1. (10%) Find the determinant of the matrix A:

$$A=\begin{bmatrix}3 & 5 & -2 & 6\\ 1 & 2 & -1 & 1\\ 2 & 4 & 1 & 5\\ 3 & 7 & 5 & 3\end{bmatrix}.$$

2. (10%) Solve the given matrix equation for X:

$$\begin{bmatrix}1 & 2 & 3\\ 3 & 7 & 6\\ 1 & 0 & 8\end{bmatrix}X=\begin{bmatrix}1 & 4\\ 0 & -1\\ -3 & 6\end{bmatrix}.$$

3. (10%) Find a basis for and the dimension of the solution space of the homogeneous system:

$$\begin{aligned}2x_1+2x_2-x_3\qquad\quad+x_5&=0\\ -x_1-x_2+2x_3-3x_4+x_5&=0\\ x_1+x_2-2x_3\qquad\quad-x_5&=0\\ x_3+\ x_4+x_5&=0\end{aligned}$$

4. (10%) Find the eigenvalues and corresponding eigenvectors of the matrix A:

$$A=\begin{bmatrix}5 & 4 & 2\\ 4 & 5 & 2\\ 2 & 2 & 2\end{bmatrix}.$$

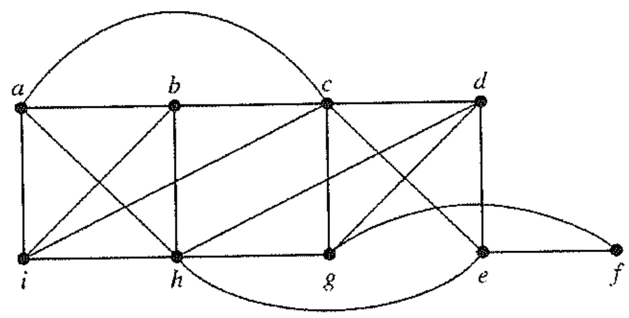
5. (10%) If A is a symmetric $n\times n$ matrix and x is an $n\times 1$ column vector of variables, prove that $\mathrm{x}^{\mathrm{T}}A\mathrm{x}$ is positive definite if and only if all eigenvalues of A are positive.

6. (5%) If we take all people as the universe, write the proposition in symbols using predicates and quantifiers (Universal Quantifier or Existential Quantifier) of the sentence: "Everyone who visited France stayed in Paris."

7. a) (7%) Draw the graph represented by the incident matrix G. b) (7%) Use depth-first-search to produce a spanning tree of the graph in a). Choose the first vertex in the matrix as the root of the tree.

$$G=\begin{bmatrix} 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

8. (6%) Determine whether the following graph has an Euler circuit. Construct such a circuit when one exists. If no Euler circuit exists, give your reason.



9. (5%) Show that at least four of any 22 days must fall on the same day of the week.

10. If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides $a-b$. We use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to b modulo m.
(a) (5%) Find an inverse of 4 modulo 9, i.e., find all integers x such that $4x \equiv 1 \pmod{9}$.
(b) (5%) Find all integers y that satisfy the congruence $4y \equiv 5 \pmod{9}$.

11. (10%) Find the solution to the recurrence relation $a_n = 7a_{n-1} - 10a_{n-2}$ for $n > 1$ with initial conditions $a_0 = 2$, $a_1 = 1$.