

Homework 4

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1. *Collaborators and sources:* Guanghao Ye, Zixuan Xu.

*Proof.*

□

2. (a) *Collaborators and sources*: none.

*Proof.* Note that  $\mathbb{1}_{\text{test accepts}} = (1 + f(x)f(y)f(z))/2$ . By the Fourier transform of  $f$  and by linearity of expectation,

$$\begin{aligned}\mathbb{E}[f(x)f(y)f(z)] &= \mathbb{E} \left[ \left( \sum_{S \subset [n]} \hat{f}(S) \chi_S(x) \right) \left( \sum_{T \subset [n]} \hat{f}(T) \chi_T(y) \right) \left( \sum_{U \subset [n]} \hat{f}(U) \chi_U(z) \right) \right] \\ &= \sum_{S, T, U \subset [n]} \hat{f}(S) \hat{f}(T) \hat{f}(U) \mathbb{E} [\chi_S(x) \chi_T(y) \chi_U(x \circ y \circ w)].\end{aligned}$$

Let  $S, T, U \subset [n]$ . For all  $i \in [n]$ , since  $x_i, y_i \in \{\pm 1\}$ , then  $x_i^2 = y_i^2 = 1$ . Hence,

$$\begin{aligned}\chi_S(x) \chi_T(y) \chi_U(x \circ y \circ w) &= \left( \prod_{i \in S} x_i \right) \left( \prod_{i \in T} y_i \right) \left( \prod_{i \in U} x_i y_i w_i \right) \\ &= \left( \prod_{i \in S \cap U} x_i^2 \right) \left( \prod_{i \in T \cap U} y_i^2 \right) \left( \prod_{i \in S \Delta U} x_i \right) \left( \prod_{i \in T \Delta U} y_i \right) \left( \prod_{i \in U} w_i \right) \\ &= \chi_{S \Delta U}(x) \chi_{T \Delta U}(y) \chi_U(w).\end{aligned}$$

If  $S = T = U$ , since  $w_1, \dots, w_n$  are all chosen independently and since  $\mathbb{E}[w_i] = (-1) \cdot \delta + 1 \cdot (1 - \delta) = 1 - 2\delta$  for all  $i \in [m]$ , then

$$\mathbb{E} [\chi_{S \Delta U}(x) \chi_{T \Delta U}(y) \chi_U(w)] = \mathbb{E} \left[ \prod_{i \in S} w_i \right] = \prod_{i \in S} \mathbb{E} [w_i] = (1 - 2\delta)^{|S|}.$$

Now, suppose that either  $S \neq U$  or  $T \neq U$ . WLOG assume that  $S \neq U$ . Then  $S \Delta U \neq \emptyset$ . Let  $j \in S \Delta U$ . For  $x \in \{\pm 1\}^n$ , let  $x^{\oplus j}$  be the vector obtained by flipping the  $j^{\text{th}}$  bit in  $x$ . Then we can partition  $\{\pm 1\}^n$  into (unordered) pairs  $(x, x^{\oplus j})$ . Therefore,

$$\begin{aligned}\mathbb{E} [\chi_{S \Delta U}(x)] &= \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} \chi_{S \Delta U}(x) = \frac{1}{2^n} \sum_{\text{pairs } (x, x^{\oplus j})} (\chi_{S \Delta U}(x) + \chi_{S \Delta U}(x^{\oplus j})) \\ &= \frac{1}{2^n} \sum_{\text{pairs } (x, x^{\oplus j})} \left( x_j \prod_{i \in (S \Delta U) \setminus \{j\}} x_i + (-x_j) \prod_{i \in (S \Delta U) \setminus \{j\}} x_i \right) = 0.\end{aligned}$$

Since  $x, y$  and  $w$  are chosen independently, then for all  $S, T, U \subset [n]$  such that either  $S \neq U$  or  $T \neq U$ ,

$$\mathbb{E} [\chi_{S \Delta U}(x) \chi_{T \Delta U}(y) \chi_U(w)] = \mathbb{E} [\chi_{S \Delta U}(x)] \mathbb{E} [\chi_{T \Delta U}(y)] \mathbb{E} [\chi_U(w)] = 0.$$

Therefore,

$$\begin{aligned}\mathbb{P}[\text{test accepts}] &= \mathbb{E} [\mathbb{1}_{\text{test accepts}}] = \mathbb{E} \left[ \frac{1 + f(x)f(y)f(z)}{2} \right] = \frac{1}{2} + \frac{1}{2} \mathbb{E}[f(x)f(y)f(z)] \\ &= \frac{1}{2} + \frac{1}{2} \sum_{S, T, U \subset [n]} \hat{f}(S) \hat{f}(T) \hat{f}(U) \mathbb{E} [\chi_{S \Delta U}(x) \chi_{T \Delta U}(y) \chi_U(w)] \\ &= \frac{1}{2} + \frac{1}{2} \sum_{S \subset [n]} (1 - 2\delta)^{|S|} \hat{f}(S)^3.\end{aligned}$$

This completes the proof.  $\square$

(b) *Collaborators and sources:* none.

*Proof.* Let  $f : \{\pm 1\}^n \rightarrow \{\pm 1\}$  be a dictator function. Then  $f = \chi_{\{j\}}$  for some  $j \in [n]$ . Therefore,  $\hat{f}(\{j\}) = 1$  and  $\hat{f}(S) = 0$  for all  $S \subset [n]$  with  $S \neq \{j\}$ . By part (a),

$$\begin{aligned}
\mathbb{P}[\text{test accepts}] &= \frac{1}{2} + \frac{1}{2} \sum_{S \subset [n]} (1 - 2\delta)^{|S|} \hat{f}(S)^3 \\
&= \frac{1}{2} + \frac{1}{2} \left( (1 - 2\delta)^{|\{j\}|} \hat{f}(\{j\})^3 + \sum_{\substack{S \subset [n] \\ S \neq \{j\}}} (1 - 2\delta)^{|S|} \hat{f}(S)^3 \right) \\
&= \frac{1}{2} + \frac{1}{2} \left( (1 - 2\delta)^1 \cdot 1^3 + \sum_{\substack{S \subset [n] \\ S \neq \{j\}}} (1 - 2\delta)^{|S|} \cdot 0^3 \right) \\
&= \frac{1}{2} + \frac{1}{2}(1 - 2\delta) = 1 - \delta.
\end{aligned}$$

This completes the proof. □