GR report

1. 我們要求 spherical symmetric solution.

所以我們知道我們的 metric 的形式如下

$$d\tau^2 = -C(r,t)\,dt^2 + D(r,t)\,dr^2 + E(r,t)\,dr\,dt + F(r,t)\,r^2 d\theta^2$$

(∵ 若有 $dr\,d\theta$ or $dt\,d\theta$ 的 term we can make $\theta' = -\theta$ 的變數代換得到 $dr\,d\theta$、$dt\,d\theta$ 的 term 只能 vanishing)

首先我們作 $r' = \sqrt{F}\,r$ 的變換

$$d\tau^2 = -C'(r',t)\,dt^2 + D'(r',t)\,dr'^2 + E'(r',t)\,dr'\,dt + r'^2 d\theta^2$$

為了消掉 E', $dr'dt$ 的 term, we need to define 新的 t'

$$dt' = \eta(r',t)\left[C'(r',t)\,dt - E'(r',t)\,dr'\right].$$

where $\eta(r',t)$ 是積分因子 讓我們真的能找到 t' or 讓

$\eta(r',t)\left[C'(r',t)\,dt - E'(r',t)\,dr'\right]$ 是 exact form.

$$\left(\frac{\partial}{\partial r'}\left[\eta(r',t)\,C'(r',t)\right] = -\frac{\partial}{\partial t}\left[\eta(r',t)\,E'(r',t)\right] \right)$$

所以 $d\tau^2 = -\eta^{-2} C^{-1}\,dt'^2 + (D + C^{-1}E^2)\,dr'^2 - r'^2 d\theta^2$

整理後我們知道在經過一定的變數代換後

$$d\tau^2 = -B\,dt_{new}^2 + A\,dr_{new}^2 - r_{new}^2\,d\theta^2$$

(注意 A、B depend on t and r)

而上述是最廣的結果，我們可以考慮 stationary 的 case

則是我們的 metric 的係數只 depend on r 的 case

$$d\tau^2 = -F(r)dt^2 + 2rE(r)drdt + r^2D(r)dr^2 + C(r)(dr^2 + r^2d\theta^2)$$

define $t' = t + \Phi(r)$ 其中 $\Phi(r)$ satisfy $\frac{d\Phi}{dr} = -\frac{rE(r)}{F(r)}$

則可以消到 $dr\,dt$ term.

$$\Rightarrow d\tau^2 = -F(r)dt'^2 + \cancel{2F(r)\frac{d\Phi}{dr}drdt} + \left(\frac{d\Phi}{dr}\right)^2 F(r)dr^2 + \cancel{2rE(r)drdt} + r^2D(r)dr^2$$

$$+ C(r)(dr^2 + r^2d\theta^2)$$

$$= -F(r)dt'^2 + r^2\left(D + \frac{E^2}{F^2}\right)dr^2 + C(r)(dr^2 + r^2d\theta^2)$$

We denote $r^2\left(D + \frac{E^2}{F^2}\right) = G(r)$

we define $r' = \exp\int\left[1 + \frac{G(r)}{C(r)}\right]^{\frac{1}{2}}\frac{dr}{r}$

$$\Rightarrow dr' = r'\left(\frac{1}{r}\left[1 + \frac{G(r)}{C(r)}\right]^{\frac{1}{2}}\right)dr \Rightarrow dr'^2 = r'^2\left(\frac{1}{r^2}\left(1 + \frac{G}{C}\right)\right)dr^2$$

$$\Rightarrow d\tau^2 = -F'(r')dt'^2 + \frac{C(r)r^2}{r'^2}(dr'^2 + r'^2d\theta^2)$$

$$= -F'(r')dt'^2 + H(r')(dr'^2 + r'^2d\theta^2)$$

所以我們的 metric 可以寫成以上形式.

接下來我們來 Reproduce 文章的結果!!!(for static case)

②

$$g_{ij} = \left(\begin{array}{c|cc} -N^2(r) & 0 & 0 \\ \hline 0 & \gamma_{ij}(\vec{r}) & \\ 0 & & \end{array}\right) \qquad \sqrt{-g} = N\sqrt{\gamma}$$

γ = determinants γ_{ij}

g = determinats $g_{\mu\nu}$.

Let $S := S_{E\text{-}H} = \int d^2x \sqrt{\gamma} NR$ 　R是只有考慮 spatial 2维度时的 scalar curvature.

$$\delta S = \int dx^2 [(\delta N)\sqrt{\gamma} R + N(\delta\sqrt{\gamma})R + \underline{N\sqrt{\gamma}\,\delta R}]$$

$$= \int dx^2 \left\{\left[\delta\sqrt{-g_{00}}\sqrt{\gamma} + N\frac{\sqrt{\gamma}}{2}\gamma^{ij}(\delta\gamma_{ij})\right]R + \underline{N\sqrt{\gamma}(\delta g^{ij})R_{ij} + N\sqrt{\gamma}g^{ij}(\delta R_{ij})}\right.$$

$$= \int dx^2 \left(\sqrt{\gamma}\left[\frac{N}{2}g^{ij}(\delta g_{ij}) + \left(-\frac{\delta g_{00}}{2N}\right)\right]R + N\sqrt{\gamma}\,\delta R\right)$$

$$N = \sqrt{-g_{00}} \Rightarrow \delta N = -\frac{\delta g_{00}}{2N} \qquad \gamma_{ij} = g_{ij} \qquad \delta\sqrt{\gamma} = \frac{1}{2}\sqrt{\gamma}\gamma^{ij}(\delta\gamma_{ij})$$

$$\delta g_{ij} = -g_{jk}g_{jl}(\delta g^{kl})$$

$$\delta g_{00} = \delta(-N^2) = -2N\delta N$$

$$\delta g^{00} = \delta\left(-\frac{1}{N^2}\right) = \frac{2}{N^3}(\delta N)$$

$$\delta g_{00} = -N^4(\delta g^{00})$$

$$\delta R_{ij} = \delta(R^{\rho}{}_{i\rho j}) = \nabla_\rho(\delta\Gamma^{\rho}_{ji}) - \nabla_j(\delta\Gamma^{\rho}_{\rho i})$$

$$\delta\Gamma^{\lambda}_{\mu\nu} = \delta g^{\lambda\rho}g_{\rho\alpha}\Gamma^{\alpha}_{\mu\nu} + \frac{1}{2}g^{\lambda\rho}(\partial_\mu\delta g_{\nu\rho} + \partial_\nu\delta g_{\mu\rho} - \partial_\rho\delta g_{\mu\nu})$$

$$= \frac{1}{2}g^{\lambda\rho}(\nabla_\mu\delta g_{\nu\rho} + \nabla_\nu\delta g_{\mu\rho} - \nabla_\rho\delta g_{\mu\nu})$$

$$= -\frac{1}{2}\left(g_{\nu\alpha}\nabla_\mu\delta g^{\alpha\lambda} + g_{\mu\alpha}\nabla_\nu\delta g^{\alpha\lambda} - g_{\mu\alpha}g_{\nu\beta}\nabla^\lambda\delta g^{\alpha\beta}\right)$$

$$g^{\mu\nu}\delta\Gamma^{\sigma}_{\mu\nu} = -\nabla_\alpha\delta g^{\alpha\sigma} + \frac{1}{2}g_{\alpha\beta}\nabla^\sigma\delta g^{\alpha\beta}$$

$$g^{\mu\sigma}\delta\Gamma^{\lambda}_{\lambda\mu} = -\frac{1}{2}g_{\alpha\beta}\nabla^\sigma\delta g^{\alpha\beta}$$

$$\delta R = R_{ij}\delta g^{ij} - D_iD_j\delta g^{ij} + g_{ij}D^2\delta g^{ij} \qquad D_i = \nabla_i$$

$$D^2 = \Delta$$

$$\Rightarrow \delta S = \int d^2x \left\{ N\sqrt{\gamma}\,(\delta g^{00}) \frac{RN^2}{2} - \frac{N\sqrt{\gamma}}{2} g_{ij} R\,\delta g^{ij} + N\sqrt{\gamma}\,\delta R \right]$$

$$= \int d^2x \sqrt{-g} \left[\frac{RN^2}{2}(\delta g^{00}) - \frac{g_{ij}R}{2}\delta g^{ij} + R_{ij}\,\delta g^{ij} \right] \quad (i,j=1,2)$$

$$+ \sqrt{\gamma} N \left(\frac{1}{N}\left(g_{ij}D^2 N - D_i D_j N \right) \delta g^{ij} \right)$$

$\underbrace{-\frac{g_{ij}R}{2}\delta g^{ij} + R_{ij}\,\delta g^{ij}}_{=0}$

$\left(\because (g_{ij}D^2 - D_iD_j)\gamma = 0 \right)$

且 $\frac{g_{ij}R}{2} + R_{ij} = 0$ 直接計算得

$$= \int dx^2 \sqrt{-g} \left\{ \frac{RN^2}{2}(\delta g^{00}) - \frac{1}{N}(D_iD_j - \gamma_{ij}D^2) N\,\delta g^{ij} \right]$$

$$\Rightarrow G_{00} = \frac{RN^2}{2} \quad G_{i0} = 0 \quad G_{ij} = -\frac{1}{N}(D_iD_j - \gamma_{ij}D^2)N$$

This term 在以前會消失因為平時考慮正常的 action 時這一項會是全微分，所以是0。

其中 $R = \dfrac{\gamma_{11}\left(\frac{\partial\gamma_{22}}{\partial r}\right)^2 + \left(\gamma_{22}\frac{\partial\gamma_{11}}{\partial r} - 2\gamma_{12}\frac{\partial\gamma_{12}}{\partial r}\right)\frac{\partial\gamma_{22}}{\partial r} + 2\left(\gamma_{12}^2 - \gamma_{11}\gamma_{22}\right)\frac{\partial^2\gamma_{22}}{\partial r^2}}{2\left(\gamma_{12}^4 - 2\gamma_{11}\gamma_{12}^2\gamma_{22} + \gamma_{11}^2\gamma_{22}^2\right)}$

是 $\begin{pmatrix}\gamma_{11} & \gamma_{12}\\ \gamma_{12} & \gamma_{22}\end{pmatrix}$ 的 scalar curvature

值得注意的是 在2維時 我們的 $R_{ij} - \frac{g_{ij}}{2}R \equiv 0$ 換言之 在2維時我們的 Einstein tensor 會 vanishing，所以前面 G_{ij} 計算中的 $R_{ij} - \frac{g_{ij}}{2}R = R_{ij} - \frac{\gamma_{ij}}{2}R = 0$ $\quad g_{ij} = \gamma_{ij}$ for $i,j=1,2$

所以 G_{ij} 只剩 $-\frac{1}{N}(D_iD_j - \gamma_{ij}D^2)N$.

G_{ij} 就是正常的 2+1 維中，$g_{ij} = \left(\begin{array}{c|cc} -N^2(r) & 0 & 0 \\ \hline 0 & \multicolumn{2}{c}{\multirow{2}{*}{$\gamma_{ij}(r)$}} \\ 0 & & \end{array}\right)$

的 Einstein tenor 剛好會等於 $\frac{RN^2}{2} = G_{00}$

③ 所以我們解 point mass 在 2+1 維的 spherical symmetric solution.

Einstein equation

$$G_{\alpha\beta} = 8\pi G T_{\alpha\beta}$$

G is Einstein tensor
T is stress-energy tensor

而考慮 static 的情況 so $g_{0i} = 0 = g_{i0}$

So we let $g_{ij} = \begin{pmatrix} -N^2(r) & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{21} & \gamma_{22} \end{pmatrix}$

但我們由 spherical symmetry, we consider g as following

$g_{ij} = \begin{pmatrix} -N^2 & 0 & 0 \\ 0 & \phi & 0 \\ 0 & 0 & \phi r^2 \end{pmatrix}$ 而考慮 a point mass at origin. with mass M.

$\Rightarrow T^{00} \propto M\delta^2(r)$ $T^{0i} = T^{i0} = T^{ij} = 0$ $\forall i,j \in \{1,2\}$

經計算 可以先算出在此 metric 下的

$\Gamma^t_{tr} = \dfrac{\frac{\partial N}{\partial r}}{N(r)}$ $\Gamma^r_{tt} = \dfrac{N\frac{\partial N}{\partial r}}{\phi(r)}$ $\Gamma^r_{rr} = \dfrac{\frac{\partial\phi}{\partial r}}{2\phi}$ $\Gamma^r_{\theta\theta} = -\dfrac{r^2\frac{\partial\phi}{\partial r} + 2r\phi}{2\phi}$

$\Gamma^\theta_{r\theta} = \dfrac{r\frac{\partial\phi}{\partial r} + 2\phi}{2r\phi}$

$$\Gamma^i_{kl} = \frac{1}{2} g^{ij}\left(\frac{\partial g_{jk}}{\partial x_l} + \frac{\partial g_{jl}}{\partial x_k} - \frac{\partial g_{kl}}{\partial x_j}\right)$$

接下來我們用 2 種方法計算出我們的 G_{ij}

1. 我們用第一章得到的 $G_{00}=\frac{RN^2}{2}$ $G_{0i}=0$ $G_{ij}=\frac{1}{N}(\gamma_{ij}D^2-D_iD_j)N$ 來計算。

直接用2維的 Scalar curvature 的計算可以得到

(這邊主要是對 $\begin{pmatrix}\phi & 0\\ 0 & \phi r^2\end{pmatrix}$ 作計算)

$$R=\frac{r(\phi_r)^2-r\phi\phi_{rr}-\phi\phi_r}{r\phi^3}$$

而 $D_iD_jN=\partial_i\partial_jN-\Gamma^k_{ij}\partial_kN \Rightarrow D_1D_1N=\partial_r\partial_rN-\Gamma^r_{rr}\partial_rN$

$$=\frac{\partial^2N}{\partial r^2}-\frac{\phi_r}{2\phi}N_r$$

$$D^2=g^{ab}D_aD_b$$

$$D_2D_2N=\cancel{\partial_\theta\partial_\theta N}-\Gamma^r_{\theta\theta}\partial_rN$$

$$=\left(\frac{r^2\phi_r+2r\phi}{2\phi}\right)N_r$$

$$\Rightarrow \frac{1}{N}(g_{ij}D^2-D_iD_j)N=G_{ij}$$

$$D^2N=\frac{1}{\phi}\left(N_{rr}-\cancel{\frac{\phi_rN_r}{2\phi}}\right)+\frac{N_r}{\phi r^2}\left(\frac{\cancel{r^2\phi_r}+2r\phi}{2\phi}\right)$$

$$=\frac{1}{\phi}\left(N_{rr}+\frac{1}{r}N_r\right)$$

$$\Rightarrow G_{11}=\frac{1}{N}\left(\phi\cdot\frac{1}{\phi}\left(N_{rr}+\frac{1}{r}N_r\right)-\left(N_{rr}+\frac{\phi_rN_r}{2\phi}\right)\right)$$

$$=\frac{N_r}{rN}+\frac{\phi_rN_r}{2\phi N}$$

$$G_{22}=\frac{1}{N}\left(r^2\left(N_{rr}+\cancel{\frac{N_r}{r}}\right)-\frac{r^2\phi_r+\cancel{2r\phi}}{2\phi}N_r\right)$$

$$=\frac{1}{N}\left(r^2N_{rr}-\frac{r^2\phi_rN_r}{2\phi}\right)$$

$$G_{00}=\frac{rN^2\phi_r^2-rN^2\phi\phi_{rr}-N^2\phi\phi_r}{2r\phi^3}=\frac{N^2R}{2}$$

2. 可以與以下我們是用最正規(2+1)D的 metric 然後直接計算其 G_{ij}，會發現結果一樣

$$G_{ij}=R^{(3)}_{ij}-\frac{1}{2}g_{ij}R^{(3)}$$ (這裡 R_{ij} 和 R 是考慮 2+1 維的曲率)

這是使用程式計算 2+1 維的

$$G_{ij}=\begin{pmatrix} \frac{rN^2\left(\frac{\partial\phi}{\partial r}\right)^2-rN^2\phi\frac{\partial^2\phi}{\partial r^2}-N^2\phi\frac{\partial\phi}{\partial r}}{2r\phi^3} & 0 & 0 \\ 0 & \frac{r\frac{\partial N}{\partial r}\frac{\partial\phi}{\partial r}+2\phi\frac{\partial N}{\partial r}}{2rN\phi} & 0 \\ 0 & 0 & \frac{2r^2\phi\frac{\partial^2 N}{\partial r^2}-r^2\frac{\partial N}{\partial r}\frac{\partial\phi}{\partial r}}{2N\phi} \end{pmatrix}$$

用3維的計算

$$=8\pi G T_{ij}$$

而除了 T_{00} 以外 $T_{ij}\equiv 0$

$$\Rightarrow \left(r\frac{\partial\phi}{\partial r}+2\phi\right)\frac{\partial N}{\partial r}\equiv 0\ ,\quad r^2\left(2\phi\frac{\partial^2 N}{\partial r^2}-\frac{\partial N}{\partial r}\frac{\partial\phi}{\partial r}\right)\equiv 0$$

若 $\frac{\partial N}{\partial r}\neq 0 \Rightarrow r\frac{\partial\phi}{\partial r}+2\phi\equiv 0 \Rightarrow \frac{\partial\phi}{\partial r}=-\frac{2\phi}{r} \Rightarrow \ln\phi=-2\ln r+c$

$$\Rightarrow \phi=\frac{c}{r^2}$$

$$2\phi\frac{\partial^2 N}{\partial r^2}-\frac{\partial N}{\partial r}\frac{\partial\phi}{\partial r}=\frac{2c}{r^2}\frac{\partial^2 N}{\partial r^2}+\frac{2c}{r^3}\frac{\partial N}{\partial r}=\frac{2c}{r^2}\left(\frac{\partial^2 N}{\partial r^2}+\frac{1}{r}\frac{\partial N}{\partial r}\right)$$

$\Rightarrow \Delta N\equiv 0 \Rightarrow$ Let $\frac{dN}{dr}=\alpha \quad \frac{d\alpha}{dr}+\frac{\alpha}{r}=0 \Rightarrow \frac{d\alpha}{dr}=-\frac{\alpha}{r}$

$$\frac{N^2\left(r\cdot\frac{4c^2}{r^6}-r\frac{c}{r^2}\cdot\frac{6c}{r^4}+\frac{c}{r^2}\frac{2c}{r^3}\right)}{2r\left(\frac{c}{r^2}\right)^3}=0$$

$\Rightarrow \ln\alpha=-\ln r+C_1$

$\Rightarrow \alpha=\frac{C_1}{r}$

$\frac{dN}{dr}=\frac{C_1}{r}$

$N=C_1\ln r+B$

$$G_{22}=\frac{2r^2\frac{c}{r^2}\frac{\partial^2 N}{\partial r^2}-r^2\cdot\frac{(-2)c}{r^3}\frac{\partial N}{\partial r}}{2(C_1\ln r+B)\frac{c}{r^2}}\quad\left(\because \Delta N=C_1 2\pi\delta^2(\vec r)\right)$$

$$=\frac{4\pi r^2 C_1\delta^2(\vec r)}{2(C_1\ln r+B)}$$

Then G_{22} is not identical 0, $\rightarrow\leftarrow$

So we have $N \equiv C$ for some constant $C \in \mathbb{R}$.

而另一個簡單的看法可以由

$G_{ij} \equiv 0 = -\frac{1}{2N}(D_iD_j - \gamma_{ij}D^2)N \Rightarrow D_iD_jN \equiv 0 \quad D^2N \equiv 0$

$\Rightarrow N$ is a constant.

So WLOG $N \equiv 1$

$$g_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & \phi & 0 \\ 0 & 0 & \phi r^2 \end{pmatrix}$$

$$\Gamma^r{}_{rr} = \frac{\frac{\partial \phi}{\partial r}}{2\phi}$$

$$\Gamma^r{}_{\theta\theta} = \frac{-\frac{\partial\phi}{\partial r}r^2 - 2r\phi}{2\phi}$$

$$\Gamma^\theta{}_{r\theta} = \frac{\frac{\partial\phi}{\partial r}r + 2\phi}{2\phi r}$$

$$G_{ij} = \begin{pmatrix} \dfrac{r\left(\frac{\partial\phi}{\partial r}\right)^2 - r\phi\frac{\partial^2\phi}{\partial r^2} - \phi\frac{\partial\phi}{\partial r}}{2r\phi^3} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

而 $\delta^2_{polar}(\vec{r})$ 代表在 polar coordinate (r,θ) 下的 delta function.

$\int_{\mathbb{R}^2} \delta^2_{\mathbb{R}^2}(\vec{r})\,dx\,dy = 1$

$= \int_0^\infty \int_0^{2\pi} \sqrt{g_{polar}}\,\delta^2_{polar}(\vec{r})\,dr\,d\theta$

而 $T^{\mathbb{R}^2}_{00} = M\,\delta^2_{\mathbb{R}^2}(\vec{r}) = \frac{M\,\delta'(r)}{2\pi r}$ on 一般的 $\mathbb{R}^2 = \int_0^\infty \int_0^{2\pi} \sqrt{g_{now}}\,\delta^2_{now}(\vec{r})\,dr\,d\theta$

$\Rightarrow \delta^2_{polar}(\vec{r}) = \frac{\delta'(r)}{2\pi r}$

而 $T^{(Now)}_{00} = M\,\delta^2_{(Now)}(\vec{r}) = \frac{M}{\phi}\delta^2_{\mathbb{R}^2}(\vec{r})$

$\Rightarrow \delta^2_{now}(\vec{r}) = \frac{\delta'(r)}{2\pi r\phi}$

其中 $g'_{polar} = \begin{pmatrix} 1 & 0 \\ 0 & r^2 \end{pmatrix}$ $g'_{now} = \begin{pmatrix} \phi & 0 \\ 0 & \phi r^2 \end{pmatrix}$

而 $\frac{\delta'(r)}{2\pi r} = \delta^2_{\mathbb{R}^2}(\vec{r})$

$\Rightarrow \delta^2_{now}(\vec{r}) = \frac{1}{\phi}\delta^2_{\mathbb{R}^2}(\vec{r})$

$\Rightarrow \sqrt{g'_{polar}} = r \quad \sqrt{g'_{now}} = \phi r$ 而 $\sqrt{g'_{now}}\,\delta^2_{now}(\vec{r}) = \sqrt{g_{polar}}\,\delta^2_{polar}(\vec{r})$

$$T_{00} = M\frac{r}{\phi r}\delta^2_{\mathbb{R}^2}(\vec{r}) = \frac{M}{\phi}\delta^2_{\mathbb{R}^2}(\vec{r})$$

$$G_{00} = -\frac{1}{2}\frac{\Delta(\ln\phi)}{\phi} = \frac{8\pi G M\delta^2_{\mathbb{R}^2}(\vec{r})}{\phi} = 4GM\,\Delta_{(\mathbb{R}^2)}\ln r$$

$\Rightarrow \ln\phi = -8GM\ln r + \ln C$

$\Rightarrow \phi = C r^{-8GM}$ 而 we can rescale r

$r' = r C^{-\frac{1}{8GM}}$

$\phi = r'^{-8GM}$

$$\Rightarrow g_{ij} = \begin{pmatrix} -1 & 0 & 0 \\ 0 & r^{-8GM} & 0 \\ 0 & 0 & r^{-8GM+2} \end{pmatrix}$$

Let $\alpha = 1-4GM$ $\quad \theta' = \alpha\theta \quad \rho = \alpha^{-1} r^{\alpha}$

$d\theta' = \alpha\, d\theta \qquad d\rho = r^{-4GM} dr$

$d\tau^2 = -dt^2 + d\rho^2 + \rho^2 d\theta'^2$

Now $0 \le \theta' \le 2\pi\alpha$ and the space is a cone,

the space which is metrically flat except at one point, 原點.

而 分幾個 case

1. $\alpha > 0 \Leftrightarrow m < \frac{1}{4G} \Leftrightarrow$ metric near the origin is regular

$$\because \int_0^R r^{-4GM} dr = \left.\frac{r^{-4GM+1}}{1-4GM}\right|_0^R = \frac{R^{\alpha}}{1-4GM}$$

$\because \alpha > 0$ so it's finite and well-defined.

2. $\alpha<0$, $m>\frac{1}{4G}$, then metric near origin is singular.

$$\left(\because \int_0^R r^{-4GM}\,dr = \frac{r^\alpha}{\alpha}\Big|_0^R \quad \text{But } \alpha<0 \quad \frac{r^\alpha}{\alpha}\Big|_\varepsilon^R = \frac{R^\alpha}{\alpha}-\frac{\varepsilon^\alpha}{\alpha} \right.$$

$$\left. \text{as } \varepsilon\to 0 \quad \Rightarrow \text{distance} \nearrow \infty. \right)$$

注意到 when $\alpha\to-\alpha$, $r\to\frac{1}{r}$ 時 他們 equivalent.

When $m>\frac{1}{4G}$ $\alpha<0$ $1-4GM=\alpha \Leftrightarrow \alpha'=4GM-1=1-4GM'>0$

$$\Rightarrow M'=\frac{1}{2G}-M$$

$$\Rightarrow r'=\frac{1}{r}$$

case 3.

So 當 $\frac{1}{2G}>m>\frac{1}{4G}$ 在原點 和 $M'=\frac{1}{2G}-M$ 在無限遠點 是一樣的效果.

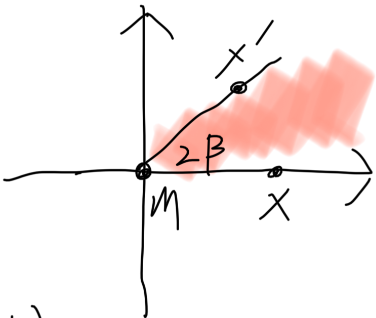
case 4. $4GM=1$ $(\alpha=0)$ $g_{ij}=\begin{pmatrix} -1 & 0 & 0 \\ 0 & r^{-2} & 0 \\ 0 & 0 & 1 \end{pmatrix}$

$$\Rightarrow ds^2=-dt^2+d(\ln r)^2+d\theta^2 \quad -\infty<\ln r<\infty \quad 0\le\theta\le 2\pi$$

This a strip on the Cartesian coordinate $(\ln r,\theta)$

④ 而當有兩個以上的粒子時，也可以透過一些等價系統的技巧來求解

在一個粒子的case時



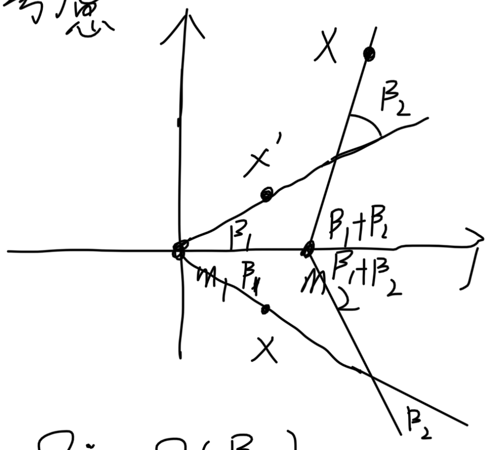
X和X'被identify成同一點 紅色區域會被移除

$(\because \theta=\alpha\theta$ 所以 θ 只從 $(0,2\pi\alpha)$ 所以有 $2\pi(1-\alpha)$ 的角度 被挖掉 其中 $\beta=\pi(1-\alpha)=4\pi GM$

所以 $x' = \Omega(\beta) x$, $t' = t$ $\quad \Omega(\beta) = \begin{pmatrix} \cos 2\beta & \sin 2\beta \\ -\sin 2\beta & \cos 2\beta \end{pmatrix}$

static的 $\qquad \beta = 4\pi G M = \pi(1-\alpha)$

而兩個粒子的情況則如下，考慮



m_1在 origin $\quad m_2$在 a 點

$\beta_1 = 4\pi G M_1 \quad \beta_2 = 4\pi G M_2$

$x' = \Omega(\beta_1) x$

$x'' = \Omega(\beta_1)\left(a + \Omega(\beta_2)(x-a)\right)$ Let $\Omega_i = \Omega(\beta_i)$

$= \Omega_1 a + \Omega_1\Omega_2 x - \Omega_1\Omega_2 a$

Let $b = \frac{\sin\beta_2}{\sin(\beta_1+\beta_2)} \Omega_1^{\frac{1}{2}} a$

$$b - \Omega_1\Omega_2 b = \frac{\sin\beta_2}{\sin(\beta_1+\beta_2)} \begin{pmatrix} \cos\beta_1 - \cos 3\beta_1+2\beta_2 & \sin\beta_1 - \sin 3\beta_1+2\beta_2 \\ -\sin\beta_1 + \sin 3\beta_1+2\beta_2 & \cos\beta_1 - \cos 3\beta_1+\beta_2 \end{pmatrix} a$$

$$= \frac{\sin\beta_2}{\sin(\beta_1+\beta_2)} \begin{pmatrix} 2\sin(2\beta_1+\beta_2)\sin(\beta_1+\beta_2) & -2\cos(2\beta_1+\beta_2)\sin(\beta_1+\beta_2) \\ 2\cos(2\beta_1+\beta_2)\sin(\beta_1+\beta_2) & 2\sin(2\beta_1+\beta_2)\sin(\beta_1+\beta_2) \end{pmatrix} a$$

$$= \begin{pmatrix} 2\sin(2\beta_1+\beta_2)\sin\beta_2 & -2\cos(2\beta_1+\beta_2)\sin\beta_2 \\ 2\cos(2\beta_1+\beta_2)\sin\beta_2 & 2\sin(2\beta_1+\beta_2)\sin\beta_2 \end{pmatrix} a$$

$$= \begin{pmatrix} \cos 2\beta_1 - \cos(2\beta_1+2\beta_2) & \sin 2\beta_1 - \sin(2\beta_1+2\beta_2) \\ -\sin 2\beta_1 + \sin(2\beta_1+2\beta_2) & \cos 2\beta_1 - \cos(2\beta_1+2\beta_2) \end{pmatrix} a$$

$$= (\Omega_1 - \Omega_1\Omega_2) a$$

$\Rightarrow x'' = b - \Omega_1\Omega_2 b$ so we can 讓這雙 particle 的系統等價於有一個 M_1+M_2 的粒子在 b 點的系統。

⑤ 而 horizon event 不存在的原因主要是我們的 metric 除了 0 點是一個 singularity 以外，其餘地方都是良好定義的，所以不會有 horizon event 的發生。

而 test particle 的加速度

$$\frac{d^2x^\mu}{d\tau^2}+\Gamma^\mu_{\alpha\beta}\frac{dx^\alpha}{d\tau}\frac{dx^\beta}{d\tau}=0 \quad 而\ g_{ij}=\begin{pmatrix} -1 & 0 & 0 \\ 0 & r^{-8GM} & 0 \\ 0 & 0 & r^{-8GM+2} \end{pmatrix}$$

$$\phi=r^{-8GM}$$

$$\Gamma^r_{rr}=\frac{\frac{\partial\phi}{\partial r}}{2\phi}=\frac{-8GM}{2r}$$

$$\Gamma^r_{\theta\theta}=\frac{-\frac{\partial\phi}{\partial r}r^2-2r\phi}{2\phi}=\frac{r(8GM+2)}{2}$$

其餘都是 0。

$$\Gamma^\theta_{r\theta}=\frac{\frac{\partial\phi}{\partial r}r+2\phi}{2\phi r}=\frac{8GM+2}{2r}$$

$$\Rightarrow \frac{d^2x^r}{d\tau^2}+\Gamma^r_{rr}\frac{dx^r}{d\tau}\frac{dx^r}{d\tau}+\Gamma^r_{\theta\theta}\frac{dx^\theta}{d\tau}\frac{dx^\theta}{d\tau}=0$$

$$\Rightarrow \frac{d^2x^r}{d\tau^2}-\frac{4GM}{r}\left(\frac{dx^r}{d\tau}\right)^2+4(GM+1)r\left(\frac{dx^\theta}{d\tau}\right)^2=0$$

$$\Rightarrow \frac{d^2x^\theta}{d\tau^2}+\frac{2\cdot(4GM+1)}{r}\frac{dx^r}{d\tau}\frac{dx^\theta}{d\tau}=0$$

So If test particle is at rest at first then test particle 不會有任何加速度

而如果是一個一開始靜止的粒子，他的 geodesic equation 只有 Γ^i_{00} 的項

有意義 ($\because \dot{x}^i=0 \quad i=1,2$)

$\ddot{x}^i+\Gamma^i_{00}=0$ 但 $\because g_{00}=-1$ 且 $g_{i0}=0 \Rightarrow \Gamma^i_{00}=0 \quad \forall i=1,2.$

總結

1. 我們先討論我們一般情況下我們metric可以經過適當變數代換調整成好看形式

2. Reproduce some result of paper，從 Eienstein action 出發照文章計算出與原文有一點差異的 G_{ij}，而我們用 $\int d^2x \sqrt{\gamma} N R^{(2)}$ 作變分得到的 G_{ij} 和直接對 $g_{ij}=\begin{pmatrix} -N^2 & 0 & 0 \\ 0 & \gamma_{11} & \gamma_{12} \\ 0 & \gamma_{21} & \gamma_{22} \end{pmatrix}$ 用 $G_{ij}=R^{(3)}_{ij}-\frac{1}{2}g_{ij}R^{(3)}$ 會得到一樣結果

3. 一開始先驗証在 $g_{ij}=\begin{pmatrix} -N^2 & 0 & 0 \\ 0 & \phi & 0 \\ 0 & 0 & r^2\phi \end{pmatrix}$ 下第2部分的結果是正確的，後來用和原文相似作法（把細節補齊後）得到我們metric的解 for one point mass 的情況.

4. 解釋若有多個point mass的情況我們該如何處理

5. 解釋 horizon event 不存在以及 conial cingular 和一個 test particle 的加速度。