6.842 Randomness and Computation	March 2, 2022

Lectures on Random Walks

Lecturer: Ronitt Rubinfield | *Scribe: Yuchong Pan*

1 Definitions

Definition 1. Let Ω be a set of states (throughout this note, Ω is finite). A sequence of random walks $X_0, X_1, \ldots \in \Omega$ is a *Markov chain* if it satisfies the *Markovian property*, i.e., for each $t \in \mathbb{N}$ and for all $x_1, \ldots, x_t, y \in \Omega$,

$$\mathbb{P}\left[X_{t+1} \mid X_1 = x_1, \ldots, X_t = x_t\right] = \mathbb{P}\left[X_{t+1} = y \mid X_t = x_t\right].$$

WLOG, we assume that transitions are independent of time. For $x, y \in \Omega$, let

$$P(x,y) = \mathbb{P}\left[X_{t+1} = y \mid X_t = x\right].$$

Interpreted as a matrix, P is called the *transition matrix* of the Markov chain. We can also interpret the transition matrix P as a weighted directed graph with vertex set Ω such that the weight on $(i,j) \in \Omega^2$ equals $P(i,j)$.

A random walk on a directed graph is a special case of Markov chains.

Definition 2. A *random walk* on a directed graph $G = (V, E)$ is a sequence $S_1, S_2, \ldots \in V$ such that S_{t+1} is picked uniformly in $N^+(S_t)$, i.e., the transition matrix P is defined so that for $x, y \in V$,

$$P(x,y) = \begin{cases} \frac{1}{d^+(x)}, & \text{if } (x,y) \in E, \\ 0, & \text{otherwise.} \end{cases}$$

Definition 3. An $n \times n$ matrix P is called a *stochastic matrix* if for all $i \in [n]$,

$$\sum_{i=1}^{n} P(i,j) = 1.$$

For each $t \in \mathbb{N}$, let $P_t(x,y)$ ve the transition probability from x to y for t steps. Then for all $x, y \in \Omega$ and $t \in \mathbb{N}$,

$$P^t(x,y) = \begin{cases} P(x,y), & \text{if } t = 1, \\ \sum_{z \in \Omega} P(x,z) P^{t-1}(z,y) & \text{if } t > 1. \end{cases}$$

Interpreted as matrix multiplication, for each $t \in \mathbb{N}$ with $t > 1$,

$$P^t = P \cdot P^{t-1}.$$

Let $\Pi^{(0)} = (\Pi_1^{(0)}, \ldots, \Pi_n^{(0)})$ be the initial distribution, where $\Pi_i^{(0)}$ is the probability of starting at vertex i for each $i \in [n]$.[1] Let $\Pi^{(t)}$ be the distribution after t steps for each $t \in \mathbb{N}$. For each $t \in \mathbb{N}$,

$$\Pi^{(t)} = \Pi^{(0)} P^t.$$

[1] WLOG, we assume $\Omega = [n]$.

Definition 4. A distribution Π^* is called a *stationary distribution* of a Markov chain with state set Ω and transition matrix P if for all $x \in \Omega$,

$$\Pi^*(x) = \sum_{y \in \Omega} \Pi^*(y) P(y, x).$$

Definition 5. A Markov chain with state set Ω and transition matrix P is said to be *irreducible* if for all $x, y \in \Omega$, there exists $t \in \mathbb{N}$ such that $P^t(x, y) > 0$.

Definition 6. A Markov chain with state set Ω and transition matrix P is said to be *aperiodic* if for all $x \in \Omega$,

$$\gcd \left\{ t \in \mathbb{N} : p^t(x, x) > 0] \right\} = 1.$$

Definition 7. A Markov chain with state set Ω and transition matrix P is said to be *ergodic* if there exists $t^* \in \mathbb{N}$ such that for all $t \in \mathbb{N}$ with $t > t^*$ and for all $x, y \in \Omega$, we have $P^t(x, y) > 0$.

Theorem 8. *Every ergodic Markov chain has a unique stationary distribution.*

In the special case of a random walk on an undirected graph $G = (V, E)$, the stationary distribution $\Pi^* = (\Pi^*_1, \ldots, \Pi^*_n)$ is given by $\Pi^*_i = d(i)/(2|E|)$ for all $i \in [n]$. Therefore, for a random walk on a d-regular graph or on a directed graph with each in-degree and each out-degree equal to d, the stationary distribution is uniform; this is not true in general directed graphs.

2 Hitting Time, Cover Time and Commute Time

Definition 9. Consider a random walk on a graph $G = (V, E)$. For $x, y \in V$, the *hitting time* $H_{x,y}$ is defined to be the expected number of steps to go from x to y. For each $x \in V$, we call $H_{x,x}$ the *recurrence time* for x.

Theorem 10. *Consider a random walk on a graph $G = (V, E)$ with stationary distribution Π^*. For each $x \in V$,*

$$h_{x,x} = \frac{1}{\Pi_*(x)}.$$

Proof sketch. Consider a very long walk. Then a $\Pi^*(x)$ fraction of the positions are x. Then the average gap between the occurrences of x is $h_{x,x} = \Pi^*(x)^{-1}$. $\square$

Definition 11. Consider a random walk on a graph $G = (V, E)$. For $u \in V$, the *cover time* $C_u(G)$ is defined to be the expected steps from u to visit all states in Ω. Define $C(G) = \max_{u \in V} C_u(G)$.

Following are several examples of the cover time:

- $C(K_n) = \Theta(n \log n)$, where K_n is the complete graph on n vertices with a self-loop at each vertex. This can be proved by a coupon collector argument.
- $C(L_n) = \Theta(n^2)$, where L_n is the n-vertex line graph with a self-loop at each vertex.
- $C(\text{lollipop}_n) = \Theta(n^3)$, where lollipop_n is an n-vertex lollipop vertex formed by $L_{n/2}$ and $K_{n/2}$ joined at a vertex. This is illustrated in Figure 1.

Theorem 12. *Let G be an undirected graph. Then*[2]

$$C(G) \leq O(mn).$$

[2] When the context is clear, we denote $m = |E|$ in a graph $G = (V, E)$.

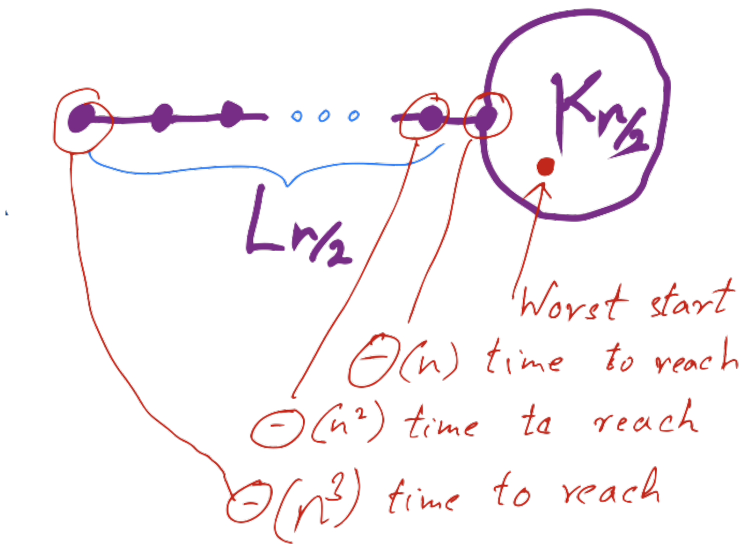

Figure 1: A lollipop graph lollipop$_n$ and its cover time.

Definition 13. Consider a random walk on a graph $G = (V, E)$. For $x, y \in V$, the *commute time* $C_{x,y} = C_{x,y}(G)$ is defined to be the expected number of steps for the random walk to start at x, hit y and return to x.

Proposition 14. *For $x, y \in V$,*

$$C_{x,y} = h_{x,y} + h_{y,x}.$$

Proof. This is due to linearity of expectation. □

Lemma 15. *Consider a random walk on a connected undirected graph $G = (V, E)$. For each $(x, y) \in E$,*

$$C_{x,y} \leq O(m).$$

Proof. Construct a graph G' by adding a self-loop at each vertex with probability $1/2$. Let $x, y \in V$. We claim that $C_{x,y}(G') = 2C_{x,y}(G)$. To see this, for each path from x to y in G', removing the self-loops in the path gives a path in G, and the expected fraction of self-loops in the path is $1/2$. Then G' is ergodic. This implies that there exists a unique stationary distribution Π^*.

Consider a walk $u_1, u_2, \ldots$, where $u_i \in V$ and $(u_i, u_{i+1}) \in E$ for each $i \in \mathbb{N}$. We look for commutes of the form

$$x \to y \to \ldots \to x \to y.$$

For each $i \in \mathbb{N}$,

$$\mathbb{P}\left[u_i = x, u_{i+1} = y\right] = \mathbb{P}\left[u_i = x\right] \cdot \mathbb{P}\left[u_{i+1} = y \mid u_i = x\right] = \frac{d(x)}{2m} \cdot \frac{1}{d(x)} = \frac{1}{2m}.$$

Therefore, the expected fraction of $x \to y$ equals $1/(2m)$. This implies that the expected gap between the $(x \to y)$'s equals $2m$. This proves that $C_{x,y}(G) = O(m)$. □

Proof of Theorem 12. Let T be a spanning tree of G. Let $(v_0, v_1, \ldots, v_{2n-2})$ be a DFS traversal of T. For instance, $(1, 2, 3, 2, 4, 2, 1, 5, 1)$ is a DFS traversal of the tree given in Figure 2. Then

$$C(G) \leq \sum_{i=0}^{2n-3} h_{v_i, v_{i+1}} = \sum_{(x,y) \in E(T)} C_{x,y} \leq (n-1) \cdot O(m) = O(mn).$$

This completes the proof.

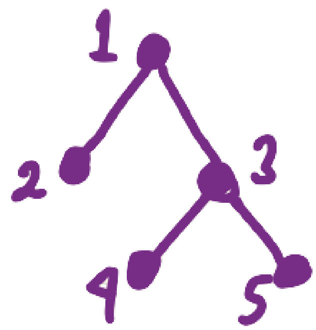

Figure 2: $(1, 2, 3, 2, 4, 2, 1, 5, 1)$ is a DFS traversal of the tree in the figure.

$\square$