$$S=\frac{1}{16\pi G}\left[\int_{\mathcal{M}}\tilde{\hat{R}}_{ab}\wedge\theta^a\wedge\theta^b-\frac{2\Lambda}{4!}\varepsilon_{abcd}\theta^a\wedge\theta^b\wedge\theta^c\wedge\theta^d\right]$$

$$\delta_\theta S=\frac{1}{16\pi G}\Big\{\Big[\int_{\mathcal{M}}\Big(\tilde{\hat{R}}_{ab}\wedge\delta\theta^a\wedge\theta^b+\tilde{\hat{R}}_{ab}\wedge\theta^a\wedge\delta\theta^b\Big)\Big]$$

$$-\frac{2\Lambda}{4!}\varepsilon_{abcd}\Big[\delta\theta^a\wedge\theta^b\wedge\theta^c\wedge\theta^d+\theta^a\wedge\delta\theta^b\wedge\theta^c\wedge\theta^d+\theta^a\wedge\theta^b\wedge\delta\theta^c\wedge\theta^d+\theta^a\wedge\theta^b\wedge\theta^c\wedge\delta\theta^d\Big]\Big\}$$

$$=\frac{1}{16\pi G}\Big\{\Big[\int_{\mathcal{M}}-2\tilde{\hat{R}}_{ab}\wedge\theta^b\wedge\delta\theta^a-\frac{8\Lambda}{4!}\varepsilon_{abcd}\ \delta\theta^a\wedge\theta^b\wedge\theta^c\wedge\theta^d\Big]\Big\}$$

$$=\frac{1}{16\pi G}\Big\{\Big[\int_{\mathcal{M}}\Big(-2\tilde{\hat{R}}_{ab}\wedge\theta^b+\frac{8\Lambda}{4!}\varepsilon_{abcd}\ \theta^b\wedge\theta^c\wedge\theta^d\Big)\wedge\delta\theta^a\Big]\Big\}$$

$\Rightarrow$ EOM

$$\tilde{\hat{R}}_{ab}\wedge\theta^b-\frac{4\Lambda}{4!}\varepsilon_{abcd}\,\theta^b\wedge\theta^c\wedge\theta^d=0$$

$$\Rightarrow\frac{1}{4}\varepsilon_{abcd}R^{cd}{}_{ij}\,\theta^i\wedge\theta^j\wedge\theta^b-\frac{\Lambda}{3!}\varepsilon_{abcd}\,\theta^b\wedge\theta^c\wedge\theta^d=0$$

$$\Rightarrow\frac{1}{4}\varepsilon_{abcd}R^{cd}{}_{ij}\,\theta^b\wedge\theta^i\wedge\theta^j-\frac{\Lambda}{3!}\varepsilon_{abij}\,\theta^b\wedge\theta^i\wedge\theta^j=0$$

$$\Rightarrow\frac{1}{4}\varepsilon_{abcd}R^{cd}{}_{ij}-\frac{\Lambda}{3!}\varepsilon_{abij}=0$$

$$\times 4\varepsilon^{kbij}\quad\Rightarrow\varepsilon^{kbij}\varepsilon_{abcd}R^{cd}{}_{ij}-\frac{4\Lambda}{3!}\varepsilon^{kbij}\varepsilon_{abij}=0$$

$$\Rightarrow\varepsilon^{bkij}\varepsilon_{bacd}R^{cd}{}_{ij}-\frac{4\Lambda}{3!}\varepsilon^{kbij}\varepsilon_{abij}=0$$

$$\Rightarrow-\delta^{kij}_{acd}R^{cd}{}_{ij}-\frac{4\Lambda}{3!}\left(-3!\,\delta^k_a\right)=0$$

$$\Rightarrow-2\delta^k_aR+4R^k{}_a+4\Lambda\delta^k_a=0$$

$$\Rightarrow R^k{}_a-\frac{1}{2}\delta^k{}_aR+\Lambda\delta^k_a=0$$

$$\Rightarrow R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R + \Lambda g_{\mu\nu} = 0$$

Then we define $\tilde{R}_{ab} - \frac{\Lambda}{3!} \varepsilon_{abcd}\, \theta^c \wedge \theta^d = \tilde{G}_{ab}$

Then $G^{ab} := \frac{1}{2} \varepsilon^{abcd} \tilde{G}_{cd} = \frac{1}{2} \varepsilon^{abcd} \left(\tilde{R}_{cd} - \frac{\Lambda}{3!} \varepsilon_{cdij}\, \theta^i \wedge \theta^j \right)$

$$= R^{ab} - \frac{\Lambda}{2 \cdot 3!} \varepsilon^{abcd} \varepsilon_{cdij}\, \theta^i \wedge \theta^j$$

$$= R^{ab} - \frac{\Lambda}{12} \left(-2 \delta^a_i \delta^b_j + 2 \delta^a_j \delta^b_i \right) \theta^i \wedge \theta^j$$

$$= R^{ab} + \frac{\Lambda}{3!} \left(\theta^a \wedge \theta^b - \theta^b \wedge \theta^a \right)$$

$$= R^{ab} + \frac{2\Lambda}{3!}\, \theta^a \wedge \theta^b$$

The Bianchi Identity

$R_{ab} \wedge \theta^b = 0 \Leftrightarrow R_{ab} + \frac{2\Lambda}{3!} \theta_a \wedge \theta_b \wedge \theta^b = 0$ where $\theta_a := \eta_{ab} \theta^b$

$$\Leftrightarrow G_{ab} \wedge \theta^b = 0$$

EOM

$$\tilde{R}_{ab} \wedge \theta^b = 0 \Rightarrow \tilde{G}_{ab} \wedge \theta^b = 0$$

Then the so(2) duality still holds.

$$\begin{pmatrix} G_{ab} \\ \tilde{G}_{ab} \end{pmatrix} \rightarrow \begin{pmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} G_{ab} \\ \tilde{G}_{ab} \end{pmatrix}$$