

# Notes: Fubini extensions for instantaneous matching (Loeb-space construction)

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# 1 Fubini extensions for instantaneous matching (Loeb-space construction)

## 1.1 Quick map to the source papers (Sun 2006; Duffie–Sun 2007)

This note is distilled primarily from:

- Y. Sun (2006), *The exact law of large numbers via Fubini extension and characterization of insurable risks* (JET 126), and
- D. Duffie and Y. Sun (2007), *Existence of independent random matching* (AAP 17).

To make it easy to find the original statements, we record the main correspondences:

- **Fubini extension definition:** this note’s Definition 1.3 is **Sun (2006), Definition 2.2** (and also **Duffie–Sun (2007), Definition 2.1**).
- **Incompatibility (usual product measurability + independence):** mentioned in §1 §1.3 is **Sun (2006), Proposition 2.1**.
- **Rich product probability space:** this note’s informal “richness” definition in §1.9 corresponds to **Sun (2006), Definition 5.1**.
- **Universality via uniform r.v.’s:** the “uniform is enough” remark in §1.9 is backed by **Sun (2006), Proposition 5.3**.
- **Random full matching & independence-in-types:** the matching construction in §1.7 is modeled on **Duffie–Sun (2007), Definition 2.3** and **Theorem 2.4**.

## 1.2 Aim and scope

Many economic and probabilistic models posit a *continuum* of agents, indexed by an atomless probability space  $(I, \mathcal{I}, \lambda)$ , together with *instantaneous random matching* at a given time (or a single random matching draw), typically represented as a random involution  $\varphi : I \times \Omega \rightarrow I$  satisfying  $\varphi(\varphi(i, \omega), \omega) = i$ .

The foundational difficulty is that, for nontrivial processes  $(f_i)_{i \in I}$ , *independence* of the family and *joint measurability* of the mapping  $(i, \omega) \mapsto f_i(\omega)$  with respect to the usual product  $\sigma$ -algebra  $\mathcal{I} \otimes \mathcal{F}$  are, in general, incompatible except in degenerate cases. This obstruction goes back at least to Doob and is made precise in the modern literature via results such as: if  $f : I \times \Omega \rightarrow X$

is  $\mathcal{I} \otimes \mathcal{F}$ -measurable and essentially pairwise independent, then almost all coordinates  $f_i$  are essentially constant (**Sun (2006), Proposition 2.1**).

The goal of this note is to give a *minimalist, self-contained, rigorous* construction of a probability space rich enough to support:

- a continuum of (essentially) independent random objects with the *Fubini property*, and
- an *instantaneous* random matching map with the independence-in-types properties used in economics and genetics.

The construction is based on *nonstandard analysis* and *Loeb measures*, following the approach underlying Sun (2006) and Duffie–Sun (2007).

### 1.3 Measure-theoretic target: Fubini extensions

**Definition 1.1** (Fubini extension). Let  $(I, \mathcal{I}, \lambda)$  and  $(\Omega, \mathcal{F}, P)$  be probability spaces. A probability space

$$(I \times \Omega, \mathcal{W}, Q)$$

is a *Fubini extension* of the usual product  $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$  if:

1.  $\mathcal{I} \otimes \mathcal{F} \subseteq \mathcal{W}$  and  $Q$  extends  $\lambda \otimes P$ ; and
2. for every  $Q$ -integrable  $g : I \times \Omega \rightarrow \mathbb{R}$ , the sections  $g_i(\omega) := g(i, \omega)$  and  $g_\omega(i) := g(i, \omega)$  are integrable for  $\lambda$ -a.e.  $i$  and  $P$ -a.e.  $\omega$ , and

$$\int g dQ = \int_I \left( \int_\Omega g_i dP \right) d\lambda = \int_\Omega \left( \int_I g_\omega d\lambda \right) dP.$$

This is exactly **Sun (2006), Definition 2.2**. The same definition is also used in **Duffie–Sun (2007), Definition 2.1**. Following Sun (2006), we often denote such a space by  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$  to emphasize the retention of Fubini’s theorem beyond  $\mathcal{I} \otimes \mathcal{F}$ .

**Remark 1.1** (Why we need an extension). If we insist on  $\mathcal{W} = \overline{\mathcal{I} \otimes \mathcal{F}}$  (the completion of the usual product) then, for many independence hypotheses that economists want (e.g. i.i.d. types, independent match outcomes), one cannot have a jointly measurable process unless it is essentially constant. A Fubini extension enlarges the measurable sets on  $I \times \Omega$  so that one can have *joint measurability* of useful processes while retaining the ability to interchange integrals.

## 1.4 A minimalist introduction to nonstandard analysis

We give a concrete “toolkit” sufficient for Loeb measures and hyperfinite constructions.

### 1.4.1 Ultrafilters and ultrapowers (one standard construction)

**Definition 1.2** (Filter on a set). Let  $S$  be a nonempty set. A *filter*  $\mathcal{F}$  on  $S$  is a nonempty collection of subsets of  $S$  such that:

1.  $\emptyset \notin \mathcal{F}$ ;
2. if  $A \in \mathcal{F}$  and  $A \subseteq B \subseteq S$ , then  $B \in \mathcal{F}$  (upward closed);
3. if  $A, B \in \mathcal{F}$ , then  $A \cap B \in \mathcal{F}$  (closed under finite intersections).

**Remark 1.2** (Existence of free ultrafilters (via Zorn/AC)). There is no explicit “formula” example of a free ultrafilter on  $\mathbb{N}$ . Its existence is usually proved using the *Ultrafilter Lemma* (every proper filter extends to an ultrafilter), which follows from Zorn’s lemma and hence from the Axiom of Choice. Concretely, start from the *Fréchet filter* of cofinite sets  $\mathcal{F}_{\text{cof}} = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$ , extend it to an ultrafilter  $\mathcal{U} \supseteq \mathcal{F}_{\text{cof}}$ , and then  $\mathcal{U}$  must be *free* (non-principal) because it contains no finite sets. For the purposes of this note, the key point is simply: we may *fix* one free ultrafilter  $\mathcal{U}$  and treat its elements as “ $\mathcal{U}$ -large” subsets of  $\mathbb{N}$ .

Let  $\mathbb{N}$  be the natural numbers. A (free) *ultrafilter*  $\mathcal{U}$  on  $\mathbb{N}$  is a collection of subsets of  $\mathbb{N}$  satisfying:

1.  $\emptyset \notin \mathcal{U}$ , and if  $A \in \mathcal{U}$  and  $A \subseteq B$ , then  $B \in \mathcal{U}$ ;
2. if  $A, B \in \mathcal{U}$ , then  $A \cap B \in \mathcal{U}$ ;
3. for every  $A \subseteq \mathbb{N}$ , exactly one of  $A$  and  $\mathbb{N} \setminus A$  lies in  $\mathcal{U}$ ;
4.  $\mathcal{U}$  is *free*: no finite subset of  $\mathbb{N}$  is in  $\mathcal{U}$ .

Fix such a (free) ultrafilter  $\mathcal{U}$ . Intuitively, members of  $\mathcal{U}$  are the subsets of  $\mathbb{N}$  that we declare to be  *$\mathcal{U}$ -large* (a strengthened notion of “for almost all  $n$ ”).

The *ultrapower*  ${}^*\mathbb{R}$  is defined as the quotient  $\mathbb{R}^{\mathbb{N}} / \sim$ , where for two real sequences  $(x_n)$  and  $(y_n)$  we declare

$$(x_n) \sim (y_n) \iff \{n \in \mathbb{N} : x_n = y_n\} \in \mathcal{U}.$$

Note that this definition uses *one specific subset* of  $\mathbb{N}$ , namely the equality set  $\{n : x_n = y_n\}$ ; it does *not* require any condition to hold for all sets in  $\mathcal{U}$ .

We write  ${}^*x$  for the equivalence class of the constant sequence  $(x, x, \dots)$ . This gives an embedding  $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$ . Moreover, we can add and multiply equivalence classes by doing so coordinatewise on representative sequences, making  ${}^*\mathbb{R}$  into a field containing  $\mathbb{R}$  as a subfield (hence the phrase “identifying  $\mathbb{R}$  with a subfield of  ${}^*\mathbb{R}$ ”).

#### 1.4.2 Infinitesimals and unlimited hyperintegers

A hyperreal  $\varepsilon \in {}^*\mathbb{R}$  is *infinitesimal* if  $|\varepsilon| < 1/n$  for all  $n \in \mathbb{N}$ . A hyperinteger  $N \in {}^*\mathbb{N}$  is *unlimited* if  $N > n$  for all  $n \in \mathbb{N}$ . Existence of unlimited hyperintegers follows from the freeness of  $\mathcal{U}$ .

#### 1.4.3 Standard part

Every finite hyperreal  $x \in {}^*\mathbb{R}$  (i.e.  $|x| < n$  for some  $n \in \mathbb{N}$ ) is infinitely close to (i.e., their difference is infinitesimal) a unique real number, called its *standard part* and denoted  $\text{st}(x) \in \mathbb{R}$ . See Appendix A for a complete existence-and-uniqueness proof (using the completeness of  $\mathbb{R}$ ).

#### 1.4.4 Internal sets and hyperfinite sets

The ultrapower construction applies not only to numbers but also to sets. We record a concrete “sequence-mod- $\mathcal{U}$ ” description that will be used later for Loeb measures and random matching.

**Internal sets via sequences.** Fix a base set  $S$  (e.g.  $S = \mathbb{N}$  or  $S = \mathbb{R}$ ). Consider sequences of subsets  $(A_n)_{n \in \mathbb{N}}$  with  $A_n \subseteq S$ . Define an equivalence relation on such sequences by

$$(A_n) \approx (B_n) \iff \{n \in \mathbb{N} : A_n = B_n\} \in \mathcal{U}.$$

An *internal subset* of  ${}^*S$  is, informally, an equivalence class  $[(A_n)]$  of such a sequence. This is the set-analogue of defining hyperreals as equivalence classes of real sequences.

**Membership is also “ $\mathcal{U}$ -almost sure”.** Elements of  ${}^*S$  are themselves equivalence classes of sequences  $(s_n)$  with  $s_n \in S$ . If  $x = [(s_n)] \in {}^*S$  and  $A = [(A_n)]$  is an internal subset of  ${}^*S$ , then the intended meaning of

$$x \in A$$

is:

$$\{n \in \mathbb{N} : s_n \in A_n\} \in \mathcal{U}.$$

Thus internal sets are precisely those for which membership can be checked coordinatewise on representatives, “for  $\mathcal{U}$ -almost all  $n$ ”.

**Hyperfinite sets.** The special internal sets that behave like finite sets are called *hyperfinite*.

**Definition 1.3** (Hyperfinite set). A set  $H$  is *hyperfinite* if:

1.  $H$  is *internal*, and
2. there exists some  $N \in {}^*\mathbb{N}$  and an *internal bijection*

$$f : \{1, 2, \dots, N\} \longrightarrow H.$$

Equivalently,  $H$  is internal and has an *internal cardinality*  $|H| \in {}^*\mathbb{N}$ .

**The canonical hyperfinite initial segment**  $I = \{1, \dots, N\}$ . Let  $N = [(N_n)] \in {}^*\mathbb{N}$  be a (possibly unlimited) hyperinteger. Define a sequence of standard finite sets

$$I_n := \{1, 2, \dots, N_n\} \subseteq \mathbb{N}.$$

Then the internal set

$$I := \{1, 2, \dots, N\} \subset {}^*\mathbb{N}$$

can be represented concretely as the equivalence class  $I = [(I_n)]$ . Membership becomes: for  $k = [(k_n)] \in {}^*\mathbb{N}$ ,

$$k \in I \iff \{n : 1 \leq k_n \leq N_n\} \in \mathcal{U}.$$

Internally,  $I$  is finite-like: by transfer,  $|\{1, \dots, N\}| = N$  in the internal sense, so  $I$  is hyperfinite with internal cardinality  $N$ . If  $N$  is unlimited, then externally (in ordinary set theory)  $I$  is not a finite set, but it retains enough finite combinatorics to support uniform counting, random matchings, etc., which will later be converted to standard measure-theoretic objects via Loeb measure.

#### 1.4.5 Transfer principle (informal statement)

The *transfer principle* says that any first-order statement true of the standard structure (e.g.  $\mathbb{R}, \mathbb{N}$ ) is also true for its nonstandard extension (e.g.  ${}^*\mathbb{R}, {}^*\mathbb{N}$ ) when all objects are replaced by their star-images. In practice, this lets one use finite combinatorial/probabilistic reasoning on hyperfinite sets as if they were genuinely finite.

## 1.5 Loeb measures

### 1.5.1 Internal finitely additive measures

Let  $I$  be hyperfinite and let  $\mathcal{I}_0$  be its *internal power set* (the family of internal subsets of  $I$ ). Define the internal counting probability measure  $\lambda_0 : \mathcal{I}_0 \rightarrow {}^*[0, 1]$  by

$$\lambda_0(A) := \frac{|A|}{|I|} = \frac{|A|}{N}.$$

This is internally countably additive (in the internal sense) and behaves like the uniform distribution on a finite set.

### 1.5.2 Loeb's theorem (construction of a standard probability space)

**Theorem 1.1** (Loeb measure). *From an internal probability space  $(I, \mathcal{I}_0, \lambda_0)$  one can construct a standard probability space  $(I, \mathcal{I}, \lambda)$ , called its Loeb space, such that:*

- $\mathcal{I}$  is a  $\sigma$ -algebra on the underlying set  $I$  containing  $\mathcal{I}_0$ ;
- $\lambda$  is a countably additive probability measure on  $\mathcal{I}$ ; and
- for every  $A \in \mathcal{I}_0$ , one has  $\lambda(A) = \text{st}(\lambda_0(A))$ .

**Remark 1.3** (Atomlessness). If  $N$  is unlimited, then the Loeb counting measure  $\lambda$  is atomless: singleton sets have Loeb measure 0. Thus Loeb counting spaces provide canonical atomless probability spaces of agents.

## 1.6 Loeb products and the Fubini property (Keisler's theorem)

Let  $(I, \mathcal{I}_0, \lambda_0)$  and  $(\Omega, \mathcal{F}_0, P_0)$  be internal probability spaces with hyperfinite underlying sets. Their internal product space is  $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$ . Let  $(I, \mathcal{I}, \lambda)$  and  $(\Omega, \mathcal{F}, P)$  be the corresponding Loeb spaces.

**Theorem 1.2** (Keisler's Fubini theorem for Loeb products, informal). *The Loeb measure of the internal product measure, denoted  $\lambda \boxtimes P$ , makes*

$$(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$$

*into a Fubini extension of the usual product  $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ . In particular, for every  $\lambda \boxtimes P$ -integrable function  $g$  that is  $\mathcal{I} \boxtimes \mathcal{F}$ -measurable, iterated integrals exist and satisfy the Fubini equalities.*



**Remark 1.4** (What is extended?). The key point is that  $\mathcal{I} \boxtimes \mathcal{F}$  typically strictly contains  $\mathcal{I} \otimes \mathcal{F}$ . This is *exactly* what allows us to accommodate independent processes and random matchings that cannot live on the usual product  $\sigma$ -algebra.

## 1.7 Instantaneous (static) random matching on a continuum: hyperfinite construction

We now construct an *instantaneous* random full matching with strong measure-preserving properties and the independence-in-types feature used in applications. This follows the Loeb-space proof scheme used in **Duffie–Sun (2007, Section 4.1)**, which proves **Duffie–Sun (2007), Theorem 2.4**.

### 1.7.1 Step 1: hyperfinite agents

Fix an unlimited even hyperinteger  $N \in {}^*\mathbb{N}$  and let  $I := \{1, 2, \dots, N\}$  with internal counting measure  $\lambda_0$ . Let  $(I, \mathcal{I}, \lambda)$  be the Loeb space (atomless).

### 1.7.2 Step 2: hyperfinite sample space of perfect matchings

Let  $\Omega$  be the internal set of all *perfect matchings* of  $I$ , meaning fixed-point-free involutions  $\omega : I \rightarrow I$  satisfying  $\omega(\omega(i)) = i$  and  $\omega(i) \neq i$  for all  $i$ . (Equivalently, partitions of  $I$  into  $N/2$  disjoint unordered pairs.) Endow  $\Omega$  with internal counting probability  $P_0$  (uniform over matchings), and let  $(\Omega, \mathcal{F}, P)$  be its Loeb space.

### 1.7.3 Step 3: the matching map and measurability

Define  $\varphi : I \times \Omega \rightarrow I$  by  $\varphi(i, \omega) := \omega(i)$ . This map is internal, hence Loeb measurable with respect to the Loeb product  $\sigma$ -algebra  $\mathcal{I} \boxtimes \mathcal{F}$ . Moreover, for each fixed  $\omega$ , the section  $\varphi_\omega(\cdot) = \omega(\cdot)$  is a bijection of  $I$  preserving  $\lambda$ .

### 1.7.4 Step 4: measure-preservation and “no mass on individuals”

Because  $\lambda$  is atomless, for fixed  $i \in I$  and fixed  $j \in I$ , one has  $P(\varphi(i, \cdot) = j) = 0$ . Intuitively, in the hyperfinite model,

$$P_0(\varphi(i, \cdot) = j) = \frac{1}{N-1} \quad (j \neq i),$$

and taking standard parts yields 0. This formalizes the economic intuition that in a continuum population, the probability of matching with any *particular* counterparty is zero.

### 1.7.5 Step 5: independence in types (how it emerges)

Let  $S = \{1, \dots, K\}$  be a finite type set, and let  $a : I \rightarrow S$  be any (Loeb) measurable type function with type distribution  $\rho \in \Delta(S)$  given by  $\rho(k) = \lambda(\{i : a(i) = k\})$ . Define the induced *partner-type* process  $g : I \times \Omega \rightarrow S$  by

$$g(i, \omega) := a(\varphi(i, \omega)).$$

In the terminology of **Duffie–Sun (2007)**,  $g$  is the “type process” induced by a random full matching (cf. **Definition 2.3**) and “independent in types” means that  $g$  is essentially pairwise independent (cf. **Definition 2.2** and **Definition 2.3(4)**). The crucial existence result is **Duffie–Sun (2007), Theorem 2.4**, which provides a random full matching that is independent in types (indeed, “universal” across finite type functions).

## 1.8 From Loeb products to Fubini extensions suitable for independent processes

The Loeb product construction above yields a probability space  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$  that:

- extends the usual product  $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ ;
- satisfies the Fubini property by Keisler’s theorem; and
- supports a jointly measurable matching map  $\varphi$ .

This is precisely a *Fubini extension* in the sense of Definition 1.3.

## 1.9 Richness: hosting essentially i.i.d. families (Sun 2006)

For applications beyond matching (e.g. idiosyncratic shocks, heterogeneous Markov chains), we need the product space to be *rich* enough to support measurable processes with essentially pairwise independent coordinates and prescribed marginals.

**Definition 1.4** (Rich product probability space (informal)). A Fubini extension  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$  is called *rich* if it supports an  $\mathcal{I} \boxtimes \mathcal{F}$ -measurable process  $U : I \times \Omega \rightarrow [0, 1]$  such that  $(U_i)_{i \in I}$  is essentially pairwise independent and each  $U_i$  is uniformly distributed on  $[0, 1]$ .

**Remark 1.5** (Why “uniform” is enough). Sun formalizes richness as: existence of an essentially pairwise independent family of uniform  $[0, 1]$  random

variables on the Fubini extension (**Sun (2006), Definition 5.1**). The universality claim—constructing essentially pairwise independent processes with essentially arbitrary prescribed marginal laws on a Polish space—is **Sun (2006), Proposition 5.3**. Practically: given a uniform family, one can generate essentially pairwise independent families with *any prescribed distributions* by measurable transforms (inverse CDF method), provided the target distributions are Borel on a Polish space.

### 1.10 Putting it together for instantaneous matching

The Loeb-space construction gives a concrete recipe for building the probability space used for instantaneous matching:

1. Choose an unlimited hyperfinite even  $N$  and set  $I = \{1, \dots, N\}$ ; take its Loeb space  $(I, \mathcal{I}, \lambda)$  as the agent space.
2. Let  $\Omega$  be the hyperfinite set of perfect matchings of  $I$ , with uniform internal counting measure, and take its Loeb space  $(\Omega, \mathcal{F}, P)$  as the sample space.
3. Form the Loeb product space  $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ , which is a Fubini extension.
4. Define the matching map  $\varphi(i, \omega) = \omega(i)$  (jointly measurable on the extension) and, for any type function  $a$ , define partner-type process  $g(i, \omega) = a(\varphi(i, \omega))$ .
5. Use the essential pairwise independence properties of  $g$  (Duffie–Sun, 2007) and the exact law of large numbers on a Fubini extension (Sun, 2006) to deduce deterministic aggregation results (e.g. deterministic cross-sectional partner-type frequencies).

### 1.11 Bibliographic notes (minimal)

The definition and systematic use of Fubini extensions and exact law of large numbers are developed in:

Y. Sun, *The exact law of large numbers via Fubini extension and characterization of insurable risks*, Journal of Economic Theory 126 (2006), 31–69.

The Loeb-space (nonstandard) existence construction of independent random matching in a continuum population is developed in:

D. Duffie and Y. Sun, *Existence of independent random matching*, Annals of Applied Probability 17 (2007), 386–419.

## A Existence and uniqueness of the standard part

In the main text we used the fact that every *finite* hyperreal is infinitely close to a unique real number. For completeness, we record a standard proof based on the least-upper-bound property of  $\mathbb{R}$ .

**Theorem A.1** (Standard part of a finite hyperreal). *Let  $x \in {}^*\mathbb{R}$  be finite, i.e.  $|x| < n$  for some  $n \in \mathbb{N}$ . Then there exists a unique  $r \in \mathbb{R}$  such that  $x - {}^*r$  is infinitesimal. We denote this real number by  $\text{st}(x) = r$ .*

*Proof.* Define

$$A := \{q \in \mathbb{Q} : {}^*q < x\}.$$

We first check that  $A$  is nonempty and bounded above in  $\mathbb{R}$ . Since  $|x| < n$  for some  $n \in \mathbb{N}$ , we have  ${}^*(-n) < x$ , hence  $-n \in A$ , so  $A \neq \emptyset$ . Also  ${}^*q < x$  implies  $q < n$ , so  $A \subset (-\infty, n)$ , hence  $A$  is bounded above. By completeness of  $\mathbb{R}$ , the supremum

$$r := \sup A \in \mathbb{R}$$

exists.

*Claim:*  $x \approx {}^*r$ , i.e.  $|x - {}^*r| < 1/m$  for every  $m \in \mathbb{N}$ . Fix  $m \in \mathbb{N}$ .

(i) *Show  $x < {}^*(r + 1/m)$ .* If instead  $x \geq {}^*(r + 1/m)$ , choose a rational  $q$  with  $r < q < r + 1/m$  (density of  $\mathbb{Q}$ ). Then  ${}^*q \leq x$ , so  $q \in A$ , contradicting  $q > r = \sup A$ . Hence  $x < {}^*(r + 1/m)$ , equivalently  $x - {}^*r < 1/m$ .

(ii) *Show  $x > {}^*(r - 1/m)$ .* If instead  $x \leq {}^*(r - 1/m)$ , then for every rational  $q < r - 1/m$  we have  ${}^*q < x$ , so  $q \in A$ . This would force  $\sup A \leq r - 1/m$ , contradicting  $\sup A = r$ . Hence  $x > {}^*(r - 1/m)$ , equivalently  ${}^*r - x < 1/m$ .

Combining (i) and (ii) gives  $|x - {}^*r| < 1/m$ . Since  $m$  was arbitrary,  $x - {}^*r$  is infinitesimal, proving existence.

*Uniqueness.* If  $x \approx {}^*r$  and  $x \approx {}^*s$  for  $r, s \in \mathbb{R}$ , then  ${}^*(r - s) = ({}^*r - x) + (x - {}^*s)$  is infinitesimal. But  $r - s$  is a real number, and the only real infinitesimal is 0, so  $r = s$ .  $\square$