Hidden zeros for particle/string amplitudes and the unity of colored scalars, pions and gluons

Goal: 1. Introduce $\mathrm{Tr}(\phi^3)$ and ABHY.

Understand the zero and factorization of amplitude of $\mathrm{Tr}(\phi^3)$ theory.

2. Stringy amplitude of $\mathrm{Tr}(\phi^3)$.

See the zero and factorization in the stringy amplitude,

$$I^{\delta}_{2n} := \int_{\mathbb{R}^{2n-3}_{>0}} \prod_{I=1}^{2n-3} \frac{dy_I}{y_I} \prod_{(a,b)} u_{a,b}^{\alpha' X_{a,b}} \left(\frac{\prod_{(e,e)} u_{e,e}}{\prod_{(o,o)} u_{o,o}} \right)^{\alpha'\delta}$$

3. Deformation of stringy amplitude of $\mathrm{Tr}(\phi^3)$

Discuss the 4-point amplitude to try to understand which theory is the different δ amplitude describe.

4. $\alpha'\delta \in (0,1) \Rightarrow$ Descibing the NLSM

5. $\alpha'\delta = 1 \Rightarrow$ YMS theory (gluons and adjoint scalars)

ref: 2312.16282 Hidden zero
2401.00041 Scalar-Scaffold gluons
1912.08707 Stringy canonical form
1711.09102 ABHY

Hidden zero.

$\mathrm{Tr}(\phi^3)$ and Associahedron.

Colored massless scalars interacting via cubic interaction.

$L_{\mathrm{Tr}(\phi^3)} = \mathrm{Tr}(\partial\phi)^2 + g\,\mathrm{Tr}(\phi^3)$ ϕ is $N\times N$ matrix.

The amplitude is a function of the Lorentz invariant dot product $P_i \cdot P_j \Rightarrow \frac{n(n-1)}{2} - \underset{P_i^2=0}{n}$ invariant massless.

$C_{i,j} := -2P_i \cdot P_j$

Consider another basis.

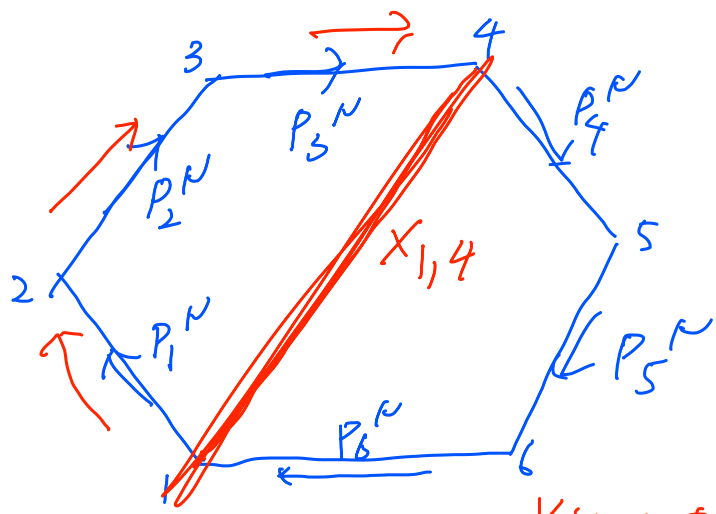
$X_{i,j} := (P_i + P_{i+1} \cdots P_{j-1})^2$

But $X_{i,i+1} = 0 \rightarrow 1 \le i \le n-1$

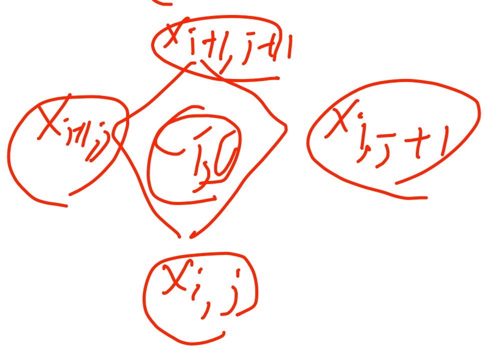
$X_{n,1} = 0$

$\downarrow \frac{n(n-1)}{2} - n$

$\Rightarrow \frac{n(n-1)}{2} - n$ and $X_{i,j}$ are independent



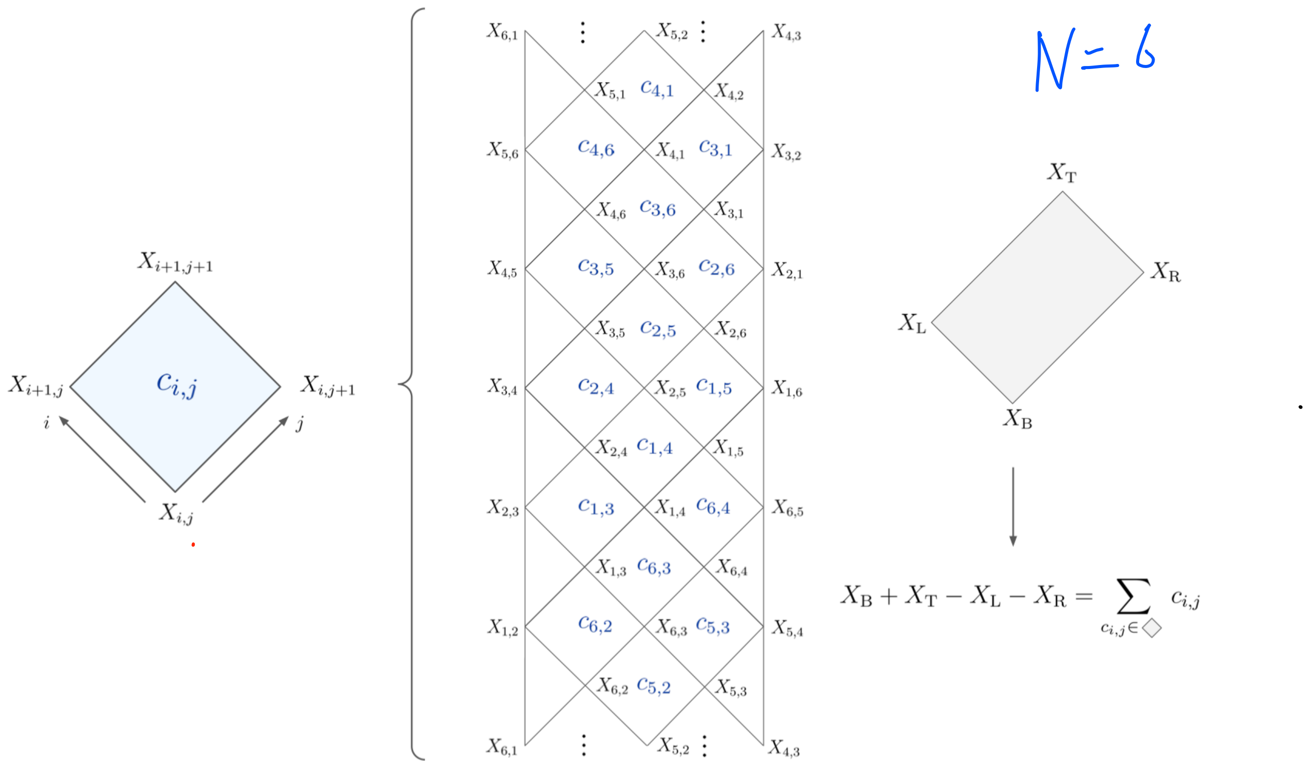
$X_{1,4} = (P_1 + P_2 + P_3)^2$



$\Rightarrow X_{i,j}$ is a basis of Kinematic invariants

Note that $C_{i,j} = -2P_i \cdot P_j = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}$

In order to see this basis more clearly, we introduce the kinematic mesh.



But we can see that $X_{i,j} = X_{j,i}$ and $C_{i,j} = C_{j,i}$

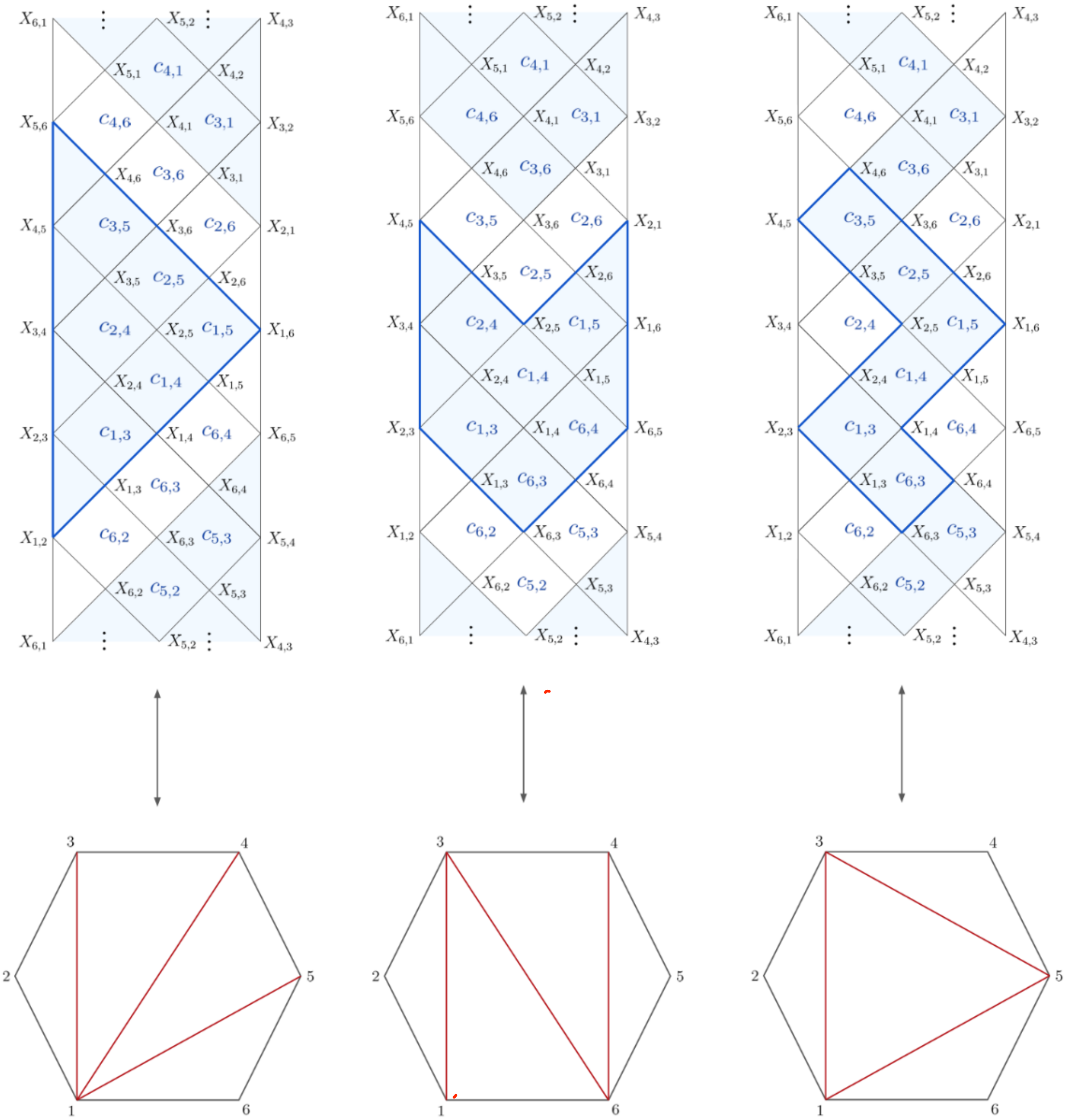
We want to find a basis by considering a minimal subregion of the mesh.

The subreigion are one-to-one correspondence with a triangulation of n-gon.

Process 1. Given a triangulation T (there are n-3 (ij))

2. $(i,j) \in T$ then we don't pick $C_{i-1,j-1}$.

3. The left connected area is the subregion we want.



ABHY

We know that kinematic K_n for n massless momenta p_i

$$\dim K_n = \frac{n(n-1)}{2} - n = \frac{n(n-3)}{2}$$

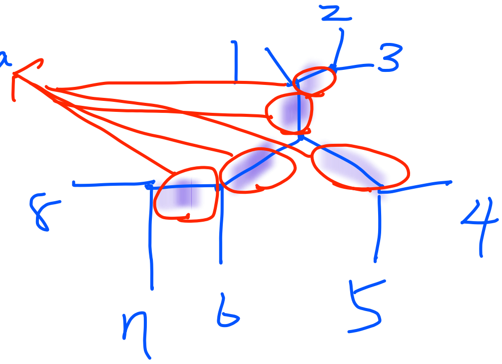
Plannar scattering amplitude.

$$\mathrm{sign}(g) \bigwedge_{a=1}^{n-3} d\log X_{i_a, j_a} = \frac{dX_{i_a,j_a}}{X_{i_a,j_a}}$$

K_n is the $\mathbb{R}$-vector space with basis $s_{i,j}$ $1 \le i < j \le n$.

$s_{i,j} := 2p_i \cdot p_j$

$$\Omega^{(n-3)}_{n}(K_n) := \sum_{\text{planar } g} \text{sign}(g) \bigwedge_{a=1}^{n-3} d\log X_{i_a, j_a}$$



This $\Omega_n^{(n-3)}$ is differential form on K_n

And we try to use this $\Omega_n^{(n-3)}$ to construct the n-point amplitude.

Define 1, $\Delta^+ := \{X_{i,j} \geq 0 \ \forall 1 \leq i < j \leq n\} \subseteq K_n$

2, Pick a subregion of kinematic mesh. Then we can fix a basis of planar variable X_{i_a, j_a} $a \in \{1, \cdots, n-3\}$ $((i_a, j_a) \in T)$

3, $H_n := \{C_{i,j} = X_{i,j} + X_{i+1,j+1} - X_{i,j+1} - X_{i+1,j}\}$ for all $C_{i,j}$ in the given mesh

There are $\frac{n(n-1)}{2} - n - (n-3)$ $C_{i,j}$ condition.

$\Rightarrow \dim H_n = \dim K_n - \left[\frac{n(n-1)}{2} - n - (n-3)\right]$

$= n-3.$

$H_n \subseteq K_n$

4. Impose all $c_{i,j} > 0$ in the mesh.

$$A_n := H_n \cap \Delta^+ \Rightarrow A_n \xhookrightarrow{i} K_n$$

We claim the following two statements

1. $A_n \xhookrightarrow{i} K_n$ $\quad \Omega_n(A_n) = m_n\, d^{n-3}X$

$\Omega_n(A_n) \xleftarrow{\text{pullback}} \Omega_n^{(n-3)}(K_n)$ $\quad m_n$ is the amplitude

$\quad \underline{d}^{n-3}X = \text{sign}(g) \bigwedge_{a=1}^{n-3} dX_{i_a, j_a}$

$\quad (i_a, j_a) \in T \Leftrightarrow X_{i_a, j_a} \in$ basis

$||$

$m_n \underline{d^{n-3}X}$

$\Downarrow$

$dX_{i_a, j_a} \in T$

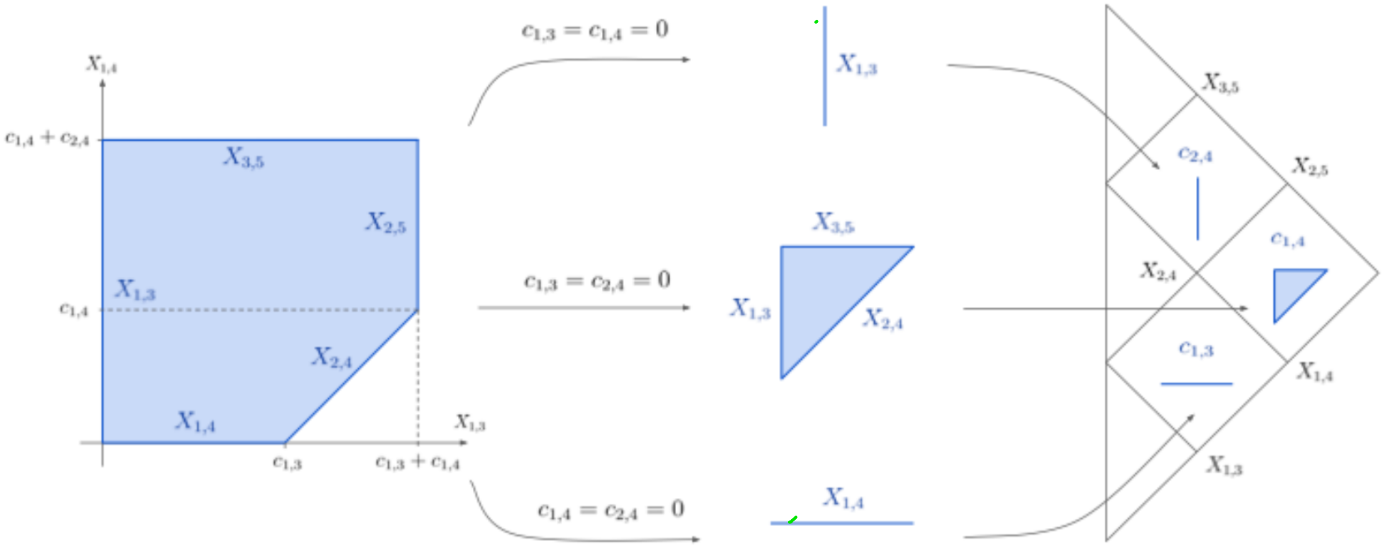
2. $\Omega_n(A_n) = \sum\limits_{Z \in \text{vertex}} \text{sign}(Z) \bigwedge_{a=1}^{n-3} d\log X_{i_{Z_a}, j_{Z_a}}$

$X_{i_{Z_a}, j_{Z_a}} = 0$ are facets which form the vertex Z.

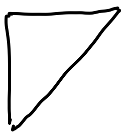
The result we may use:

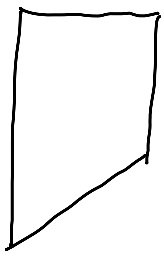
1. If H_n collapse $\Leftrightarrow$ $\dim H_n < n-3$ $\Rightarrow$ Amplitude $= 0$.

2. Associahedron can be constructed by the Minkowski sum

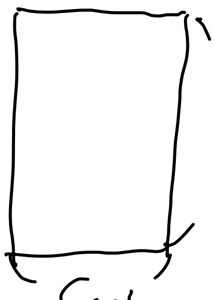


$$\text{Associahedron} = \underbrace{\sum_{(i,j)}}_{\text{Minkowski sum}} C_{i,j}\ \text{Associahedron of}\begin{pmatrix} C_I = 0 \text{ except} \\ C_{i,j} = 1 \end{pmatrix}$$

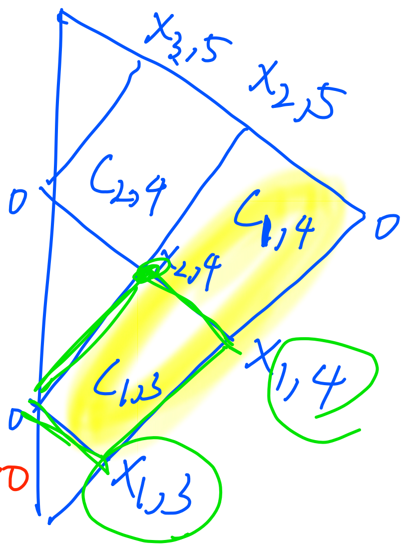
$C_{1,3}=0 \iff C_{1,4}$  $+\, C_{2,4}$ |

$=$ 

$C_{1,4}=0 \iff C_{2,4}$ | $+\, C_{1,3}$ —

$=$ 

Now we are ready to see the zero of amplitude.



Example 5-point.

$\{X_{1,3}, X_{1,4}, C_{1,3}, C_{1,4}, C_{2,4}\}$

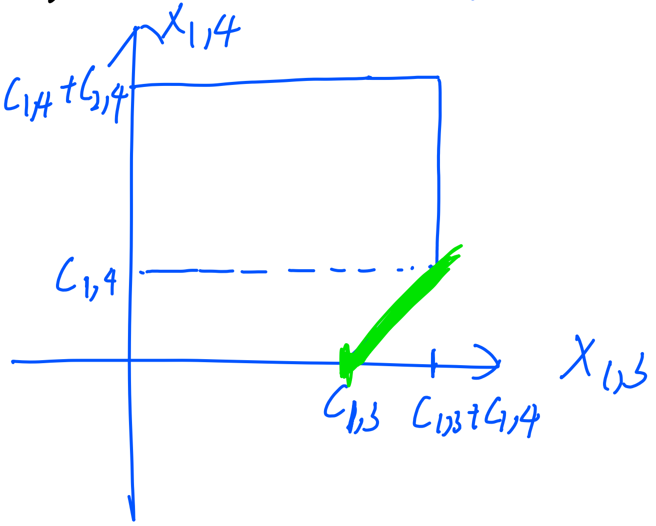
$X_{1,3} > 0$

$X_{1,4} > 0$

$X_{2,4} > 0 \iff X_{2,4} + X_{1,3} - X_{1,4} = C_{1,3} \Rightarrow X_{1,4} - X_{1,3} - C_{1,3} > 0$

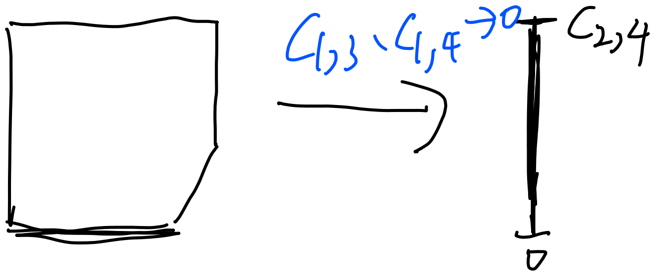
$X_{2,5} > 0 \iff X_{2,5} + X_{1,3} = C_{1,3} + C_{1,4} \Rightarrow C_{1,3} + C_{1,4} - X_{1,3} > 0$

$X_{3,5} > 0 \iff X_{3,5} + X_{1,4} = C_{1,4} + C_{2,4} \Rightarrow C_{1,4} + C_{2,4} - X_{1,4} > 0.$

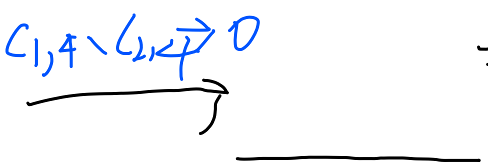


Now consider to turn off some $C_{i,j}$.

$\Rightarrow$

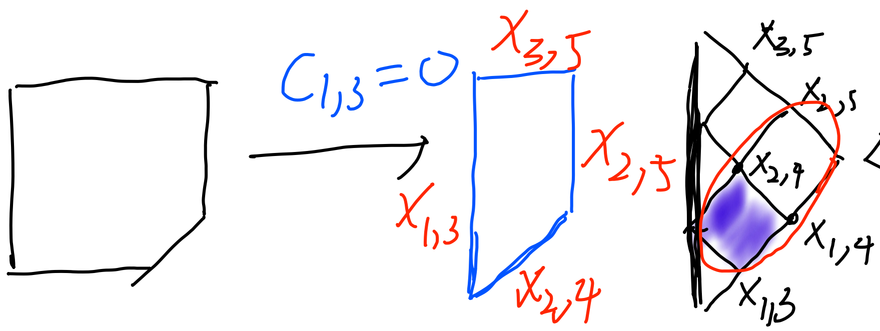


$\Rightarrow$ Amplitude $= 0$



$\Rightarrow$ Amplitude $= 0$

The interesting thing is that if we turn on one of $c_{i,j}$ just before the Amplitude $\to 0$,

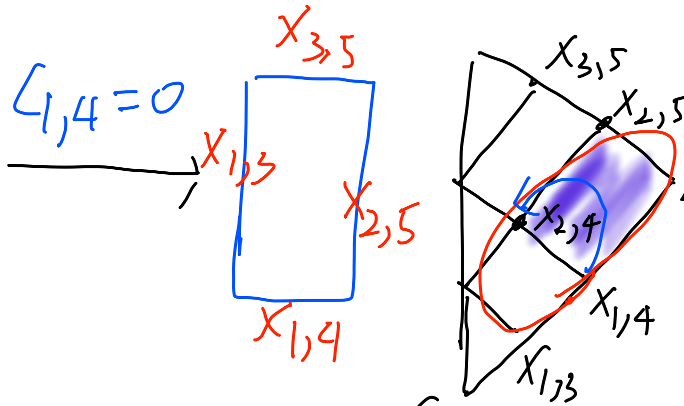


$$\frac{C_{1,3}}{X_{1,3}X_{2,5}}$$

$$\Leftarrow \left(\frac{1}{X_{1,3}}+\frac{1}{X_{2,5}}\right)\times A_4(X_{2,4},X_{3,5})$$

$X_{1,4}=X_{2,4}+X_{1,3}$

$\Rightarrow X_{1,4}$ is redundant



$$\Leftarrow \left(\frac{1}{X_{1,3}}+\frac{1}{X_{2,5}}\right)\times A_4(X_{1,4},X_{3,5}).$$

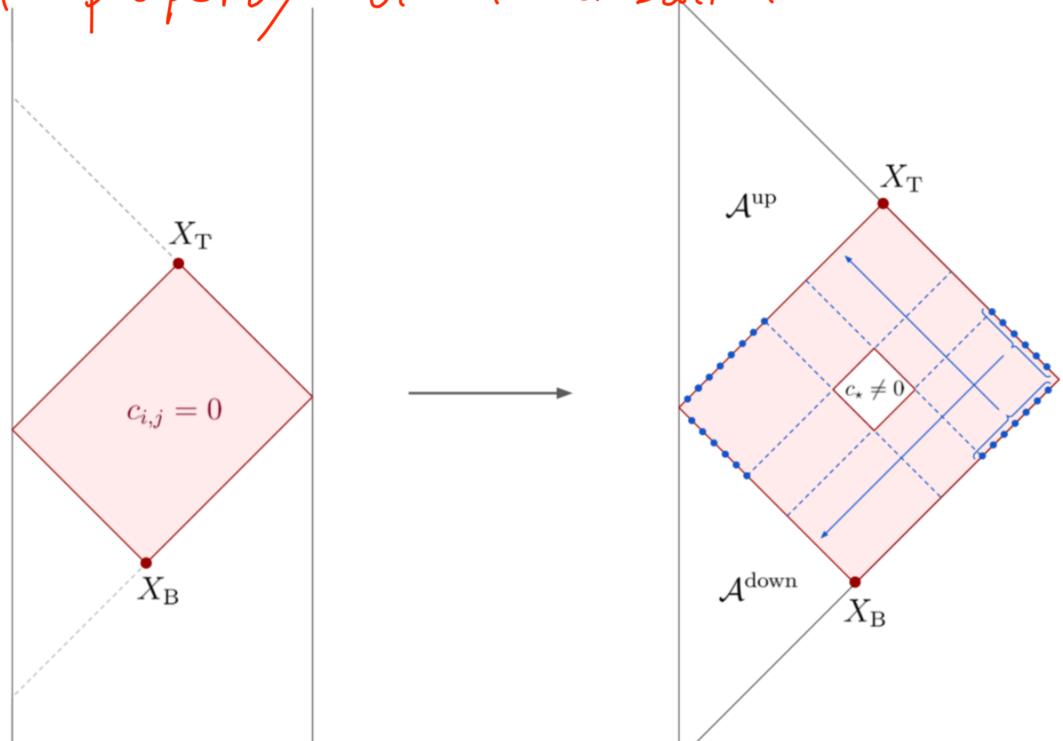
$X_{2,5}+X_{1,4}-X_{2,4}=0$

$\Rightarrow X_{2,4}=X_{1,4}+X_{2,5}$

so $X_{2,4}>0$ is redundant

$\Rightarrow X_{2,4}$ doesn't exist in Amplitude.

General property of factorization.



$$A_n(C_* \neq 0) = \left(\frac{1}{X_B} + \frac{1}{X_T}\right) \times A^{UP} \times A^{down}$$

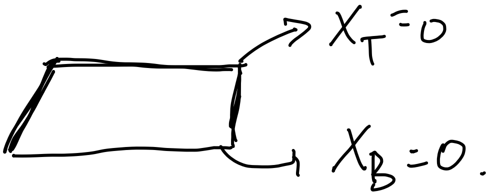
Why we can have these factorization.

1. We notice that $X_T + X_B + 0 + 0 = C_*$

(when $C_* = 0$ which means $X_B + X_T = 0 \Rightarrow X_B = 0$)

$\Rightarrow$ Associahedron collapse in X_B axis.

Then, when we turn on C_*, we have a sandwich like associahedron. And the two facets form the upper and lower bound of associahedron



$\Rightarrow \left(\frac{1}{X_B} + \frac{1}{X_T}\right)$ factor.

2. Why the remaining part can be separate as A^{UP} and A^{down}.

We can see that the facet condition in A^{UP} are indep with the facet condition in A^{down}.

So the remaining Amplitude can be factorized into A^{UP} and A^{down}.

$$\because \Omega(A_n) = \sum_{\text{vertex}} \text{sign}(Z) \bigwedge_{a=1}^{n-3} d\log X_{i_a, j_a} \qquad \bigcap_{a=1}^{n-3} \{X_{i_a, j_a} = 0\} = \{Z\}$$

But now each $Z = \bigcap_{a=1}^{n'} \{X_{i_a, j_a} = 0\}_{\text{in } A^{UP}} \cap \bigcap_{b=1}^{m'} \{X_{i_b, j_b} = 0\}_{\text{in } A^{down}}$.

The facets of associahedron are not lost in this kinematic limit will enter in A^{up} or A^{down}.

First, the facet $X_{i,j} \geq 0$ for $X_{i,j}$ in the red rectangle would be redundent condition, so don't enter the A^{up} or A^{down}.

So the remaining condition is those in the boundary of red rectangle.

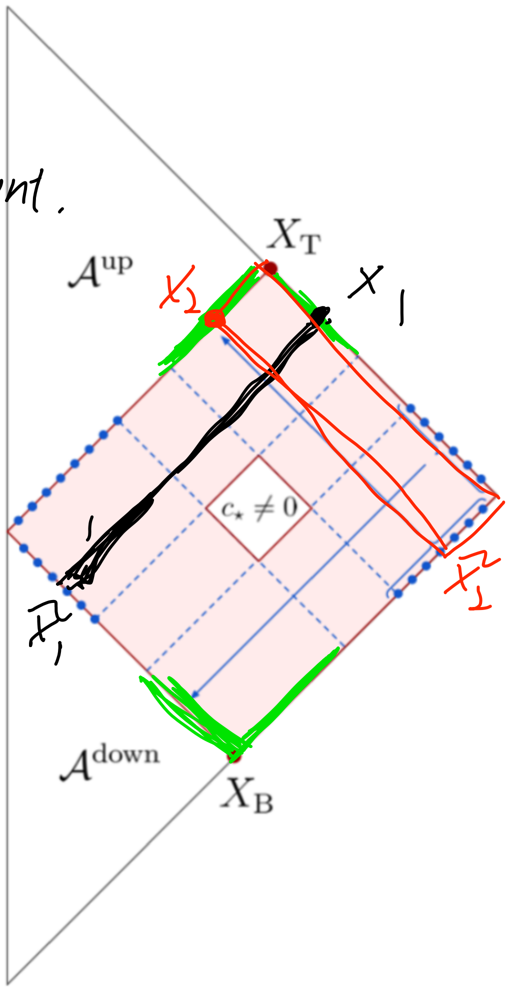
Second, the green line are redundant.

$$X_T + \hat{X}_1 - X_1 - 0 = 0$$

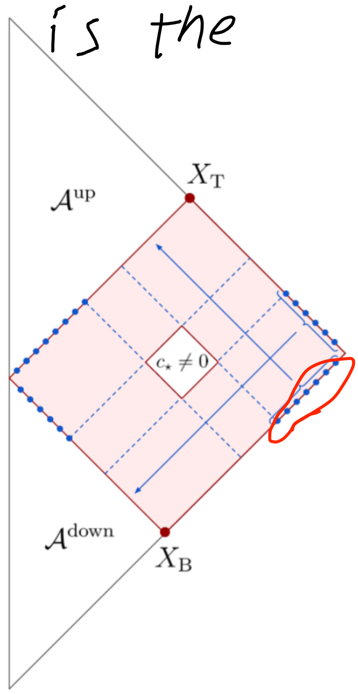
$\Rightarrow X_1 = X_T + \hat{X}_1 \Rightarrow X_1 > 0$ is redudant.

$$X_T + \hat{X}_2 - X_2 - 0 = 0$$

$\Rightarrow X_2 = X_T + \hat{X}_2 \Rightarrow \underline{X_2 > 0}$ is redundant.



Third, so the facets in A^{up} is the $X_{i,j}$ in A^{up} and these .



Stringy canonical form.

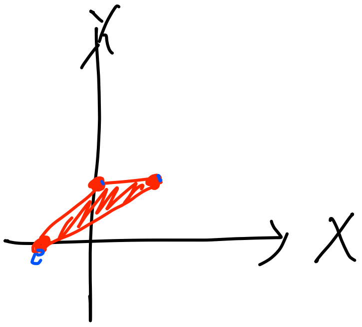
$$I_p(\vec{X}, c) = (\alpha')^d \int_0^\infty \prod_{i=1}^{d} \frac{dy_i}{y_i}\, \underbrace{y_i^{\alpha' X_i}\, p(\vec{y})^{-\alpha' C}}_{\text{regulator}}$$

Assume $p(\vec{y}) = \sum_\alpha a_\alpha \vec{y}^{\,n_\alpha}$ $\left(\vec{y}^{\,n_\alpha} := y_1^{n_{\alpha,1}} \times \cdots \times y_d^{n_{\alpha,d}} \quad \text{with } a_\alpha > 0,\ n_\alpha \in \mathbb{Z}^d \right)$

Define : Newton polytope $N[p(x)]$.

$$N[p(x)] := \left\{ \sum_\alpha \lambda_\alpha \vec{n}_\alpha \,\middle|\, \lambda_\alpha \geq 0 \quad \sum_\alpha \lambda_\alpha = 1 \right\}$$

ex: $N[\frac{1}{x} + y + xy] =$



Claim 1. $I_p(\vec{X}, c) = \alpha'^d \int_0^\infty \prod_{i=1}^{d} \frac{dy_i}{y_i} y_i^{\alpha' X_i} p(\vec{y})^{-\alpha' C}$ is converge

$\Longleftrightarrow$ 1. Newton polytope $P := CN[p(\vec{y})]$ is top-dim

2. $\vec{X}$ inside P. $\vec{X} = (X_1, X_2, \cdots, X_d)$

P is an d-dimension polytope.

And $\lim_{\alpha' \to 0} I_p(\vec{X}, c) = \underline{\Omega}(CN[p(\vec{y})]; \vec{X})$.

The amplitude with the Associahedron is $P = CN[p(\vec{y})]$.

Claim 2. $I_{\{p\}}(X, \{c\}) = (\alpha')^d \int_{\mathbb{R}^d_{>0}} \prod_{i=1}^{d} \frac{dy_i}{y_i} y_i^{\alpha' X_i} \prod_I P_I(\vec{y})^{-\alpha' c_I}$

is converge $\Longleftrightarrow$ X inside the Newton Polytope.

$P := \sum_I c_I N[P_I(\vec{y})]$ (Minkowski sum)

$= \{ \sum_I c_I u_I \mid u_I \in N[P_I(\vec{y})] \}$

and $\lim_{\alpha' \to 0} I_{\{p\}}(\vec{X}, \{c\}) = \Omega(P; \vec{X})$.

Claim 3. ABHY Associahedron A_{n-3}

$A_{n-3} = \sum_{1 \leq i < j-1 < n-1} c_{ij} A_{ij}$ (Minkowski sum)

$A_{ij} := \lim_{\substack{c_{ij} \to 1 \\ \text{else} = 0}} A_{n-3}$.

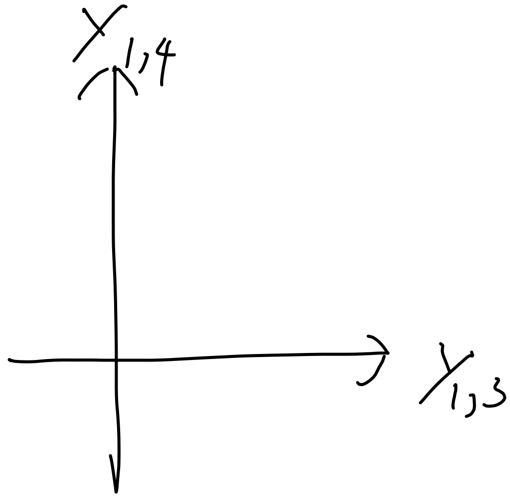
field theory stringy amplitude

Associahedron. $\longleftrightarrow$ Newton polytope

Ω (Newton polytope) $\longleftarrow$ $I_n(\alpha' \to 0)$

Then we use $n=5$ as example to determine the stringy canonical form.

We use $X_{1,3}$、$X_{1,4}$ as basis:



$X_{1,3} \geq 0$

$X_{1,4} \geq 0$

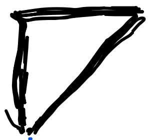
$X_{2,4} \geq 0 \Leftrightarrow C_{1,3} - X_{1,3} + X_{1,4} \geq 0$

$X_{2,5} \geq 0 \Leftrightarrow C_{1,3} + C_{1,4} - X_{1,3} \geq 0$

$X_{3,5} \geq 0 \Leftrightarrow C_{1,4} + C_{2,4} - X_{1,4} \geq 0$

$\Rightarrow A_{13} \Leftrightarrow C_{1,4}$、$C_{2,4} = 0$ $\Rightarrow P_{1,3} = 1 + Y_{1,3}$

$A_{14} \Leftrightarrow C_{1,3}$、$C_{2,4} = 0$

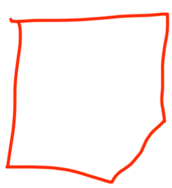


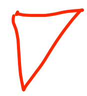
$\Rightarrow P_{1,4} = 1 + Y_{1,4} + Y_{1,3}Y_{1,4}$

$A_{2,4} \Leftrightarrow C_{1,3}$、$C_{1,4} = 0$ $\Rightarrow P_{2,4} = 1 + Y_{1,4}$

$C_{1,3}$ + $C_{1,4}$  + $C_{2,4}$ =

Associahedron



$C_{1,3} N[P_{1,3}] = C_{1,3}$
+
$C_{1,4} N[P_{1,4}] = C_{1,4}$ 
+
$C_{2,4} N[P_{2,4}] = C_{1,4}$

$$\Rightarrow I_5 = \int \frac{dY_{1,3}}{Y_{1,3}} \frac{dY_{1,4}}{Y_{1,4}} Y_{1,3}^{\alpha' X_{1,3}} Y_{1,4}^{\alpha' X_{1,4}} (1+Y_{1,3})^{-\alpha' C_{1,3}} (1+Y_{1,4}+Y_{1,3}Y_{1,4})^{-\alpha' C_{1,4}} (1+Y_{1,4})^{-\alpha' C_{2,4}}$$

String amplitude.

$$I_n^{Tr(\phi^3)} = \int_{\mathbb{R}^{n-3}_{>0}} \prod_{I=1}^{n-3} \frac{dY_I}{Y_I} Y_I^{\alpha' X_I} \prod_{i,j} F_{i,j}(Y)^{-\alpha' c_{i,j}}$$

For the ray-like triangulization $\{Y_{1,3}, Y_{1,4}, \ldots, Y_{1,n-1}\}$

$$\Rightarrow F_{i,j} = 1 + Y_{1,j} + Y_{1,j} Y_{1,j-1} + \ldots \ldots + Y_{1,j} \cdots Y_{1,i+2}$$

First, look at the zero of the string amplitude.

Example: $n=4$

$$I_4^{Tr(\phi^3)} = \int_{\mathbb{R}>0} \frac{dY_{1,3}}{Y_{1,3}} Y_{1,3}^{\alpha' X_{1,3}} (1+Y_{1,3})^{-\alpha' c_{1,3}}$$

$$= \frac{\Gamma(\alpha' X_{1,3}) \Gamma(\alpha'(c_{1,3} - X_{1,3}))}{\Gamma(\alpha' c_{1,3})}$$

$$= \frac{\Gamma(\alpha' X_{1,3}) \Gamma(\alpha' X_{2,4})}{\Gamma(\alpha' c_{1,3})}$$

As $\alpha' \to 0$ field-theory limit

$$I_4^{Tr(\phi^3)} \to \frac{c_{1,3}}{\alpha' X_{1,3} X_{2,4}}$$

So we can see that $c_{1,3}$ turn off

Then $I_4^{Tr(\phi^3)} = 0$.

We can see that when $\alpha' c_{1,3}$ is non-positive integer the $I_4^{Tr(\phi^3)}$ vanishes.

Set $\alpha' c_{1,3} = -n$ $\quad n \in \mathbb{N}_0$

$$I_4^{Tr(\phi^3)} \to \sum_{k=0}^{n} \binom{n}{k} \underbrace{\int_{\mathbb{R}>0} \frac{dY_{1,3}}{Y_{1,3}} Y_{1,3}^{\alpha' X_{1,3} + k}}_{0} = 0$$

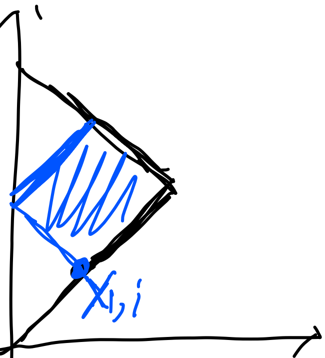
Each integral is divergent, however, they all vanish by analytic continuation.

$$\int_{\mathbb{R}>0} \frac{dy}{y} y^{\alpha' X} = \int_0^1 \frac{dy}{y} y^{\alpha' X} + \int_1^\infty \frac{dy}{y} e^{\alpha' X}$$

$$= -\left.\frac{y^{\alpha' X}}{\alpha' X}\right|_{y=0} + \left.\frac{y^{\alpha' X}}{\alpha' X}\right|_{y=\infty}$$

First part $\alpha' X > 0$
Second part $\alpha' X < 0$ $\Rightarrow 0 + 0 = 0$

Generally, if we use the ray-like triangulization, then the intergrand $F_{i,j}$ contain Y_{1i} are inside the casual diamond.



$\because$ $1 \le a \le i-2$ $\quad F_{i,j} = 1 + Y_{1,j} + \cdots + Y_{1,j} \cdots Y_{i+2}$
$i \le b \le n-1$

Then if we set $C_{i,j} = -n_{a,b}$ $\quad n_{a,b} \in \mathbb{N}_0$.

$$\Rightarrow I_n^{Tr(\phi^3)} \to \sum_{k_{a_1,b_1} \cdots k_{a_N,b_N}}^{n_{a_1,b_1} \cdots n_{a_N,b_N}} (\text{remaining}) \int_{\mathbb{R}>0} \frac{dY_{1i}}{Y_{1,i}} Y_{1,i}^{\alpha' X_{1,i} + k_{a_1,b_1} \cdots k_{a_N,b_N}} = 0$$

Since the zero causal diamond is uniquely determined by the X_B anchored on.

The total number of zeros for $Tr(\phi^3)$ is $\frac{n(n-3)}{2}$.

(Because if we set $X_{i,j} = 0 \Rightarrow$ polytope collapse
$\Rightarrow$ canonical form $= 0$.

Factorization around zero

Field theory we use the Minkowski-sum to see the zero and factorization.

Stringy case we use the Minkowski-sum of Newton polytope of the F-polynomials.

EX : $n=5$

$$I_5^{Tr(\phi^3)} = \int_0^\infty \prod_{i=3}^{4} \frac{dY_{1,i}}{Y_{1,i}} Y_{1,i}^{\alpha' X_{1,i}} \prod_{i<j} F_{i,j}(\vec{Y})^{-\alpha' c_{i,j}}$$

$$= \int_0^\infty \frac{dY_{1,3}}{Y_{1,3}} \frac{dY_{1,4}}{Y_{1,4}} Y_{1,3}^{\alpha' X_{1,3}} Y_{1,4}^{\alpha' X_{1,4}} (1+Y_{1,3})^{-\alpha' c_{1,3}} (1+Y_{1,4})^{-\alpha' c_{2,4}} (1+Y_{1,4}+Y_{1,3}Y_{1,4})^{-\alpha' c_{1,4}}$$

If $c_{1,3} = c_{1,4} = 0$

$$\Rightarrow I_5 = \underbrace{\int_0^\infty \frac{dY_{1,3}}{Y_{1,3}} Y_{1,3}^{\alpha' X_{1,3}}}_{0} \int_0^\infty \frac{dY_{1,4}}{Y_{1,4}} Y_{1,4}^{\alpha' X_{1,4}} (1+Y_{1,4})^{-\alpha' c_{2,4}}$$

If $c_{1,4} \neq 0$

$$I_5 \to \int_0^\infty \frac{dY_{1,3}}{Y_{1,3}} Y_{1,3}^{\alpha' X_{1,3}} \int_0^\infty \frac{dY_{1,4}}{Y_{1,4}} Y_{1,4}^{\alpha' X_{1,4}} (1+Y_{1,4})^{-\alpha' c_{2,4}} (1+Y_{1,4}+Y_{1,3}Y_{1,4})^{-\alpha' c_{1,4}}$$

$$= \int_0^\infty \frac{dY_{1,3}}{Y_{1,3}} Y_{1,3}^{\alpha' X_{1,3}} \int_0^\infty \frac{dY_{1,4}}{Y_{1,4}} Y_{1,4}^{\alpha' X_{1,4}} (1+Y_{1,4})^{-\alpha'(c_{2,4}+c_{1,4})} \left(1+\frac{Y_{1,3}Y_{1,4}}{(1+Y_{1,4})}\right)^{-\alpha' c_{1,4}}$$

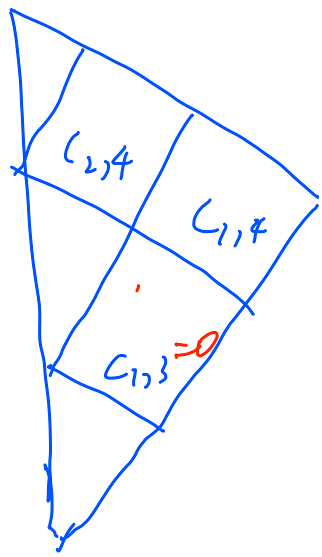
let $\tilde{y}_{1,3} = \dfrac{y_{1,3}\, y_{1,4}}{1+y_{1,4}}$

$$\Rightarrow I_5 = \int_0^\infty \frac{d\tilde{y}_{1,3}}{\tilde{y}_{1,3}}\, \tilde{y}_{1,3}^{\,\alpha' X_{1,3}} \left(1+\tilde{y}_{1,3}\right)^{-\alpha' c_{1,4}}$$

$$\int_0^\infty \frac{dy_{1,4}}{y_{1,4}}\, y_{1,4}^{\,\alpha'(X_{1,4}-X_{1,3})} \left(1+y_{1,4}\right)^{-\alpha'(c_{2,4}+c_{1,4}-X_{1,3})}$$

$$= I_4\left(\alpha' X_{1,3}, \alpha'(c_{1,4}-X_{1,3})\right) \times I^{NP}\left(\alpha'(X_{1,4}-X_{1,3}),\ \alpha'(c_{2,4}+c_{1,4}-X_{1,4})\right)$$

$$= I_4\left(\alpha' X_{1,3}, \alpha' X_{2,5}\right) \times I_4^{NP}\left(\alpha' X_{2,4}, \alpha' X_{3,5}\right).$$

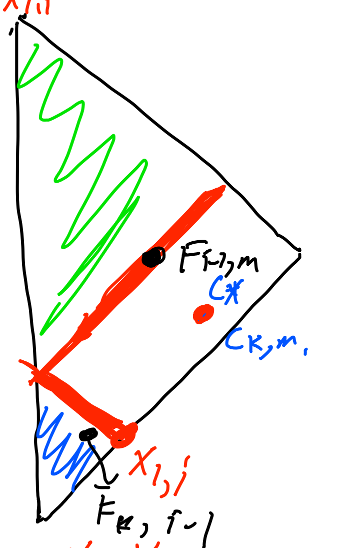


General proof of factorization.

$$I_n = \int \prod_{l=3}^{i-1} \frac{dy_{1,l}}{y_{1,l}} y_{1,l}^{\alpha' X_{1,l}} \int \prod_{j=i+1}^{n-1} \frac{dy_{1,j}}{y_{1,j}} y_{1,j}^{\alpha' X_{1,j}} \int \frac{dy_{1,i}}{y_{1,i}} y_{1,i}^{\alpha' X_{1,i}}$$

$$\prod_{1 \le a < b-1 \le i-1} F_{a,b}(\vec{y})^{-\alpha' c_{a,b}} \prod_{i-1 \le e < f-1 < n-1} F_{e,f}(\vec{y})^{-\alpha' c_{e,f}}$$

$$F_{k,m}(\vec{y})^{-\alpha' c_{k,m}}$$



1. $$\int \frac{dy_{1,i}}{y_{1,i}} \left(A + y_{1,i} B\right)^{-\alpha' c_{k,m}}$$

let $\tilde{y}_{1,i} = \frac{B}{A} y_{1,i}$

$$F_{k,m} = 1 + y_{1,m} + y_{1,m} y_{1,m-1} + \cdots + y_{1,m} \cdots y_{k+2}$$

$$= 1 + y_{1,m} + y_{1,m} y_{1,m-1} \cdots + y_{1,m} \cdots y_{1,i+1}$$
$$+ y_{1,i} \left(y_{1,m} \cdots y_{1,i+1} + y_{1,m} \cdots y_{1,i+1} y_{i-1} + y_{1,m} \cdots y_{1,i+1} y_{1,i-1} \cdots y_{k+2} \right)$$

$$= \underbrace{F_{i-1,m}}_{A} + y_{1,i} \underbrace{\left(F_{k,i-1} \prod_{p=i+1}^{m} y_{1,p} \right)}_{B}.$$

$$= A^{-\alpha' c_{k,m} + \alpha' X_{1,i}} B^{-\alpha' X_{1,i}} \int \frac{d\tilde{y}_{1,i}}{\tilde{y}_{1,i}} \tilde{y}_{1,i}^{\alpha' X_{1,i}} (1 + \tilde{y}_{1,i})^{-\alpha' c_{k,m}}$$

$$= A^{-\alpha' c_{k,m} + \alpha' X_{1,i}} B^{-\alpha' X_{1,i}} \times I_4\left(\alpha' X_{1,i}, \alpha' (c_{k,m} - X_{1,i})\right)$$

$$\| $$

$$I_4 (\alpha' X_s, \alpha' X_T)$$

Then plugging this back.

So we can see the factor A that shift the $c_{e,f}$

$c'_{e,f} = c_{e,f}$ except for $c'_{i-1,m} = c_{i-1,m} + c_{k,m} - X_{1,i}$

The factor B shift the $c_{a,b}$ and $X_{1,j}$

$c'_{a,b} = c_{a,b}$ except for $c'_{k,i-1} = c_{k,i-1} + X_{1,i}$.

$X'_{1,j} = X_{1,j}$ except for $X'_{1,j} = X_{1,j} - X_{1,i}$ for $j = i+1, \ldots, m$

$$\Rightarrow I_n = \int \prod_{l=3}^{i-1} \frac{dY_{1,l}}{Y_{1,l}} Y_{1,l}^{\alpha' X_{1,l}} \prod_{1\le a<b\le i-1} F_{a,b}^{-\alpha' C_{a,b}} \times \int \prod_{j=i+1}^{n-1} \frac{dY_{1,j}}{Y_{1,j}} Y_{1,j}^{\alpha' X_{1,j}} \prod_{i-1\le e<f-1<n-1} F_{e,f}^{-\alpha' C_{e,f}}$$

$$\times I_4(\alpha' X_{1,i}, \alpha'(C_{k,m} - X_{1,i}))$$

Notice: We shift some C、X's, the relation between X、C are broken.

But for the A^{down}, the shift of C is equivalent to let $X_{l,i} \to X_{l,i} + X_{1,i} = X_{l,n}$ for $l = 2 \cdots k$.

And set $P_I = -\sum_{j=1}^{i-1} P_j$

$$\text{Then } \int \prod_{l=3}^{i-1} \frac{dY_{1,l}}{Y_{1,l}} Y_{1,l}^{\alpha' X_{1,l}} \prod_{1\le a<b-1\le i-1} F_{a,b}^{-\alpha C_{a,b}}$$

$$= I_i^{Tr(\phi^3)}(1, 2, \cdots, i-1, I) \Big|_{X_{l,i} \to X_{l,i} + X_{1,i} = X_{l,n} \text{ for } l=2\cdots k}$$

Similarly

$$I^{NP}_{n-i+2} = \int \prod_{j=i+1}^{n-1} \frac{dY_{1,j}}{Y_{1,j}} Y_{1,j}^{\alpha' X_{1,j}} \prod_{i-1\le e<f-1<n-1} F_{e,f}(Y)^{-\alpha' C_{e,f}}$$

$$= \int \prod_{j=i+1}^{n-1} \frac{dY_{i-1,j}}{Y_{i-1,j}} Y_{i-1,j}^{\alpha' X'_{i-1,j}} \prod_{i-1\le e<f-1<n-1} F_{e,f}(Y)^{-\alpha' C_{e,f}}$$

$$= I_{n-i+2}(i, \cdots n, J) \Big|_{X_{i-1,j} \to X_{i-1,j} - X_{i-1,n} = X_{1,j} \text{ for } j=m\ldots,n-1}$$

$$P_J = -\sum_{j=i}^{n} P_j$$

Result

$$I_n^{\mathrm{Tr}(\phi^3)} \to I_i^{\mathrm{down}} \times I_{n-i+2}^{\mathrm{up}} \times I_4(\alpha' X_{1,i}, \alpha'(c_{km} - X_{1,i}))$$

$$A^{\mathrm{down}} : X_{l,i} \to X_{l,i} + X_{1,i} = X_{l,n} \quad \text{for } l = 2 \ldots k$$

$$A^{\mathrm{up}} : X_{i-1,j} \to X_{i-1,j} - X_{i-1,n} = X_{1,j} \quad \text{for } j = m \ldots n-1$$