※ 注意：請於試卷內之「非選擇題作答區」作答，並應註明作答之題號。

From Problem 1 to Problem 4, your answer will be considered correct only if all the true statements are selected.

1. (5%) Let $A, B \in M_{n\times n}(F)$, where $F = \boldsymbol{C}$ or $\boldsymbol{R}$. Which of the following statements are true?
   (a) $tr(AB) = tr(BA)$
   (b) $tr(AB) = tr(A)tr(B)$
   (c) $tr(B^{-1}AB) = tr(A)$
   (d) $tr(A^k) = (tr(A))^k$
   (e) $tr(A \pm B) = tr(A) \pm tr(B)$

2. (5%) Let $A, B \in M_{n\times n}(F)$, where $F = \boldsymbol{C}$ or $\boldsymbol{R}$. Which of the following statements are true?
   (a) $det\,(AB) = det(BA)$
   (b) $det(AB) = det(A)det(B)$
   (c) $det(B^{-1}AB) = det(A)$
   (d) $det(A^k) = (det\,(A))^k$
   (e) $det(A \pm B) = det(A) \pm det(B)$

3. (5%) Which of the following statements are true?
   (a) If A and B are invertible matrices, then the matrix $C = A^{-1} + B^{-1}$ is also invertible.
   (b) If R is a rectangular matrix, then $A = R^T R$ is positive definite.
   (c) Every orthogonal set is linearly independent.
   (d) $T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}$, where T_{11} is a $p\times p$ matrix, T_{22} is a $q\times q$ matrix, and T_{12} is a $p\times q$ matrix. The set of eigenvalues of T is the union of the sets of eigenvalues of T_{11} and T_{22}.
   (e) If $T: V \to W$ is a linear transformation with V and W are vector spaces over a field F. Then, *ker(T)* is a vector space.

4. (10%) Let vectors $\boldsymbol{x} \in \boldsymbol{R}^n$ be represented as $\boldsymbol{x} = (x_1, x_2, \ldots, x_n)^T$. Which of the following sets is a subspace of $\boldsymbol{R}^n$?
   (a) $\{\boldsymbol{x} \in \boldsymbol{R}^n \mid \sum_{i=1}^{n} x_i^2 = n^2\}$
   (b) $\{\boldsymbol{x} \in \boldsymbol{R}^n \mid x_1 = 1\}$
   (c) $\{\boldsymbol{x} \in \boldsymbol{R}^n \mid \sum_{i=1}^{n} \alpha_i x_i = min_i(\alpha_i)\}$ for some known constants $\alpha_i \in \boldsymbol{R}$.
   (d) $\{\boldsymbol{x} \in \boldsymbol{R}^n \mid x_{2i} = x_{2i-1}, \forall i = 1, \ldots, n/2\}$, where n is assumed to be even in this case.
   (e) $\{\boldsymbol{x} \in \boldsymbol{R}^n \mid x_{2i} - x_{2i-1} = 1, \forall i = 1, \ldots, n/2\}$, where n is assumed to be even in this case.

5. (5%) Let N be an $n\times n$ matrix over $\boldsymbol{R}$ or $\boldsymbol{C}$ such that $N^k = 0$ for some integer k. Is I+N invertible? If yes, find its inverse. If not, provide the reason why.

6. (5%) Define an inner product in $\boldsymbol{R}^2$ as $\langle u, v\rangle = \frac{1}{4}u_1 v_1 + \frac{1}{9}u_2 v_2$. What is the equation of the unit circle?

7. (5%) Please find the eigenvectors of $(I + A)^{100}$ given $A = \begin{bmatrix} -4 & -5 \\ 10 & 11 \end{bmatrix}$.

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8. (10%) Please find the determinant of A.

$$A = \begin{pmatrix} 1+x_1 & x_2 & x_3 & \cdots & x_n \\ x_1 & 1+x_2 & x_3 & \cdots & x_n \\ \vdots & \vdots & 1+x_3 & \ddots & \vdots \\ x_1 & x_2 & x_3 & \cdots & 1+x_n \end{pmatrix}$$

9. (10%) Consider

$$x_1 + x_2 + \text{L} + x_n < r,$$

where $x_i \geq 0$ for $1 \leq i \leq n$. When $n = 4$ and $r = 8$, the number of nonnegative integer solutions is _________.

10. (5%) The cycle decomposition of the permutation

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 6 & 5 & 8 & 3 & 1 & 7 & 4 \end{pmatrix}$$

is __________________.

11. (10%) The solution to the recurrence equation

$$a_n = 2a_{n-1} + n$$

with $a_0 = 4$ is $a_n =$ _________.

12. (10%) The generating function in partial fraction decomposition for the above recurrence equation is _________. (Note that expressions like

$$\frac{x-8}{(x-3)^2} - \frac{9}{x-1}$$

are *not* partial fraction decompositions.)

13. (5%) Let $p(m)$ denote the number of partitions of $m \in \mathbb{Z}^+$. For example, the number of partitions of $m = 5$ is $p(5) = 7$:

5, 4+1, 3+2, 3+1+1, 2+2+1, 2+1+1+1, 1+1+1+1+1

The generating function for $\{p(n)\}_{n=0,1,\text{K}}$ is _________.

14. (5%) A Boolean function f is **self-dual** if

$$f(x_1, x_2, \ldots, x_m) = f(\neg x_1, \neg x_2, \ldots, \neg x_m).$$

There are _________ self-dual Boolean functions of m variables.

15. (5%) For any binary tree with n nodes and i internal nodes, the relation between n and i is _________ $\leq i$.

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