2 ways to see the rule of curves.

$$L = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \qquad R = \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix}$$

第一種 用階堤圖來看

Denote $w_{i,j} = i\, L\, y_1\, R\, y_2\, L\, y_3\, R\, j$

$i \; y_1 \; y_2 \; y_3 \; j$

Define $F^{++}_{w_{i,j}}$ = subset include beginning and end

$F^{+-}_{w_{i,j}}$ = subset include begin, exclude end

$F^{-+}_{w_{i,j}}$ = exclude begin, include end

$F^{--}_{w_{i,j}}$ = exclude both begin and end

$$F^{++}_{w_{i,j}} = \underset{i \quad j}{1} + \underset{i \; y_2 \; j}{y_2} + \underset{i \; y_1 \; y_2 \; j}{y_1 y_2} + \underset{i \; y_2 \; y_3 \; j}{y_2 y_3} + \underset{i \; y_1 \; y_2 \; y_3 \; j}{y_1 y_2 y_3}$$

F poly

all possible valid subset

$$F^{+-}_{w_{i,j}} = \underset{i}{1} + \underset{i \; y_2}{y_2} + \underset{i \; y_1 \; y_2}{y_1 y_2}$$

$$F^{-+}_{w_{i,j}} = \underset{j}{1} + \underset{y_2 \; j}{y_2} + \underset{y_2 \; y_3 \; j}{y_2 y_3}$$

$$F^{--}_{w_{i,j}} = \underset{\text{empty}}{1} + \underset{y_2}{y_2}$$

這邊跟 all loop chapter 5 的討論相似.

然後 $u_{i,j}$ definition 是 $u_{i,j} = \dfrac{F^{+-}_{w_{i,j}} \cdot F^{-+}_{w_{i,j}}}{F^{++}_{w_{i,j}} F^{--}_{w_{i,j}}}$.

第二種

而我們可以用矩陣方式一步步構造 F^{++}、F^{+-}、F^{-+}、F^{--}

我們 Let $M_w = \begin{pmatrix} F_w^{++} & F_w^{+-} \\ F_w^{-+} & F_w^{--} \end{pmatrix}$

考慮 Word w

Case1: $w = YRw'$ $\qquad w = Y_{w'}$

$$\Rightarrow F_w^{++} = Y \cdot F_{w'}^{++}$$

$$F_w^{+-} = Y \cdot F_{w'}^{+-}$$

$$F_w^{-+} = F_{w'}^{++} + F_{w'}^{-+}$$

$$F_w^{--} = F_{w'}^{+-} + F_{w'}^{--}$$

$$\Rightarrow \begin{pmatrix} F_w^{++} & F_w^{+-} \\ F_w^{-+} & F_w^{--} \end{pmatrix} = \begin{pmatrix} Y & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{w'}^{++} & F_{w'}^{+-} \\ F_{w'}^{-+} & F_{w'}^{--} \end{pmatrix}$$

$$= \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} F_{w'}^{++} & F_{w'}^{+-} \\ F_{w'}^{-+} & F_{w'}^{--} \end{pmatrix}$$

$$M_w = \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 1 & 1 \end{pmatrix} M_{w'}$$

case 2. $w = Y\ell w'$ $\qquad w = {}_{Y}{}^{w'}$

$$\Rightarrow F_w^{++} = Y F_{w'}^{++} + Y F_{w'}^{-+}$$

$$F_w^{+-} = Y F_{w'}^{+-} + Y F_{w'}^{--}$$

$$F_w^{-+} = F_{w'}^{-+}$$

$$F_w^{--} = F_{w'}^{--}$$

$$\begin{pmatrix} F_w^{++} & F_w^{+-} \\ F_w^{-+} & F_w^{--} \end{pmatrix} = \begin{pmatrix} Y & Y \\ 0 & 1 \end{pmatrix} \begin{pmatrix} F_{w'}^{++} & F_{w'}^{+-} \\ F_{w'}^{-+} & F_{w'}^{--} \end{pmatrix}$$

$$\Rightarrow M_w = \begin{pmatrix} Y & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} M_{w'}$$

So given word W we can 用上面的方法一次構造 F_w^{++}、F_w^{+-}、F_w^{-+}、F_w^{--} By M_w 的方式

$$W = i \quad R \quad Y_1 \quad L \quad Y_2 \quad L \quad Y_3 \quad R \quad Y_4 \quad j$$

$$M_w = \begin{pmatrix}1&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}Y_1&0\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}Y_2&0\\0&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}Y_3&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}\begin{pmatrix}Y_4&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\0&1\end{pmatrix}$$

Summary：可以用數 subset 的方式看 或是用矩陣相乘 是等價的.

$F_{i,j}$ in new representation

1. Consider $W_{i+1,j+1}$ $\quad W_{i+1,j+1} = i{+}1\, R\, Y_1\, L\, Y_2\, R\, Y_3\, L\, Y_4$
2. Trim $W_{i+1,j+1}$ $\Rightarrow W^{Trim}_{i+1,j+1} = L\,\underline{Y_2 R} = L\,\underline{W'}$
3. $F_{i,j} = (1,1)\,\underline{\begin{pmatrix}Y_2&0\\0&1\end{pmatrix}\begin{pmatrix}1&0\\1&1\end{pmatrix}}\begin{pmatrix}1\\0\end{pmatrix}$ 照定義 New 那篇

$$= (1,1)\; M_{W'}\begin{pmatrix}1\\0\end{pmatrix}$$

$$= (1,0)\underbrace{\begin{pmatrix}1&1\\0&1\end{pmatrix}}_{L} M_{W'}\begin{pmatrix}1\\0\end{pmatrix}$$

因為 Trim 完後 第一個字一定是 L 所以一定可以這樣討論。

$$= (1,0)\, M_{w^{Trim}_{i+1,j+1}} \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

$$= (1,0) \begin{pmatrix} F^{++}_{w^{Trim}_{i+1,j+1}} \\ F^{-+}_{w^{Trim}_{i+1,j+1}} \end{pmatrix}$$

$$= F^{++}_{w^{Trim}_{i+1,j+1}}$$

Summary $F_{i,j}$ follow New representation 的定義可以推得 $F^{++}_{w^{trim}_{i+1,j+1}}$.

而用圖像式來看那 $F^{++}_{w^{Trim}_{i+1,j+1}}$ 就是把路對應的 valid subset 取和，而起終點交換只是把路對應的高低圖對最右邊做鏡像。

$$W_{2,7} = 2\, L\, Y_{13}\, R\, Y_{14}\, R\, Y_{16}\, L\, Y_{68}\, R\, 7$$

$$= 2^{Y_{13}}{}_{Y_{14}}{}_{Y_{16}}{}^{Y_{68}}\, 7$$

$$W_{7,2} = 7\, L\, Y_{68}\, R\, Y_{16}\, L\, Y_{14}\, L\, Y_{13}\, R\, 2$$

$$= 7^{Y_{68}}{}_{Y_{16}}{}^{Y_{14}}{}^{Y_{13}}\, 2$$

鏡像。

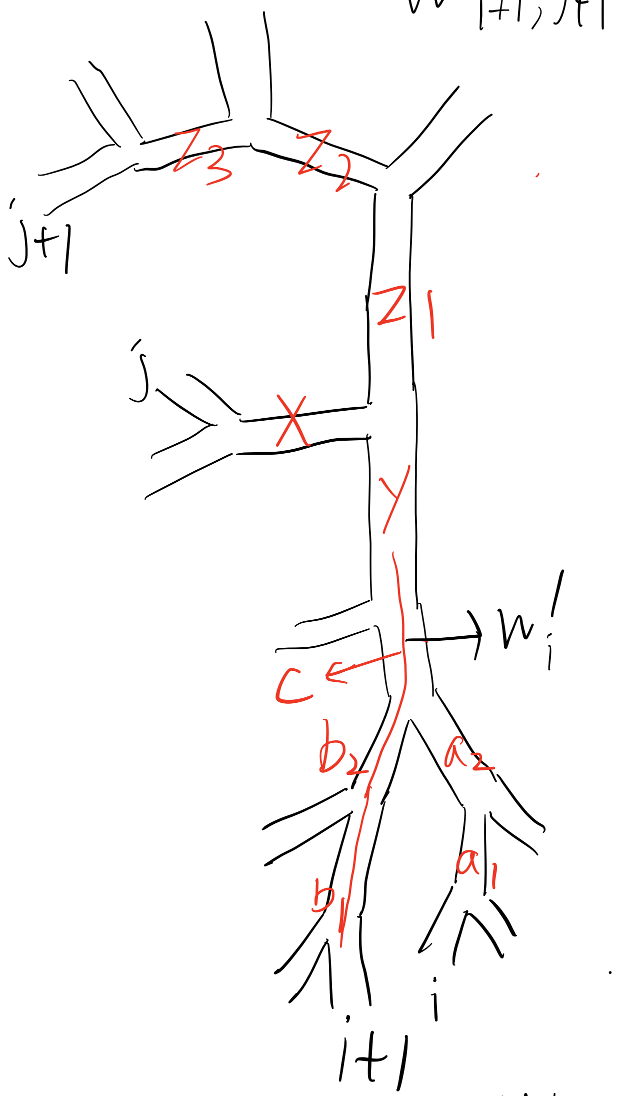
2、回到 $u_{i,j} = \dfrac{F^{+-}_{w_{i,j}} F^{-+}_{w_{i,j}}}{F^{++}_{w_{i,j}} F^{--}_{w_{i,j}}}$

先分析 $W_{i,j}$ 之間的關係

$W_{i,j+1}$

$W_{i+1,j}$

$W_{i+1,j+1}$



$W_{i,j} = w_i' \, YLXRj$

$W_{i,j+1} = w_i' \, YRz_1Lz_2Lz_3L(j+1)$

尾巴是j+1和j之間的關係是

把$W_{i,j}$中從j倒著走直到遇到第一個L，然後把那個L改成R後後面一路走L直到j+1

$W_{i,j} = w_i' \, y \boxed{L} X \underline{R} \, j$

$W_{i,j+1} = w_i' \, y \, R z_1 \underline{L} z_2 \underline{L} z_3 \underline{L} \, j+1$

而相同尾不同頭之間的關係

$W_{i,j} = i L a_1 L a_2 R C W_j$

$W_{i+1,j} = i{+}1 R b_1 R b_2 L C W_j$

i到 i+1 的方法

先從 i 出發一直往後直到第一個R 把他改成L 以後一直往前補R直到 i+1 R R R ⋯ L C

$W_{i,j} = i L a_1 L a_2 \boxed{R} C W_j$

$W_{i+1,j} = \underline{i{+}1} R b_1 R b_2 \boxed{L} C W_j$

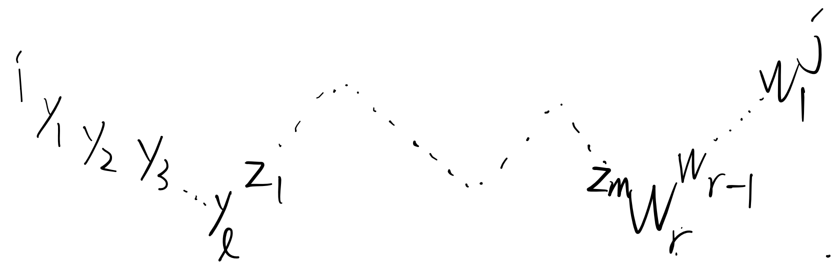
分析 $W^{Trim}_{i+1,j}$、$W^{Trim}_{i,j+1}$、$W^{Trim}_{i+1,j+1}$ 跟 $W^{Trim}_{i,j}$ 的關係

1. $W_{i+1,j}$ 是把 $W_{i,j}$ 從左數來第一個 R 改成 L 前面補上一堆R.
$W^{Trim}_{i+1,j}$ 的左邊會全刪掉直到那個做變更的L.
⇒ $W^{Trim}_{i+1,j}$ 會是 $W_{i,j}$ 右邊把連續L刪掉，而左邊是一路刪掉直到第一個 R 把他變L.

2. $W^{Trim}_{i,j+1}$ 把 $W_{i,j}$ 左邊連續R刪掉，而右邊倒著一路刪掉直到第一個L變成R

3. $W^{Trim}_{i+1,j+1}$ 把左邊一路刪掉到第一個R改成L 再把右邊一路倒著刪直到第一個L然後將其改成R

Consider $W_{i,j} =$



$$= i R y_1 R \cdots R y_l \underline{L} z_1 \cdots\cdots z_m \underline{R} w_r L w_{r-1} \cdots L w_1 L j$$

在此 case $i+1$ 開頭的就是

$$i+1 \, \underline{L} y_1 R y_2 R y_3 \cdots\cdots$$

而 $j+1$ 結尾的就是

$$\cdots\cdots \quad w_r L w_{r-1} L \cdots\cdots L w_1 \underline{R} \, j+1$$

Denote $W^m = L z_1 \cdots\cdots z_m R$ 把所有連續R開頭去掉 L結尾去掉

$W^l = \underline{L} y_1 R \cdots\cdots R y_l W^m$ 把所有連續L結尾去掉 把第一個遇到的開頭R改成L

$W^r = W^m w_r L w_{r-1} \cdots w_1 \underline{R}$ 把所有連續R開頭去掉 把倒著數第一個遇到的L改成R.

$W^{l,r} = \underline{L} y_1 R \cdots R y_l W^m w_r L w_{r-1} \cdots w_1 \underline{R}$

把第一個遇到的開頭R改成L 把倒著數第一個遇到的L改成R.

By all loop F^{++} F^{+-} F^{-+} F^{--} definition

$$\Rightarrow F^{++}_{W_{i,j}} = \prod_{\alpha=1}^{l} y_\alpha \prod_{\beta=1}^{r} w_\beta \, F^{++}_{W^m} \qquad F^{+-}_{W_{i,j}} = \prod_{\alpha=1}^{l} y_\alpha \, F^{++}_{W^r}$$

$$F^{-+}_{W_{i,j}} = \prod_{\beta=1}^{r} w_\beta \, F^{++}_{W^l} \qquad F^{--}_{W_{i,j}} = F^{++}_{W^{l,r}}$$

$$W^m = L Z_1 \cdots\cdots Z_m R$$

$$W^l = \underline{L} Y_1 R \cdots\cdots R Y_l W^m$$

$$W^r = W^m W_r L W_{r-1} \cdots W_1 \underline{R}$$

$$W^{l,r} = \underline{L} Y_1 R \cdots R Y_l W^m W_r L W_{r-1} \cdots\cdots W_1 \underline{R}$$

但注意

$$W^{Trim}_{i,j} = W^m \qquad F^{++}_{W_{i,j}} = \prod_{\alpha=1}^{l} Y_\alpha \prod_{\beta=1}^{r} w_\beta F^{++}_{W^{Trim}_{i,j}} = \prod_{\alpha=1}^{l} Y_\alpha \prod_{\beta=1}^{r} w_\beta F_{i-1,j-1}$$

$$W^{Trim}_{i+1,j} = W^l \qquad \Rightarrow F^{+-}_{W_{i,j}} = \prod_{\alpha=1}^{l} Y_\alpha F^{+-}_{W^{Trim}_{i,j+1}} = \prod_{\alpha=1}^{l} Y_\alpha F_{i-1,j}$$

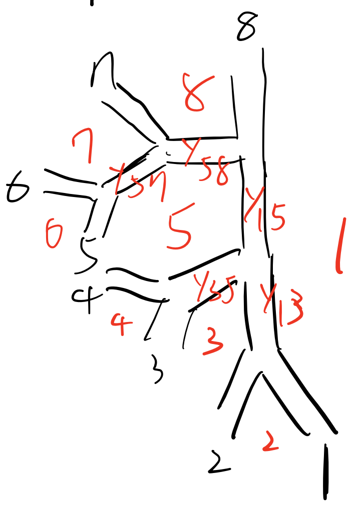
$$W^{Trim}_{i,j+1} = W^r \qquad F^{-+}_{W_{i,j}} = \prod_{\beta=1}^{r} Y_\beta F^{++}_{W^{Trim}_{i+1,j}} = \prod_{\beta=1}^{r} w_\beta F_{i,j-1}$$

$$W^{Trim}_{i+1,j+1} = W^{l,r} \qquad F^{--}_{W_{i,j}} = F^{++}_{W^{Trim}_{i+1,j+1}} = F_{i,j}$$

$$\Rightarrow U_{i,j} = \frac{F^{+-}_{W_{i,j}} F^{-+}_{U_{i,j}}}{F^{++}_{W_{i,j}} F^{--}_{U_{i,j}}} = \frac{\prod_{\alpha=1}^{l} Y_\alpha F_{i-1,j} \prod_{\beta=1}^{r} W_\beta F_{i,j-1}}{\prod_{\alpha=1}^{l} Y_\alpha \prod_{\beta=1}^{r} W_\beta F_{i-1,j-1} \cdot F_{i,j}}$$

$$= \frac{F_{i-1,j} F_{i,j-1}}{F_{i-1,j-1} F_{i,j}} .$$

Special case when $(i,j) \in T$.



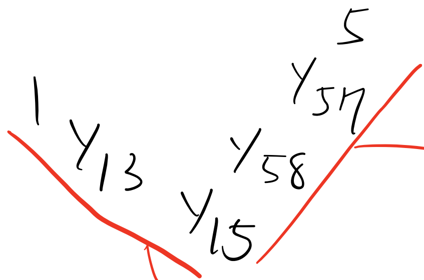
$\Rightarrow (1,3)、(1,5)、(3,5)、(5,8)、(5,7) \in T$

Consider $u_{1,5}$ $(1,5) \in T$.

$$\Rightarrow W_{1,5} = 1 R y_{13} R y_{15} L y_{58} L y_{57} L 5.$$

特色是 $W_{1,5}$ 在 $y_{1,5}$ 前是連續 R

後是連續 L



$$F^{++}_{w_{1,5}} = y_{13} y_{15} y_{58} y_{57} = \frac{(y_{13} y_{15})(y_{57} y_{58} y_{15})}{y_{15}} \quad F_{8,4} = 1$$

$$F^{+-}_{w_{1,5}} = (y_{13} y_{15}) F_{8,5} \quad (\because F_{1-1,5} = F_{8,5})$$

$$F^{-+}_{w_{1,5}} = (y_{57} y_{58} y_{15}) F_{1,4}$$

$$F^{--}_{w_{1,5}} = F_{1,5}$$

$$\Rightarrow u_{1,5} = \frac{(y_{13} y_{15}) F_{8,5} \cdot (y_{57} y_{58} y_{15}) F_{14}}{\frac{(y_{13} y_{15})(y_{57} y_{58} y_{15})}{y_{15}} F_{8,4} \cdot F_{1,5}}$$

$$= y_{1,5} \frac{F_{8,5} F_{14}}{F_{8,4} F_{1,5}} = y_{1,5} \frac{F_{1-1,5} F_{1,5-1}}{F_{1-1,5-1} F_{1,5}} \quad \square$$

$$\Rightarrow W^{Trim}_{i,j+1} = W'^{Trim,l}_{i} YR$$

$$W^{Trim}_{i,j} = W'^{Trim,l}_{i} YLXR$$

$$W^{Trim}_{i+1,j+1} = W'^{Trim,l}_{i+1} YR$$

$$W^{Trim}_{i+1,j+1} = W'^{Trim,l}_{i+1} YLXR$$