1. (10%) Let $\mathbf{u} = (3, -1, 2)$ and $\mathbf{a} = (1, 3, 0)$. Find the vector component of $\mathbf{u}$ along $\mathbf{a}$ and the vector component of $\mathbf{u}$ orthogonal to $\mathbf{a}$.

2. In each part, determine whether the given vectors span R^3. Please provide an explanation for each of your answers.
   a) (5%) $\mathbf{v}_1 = (1, 2, 3)$, $\mathbf{v}_2 = (2, 0, 0)$, $\mathbf{v}_3 = (-2, 1, 0)$
   b) (5%) $\mathbf{v}_1 = (2, -1, 2)$, $\mathbf{v}_2 = (4, 1, 3)$, $\mathbf{v}_3 = (2, 2, 1)$
   c) (5%) $\mathbf{v}_1 = (3, 2, 4)$, $\mathbf{v}_2 = (-3, -1, 0)$, $\mathbf{v}_3 = (0, 1, 4)$, $\mathbf{v}_4 = (0, 2, 8)$

3. (5%) Let A be a 2×2 matrix such that $\binom{1}{1}$ is an eigenvector for A with eigenvalue 2 and $\binom{2}{3}$ is an eigenvector for A with eigenvalue 1.
   If $v = \binom{3}{4}$, compute $A^3 v$.

4.
   a) (5%) Find an LU-Decomposition of the following matrix $\begin{bmatrix} 2 & 6 & 2 \\ -3 & -8 & 0 \\ 4 & 9 & 2 \end{bmatrix}$
   b) (5%) Use the solution of part(a) to solve the following system of linear equations

$$\begin{aligned} 2x_1 + 6x_2 + 2x_3 &= 2 \\ -3x_1 - 8x_2 &= 2 \\ 4x_1 + 9x_2 + 2x_3 &= 3 \end{aligned}$$

   (No credit for other methods)

5.
   a) (5%) Find an orthonormal basis for the subspace of R^3 spanned by

$$\left\{\begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}1\\2\\3\end{bmatrix}\right\}.$$

   b) (5%) Extend the basis you found in part(a) to an orthonormal basis for R^3.

6. If a and b are integers and m is a positive integer, then a is congruent to b modulo m if m divides $a - b$. We use the notation $a \equiv b \pmod{m}$ to indicate that a is congruent to b modulo m.
   a) (5%) Find an inverse of 5 modulo 43.
   b) (5%) Solve the congruence $5x \equiv 8 \pmod{43}$

7. (10%) A string that contains only 0s and 1s is called a binary string. Find a recurrence relation for the number of bit sequences (binary strings) of length n with an even number of 1s.

8. (5%) A simple graph is called regular if every vertex of this graph has the same degree. How many vertices does a regular graph of degree 6 with 99 edges have?

9. Show that if n is an integer and $n^3 + 5$ is odd, then n is even using
   a) (5%) a proof by contraposition.
   b) (5%) a proof by contradiction.

10. (5%) Give the definition of **incidence matrices** of a graph as follows. Let $G = (V, E)$ be an undirected graph. Suppose that $v_1, v_2, \ldots, v_n$ are the vertices and $e_1, e_2, \ldots, e_m$ are the edges of G. Then the incidence matrix with respect to this ordering of V and E is the $n \times m$ matrix $\mathbf{M} = [m_{ij}]$, where

$$m_{ij} = \begin{cases} 1, \text{when edge } e_j \text{ is incident with } v_i, \\ 0, \text{otherwise.} \end{cases}$$

Determine whether the given pair of graphs G_1 and G_2 represented by incidence matrices is isomorphic or not. Give your reasons.

$$G_1 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}, \quad G_2 = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}.$$

11. (10%) Show that the set of positive rational numbers is countable.