Notes: Continuous-time random matching (Duffie–Qiao–Sun 2025)

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Contents

1 Purpose and a quick map to the paper

These notes summarize the setup and construction strategy of:

> D. Duffie, L. Qiao, and Y. Sun (2025), *Continuous time random matching*, The Annals of Applied Probability 35(3), 1755–1790.

The goal is to understand (i) the continuous-time model objects, (ii) the deterministic ODE law for cross-sectional type distributions, and (iii) the probability-space/Fubini-extension construction used to make the model rigorous.

Where to look in DLQ (2025). In the extracted text ‘paper_text/aap_2025_dqs_extracted.txt‘:

- Model primitives and objects $(p_0, \eta, \theta, \varsigma)$, definition of $R(p)$, and ODE: around "Model and results" and equation (2).
- Formal model definition (random type process α and matching process φ, measurability, RCLL, etc.): **Definition 1**.
- Main existence/properties theorem: **Theorem 1**.
- Nonstandard machinery overview: **Section 4**.
- The "generalized Keisler" ingredient (Loeb transition probabilities and generalized Fubini): **Section 4.4 and Section 5**, with explicit reference to Duffie–Sun (2007) for the generalized Fubini theorem for Loeb transition probabilities.

2 Minimal background: CTMCs, generators, and intensities (as used in DLQ)

DLQ's model is built so that each agent's type evolves as a *continuous-time Markov chain* (CTMC) on a finite state space $S = \{1, \dots, K\}$. We record the minimal definitions used in their statements.

Definition 2.1 (Generator / intensity matrix on a finite state space)**.** Let $S = \{1, \dots, K\}$. A *generator* (or *intensity matrix*) is a matrix $R = (R_{kr})_{k,r \in S}$ such that

- $R_{kr} \geq 0$ for $k \neq r$,
- $R_{kk} = -\sum_{r \neq k} R_{kr}$ (row sums equal 0).

For a CTMC $X(t)$ on S with generator R, the instantaneous transition rate from state k to $r \neq k$ at time t is R_{kr}.

Remark 2.1 (Kolmogorov forward equation)**.** If $\pi(t)$ is the row vector of state probabilities $\pi_k(t) = \mathbb{P}(X(t) = k)$, then $\pi(t)$ solves the ODE

$$\frac{d}{dt}\pi(t) = \pi(t)R.$$

DLQ uses a version of this equation with a *state-dependent* generator $R(p_t)$ that depends on the cross-sectional distribution p_t.

Remark 2.2 (Counting processes and martingale characterization (informal))**.** DLQ encodes "$k \to r$ transitions occur at intensity $R_{kr}(p_t)$" via counting processes $C_{ikr}(t)$ and compensated processes

$$M_{ikr}(t) = C_{ikr}(t) - \int_0^t \mathbf{1}\{\alpha(i,s) = k\} R_{kr}(p_s)\, ds,$$

which are (local) martingales. You do not need deep martingale theory for the big picture; this is mainly a standard way to define/verify intensities.

3 DLQ (2025) model: primitives, processes, and the ODE

3.1 Agent space, sample space, and Fubini extension

DLQ assumes an atomless agent space $(I, \mathcal{I}, \lambda)$ with $\lambda(I) = 1$ and a probability space $(\Omega, \mathcal{F}, P)$ for randomness. Because joint measurability and independence are incompatible on the usual product, they work on a *Fubini extension*

$$(I \times \Omega,\ \mathcal{I} \boxtimes \mathcal{F},\ \lambda \boxtimes P),$$

in the sense of Sun (2006, Definition 2.2) (DLQ repeats the definition early in Section 2).

3.2 Primitives

Fix $K \geq 2$ and let $S = \{1, \ldots, K\}$ and let Δ be the simplex of probability measures on S. DLQ's primitives are:

- An initial cross-sectional type distribution $p_0 \in \Delta$.
- Mutation intensities $\eta_{kl} \in [0, \infty)$ for $k \neq l$.

- Matching intensities $\theta_{kl} : \Delta \to \mathbb{R}_+$ satisfying mass-balance

$$p_k \theta_{kl}(p) = p_l \theta_{lk}(p),$$

  and a Lipschitz condition on $p_k \theta_{kl}(p)$ (DLQ uses this to ensure well-posedness of the ODE).

- Match-induced type-change distributions $\varsigma_{kl} \in \Delta$ (the new type distribution for a type-k agent matched to a type-l agent).

3.3 The induced generator $R(p)$ and the deterministic ODE

Define $R(p)$ (a generator on S) by combining mutation and match-induced type change:

$$R_{kr}(p) = \eta_{kr} + \sum_{l=1}^{K} \theta_{kl}(p)\, \varsigma_{kl}(r), \qquad k \neq r, \quad R_{kk}(p) = -\sum_{r \neq k} R_{kr}(p).$$

DLQ's key deterministic law is that the (expected, and in fact realized) cross-sectional type distribution solves

$$\frac{d}{dt} x(t) = x(t) R(x(t)).$$

4 Continuous-time matching objects (DLQ Definition 1) and main result (Theorem 1)

DLQ defines two coupled processes on $I \times \Omega \times \mathbb{R}_+$:

- a *type process* $\alpha : I \times \Omega \times \mathbb{R}_+ \to S$ (RCLL in t),
- a *matching process* $\varphi : I \times \Omega \times \mathbb{R}_+ \to I$ (RCLL in t),

such that for each fixed (ω, t) the map $i \mapsto \varphi(i, \omega, t)$ is an involution (pairwise matching) and is measure-preserving in i for λ.

Remark 4.1 (Interpretation of $\varphi(i, \omega, t)$)**.** DLQ's $\varphi(i, \omega, t) = j$ means: as of time t in state ω, agent j is the *last* partner met by i (and symmetrically i is the last partner met by j), with no intervening meetings since that last meeting time.

DLQ's **Theorem 1** asserts existence of such a system on some Fubini extension, and that:

- the realized cross-sectional distribution $p_t(\omega)$ is almost surely equal to its expectation and solves the ODE,
- the cross-sectional type process is a Markov chain with generator $R(\bar{p}_t)$,
- match counts aggregate deterministically at rate $\bar{p}_{t,k}\theta_{kl}(\bar{p}_t)$,
- there exist stationary distributions p^* satisfying $p^* R(p^*) = 0$.

5 How DLQ constructs the model (roadmap)

At a high level, DLQ proceeds in three layers:

1. **Finite agent, discrete time approximation.** Build a finite-agent dynamic matching model over many short periods (Sections 3.1–3.3), and derive quantitative bounds controlling dependence and cumulative errors across periods.
2. **Hyperfinite time grid via nonstandard analysis.** Replace the number of periods by a hyperfinite integer M^2 and the step length by an infinitesimal $1/M$, producing a hyperfinite-time internal model (Section 5).
3. **Loeb-ization and exact LLN.** Take Loeb measures to obtain a standard atomless agent space and a Fubini extension; then apply the exact law of large numbers for a continuum of stochastic processes (Sun 2006, Theorem 2.16) to obtain deterministic cross-sectional evolution.

6 The "generalized Keisler" ingredient: Loeb transition probabilities

Continuous time (built from hyperfinite discrete time) naturally leads to an *iterated* randomization across periods. This is most cleanly handled by *transition probabilities* and their products. DLQ introduces Loeb transition probabilities (Section 4.4) and uses a generalized Fubini theorem for them. The generalized Fubini result they need is proved in Duffie–Sun (2007) (DLQ cites this explicitly).

Definition 6.1 (Internal transition probability (informal))**.** Let $(X, \mathcal{X}_0)$ and $(Y, \mathcal{Y}_0)$ be internal measurable spaces (typically hyperfinite with internal power sets). An *internal transition probability* is an internal map that assigns to each $x \in X$ an internal probability measure on $(Y, \mathcal{Y}_0)$.

Lemma 6.1 (Generalized Fubini for Loeb transition probabilities (informal))**.** *Let λ be a Loeb measure on an internal space I and let $P = \{P_i : i \in I\}$ be a* Loeb transition probability *(a family of Loeb measures induced from an internal transition probability). Let $\tau = \lambda \boxtimes P$ denote the associated Loeb product measure on $I \times \Omega$. Then for every τ-integrable function f on $I \times \Omega$,*

$$\int_{I\times\Omega} f\, d\tau = \int_I \left(\int_\Omega f(i,\omega)\, dP_i(\omega) \right) d\lambda(i),$$

with appropriate measurability/integrability of sections. (This is the transition-probability analogue of Keisler's Fubini theorem for two Loeb measures.)

Remark 6.1 (Why this is needed beyond Keisler (two-factor) Fubini)**.** In static matching, the sample measure on Ω does not depend on i, and Keisler's Loeb-product Fubini theorem suffices. In dynamic/hyperfinite-time constructions, the period-n randomization may depend on past states (hence on the current "state" variable), leading naturally to transition kernels. One then needs a Fubini theorem compatible with *kernel products*, and eventually an Ionescu–Tulcea type theorem to build the product/path measure.

7 Connections to Duffie–Sun (2007) and Sun (2006)

DLQ (2025) is continuous-time, but the logic is consistent with the earlier discrete-time/static program:

- Duffie–Sun (2007) provides the Loeb-space construction for static/discrete-time matching and develops generalized Fubini/Ionescu–Tulcea type results for Loeb transition probabilities (their Section 5), which DLQ cites and builds on.
- Sun (2006) provides the Fubini extension framework and the exact LLN for a continuum of stochastic processes (Theorem 2.16), which DLQ invokes to pass from independent agent-level dynamics to deterministic cross-sectional dynamics.

8 What to read next in DLQ (2025)

For a first deep read focused on construction:

- **Section 2**: formal definition of the model objects and Fubini extension.
- **Section 3**: finite-agent discrete approximation (where the technical estimates live; many proofs are in the supplement).

- **Section 4.4 and Section 5**: Loeb transition probabilities and the hyperfinite-time-to-continuous-time construction (the "generalized Keisler" part).
- **Sun (2006), Theorem 2.16**: exact LLN for stochastic processes, used to deduce that cross-sectional paths are deterministic almost surely.