1. Determine the values of a for which the system has
   a) (5%) no solutions;
   b) (5%) exactly one solution;
   c) (5%) infinitely many solutions.

$$\begin{aligned} x + 2y - 3z &= 4 \\ 3x - y + 5z &= 2 \\ 4x + y + (a^2 - 2)z &= a + 4 \end{aligned}$$

2. The scalar triple product of $\mathbf{u} = (u_1, u_2, u_3)$, $\mathbf{v} = (v_1, v_2, v_3)$, and $\mathbf{w} = (w_1, w_2, w_3)$ can be calculated from the formula

$$\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}.$$

   Suppose that $\mathbf{u}\cdot(\mathbf{v} \times \mathbf{w}) = -4$. Find
   a) (5%) $\mathbf{u}\cdot(\mathbf{w} \times \mathbf{v})$
   b) (5%) $(\mathbf{v} \times \mathbf{w})\cdot\mathbf{u}$

3. (10%) Find the orthogonal projection of $\mathbf{u}$ on the subspace of R^3 spanned by the vector $\mathbf{v}_1$ and $\mathbf{v}_2$

$$\mathbf{u} = (1, -6, 1); \quad \mathbf{v}_1 = (-1, 2, 1); \quad \mathbf{v}_2 = (2, 2, 4)$$

4. Consider the following matrix

$$A = \begin{bmatrix} 2 & -1 \\ -2 & 1 \end{bmatrix}$$

   a) (5%) Find the eigenvalues and eigenvectors of $A^T A$.
   b) (7%) Compute the Singular Value Decomposition of A.
   c) (3%) Compute the rank 1 approximation of A.

5. Read the following conclusion "*Someone who passed the final exam has not read the book.*" and the premises
"*A student in this class has not read the book.*"
"*Everyone in this class passed the final exam.*"
   a) (5%) Let $C(x)$ denote "x is in this class", $B(x)$ denote "x has read the book" and $P(x)$ denote "x passed the final exam". Translate the premises and conclusion into symbolic form.
   b) (10%) Use the rules of inference to construct a valid argument showing that the conclusion follows from the premises.

6. Given the following definitions:
Definition 1: The ***height h(T)*** of a full binary tree T is defined recursively as follows:
   - BASIS STEP: The height of a full binary tree T consisting of only a root r is $h(T) = 0$.
   - RECURSIVE STEP: If T_1 and T_2 are full binary trees, then the full binary tree $T = T_1 \cdot T_2$ has height $h(T) = 1 + \max(h(T_1), h(T_2))$.

Definition 2: The ***number* of vertices *n(T)*** of a full binary tree T satisfies the following recursive formula:
   - BASIS STEP: The number of vertices of a full binary tree T consisting of only a root r is $n(T) = 1$.
   - RECURSIVE STEP: If T_1 and T_2 are full binary trees, then the full binary tree $T = T_1 \cdot T_2$ has the number of vertices $n(T) = 1 + n(T_1) + n(T_2)$.

Prove that if T is a full binary tree, then $n(T) \leq 2^{h(T)+1} - 1$ by induction.
   a) (5%) Provide the basis step of induction.
   b) (5%) Provide the inductive hypothesis and the inductive step to complete the induction.

7. (10%) If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides $a - b$. We use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to b modulo m. Find all solutions to the system of congruences.
$x \equiv 1 \pmod{2}$
$x \equiv 2 \pmod{3}$
$x \equiv 3 \pmod{5}$
$x \equiv 4 \pmod{11}$

8. (10%) How many binary strings of length 10 contain either five consecutive 0s or five consecutive 1s.

9. (5%) What is the relationship between the sum of the degrees of the vertices in an undirected graph and the number of edges in this graph? Explain why this relationship holds.