國立交通大學 103 學年度碩士班考試入學試題

科目：線性代數與離散數學(1002)　　考試日期:103年2月15日 第 2 節

系所班別：資訊聯招　　

【不可使用計算機】＊作答前請先核對試題、答案卷(試卷)與准考證之所組別與考科是否相符！！

請在答案紙上按題號依序作答，並清楚標示題號。每題分數以小括弧數字表示，總分共計 100 分。Please write down your answers with the question numbers on the **ANSWER PAPER** in order. The credit points for each problem are indicated by the number in parentheses, and the total points are 100.

1 (10 points) Find a best approximation (in the least-squares error sense) to $y = x^3$ by a straight line between $x = -1$ and $x = 3$. (Note: $C[a, b]$ denotes the set of all real-valued functions that are defined and continuous on the closed interval $[a, b]$. The inner product is defined as $\langle f, g \rangle = \int_a^b f(x)g(x)dx$.)

2 (5 points) Calculate the following matrix-vector multiplication:

$$\begin{bmatrix} \cos(\pi/16) & \sin(\pi/16) \\ -\sin(\pi/16) & \cos(\pi/16) \end{bmatrix} \begin{bmatrix} \cos(9\pi/16) & \sin(9\pi/16) \\ -\sin(9\pi/16) & \cos(9\pi/16) \end{bmatrix} \begin{bmatrix} \cos(2\pi/16) & -\sin(2\pi/16) \\ \sin(2\pi/16) & \cos(2\pi/16) \end{bmatrix} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

(Hint: $\begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ is a clockwise rotation (linear transformation) of vector $\begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$ by an angle θ.)

3 (10 points) Let **A** and **B** be similar matrices, please show that det(**A**)=det(**B**) and also show that if λ is any scalar, then det(**A**-λ**I**)=det(**B**-λ**I**).

4 Consider the matrix $\mathbf{A} = \begin{bmatrix} 1 & 2 & 1 & 2 \\ 1 & 2 & 2 & 5 \\ 0 & 0 & 1 & 3 \end{bmatrix}$.

(a) (2 points) Compute the dimensions of the column space $\mathrm{C}(\mathbf{A})$ and the null-space $\mathrm{N}(\mathbf{A})$.

(b) (2 points) Give an orthogonal basis for $\mathrm{C}(\mathbf{A})$ and $\mathrm{N}(\mathbf{A})$, respectively.

(c) (2 points) Find a projection matrix that projects any vector $b \in R^3$ onto the column space $\mathrm{C}(\mathbf{A})$.

(d) (4 points) Find the least-squares solutions to the problem $\mathbf{Ax} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}$.

(e) (5 points) Find the minimum-length, least-squares solution to the problem

$$\mathbf{Ax} = \begin{bmatrix} -1 \\ 3 \\ 1 \end{bmatrix}.$$

5 Let $\mathbf{A}_n = \begin{bmatrix} 3 & 1 & 0 & 0 & \cdots & 0 \\ 1 & 3 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 3 & 1 & \cdots & 0 \\ 0 & 0 & 1 & 3 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \cdots & 3 \end{bmatrix}_{n\times n}$ be the triple diagonal matrix

(a) (4 points) Express the determinant $\det(\mathbf{A}_n)$ in terms of $\det(\mathbf{A}_{n-1})$ and $\det(\mathbf{A}_{n-2})$

(b) (6 points) Evaluate $\det(\mathbf{A}_n)$ as a function of n.

6 A strict weak ordering is a binary relation on a set that is (1) irreflexive, (2) transitive, and (3) asymmetric in which (4) the relation of incomparability is transitive.

(a) (3 points) One of the four conditions is redundant, i.e. one of them can be implied by the others. Which condition is redundant? What is the smallest set of other conditions that can imply the redundant one?

(b) (3 points) For each relation below, determine if it is a strict weak ordering.
(1) $<$ on the set $\mathbf{Z}$ of integers
(2) $\subsetneq$ on the power set of $\mathbf{Z}$
(3) $\prec$ on $\mathbf{Z}$ defined by $a \prec b$ iff $a \bmod 2014 < b \bmod 2014$

(c) (3 points) C++ standard template library provides several sorting algorithms that can be used to order a sequence of elements with respect to a user-defined strict weak ordering R in the sense that the sorted sequence $\{a_i\}$ satisfies the property: $(a_j, a_i) \notin R$ whenever $i < j$.
Now, assume that
"ios", "wp8", "htc", "imac", "ipod", "ipad", "iphone", "google", "itunes"
is a sequence of nine strings sorted by a C++ sorting algorithm with respect to some strict weak ordering R. Then, for what condition will $(s,t) \in R$, where s and t are strings?

(d) (3 points) Among the nine strings mentioned in part (c), which, if any, are incomparable with respect to the strict weak ordering R?

7 The line graph $L(G) = (V', E')$ of an undirected simple graph $G = (V, E)$ is defined by

(1) $V' = E$, and

(2) $E' = \{((u,v),(v,w)) | (u,v) \in E, (v,w) \in E, u \neq w\}$

Put another way, (1) the edges of G are the vertices of $L(G)$, and (2) there is an edge connecting vertices (u,v) and (v,w) in $L(G)$ iff the corresponding edges share the same endpoint v in G.

(a) (3 points) For each statement below, determine if it is true.

(1) If G has an Euler circuit, then $L(G)$ has a Hamiltonian circuit.

(2) If G is disconnected, then $L(G)$ is disconnected.

(3) If G is a cycle graph, then G and $L(G)$ are isomorphic.

(b) (4 points) How many edges are there in $L(G)$? Hint: Count the degrees

8 (6 points) Given the recurrence relations

$a_n = 2b_{n-1} + 1$

$b_n = a_{n-1} + 2b_{n-1} + 2 \qquad n \geq 1$

with the boundary conditions $a_0 = 0$ and $b_0 = 0$.

What is the solution of a_n?

Sufficè it to write down the solution; no computations are required.

9. (a) (5 points). Define $F(x) = x/2$ if x is even. Otherwise, $F(x) = F(F(5x+3))$. Prove that we can compute $F(x)$ in a finite amount of time for any positive integer x.

9. (b) (5 points) Please translate Goldbach's conjecture into a logical formala using $\forall$, $\exists$, *and*, *or*, *not*, *implies* and *if-and-only-if*, and the four basic integer arithmetic operators (+, -, *, /). You need to state Goldbach's conujecture in Chinese or English first.

9. (c) (5 points) Show that there are more irrational numbers than rational numbers.

9. (d) (7 points) Find the prime factors of 820307. Hint. There are two factors that are roughly equivalent.

9. (e) (3 points) Please find an irrational number that is between 23.3756 and 23.3757.