Notes: Continuous-time random matching (Duffie–Qiao–Sun 2025)

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January 9, 2026

Contents

1 Purpose and a quick map to the paper

These notes summarize the setup and construction strategy of:

> D. Duffie, L. Qiao, and Y. Sun (2025), *Continuous time random matching*, The Annals of Applied Probability 35(3), 1755–1790.

The goal is to understand (i) the continuous-time model objects, (ii) the deterministic ODE law for cross-sectional type distributions, and (iii) the probability-space/Fubini-extension construction used to make the model rigorous.

Where to look in DLQ (2025). In the extracted text ‘paper_text/aap_2025_dqs_extracted.txt‘:

- Model primitives and objects $(p_0, \eta, \theta, \varsigma)$, definition of $R(p)$, and ODE: around “Model and results” and equation (2).
- Formal model definition (random type process α and matching process φ, measurability, RCLL, etc.): **Definition 1**.
- Main existence/properties theorem: **Theorem 1**.
- Nonstandard machinery overview: **Section 4**.
- The “generalized Keisler” ingredient (Loeb transition probabilities and generalized Fubini): **Section 4.4 and Section 5**, with explicit reference to Duffie–Sun (2007) for the generalized Fubini theorem for Loeb transition probabilities.

2 Minimal background: CTMCs, generators, and intensities (as used in DLQ)

DLQ’s model is built so that each agent’s type evolves as a *continuous-time Markov chain* (CTMC) on a finite state space $S = \{1, \ldots, K\}$. We record the minimal definitions used in their statements.

Definition 2.1 (Generator / intensity matrix on a finite state space)**.** Let $S = \{1, \ldots, K\}$. A *generator* (or *intensity matrix*) is a matrix $R = (R_{kr})_{k,r\in S}$ such that

- $R_{kr} \geq 0$ for $k \neq r$,
- $R_{kk} = -\sum_{r\neq k} R_{kr}$ (row sums equal 0).

For a CTMC $X(t)$ on S with generator R, the instantaneous transition rate from state k to $r \neq k$ at time t is R_{kr}.

Remark 2.1 (Kolmogorov forward equation)**.** If $\pi(t)$ is the row vector of state probabilities $\pi_k(t) = \mathbb{P}(X(t) = k)$, then $\pi(t)$ solves the ODE

$$\frac{d}{dt}\pi(t) = \pi(t)R.$$

DLQ uses a version of this equation with a *state-dependent* generator $R(p_t)$ that depends on the cross-sectional distribution p_t.

Remark 2.2 (Counting processes and martingale characterization (informal))**.** DLQ encodes "$k \to r$ transitions occur at intensity $R_{kr}(p_t)$" via counting processes $C_{ikr}(t)$ and compensated processes

$$M_{ikr}(t) = C_{ikr}(t) - \int_0^t \mathbf{1}\{\alpha(i,s) = k\} R_{kr}(p_s)\, ds,$$

which are (local) martingales. You do not need deep martingale theory for the big picture; this is mainly a standard way to define/verify intensities.

3 DLQ (2025) model: primitives, processes, and the ODE

3.1 Agent space, sample space, and Fubini extension

DLQ assumes an atomless agent space $(I, \mathcal{I}, \lambda)$ with $\lambda(I) = 1$ and a probability space $(\Omega, \mathcal{F}, P)$ for randomness. Because joint measurability and independence are incompatible on the usual product, they work on a *Fubini extension*

$$(I \times \Omega,\ \mathcal{I} \boxtimes \mathcal{F},\ \lambda \boxtimes P),$$

in the sense of Sun (2006, Definition 2.2) (DLQ repeats the definition early in Section 2).

3.2 Time domain and measurability conventions

DLQ's time domain is $\mathbb{R}_+ = [0, \infty)$ equipped with its Borel σ-algebra $\mathcal{B}(\mathbb{R}_+)$. The model's primitive objects and solution objects are formulated on the *triple* product

$$I \times \Omega \times \mathbb{R}_+,$$

with measurability taken with respect to the product σ-algebra $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}(\mathbb{R}_+)$.

Remark 3.1 (Which measure do we use on $\mathbb{R}_+$?)**.** There are two distinct roles of time:

- **Measurability/topology:** for processes like $\alpha(i, \omega, t)$ and $\varphi(i, \omega, t)$ one only needs the Borel σ-algebra and path regularity (RCLL/càdlàg).
- **Integration over time:** whenever DLQ writes $\int_0^t (\cdots)\, ds$, the measure is the *Lebesgue measure* ds on $\mathbb{R}_+$.

Since Lebesgue measure on $\mathbb{R}_+$ is σ-finite, all time-integrals can be understood locally on each finite horizon $[0, T]$ and then extended by letting $T \uparrow \infty$. This is also the natural setup for DSNC: prove measurability/LLN/ODE claims on finite horizons and then take an increasing limit.

3.3 Primitives

Fix $K \geq 2$ and let $S = \{1, \ldots, K\}$ and let Δ be the simplex of probability measures on S. DLQ's primitives are:

- An initial cross-sectional type distribution $p_0 \in \Delta$.
- Mutation intensities $\eta_{kl} \in [0, \infty)$ for $k \neq l$.
- Matching intensities $\theta_{kl} : \Delta \to \mathbb{R}_+$ satisfying mass-balance

  $$p_k \theta_{kl}(p) = p_l \theta_{lk}(p),$$

  and a Lipschitz condition on $p_k \theta_{kl}(p)$ (DLQ uses this to ensure well-posedness of the ODE).
- Match-induced type-change distributions $\varsigma_{kl} \in \Delta$ (the new type distribution for a type-k agent matched to a type-l agent).

3.4 The induced generator $R(p)$ and the deterministic ODE

Define $R(p)$ (a generator on S) by combining mutation and match-induced type change:

$$R_{kr}(p) = \eta_{kr} + \sum_{l=1}^{K} \theta_{kl}(p)\, \varsigma_{kl}(r), \qquad k \neq r, \quad R_{kk}(p) = -\sum_{r \neq k} R_{kr}(p).$$

DLQ's key deterministic law is that the (expected, and in fact realized) cross-sectional type distribution solves

$$\frac{d}{dt} x(t) = x(t) R(x(t)).$$

4 Continuous-time matching objects (DLQ Definition 1) and main result (Theorem 1)

DLQ defines two coupled processes on $I \times \Omega \times \mathbb{R}_+$:

- a *type process* $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$ (RCLL in t),
- a *matching process* $\varphi : I \times \Omega \times \mathbb{R}_+ \to I$ (RCLL in t),

such that for each fixed (ω, t) the map $i \mapsto \varphi(i, \omega, t)$ is an involution (pairwise matching) and is measure-preserving in i for λ.

Remark 4.1 (Measurability level for the processes (as in DLQ Definition 1))**.** DLQ's Definition 1 imposes $(\mathcal{I} \boxtimes \mathcal{F}) \otimes \mathcal{B}(\mathbb{R}_+)$-measurability for α (and appropriate measurability for φ), and RCLL/càdlàg path regularity in t for almost every agent/sample point. For each fixed t, the section $\varphi_t(\cdot, \cdot) := \varphi(\cdot, \cdot, t)$ is a random matching on the Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$: for P-a.e. ω, the map $i \mapsto \varphi(i, \omega, t)$ is a measure-preserving involution.

Remark 4.2 (Interpretation of $\varphi(i, \omega, t)$)**.** DLQ's $\varphi(i, \omega, t) = j$ means: as of time t in state ω, agent j is the *last* partner met by i (and symmetrically i is the last partner met by j), with no intervening meetings since that last meeting time.

DLQ's **Theorem 1** asserts existence of such a system on some Fubini extension, and that:

- the realized cross-sectional distribution $p_t(\omega)$ is almost surely equal to its expectation and solves the ODE,
- the cross-sectional type process is a Markov chain with generator $R(\bar{p}_t)$,
- match counts aggregate deterministically at rate $\bar{p}_{t,k}\theta_{kl}(\bar{p}_t)$,
- there exist stationary distributions p^* satisfying $p^* R(p^*) = 0$.

5 How DLQ constructs the model (roadmap)

DLQ's existence proof is long because *continuous time* forces you to control the cumulative effect of small dependence across *many* periods. Here is a more detailed roadmap than the one-page overview typically given in summaries.

5.1 Layer 1: a finite-agent, small-step discrete-time model (DLQ Section 3; proofs in the supplement)

Fix a large integer M, a finite population of $\bar{M}$ agents, and a finite type space $S = \{1, \ldots, K\}$. DLQ constructs a discrete-time model on a fine grid with step size on the order of $1/M$. Within each "period" there are three sequential stages:

- **Mutation step:** each type-k agent mutates to type l with probability $O(1/M)$ so that mutation becomes a Poisson intensity η_{kl} in the limit.
- **Matching step:** conditional on the current cross-sectional distribution, each type-k agent meets a type-l partner with probability $O(1/M)$, scaled to approximate the intended matching intensity $\theta_{kl}(p)$.
- **Match-induced type change:** conditional on partner types (k, l), an agent's post-meeting type is drawn from ς_{kl}.

What must be proved in the finite model. Because matching is *without replacement* and repeats over many periods, partner draws are not independent across agents, and dependence can accumulate across time. DLQ therefore proves a collection of quantitative lemmas in Section 3, with proofs in the supplement:

- **Within-period matching accuracy:** the realized type-by-type matching frequencies are close to their targets (up to vanishing rounding errors).
- **Approximate conditional independence / approximate Markov property:** one-step conditional laws of finite-agent observables factorize up to small errors.
- **Cumulative error control across many steps:** Appendix A provides general lemmas for controlling the probability of multiple "rare" events over many steps; the supplement applies them to the model's jump counts and dependence terms.

This is the technical core: the errors must be small enough that when you later take a hyperfinite limit (infinitely many steps), they still wash out.

5.2 Layer 2: transfer to a hyperfinite time grid (DLQ Section 5)

Replace the large integer M by an *unlimited hyperinteger* and transfer the entire finite construction to obtain an *internal* model on:

- a hyperfinite agent set I with internal counting measure λ_0, and
- a hyperfinite time grid with infinitesimal step length $1/M$.

At this stage, DLQ has an internally well-defined discrete-time system whose one-step transition probabilities encode the intended continuous-time intensities, up to infinitesimal errors.

5.3 Layer 3: Loeb-ization + generalized Fubini + exact LLN (DLQ Section 4.4; DS07 Section 5; Sun 2006)

Convert the internal (hyperfinite) probability objects into standard ones via Loeb measures. The key technical ingredients are:

- **Generalized Fubini for Loeb transition probabilities (DLQ Section 4.4):** this yields a joint agent–sample measure even when the sample law depends on an agent/state via a kernel.
- **Infinite kernel products (generalized Ionescu–Tulcea) for Loeb transition probabilities (Duffie–Sun 2007, Section 5):** this constructs a path-space law from a sequence of kernels.
- **Exact LLN for stochastic processes on a Fubini extension (Sun 2006, Theorem 2.16):** once essential pairwise independence is obtained, cross-sectional paths become deterministic a.s., and the deterministic limit solves the ODE.

6 The "generalized Keisler" ingredient: Loeb transition probabilities

Continuous time (built from hyperfinite discrete time) naturally leads to an *iterated* randomization across periods. This is cleanly expressed using *transition probabilities* (kernels) rather than a single product measure. DLQ therefore uses the Loeb-transition-probability technology developed in Duffie–Sun (2007, Section 5).

6.1 Internal transition probabilities and their Loeb product (DLQ Section 4.4)

We keep this section at the level needed for DLQ, and refer to the static DS07 note for nonstandard basics (internal sets, Loeb measures, etc.).

Definition 6.1 (Internal transition probability)**.** Let $(I, \mathcal{I}_0, \lambda_0)$ be a hyperfinite internal probability space where $\mathcal{I}_0$ is the internal power set on a hyperfinite set I. Let Ω be a hyperfinite internal set with internal power set $\mathcal{F}_0$. An *internal transition probability* is an internal function P_0 that assigns to each $i \in I$ a hyperfinite internal probability measure P_{0i} on $(\Omega, \mathcal{F}_0)$.

Definition 6.2 (Internal product transition probability)**.** Given λ_0 and an internal transition probability $P_0 = \{P_{0i}\}_{i \in I}$, define an internal probability measure τ_0 on $(I \times \Omega,\ \mathcal{I}_0 \otimes \mathcal{F}_0)$ by

$$\tau_0(\{(i,\omega)\}) := \lambda_0(\{i\})\, P_{0i}(\{\omega\}).$$

Let λ be the Loeb measure associated with λ_0, let P_i be the Loeb measure associated with P_{0i}, and let τ be the Loeb measure associated with τ_0. The collection $P = \{P_i\}_{i \in I}$ is called a *Loeb transition probability*, and τ is the *Loeb product transition probability* of λ and P (DLQ's terminology). DLQ also denotes τ by $\lambda \boxtimes P$.

6.2 Generalized Fubini theorem (DLQ Proposition 4; proved in Duffie–Sun 2007)

Theorem 6.1 (Generalized Fubini for a Loeb transition probability (DLQ Proposition 4, informal))**.** *Let f be a real-valued τ-integrable function on $(I \times \Omega,\ \sigma(\mathcal{I}_0 \otimes \mathcal{F}_0),\ \tau)$. Then:*

1. *for λ-a.e. i, the section $f_i(\cdot) = f(i, \cdot)$ is $\sigma(\mathcal{F}_0)$-measurable and integrable on $(\Omega, \sigma(\mathcal{F}_0), P_i)$;*
2. *the function $i \mapsto \int_\Omega f_i(\omega)\, dP_i(\omega)$ is integrable on $(I, \sigma(\mathcal{I}_0), \lambda)$;*
3. *the Fubini identity holds:*

$$\int_{I \times \Omega} f(i,\omega)\, d\tau(i,\omega) = \int_I \left(\int_\Omega f(i,\omega)\, dP_i(\omega) \right) d\lambda(i).$$

Remark 6.1 (Why this is needed beyond Keisler (two-factor) Fubini)**.** In static matching, the sample measure on Ω does not depend on i, and Keisler's Loeb-product Fubini theorem suffices. In dynamic/hyperfinite-time constructions, the period-n randomization may depend on past states (hence on the current "state" variable), leading naturally to transition kernels. One then needs a Fubini theorem compatible with *kernel products*, and eventually an Ionescu–Tulcea type theorem to build the product/path measure.

Remark 6.2 (How this ensures existence of a suitable extension (big picture))**.** For a *single* kernel $P = \{P_i\}$, the generalized Fubini theorem above ensures that $\lambda \boxtimes P$ behaves like a "product" measure with respect to iterated integration, even though the second factor depends on the first.

For *dynamic* matching, one needs an *infinite* composition of kernels (one per randomization step) to construct a path-space probability measure. Duffie–Sun (2007, Section 5) proves a generalized Ionescu–Tulcea theorem for Loeb transition probabilities and shows that the resulting infinite product yields a Fubini extension on the joint agent–path space. This is the core existence machinery DLQ calls "generalized Keisler."

7 Connections to Duffie–Sun (2007) and Sun (2006)

DLQ (2025) is continuous-time, but the logic is consistent with the earlier discrete-time/static program:

- Duffie–Sun (2007) provides the Loeb-space construction for static/discrete-time matching and develops generalized Fubini/Ionescu–Tulcea type results for Loeb transition probabilities (their Section 5), which DLQ cites and builds on.
- Sun (2006) provides the Fubini extension framework and the exact LLN for a continuum of stochastic processes (Theorem 2.16), which DLQ invokes to pass from independent agent-level dynamics to deterministic cross-sectional dynamics.

8 What to read next in DLQ (2025)

For a first deep read focused on construction:

- **Section 2**: formal definition of the model objects and Fubini extension.
- **Section 3**: finite-agent discrete approximation (where the technical estimates live; many proofs are in the supplement).
- **Section 4.4 and Section 5**: Loeb transition probabilities and the hyperfinite-time-to-continuous-time construction (the "generalized Keisler" part).
- **Sun (2006), Theorem 2.16**: exact LLN for stochastic processes, used to deduce that cross-sectional paths are deterministic almost surely.