國立中興大學112學年度碩士班招生考試試題

科目： 離散數學與線性代數　　系所： 資訊科學與工程學系 甲組

本科目不得使用計算機　　本科目試題共 3 頁

請於答案卷上作答，否則不予計分。

1. (10%) True and False. Mark '**O**' if the statement is True and mark '**X**' if it is False.

   (a) _____ Suppose $A \in R^{n\times n}$. If $trace(A) = \alpha$, then $trace(A^{-1}) = 1 / \alpha$

   (b) _____ Suppose $A \in R^{n\times n}$. If $A^2 = A$, then $A = I$ $(identiy\ matrix)$ $or\ A = 0$ $(zero\ matrix)$.

   (c) _____ If V is a subspace of dimension k, then any generating set for V contains exactly k vectors.

   (d) _____ Suppose that A and B are both $n \times n$ invertible matrices, if A and B are symmetric matrices, then AB is symmetric.

   (e) _____ Suppose that A and B are both $n \times n$ invertible matrices, if $AB = I$, then the inverse of B is A.

   (f) _____ Suppose the matrices $A, B \in R^{n\times n}$. If B is obtained from A by adding a multiple of some row to a different row, then determinant of A is equal to the determinant of B.

   (g) _____ If the equation $Ax = b$ is inconsistent, then the rank of $[A|b]$ is greater than the rank of A.

   (h) _____ If $A \in R^{m\times n}$ (A is an m by n matrix), then A^TA is a symmetric matrix, but $(A + A^T)$ is a skew symmetric matrix.

   (i) _____ If A is an m by n matrix and A is diagonalizable, then A must have n distinct eigenvalues

   (j) _____ If R is an $n \times n$ matrix in reduced row echelon form that has rank n, then $R = I_n$.

2. (8%) Find the eigenvalues and eigenvectors for the matrix $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$

3. (8%) Suppose $T: R^2 \to R^2$ is a linear transformation such that

   $T\left(\begin{bmatrix} 3 \\ 2 \end{bmatrix}\right) = \begin{bmatrix} -4 \\ 11 \end{bmatrix}$ and $T\left(\begin{bmatrix} -4 \\ -4 \end{bmatrix}\right) = \begin{bmatrix} 4 \\ -8 \end{bmatrix}$

   (a) (4%) Find the standard matrix of linear transformation T.
   (b) (4%) Please find a generating set for the range of this linear transformation T.

4. (8%)Consider the following linear equations system.

$$\begin{cases} x - y + 3z = 1 \\ -x + 2y - 3z = 4 \\ 3x - 3y + k^2 z = k \end{cases}$$

Write the linear equations system in the form of **Ax=b**. Apply the Gaussian elimination algorithm to transform the augmented matrix **[A | b]** into the row echelon form (or the row reduced echelon form). Then find the value of k when there is a unique solution. (**x** has a unique solution)

5. (8%) Given a vector space W, which is spanned from the set $S = \{v_1, v_2, v_3\}$,

where $v_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $v_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 1 \end{bmatrix}$, $v_3 = \begin{bmatrix} 1 \\ 1 \\ 2 \\ 1 \end{bmatrix}$.

Apply **Gram-Schmidt** algorithm to convert **set** S into the Orthonomal set S'

6. (8%) Given $v = \begin{bmatrix} 3 \\ 2 \\ 6 \end{bmatrix} \in R^3$, Please find the projection vector of v in the subspace W,

where $W = \left\{ \begin{bmatrix} a \\ b \\ b \end{bmatrix} \middle| a, b \in R \right\}$

7. Given a sequence 2, 3, 7, 15, 31, 63, ...

   (a) Please state the recurrence relation for the sequence. (3%)
   (b) Please find the generating function for the sequence (5%).

8. Given the following recurrence relation, please pick the correct statements. (5%)

   (a) $T(n)= 3T(n/4)+\Theta(n^2)$, $T\in\Theta(n^2)$
   (b) $T(n)= 5T(n/2)+\Theta(n^2)$, $T\in\Theta(n^2)$
   (c) $T(n)= 2T(n/4)+\Theta(n)$, $T\in\Theta(n)$
   (d) $T(n)= 2T(n/4)+\Theta(n^2\log n)$, $T\in\Theta(n^2\log n)$

9. How can we make \$12.74 using 10-cent and 16-cent stamps? What is the smallest and largest number of stamps we could use? (10%)

10. Prove that $\sqrt{13}$ is irrational. (5%)

11. Let G = (V, T, S, P) be the phrase-structure grammar with V = {0, 1, A, S}, T = {0, 1} and the set of productions P consisting of $S \to 1S$, $S \to 10A$, $A \to 0A$, and $A \to 0$.

   (a) Show that 1111000 belongs to the language generated by G (5%)
   (b) What is the language generated by G (5%)

12. True or False (12%; 2% each):

(a) _____ Suppose you deal 52 regular playing cards into 13 piles of 4 cards each. You can always select one card from each pile to get one of each of the 13 card values Ace, 2, 3, . . . , 10, Jack, Queen, and King.
(b) _____ A graph has an Euler circuit if and only if the degree of every vertex is even.
(c) _____ Every tree is a bipartite graph.
(d) _____ If a graph has its degree sequence is (4, 4, 3, 3, 3, 2, 1), the number of edges in the graph 20.
(e) _____ The minimum number of bits to encode a source with symbols {A, B, C} and Prob(A)=0.5, Prob(B)=0.25, and Prob(C)=0.25 is 2 bits.
(f) _____ The statements $(P \vee Q) \to R$ and $(P \to R) \vee (Q \to R)$ are logically equivalent.