$

Because the assumption of M is $-b^2 \leq K_M \leq -a^2$, so we have $Ric(M) \geq -(n-1)b^2 \Rightarrow$ From the corollary 1.2 in the textbook, we have $\Delta\rho \leq \frac{n-1}{\rho}(1+b\rho) \Rightarrow \rho\Delta\rho \leq (n-1)(1+b\rho) \leq (n-1)(1+b)$ for $\rho \leq 1$. And from $|\nabla\rho| = 1$.

Then it's easy to see that $|\nabla u|$ and $|\Delta u|$ is bounded and $|\nabla v|$ and $|\Delta v|$ is bounded.

On the other hand, with volume comparison theorem in textbook (1.20) we have $v(x) = \int_M \chi(\rho_x^2(y))\,dy \geq Vol\, B_x(\frac{1}{2}) \geq C$ (some constant)

Then we have $\Delta\left(\frac{u}{v}\right)$ is bounded

So $\Delta\overline{\varphi}(x_0) \leq \int_M \Delta\left(\frac{u}{v}\right)(\varphi(y)-\varphi(x_0))dy\Big|_{x=x_0}$

$\leq C\, \mathrm{osc}_{B_x(1)}\varphi = O(e^{-a\rho(x)})$

We notice that $|\Delta(\frac{u}{v})|=0$ when $\rho_y(x)>1$ and $\Delta(\frac{u}{v})$ is bounded in $\rho_y(x)\leq 1$, so we can get the result.

$\square$

3. $g(x)=e^{-\delta\rho(x)}$ $\quad \Delta g = \nabla\cdot(-\delta e^{-\delta\rho(x)}\nabla\rho) = \delta^2 e^{-\delta\rho(x)}|\nabla\rho|^2 - \delta e^{-\delta\rho(x)}\Delta\rho$

Then from the homework 2, we have the lower bound of laplacian of ρ under the condition that each sectional curvature K_M with $K_M\leq -a^2$. We have $\Delta\rho \geq (n-1)a\coth a\rho \geq (n-1)a = C_1>0$

So $\Delta g \leq e^{-\delta\rho(x)}(\delta^2|\nabla\rho|^2 - \delta C_1)$ and when δ is small enough we can have $\Delta g<0$. ($\because C_1$ is independent with δ.)

Because $\Delta g<0$ and $\Delta\overline{\varphi} = O(e^{-a\rho(x)})$, so we can find a $\alpha>0$ such that $\Delta(\alpha g)\leq -|\Delta\overline{\varphi}| \Leftrightarrow \Delta(\overline{\varphi}+\alpha g)\leq 0$ and $\Delta(\overline{\varphi}-\alpha g)\geq 0$,

(The reason why we can find such α is $\Delta\overline{\varphi}$ decay fast than $|\Delta g|$.)

Then from the Perron method, we can construct u is harmonic function with $\overline{\varphi}-\alpha g\leq u\leq\overline{\varphi}+\alpha g$. $\square$

4. $|u-\varphi|(x) \leq |u-\overline{\varphi}+\overline{\varphi}-\varphi|(x) \leq |u-\overline{\varphi}|(x)+|\overline{\varphi}-\varphi|(x)$

$\leq \alpha g(x) + |\overline{\varphi}-\varphi|(x)$

$\leq Ce^{-\delta\rho(x)} + Ce^{-a\rho(x)}$ (from 2. and 3.)

$\to 0$ as $\rho(x)\to\infty$, so u satisify the boundary condition.

$\square$

3. Poisson Kernel and Harnack Inequality

In this section, we introduce the Poisson kernel and prove the existence and uniqueness of Poisson kernel.

Before we define Poisson kernel, we dicuss the Green function on M. From the Appendix of chapter 2 in the textbook, we have the following theorem A.1

Theorem A.1: Let M be a noncompact, complete, Riemannian manifold suppose that there exist $\varphi \in C^2(M)$ satisisfing $\varphi > 0$, $\Delta\varphi \leq 0$ and $\varphi \to 0$ as $\rho(x) \to \infty$. Then there exist an entire green function on M.

First, our manifold M satisisfies the condition of Theorem A.1.

Secondly, oberserve $e^{-k\rho}$, $\Delta e^{-k\rho} = e^{-k\rho} k(k|\nabla\rho|^2 - \Delta\rho)$. so when k is small enough and $k > 0$, we have $\Delta e^{-k\rho} < 0$. Then we find the $\varphi = e^{-k\rho}$ satisisfing the condition Theorem A.1. So we have the entire green function on M.

However, with Corollary 2.1 in the textbook, we have $\Delta\rho \leq \frac{n-1}{\rho}(1+b\rho)$. Then when $\rho \geq 1$, $\Delta\rho \leq (b+1)(n-1)$. So $\Delta e^{-k\rho} > 0$ for $\rho \geq 1$ when k is large enough.

Summary the above discussion, for a small enough $\delta > 0$, we have $\Delta e^{-\delta\rho} < 0$ on M and $\Delta e^{-\frac{\rho}{\delta}} > 0$ on $M \setminus B_o(1)$ and with Theorem A.1 we have the entire green function on M.

Then we recall some property of green function.

1. $G(x,y) = G(y,x)$ and for a fixed $y \in M$, $\Delta_x G(x,y) = 0$ $\forall x \in M \setminus \{y\}$
2. $G(x,y) \geq 0$, $\forall x, y \in M$
3. As $x \to y$, $G(x,y) = \begin{cases} \rho_y^{2-n}(x)(1+O(1)) & \text{if } n > 2 \\ -(\log \rho_y(x))(1+O(1)) & \text{if } n = 2 \end{cases}$

For any $y\in M$, let $\sup_{\partial B_y(1)} G(x,y)e^{\delta \rho_y(x)} = c_1 > 0$, $\inf_{\partial B_y(1)} G(x,y)e^{\frac{\rho_y(x)}{\delta}} = c_2 > 0$

$c_1 \Delta e^{-\delta \rho_y} < \Delta G = 0 < c_2 \Delta e^{-\frac{\rho_y}{\delta}}$

$\Rightarrow c_1 e^{-\delta \rho_y} \geq G \geq c_2 e^{-\frac{1}{\delta}\rho_y}$ on $\partial B_y(1)$.

From maximum principle we have $c_2 e^{-\frac{1}{\delta}\rho_y(x)} \leq G(x,y) \leq c_1 e^{-\delta \rho_y(x)}$, $\forall x \in M \setminus B_y(1)$

So it's easy to see that $G(x,y) \longrightarrow 0$ uniformly as $\rho_y(x) \to \infty$, we see that

$G(\cdot,y)$ can be continuously extended to $\bar{M}\setminus\{y\}$ with $G(\cdot,y)|S(\infty) = 0$.

Let $O\in M$ be the base point. For fixed $y \in M\setminus\{O\}$ consider that,

$h_y(x) = \frac{G(x,y)}{G(O,y)}$ which is a positive harmonic function for $x\in M\setminus\{y\}$

with $h_y(O) = 1$ and having a unique pole y. We call $h_y(x)$ a positive

harmonic function normalized at O.

Now we can start to define and construct Poisson kernel.

Define: A harmonic function $P_\xi(x)$ on M is called a Poisson kernel normalized at O for $\xi \in S(\infty)$ iff it satisifies.

1. $P_\xi(x) > 0$ $\forall x \in M$ 2. $P_\xi(O) = 1$ 3. $P_\xi \in C^0(\bar{M}\setminus\xi)$ and $P_\xi|S(\infty)\setminus\{\xi\} = 0$.

However, in the construction of Poisson kernel we may need the following two Harnack Inequalities which we will prove them in the appendix.

Theorem 3.1: Fixing $O\in M$ and given an $v\in T_O M$, let h be a positive harmonic function on $C_O(v,\frac{\pi}{4})$ such that it is continuous in the closure of $C_O(v,\frac{\pi}{4})$ and vanishs on $\bar{C}_O(v,\frac{\pi}{4}) \cap S(\infty)$.

(Note that $C_O(v,\theta)$ and $T_O(v,\theta,R)$ are defined in the last of Section 1.)

Then we have the following estimate

$h(x) \le C_1 h(0') e^{-C_2 \rho(x)}$ $\quad \forall x \in T_0(V,\frac{\pi}{8},1)$ (where C_1, C_2 are positive and only depending on n, a, b)

$0' = \exp_0(V)$

Theorem 3.2: Fixing $0 \in M$ and given an $V \in T_0M$, Let u, w be two positive harmonic function on $C_0(V,\frac{\pi}{4})$ such that they are continuous in the closure of $C_0(V,\frac{\pi}{4})$ and vanishs on $\overline{C_0}(V,\frac{\pi}{4}) \cap S(\infty)$.

Then we have $C_1^{-1}\frac{u(0')}{w(0')} \le \frac{u(x)}{w(x)} \le C_1 \frac{u(0')}{w(0')}$, $\forall x\in T_0(V,\frac{\pi}{8},1)$

(where C_1 is positive number depending only on n, a, b).

Now, the following propositions are about the existence and uniqueness of Poisson kernel.

Proposition 3.3: A Poisson kernel exists for any $\xi \in S(\infty)$.

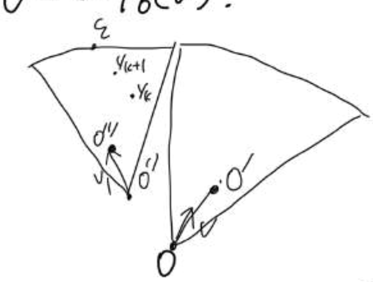
(pf) Let $h_y(x)$ be the positive harmonic function normalized at 0 with pole y be $\frac{G(x,y)}{G(0,y)}$. Let $\{y_k\}$ be sequence such that $y_k \to \xi \in S(\infty)$.

We show that along a subsequence h_{y_k} converge to Poisson kernel P_ξ.

First, if we pick $V\in T_0M$ with $\xi \notin \overline{C_0}(V,\frac{\pi}{4})\cap S(\infty)$, then for k is large enough h_{y_k} is positive harmonic function on $C_0(V,\frac{\pi}{4})$ and h_{y_k} vanishs on $\overline{C_0}(V,\frac{\pi}{4})\cap S(\infty)$. ($\because G(x,y_k) \longrightarrow 0$ as $\rho_{y_k}(x)\to\infty$).)

Claim: $h_{y_k}(0') \le C$ for k is large enough where $0'=\exp_0(V)$.

(pf) $h_{y_k}(0') = \dfrac{G(0',y_k)}{G(0,y_k)}$



We use Theorem 3.2 and set $0''$ on $\partial B_0(1)$

satisitying $0'' \notin \overline{C_0}(V,\frac{\pi}{4})$ and pick $V_1 \in T_{0''}(M)$ such that $y_k \in \overline{C_{0''}}(V_1,\frac{\pi}{4})$

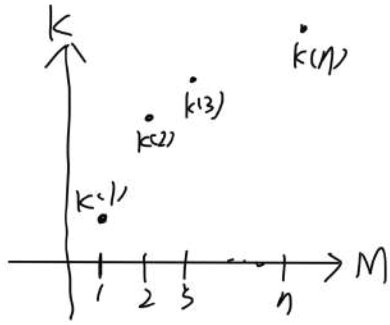
When k is large enough. Then set $u(x)=G(O',x)$, $v(x)=G(O,x)$ which are harmonic function on $\overline{C}_{O''}(V_1,\frac{\pi}{4})$.

Then we have $C_1^{-1}\frac{u(O''')}{v(O''')}\leq\frac{u(x)}{v(x)}\leq C_1\frac{u(O''')}{v(O''')}$, but it's easy to see that $\frac{u(O''')}{v(O''')}$ is bounded, then $\frac{u(Y_k)}{v(Y_k)}=h_{Y_k}(O')$ is bounded by $C_3\leq h_x(O')\leq C_2$, $\forall x\in T_{O''}(V_1,\frac{\pi}{8},1)$ But $Y_k\in T_{O''}(V_1,\frac{\pi}{8},1)$ for k is large enough, then we get $h_{Y_k}(O')\leq C_2$. $\square$

Since $h_{Y_k}(O)=1$ and $O'=\exp_O(V)$, then $h_{Y_k}(O')\leq C$ by the above lemma.

$\Rightarrow$ From theorem 3.1 $\Rightarrow h_{Y_k}(x)\leq C_3 e^{-C_2\rho(x)}$ $\forall x\in T_O(V,\frac{\pi}{8},1)$

Secondly, by $h_{Y_k}(O)=1$ and Harnack inequality, we have for each integer $m>0$, $0<h_{Y_k}(x)<C(m)$ for $\rho(x)<m$ and all large enough k ($k\geq k(m)$) so that $\rho(Y_k)>m$.



Then we get $\|h_{Y_k}\|_{C^2(\overline{B}_O(m))}\leq C_1(m)$

Then we pick the subsequence $\{Y_{k(i)}\}_{i=1}^{\infty}$ of $\{Y_k\}_{k=1}^{\infty}$, $h_{Y_{k(i)}}$ converge in $C^2(\overline{B}_O(m))$ for all $m\in\mathbb{N}^+$ to a non-negative harmonic function P_ξ satisifing $P_\xi(O)=1$. And from maximum priciple, we know P_ξ is positive in M. Then using the upper bound in the first part, we know P_ξ is continuous on $\overline{M}\setminus\{\xi\}$ and vanish on $S(\infty)\setminus\{\xi\}$. Therefore we construct the Poisson kernel P_ξ. $\square$

Remark. The result of theorem 2.1, which leads to the existence of harmonic measure ω^x with $x\in\overline{M}$, which is the unique positive Borel measure on $S(\infty)$ satisifying

$Hf(X) = \int_{S(\infty)} f(Q)\,d\omega^X(Q)$ $\forall f \in C^0(S(\infty))$ where Hf is the unique harmonic function with f as its boundary value.

Then we can also define $\omega_E(X) = \int_{S(\infty)} \chi_E(Q)\,d\omega^X(Q)$ (where χ_E is characteristic function for given Borel subset $E \subset S(\infty)$.) which $\omega_E(X)$ is the unique positive harmonic function with boundary value χ_E.

In this point of view, we can construct Poisson kernel by the following steps. Define $\Delta_i = \overline{C}_0(\xi, \delta_i) \cap S(\infty)$ for $i \in \mathbb{N}$ and let $\delta_i \to 0$.

So we have $\Delta_{i+1} \subset \Delta_i$ and $\bigcap_{i \geq 1} \Delta_i = \{\xi\} \in S(\infty)$. Then define

$u_i(X) = \dfrac{\omega_{\Delta_i}(X)}{\omega_{\Delta_i}(O)}$, we can prove a subsequence of u_i converge to P_ξ in a similar way of the proof of Proposition 3.3.

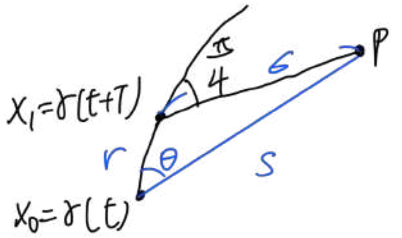
Before dicussing the uniqueness and the continuous dependence on ε of P_ε, we prove some useful lemmas.

Lemma 3.4 Assume $-b^2 \le K_M \le -a^2$. Let γ be geodesic ray in M starting from $O \in M$. Then there are constant $T>0$ depending only on n, a, b such that for any $t \ge 0$ we have $C_{\gamma(t+T)}(\frac{\pi}{4}) \subset C_{\gamma(t)}(\frac{\pi}{8})$.

(pf) Let $x_0 = \gamma(t)$ $x_1 = \gamma(t+T)$. Then, for any $p \in \partial C_{x_1}(\frac{\pi}{4})$

Then $\theta(\overline{x_1 x_0}, \overline{x_1 P}) = \frac{3\pi}{4}$ (The notation is defined in 1.1 section)

Now we need to prove for large enough T, then $\theta(\overline{x_0 P}, \overline{x_0 x_1}) \le \frac{\pi}{8}$ $\forall p \in \partial C_{x_1}(\frac{\pi}{4})$.



Set $r = \rho(x_0, x_1)$ $s = \rho(x_0, P)$ $\sigma = \rho(x_1, P)$ $\theta(\overline{x_0 P}, \overline{x_0 x_1}) = \theta$.

From the Proposition 1.1.1, we have $\cos\theta \ge \coth ar \coth as - \frac{\cosh a\sigma}{\sinh ar \sinh as}$

Notice that in triangle $\Delta x_0 x_1 P$, $\theta(\overline{x_1 x_0}, \overline{x_1 P})$ is the biggest angle, (Because the total angle of triangle on the negatively curved manifold $<\pi$)

so $s > r$ and $s > \sigma$.

Then
$$\frac{\cosh a\sigma}{\sinh ar \sinh as} = \frac{\cosh a\sigma}{\cosh as} \frac{\cosh as}{\sinh as} \frac{1}{\sinh ar}$$
$$\le \coth as \cdot \frac{1}{\sinh ar} \qquad (1)$$

Let $r \to \infty$ $\sinh ar \to \infty$ $\coth as \to 1$ ($\because s > r$, P changes when r is change by definition.)

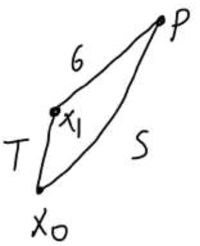
Using (1), so $\cos\theta \ge \coth ar \coth as - \frac{\cosh as}{\sinh ar}$ as $r \to \infty$

$$\cos\theta \to 1 - 0 \Rightarrow \theta \to 0$$

So when we pick large enough $T > 0$, then for any $r \ge T$, we have $\cos\theta > \cos\frac{\pi}{8} \Rightarrow \theta < \frac{\pi}{8}$ $\square$

Lemma 3.5. As the notation in Lemma 3.4, let $x_0 = \gamma(t)$, $x_1 = \gamma(t+T)$ and $p \in \partial C_{x_1}(\frac{\pi}{4})$. Then there exists a constant C such that for sufficently large T we have the following inequality.

$$\max_{p \in \partial C_{x_1}(\frac{\pi}{4})} \theta(\overline{x_0x_1}, \overline{x_0p}) \geq e^{-CbT}$$



(pf) Set $\theta = \theta(\overline{x_0x_1}, \overline{x_0p})$, $\alpha = (\overline{px_1}, \overline{px_0})$, $s = \rho(x_0, p)$, $6 = \rho(x_1, p)$.

Notice also we have $\theta(\overline{x_1x_0}, \overline{x_1p}) = \frac{3\pi}{4}$ $\rho(x_0, x_1) = T$.

From Proposition 1.1.1, we have $\cos\theta$ $\coth ar \coth as - \dfrac{\cosh a6}{\sinh ar \sinh as}$

Since that the total angle of triangle on the negatively curved manifold is smaller than π, so $\alpha < \pi - \frac{3}{4}\pi = \frac{\pi}{4}$.

$$\cos\frac{\pi}{4} < \cos\alpha \leq \frac{\cosh b6 \cosh bs - \cosh bT}{\sinh bs \sinh b6}$$

$$\cos\frac{\pi}{4} \frac{\sinh bs \sinh b6}{\cosh bs} \leq \cosh b6 - \frac{\cosh bT}{\cosh bs} \Rightarrow \cos\frac{\pi}{4} \frac{\sinh bs \sinh b6}{\cosh bs} + \frac{\cosh bT}{\cosh bs} \leq \cosh b6$$

$$\begin{aligned}
\cos\theta \sinh bT \sinh bs &\leq \cosh bT \cosh bs - \cosh b6 \\
&\leq \cosh bT \cosh bs - \frac{\cosh bT}{\cosh bs} - \cos\frac{\pi}{4} \frac{\sinh bs \sinh b6}{\cosh bs} \\
&= \cosh bT \frac{\sinh^2 bs}{\cosh bs} - \frac{\cos\frac{\pi}{4} \sinh bs \sinh b6}{\cosh bs}
\end{aligned}$$

$$\cos\theta \leq \coth bT \tanh bs - \frac{\cos\frac{\pi}{4} \sinh b6}{\sinh bT \cosh bs} \quad (1)$$

Since $\theta(\overline{x_1x_0}, \overline{x_1p}) = \frac{3}{4}\pi$ is biggest angle, so $s > 6 > s - T$, fixing T we have

$$\frac{\sinh b6}{\sinh bs} \to 1 \text{ as } s \to \infty$$

From (1) and let $\theta_\infty = \lim_{s\to\infty} \theta$, Then we get the following result,

$(\because s \to \infty \ \tanh bs \to 1)$

$$1-\frac{\theta_\infty^2}{2} \le \cos\theta_\infty \le \coth bT - \frac{\sqrt{2}}{2\sinh bT} = \frac{1+e^{-2bT}}{1-e^{-2bT}} - \frac{\sqrt{2}\,e^{-bT}}{1-e^{-2bT}}$$

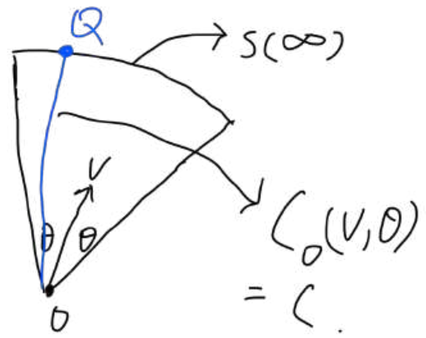
as $T\to\infty$ $\quad 1-\frac{\theta_\infty^2}{2} \le 1+\left(2e^{-2bT}-\sqrt{2}e^{-bT}\right)\left(1-e^{-2bT}\right)^{-1}$

$$\Rightarrow \frac{\theta_\infty^2}{2} \ge \left(\sqrt{2}e^{-bT}-2e^{-2bT}\right)\left(1+e^{-2bT}\right)$$

$$\theta_\infty > e^{-CbT} \text{ for some } C\in\mathbb{R}. \quad \square$$

Lemma 3.6 For a fixed $O\in M$, and given $V\in T_OM$, let u, w be positive Harmonic function on $C_O(V,\theta)$ (denote as C) satisfying $u|C\cap S(\infty) = w|C\cap S(\infty)=0$

Then $\frac{u}{w}$ is C^α continuous in the interior of $C_O(V,\theta)\cap S(\infty)$, where $\alpha\in(0,1)$ depend only on n, a, b.



(pf) Let Q be an interior point of $C\cap S(\infty)$.

Let $\sigma(t)$ be geodesic ray $\overline{OQ}$. Since Q is interior point of $C\cap S(\infty)$, there exist $\delta>0$ such that $C_O(\delta)=C_O(\sigma'(0),\delta)\subset C$. By Lemma 3.5 $\exists T>0$ such that, $C_{\sigma((k+1)T)}(2\delta)\subset C_{\sigma(kT)}(\delta)$ $\forall k=0,1,2,3\cdots$ ($\because T$ is only depend on n, a, b)

Let $C_k = C_{\sigma(kT)}(\delta)$ Then $\{C_k\}$ forms a family of decreasing open neighorhoods of Q and $\bigcap_{k=1}^{\infty} C_k=\{Q\}$.

Let $\varphi_i := \inf_{C_i}\frac{u}{w}$ $\quad \psi_i := \sup_{C_i}\frac{u}{w}$, Then $\psi_i \ge \frac{u}{w}\ge \varphi_i$ in C_i.

Then from the theorem 3.2 (Harnack inequality) there exists $C_1>0$, such that $\psi_i\le C_1\varphi_i$ $\forall i=1,2,3\cdots$ when T is chosen large enough.

Set $u_i = u-\varphi_i w$. Similarly from the theorem 3.2, then there exist $K=k(n,a,b)$ such that $\sup_{C_{i+1}}\frac{u_i}{w}\le K\inf_{C_{i+1}}\frac{u_i}{w} \Rightarrow \psi_{i+1}-\varphi_i\le k(\varphi_{i+1}-\varphi_i)$

Set $u_i' = \psi_i W - u$, we would get $\sup\limits_{C_{i+1}} \frac{u_i'}{W} \leq k' \inf\limits_{C_{i+1}} \frac{u_i}{W} \Rightarrow (\psi_i - \varphi_{i+1}) \leq K(\psi_i - \psi_{i+1})$

Then pick the bigger one of $\{k, k'\}$ as K.

Let $w_i = \psi_i - \varphi_i = \sup\limits_{C_i} \frac{u}{W} - \inf\limits_{C_i} \frac{u}{w} = \operatorname{osc}\limits_{C_i} \left(\frac{u}{W}\right)$

But with the above two inequality, we add both of them we get the following inequality.

$(\psi_{i+1} - \varphi_i) + (\varphi_i - \varphi_{i+1}) = w_{i+1} + w_i \leq K(w_i - w_{i+1})$

$\Rightarrow w_{i+1} \leq \frac{K-1}{K+1} w_i = \varepsilon w_i$ where $\varepsilon = \frac{K-1}{K+1} < 1 \Rightarrow w_i \leq \varepsilon^i w_0$.

$\Rightarrow \operatorname{osc}_{C_i}\left(\frac{u}{W}\right) \leq \varepsilon^i \sup\limits_{C_0}\left(\frac{u}{W}\right) = e^{-iC_3} \sup\limits_{C_0}\left(\frac{u}{w}\right)$ (Notice that this i is index.)

Denote $\theta_i := \inf\{\theta \in \mathbb{R} \mid C_i \subseteq C_0(\sigma(0), \theta)\}$

From lemma 3.5, $\theta_i \geq e^{-\alpha b i T}$ for large enough i and α is constant.

$\operatorname{osc}_{C_i}\left(\frac{u}{w}\right) \leq \theta_i^{\alpha} \sup\limits_{C_0}\left(\frac{u}{w}\right)$ (where $\alpha = \frac{C_3}{\alpha b T} > 0$)

But from theorem 3.2, we know $\sup\limits_{C_0}\left(\frac{u}{w}\right)$ is bounded.

Then $\left|\frac{u}{W}(y) - \frac{u}{w}(y')\right| \leq C\theta_i^{\alpha} \quad \forall y, y' \in C_i \quad (1)$

Then $\frac{u}{W}$ is bounded $\Rightarrow \lim\limits_{y \to Q} \frac{u}{W}(y)$ exists. $\Rightarrow \frac{u}{w}$ can be extended to interior of $\overline{C} \cap S(\infty)$.

And Moreover, with (1) we know that if $V_1, V_2 \in \operatorname{int}(\overline{C} \cap S(\infty))$ we have. $\left|\frac{u}{W}(V_1) - \frac{u}{w}(V_2)\right| \leq C \cdot \theta(V_1, V_2)^{\alpha}$ #

Then we dicuss the uniqueness and the continuous dependence on ξ of P_ξ.

Theorem 3.7 $P(X,\xi)$ is a Poisson kernel for $\xi \in S(\infty)$. Then for a fixed X we have $|P(X,\xi)-P(X,\xi')| \leq C\,\theta(\xi,\xi')^\alpha$

(pf) If we assume the result of theorem 3.8 which Poisson kernel is unique. Then we know $P(X,\xi) = \lim\limits_{Y\to\xi} \dfrac{G(X,Y)}{G(O,Y)}$

Then set $u(Y)=G(X,Y)$ $w(Y)=G(O,Y)$ and use the lemma 3.6 we get $|P(X,\xi)-P(X,\xi')| \leq C\,\theta(\xi,\xi')^\alpha$ $\square$

Theorem 3.8 For any $\xi \in S(\infty)$, the Poisson kernel $P(X,\xi)$ normalized at O is unique.

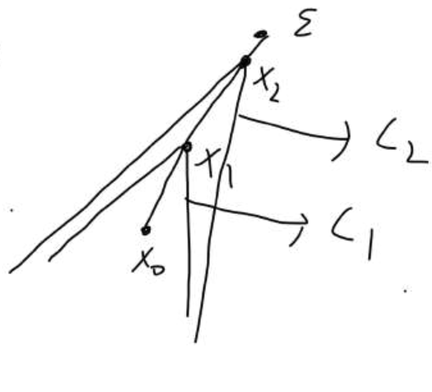
(pf) Suppose that Poisson kernel is not unique, let f, g be the poisson kernel for $\xi \in S(\infty)$ and satisify $f(O)=g(O)=1$

Let $\sigma(t)$ be the geodesic ray from O to ξ.

$\sigma(0)=O=X_0$, $X_k=\sigma(kT)$ for $k=1,2,3\cdots$ where $T>0$ is large enough and be determined by lemma 3.4.

We use the different notation $C_k(\delta) := C_{X_k}(-\sigma'(kT),\delta)$

From lemma 3.4 we have $C_{k-1}(\frac{\pi}{4}) \subset C_k(\frac{\pi}{8})$



(Notice this is a little different from the case we meet before, we pick the inverse direction of tangent vector.)

So in this case, $\varepsilon \notin \overline{C}_k(\frac{\pi}{4}) \cap S(\infty)$

Then $f = g = 0$ on $\overline{C}_k(\frac{\pi}{4}) \cap S(\infty)$ $\forall k = 1, 2, 3 \ldots$

Apply theorem 3.2, there exist $C > 0$ is independent of k st

$$\sup_{C_k(\frac{\pi}{4})} \frac{f(x)}{g(x)} \leq C \frac{f(0)}{g(0)} = C \quad \text{and} \quad \inf_{C_k(\frac{\pi}{4})} \frac{f(x)}{g(x)} \geq C^{-1}$$

Follow the procedure of the proof of lemma 3.6.

Let $\psi_k = \sup_{C_k(\frac{\pi}{4})} \frac{f(x)}{g(x)}$ $\varphi_k = \inf_{C_k(\frac{\pi}{4})} \frac{f(x)}{g(x)}$

Let $f_k = f - \varphi_k g$ $\Rightarrow \exists C_1 > 0$ $\sup_{C_{k-1}} \frac{f_k}{g} \leq C_1 \inf_{C_{k-1}} \frac{f_k}{g}$ $\left(\because C_{k-1}(\frac{\pi}{4}) \subseteq C_k(\frac{\pi}{4}) \text{ This is different} \right)$

such that $(\psi_{k-1} - \varphi_k) \leq C_1 (\varphi_{k-1} - \varphi_k)$ ①

Let $f_k' = \psi_k g - f$ $\Rightarrow \exists C_2 > 0$ $\sup_{C_{k-1}} \frac{f_k'}{g} \leq C_2 \inf_{C_{k-1}} \frac{f_k'}{g}$

st $(\psi_k - \varphi_{k-1}) \leq C_2 (\psi_k - \psi_{k-1})$ ②

$\Rightarrow$ Pick $C_3 = \max\{C_1, C_2\} > 0$

Let $w_i = \psi_i - \varphi_i = \operatorname{osc}_{C_i(\frac{\pi}{4})} \left(\frac{f}{g}\right)$

Consider ① + ② $\Rightarrow w_k + w_{k-1} \leq C_3 (w_k - w_{k-1})$

$\Rightarrow w_{k-1} \leq \frac{C_3 - 1}{C_3 + 1} w_k$ $\Rightarrow w_0 \leq \varepsilon^k w_k$, let $\varepsilon = \frac{C_3 - 1}{C_3 + 1} < 1$.

$\operatorname{osc}_{C_0(\frac{\pi}{4})} \left(\frac{f}{g}\right) \leq \varepsilon^k \sup_{C_k(\frac{\pi}{4})} \left(\frac{f}{g}\right) \leq C \varepsilon^k$ where $\varepsilon < 1$, C is

As $k \to \infty$, then we have $\operatorname{osc}_{C_0(\frac{\pi}{4})} \left(\frac{f}{g}\right) \longrightarrow 0 \Rightarrow f \equiv g$ on $C_0(\frac{\pi}{4})$

Then from maximum principle, we have $f \equiv g$ on M.

So Poisson kernel is unique. $\square$

4. Martin boundary and Martin Integral Representation

Define: The sequence $Y=\{y_i\}_{i=1}^{\infty}$ is called fundamental if $\{h_{y_i}\}$ converge to a harmonic function h_Y. (where $y_i \in M, \forall i$).

Define: Two fundamental sequene Y_1 and Y_2 is called equivalent if the limiting harmonic functions $h_{Y_1}=h_{Y_2}$.

Define: Martin boundary $\mathcal{M}$ of M is the set of all equivalence classes of fundamental sequence which have no limit points in M.
Since each $[Y] \in \mathcal{M}$ corresponds to a unique harmonic function h_Y, so Martin boundary $\mathcal{M}$ can be identifying as the set of such h_Y.

Then they define that $\rho([Y_1],[Y_2]=\sup_{B_0(1)}|h_{Y_1}(x)-h_{Y_2}(x)|$ for whole $\tilde{M}=M\cup\mathcal{M}$. But I don't understand why it is well-defined for $[Y_1]$ and $[Y_2]\in M$, I think it's ok for $[Y_1]$ and $[Y_2]\in\mathcal{M}$
(Ref: textbook chapter 2 (3.2) or Positive harmonic function on complete manifolds of negative curvature By Michael T. Anderson and Richard Scheon.)

Theorem 4.1 There exist a homeomorphism $\Phi:\mathcal{M}\longrightarrow S(\infty)$.
Moreover, Φ^{-1} is Hölder continuous.

(pf) Let $Y=\{y_i\}$ be a fundamental sequence which represents a point $[Y]$ of $\mathcal{M}$. Then $h_{y_i}(x)\to h_Y(x)$. From the theorem 3.8 we know that the poisson kernel is unique, so different limit points of Y in $S(\infty)$ would correpond to different harmonic functions (or Poisson kernel).

Hence, the sequence Y can have only one limit point $Y_\infty \in S(\infty)$

Then we can define $\Phi: \mathcal{M} \to S(\infty)$.

$$h_Y \to Y_\infty$$

Φ is surjective, since for any sequence convergeing to any given $\xi \in S(\infty)$ contain a fundamental subsequence.

Φ is continuous. Let $\{h_i\}$ be a sequence in $\mathcal{M}$ such that $h_i \to h \in \mathcal{M}$. Assume that $\Phi(h_i) = Q_i \in S(\infty)$ and $\Phi(h) = Q \in S(\infty)$

With above discussion, we know h_i is Poisson kernel at Q_i and h is Poisson kernel at Q. If $\{Q_i\}$ has two different limit points, then $\{h_i\}$ will converge along subsequences to two different kernel functions, a contradiction to the fact that $h_i \to h$. So we must have $Q_i \to Q$.

Φ is one to one. Let $Y = \{Y_i\}$ and $Z = \{Z_i\}$ be two fundamental sequences representing $[Y] \simeq h_Y$ and $[Z] \simeq h_Z$ in $\mathcal{M}$ with $\Phi(h_Y) = \Phi(h_Z) = Q \in S(\infty)$.

But we know that Poisson kernel with the same $Q \in S(\infty)$ is unique, so $h_Y = h_Z$

Now we have Φ is homeomorphism between $\mathcal{M}$ and $S(\infty)$.

Then we can regard $\mathcal{M}$ as the set of Poisson kernel h_ξ with $\xi \in S(\infty)$.

In order to prove that Φ^{-1} is α Hölder continuous, We apply the theorem 3.7, we have for $\varepsilon, \eta \in S(\infty)$.

$|P(x,\varepsilon)-P(x,\eta)| = |h_\varepsilon(x)-h_\eta(x)| \le C_x \theta(\varepsilon,\eta)^\alpha$ for a fixed $x \in M$.

$\Rightarrow |P(x,\varepsilon)-P(x,\eta)| \le C\theta(\varepsilon,\eta)^\alpha \ \forall x \in B_0(1)$

Let $C = \sup_{B_0(1)} C_x$, this must exist because $|P(x,\xi)-P(x,\eta)|$ is continous on x.

where C, α are positive constants depending only on n, a, b.

This is equivalent to say that the

$\rho_{\mathcal{M}}(h_\varepsilon, h_\eta) = \rho_{\mathcal{M}}(\Phi^{-1}(\varepsilon), \Phi^{-1}(\eta)) \le C\,\theta(\varepsilon,\eta)^\alpha$

Then $\Phi^{-1}: S(\infty) \to \mathcal{M}$ is C^α $\square$

(This use the metric we define above. But I think this metric is OK on $\mathcal{M}$)

Before we prove the representation formula for positive harmonic function on M with Martin Boundary, we need to recall the harmonic measure w^x for $x \in \overline{M} = M \cup S(\infty)$, which is a positive Borel measure on $S(\infty)$ satisfying

$(Hf)(x) = \int_{S(\infty)} f(Q)\, dw^x(Q) \quad \forall f \in C^0(S(\infty))$

Firstly, notice that when $f \equiv 1$ on $S(\infty)$, then $Hf \equiv 1$ on $\overline{M}$ then we know that total measure of w^x on $S(\infty)$ is 1 for all $x \in \overline{M}$.

Secondly, for any Borel set E of $S(\infty)$, the bounded function

$W_E(x) := w^x(E) = \int_\infty \chi_E(Q)\, dw^x(Q)$ (where χ_E is characteristic function of E)

Then we can get the different expression of Poisson kernel.

From the some result of measure theorem, the Radon–Nikodym derivative of the above measure.

$K(X,Q) = \frac{d\omega^X}{d\omega^0}(Q) = \lim_{i\to\infty} \frac{\omega^X(\Delta_i)}{\omega^0(\Delta_i)}$ exist for almost all $Q \in S(\infty)$

(Where O is the fixed point, and Δ_i is the neighborhood of Q on $S(\infty)$ with $\bigcap_{i=1}^{\infty} \Delta_i = \{Q\}$ and $\Delta_{i+1} \subseteq \Delta_i$ $\forall i = 1, 2, 3 \cdots$.)

But notice that $\frac{\omega^X(\Delta_i)}{\omega^0(\Delta_i)} = \frac{W_{\Delta_i}(X)}{W_{\Delta_i}(O)} \xrightarrow{i\to\infty} P(X,Q) = K(X,Q)$

(According to the Remark after Proposition 3.3 and the uniqueness of Poisson kernel.)

With the above discussion, naively we can have the following guessing.

$$Hf(X) = \int_{S(\infty)} f(Q)\, d\omega^X(Q) = \int_{S(\infty)} f(Q) \frac{d\omega^X}{d\omega^0}(Q)\, d\omega^0(Q) = \int_{S(\infty)} f(Q) K(X,Q)\, d\omega^0(Q)$$

$$= \int_{S(\infty)} f(Q) P(X,Q)\, d\omega^0(Q) \qquad (\text{for some measure } \nu \text{ on } S(\infty))$$

$$= \int_{S(\infty)} P(X,Q) [f(Q)\, d\omega^0(Q)] = \int_{S(\infty)} P(X,Q)\, d\nu(Q)$$

This Integral Representation was found for the case of harmonic function on unit disk by Herglotz.

Then it was improved by Martin, this Integral representation can applied to the case on bounded domain of $\mathbb{R}^n$.

And now we assume the Martin result, and we now generalize the case to M.

Theorem 4.2 Let u be a positive harmonic function on M. Then there exist an unique Borel measure ν on $S(\infty)$ such that

$u(X) = \int_{S(\infty)} P_\xi(X)\, d\nu(\xi) = \int_{S(\infty)} P(X,\xi)\, d\nu(\xi)$ (Denote $P_\xi(X)$ as $P(X,\xi)$).

(pf) Fixing $O \in M$ and applying the classical Martin representation formula to bounded domain $B_O(R)$ we have

$u(x) = \int_{S_O(R)} u(Q) P_R(x,Q)\, dw_R(Q)$ (where $P_R(x,Q)$ is Poisson kernel for $B_O(R)$, and $S_O(R)$ is the sphere of radius R in $T_O M$.

(We identified the boundary of $B_O(R)$ as the sphere in $\mathbb{R}^n$ via exponential map, so we can use the classical martin Integral formula.)

And w_R is the harmonic measure on $S_O(R)$.

Notice that $P_R(O,Q) = 1$ by the definition of Poisson kernel.

Then $u(O) = \int_{S_O(R)} u(Q)\, dw_R(Q)$

Consider $\mu_R = u|_{S_O(R)} \cdot w_R$. Now we identify $S_O(R)$ with $S(\infty)$

Then we identify μ_R as a measure on $S(\infty)$.

From $u(O) = \int_{S_O(R)} u(Q)\, dw_R(Q)$, we know that μ_R has the same total measure $u(O)$ for each R.

Then from a result in measure theory that there exist a sequence $R_i \to \infty$ such that μ_{R_i} converge weakly to some positive Borel measure μ with finite total measure.

Now we have constructed the Borel measure μ in the Martin Integral formula.

On the other hand, we need to study the Poisson kernel.

We claim the sequence $\{P_{R_i}(X, Q)\}$ converge along some subsequence to a kernel function at Q for fixed $Q \in S(\infty)$.

But from the uniqueness of Poisson kernel, we know that subsequence $\{P_{R_i}(X, Q)\}$ would just converge to $P(X, Q)$.

$\because$ we have identify $S_0(R)$ as $S(\infty)$, so the Q on $S_0(R) \leftrightarrow Q$ on $S(\infty)$. So the Q in $P_{R_i}(X, Q)$ is the Q on $S_0(R_i)$ which correspond $Q \in S(\infty)$.

To prove the above claim, consider $P_{R_i}(X, Q)$ in any cone $C = C_0(V, \frac{\pi}{4})$ which doesn't containing the geodesic $\overline{OQ}$.

Then extended $P_{R_i}(X, Q)$ to a subharmonic functin η_i on C by setting $\eta_i = P_{R_i}$ on $C \cap B_0(R_i)$ and $\eta_i \equiv 0$ on $C \setminus B_0(R_i)$

We can solve the dirichelet problem.

$\Delta h_i = 0$ in C $\quad h_i = \eta_i$ on ∂C.

By maximum principle and Theorem 3.1, on $T_0(\frac{\pi}{8}, 1)$ we have the estimate $\eta_i \leq h_i \leq C_1 e^{-C_2 \rho(X)} h_i(O')$.

Then by the theorem 3.2, $h_i(O')$ is uniformly bounded.

Notice that $\{\eta_i\}$ vanish continuously in $C(\frac{\pi}{8}) \cap S(\infty)$.

By construction, we can see that $\{P_{R_i}\}$ coincide with $\{\eta_i\}$

Therefore, any limit function of $\{P_{R_i}\}$ vanishing continuously on $S(\infty) \setminus \{Q\}$, then it must be a kernel function.

So we know $P_{R_i}(X, Q) \longrightarrow P(X, Q)$. It's also clear for a fixed $X \in M$, this convergence is uniform in Q.

Thus, since $\mu_{R_i} \to \mu$ weakly, we have

$$u(X) = \int_{S_0(R_i)} u(Q)\, P_{R_i}(X,Q)\, d\omega_{R_i}(Q)$$

$$\to \int_{S(\infty)} P(X,Q)\, d\mu(Q).$$

Then we have done the existence of Martin Integral formula.

The last thing that we need to do is to prove the uniqueness of Borel measure μ.

Suppose there exists another Borel measure ν satisifying

$$u(X) = \int_{S(\infty)} P(X,Q)\, d\nu(Q)$$

Let E be any closed subset of $S(\infty)$. We are going to show that $\mu(E) \le \nu(E)$. (Because with this by the symmetry of μ and ν, we have $\nu(E) \le \mu(E)$. Then $\mu(E) = \nu(E)$ for any E. $\Rightarrow \nu = \mu$)

(pf) $\mu(E) = \lim_{i\to\infty} \int_E u(Q)\, d\omega_{R_i}(Q)$

$$= \lim_{i\to\infty} \int_E \int_{S(\infty)} P(Q,Q')\, d\nu(Q')\, d\omega_{R_i}(Q)$$

in

$$= \lim_{i\to\infty} \int_{S(\infty)} \left[\int_E P(Q,Q')\, d\omega_{R_i}(Q)\right] d\nu(Q')$$

Note that since P is a kernel function, then

$F_i(Q') := \int_E P(Q,Q')\, d\omega_{R_i}(Q) \to 0$ as $R_i \to \infty$ ($\because P(\cdot,Q')$ vanishing on $S(\infty)\setminus Q'$)

when $Q' \in S(\infty)\setminus V$ (where V is any open set satisifying $E \subseteq V \subseteq S(\infty)$)

Since $F_i \le 1$, we have $\mu(E) \le \nu(V)$. But V is arbitrary open set containing E, so $\mu(E) \le \nu(E)$, then we done $\square$

5. Proof of Harnack inequality.

In this section we are going to prove the Theorem 3.1 and Theorem 3.2.

Recall some notation and setting.

M is simply connected, complete, n-dimensional Riemannian manifold with $-b^2 \leq K_M \leq -a^2$, $a, b > 0$. Fix a base point $O \in M$.

$C_O(\delta) = C_O(V, \delta)$ is a cone of angle δ about $V \in T_O M$.

$T_O(\delta, R) = C_O(\delta) \setminus B_O(R)$ is trucated cone.

$O' = \exp_O(V)$.

Theorem 3.1: Fixing $O \in M$ and given an $V \in T_O M$, let h be a positive harmonic function on $C_O(V, \frac{\pi}{4})$ such that it is continuous in the closure of $C_O(V, \frac{\pi}{4})$ and vanishs on $\overline{C_O(V, \frac{\pi}{4})} \cap S(\infty)$.

Then we have the following estimate

$h(x) \leq C_1 h(O') e^{-C_2 \rho(x)} \quad \forall x \in T_O(V, \frac{\pi}{8}, 1)$ (where C_1, C_2 are positive and only depending on n, a, b)

$O' = \exp_O(V)$

(pf) We first prove two lemmas.

Lemma 5.1. For any $\beta \in [0,1)$, there exist $\varphi \in C^\infty(M)$ and a constant $R_0 > 0$ such that

(i) $|D\varphi| + |D^2\varphi| \leq C_1 e^{-C_2 \rho(x)} \quad \forall x \in T(\frac{\pi}{4}, R_0)$

(ii) $\varphi = 1$ on $\partial C_O(\frac{\pi}{4}) \setminus B_O(R_0)$

(iii) $\varphi = \beta$ on $T(\frac{\pi}{8}, R_0)$

(pf) Let $\psi: [0, \pi] \to \mathbb{R}$ be a continuous piecewise linear function such that $\psi = 1$ on $[\frac{\pi}{4}-\varepsilon, \frac{\pi}{4}]$ and $\psi = \beta$ on $[0, \frac{\pi}{8}+\varepsilon]$ where $\varepsilon > 0$ is small. We will consider $\psi(\theta)$ as a function of $\eta \in S_0(1) = S(\infty)$ with $\theta = \angle(\eta, V)$. Using the normal polar coordinates (r, θ) at O, we extend ψ to $M \setminus \{O\}$ by $\psi(r, \theta) = \psi(\theta)$.

As the proof of Theorem 2.1, let $\chi \in C_0^\infty(\mathbb{R})$ be cut-off function. and take the average φ of ψ as follows.

$$\varphi(x) := \frac{\int_M \chi(\rho^2(x,y))\psi(y)\,dy}{\int_M \chi(\rho^2(x,y))\,dy} \quad \left(\text{Where } 0 \le \chi \le 1 \quad \begin{matrix} \chi(t) = 0 \text{ for } |t| \ge 1 \\ \chi(t) = 1 \text{ for } |t| \le \frac{1}{2} \end{matrix} \right)$$

Then we check that φ satisifies the three conditions in Lemma 5.1.

(i) Recall the fact we have used in Rauch Comparison Theorem.

$$a \coth a\rho (g - d\rho \otimes d\rho) \le D^2\rho \le b \coth b\rho (g - d\rho \otimes d\rho)$$

(From the assumption $-b^2 \le K_M \le -a^2$)

Just as the part 2 in the proof of Theorem 2.1

$$D^2\chi(\rho^2) = \chi'' d\rho \otimes d\rho + \chi' D^2\rho$$

$\|D^2\varphi\|(x_0) = \|D^2[\varphi - \psi(x_0)]\|(x_0) \le C_1 \operatorname{osc}_{B_{x_0}(1)}\psi$, while $\operatorname{osc}_{B_{x_0}(1)}\psi = O(e^{-\alpha\rho(x)})$

$\|D\varphi\|(x_0)$ is bounded. (In the proof of Theorem 2.1)

So we have $|D\varphi| + |D^2\varphi| \le C_1 e^{-C_2\rho(x)} \quad \forall x \in T(\frac{\pi}{4}, R_0)$

(ii) Let $y \in B_x(1)$ and let θ be the angle between $\overline{Ox}$ and $\overline{Oy}$ at O. Then by proposition 1.1.2, we have $\theta \le C_1 e^{-\alpha\rho(x)}$.

We can find $R_0 > 0$ such that if $\rho(x) \ge R_0$, $\theta \le \varepsilon$. It follows that when $x \in \partial C_0(\frac{\pi}{4})$ and $\rho(x) \geq R_0+1$, we have

$$\varphi(x) = \frac{\int_{B_x(1)} \chi(\rho^2(x,y))\,dy}{\int_M \chi(\rho^2(x,y))\,dy} = 1$$ since for $y \in B_x(1)$, $\theta(\overline{Ox}, \overline{Oy}) \leq \varepsilon$ and $\psi(y) = 1$.

(iii) Follow the same way of (ii), let θ be angle of $\overline{Ox}$ and $\overline{Oy}$ at O, and choose $R_0 > 0$ is large enough such that, $\theta \leq \varepsilon$.

Then $$\varphi(x) = \frac{\int_{B_x(1)} \chi(\rho^2(x,y)) \cdot \beta\,dy}{\int_M \chi(\rho^2(x,y))\,dy} = \beta$$ on $T(\frac{\pi}{8}, R_0)$.

(Note that the R_0 is chosen that satisfy the condition (ii) and (iii)).

$\square$

Lemma 5.2. Let u be positive harmonic function which vanishes on $\overline{\partial C_0(\frac{\pi}{4})} \cap S(\infty)$. There exist $\delta > 0$ and $R_0 > 0$ such that.

$$u(x) \leq C_1 e^{-\delta\rho(x)} \sup_{\partial C_0(\frac{\pi}{4})} u, \quad \forall x \in T(\tfrac{\pi}{8}, R_0)$$

pf) By lemma 5.1, there exist φ satisfying the condition (i)-(iii). with $\beta = 0$.

From (i) $\Rightarrow$ $\Delta\varphi \leq C_1 e^{-C_2\rho(x)}$ $\forall x \in C_0(\frac{\pi}{4})$

Let $f = \varphi + Ce^{-\delta\rho(x)}$ where C and δ are positive constant to be determined.

Notice that $\Delta f = \Delta\varphi + C\Delta e^{-\delta\rho(x)} \leq C_1 e^{-C_2\rho(x)} - C\delta(C_3 - \delta)e^{-\delta\rho(x)}$.

(where $C_3 > 0$ depend only on $a \cdot b$.) (Which we have used in the proof of Theorem 2.1 step 3)

Therefore, if we take δ sufficently small and C is large enough,

$\Rightarrow \Delta f \leq 0$.

On the other hand, we have $f \geq 1$ on $\partial C_0(\frac{\pi}{4})$. To check this,
We use the condition (ii) in lemma 5,1, we have

$f(x) = 1 + e^{-\delta\rho} \geq 1$ if $\rho(x) \geq R_0$

When $\rho(x) \leq R_0$, we have $f(x) \geq \varphi(x) + Ce^{-\delta R_0}$.

So we choose C is large enough such that $\varphi(x) + Ce^{-\delta R_0} \geq 1$ for $\rho(x) \leq R_0$.

Also from the condition (iii), we have $f(x) = Ce^{-\delta\rho(x)}$.

Now, let $\bar{u} = u/m$ where $m = \sup\limits_{\partial C_0(\frac{\pi}{4})} u$. Then $\bar{u}$ is harmonic with

$\Delta(\bar{u} - f) \geq 0$

$\bar{u} - f \leq 0$ on $\partial C_0(\frac{\pi}{4})$ $\because$ $\bar{u} \leq 1$ on $\partial C_0(\frac{\pi}{4})$ But $f \geq 1$ on $\partial C_0(\frac{\pi}{4})$.

Then by maximum priciple, we have $\bar{u} \leq f(x) = Ce^{-\delta\rho(x)}$ on $T(\frac{\pi}{8}, R_0)$

$\Rightarrow u \leq Ce^{-\delta\rho(x)} \sup\limits_{\partial C_0(\frac{\pi}{4})} u$. $\square$

Now we are ready to prove theorem 3,1.
It follow the following 3 steps.

1. Let $\varphi \in C^\infty(M)$ satisfying lemma 5.1 with $\beta = \frac{1}{2}$.

From (i) condition, there exist C_1, C_2 such that

$|\nabla\varphi| + |\nabla^2\varphi| \leq C_1 e^{-C_2\rho(x)}$ $\forall x \in C_0(\frac{\pi}{4})$

and from the assumption $\beta = \frac{1}{2}$, we have $\frac{1}{2} \leq \varphi \leq 1$.

Consider the function $u^\varphi = e^{\varphi \ln u}$. By some computation we have

$\nabla u^\varphi = e^{\varphi \ln u} \cdot \nabla(\varphi \ln u) = u^\varphi(\nabla\varphi \ln u + \varphi \nabla \ln u)$

$$\Delta u^{\varphi}=\nabla\cdot(\nabla u^{\varphi})=u^{\varphi}|\nabla\varphi\ln u+\varphi\nabla\ln u|^2+u^{\varphi}(\Delta\varphi\ln u+2\nabla\varphi\cdot\nabla\ln u)+\varphi(\Delta\ln u))$$

$$=u^{\varphi}\left(|\nabla\varphi\ln u+\varphi\nabla\ln u|^2+\Delta\varphi\ln u+2\nabla\varphi\cdot\nabla\ln u+\varphi\left(\cancel{\frac{\Delta u}{u}}-|\nabla\ln u|^2\right)\right)$$

($\because$ u is harmonic function, so $\Delta u=0$. Then the $\frac{\Delta u}{u}$ term is 0.)

WLOG, we let $u(O)=1$ by replace u with $\frac{u}{u(O)}$.

Then by the gradient esimate $|\nabla\ln u|\leq C$

$\Rightarrow \ln u=\ln u-\ln u(O)=\int_0^x|\nabla\ln u(s)|ds\leq C\rho(x)$

Then we use the inequality to give estimate Δu^{φ}.

$$\Delta u^{\varphi}=u^{\varphi}(|\nabla\varphi|^2\ln^2 u+\underline{\varphi^2|\nabla\ln u|^2}+2\varphi\ln u(\nabla\varphi\cdot\nabla\ln u))$$

$$+u^{\varphi}(\Delta\varphi\ln u+2(\nabla\varphi\cdot\nabla\ln u)-\underline{\varphi|\nabla\ln u|^2})$$

$$\leq u^{\varphi}(\underline{(\varphi^2-\varphi)|\nabla\ln u|^2}+C_1e^{-C_2\rho})$$

$\because$ $|\nabla\ln u|\leq C$

$|\nabla\varphi|\leq\alpha e^{-\beta\rho}$

$|\Delta\varphi|\leq\alpha_1 e^{-\beta_1\rho}$

$\ln u\leq C\rho$

$\rho^k e^{-\alpha\rho}\leq Ce^{-\alpha_1\rho}$ with $0<\alpha_1<\alpha$, for some $C>0$

$$\leq u^{\varphi}C_1e^{-C_2\rho}\quad(\because \tfrac{1}{2}\leq\varphi\leq 1\Rightarrow\varphi^2-\varphi\leq 0)$$

$$=C_1e^{-C_2\rho}u^{\varepsilon}u^{\varphi-\varepsilon}$$

$$\leq C_1e^{-C_2\rho}e^{\varepsilon C\rho}u^{\varphi-\varepsilon}\quad(\because\ u^{\varepsilon}=e^{\varepsilon\ln u}\leq e^{\varepsilon C\rho})$$

$$=C_1e^{-C_3\rho}u^{\varphi-\varepsilon}$$

(where $\varepsilon>0$ and is small enough and $C_3=C_2-\varepsilon C(>0)$)

Let $\alpha=1-\frac{\varepsilon}{2}\in(0,1)\Rightarrow u^{\varphi-\varepsilon}\leq u^{\alpha}+1$

(Since $0<\frac{1}{2}-\varepsilon<\varphi-\varepsilon\leq 1-\varepsilon<\alpha$, we see that $u^{\varphi-\varepsilon}\leq 1$ if $u\leq 1$

and $u^{\varphi-\varepsilon}<u^{\alpha}$ if $u\geq 1$.

Therefore, we have $\Delta u^{\varphi} \leq C_1 e^{-C_3\rho}(u^\alpha+1) \quad \forall x \in C_0(\frac{\pi}{4})$

2. Then consider $\Delta(e^{-\delta\rho}u^\alpha)$. $(\exists C_4>0)$

$$\Delta(e^{-\delta\rho}u^\alpha) = u^\alpha \Delta e^{-\delta\rho} + 2\nabla e^{-\delta\rho}\cdot\nabla u^\alpha + e^{-\delta\rho}\Delta u^\alpha$$

$$\leq -\delta(C_4-\delta)e^{-\delta\rho}u^\alpha + 2\delta e^{-\delta\rho}u^{\alpha-1}|\nabla u| - \alpha(1-\alpha)e^{-\delta\rho}u^{\alpha-2}|\nabla u|^2$$

$$= e^{-\delta\rho}u^\alpha\left[-\delta(C_4-\delta) + 2\delta|\nabla \ln u| - \alpha(1-\alpha)|\nabla \ln u|^2\right]$$

Then using the inequlity $2Bt - At^2 \leq B^2/A$ with $|\nabla \ln u| = t$

$$\Rightarrow \Delta(e^{-\delta\rho}u^\alpha) \leq e^{-\delta\rho}u^\alpha\left[-\delta(C_4-\delta) + \frac{\delta^2}{\alpha(1-\alpha)}\right] \quad \left(\because \text{In order to replace } |\nabla \ln u| \text{ which is not easy to control}\right).$$

$$\leq -\delta C_5 e^{-\delta\rho}u^\alpha \quad (\text{for } \delta \text{ is small enough})$$

$$\leq -\delta C_5 e^{-\delta\rho} \quad (\text{for a smaller } \delta)$$

Then we can construct a superharmonic function f.

$f = u^{\varphi} + C_6 e^{-\delta\rho}(u^\alpha+1)$ (with large enough $C_6>0$.)

$$\left(\because \Delta f \leq \Delta u^{\varphi} + C_6\cdot(-\delta C_5 e^{-\delta\rho})\right.$$

$$\left.\leq C_1 e^{-C_3\rho}(u^\alpha+1) - \delta C_5 C_6 e^{-\delta\rho} \leq 0 \quad \left(\text{If } \delta \text{ is small enough and } C_6 \text{ is large enough}\right)\right)$$

3. We estimate u by compare u with f.

Notice that $\varphi \equiv 1$ on $\partial C_0(\frac{\pi}{4})$. So $u \leq f$ on $\partial C_0(\frac{\pi}{4})$.

Since $\Delta(u-f) \geq 0$, so by maximum principle, $u \leq f = u^{\varphi} + C_6 e^{-\delta\rho}(u^\alpha+1)$ in $C_0(\frac{\pi}{4})$.

Since $\varphi \equiv \frac{1}{2}$ on $T(\frac{\pi}{8}, R_0)$, so we have.

$$u \leq u^{\frac{1}{2}} + C_6 e^{-\delta\rho}(u^\alpha+1) \leq \frac{1}{4}u + 1 + C_6(u^\alpha+1) \quad x \in T(\tfrac{\pi}{8}, R_0)$$

$$\left(\because u^{\frac{1}{2}} = 2\cdot\left(\tfrac{1}{2}u^{\frac{1}{2}}\cdot 1\right) \leq \left(\tfrac{u^{\frac{1}{2}}}{2}\right)^2 + 1^2\right)$$

Since $\alpha<1$, so $\frac{3}{4}u \le (1+C_6)+C_6u^{\alpha}$. So u is bounded above on $T(\frac{\pi}{8}, R_0)$. Then recall that we have gradient estimate of harmonic function in the textbook Theorem 3.1. We can have that u is bounded on $C_0(\frac{\pi}{8})$.

Then from the lemma 5.2

$$u(x) \le C e^{-\delta\rho(x)} \sup_{\partial C_0(\frac{\pi}{8})} u \le \bar{C} e^{-\delta\rho(x)} \quad \text{for } x\in T(\tfrac{\pi}{16}, R_0)$$

Last, we recall that we first replace u by $u/u(o')$.

So we get the general result.

$$u(x) \le \bar{C} e^{-\delta\rho(x)} u(o') \quad \forall x\in T(\tfrac{\pi}{16}, R_0)$$

By the gradient estimate, we can extend $T(\frac{\pi}{16}, R_0)$ to $T(\frac{\pi}{16}, 1)$.

$\square$

Remark: In the proof, the angle θ of $T(\theta, 1)$ is arbitrary chosen. In fact, we know that if $\theta\in(0,\frac{\pi}{4})$ then the such estimate is still hold.

Theorem 3.2: Fixing $O\in M$ and given an $v\in T_oM$, Let u, w be two positive harmonic function on $C_0(v,\frac{\pi}{4})$ such that they are continuous in the closure of $C_0(v,\frac{\pi}{4})$ and vanishs on $\bar{C}_0(v,\frac{\pi}{4})\cap S(\infty)$.

Then we have $C_1^{-1}\frac{u(o')}{w(o')} \le \frac{u(x)}{w(x)} \le C_1\frac{u(o')}{w(o')}$, $\forall x\in T_o(v,\frac{\pi}{8},1)$

(where C_1 is positive number depending only on n, a, b).

(pf) Let $0<\theta_2<\theta_1<\frac{\pi}{4}$, we prove the above inequality on $T(\theta_2,1)$. WLOG assume $u(0')=v(0')=1$. The main idea of the proof is to construct a superharmonic function F such that

(i) $\Delta F \leq 0 \quad \forall x \in T(\theta_1, R_0)$

(ii) $F \geq V$ on $\partial T(\theta_1, R_0)$ ①

(iii) $F \leq Cu$ on $T_2(\theta_2, R_0)$

Then with this F, by maximum principle then imply

$V \leq F$ on $T(\theta_1, R_0)$ (by (ii))

$V \leq Cu$ on $T(\theta_2, R_0)$ (by (iii))

Then from the gradient estimate, we can extend $T(\theta_2, R_0)$ to $T(\theta_2,1)$ with a larger C. Since we set $\frac{u(0')}{v(0')}=1$ then

$V \leq Cu \Leftrightarrow C^{-1}\frac{u(0')}{v(0')} \leq \frac{u}{V}$. we can get the $C\frac{V(0')}{u(0')} \geq \frac{V}{u}$

Then exchange u and $V \Rightarrow C\frac{u(0')}{V(0')} \geq \frac{u}{V}$ then done.

So our goal is to construct such F.

Firstly, we set up our notation. Let $C_1, C_2, \cdots, \alpha_1, \alpha_2 \cdots$ be the positive contants depending only on n, a, b.

Secondly, by theorem 2.1, $\exists\, C_1, \alpha_1 > 0$ st

$u, v \leq C_1 e^{-\alpha_1 \rho(x)}$ Now we have set $u(0')$ and $V(0')=1$.

On the other hand, by the gradient estimate $|\nabla \log u| \leq C$

$u(x), V(x) \geq C_2 e^{-\alpha_2 \rho(x)} \quad \forall x \in T(\theta_1, 1)$

Therefore, for $x \in T(\theta_1, 1)$.

$V(x) \leq C_1 e^{-\alpha_1 \rho(x)} \leq C_3 u^{-\lambda_0}$ where $\lambda_0 = \frac{\alpha_1}{\alpha_2} < 1$. ②

By lemma 5.1, there exists $\varphi \in C^\infty(M)$ with $\varepsilon \leq \varphi \leq 1$ (where ε is small positive number to be determined later) and satisitying

for $R_0 > 0$.

(i) $|\nabla \varphi| + |\nabla^2 \varphi| < C_4 e^{-C_5 \rho(x)}$

(ii) $\varphi = 1$ on $T(\theta_2, R_0)$

(iii) $\varphi = \varepsilon$ on $\partial C_0(\theta_1) \setminus B_0(R_0)$

Let $\lambda = 1 - (1-\varphi)^{2s} - (1-\varphi)^s e^{-\delta \rho}$

(where s and δ are positive constants to be determined)

Notice that

$\lambda = 1$ on $T(\theta_2, R_0)$

$\lambda = 1 - (1-\varepsilon)^{2s} - (1-\varepsilon)^s e^{-\delta\rho}$ on $\partial C_0(\theta_1) \setminus B_0(R_0)$

For a fixed s, ε, δ, if R_0 is sufficiently large we have

$\frac{1}{2}(1 - (1-\varepsilon)^{2s}) \leq \lambda \leq 1 - (1-\varepsilon)^{2s}$ on $\partial C_0(\theta_1) \setminus B_0(R_0)$

For a fixed s, if ε is small enough we have

$\lambda \leq \lambda_0 < 1$ on $\partial C_0(\theta_1) \setminus B_0(R_0)$.

Since $u \leq C_1 e^{-\alpha \rho}$, for big R_0 we have by ② that

$u^\lambda \geq u^{\lambda_0} \geq C_3^{-1} V$ on $\partial C_0(\theta_1) \setminus B_0(R)$.

We summary some properties of u^λ:

1. $u^\lambda \geq C_3^{-1} V$ on $\partial C_0(\theta_1) \setminus B_0(R) \Rightarrow u^\lambda \geq C_9 V$ on $\partial T(\theta_1, R_0)$

2. $u^\lambda = u$ on $T(\theta_2, R_0)$

Now, notice that u^λ we have the property (ii) and (iii) of

①, so we try to modify u^λ such that satisify the (i) of ① then we done.

Thirdly, let $\eta = -\log u$. Since for large ρ we have

$C_2 e^{-\alpha_2 \rho} \leq u \leq C_1 e^{-\alpha_1 \rho}$

So we have η satisifies $C_{10}\rho \geq \eta \geq C_{11}\rho$.

From $\Delta u = 0$ and $|\nabla \log u| \leq C$, $|\nabla \eta| \leq C_{12}$, $\Delta \eta = |\nabla \eta|^2 \leq C_{13}$

Set $F(x) = \psi(\eta(x) - e^{-\beta\rho(x)}) u(x)^{\lambda(x)}$

where β is a positive number to be determined, $\psi \in C^\infty(\mathbb{R})$ satisfies $1 \leq \psi \leq C_{13}$, $\psi'' \leq 0$, and $1 \leq \psi'(t) t^2 \leq 2$ for t sufficiently large.

Noticing that $\eta \sim C\rho$, so $1 \leq \psi'(t) t^2 \leq 2 \Rightarrow C_{15} \leq \rho^2 \psi'(\eta - e^{-\beta\rho}) \leq C_{14}$

($\because$ when t is sufficiently large $\Leftrightarrow \eta - e^{-\beta\rho}$ is large enough
$\Leftrightarrow \because \eta \sim C\rho \Rightarrow \rho$ is large enough.

so $t^2 = (\eta - e^{-\beta\rho})^2 \sim \eta^2 \propto \rho^2$)

Since ψ is positive and bounded, $F = \psi u^\lambda$ satisifies the boundary condition of (ii) and (iii) in ①.

Then we claim that there exists a proper β such that $\Delta F \leq 0$, for $\rho(x)$ is large enough.

(pf) $\Delta F = \Delta \psi\, u^\lambda + 2 \nabla \psi \cdot \nabla u^\lambda + \psi \Delta u^\lambda$

Look at $\Delta \psi$, since $\psi'' \leq 0$ we have $\Delta \psi \leq \psi'(\Delta \eta - e^{-\beta\rho}(\beta^2 - \beta \Delta \rho))$

$\leq \psi'(|\nabla \log u|^2 - \frac{1}{2}\beta^2 e^{-\beta\rho})$

(when ρ is large enough)

But we also know that $|\Delta\rho| \leq \frac{k_1}{\rho}(k_2+k_3\rho)$ for some constant

(Recall Cor 1.2 in chapter 1 textbook)

When ρ is large enough $\Rightarrow |\Delta\rho| \leq C$ for some C.

Now we first assume β is suffieciently large.

$$\Rightarrow \nabla\psi\cdot\nabla u^\lambda = \psi' u^\lambda(-\nabla\log u - \beta e^{-\beta\rho}\nabla\rho)(\log u\nabla\lambda + \lambda\nabla\log u)$$

$$\leq \psi' u^\lambda[-\lambda|\nabla\log u|^2 + C_{15}(\rho|\nabla\lambda||\nabla\log u| + \beta\rho e^{-\beta\rho}|\nabla\lambda| + \beta e^{-\beta\rho})]$$

$|\nabla\rho|=1$, $|\log u| \leq C\rho$, $|\nabla\log u| \leq C$

Then combine the above inequlity that we have.

$$\Delta F = \psi\Delta u^\lambda + 2\nabla\psi\cdot\nabla u^\lambda + \Delta\psi\cdot u^\lambda$$

$$\leq \psi\Delta u^\lambda + \psi' u^\lambda[-2\lambda|\nabla\log u|^2 + 2C_{15}\rho|\nabla\lambda||\nabla\log u| + 2C_{15}\beta\rho e^{-\beta\rho}|\nabla\lambda| + 2\beta e^{-\beta\rho}]$$

$$+ u^\lambda\psi'(|\nabla\log u|^2 - \tfrac{1}{2}\beta^2 e^{-\beta\rho})$$

$$\leq \psi\Delta u^\lambda + \psi' u^\lambda[(1-2\lambda)|\nabla\log u|^2 - C_{17}e^{-\beta\rho} + C_{16}\rho|\nabla\lambda||\nabla\log u|]$$

($\because |\nabla\log u| \leq C$ so we find C_{16} and when β and ρ are large enough)

$$|\nabla\lambda| = |(-2s)\cdot(-1)\cdot(1-\varphi)^{2s-1}\nabla\varphi + (-s)\cdot(1-\varphi)^{s-1}\cdot(-1)e^{-\delta\rho}\nabla\varphi + \delta(1-\varphi)^s e^{-\delta\rho}\nabla\rho|$$

$$\leq |(2s(1-\varphi)^{2s-1} + s(1-\varphi)^{s-1}e^{-\delta\rho})\nabla\varphi| + |\delta(1-\varphi)^s e^{-\delta\rho}\nabla\rho|$$

$$\leq K_1|\nabla\varphi| + K_2 e^{-\delta\rho} \qquad \because |\nabla\varphi| + |\nabla^2\varphi| < C_4 e^{-C_5\rho(x)}$$

$$|\nabla\varphi| < C_4 e^{-C_5\rho(x)}$$

$$\leq C_6 e^{-C_7\rho}$$

So $\rho|\nabla\lambda| \to 0$ as $\rho\to\infty$.

Next, we look at the term Δu^λ.

$$\Delta u^\lambda = \nabla\cdot(\nabla u^\lambda) = \nabla\cdot\left(u^\lambda(\nabla\lambda\cdot\log u + \lambda\nabla\log u)\right)$$

$$= u^\lambda\left[(\log u\nabla\lambda + \lambda\nabla\log u)\cdot(\log u\nabla\lambda + \lambda\nabla\log u) + \Delta\lambda\log u + 2\nabla\lambda\cdot\nabla\log u + \lambda\Delta\log u\right]$$

$$= u^\lambda\left[(\log u)^2|\nabla\lambda|^2 + \underline{(2\lambda\log u + 2)}(\nabla\lambda\cdot\log u) + \lambda(\lambda-1)|\nabla\log u|^2 + \Delta\lambda\log u + \lambda\Delta\log u\right]$$

$$\leq u^\lambda\left[C_{18}\rho^2|\nabla\lambda|^2 + \underline{C_{19}}\rho|\nabla\lambda||\nabla\log u| + (\Delta\lambda)\log u + \lambda(\lambda-1)|\nabla\log u|^2\right]$$

$$\left(\because |2\lambda\log u + 2| \leq C_{19}\rho,\ \lambda = 1-(1-\varphi)^{2s} - (1-\varphi)^s e^{-s\rho} \leq 1\right)$$

We consider two cases for $\lambda \geq \frac{3}{4}$ and $\lambda < \frac{3}{4}$

1. At points where $\lambda \geq \frac{3}{4}$, we have

$$C_{19}\psi\rho|\nabla\lambda||\nabla\log u| \leq (2\lambda-1)\psi'|\nabla\log u|^2 + C_{20}\psi\rho^4|\nabla\lambda|^2$$

(since $\psi'(t) \geq t^2$ and ψ is bounded $\because t\sim\rho$

$$C_{19}\psi\rho|\nabla\lambda||\nabla\log u| \leq K\rho|\nabla\lambda||\nabla\log u| \leq k_1^2(\rho|\nabla\log u|)^2 + \frac{k^2}{2k_1^2}|\nabla\lambda|^2$$

$$\leq \left(\frac{\psi'}{2}|\nabla\log u|^2 + \frac{k^2}{2k_1^2}|\nabla\lambda|^2\right) \leq (2\lambda-1)\psi'|\nabla\log u|^2 + C_{20}\rho^4|\nabla\lambda|^2$$

$\because \rho$ is large enough)

$$\Delta F \leq -C_{17}\psi' u^\lambda e^{-\beta\rho} + \psi u^\lambda\left[C_{18}\rho^2|\nabla\lambda|^2 + C_{20}\rho^4|\nabla\lambda|^2 + \Delta\lambda\log u\right].$$

2. We consider the case $\lambda < \frac{3}{4}$, since $\lambda \geq (1-(1-\varepsilon)^{2s})/2 \geq \frac{\varepsilon}{2}$, we see $\lambda(1-\lambda) \geq \frac{\varepsilon}{8}$.

Then $C_{19}\rho|\nabla\lambda||\nabla\log u| \leq \lambda(1-\lambda)|\nabla\log u|^2 + C_{21}\varepsilon^{-1}\rho^2|\nabla\lambda|^2$

$$\left(C_{19}\rho|\nabla\lambda||\nabla\log u| \leq \frac{\varepsilon}{8}|\nabla\log u|^2 + \frac{4}{\varepsilon}C_{19}^2\rho^2|\nabla\lambda|^2 \leq \lambda(1-\lambda)|\nabla\log u|^2 + C_{21}\varepsilon^{-1}\rho^2|\nabla\lambda|^2\right)\left(\because AB \leq \frac{A^2+B^2}{2}\right)$$

$$\Rightarrow \Delta F \leq -C_{17}\psi' u^\lambda e^{-\beta\rho} + \psi u^\lambda\left[C_{18}\rho^2|\nabla\lambda|^2 + C_{21}\varepsilon^{-1}\rho^2|\nabla\lambda|^2 + \Delta\lambda\log u\right]$$

Then we can see that $\forall x\in T(\theta_1, R_0)$ for sufficient large R_0

$$\Rightarrow \Delta F \leq -C_{17}\psi' u^\lambda e^{-\beta\rho} + \psi u^\lambda\left[C_{22}\varepsilon^{-1}\rho^4|\nabla\lambda|^2 + \Delta\lambda\log u\right] \text{ for each case.}$$

Now we consider the term $(\Delta\lambda)\log u$.

$$(\Delta\lambda)\log u = \log u[-\Delta(1-\varphi)^{2s} - \Delta(1-\varphi)^s e^{-\delta\rho} - 2\nabla(1-\varphi)^s\cdot\nabla e^{-\delta\rho} - (1-\varphi)^s\Delta e^{-\delta\rho}]$$

$$\leq |\log u|[|\Delta(1-\varphi)^{2s}| + |\Delta(1-\varphi)^s| e^{-\delta\rho} + 2|\nabla(1-\varphi)^s||\nabla e^{-\delta\rho}|] - (1-\varphi)^s\Delta e^{-\delta\rho}\log u.$$

$$\leq C_{23}e^{-C_{24}\rho}(1-\varphi)^{s-1} - C_{25}\rho\delta(1-\varphi)^s e^{-\delta\rho}$$

$\left(\begin{array}{l} 1.\ \varepsilon\leq\varphi\leq 1 \Rightarrow (1-\varphi)^{2s}\leq(1-\varphi)^s \\ 2.\ |\log u|\sim C\rho \end{array}\right)$

Notice, when ε is chosen we can find R_0 so that $\forall\rho\geq R_0$,

$$C_{22}\varepsilon^{-1}\rho^4|\nabla\lambda|^2 \leq C_{26}e^{-C_{27}\rho}(1-\varphi)^{s-1} + C_{28}\rho^4\varepsilon^{-1}e^{-2\delta\rho}(1-\varphi)^{2s}$$

$$\leq C_{29}e^{-C_{30}\rho}(1-\varphi)^{s-1} \quad \left(\because e^{-ax}x^k < e^{-bx} \text{ when } x\to\infty \text{ If } 0<b<a,\ k>0\right)$$

Now, summary all results we get the estimation about ΔF.

$$\Delta F \leq -C_{19}\psi' u^\lambda e^{-\beta\rho} + \psi u^\lambda[C_{31}e^{-C_{32}\rho}(1-\varphi)^{s-1} - C_{25}\rho\delta(1-\varphi)^s e^{-\delta\rho}]$$

$$= -C_{19}\psi' u^\lambda e^{-\beta\rho} + \psi u^\lambda(1-\varphi)^{s-1}[C_{31}e^{-C_{32}\rho} - C_{25}\rho\delta(1-\varphi)e^{-\delta\rho}]$$

If $C_{31}e^{-C_{32}\rho}\leq C_{25}\rho\delta(1-\varphi)e^{-\delta\rho}$, $\Delta F\leq 0$ then done.

Otherwise $C_{31}e^{-C_{32}\rho} > C_{25}\rho\delta(1-\varphi)e^{-\delta\rho}$ when ρ is large enough.

then $0<1-\varphi<C e^{-(C_{32}-\delta)\rho}$ for some C.

Then $\Delta F \leq -C_{19}\psi' u^\lambda e^{-\beta\rho} + \psi u^\lambda C_{33}e^{-[C_{32}\rho + (s-1)(C_{32}-\delta)\rho]}$

Therefore if $\delta < C_{32}$ and $\beta < C_{32} + (s-1)(C_{32}-\delta)$ then we have $\Delta F\leq 0$ for $\rho\geq R_0$ $\square$