Notes: Fubini extensions for instantaneous matching (Loeb-space construction)

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1 Fubini extensions for instantaneous matching (Loeb-space construction)

1.1 Quick map to the source papers (Sun 2006; Duffie–Sun 2007)

This note is distilled primarily from:

- Y. Sun (2006), *The exact law of large numbers via Fubini extension and characterization of insurable risks* (JET 126), and
- D. Duffie and Y. Sun (2007), *Existence of independent random matching* (AAP 17).

To make it easy to find the original statements, we record the main correspondences:

- **Fubini extension definition**: this note's Definition 1.3 is **Sun (2006), Definition 2.2** (and also **Duffie–Sun (2007), Definition 2.1**).
- **Incompatibility (usual product measurability + independence)**: mentioned in §1 §1.3 is **Sun (2006), Proposition 2.1**.
- **Rich product probability space**: this note's informal "richness" definition in §1.9 corresponds to **Sun (2006), Definition 5.1**.
- **Universality via uniform r.v.'s**: the "uniform is enough" remark in §1.9 is backed by **Sun (2006), Proposition 5.3**.
- **Random full matching & independence-in-types**: the matching construction in §1.7 is modeled on **Duffie–Sun (2007), Definition 2.3** and **Theorem 2.4**.

1.2 Aim and scope

Many economic and probabilistic models posit a *continuum* of agents, indexed by an atomless probability space $(I, \mathcal{I}, \lambda)$, together with *instantaneous random matching* at a given time (or a single random matching draw), typically represented as a random involution $\varphi : I \times \Omega \to I$ satisfying $\varphi(\varphi(i, \omega), \omega) = i$.

The foundational difficulty is that, for nontrivial processes $(f_i)_{i \in I}$, *independence* of the family and *joint measurability* of the mapping $(i, \omega) \mapsto f_i(\omega)$ with respect to the usual product σ-algebra $\mathcal{I} \otimes \mathcal{F}$ are, in general, incompatible except in degenerate cases. This obstruction goes back at least to Doob and is made precise in the modern literature via results such as: if $f : I \times \Omega \to X$

is $\mathcal{I} \otimes \mathcal{F}$-measurable and essentially pairwise independent, then almost all coordinates f_i are essentially constant (**Sun (2006), Proposition 2.1**).

The goal of this note is to give a *minimalist, self-contained, rigorous* construction of a probability space rich enough to support:

- a continuum of (essentially) independent random objects with the *Fubini property*, and
- an *instantaneous* random matching map with the independence-in-types properties used in economics and genetics.

The construction is based on *nonstandard analysis* and *Loeb measures*, following the approach underlying Sun (2006) and Duffie–Sun (2007).

1.3 Measure-theoretic target: Fubini extensions

Definition 1.1 (Fubini extension)**.** Let $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ be probability spaces. A probability space

$$(I \times \Omega,\ \mathcal{W},\ Q)$$

is a *Fubini extension* of the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if:

1. $\mathcal{I} \otimes \mathcal{F} \subseteq \mathcal{W}$ and Q extends $\lambda \otimes P$; and
2. for every Q-integrable $g : I \times \Omega \to \mathbb{R}$, the sections $g_i(\omega) := g(i, \omega)$ and $g_\omega(i) := g(i, \omega)$ are integrable for λ-a.e. i and P-a.e. ω, and

$$\int g\, dQ = \int_I \Big(\int_\Omega g_i\, dP \Big)\, d\lambda = \int_\Omega \Big(\int_I g_\omega\, d\lambda \Big)\, dP.$$

This is exactly **Sun (2006), Definition 2.2**. The same definition is also used in **Duffie–Sun (2007), Definition 2.1**. Following Sun (2006), we often denote such a space by $(I \times \Omega,\ \mathcal{I} \boxtimes \mathcal{F},\ \lambda \boxtimes P)$ to emphasize the retention of Fubini's theorem beyond $\mathcal{I} \otimes \mathcal{F}$.

Remark 1.1 (Why we need an extension)**.** If we insist on $\mathcal{W} = \overline{\mathcal{I} \otimes \mathcal{F}}$ (the completion of the usual product) then, for many independence hypotheses that economists want (e.g. i.i.d. types, independent match outcomes), one cannot have a jointly measurable process unless it is essentially constant. A Fubini extension enlarges the measurable sets on $I \times \Omega$ so that one can have *joint measurability* of useful processes while retaining the ability to interchange integrals.

1.4 A minimalist introduction to nonstandard analysis

We give a concrete "toolkit" sufficient for Loeb measures and hyperfinite constructions.

1.4.1 Ultrafilters and ultrapowers (one standard construction)

Definition 1.2 (Filter on a set)**.** Let S be a nonempty set. A *filter* $\mathcal{F}$ on S is a nonempty collection of subsets of S such that:

1. $\emptyset \notin \mathcal{F}$;
2. if $A \in \mathcal{F}$ and $A \subseteq B \subseteq S$, then $B \in \mathcal{F}$ (upward closed);
3. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ (closed under finite intersections).

Remark 1.2 (Existence of free ultrafilters (via Zorn/AC))**.** There is no explicit "formula" example of a free ultrafilter on $\mathbb{N}$. Its existence is usually proved using the *Ultrafilter Lemma* (every proper filter extends to an ultrafilter), which follows from Zorn's lemma and hence from the Axiom of Choice. Concretely, start from the *Fréchet filter* of cofinite sets $\mathcal{F}_{\text{cof}} = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$, extend it to an ultrafilter $\mathcal{U} \supseteq \mathcal{F}_{\text{cof}}$, and then $\mathcal{U}$ must be *free* (non-principal) because it contains no finite sets. For the purposes of this note, the key point is simply: we may *fix* one free ultrafilter $\mathcal{U}$ and treat its elements as "$\mathcal{U}$-large" subsets of $\mathbb{N}$.

Let $\mathbb{N}$ be the natural numbers. A (free) *ultrafilter* $\mathcal{U}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ satisfying:

1. $\emptyset \notin \mathcal{U}$, and if $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$;
2. if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$;
3. for every $A \subseteq \mathbb{N}$, exactly one of A and $\mathbb{N} \setminus A$ lies in $\mathcal{U}$;
4. $\mathcal{U}$ is *free*: no finite subset of $\mathbb{N}$ is in $\mathcal{U}$.

Fix such a (free) ultrafilter $\mathcal{U}$. Intuitively, members of $\mathcal{U}$ are the subsets of $\mathbb{N}$ that we declare to be *$\mathcal{U}$-large* (a strengthened notion of "for almost all n").

The *ultrapower* ${}^*\mathbb{R}$ is defined as the quotient $\mathbb{R}^{\mathbb{N}}/\sim$, where for two real sequences (x_n) and (y_n) we declare

$$(x_n) \sim (y_n) \quad \Longleftrightarrow \quad \{n \in \mathbb{N} : x_n = y_n\} \in \mathcal{U}.$$

Note that this definition uses *one specific subset* of $\mathbb{N}$, namely the equality set $\{n : x_n = y_n\}$; it does *not* require any condition to hold for all sets in $\mathcal{U}$.

We write ${}^*x$ for the equivalence class of the constant sequence $(x, x, \dots)$. This gives an embedding $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$. Moreover, we can add and multiply equivalence classes by doing so coordinatewise on representative sequences, making ${}^*\mathbb{R}$ into a field containing $\mathbb{R}$ as a subfield (hence the phrase "identifying $\mathbb{R}$ with a subfield of ${}^*\mathbb{R}$").

1.4.2 Infinitesimals and unlimited hyperintegers

A hyperreal $\varepsilon \in {}^*\mathbb{R}$ is *infinitesimal* if $|\varepsilon| < 1/n$ for all $n \in \mathbb{N}$. A hyperinteger $N \in {}^*\mathbb{N}$ is *unlimited* if $N > n$ for all $n \in \mathbb{N}$. Existence of unlimited hyperintegers follows from the freeness of $\mathcal{U}$.

1.4.3 Standard part

Every finite hyperreal $x \in {}^*\mathbb{R}$ (i.e. $|x| < n$ for some $n \in \mathbb{N}$) is infinitely close to (i.e., their difference is infinitesimal) a unique real number, called its *standard part* and denoted $\mathrm{st}(x) \in \mathbb{R}$. See Appendix A for a complete existence-and-uniqueness proof (using the completeness of $\mathbb{R}$).

1.4.4 Internal sets and hyperfinite sets

The ultrapower construction applies not only to numbers but also to sets. We record a concrete "sequence-mod-$\mathcal{U}$" description that will be used later for Loeb measures and random matching.

Internal sets via sequences. Fix a base set S (e.g. $S = \mathbb{N}$ or $S = \mathbb{R}$). Consider sequences of subsets $(A_n)_{n\in\mathbb{N}}$ with $A_n \subseteq S$. Define an equivalence relation on such sequences by

$$(A_n) \approx (B_n) \quad \iff \quad \{n \in \mathbb{N} : \ A_n = B_n\} \in \mathcal{U}.$$

An *internal subset* of ${}^*S$ is, informally, an equivalence class $[(A_n)]$ of such a sequence. This is the set-analogue of defining hyperreals as equivalence classes of real sequences.

Membership is also "$\mathcal{U}$-almost sure". Elements of ${}^*S$ are themselves equivalence classes of sequences (s_n) with $s_n \in S$. If $x = [(s_n)] \in {}^*S$ and $A = [(A_n)]$ is an internal subset of ${}^*S$, then the intended meaning of

$$x \in A$$

is:

$$\{n \in \mathbb{N} : \ s_n \in A_n\} \in \mathcal{U}.$$

Thus internal sets are precisely those for which membership can be checked coordinatewise on representatives, "for $\mathcal{U}$-almost all n".

Hyperfinite sets. The special internal sets that behave like finite sets are called *hyperfinite*.

Definition 1.3 (Hyperfinite set)**.** A set H is *hyperfinite* if:

1. H is *internal*, and

2. there exists some $N \in {}^*\mathbb{N}$ and an *internal bijection*

$$f : \{1, 2, \dots, N\} \longrightarrow H.$$

Equivalently, H is internal and has an *internal cardinality* $|H| \in {}^*\mathbb{N}$.

The canonical hyperfinite initial segment $I = \{1, \dots, N\}$. Let $N = [(N_n)] \in {}^*\mathbb{N}$ be a (possibly unlimited) hyperinteger. Define a sequence of standard finite sets

$$I_n := \{1, 2, \dots, N_n\} \subseteq \mathbb{N}.$$

Then the internal set

$$I := \{1, 2, \dots, N\} \subset {}^*\mathbb{N}$$

can be represented concretely as the equivalence class $I = [(I_n)]$. Membership becomes: for $k = [(k_n)] \in {}^*\mathbb{N}$,

$$k \in I \quad \Longleftrightarrow \quad \{n : \ 1 \le k_n \le N_n\} \in \mathcal{U}.$$

Internally, I is finite-like: by transfer, $|\{1, \dots, N\}| = N$ in the internal sense, so I is hyperfinite with internal cardinality N. If N is unlimited, then externally (in ordinary set theory) I is not a finite set, but it retains enough finite combinatorics to support uniform counting, random matchings, etc., which will later be converted to standard measure-theoretic objects via Loeb measure.

1.4.5 Transfer principle (informal statement)

The *transfer principle* says that any first-order statement true of the standard structure (e.g. $\mathbb{R}$, $\mathbb{N}$) is also true for its nonstandard extension (e.g. ${}^*\mathbb{R}$, ${}^*\mathbb{N}$) when all objects are replaced by their star-images. In practice, this lets one use finite combinatorial/probabilistic reasoning on hyperfinite sets as if they were genuinely finite.

1.5 Loeb measures

1.5.1 Internal finitely additive measures

Let I be hyperfinite and let $\mathcal{I}_0$ be its *internal power set* (the family of internal subsets of I). Define the internal counting probability measure $\lambda_0 : \mathcal{I}_0 \to {}^*[0,1]$ by

$$\lambda_0(A) \coloneqq \frac{|A|}{|I|} = \frac{|A|}{N}.$$

This is internally countably additive (in the internal sense) and behaves like the uniform distribution on a finite set.

1.5.2 Loeb's theorem (construction of a standard probability space)

Theorem 1.1 (Loeb measure)**.** *From an internal probability space $(I, \mathcal{I}_0, \lambda_0)$ one can construct a* standard *probability space $(I, \mathcal{I}, \lambda)$, called its* Loeb space*, such that:*

- *$\mathcal{I}$ is a σ-algebra on the underlying set I containing $\mathcal{I}_0$;*
- *λ is a countably additive probability measure on $\mathcal{I}$; and*
- *for every $A \in \mathcal{I}_0$, one has $\lambda(A) = \mathrm{st}(\lambda_0(A))$.*

Remark 1.3 (Atomlessness)**.** If N is unlimited, then the Loeb counting measure λ is atomless: singleton sets have Loeb measure 0. Thus Loeb counting spaces provide canonical atomless probability spaces of agents.

1.6 Loeb products and the Fubini property (Keisler's theorem)

Let $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega, \mathcal{F}_0, P_0)$ be internal probability spaces with hyperfinite underlying sets. Their internal product space is $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$. Let $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ be the corresponding Loeb spaces.

Theorem 1.2 (Keisler's Fubini theorem for Loeb products, informal)**.** *The Loeb measure of the internal product measure, denoted $\lambda \boxtimes P$, makes*

$$(I \times \Omega,\ \mathcal{I} \boxtimes \mathcal{F},\ \lambda \boxtimes P)$$

into a Fubini extension *of the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. In particular, for every $\lambda \boxtimes P$-integrable function g that is $\mathcal{I} \boxtimes \mathcal{F}$-measurable, iterated integrals exist and satisfy the Fubini equalities.*

Remark 1.4 (What is extended?)**.** The key point is that $\mathcal{I} \boxtimes \mathcal{F}$ typically strictly contains $\mathcal{I} \otimes \mathcal{F}$. This is *exactly* what allows us to accommodate independent processes and random matchings that cannot live on the usual product σ-algebra.

1.7 Instantaneous (static) random matching on a continuum: hyperfinite construction

We now construct an *instantaneous* random full matching with strong measure-preserving properties and the independence-in-types feature used in applications. This follows the Loeb-space proof scheme used in **Duffie–Sun (2007, Section 4.1)**, which proves **Duffie–Sun (2007), Theorem 2.4**.

1.7.1 Step 1: hyperfinite agents

Fix an unlimited even hyperinteger $N \in {}^*\mathbb{N}$ and let $I \coloneqq \{1, 2, \dots, N\}$ with internal counting measure λ_0. Let $(I, \mathcal{I}, \lambda)$ be the Loeb space (atomless).

1.7.2 Step 2: hyperfinite sample space of perfect matchings

Let Ω be the internal set of all *perfect matchings* of I, meaning fixed-point-free involutions $\omega : I \to I$ satisfying $\omega(\omega(i)) = i$ and $\omega(i) \neq i$ for all i. (Equivalently, partitions of I into $N/2$ disjoint unordered pairs.) Endow Ω with internal counting probability P_0 (uniform over matchings), and let $(\Omega, \mathcal{F}, P)$ be its Loeb space.

1.7.3 Step 3: the matching map and measurability

Define $\varphi : I \times \Omega \to I$ by $\varphi(i, \omega) \coloneqq \omega(i)$. This map is internal, hence Loeb measurable with respect to the Loeb product σ-algebra $\mathcal{I} \boxtimes \mathcal{F}$. Moreover, for each fixed ω, the section $\varphi_\omega(\cdot) = \omega(\cdot)$ is a bijection of I preserving λ.

1.7.4 Step 4: measure-preservation and "no mass on individuals"

Because λ is atomless, for fixed $i \in I$ and fixed $j \in I$, one has $P(\varphi(i, \cdot) = j) = 0$. Intuitively, in the hyperfinite model,

$$P_0(\varphi(i, \cdot) = j) = \frac{1}{N - 1} \quad (j \neq i),$$

and taking standard parts yields 0. This formalizes the economic intuition that in a continuum population, the probability of matching with any *particular* counterparty is zero.

1.7.5 Step 5: independence in types (how it emerges)

Let $S = \{1, \dots, K\}$ be a finite type set, and let $a : I \to S$ be any (Loeb) measurable type function with type distribution $\rho \in \Delta(S)$ given by $\rho(k) = \lambda(\{i : a(i) = k\})$. Define the induced *partner-type* process $g : I \times \Omega \to S$ by

$$g(i, \omega) \coloneqq a(\varphi(i, \omega)).$$

In the terminology of **Duffie–Sun (2007)**, g is the "type process" induced by a random full matching (cf. **Definition 2.3**) and "independent in types" means that g is essentially pairwise independent (cf. **Definition 2.2** and **Definition 2.3(4)**). The crucial existence result is **Duffie–Sun (2007), Theorem 2.4**, which provides a random full matching that is independent in types (indeed, "universal" across finite type functions).

1.8 From Loeb products to Fubini extensions suitable for independent processes

The Loeb product construction above yields a probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ that:

- extends the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$;
- satisfies the Fubini property by Keisler's theorem; and
- supports a jointly measurable matching map φ.

This is precisely a *Fubini extension* in the sense of Definition 1.3.

1.9 Richness: hosting essentially i.i.d. families (Sun 2006)

For applications beyond matching (e.g. idiosyncratic shocks, heterogeneous Markov chains), we need the product space to be *rich* enough to support measurable processes with essentially pairwise independent coordinates and prescribed marginals.

Definition 1.4 (Rich product probability space (informal))**.** A Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is called *rich* if it supports an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process $U : I \times \Omega \to [0, 1]$ such that $(U_i)_{i \in I}$ is essentially pairwise independent and each U_i is uniformly distributed on $[0, 1]$.

Remark 1.5 (Why "uniform" is enough)**.** Sun formalizes richness as: existence of an essentially pairwise independent family of uniform $[0, 1]$ random variables on the Fubini extension (**Sun (2006), Definition 5.1**). The universality claim—constructing essentially pairwise independent processes with essentially arbitrary prescribed marginal laws on a Polish space—is **Sun (2006), Proposition 5.3**. Practically: given a uniform family, one can generate essentially pairwise independent families with *any prescribed distributions* by measurable transforms (inverse CDF method), provided the target distributions are Borel on a Polish space.

1.10 Putting it together for instantaneous matching

The Loeb-space construction gives a concrete recipe for building the probability space used for instantaneous matching:

1. Choose an unlimited hyperfinite even N and set $I = \{1, \ldots, N\}$; take its Loeb space $(I, \mathcal{I}, \lambda)$ as the agent space.

2. Let Ω be the hyperfinite set of perfect matchings of I, with uniform internal counting measure, and take its Loeb space $(\Omega, \mathcal{F}, P)$ as the sample space.

3. Form the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, which is a Fubini extension.

4. Define the matching map $\varphi(i, \omega) = \omega(i)$ (jointly measurable on the extension) and, for any type function a, define partner-type process $g(i, \omega) = a(\varphi(i, \omega))$.

5. Use the essential pairwise independence properties of g (Duffie–Sun, 2007) and the exact law of large numbers on a Fubini extension (Sun, 2006) to deduce deterministic aggregation results (e.g. deterministic cross-sectional partner-type frequencies).

1.11 Bibliographic notes (minimal)

The definition and systematic use of Fubini extensions and exact law of large numbers are developed in:

> Y. Sun, *The exact law of large numbers via Fubini extension and characterization of insurable risks*, Journal of Economic Theory 126 (2006), 31–69.

The Loeb-space (nonstandard) existence construction of independent random matching in a continuum population is developed in:

> D. Duffie and Y. Sun, *Existence of independent random matching*, Annals of Applied Probability 17 (2007), 386–419.

A Existence and uniqueness of the standard part

In the main text we used the fact that every *finite* hyperreal is infinitely close to a unique real number. For completeness, we record a standard proof based on the least-upper-bound property of $\mathbb{R}$.

Theorem A.1 (Standard part of a finite hyperreal). *Let $x \in {}^*\mathbb{R}$ be finite, i.e. $|x| < n$ for some $n \in \mathbb{N}$. Then there exists a unique $r \in \mathbb{R}$ such that $x - {}^*r$ is infinitesimal. We denote this real number by $\mathrm{st}(x) = r$.*

Proof. Define
$$A := \{q \in \mathbb{Q} : \ {}^*q < x\}.$$
We first check that A is nonempty and bounded above in $\mathbb{R}$. Since $|x| < n$ for some $n \in \mathbb{N}$, we have ${}^*(-n) < x$, hence $-n \in A$, so $A \neq \emptyset$. Also ${}^*q < x$ implies $q < n$, so $A \subset (-\infty, n)$, hence A is bounded above. By completeness of $\mathbb{R}$, the supremum
$$r := \sup A \in \mathbb{R}$$
exists.

Claim: $x \approx {}^*r$, i.e. $|x - {}^*r| < 1/m$ for every $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$.

(i) Show $x < {}^*(r + 1/m)$. If instead $x \geq {}^*(r + 1/m)$, choose a rational q with $r < q < r + 1/m$ (density of $\mathbb{Q}$). Then ${}^*q \leq x$, so $q \in A$, contradicting $q > r = \sup A$. Hence $x < {}^*(r + 1/m)$, equivalently $x - {}^*r < 1/m$.

(ii) Show $x > {}^*(r - 1/m)$. If instead $x \leq {}^*(r - 1/m)$, then for every rational $q < r - 1/m$ we have ${}^*q < x$, so $q \in A$. This would force $\sup A \leq r - 1/m$, contradicting $\sup A = r$. Hence $x > {}^*(r - 1/m)$, equivalently ${}^*r - x < 1/m$.

Combining (i) and (ii) gives $|x - {}^*r| < 1/m$. Since m was arbitrary, $x - {}^*r$ is infinitesimal, proving existence.

Uniqueness. If $x \approx {}^*r$ and $x \approx {}^*s$ for $r, s \in \mathbb{R}$, then ${}^*(r-s) = ({}^*r - x) + (x - {}^*s)$ is infinitesimal. But $r - s$ is a real number, and the only real infinitesimal is 0, so $r = s$. $\square$