Notes: Fubini extensions for instantaneous matching (Loeb-space construction)

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1 Fubini extensions for instantaneous matching (Loeb-space construction)

1.1 Quick map to the source papers (Sun 2006; Duffie–Sun 2007)

This note is distilled primarily from:

- Y. Sun (2006), *The exact law of large numbers via Fubini extension and characterization of insurable risks* (JET 126), and
- D. Duffie and Y. Sun (2007), *Existence of independent random matching* (AAP 17).

To make it easy to find the original statements, we record the main correspondences:

- **Fubini extension definition**: this note's Definition 1.3 is **Sun (2006), Definition 2.2** (and also **Duffie–Sun (2007), Definition 2.1**).
- **Incompatibility (usual product measurability + independence)**: mentioned in §1 §1.3 is **Sun (2006), Proposition 2.1**.
- **Rich product probability space**: this note's informal "richness" definition in §1.9 corresponds to **Sun (2006), Definition 5.1**.
- **Universality via uniform r.v.'s**: the "uniform is enough" remark in §1.9 is backed by **Sun (2006), Proposition 5.3**.
- **Random full matching & independence-in-types**: the matching construction in §1.7 is modeled on **Duffie–Sun (2007), Definition 2.3** and **Theorem 2.4**.

1.2 Aim and scope

Many economic and probabilistic models posit a *continuum* of agents, indexed by an atomless probability space $(I, \mathcal{I}, \lambda)$, together with *instantaneous random matching* at a given time (or a single random matching draw), typically represented as a random involution $\varphi : I \times \Omega \to I$ satisfying $\varphi(\varphi(i, \omega), \omega) = i$.

The foundational difficulty is that, for nontrivial processes $(f_i)_{i \in I}$, *independence* of the family and *joint measurability* of the mapping $(i, \omega) \mapsto f_i(\omega)$ with respect to the usual product σ-algebra $\mathcal{I} \otimes \mathcal{F}$ are, in general, incompatible except in degenerate cases. This obstruction goes back at least to Doob and is made precise in the modern literature via results such as: if $f : I \times \Omega \to X$ is $\mathcal{I} \otimes \mathcal{F}$-measurable and essentially pairwise independent, then almost all coordinates f_i are essentially constant (**Sun (2006), Proposition 2.1**).

The goal of this note is to give a *minimalist, self-contained, rigorous* construction of a probability space rich enough to support:

- a continuum of (essentially) independent random objects with the *Fubini property*, and
- an *instantaneous* random matching map with the independence-in-types properties used in economics and genetics.

The construction is based on *nonstandard analysis* and *Loeb measures*, following the approach underlying Sun (2006) and Duffie–Sun (2007).

1.3 Measure-theoretic target: Fubini extensions

Definition 1.1 (Fubini extension)**.** Let $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ be probability spaces. A probability space

$$(I \times \Omega,\ \mathcal{W},\ Q)$$

is a *Fubini extension* of the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if:

1. $\mathcal{I} \otimes \mathcal{F} \subseteq \mathcal{W}$ and Q extends $\lambda \otimes P$; and
2. for every Q-integrable $g : I \times \Omega \to \mathbb{R}$, the sections $g_i(\omega) := g(i, \omega)$ and $g_\omega(i) := g(i, \omega)$ are integrable for λ-a.e. i and P-a.e. ω, and

$$\int g\, dQ = \int_I \Big(\int_\Omega g_i\, dP \Big)\, d\lambda = \int_\Omega \Big(\int_I g_\omega\, d\lambda \Big)\, dP.$$

This is exactly **Sun (2006), Definition 2.2**. The same definition is also used in **Duffie–Sun (2007), Definition 2.1**. Following Sun (2006), we often denote such a space by $(I \times \Omega,\ \mathcal{I} \boxtimes \mathcal{F},\ \lambda \boxtimes P)$ to emphasize the retention of Fubini's theorem beyond $\mathcal{I} \otimes \mathcal{F}$.

Remark 1.1 (Why we need an extension)**.** If we insist on $\mathcal{W} = \overline{\mathcal{I} \otimes \mathcal{F}}$ (the completion of the usual product) then, for many independence hypotheses that economists want (e.g. i.i.d. types, independent match outcomes), one cannot have a jointly measurable process unless it is essentially constant. A Fubini extension enlarges the measurable sets on $I \times \Omega$ so that one can have *joint measurability* of useful processes while retaining the ability to interchange integrals.

1.4 A minimalist introduction to nonstandard analysis

We give a concrete "toolkit" sufficient for Loeb measures and hyperfinite constructions.

1.4.1 Ultrafilters and ultrapowers (one standard construction)

Let $\mathbb{N}$ be the natural numbers. A (free) *ultrafilter* $\mathcal{U}$ on $\mathbb{N}$ is a collection of subsets of $\mathbb{N}$ satisfying:

1. $\emptyset \notin \mathcal{U}$, and if $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$;
2. if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$;
3. for every $A \subseteq \mathbb{N}$, exactly one of A and $\mathbb{N} \setminus A$ lies in $\mathcal{U}$;
4. $\mathcal{U}$ is *free*: no finite subset of $\mathbb{N}$ is in $\mathcal{U}$.

Fix such a $\mathcal{U}$. The *ultrapower* ${}^*\mathbb{R}$ is defined as the quotient $\mathbb{R}^{\mathbb{N}}/\sim$, where $(x_n) \sim (y_n)$ iff $\{n : x_n = y_n\} \in \mathcal{U}$. We write ${}^*x$ for the class of the constant sequence $(x, x, \dots)$, identifying $\mathbb{R}$ with a subfield of ${}^*\mathbb{R}$.

1.4.2 Infinitesimals and unlimited hyperintegers

A hyperreal $\varepsilon \in {}^*\mathbb{R}$ is *infinitesimal* if $|\varepsilon| < 1/n$ for all $n \in \mathbb{N}$. A hyperinteger $N \in {}^*\mathbb{N}$ is *unlimited* if $N > n$ for all $n \in \mathbb{N}$. Existence of unlimited hyperintegers follows from the freeness of $\mathcal{U}$.

1.4.3 Standard part

Every finite hyperreal $x \in {}^*\mathbb{R}$ (i.e. $|x| < n$ for some $n \in \mathbb{N}$) is infinitely close to a unique real number, called its *standard part* and denoted $\mathrm{st}(x) \in \mathbb{R}$.

1.4.4 Internal sets and hyperfinite sets

In the ultrapower approach, an *internal set* is (informally) a set that is represented by a sequence of standard sets modulo $\mathcal{U}$. Of special importance are *hyperfinite* sets: internal sets that behave like finite sets internally. For example, for an unlimited hyperinteger $N \in {}^*\mathbb{N}$, the set

$$I := \{1, 2, \ldots, N\} \subset {}^*\mathbb{N}$$

is hyperfinite: internally it has N elements, although (externally) it has the cardinality of the continuum.

1.4.5 Transfer principle (informal statement)

The *transfer principle* says that any first-order statement true of the standard structure (e.g. $\mathbb{R}$, $\mathbb{N}$) is also true for its nonstandard extension (e.g. ${}^*\mathbb{R}$, ${}^*\mathbb{N}$) when all objects are replaced by their star-images. In practice, this lets one use finite combinatorial/probabilistic reasoning on hyperfinite sets as if they were genuinely finite.

1.5 Loeb measures

1.5.1 Internal finitely additive measures

Let I be hyperfinite and let $\mathcal{I}_0$ be its *internal power set* (the family of internal subsets of I). Define the internal counting probability measure $\lambda_0 : \mathcal{I}_0 \to {}^*[0, 1]$ by

$$\lambda_0(A) := \frac{|A|}{|I|} = \frac{|A|}{N}.$$

This is internally countably additive (in the internal sense) and behaves like the uniform distribution on a finite set.

1.5.2 Loeb's theorem (construction of a standard probability space)

Theorem 1.1 (Loeb measure)**.** *From an internal probability space $(I, \mathcal{I}_0, \lambda_0)$ one can construct a* standard *probability space $(I, \mathcal{I}, \lambda)$, called its* Loeb space*, such that:*

- *$\mathcal{I}$ is a σ-algebra on the underlying set I containing $\mathcal{I}_0$;*
- *λ is a countably additive probability measure on $\mathcal{I}$; and*
- *for every $A \in \mathcal{I}_0$, one has $\lambda(A) = \mathrm{st}(\lambda_0(A))$.*

Remark 1.2 (Atomlessness)**.** If N is unlimited, then the Loeb counting measure λ is atomless: singleton sets have Loeb measure 0. Thus Loeb counting spaces provide canonical atomless probability spaces of agents.

1.6 Loeb products and the Fubini property (Keisler's theorem)

Let $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega, \mathcal{F}_0, P_0)$ be internal probability spaces with hyperfinite underlying sets. Their internal product space is $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$. Let $(I, \mathcal{I}, \lambda)$ and $(\Omega, \mathcal{F}, P)$ be the corresponding Loeb spaces.

Theorem 1.2 (Keisler's Fubini theorem for Loeb products, informal)**.** *The Loeb measure of the internal product measure, denoted $\lambda \boxtimes P$, makes*

$$(I \times \Omega,\ \mathcal{I} \boxtimes \mathcal{F},\ \lambda \boxtimes P)$$

into a Fubini extension *of the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. In particular, for every $\lambda \boxtimes P$-integrable function g that is $\mathcal{I} \boxtimes \mathcal{F}$-measurable, iterated integrals exist and satisfy the Fubini equalities.*

Remark 1.3 (What is extended?)**.** The key point is that $\mathcal{I} \boxtimes \mathcal{F}$ typically strictly contains $\mathcal{I} \otimes \mathcal{F}$. This is *exactly* what allows us to accommodate independent processes and random matchings that cannot live on the usual product σ-algebra.

1.7 Instantaneous (static) random matching on a continuum: hyperfinite construction

We now construct an *instantaneous* random full matching with strong measure-preserving properties and the independence-in-types feature used in applications. This follows the Loeb-space proof scheme used in **Duffie–Sun (2007, Section 4.1)**, which proves **Duffie–Sun (2007), Theorem 2.4**.

1.7.1 Step 1: hyperfinite agents

Fix an unlimited even hyperinteger $N \in {}^*\mathbb{N}$ and let $I := \{1, 2, \dots, N\}$ with internal counting measure λ_0. Let $(I, \mathcal{I}, \lambda)$ be the Loeb space (atomless).

1.7.2 Step 2: hyperfinite sample space of perfect matchings

Let Ω be the internal set of all *perfect matchings* of I, meaning fixed-point-free involutions $\omega : I \to I$ satisfying $\omega(\omega(i)) = i$ and $\omega(i) \neq i$ for all i. (Equivalently, partitions of I into $N/2$ disjoint unordered pairs.) Endow Ω with internal counting probability P_0 (uniform over matchings), and let $(\Omega, \mathcal{F}, P)$ be its Loeb space.

1.7.3 Step 3: the matching map and measurability

Define $\varphi : I \times \Omega \to I$ by $\varphi(i, \omega) := \omega(i)$. This map is internal, hence Loeb measurable with respect to the Loeb product σ-algebra $\mathcal{I} \boxtimes \mathcal{F}$. Moreover, for each fixed ω, the section $\varphi_\omega(\cdot) = \omega(\cdot)$ is a bijection of I preserving λ.

1.7.4 Step 4: measure-preservation and "no mass on individuals"

Because λ is atomless, for fixed $i \in I$ and fixed $j \in I$, one has $P(\varphi(i, \cdot) = j) = 0$. Intuitively, in the hyperfinite model,

$$P_0(\varphi(i, \cdot) = j) = \frac{1}{N - 1} \quad (j \neq i),$$

and taking standard parts yields 0. This formalizes the economic intuition that in a continuum population, the probability of matching with any *particular* counterparty is zero.

1.7.5 Step 5: independence in types (how it emerges)

Let $S = \{1, \ldots, K\}$ be a finite type set, and let $a : I \to S$ be any (Loeb) measurable type function with type distribution $\rho \in \Delta(S)$ given by $\rho(k) = \lambda(\{i : a(i) = k\})$. Define the induced *partner-type* process $g : I \times \Omega \to S$ by

$$g(i, \omega) := a(\varphi(i, \omega)).$$

In the terminology of **Duffie–Sun (2007)**, g is the "type process" induced by a random full matching (cf. **Definition 2.3**) and "independent in types" means that g is essentially pairwise independent (cf. **Definition 2.2** and **Definition 2.3(4)**). The crucial existence result is **Duffie–Sun (2007), Theorem 2.4**, which provides a random full matching that is independent in types (indeed, "universal" across finite type functions).

1.8 From Loeb products to Fubini extensions suitable for independent processes

The Loeb product construction above yields a probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ that:

- extends the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$;
- satisfies the Fubini property by Keisler's theorem; and
- supports a jointly measurable matching map φ.

This is precisely a *Fubini extension* in the sense of Definition 1.3.

1.9 Richness: hosting essentially i.i.d. families (Sun 2006)

For applications beyond matching (e.g. idiosyncratic shocks, heterogeneous Markov chains), we need the product space to be *rich* enough to support measurable processes with essentially pairwise independent coordinates and prescribed marginals.

Definition 1.2 (Rich product probability space (informal))**.** A Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is called *rich* if it supports an $\mathcal{I} \boxtimes \mathcal{F}$-measurable process $U : I \times \Omega \to [0, 1]$ such that $(U_i)_{i \in I}$ is essentially pairwise independent and each U_i is uniformly distributed on $[0, 1]$.

Remark 1.4 (Why "uniform" is enough)**.** Sun formalizes richness as: existence of an essentially pairwise independent family of uniform $[0, 1]$ random variables on the Fubini extension (**Sun (2006), Definition 5.1**). The universality claim—constructing essentially pairwise independent processes with essentially arbitrary prescribed marginal laws on a Polish space—is **Sun (2006), Proposition 5.3**. Practically: given a uniform family, one can generate essentially pairwise independent families with *any prescribed distributions* by measurable transforms (inverse CDF method), provided the target distributions are Borel on a Polish space.

1.10 Putting it together for instantaneous matching

The Loeb-space construction gives a concrete recipe for building the probability space used for instantaneous matching:

1. Choose an unlimited hyperfinite even N and set $I = \{1, \ldots, N\}$; take its Loeb space $(I, \mathcal{I}, \lambda)$ as the agent space.
2. Let Ω be the hyperfinite set of perfect matchings of I, with uniform internal counting measure, and take its Loeb space $(\Omega, \mathcal{F}, P)$ as the sample space.

3. Form the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, which is a Fubini extension.
4. Define the matching map $\varphi(i, \omega) = \omega(i)$ (jointly measurable on the extension) and, for any type function a, define partner-type process $g(i, \omega) = a(\varphi(i, \omega))$.
5. Use the essential pairwise independence properties of g (Duffie–Sun, 2007) and the exact law of large numbers on a Fubini extension (Sun, 2006) to deduce deterministic aggregation results (e.g. deterministic cross-sectional partner-type frequencies).

1.11 Bibliographic notes (minimal)

The definition and systematic use of Fubini extensions and exact law of large numbers are developed in:

Y. Sun, *The exact law of large numbers via Fubini extension and characterization of insurable risks*, Journal of Economic Theory 126 (2006), 31–69.

The Loeb-space (nonstandard) existence construction of independent random matching in a continuum population is developed in:

D. Duffie and Y. Sun, *Existence of independent random matching*, Annals of Applied Probability 17 (2007), 386–419.