6.842 Randomness and Computation April 27, 2022

Homework 5

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1. *Collaborators and sources:* none.

Proof. Let $f : \{\pm 1\}^n \to \{\pm 1\}$. Let $\varepsilon \in (0, 1/2)$. Then

$$\begin{aligned} NS_\varepsilon(f) &= \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{P}} \left[f(x) \neq f\left(N_\varepsilon(x)\right) \right] = \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{P}} \left[f(x) f\left(N_\varepsilon(x)\right) = -1 \right] \\ &= \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[\frac{1}{2} - \frac{1}{2} f(x) f\left(N_\varepsilon(x)\right) \right] = \frac{1}{2} - \frac{1}{2} \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[f(x) f\left(N_\varepsilon(x)\right) \right] \\ &= \frac{1}{2} - \frac{1}{2} \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[\left(\sum_{S \subset [n]} \hat{f}(S) \chi_S(x) \right) \left(\sum_{T \subset [n]} \hat{f}(T) \chi_T\left(N_\varepsilon(x)\right) \right) \right] \\ &= \frac{1}{2} - \frac{1}{2} \sum_{S, T \subset [n]} \hat{f}(S) \hat{f}(T) \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[\chi_S(x) \chi_T\left(N_\varepsilon(x)\right) \right]. \end{aligned}$$

For all $x \in \{\pm 1\}^n$ and $i \in [n]$, we denote by x_i and $N_\varepsilon(x)_i$ the i^{th} coordinates of x and $N_\varepsilon(x)$, respectively. For all $S \subset [n]$,

$$\begin{aligned} \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[\chi_S(x) \chi_S\left(N_\varepsilon(x)\right) \right] &= \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[\left(\prod_{i \in S} x_i \right) \left(\prod_{i \in S} N_\varepsilon(x)_i \right) \right] = \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[\prod_{i \in S} x_i N_\varepsilon(x)_i \right] \\ &= \prod_{i \in S} \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[x_i N_\varepsilon(x)_i \right] = \left(\varepsilon \cdot (-1) + (1 - \varepsilon) \cdot 1 \right)^{|S|} \qquad (1) \\ &= (1 - 2\varepsilon)^{|S|}. \end{aligned}$$

Note that (1) is due to the independence of each bit in $N_\varepsilon(x)$ and the fact that each bit of x uniformly chosen from $\{\pm 1\}^n$ is uniform in $\{\pm 1\}$. For all $S, T \subset [n]$ with $S \neq T$,

$$\begin{aligned} & \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[\chi_S(x) \chi_T\left(N_\varepsilon(x)\right) \right] \\ &= \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[\left(\prod_{i \in S} x_i \right) \left(\prod_{i \in T} N_\varepsilon(x)_i \right) \right] \\ &= \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[\left(\prod_{i \in S \cap T} x_i N_\varepsilon(x)_i \right) \left(\prod_{i \in S \setminus T} x_i \right) \left(\prod_{i \in T \setminus S} N_\varepsilon(x)_i \right) \right] \\ &= \left(\prod_{i \in S \cap T} \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[x_i N_\varepsilon(x)_i \right] \right) \left(\prod_{i \in S \setminus T} \underset{x \in \{\pm 1\}^n}{\mathbb{E}} \left[x_i \right] \right) \left(\prod_{i \in T \setminus S} \underset{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}}{\mathbb{E}} \left[N_\varepsilon(x)_i \right] \right). \qquad (2) \end{aligned}$$

Note that (2) is again due to the independence of each bit in $N_\varepsilon(x)$. For $S, T \subset [n]$ with $S \neq T$, either $S \setminus T \neq \emptyset$ or $T \setminus S \neq \emptyset$. Note that each bit of x uniformly chosen from $\{\pm 1\}^n$ is uniform in $\{\pm 1\}$. Therefore, if $S \setminus T \neq \emptyset$,

$$\prod_{i \in S \setminus T} \mathop{\mathbb{E}}_{x \in \{\pm 1\}^n} [x_i] = \left(\mathop{\mathbb{E}}_{b \in \{\pm 1\}} [b] \right)^{|S \setminus T|} = 0^{|S \setminus T|} = 0.$$

Moreover, if $T \setminus S \neq \emptyset$,

$$\prod_{i \in T \setminus S} \mathop{\mathbb{E}}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} [N_\varepsilon(x)_i] = \left(\frac{1}{2}(\varepsilon(-1) + (1-\varepsilon) \cdot 1) + \frac{1}{2}(\varepsilon \cdot 1 + (1-\varepsilon)(-1)) \right)^{|T \setminus S|} = 0^{|T \setminus S|} = 0.$$

Therefore, for all $S, T \subset [n]$ with $S \neq T$,

$$\mathop{\mathbb{E}}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} [\chi_S(x) \chi_T(N_\varepsilon(x))] = 0.$$

It follows that

$$NS_\varepsilon(f) = \frac{1}{2} - \frac{1}{2} \sum_{S, T \subset [n]} \hat{f}(S) \hat{f}(T) \mathop{\mathbb{E}}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} [\chi_S(x) \chi_T(N_\varepsilon(x))] = \frac{1}{2} - \frac{1}{2} \sum_{S \subset [n]} \hat{f}(S)^2 (1 - 2\varepsilon)^{|S|}.$$

This completes the proof. $\square$

2. (a) *Collaborators and sources:* none.

Proof. Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be monotone. Let $i \in [n]$. WLOG, assume $i = 1$. Then

$$\begin{aligned}
\hat{f}(\{1\}) &= \langle f, \chi_{\{1\}} \rangle \\
&= \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} f(x) \chi_{\{1\}}(x) \\
&= \frac{1}{2^n} \sum_{x = (x_1, \ldots, x_n) \in \{\pm 1\}^n} f(x) x_1 \\
&= \frac{1}{2^n} \left(\sum_{\substack{x = (x_1, \ldots, x_n) \in \{\pm 1\}^n \\ x_1 = 1}} f(x) \cdot 1 + \sum_{\substack{x = (x_1, \ldots, x_n) \in \{\pm 1\}^n \\ x_1 = -1}} f(x) \cdot (-1) \right) \\
&= \frac{1}{2^n} \left(\sum_{x' \in \{\pm 1\}^{n-1}} f\left(1, x'\right) - \sum_{x' \in \{\pm 1\}^{n-1}} f\left(-1, x'\right) \right) \\
&= \frac{1}{2^n} \sum_{x' \in \{\pm 1\}^{n-1}} \left(f\left(1, x'\right) - f\left(-1, x'\right) \right) \\
&= \frac{1}{2^n} \sum_{\substack{x' \in \{\pm 1\}^{n-1} \\ f(1, x') \neq f(-1, x')}} \left(f\left(1, x'\right) - f\left(-1, x'\right) \right).
\end{aligned}$$

Since f is monotone, then $f(1, x') \geq f(-1, x')$ for all $x' \in \{\pm 1\}^{n-1}$. Hence, for all $x' \in \{\pm 1\}^{n-1}$, if $f(1, x') \neq f(-1, x')$, then $f(1, x') = 1$ and $f(-1, x') = -1$, so $f(1, x') - f(-1, x') = 1 - (-1) = 2$. Therefore,

$$\begin{aligned}
\hat{f}(\{1\}) &= \frac{1}{2^n} \sum_{\substack{x' \in \{\pm 1\}^{n-1} \\ f(1, x') \neq f(-1, x')}} 2 \\
&= \frac{1}{2^n} \cdot 2 \left| \left\{ x' \in \{\pm 1\}^{n-1} : f\left(1, x'\right) \neq f\left(-1, x'\right) \right\} \right| \\
&= \frac{1}{2^{n-1}} \left| \left\{ x' \in \{\pm 1\}^{n-1} : f\left(1, x'\right) \neq f\left(-1, x'\right) \right\} \right|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathit{Inf}_1(f) &= \mathop{\mathbb{P}}_{x \in \{\pm 1\}^n} \left[f(x) \neq f\left(x^{\oplus 1}\right) \right] \\
&= \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} \mathbb{1} \left[f(x) \neq f\left(x^{\oplus 1}\right) \right] \\
&= \frac{1}{2^n} \cdot 2 \sum_{x' \in \{\pm 1\}^{n-1}} \mathbb{1} \left[f\left(1, x'\right) \neq f\left(-1, x'\right) \right] \\
&= \frac{1}{2^{n-1}} \left| \left\{ x' \in \{\pm 1\}^{n-1} : f\left(1, x'\right) \neq f\left(-1, x'\right) \right\} \right| \\
&= \hat{f}(\{1\}).
\end{aligned}$$

This completes the proof. $\square$

(b) *Collaborators and sources:* none.

Proof. Let $n \in \mathbb{N}$ be odd. Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be the majority function, i.e., $f(x) = \text{sign}(\sum_{i=1}^n x_i)$ for all $x = (x_1, \dots, x_n) \in \{\pm 1\}^n$. First, we show that f is monotone. Let $x = (x_1, \dots, x_n), y = (y_1, \dots, y_n) \in \{\pm 1\}^n$ be such that $x_i \le y_i$ for all $i \in [n]$. Then $\sum_{i=1}^n x_i \le \sum_{i=1}^n y_i$, so $f(x) = \text{sign}(\sum_{i=1}^n x_i) \le \text{sign}(\sum_{i=1}^n y_i) = f(y)$. This proves that f is monotone.

Second, let $g : \{\pm 1\}^n \to \{\pm 1\}$ be monotone. Then

$$\begin{aligned}
\mathit{Inf}(g) &= \sum_{i=1}^n \mathit{Inf}_i(g) \\
&= \sum_{i=1}^n \hat{g}(\{i\}) && \text{(part (a))} \\
&= \sum_{i=1}^n \langle g, \chi_{\{i\}} \rangle \\
&= \sum_{i=1}^n \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} g(x) \chi_{\{i\}}(x) \\
&= \frac{1}{2^n} \sum_{x = (x_1, \dots, x_n) \in \{\pm 1\}^n} g(x) \sum_{i=1}^n x_i \\
&\le \left| \frac{1}{2^n} \sum_{x = (x_1, \dots, x_n) \in \{\pm 1\}^n} g(x) \sum_{i=1}^n x_i \right| \\
&\le \frac{1}{2^n} \sum_{x = (x_1, \dots, x_n) \in \{\pm 1\}^n} |g(x)| \left| \sum_{i=1}^n x_i \right| && \text{(triangle inequality)} \\
&= \frac{1}{2^n} \sum_{x = (x_1, \dots, x_n) \in \{\pm 1\}^n} \left| \sum_{i=1}^n x_i \right|. && \text{(since } g(x) \in \{\pm 1\} \text{ for all } x \in \{\pm 1\}^n)
\end{aligned}$$

Third, since f is monotone,

$$\mathit{Inf}(f) = \frac{1}{2^n} \sum_{x = (x_1, \dots, x_n) \in \{\pm 1\}^n} f(x) \sum_{i=1}^n x_i.$$

Since n is odd, then $\sum_{i=1}^n x_i \neq 0$. If $\sum_{i=1}^n x_i < 0$, then $f(x) = \text{sign}(\sum_{i=1}^n x_i) < 0$, so

$$\mathit{Inf}(f) = \frac{1}{2^n} \sum_{x = (x_1, \dots, x_n) \in \{\pm 1\}^n} |f(x)| \left| \sum_{i=1}^n x_i \right| = \frac{1}{2^n} \sum_{x = (x_1, \dots, x_n) \in \{\pm 1\}^n} \left| \sum_{i=1}^n x_i \right|, \tag{3}$$

since $f(x) \in \{\pm 1\}$ for all $x \in \{\pm 1\}^n$. Otherwise, $\sum_{i=1}^n x_i > 0$, so $f(x) = \text{sign}(\sum_{i=1}^n x_i) > 0$, implying that (3) holds. Hence, (3) holds in both cases. It follows that $\mathit{Inf}(g) \le \mathit{Inf}(f)$ for any monotone $g : \{\pm 1\}^n \to \{\pm 1\}$, completing the proof. $\square$

3. (a) *Collaborators and sources:* none.

Proof. Let $f : \{\pm 1\}^n \to \{\pm 1\}$. Let $\varepsilon > 0$. We show that the statement holds with $C = 1$. Suppose for the sake of contradiction that $\sum_{S\subset[n],|S|\geq Inf(f)/\varepsilon} \hat{f}(S)^2 > C\varepsilon = \varepsilon$. For each $i \in [n]$, let $g_i : \{\pm 1\}^n \to \{0, \pm 1\}$ be defined by

$$g_i(x) = \frac{f(x) - f\left(x^{\oplus i}\right)}{2} = \frac{1}{2}\left(\sum_{S\subset[n]} \hat{f}(S)\chi_S(x) - \sum_{S\subset[n]} \hat{f}(S)\chi_S\left(x^{\oplus i}\right)\right)$$
$$= \frac{1}{2}\sum_{S\subset[n]} \hat{f}(S)\left(\chi_S(x) - \chi_S\left(x^{\oplus i}\right)\right).$$

Then $g_i(x)^2 = \mathbb{1}[f(x) \neq f(x^{\oplus i})]$ for all $i \in [n]$ and $x \in \{\pm 1\}^n$. Fix $i \in [n]$, $x = (x_1, \ldots, x_n) \in \{\pm 1\}^n$ and $S \subset [n]$. If $i \in S$, then

$$\chi_S(x) - \chi_S\left(x^{\oplus i}\right) = \prod_{j\in S} x_j - (-x_i)\prod_{j\in S\setminus\{i\}} x_j = x_i \prod_{j\in S\setminus\{i\}} x_j - (-x_i)\prod_{j\in S\setminus\{i\}} x_j$$
$$= (x_i - (-x_i))\prod_{j\in S\setminus\{i\}} x_j = 2x_i\prod_{j\in S\setminus\{i\}} x_j = 2\prod_{j\in S} x_j = 2\chi_S(x).$$

If $i \notin S$, then

$$\chi_S(x) - \chi_S\left(x^{\oplus i}\right) = \prod_{j\in S} x_j - \prod_{j\in S} x_j = 0.$$

Hence, for all $i \in [n]$ and $x \in \{\pm 1\}^n$,

$$g_i(x) = \frac{1}{2}\sum_{\substack{S\subset[n]\\ i\in S}} \hat{f}(S)\cdot 2\chi_S(x) = \frac{1}{2}\cdot 2\sum_{\substack{S\subset[n]\\ i\in S}} \hat{f}(S)\chi_S(x) = \sum_{\substack{S\subset[n]\\ i\in S}} \hat{f}(S)\chi_S(x).$$

For all $i \in [n]$, by the orthonormality of the Fourier basis $\{\chi_S : S \subset [n]\}$,

$$Inf_i(f) = \mathop{\mathbb{P}}_{x\in\{\pm1\}^n}\left[f(x) \neq f\left(x^{\oplus i}\right)\right] = \mathop{\mathbb{E}}_{x\in\{\pm1\}^n}\left[\mathbb{1}\left[f(x) \neq f\left(x^{\oplus i}\right)\right]\right] = \mathop{\mathbb{E}}_{x\in\{\pm1\}^n}\left[g_i(x)^2\right]$$
$$= \mathop{\mathbb{E}}_{x\in\{\pm1\}^n}\left[\left(\sum_{\substack{S\subset[n]\\ i\in S}} \hat{f}(S)\chi_S(x)\right)^2\right] = \mathop{\mathbb{E}}_{x\in\{\pm1\}^n}\left[\sum_{\substack{S\subset[n]\\ i\in S}}\sum_{\substack{T\subset[n]\\ i\in T}} \hat{f}(S)\hat{f}(T)\chi_S(x)\chi_T(x)\right]$$
$$= \mathop{\mathbb{E}}_{x\in\{\pm1\}^n}\left[\sum_{\substack{S\subset[n]\\ i\in S}} \hat{f}(S)^2\right] = \sum_{\substack{S\subset[n]\\ i\in S}} \hat{f}(S)^2.$$

Therefore,

$$Inf(f) = \sum_{i=1}^n Inf_i(f) = \sum_{i=1}^n \sum_{\substack{S\subset[n]\\ i\in S}} \hat{f}(S)^2 = \sum_{S\subset[n]}\sum_{i\in S} \hat{f}(S)^2 = \sum_{S\subset[n]} |S|\hat{f}(S)^2$$
$$\geq \sum_{\substack{S\subset[n]\\ |S|\geq\frac{Inf(f)}{\varepsilon}}} |S|\hat{f}(S)^2 \geq \frac{Inf(f)}{\varepsilon}\sum_{\substack{S\subset[n]\\ |S|\geq\frac{Inf(f)}{\varepsilon}}} \hat{f}(S)^2 > \frac{Inf(f)}{\varepsilon}\cdot\varepsilon = Inf(f),$$

a contradiction. This completes the proof. $\square$