

Notes: Fubini extensions for instantaneous matching (Loeb-space construction)

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1 Fubini extensions for instantaneous matching (Loeb-space construction)

1.1 Quick map to the source papers (Sun 2006; Duffie–Sun 2007)

This note is distilled primarily from:

- Y. Sun (2006), *The exact law of large numbers via Fubini extension and characterization of insurable risks* (JET 126), and
- D. Duffie and Y. Sun (2007), *Existence of independent random matching* (AAP 17).

To make it easy to find the original statements, we record the main correspondences:

- **Fubini extension definition:** this note’s section 1.3 is **Sun (2006), Definition 2.2** (and also **Duffie–Sun (2007), Definition 2.1**).
- **Incompatibility (usual product measurability + independence):** mentioned in §section 1 §section 1.3 is **Sun (2006), Proposition 2.1**.
- **Rich product probability space:** this note’s informal “richness” definition in §section 1.9 corresponds to **Sun (2006), Definition 5.1**.
- **Universality via uniform r.v.’s:** the “uniform is enough” remark in §section 1.9 is backed by **Sun (2006), Proposition 5.3**.
- **Random full matching & independence-in-types:** the matching construction in §section 1.7 is modeled on **Duffie–Sun (2007), Definition 2.3** and **Theorem 2.4**.

1.2 Aim and scope

Many economic and probabilistic models posit a *continuum* of agents, indexed by an atomless probability space $(I, \mathcal{I}, \lambda)$, together with *instantaneous random matching* at a given time (or a single random matching draw), typically represented as a random involution $\varphi : I \times \Omega \rightarrow I$ satisfying $\varphi(\varphi(i, \omega), \omega) = i$.

The foundational difficulty is that, for nontrivial processes $(f_i)_{i \in I}$, *independence* of the family and *joint measurability* of the mapping $(i, \omega) \mapsto f_i(\omega)$ with respect to the usual product σ -algebra $\mathcal{I} \otimes \mathcal{F}$ are, in general, incompatible except in degenerate cases. This obstruction goes back at least to Doob and is made precise in the modern literature via results such as: if $f : I \times \Omega \rightarrow X$

is $\mathcal{I} \otimes \mathcal{F}$ -measurable and essentially pairwise independent, then almost all coordinates f_i are essentially constant (**Sun (2006), Proposition 2.1**).

The goal of this note is to give a *minimalist, self-contained, rigorous* construction of a probability space rich enough to support:

- a continuum of (essentially) independent random objects with the *Fubini property*, and
- an *instantaneous* random matching map with the independence-in-types properties used in economics and genetics.

The construction is based on *nonstandard analysis* and *Loeb measures*, following the approach underlying Sun (2006) and Duffie–Sun (2007).

1.3 Measure-theoretic target: Fubini extensions

Definition 1.1 (Fubini extension). Let $(I, \mathcal{I}, \lambda)$ and (Ω, \mathcal{F}, P) be probability spaces. A probability space

$$(I \times \Omega, \mathcal{W}, Q)$$

is a *Fubini extension* of the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$ if:

1. $\mathcal{I} \otimes \mathcal{F} \subseteq \mathcal{W}$ and Q extends $\lambda \otimes P$; and
2. for every Q -integrable $g : I \times \Omega \rightarrow \mathbb{R}$, the sections $g_i(\omega) := g(i, \omega)$ and $g_\omega(i) := g(i, \omega)$ are integrable for λ -a.e. i and P -a.e. ω , and

$$\int g dQ = \int_I \left(\int_\Omega g_i dP \right) d\lambda = \int_\Omega \left(\int_I g_\omega d\lambda \right) dP.$$

This is exactly **Sun (2006), Definition 2.2**. The same definition is also used in **Duffie–Sun (2007), Definition 2.1**. Following Sun (2006), we often denote such a space by $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ to emphasize the retention of Fubini’s theorem beyond $\mathcal{I} \otimes \mathcal{F}$.

Remark 1.1 (Why we need an extension). If we insist on $\mathcal{W} = \overline{\mathcal{I} \otimes \mathcal{F}}$ (the completion of the usual product) then, for many independence hypotheses that economists want (e.g. i.i.d. types, independent match outcomes), one cannot have a jointly measurable process unless it is essentially constant. A Fubini extension enlarges the measurable sets on $I \times \Omega$ so that one can have *joint measurability* of useful processes while retaining the ability to interchange integrals.

1.4 A minimalist introduction to nonstandard analysis

We give a concrete “toolkit” sufficient for Loeb measures and hyperfinite constructions.

1.4.1 Ultrafilters and ultrapowers (one standard construction)

Definition 1.2 (Filter on a set). Let S be a nonempty set. A *filter* \mathcal{F} on S is a nonempty collection of subsets of S such that:

1. $\emptyset \notin \mathcal{F}$;
2. if $A \in \mathcal{F}$ and $A \subseteq B \subseteq S$, then $B \in \mathcal{F}$ (upward closed);
3. if $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$ (closed under finite intersections).

Remark 1.2 (Existence of free ultrafilters (via Zorn/AC)). There is no explicit “formula” example of a free ultrafilter on \mathbb{N} . Its existence is usually proved using the *Ultrafilter Lemma* (every proper filter extends to an ultrafilter), which follows from Zorn’s lemma and hence from the Axiom of Choice. Concretely, start from the *Fréchet filter* of cofinite sets $\mathcal{F}_{\text{cof}} = \{A \subseteq \mathbb{N} : \mathbb{N} \setminus A \text{ is finite}\}$, extend it to an ultrafilter $\mathcal{U} \supseteq \mathcal{F}_{\text{cof}}$, and then \mathcal{U} must be *free* (non-principal) because it contains no finite sets. For the purposes of this note, the key point is simply: we may *fix* one free ultrafilter \mathcal{U} and treat its elements as “ \mathcal{U} -large” subsets of \mathbb{N} .

Let \mathbb{N} be the natural numbers. A (free) *ultrafilter* \mathcal{U} on \mathbb{N} is a collection of subsets of \mathbb{N} satisfying:

1. $\emptyset \notin \mathcal{U}$, and if $A \in \mathcal{U}$ and $A \subseteq B$, then $B \in \mathcal{U}$;
2. if $A, B \in \mathcal{U}$, then $A \cap B \in \mathcal{U}$;
3. for every $A \subseteq \mathbb{N}$, exactly one of A and $\mathbb{N} \setminus A$ lies in \mathcal{U} ;
4. \mathcal{U} is *free*: no finite subset of \mathbb{N} is in \mathcal{U} .

Fix such a (free) ultrafilter \mathcal{U} . Intuitively, members of \mathcal{U} are the subsets of \mathbb{N} that we declare to be *\mathcal{U} -large* (a strengthened notion of “for almost all n ”).

The *ultrapower* ${}^*\mathbb{R}$ is defined as the quotient $\mathbb{R}^{\mathbb{N}} / \sim$, where for two real sequences (x_n) and (y_n) we declare

$$(x_n) \sim (y_n) \iff \{n \in \mathbb{N} : x_n = y_n\} \in \mathcal{U}.$$

Note that this definition uses *one specific subset* of \mathbb{N} , namely the equality set $\{n : x_n = y_n\}$; it does *not* require any condition to hold for all sets in \mathcal{U} .

We write *x for the equivalence class of the constant sequence (x, x, \dots) . This gives an embedding $\mathbb{R} \hookrightarrow {}^*\mathbb{R}$. Moreover, we can add and multiply equivalence classes by doing so coordinatewise on representative sequences, making ${}^*\mathbb{R}$ into a field containing \mathbb{R} as a subfield (hence the phrase “identifying \mathbb{R} with a subfield of ${}^*\mathbb{R}$ ”).

1.4.2 Infinitesimals and unlimited hyperintegers

A hyperreal $\varepsilon \in {}^*\mathbb{R}$ is *infinitesimal* if $|\varepsilon| < 1/n$ for all $n \in \mathbb{N}$. A hyperinteger $N \in {}^*\mathbb{N}$ is *unlimited* if $N > n$ for all $n \in \mathbb{N}$. Existence of unlimited hyperintegers follows from the freeness of \mathcal{U} .

1.4.3 Standard part

Every finite hyperreal $x \in {}^*\mathbb{R}$ (i.e. $|x| < n$ for some $n \in \mathbb{N}$) is infinitely close to (i.e., their difference is infinitesimal) a unique real number, called its *standard part* and denoted $\text{st}(x) \in \mathbb{R}$. See section A for a complete existence-and-uniqueness proof (using the completeness of \mathbb{R}).

1.4.4 Internal sets and hyperfinite sets

The ultrapower construction applies not only to numbers but also to sets. We record a concrete “sequence-mod- \mathcal{U} ” description that will be used later for Loeb measures and random matching.

Internal sets via sequences. Fix a base set S (e.g. $S = \mathbb{N}$ or $S = \mathbb{R}$). Consider sequences of subsets $(A_n)_{n \in \mathbb{N}}$ with $A_n \subseteq S$. Define an equivalence relation on such sequences by

$$(A_n) \approx (B_n) \iff \{n \in \mathbb{N} : A_n = B_n\} \in \mathcal{U}.$$

An *internal subset* of *S is, informally, an equivalence class $[(A_n)]$ of such a sequence. This is the set-analogue of defining hyperreals as equivalence classes of real sequences.

Membership is also “ \mathcal{U} -almost sure”. Elements of *S are themselves equivalence classes of sequences (s_n) with $s_n \in S$. If $x = [(s_n)] \in {}^*S$ and $A = [(A_n)]$ is an internal subset of *S , then the intended meaning of

$$x \in A$$

is:

$$\{n \in \mathbb{N} : s_n \in A_n\} \in \mathcal{U}.$$

Thus internal sets are precisely those for which membership can be checked coordinatewise on representatives, “for \mathcal{U} -almost all n ”.

Hyperfinite sets. The special internal sets that behave like finite sets are called *hyperfinite*.

Definition 1.3 (Hyperfinite set). A set H is *hyperfinite* if:

1. H is *internal*, and
2. there exists some $N \in {}^*\mathbb{N}$ and an *internal bijection*

$$f : \{1, 2, \dots, N\} \longrightarrow H.$$

Equivalently, H is internal and has an *internal cardinality* $|H| \in {}^*\mathbb{N}$.

The canonical hyperfinite initial segment $I = \{1, \dots, N\}$. Let $N = [(N_n)] \in {}^*\mathbb{N}$ be a (possibly unlimited) hyperinteger. Define a sequence of standard finite sets

$$I_n := \{1, 2, \dots, N_n\} \subseteq \mathbb{N}.$$

Then the internal set

$$I := \{1, 2, \dots, N\} \subset {}^*\mathbb{N}$$

can be represented concretely as the equivalence class $I = [(I_n)]$. Membership becomes: for $k = [(k_n)] \in {}^*\mathbb{N}$,

$$k \in I \iff \{n : 1 \leq k_n \leq N_n\} \in \mathcal{U}.$$

Internally, I is finite-like: by transfer, $|\{1, \dots, N\}| = N$ in the internal sense, so I is hyperfinite with internal cardinality N . If N is unlimited, then externally (in ordinary set theory) I is not a finite set, but it retains enough finite combinatorics to support uniform counting, random matchings, etc., which will later be converted to standard measure-theoretic objects via Loeb measure.

1.4.5 Transfer principle (informal statement)

The *transfer principle* says that any first-order statement true of the standard structure (e.g. \mathbb{R}, \mathbb{N}) is also true for its nonstandard extension (e.g. ${}^*\mathbb{R}, {}^*\mathbb{N}$) when all objects are replaced by their star-images. In practice, this lets one use finite combinatorial/probabilistic reasoning on hyperfinite sets as if they were genuinely finite.

1.5 Loeb measures

1.5.1 Internal finitely additive measures

Let I be hyperfinite and let \mathcal{I}_0 be its *internal power set* (the family of internal subsets of I). Define the internal counting probability measure $\lambda_0 : \mathcal{I}_0 \rightarrow {}^*[0, 1]$ by

$$\lambda_0(A) := \frac{|A|}{|I|} = \frac{|A|}{N}.$$

This is internally countably additive (in the internal sense) and behaves like the uniform distribution on a finite set.

1.5.2 Loeb's theorem (construction of a standard probability space)

Theorem 1.1 (Loeb measure). *From an internal probability space $(I, \mathcal{I}_0, \lambda_0)$ one can construct a standard probability space $(I, \mathcal{I}, \lambda)$, called its Loeb space, such that:*

- \mathcal{I} is a σ -algebra on the underlying set I containing \mathcal{I}_0 ;
- λ is a countably additive probability measure on \mathcal{I} ; and
- for every $A \in \mathcal{I}_0$, one has $\lambda(A) = \text{st}(\lambda_0(A))$.

Remark 1.3 (Atomlessness). If N is unlimited, then the Loeb counting measure λ is atomless: singleton sets have Loeb measure 0. Thus Loeb counting spaces provide canonical atomless probability spaces of agents.

1.6 Loeb products and the Fubini property (Keisler's theorem)

Let $(I, \mathcal{I}_0, \lambda_0)$ and $(\Omega, \mathcal{F}_0, P_0)$ be internal probability spaces with hyperfinite underlying sets. Their internal product space is $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$. Let $(I, \mathcal{I}, \lambda)$ and (Ω, \mathcal{F}, P) be the corresponding Loeb spaces.

Theorem 1.2 (Keisler's Fubini theorem for Loeb products, informal). *The Loeb measure of the internal product measure, denoted $\lambda \boxtimes P$, makes*

$$(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$$

into a Fubini extension of the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$. In particular, for every $\lambda \boxtimes P$ -integrable function g that is $\mathcal{I} \boxtimes \mathcal{F}$ -measurable, iterated integrals exist and satisfy the Fubini equalities.

Remark 1.4 (What is extended?). The key point is that $\mathcal{I} \boxtimes \mathcal{F}$ typically strictly contains $\mathcal{I} \otimes \mathcal{F}$. This is *exactly* what allows us to accommodate independent processes and random matchings that cannot live on the usual product σ -algebra.

1.7 Instantaneous (static) random matching on a continuum: hyperfinite construction

We now construct an *instantaneous* random full matching with strong measure-preserving properties and the independence-in-types feature used in applications. This follows the Loeb-space proof scheme used in **Duffie–Sun (2007, Section 4.1)**, which proves **Duffie–Sun (2007), Theorem 2.4**.

1.7.1 Step 1: hyperfinite agents

Fix an unlimited even hyperinteger $N \in {}^*\mathbb{N}$ and let $I := \{1, 2, \dots, N\}$ with internal counting measure λ_0 . Let $(I, \mathcal{I}, \lambda)$ be the Loeb space (atomless).

1.7.2 Step 2: hyperfinite sample space of perfect matchings

Let Ω be the internal set of all *perfect matchings* of I^1 , meaning fixed-point-free involutions $\omega : I \rightarrow I$ satisfying $\omega(\omega(i)) = i$ and $\omega(i) \neq i$ for all i . (Equivalently, partitions of I into $N/2$ disjoint unordered pairs.) Endow Ω with internal counting probability P_0 (uniform over matchings), and let (Ω, \mathcal{F}, P) be its Loeb space.

1.7.3 Step 3: the matching map and measurability

Define $\varphi : I \times \Omega \rightarrow I$ by $\varphi(i, \omega) := \omega(i)$. This map is internal, hence Loeb measurable with respect to the Loeb product σ -algebra $\mathcal{I} \boxtimes \mathcal{F}$. Moreover, for each fixed ω , the section $\varphi_\omega(\cdot) = \omega(\cdot)$ is a bijection of I preserving λ .

1.7.4 Step 4: measure-preservation and “no mass on individuals”

Because λ is atomless, for fixed $i \in I$ and fixed $j \in I$, one has $P(\varphi(i, \cdot) = j) = 0$. Intuitively, in the hyperfinite model,

$$P_0(\varphi(i, \cdot) = j) = \frac{1}{N-1} \quad (j \neq i),$$

¹One way to see that this collection is internal is via the sequence/ultrapower representation: if $I = [(I_n)]$ with finite I_n , let Ω_n be the (finite) set of perfect matchings of I_n and define $\Omega = [(\Omega_n)]$. Equivalently, since I is internal, the set I^I of internal maps $I \rightarrow I$ is internal and the condition “ ω is a fixed-point-free involution” is an internal (first-order) property, so the set of such ω is an internal subset of I^I .

and taking standard parts yields 0. This formalizes the economic intuition that in a continuum population, the probability of matching with any *particular* counterparty is zero.

1.7.5 Step 5: independence in types (how it emerges)

Let $S = \{1, \dots, K\}$ be a finite type set, and let $a : I \rightarrow S$ be any (Loeb) measurable type function with type distribution $\rho \in \Delta(S)$ given by $\rho(k) = \lambda(\{i : a(i) = k\})$. Define the induced *partner-type* process $g : I \times \Omega \rightarrow S$ by

$$g(i, \omega) := a(\varphi(i, \omega)).$$

In the terminology of **Duffie–Sun (2007)**, g is the “type process” induced by a random full matching (cf. **Definition 2.3**), and “independent in types” means that the family $(g_i)_{i \in I}$ is *essentially pairwise independent* (cf. **Definition 2.2** and **Definition 2.3(4)**).

The main existence result, **Duffie–Sun (2007), Theorem 2.4**, constructs (on a suitable Loeb/Fubini product space) a random full matching φ such that for *any* given finite type assignment $a : I \rightarrow S$, the induced partner-type process $g = a \circ \varphi$ is independent in types. At a high level, the proof uses finite matching combinatorics to show that dependence across distinct agents’ partner-type events is of order $1/N$ in an N -agent uniform matching model; passing to a hyperfinite N makes these errors infinitesimal; and the Loeb/standard-part step then yields the exact “almost-everywhere” (essential) pairwise independence statement in the continuum limit.

To situate this precisely, the construction is carried out on the *Loeb product (joint) space*

$$(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P),$$

obtained by Loeb-izing the internal product $(I \times \Omega, \mathcal{I}_0 \otimes \mathcal{F}_0, \lambda_0 \otimes P_0)$; here I is a hyperfinite agent set, Ω is the hyperfinite set of perfect matchings of I , and $\varphi(i, \omega) = \omega(i)$ and $g(i, \omega) = a(\varphi(i, \omega))$ are internal maps (hence $\mathcal{I} \boxtimes \mathcal{F}$ -measurable after Loeb-ization). Keisler’s Fubini theorem implies that $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is a *Fubini extension* of the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$, so iterated integrals over agents and randomness can be interchanged for $\lambda \boxtimes P$ -integrable functions.

The remaining substantive point in **Duffie–Sun (2007), Theorem 2.4** is the *independence-in-types* property for g . The proof proceeds by (i) analyzing a genuinely finite N -agent uniform random perfect matching and obtaining explicit bounds showing that cross-agent dependence of partner-type events is of order $1/N$; (ii) transferring these bounds to a hyperfinite N , making the dependence *infinitesimal*; and (iii) passing to the Loeb product measure on

$I \times I$ (equivalently, using $\lambda \times \lambda$ -almost-every pair (i, j)) to conclude that the limiting statement holds *exactly* in the essential sense: for λ -a.e. i , the random variables g_i and g_j are independent for λ -a.e. j .

1.8 From Loeb products to Fubini extensions suitable for independent processes

The Loeb product construction above yields a probability space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ that:

- extends the usual product $(I \times \Omega, \mathcal{I} \otimes \mathcal{F}, \lambda \otimes P)$;
- satisfies the Fubini property by Keisler’s theorem; and
- supports a jointly measurable matching map φ .

This is precisely a *Fubini extension* in the sense of section 1.3.

1.9 Richness: hosting essentially i.i.d. families (Sun 2006)

For applications beyond matching (e.g. idiosyncratic shocks, heterogeneous Markov chains), we need the product space to be *rich* enough to support measurable processes with essentially pairwise independent coordinates and prescribed marginals.

Definition 1.4 (Rich product probability space (informal)). A Fubini extension $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$ is called *rich* if it supports an $\mathcal{I} \boxtimes \mathcal{F}$ -measurable process $U : I \times \Omega \rightarrow [0, 1]$ such that $(U_i)_{i \in I}$ is essentially pairwise independent and each U_i is uniformly distributed on $[0, 1]$.

Remark 1.5 (Why “uniform” is enough). Sun formalizes richness as: existence of an essentially pairwise independent family of uniform $[0, 1]$ random variables on the Fubini extension (**Sun (2006), Definition 5.1**). The universality claim—constructing essentially pairwise independent processes with essentially arbitrary prescribed marginal laws on a Polish space—is **Sun (2006), Proposition 5.3**. Practically: given a uniform family, one can generate essentially pairwise independent families with *any prescribed distributions* by measurable transforms (inverse CDF method), provided the target distributions are Borel on a Polish space.

1.10 Putting it together for instantaneous matching

The Loeb-space construction gives a concrete recipe for building the probability space used for instantaneous matching:

1. Choose an unlimited hyperfinite even N and set $I = \{1, \dots, N\}$; take its Loeb space $(I, \mathcal{I}, \lambda)$ as the agent space.
2. Let Ω be the hyperfinite set of perfect matchings of I , with uniform internal counting measure, and take its Loeb space (Ω, \mathcal{F}, P) as the sample space.
3. Form the Loeb product space $(I \times \Omega, \mathcal{I} \boxtimes \mathcal{F}, \lambda \boxtimes P)$, which is a Fubini extension.
4. Define the matching map $\varphi(i, \omega) = \omega(i)$ (jointly measurable on the extension) and, for any type function a , define partner-type process $g(i, \omega) = a(\varphi(i, \omega))$.
5. Use the essential pairwise independence properties of g (Duffie–Sun, 2007) and the exact law of large numbers on a Fubini extension (Sun, 2006) to deduce deterministic aggregation results (e.g. deterministic cross-sectional partner-type frequencies).

1.11 Bibliographic notes (minimal)

The definition and systematic use of Fubini extensions and exact law of large numbers are developed in:

Y. Sun, *The exact law of large numbers via Fubini extension and characterization of insurable risks*, Journal of Economic Theory 126 (2006), 31–69.

The Loeb-space (nonstandard) existence construction of independent random matching in a continuum population is developed in:

D. Duffie and Y. Sun, *Existence of independent random matching*, Annals of Applied Probability 17 (2007), 386–419.

A Existence and uniqueness of the standard part

In the main text we used the fact that every *finite* hyperreal is infinitely close to a unique real number. For completeness, we record a standard proof based on the least-upper-bound property of \mathbb{R} .

Theorem A.1 (Standard part of a finite hyperreal). *Let $x \in {}^*\mathbb{R}$ be finite, i.e. $|x| < n$ for some $n \in \mathbb{N}$. Then there exists a unique $r \in \mathbb{R}$ such that $x - {}^*r$ is infinitesimal. We denote this real number by $\text{st}(x) = r$.*

Proof. Define

$$A := \{q \in \mathbb{Q} : {}^*q < x\}.$$

We first check that A is nonempty and bounded above in \mathbb{R} . Since $|x| < n$ for some $n \in \mathbb{N}$, we have $\mathbf{N}(-n) < x$, hence $-n \in A$, so $A \neq \emptyset$. Also $\mathbf{N}q < x$ implies $q < n$, so $A \subset (-\infty, n)$, hence A is bounded above. By completeness of \mathbb{R} , the supremum

$$r := \sup A \in \mathbb{R}$$

exists.

Claim: $x \approx {}^*r$, i.e. $|x - {}^*r| < 1/m$ for every $m \in \mathbb{N}$. Fix $m \in \mathbb{N}$.

(i) Show $x < \mathbf{N}(r + 1/m)$. If instead $x \geq \mathbf{N}(r + 1/m)$, choose a rational q with $r < q < r + 1/m$ (density of \mathbb{Q}). Then $\mathbf{N}q \leq x$, so $q \in A$, contradicting $q > r = \sup A$. Hence $x < \mathbf{N}(r + 1/m)$, equivalently $x - {}^*r < 1/m$.

(ii) Show $x > \mathbf{N}(r - 1/m)$. If instead $x \leq \mathbf{N}(r - 1/m)$, then for every rational $q < r - 1/m$ we have $\mathbf{N}q < x$, so $q \in A$. This would force $\sup A \leq r - 1/m$, contradicting $\sup A = r$. Hence $x > \mathbf{N}(r - 1/m)$, equivalently $\mathbf{N}r - x < 1/m$.

Combining (i) and (ii) gives $|x - {}^*r| < 1/m$. Since m was arbitrary, $x - {}^*r$ is infinitesimal, proving existence.

Uniqueness. If $x \approx {}^*r$ and $x \approx {}^*s$ for $r, s \in \mathbb{R}$, then $\mathbf{N}(r - s) = (\mathbf{N}r - x) + (x - \mathbf{N}s)$ is infinitesimal. But $r - s$ is a real number, and the only real infinitesimal is 0, so $r = s$. \square