In the superstring case, we first consider the vertex operater $V(z) = :P\psi(z) e^{iPX(z)}:$ [where the $:\psi(z_1)::\psi(z_2): = \frac{1}{z_{21}} + :\psi(z_1)\psi(z_2):$]

Then we first consider the 2n point amplitude of this vertex operator. (Notice that the $P_i^2 = \frac{1}{2\alpha'}$ $\forall i \in \{1, 2 \cdots, 2n\}$ in this case).

$$A^n = \int \frac{d^{2n}z}{SL(2,\mathbb{R})} \langle \prod_{a=1}^{2n} P_a \psi(z_a) e^{iP_a X(z_a)} \rangle$$

$$= \int \frac{d^{2n}z}{SL(2,\mathbb{R})} d^{2n}\theta \langle \prod_{a=1}^{2n} e^{iP_a X(z_a) + \theta_a P_a \psi_a(z_a)} \rangle$$

$$= \int \frac{d^{2n}z \, d^{2n}\theta}{SL(2,\mathbb{R})} \prod_{i<j} |z_j - z_i + \theta_i \theta_j|^{2\alpha' P_i P_j}$$

$$= \int \frac{d^{2n}z \, d^{2n}\theta}{SL(2,\mathbb{R})} \prod_{i<j} z_{ij}^{\alpha' s_{ij}} \left|1 + \frac{\theta_i \theta_j}{z_{ij}}\right|^{\alpha' s_{ij}}$$

$$= \int \frac{d^{2n}z \, d^{2n}\theta}{SL(2,\mathbb{R})} \prod_{i<j} z_{ij}^{\alpha' s_{ij}} \left(1 + \frac{\alpha' s_{ij}}{z_{ij}} \theta_i \theta_j\right)$$

$$= \int \frac{d^{2n}z}{SL(2,\mathbb{R})} \prod_{i<j} z_{ij}^{\alpha' s_{ij}} \, \mathrm{Pf}(A) \quad \text{where } A_{ij} = \begin{cases} \frac{\alpha' s_{ij}}{z_{ij}} & i<j \\ -A_{ji} & \text{else} \end{cases}$$

We also consider the another amplitude which replace two of n vertex operator $P\psi e^{iPX} \longrightarrow e^{-\phi} e^{iPX}$

$$\Rightarrow \int \frac{d^{2n}z}{SL(2,\mathbb{R})} \langle \prod_{a=1}^{2n \setminus \{i,j\}} P_a \psi(z_a) e^{iP_a X(z_a)} \cdot e^{-\phi(z_i)} e^{iP_i X(z_i)} e^{-\phi(z_j)} e^{iP_j X(z_j)} \rangle$$

$$= \int \frac{d^{2n}z}{SL(2,\mathbb{R})} \frac{1}{z_{ij}} \prod_{a<b} z_{ab}^{\alpha' s_{ab}} \, \mathrm{Pf}(A^{ij})$$

where A^{ij} is A with ith and jth row and column deleted.

We try to use the above "Tachyon amplitude" to construct n-point gluon's amplitude.

First, we start from 2n-"Tachyon" amplitude and we identity $2i-1$ and $2i$

particle in ith group.

Secondly, choose two particle $2i-1$th and $2j-1$th in different group, so the amplitude look like $\int \frac{d^{2n}z}{SL(2,\mathbb{R})} \langle P_1\psi e^{iP_1X(z_1)} \cdots e^{\phi(z_i)} e^{iP_iX(z_i)} \cdots e^{\phi(z_j)} e^{iP_jX(z_j)} \cdots P_{2n}\psi e^{iP_{2n}X(z_{2n})} \rangle$

Thirdly, we fuse each group to gluon by taking the residue on the

$\alpha'(P_{2i-1}+P_{2i})^2=0 \Leftrightarrow \alpha' P_{2i-1}^2 + \alpha' P_{2i}^2 + \alpha' S_{2i-1,2i} = 0 \Leftrightarrow \frac{1}{2} + \frac{1}{2} + \alpha' S_{2i-1,2i} = 0$

$\Leftrightarrow$ taking the residue on $z_{2i-1,2i}$ (as the bosonic discussion)

Then we study how to pick up the residue, we need to consider this in two case.

Case 1. two vertex in the group is $P_1\psi e^{iP_1X(z_1)}$ $P_2\psi e^{iP_2X(z_2)}$,

then when we take the residue on $z_2 - z_1 = z_{1,2}$

there are two way to contribute.

1. $P_1\psi(z_1)$ and $P_2\psi(z_2)$ contract with $P_i\psi$ in other group. Since $\alpha' S_{2i-1,2i}+1=0$, so the residue is $P_1\psi(z_2)P_2\psi(z_2)e^{i(P_1+P_2)X(z_2)}$

2. $P_1\psi(z_1)$ contract with $P_2\psi(z_2)$, then $P_1\cdot P_2 \frac{-1}{z_{1,2}} e^{iP_1X(z_1)} e^{iP_2X(z_2)}$

But we want to get the residue on $z_{1,2}$ so as the Bosonic case we get $P_1\cdot P_2\, iP_2\cdot\partial X(z_1) e^{i(P_1+P_2)X(z_1)}$.

$\Rightarrow \left[P_1\cdot\psi(z_1)\, P_2\cdot\psi(z_1) - (P_1\cdot P_2)\, iP_2\cdot\partial X(z_1)\right] e^{i(P_1+P_2)X(z_1)}$.

Identity $P_1+P_2 = k_1$ and $P_2 = \varepsilon_1$.

$\left[(k_1-\varepsilon_1)\cdot\psi(z_1)\, \varepsilon_1\cdot\psi(z_1) - \left(-\frac{1}{2\alpha'}\right) i\varepsilon_1\cdot\partial X(z_1)\right] e^{ik_1X(z_1)}$

$= \frac{1}{2\alpha'}\left[i\varepsilon_1\partial X(z_1) + 2\alpha' k_1\cdot\psi(z_1)\, \varepsilon_1\cdot\psi(z_1)\right] e^{ik_1X(z_1)}$

$:= V^0(z_1)$

Case 2. One vertex is replaced by $e^{-\phi} e^{iPX}$.

$\Rightarrow$ WLOG let the vertex's are $e^{-\phi(z_3)} e^{iP_3 X(z_3)}$ $P_4 \psi(z_4) e^{iP_4 X(z_4)}$.

Then consider the residue of $\alpha' S_{3,4} + 1 = 0$.

we find it become $e^{-\phi(z_3)} P_4 \psi(z_3) e^{i(P_3+P_4)X(z_3)}$.

Then identity $P_4 = \varepsilon_2$, $k_2 = P_3 + P_4$, and $z_3 \equiv z_2$

$$\Rightarrow e^{-\phi(z_2)} \varepsilon_2 \cdot \psi(z_2) e^{ik_2 X(z_2)}$$

$$:= V^{-1}(z_2)$$

Consequently, we get the following form after taking all residues.

$$I^{n,\text{gluons}} = \int \frac{d^n z}{SL(2,\mathbb{R})} \langle V^0(z_1) \cdots V^{-1}(z_i) \cdots V^{-1}(z_j) \cdots V^0(z_n) \rangle$$

As in bosonic case we try to study the zero in this case.

We change the variable from $z \to u \to F$.

$$\prod_C u_C^{X_C} = \prod_{i<j} z_{ij}^{\alpha' s_{ij}} \prod_{i=1}^{2n} z_{i,i+1}^{2\alpha' P_i^2} \prod_{i=1}^{2n} z_{i,i+2}^{-\alpha' P_i^2}$$

$$\Rightarrow A^{2n,a,b} = \int \frac{d^{2n} z}{SL(2,\mathbb{R})} \prod_{i<j} z_{ij}^{\alpha' s_{ij}} \cdot \frac{1}{z_{a,b}} Pf(A^{a,b})$$

$$= \int \frac{d^{2n} z}{SL(2,\mathbb{R})} \frac{\prod_{i=1}^{2n} z_{i,i+2}^{\frac{1}{2}}}{\prod_{i=1}^{2n} z_{i,i+1}} \prod_C u_C^{\alpha' X_C} \frac{1}{z_{a,b}} Pf(A^{a,b})$$

$$= \int \prod_{d=1}^{2n-3} \frac{dy_{i_d,j_d}}{y_{i_d,j_d}} \prod_C u_C^{\alpha' X_C} \prod_{i=1}^{2n} z_{i,i+2}^{\frac{1}{2}} \cdot \frac{1}{z_{a,b}} Pf(A^{a,b})$$

$$= \int \prod_{d=1}^{2n-3} \frac{dy_{i_d,j_d}}{y_{i_d,j_d}} \prod_C u_C^{\alpha' X_C} \sum_{\pi \text{ partition}} \prod_{p\in\pi} \alpha' s_p \prod_\alpha u_\alpha^{n_\alpha^\pi}$$

where $n_\alpha^\pi \in \{0, \pm\frac{1}{2}, \pm 1\}$.

$$= \int \prod_{d=1}^{2n-3} \frac{dy_{i_d,j_d}}{y_{i_d,j_d}} \prod_{d=1}^{2n-3} y_{i_d,j_d}^{\alpha' X_{i_d,j_d}} \prod_{i<j} F_{i,j}^{-\alpha' C_{i,j}} \cdot \prod_{i=1}^{2n} F_{i,i+1} \prod_{i=1}^{2n} F_{i,i+2}^{-\frac{1}{2}}$$
$$\cdot \left(\sum_{\pi \text{ partition}} (-1)^\pi \prod_{p\in\pi} \alpha' s_p \prod_{d=1}^{2n-3} y_{i_d,j_d}^{n^\pi_{i_d,j_d}} \prod_{i=1}^{2n} F_{i,i+2}^{\frac{1}{2}} \cdot F_p^{-1} \right)$$

$$= \int \prod_{d=1}^{2n-3} \frac{dy_{i_d,j_d}}{y_{i_d,j_d}} \prod_{d=1}^{2n-3} y_{i_d,j_d}^{\alpha' X_{i_d,j_d}} \prod_{i<j} F_{i,j}^{-\alpha' C_{i,j}} \left(\sum_{\pi \text{ partition}} (-1)^\pi \prod_{p\in\pi} \frac{\alpha' s_p}{F_p} y_{i_d,j_d}^{n^\pi_{i_d,j_d}} \right)$$

We take the ray-like triangulation, then we consider when will make $y_{1,a}$'s intergral become scaleless.

When we set $\alpha' C_{ij} \in \mathbb{Z}_{\le 0}$ $\forall (i,j) \in N_a := \{(i,j) \mid 1 \le i \le a-2, a \le j \le n-1\}$

Then we look at each term in the integral, there are two case.

1. For some $p = (w,x) \in \pi$ and p also lies in N_a.

Then we focus on these term, when $C_{w,x} = 0$ then this term vanishes. When $\alpha' C_{w,x} \in \mathbb{Z}_-$ $\Rightarrow \alpha' C_{w,x} = -1 - d_{w,x}$, $d_{w,x} \geq 0$.

$F_{w,x}^{-\alpha' C_{w,x}} \cdot \frac{1}{F_{w,x}} = F_{w,x}^{1+d_{w,x}} \cdot \frac{1}{F_{w,x}} = F^{d_{w,x}}$ is polynomial.

$\Rightarrow$ This term become scaleless integral $\Rightarrow$ Amplitude vanish.

2. $\nexists P \in \pi$ st $P \in N_a$, then it is scaleless integral as bosonic case.

$\Rightarrow$ All terms vanishes.

Next when we want to study the zero in SYM case. We also consider the triangulation which is the same as the Bosonic case.

$$I^{2n} = \int \prod_{a=1}^{n} \frac{dY_{2a-1,2a+1}}{Y_{2a-1,2a+1}} Y^{\alpha' X_{2a-1,2a+1}} \prod_{t=1}^{n-3} \frac{dY_{1,2t+3}}{Y_{1,2t+3}} Y^{\alpha' X_{1,2t+3}} \prod_{kj} F_{i,j}^{-\alpha' C_{i,j}} \left(\sum_{\pi \text{ partition}} (-1)^{\pi} \prod_{p\in\pi} \frac{\alpha' S_p}{F_p} \prod_{a=1}^{n} Y_{2a-1,2a+1}^{n^{\pi}_{2a-1,2a+1}} \prod_{t=1}^{n-3} Y_{1,2t+3}^{n^{\pi}_{1,2t+3}} \right)$$

First, we observe that $n^{\pi}_{i_a,j_a}$ might be 0 or -1

When $p\in\pi$ and $p=(2i-1,2i)$, then when we change varinble from z to u, the $\frac{1}{z_p}=\frac{1}{z_{2i-1,2i}}$ might only come from the following u varinbles. $u_{2i-1,2i}$, $u_{2i,2i}$, $u_{2i-1,2i+1}$, $u_{2i,2i+1}$ but only $u_{2i-1,2i+1}$ exist.

So we know that the $n^{\pi}_{2a-1,2a+1} = \begin{cases} -1 & \text{iff } (2a-1,2a)\in\pi. \\ 0 & \text{else} \end{cases}$

By the discussion in Bosonic case, we know that only in the set $N_t := \{(i,j) \mid 1\le i\le 2t,\ 2t+3\le j\le 2n-2\}$ that $F_{ij}\big|_{Y_{2a-1,2a+1}=0}$ $\forall a\in\{1,2,\dots n\}$ might depend on $Y_{1,2t+3}$.

Let all $c_{ij}=0$ $\forall (i,j)\in N_t$, then we look the term in I^{2n} there are two case.

1, $\exists (i,j)\in N_t$, $(i,j)\in\pi \Rightarrow \because c_{ij}=0=-S_{ij} \Rightarrow$ this term vanish.

2. $\forall (i,j)\in N_t$, $(i,j)\notin\pi$, then in this case we know that the integrand about $F_{i,j}$, $(i,j)\in N_t$ is $\prod_{N_t} F_{i,j}^{-\alpha' C_{i,j}} = 1$ because all $c_{i,j}$ in N_t we set them to zero $\Rightarrow$ Follow the discussion in Bosonic case we get scaleless integral.

$\Rightarrow$ All term vaninshs.