Introduction to marginal triviality of the scaling limits of critical 4D Ising and ϕ^4models

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ABSTRACT: In this article, I want to give some summary of the result about what the professor Hugo Duminil-Copin who was awarded fields medal in 2022 had done. The problem is about $\lambda\phi^4$ fields over R^4 with a lattice ultraviolet cutoff, in the limit of infinite volume and vanishing lattice spacing which is Gaussian. We use Griffith-Simon class with some scaling limit we can find the relation between ϕ^4 field model and Ising model. However, we have many useful tools in Ising model. For example, random current representation enables us to calculate the correlation function and give some inequality of correlation function. With these useful inequalities, we can get the result on Ising model, then by the relation between Ising model and ϕ^4 field model, we can prove the problem. In particular, the main contribution of Hugo Duminil-Copin is the improvement of the so-called tree diagram bound by a logarithmic correction term which solves the problem in the marginal 4D case.

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1 Introduction

Quantum field theories with local interaction play an important role in the physics, so how to properly formulate this concept in math led to the programs of Constructive Quantum Field Theory. In this article, By the Osterwalder-Schrader theorem, correlation functions with the required properties may potentially be obtained through analytic continuation from those of random distributions defined over the corresponding Euclidean space that meet a number of conditions: suitable analyticity, permutation symmetry, Euclidean covariance, and reflection-positivity. Therefore, we want to construct probability averages over random distribution Φ(x), for which the expectation value of functionals F(Φ(x)) should be the following form

$$\langle F(\Phi(x))\rangle \approx \int F(\Phi)exp[-H(\Phi)]\prod_{x\in\mathbb{R}^d} d\Phi(x) \tag{1.1}$$

where $H(\Phi)$is Hamiltonian with the expressions of the form

$$H(\Phi) \approx (\Phi, A\Phi) + \int_{\mathbb{R}^d} P(\Phi(x))dx \tag{1.2}$$

with P is a polynomial with even order terms, and $(\Phi, A\Phi)$ a positive definite and reflection-positive quadratic form. For example,

$$(\Phi, A\Phi) = \int_{\mathbb{R}^d} (K|\nabla\Phi|^2(x) + b|\Phi(x)|^2)dx \tag{1.3}$$

with K,b>0. The functional $F(\Phi)$ is inteneded to apply include the smeared averages

$$T_f(\Phi) = \int_{\mathbb{R}^d} f(x)\Phi(x)dx \tag{1.4}$$

with a continuous functions of compact support $f \in C_0(\mathbb{R}^d)$. We should notice that $T_f(\Phi)$ and $\Phi(x)$ are random variables, but Φ is a random field. Then the expectation value of products of $T_{f_j}(\Phi)$ take the form.

$$\langle\prod_{j=1}^{n} T_{f_j}(\Phi)\rangle = \int_{(\mathbb{R}^d)^n} dx_1...dx_n S_n(x_1,...,x_n)\prod_{j=1}^{n} f(x_j) \tag{1.5}$$

with $S_n(x_1,...,x_n)$characterizing the probability measure on the space of distribution which corresponds to the expectation value.

$$\langle\prod_{j=1}^{n}\Phi(x_j)\rangle = S_n(x_1,...,x_n) \tag{1.6}$$

However, when our Euclidean field are Gaussian fields, by Wick's law any 2n-point correlation function can be determined by two-point function. In the physics aspect, we may interpret that the absence of interaction in this case, so we call such field trivial.

Definition 1.1 Gaussian random field
If Φ is called Gaussian field, then it satisfies the following condition. For any n>0 points $x_1, x_2...x_n \in \mathbb{R}^d$ and $y_1, y_2...y_n \in \mathbb{R}$, the joint probability with $\Phi(x_i) = y_i \forall i in 1, 2, ..., n$ may be the following equation.

$$P(y_1,...y_n)dy_1...dy_n = \frac{1}{(2\pi)(detM)^(1/2)} exp(-\frac{1}{2}(y_1...y_n)M^{-1}((y_1...y_n)^T))dy_1...dy_n \tag{1.7}$$

In particular, M is the correlation matrix.

In order to construct the non-trivial field theory, we add the Φ^4 term in the Hamitonian. However,when interpreting (1.1), one quickly encounters a number of problems. Even in the generally understood case of the Gaussian free field, with H consisting of just the quadratic term (1.3), Equation (1.1) is not to be taken literally as the measure is supported by non-differentiable functions for which the integral in the exponential is almost surely divergent. So a natural starting point toward the construction of Φ^4 functional integral in (1.1) is to consider the lattice version of such random field, which we restrict theΦ to the finite graph with vertex set

$$V_{a,R} = (a\mathbb{Z})^d \cap \Lambda_R, \Lambda_R := [-R, R]^d \tag{1.8}$$

Hamiltonian is interpreted in term of the Riemann-sum style discrete analog of integral expression with $\Phi(x)$ for any $x \in V_{a,R}$.

In particular, the fourth power addition takes the form.

$$P(\Phi(x)) = \lambda\Phi^4 - c(\lambda, a, R)\Phi^2, \tag{1.9}$$

Then we remove the cutoffs through the limit R$\nearrow \infty$ and $a \searrow 0$ and $c(\lambda, a, R)$ is allowed to stabilize the Schwinger functions $S_n(x_1, .., x_n)$ on the continuum limit scale.

However,the progression of constructive results was halted because in 1981 professor Michael Aizenman proved that for d>4 the attempt to construct Φ^4 with the condition

$$\lim_{|x-y|\to\infty} S_2(x,y) = 0 \tag{1.10}$$

only yields Gaussian fields.

2 Statement of the main result

The probability measures correspond to (1)with lattice and finite volume cutoffs take the form of statistical-mechanics Gibbs equilibrium state average

$$\langle F(\phi(x))\rangle = \frac{1}{norm}\int F(\phi)exp[-H(\phi)]\prod_{x\in\Lambda_R}\rho(d\phi_x) \tag{2.1}$$

with $H(\Phi)$ and $\rho(\phi_x)$ has the following form.

$$H(\phi) = -\sum_{(x,y)\in E_R} J_{x,y}\phi_x\phi_y, \rho(d\phi_x) = e^{\lambda{\phi_x}^4+b{\phi_x}^2}d\phi_x, \tag{2.2}$$

where $d\phi_x$ is just Lebesgue measure on $\mathbb{R}$ and $J_{x,y} = 0$ when $(x,y)\notin E_R$(Edge of graph) and J>0 otherwise. In the other word, the interaction only happen for the nearest neighbour vertices. The reason why we set $J_{x,y}$ as above is we want to describe the gradient term of Hamiltonian in(1.3). The quantities in the lattice version may be the following form.

$$T_{f,L}(\phi) := \frac{1}{\sqrt{\Sigma_L}}\sum_{x\in\mathbb{Z}^d} f(x/L)\phi_x \tag{2.3}$$

where f can be any compactly supported continuous function, and Σ_L denotes the variance of sum of spins over the box of size L. $\Sigma_L := \langle(\sum_{x\in\Lambda_L}\phi_x)^2\rangle$.

By the above probabilistic construction, the limit can be presented as a random field, and the $T_f(\Phi)$ would be the limit of the $T_{f,L}(\phi)$.

Theorem 2.1(Gaussianity of Φ^4)
For dimension d=4, any random field constructing as above method and satisfies that $\lim_{|x-y|\to\infty}S_2(x,y) = 0$, is a Gaussian field.

3 The main result in the statistical mechanics perspective

In the statistical mechanics, Ising model be the most studied model, so we want to using the method of Ising model to fix the problem we meet on the case of quantum field theory or even we find the similar statement of our main theorem 2 on the general Ising model. In fact, we can find the relation between two model. Before finding the relation between two perspective, we give some basic notations of Ising spin model. Ising spin model on $\Lambda\subset\mathbb{Z}^d$

is consisted by the variables $\{\sigma_x\}_{x\in\Lambda}$ which valued in ±1, and Hamiltonian would have the following form

$$H_{\Lambda,J,h}(\sigma) := -\sum_{\{x,y\}\in\Lambda} J_{x,y}\sigma_x\sigma_y - \sum_{x\in\Lambda} h\sigma_x. \tag{3.1}$$

The model's finite volume Gibbs equilibrium state $\langle . \rangle, \Lambda, J, h, \beta$ at inverse temperature $\beta \geq 0$ is the probability measure under which the expectation value of any function $F : \{\pm1\}^\Lambda \to \mathbb{R}$ is given by

$$\langle F\rangle_{\Lambda,J,h,\beta} := \frac{1}{Z(\Lambda,J,h,\beta)} \sum_{\sigma\in\{\pm1\}^\Lambda} F(\sigma)exp[-\beta H_{\Lambda,J,h}(\sigma)] \tag{3.2}$$

where $Z(\Lambda, J, h, \beta)$ is the model's partition function. Then we let $\Lambda \nearrow \mathbb{Z}^d$, we get the infinite volume measure of Gibbs states $\langle . \rangle_{J,h,\beta}$.

In order to match the case in field thory, we focus on the nearest neighbor ferromagnetic interaction (n.n.f.)

$$J_{x,y} = \begin{cases} J & \|x-y\| = 1 \\ 0 & \text{otherwise} \end{cases} \tag{3.3}$$

with J >0. From some results of Ising model, we know that the two point correlation function decay exponentially away from the critical point, so we can define the correlation length $\epsilon(\beta)$ as following:

$$J_{x,y} = \epsilon(\beta) := \lim_{n\to\infty} -n/log\langle\sigma_0;\sigma_{ne_1}\rangle_\beta(withe_1 = (1,0,...,0)) \tag{3.4}$$

In particular, the correlation length is $+\infty$.

With the above setting, now we are ready to show that the similarity between the Ising model's Gibbs equilibrium distribution (3.2) and the discretized functional integral(2.1). We can see that when we set $\lambda \to \infty$ and $b = 2\lambda$ in the case of field theory, then the probability measure becomes

$$\lim_{\lambda\to\infty} e^{-\lambda(\phi^2-1)^2} d\phi/Norm(\lambda) \tag{3.5}$$

We can notice that (3.5) when $\lambda \to \infty$, $e^{-\lambda(\phi^2-1)^2}$ would be the delta functions as the following equation.

$$\lim_{\lambda\to\infty} e^{-\lambda(\phi^2-1)^2} d\phi/Norm(\lambda) \sim \delta(\phi+1) + \delta(\phi-1) \tag{3.6}$$

But the probability measure in Ising case is $\frac{1}{2}[\delta(\sigma-1)+\delta(\sigma+1)]d\sigma$.

Therefore, Ising spin's distribution can be viewed as the "hard" limit of the ϕ^4 measure. The theorem (2.1) imply that for $d = 4$ any scaling limit of critical Ising model is Gaussian. Unfortunately, the case which is easier to us is Ising model not ϕ^4 model.
In order to using the results of Ising model to solve the problem, we consider the following

process.

First, we need to find relation of probability measure from Ising model to ϕ^4 model, then we can pass the problem from ϕ^4 model to the case on Ising model. We use the Griffiths-Simon (G-S) class which is a generalization of Ising model and the measures are like the Φ^4 measure $\phi(d\phi)$ in (2.2).

Second, we try to prove the statement in the easy case of n.n.f. Ising model (in four dimensions).

Third, we pass our analysis to the model's extension, in which each spin is replaced by block average of elemental Ising spins with an block ferromagnetic coupling. However, why we consider this because that the G-S class is the weak limit of the block case. Then if the block case is true, then take the limit we get the result on G-S class, and then by the first step we prove the Theorem (2.1).

4 The sketch of the proof

From the theorem which proved by the Charles M. Newman in the paper *Inequalities for Ising Models and Field Theories which Obey the Lee-Yang Theorem.*, it gives a easy way to determine whether the random field is Gaussian.

Proposition 4.1
If $U_{2m} = 0$ for any $m = 1, 2, ...$,then $U_n = 0$ for all n $>$ 2 and X is Gaussian. Where U_n is the Ursell function.

With the above proposition, then the goal become finding a degree of Ursell function and prove that such Ursell function equal zero as we taking limit. So we consider the degree 4 Ursell function as following.

$$U_4^\beta(x, y, z, t) := \langle\sigma_x\sigma_y\sigma_z\sigma_t\rangle_\beta - [\langle\sigma_x\sigma_y\rangle_\beta\langle\sigma_z\sigma_t\rangle_\beta + \langle\sigma_x\sigma_z\rangle_\beta\langle\sigma_x\sigma_t\rangle_\beta + \langle\sigma_x\sigma_t\rangle_\beta\langle\sigma_z\sigma_y\rangle_\beta] \quad (4.1)$$

The relevant question is whether $U_4^\beta(x, y, z, t)/\langle\sigma_x\sigma_y\sigma_z\sigma_t\rangle_\beta$ vanishes asymptotically for quadruples of sites at large distances, of comparable order between the pairs.
However, Gaussianity of the scaling limits for the case d $>$ 4 was previously established through the combination of the tree diagram bound

$$|U_4^\beta(x, y, z, t)| \leq 2 \sum_{u\in\mathbb{Z}^d} \langle\sigma_x\sigma_u\rangle_\beta\langle\sigma_y\sigma_u\rangle_\beta\langle\sigma_z\sigma_u\rangle_\beta\langle\sigma_t\sigma_u\rangle_\beta \quad (4.2)$$

and the Infrared Bound

$$\langle\sigma_x\sigma_y\rangle_{\beta_c} \leq \frac{C}{|x-y|^{d-2}} \quad (4.3)$$

At the heuristic level, the triviality of the scaling limit for $d > 4$ can be easily found by the following dimension counting. We consider the$U_4^\beta(x, y, z, t)/\langle\sigma_x\sigma_y\sigma_z\sigma_t\rangle_\beta$. Assume that at β_c the two-point function is of comparable values for pairs of sites at similar distances (which is false for $\beta \neq \beta_c$ at distances much larger than $\epsilon(\beta)$). Then, for the such four points at mutual distances of order L. the sum in the tree diagram bound (4.2) contributes a factor L^d while the summand has two extra correlation function factorseach factor dominated by $1/L^{d-2}(d-2(d-2)=4-d)$.Therefore the $U_4^\beta(x,y,z,t)/\langle\sigma_x\sigma_y\sigma_z\sigma_t\rangle_\beta$ is the order $O(L^{4-d})$, which for $d>4$ vanishes in the limit $L\to\infty$. Then we prove the theorem 2 in d>4. In particular, when d equals to 4, we can't get any effective upper bound.

The key point in this article is to improve the tree diagram bound, and with the such upper bound enable us to fix the problem we met in the case d=4.

Theorem 4.2(Improved tree diagram bound inequality)
For the n.n.f. Ising model in dimension $d=4$, there exist c,C>0 s.t. for every $\beta \leq \beta_c$, every $L \leq \epsilon(\beta)$ and every x,y,z,t$\in \mathbb{Z}^d$ at a distance larger than L of each other. then

$$|U_4^\beta(x,y,z,t)| \leq \frac{C}{B_L(\beta)^c}\sum_{u\in\mathbb{Z}^4}\langle\sigma_x\sigma_u\rangle_\beta\langle\sigma_y\sigma_u\rangle_\beta\langle\sigma_z\sigma_u\rangle_\beta\langle\sigma_t\sigma_u\rangle_\beta \tag{4.4}$$

where $B_L(\beta)^c$ is a bubble diagram truncated at L define as the following form

$$B_L(\beta) := \sum_{x\in\Lambda_L}\langle\sigma_0\sigma_x\rangle_\beta{}^2 \tag{4.5}$$

With this improved tree diagram bound inequality, we can follow the above argument to get the our main theorem (2.1). For convenience, we consider the following two case when $d=4$, one case is our two-points correlation function $\langle\sigma_0\sigma_x\rangle_\beta$ decay roughly of order L^{2-d}, the other is decay faster in order $L^{2-d-\eta}$ where $\eta>0$.
In the first case, $B_L(\beta)$ is order of $log(L)$, then $|U_4|/S_4 = O(logL)^{-c}$ which is zero as $L\to\infty$.
In the second case, which is just follow the argument in the case $d>4$.
Then now we had already proved the Theorem (2.1). Furthermore, instead of focusing on four-points function, the full statement of the scaling limit's gaussianity can be established through the following proposition.

Proposition 4.3
There exist $c, C>0$such that for the n.n.f. Ising model on $\mathbb{Z}^4$, every $\beta\leq\beta_c$, $L\leq\epsilon(\beta)$,$f\in C_0(\mathbb{R}^4)$, then

$$|\langle exp[zT_{f,L}(\sigma) - \frac{z^2}{2}\langle T_{f,L}(\sigma)^2\rangle_\beta]\rangle_\beta - 1| \leq \frac{C\|f\|_\infty^4 r_f^{12}}{(logL)^c}z^4 \tag{4.6}$$

where r_f the diameter of the support.

With the above proposition 4.3, we get that for $L\gg 1$ the distribution of $T_{f,L}(\sigma)$ is approximately Gaussian of variance $\langle T_{f,L}(\sigma)^2\rangle_\beta$.

5 The some skills in the proof

5.1 The Griffiths-Simon class of measures

The Griffiths-Simon class which is very important, which permits us to apply tools which is initially developed for general Ising model to study ϕ^4 functional intergral.So we gives the definition of Griffiths-Simon class as following.

Definition 5.1.1

A probability measure on $\rho(d\varphi)$ on $\mathbb{R}$ is belong to Griffiths-Simon class, if it satisfies one of the following condition

1) the expectation value with respect to ρ can be presented as

$$\int F(\varphi)\rho(d\varphi) = \sum_{\underline{\sigma}\in\{\{-1,+1\}} F(\alpha\sum_{n=1}^{N} b_n\sigma_n)e^{\sum_{n,m=1}^{N} K_{n,m}\sigma_n\sigma_m}/Norm \tag{5.1.1}$$

with some$\{b_n\} \in \mathbb{R}$ and $K_{n,m} > 0$.

2)ρ can be presented as a weak limit of probability measure as the above type,and is of sub-gaussian growth:

$$\int e^{|\varphi|^{\alpha}}\rho(d\varphi) < \infty, \alpha > 2 \tag{5.1.2}$$

In the definition of Griffiths-Simon class, we can see that the second condition correspond to when we taking scaling limit.

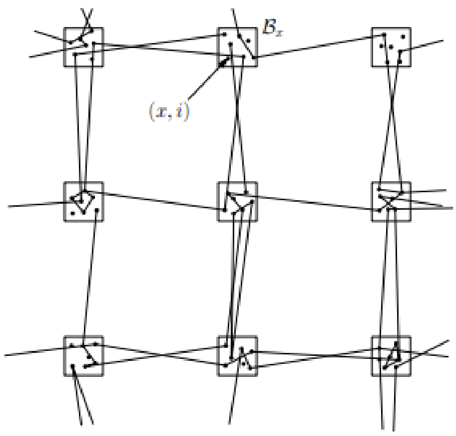

Figure 1: The decorated graph, in which the sites $x \in \Lambda$ of a graph of interest are replaced by "blocks" $\mathscr{B}_x$ of sites indexed as (x,n). The Ising "constituent spins" $\sigma_{x,n}$ are coupled pairwise through intra-block couplings $\delta_{x,y}K_{n,m}$ and inter-block couplings $J_{x,y}$. The depicted lines indicate a possible realization of the corresponding random current.

As the above picture, GS class is like we replace each site to a block of many elementary Ising spins. As we have mentioned, Simon and Griffiths had proved that taking weak limits of the GS class would be like the measure in ϕ^4 case(2.2). More percisely, the each variables $\{\phi_x\}_{x\in\Lambda}$ in ϕ^4 with the measure $\rho(d\phi) = e^{\lambda\phi_x{}^4+b\phi_x{}^2}d\phi/norm$ can be produced as

$N \to \infty$ limit of the collection of the block averages of elemental Ising spins$\{\sigma_{x,n}\}$, then the variable represented the block x has the following form.

$$\int e^{|\phi|^\alpha} \rho(d\phi) < \infty, \alpha > 2 \tag{5.1.3}$$

adding the intersite interaction H

$$H_{inner} = -\frac{g_N(\lambda, b)}{N} \sum_{x\in\Lambda} \sum_{n,m} \sigma_{x,n}\sigma_{x,m} \tag{5.1.4}$$

with suitable pair (α_N, g_N). (Notice that we use ϕ to represent the case of ϕ^4 and φ to represent case of the GS class, and by the above construction φ would have the same measure as $N \to \infty$)

Remark:With the discussion in this GS class section, we can construct the relation between ϕ^4 field theory in Euclidean space and the generalized Ising model.

5.2 Random current representation

In order to calculate the correlation of Ising model, we can use the tool *Random current representation* which express the correlation of Ising model in more geometric terms. In particular, the utility of the random current representation is the combinatorial symmetry expressed in its switching lemma, which enables to structure some of the essential truncated correlations in terms guided by the analysis of the intersection properties of the traces of random walks.

Definition 5.2.1

A current configuration $\mathbf{n}$ on Λ is an integer-valued function defined over unordered pairs $(x, y) \in \Lambda$. The current's set of source is defined as

$$\partial\mathbf{n} := \{x \in \Lambda : (-1)^{\sum_{y\in\Lambda} \mathbf{n}(x,y)} = -1\} \tag{5.2.1}$$

The weight of the current configuration $\mathbf{n}$ on Λ is defined below.

$$w(\mathbf{n}) = w_{\Lambda,J,\beta}(\mathbf{n}) := \prod_{(x,y)\subset V} \frac{(\beta J_{x,y})^{\mathbf{n}(x,y)}}{\mathbf{n}(x, y)!} \tag{5.2.2}$$

Lemma 5.2.2

For any $\beta \geq 0$ the Ising model's partition function (at h=0) can be expresses as the following sum over sourceless rand currents.

$$Z(\Lambda, \beta) := \sum_{\sigma:\{\pm 1\}^\Lambda \to \{-1,1\}} \prod_{(x,y)\in V} exp(\beta J_{x,y}\sigma_x\sigma_y) = 2^{|\Lambda|} \sum_{\partial\mathbf{n}=\emptyset} w(\mathbf{n}) \tag{5.2.3}$$

Furthermore, the Gibbs state expectation value of even products of spins can be presented as the effect on the sum of the insertion of sources at the corresponding sites:

$$\langle \prod_{x\in A} \sigma_x \rangle_{\Lambda,\beta} = \frac{\sum_{\mathbf{n}:\partial\mathbf{n}=A} w(\mathbf{n})}{\sum_{\mathbf{n}:\partial\mathbf{n}=\emptyset} w(\mathbf{n})} \tag{5.2.4}$$

(pf) From Taylor's expansion,

$$exp(\beta J_{x,y}\sigma_x\sigma_y) = \sum_{\mathbf{n}(x,y)\geq 0} \frac{(\beta J_{x,y}\sigma_x\sigma_y)^{\mathbf{n}(x,y)}}{\mathbf{n}(x,y)!} \tag{5.2.5}$$

For convenience,we also define the following quantity.

$$Z_{\Lambda,\beta}(\sigma_A) = \sum_{\sigma\in\{\pm1\}^\Lambda} \sigma_A \sum_{\sigma:\{\pm1\}^\Lambda\to\{-1,1\}} \prod_{(x,y)\in V} exp(\beta J_{x,y}\sigma_x\sigma_y) \tag{5.2.6}$$

$$Z_{\Lambda,\beta}(\sigma_A) = \sum_{\mathbf{n}} w(\mathbf{n}) \sum_{\sigma:\{\pm1\}^\Lambda\to\{-1,1\}} \prod_{(x,y)\in V} \sigma_x^{1[x\in A]+X(\mathbf{n},x)} \tag{5.2.7}$$

where $X(\mathbf{n},x) := \sum_{y\in\Lambda}\mathbf{n}_{xy}$.
The trick is to fix $x\in\Lambda$ and $\sigma\in\{\pm1\}^\Lambda$. And define the configuration $\sigma^{(x)}$ obtained from σ by reversing the spin at x. Since for a fixed $x\in\Lambda$, the map $\sigma\to\sigma^{(x)}$ is an involution, and since the contributions of $\sigma^(x)$ and σ to sum over spin configurations in (5.2.7) cancel each other when $1[x\in A]+X(\mathbf{n},x)$ is odd, so we find that

$$\sum_{\sigma:\{\pm1\}^\Lambda\to\{-1,1\}} \prod_{(x,y)\in V} \sigma_x^{1[x\in A]+X(\mathbf{n},x)} = \begin{cases} 2^{|\Lambda|} & \partial\mathbf{n}=A \\ 0 & \text{otherwise} \end{cases} \tag{5.2.8}$$

In conclusion,

$$Z_{\Lambda,\beta} = 2^{|\Lambda|}\sum_{\partial\mathbf{n}=A} w(\mathbf{n}) \tag{5.2.9}$$

In particular, when we set $A=\emptyset$, we get(5.2.3). Since $\langle\sigma_A\rangle = Z_{\Lambda,\beta}(\sigma_A)/Z(\Lambda,\beta)$, then we have done the proof of the lemma(5.2.2).

Remark: For the sourceless ($\partial\mathbf{n}=\emptyset$) configuration, can be viewed as the edge count of a multigraph which is decomposable into a union of loops. When configuration with ($\partial\mathbf{n}=A$), can be view as describing the edge count of a multigraph which is a decomposable into a collection of loops and of paths connecting pairwise the sources.

Definition 5.2.3
1) We say that x is connected to y (in $\mathbf{n}$) , $x \overset{\mathbf{n}}{\leftrightarrow} y$, if there exists a path of vertices $x=u_0,u_1,...u_k=y$ with $\mathbf{n}(u_i,u_{i+1})>0$ for every $0\leq i<k$
2) The cluster of x, denoted by $\mathbf{C_n}(x)$, is the set of vertices which is connected to x in $\mathbf{n}$.
3) For a set of vertices B, we denote by $\mathcal{F}_B$ the set of $\mathbf{n}$ satisifying that there exists a sub-current $\mathbf{m}<\mathbf{n}$ such that $\partial m=B$

Lemma 5.2.4(Switching lemma)
For any $A,B\subset\Lambda$ and any function F from the set of currents into $\mathbb{R}$,

$$\sum_{\substack{\partial\mathbf{n}_1=A\\ \partial\mathbf{n}_2=B}} F(\mathbf{n}_1+\mathbf{n}_2)w(\mathbf{n}_1)w(\mathbf{n}_2) = \sum_{\substack{\partial\mathbf{n}_1=A\triangle B\\ \partial\mathbf{n}_2=\emptyset}} F(\mathbf{n}_1+\mathbf{n}_2)w(\mathbf{n}_1)w(\mathbf{n}_2)\mathbf{1}_{\mathbf{n}_1+\mathbf{n}_2\in\mathcal{F}_B} \tag{5.2.10}$$

where $A \triangle B$ denotes the symmetric difference of sets, $A \triangle B := (A \setminus B) \cup (B \setminus A)$

With the switching lemma, we can start to calculate the correlation function, then we list some of results. The first of these is the relation.

$$\frac{\langle \sigma_A \rangle_{\Lambda,\beta} \langle \sigma_B \rangle_{\Lambda,\beta}}{\langle \sigma_A \sigma_B \rangle_{\Lambda,\beta}} \tag{5.2.11}$$

for which we denote by $\mathbf{P}^A_{\Lambda,\beta}(\mathbf{n})$ the probability distribution on random currents when the source $\partial \mathbf{n} = A$, or more explicitly

$$\mathbf{P}^A_{\Lambda,\beta}(\mathbf{n}) := \frac{2^{|\Lambda|} w(\mathbf{n})}{\langle \Pi_{x \in A} \sigma_x \rangle_{\Lambda.\beta} Z(\Lambda, \beta)} 1[\partial \mathbf{n} = A], \tag{5.2.12}$$

And define the the probability on the independent family of currents $(\mathbf{n}_1, ..., \mathbf{n}_i)$

$$\mathbf{P}^{A_1,...,A_i}_{\Lambda,\beta} := \mathbf{P}^{A_1}_{\Lambda,\beta} \otimes ... \otimes \mathbf{P}^{A_i}_{\Lambda,\beta} \tag{5.2.13}$$

Then we can take scaling limit $\Lambda \to \infty$, so we have the similar statement as following.

$$\frac{\langle \sigma_A \rangle_\beta \langle \sigma_B \rangle_\beta}{\langle \sigma_A \sigma_B \rangle_\beta} = \mathbf{P}^{A \triangle B, \emptyset}_\beta [\mathbf{n}_1 + \mathbf{n}_2 \in \mathcal{F}_B]. \tag{5.2.14}$$

Using (5.2.14), the Ursell function can be expressed as following.

$$U_4^\beta(x,y,z,t) = -2\langle \sigma_x \sigma_y \rangle_\beta \langle \sigma_z \sigma_t \rangle_\beta \mathbf{P}^{xy,zt}_\beta [\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x) \cap \mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(z) \neq \emptyset] \tag{5.2.15}$$

By (5.2.15) and the discrepancy in Wick's formula, we can get the following inequality.

$$\frac{|U_4^\beta(x,y,z,t)|}{\langle \sigma_x \sigma_y \sigma_z \sigma_t \rangle_{beta}} \leq \mathbf{P}^{xy,zt}_\beta [\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x) \cap \mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(z) \neq \emptyset] \tag{5.2.16}$$

Next, by the some monotonicity property of random currents which was proved by Aizenman in 1982, we can get

$$\mathbf{P}^{xy,zt}_\beta [\mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(x) \cap \mathbf{C}_{\mathbf{n_1}+\mathbf{n_2}}(z) \neq \emptyset] \leq \mathbf{P}^{xy,zt,\emptyset,\emptyset}_\beta [\mathbf{C}_{\mathbf{n_1}+\mathbf{n_3}}(x) \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_4}}(z) \neq \emptyset] \tag{5.2.17}$$

With the above two equation(5.2.16)(5.2.17) and switching lemma(5.2.4), we can give a simpler upper bound of Ursell function.

$$|U_4^\beta(x,y,z,t)| \leq 2\langle \sigma_x \sigma_y \rangle_\beta \langle \sigma_z \sigma_t \rangle_\beta \mathbf{P}^{xy,zt,\emptyset,\emptyset}_\beta [\mathbf{C}_{\mathbf{n_1}+\mathbf{n_3}}(x) \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_4}}(z) \neq \emptyset] \tag{5.2.18}$$

Therefore, we see that if we find the bound of the intersection probability by the expected number of intersection sites and applying the switching lemma leads directly to the tree diagram bound(4.2).

Some remarks

In (5.2.18), we need to calculate the intersection probability $\mathbf{P}^{xy,zt,\emptyset,\emptyset}_\beta [\mathbf{C}_{\mathbf{n_1}+\mathbf{n_3}}(x) \cap \mathbf{C}_{\mathbf{n_2}+\mathbf{n_4}}(z) \neq \emptyset]$, the analogy of estimate the intersection probability in the random walks give us some

help. In particular,in dimension d=4 the probability that two traces of two random walks starting at distance L of each other intersect, tends to 0(as 1/logL) but the number of intersection points is $\Omega(1)$.
Then we can know that although the intersections occur rarely, but the conditional expectation of the number of intersection sites, conditioned on there being at least one, diverge logarithmically in L. Then we try to make a similar statement that the conditional expectation of cluster size with the condition that the intersection set is non-empty, grows at least $(logL)^c$.
However, there are a lot of works in the analyzing the cluster's intersection property in the random current representation, which are very difficult and complicated, so we don't get further in this article.

6 Conclusion

In this article, we learned that there exist the relation to connect two different models, Ising model and Φ^4 model which looks very different at the first glance. The such relation is structured by the GS class with some skill of first restrict our model to the lattice and taking scaling limit.
With the GS class, we pass our problem to the generalized Ising model, and then we use the random current representation which enable as to estimate the correlation function of Ising model by calculating the intersection property of random current representation. And before we tackle on the intersection property on random current representation, we use the some results of the same property on random walk model to guess or imagine the intersection property on random current representation. Then using a lot of skills which I hadn't mention in this article to prove the improved tree bound inequality(Theorem 4.2) which is the key to prove the origin problem (Theorem 2.1).

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