

Randomization in Recent Progress on Traveling Salesman Problem

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hard ★★★★ (116)

White Mountain National Forest

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Enjoy this 8.3-mile loop trail near Jackson, New Hampshire. Generally considered a challenging route, it takes an average of 5 h 39 min to complete. This is a popular trail for birding and hiking, but you can still enjoy some solitude during quieter times of day. The best times to visit this trail are April through September.

Length 8.3 mi Elevation gain 4,340 ft Route type Loop

Hiking Bird watching Forest Views

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0.0 mi 2.0 mi 4.0 mi 6.0 mi 8.0 mi

6,651 ft 7,780 ft

Metric Traveling Salesman Problem

Metric Traveling Salesman Problem (TSP)

Input: a set V of vertices and pairwise symmetric costs
 $c : V \times V \rightarrow \mathbb{R}_+$ which satisfies the triangle inequality

Output: a shortest tour that visits each vertex exactly once

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It belongs to the most seductive problems in combinatorial optimization, thanks to a blend of complexity, applicability, and appeal to imagination.

— Lex Schrijver

The Christofides-Serdyukov Algorithm

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- 1 $T \leftarrow$ a minimum spanning tree of G
- 2 $O \leftarrow \{v \in V : \deg_T(v) \text{ is odd}\}$
- 3 $M \leftarrow$ a minimum-weight perfect matching in $G[O]$
- 4 find a Eulerian tour in $T \cup M$
- 5 shortcut repeated vertices to form a Hamiltonian tour

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the “4/3-conjecture”

... 45 Years Later

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Theorem (Karlin, Klein and Oveis Gharan, 2021)

*For some absolute constant $\varepsilon > 10^{-36}$, there exists a **randomized algorithm** that outputs a tour with expected cost at most $3/2 - \varepsilon$ times the cost of the optimum solution.*

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- We assume that the metric is a **graph metric**, i.e., there exists an unweighted graph where the metric is the shortest path metric of that graph. This special case is due to Oveis Gharan, Saberi and Singh (2011).

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- To focus on the randomization part of the proof, we assume that all proper cuts with respect to x have size at least $2 + \varepsilon$. In the general case, we exploit the structure of near-min-cuts, called the **deformable polygon representation** developed by Benczúr and Goemans.

Subtour Elimination LP

$$\text{minimize} \quad \sum_{u,v \in V} x_{\{u,v\}} c(u, v)$$

$$\text{subject to} \quad \sum_{\{u,v\} \in \delta(S)} x_{\{u,v\}} \geq 2 \quad \forall S \subsetneq V, S \neq \emptyset$$

$$\sum_{\{u,v\} \in \delta(\{v\})} x_{\{u,v\}} = 2 \quad \forall v \in V$$

$$x_{\{u,v\}} \in [0, 1] \quad \forall u, v \in V$$

O -Joins

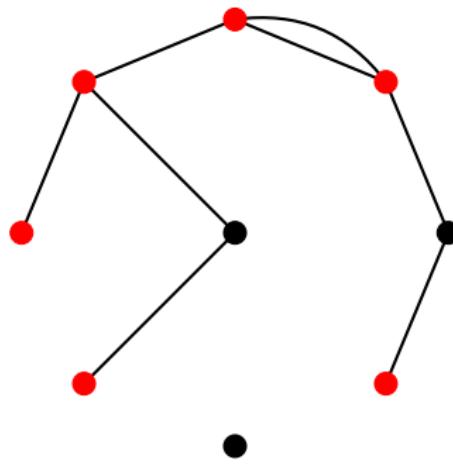
Definition (O -joins)

A set $F \subset E$ is an **O -join** if every vertex $v \in O$ has odd degree in F , and every other vertex has even degree in F .

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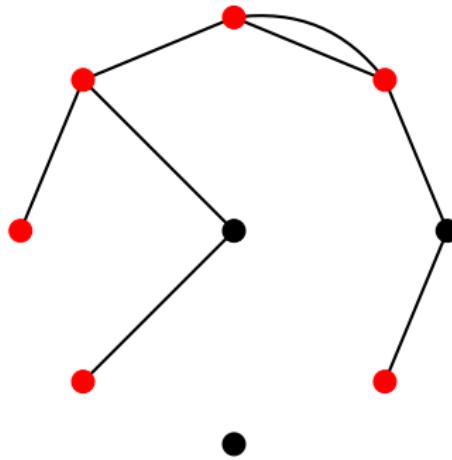
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observation: an O -join is a union of paths connecting red vertices

O -Joins

Theorem (Edmonds and Johnson, 1973)

For any graph $G = (V, E)$ with edge costs $c : E \rightarrow \mathbb{R}_+$ and for any $O \subset V$ with $|O|$ even, the minimum cost of an O -join equals the optimum value of the following LP:

$$\begin{array}{ll} \text{minimize} & c(y) \\ \text{subject to} & y(\delta(S)) \geq 1 \quad \forall S \subsetneq V, |S \cap O| \text{ odd} \\ & y_e \geq 0 \quad \forall e \in E \end{array}$$

A Randomized Rounding Algorithm

Definition (λ -uniform distributions)

For $\lambda : \mathbb{E} \rightarrow \mathbb{R}_{\geq 0}$, a λ -**uniform** distribution μ_λ on spanning trees on a graph G satisfies that for any spanning tree T ,

$$\mathbb{P}_\mu[T] \propto \prod_{e \in E(T)} \lambda_e.$$

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Algorithm (Oveis Gharan, Saberi and Singh, 2011)

- 1 $x \leftarrow$ an optimal solution to the subtour elimination LP
- 2 $\mu \leftarrow$ a z -uniform distribution with marginal $z = (1 - 1/n)x$
- 3 $T \sim \mu$
- 4 $O \leftarrow$ set of odd degree vertices in T
- 5 find a minimum cost O -join F
- 6 **return** $T \cup F$

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- Hence, $c(F) \leq c(y) = c(x)/2$.
- However, we only need to satisfy $y(\delta(S)) \geq 1$ for all $S \subset V$ with $|S \cap O|$ odd.
- The idea is to use the randomness of T to assign a slightly smaller value $y(e) = (1/2 - \varepsilon')x(e)$ to some of the edges (with constant probability) while preserving the feasibility of y .

Even Edges, Good Edges

Definition (even edges)

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Definition (good edges)

An edge e is **good** if

$$\mathbb{P}[e \text{ is even}] \geq \gamma,$$

for some constant γ to be determined later.

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Lemma

Suppose that every edge e is good. Then

$$\mathbb{E}[c(F)] \leq \left(\frac{1}{2} - \Omega(\varepsilon \cdot \gamma) \right) c(x).$$

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Proof.

- For any $e \in E$, let

$$y(e) = \begin{cases} x(e)/(2 + \varepsilon), & \text{if } e \text{ is even,} \\ x(e)/2, & \text{otherwise.} \end{cases}$$

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- For any proper S , we have $y(\delta(S)) \geq x(\delta(S))/(2 + \varepsilon) \geq 1$.
- If v has odd degree, then $y(\delta(\{v\})) = x(\delta(\{v\})) = 1$.
- Since every edge is good,

$$\mathbb{E}[c(y)] \leq \sum_{e \in E} \frac{x(e)}{2} \left(1 - \mathbb{P}[e \text{ even}] \frac{\varepsilon}{4} \right) \leq \left(\frac{1}{2} - \Omega(\varepsilon \cdot \gamma) \right) c(x).$$



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Theorem (Hoeffding, 1956)

For any Bernoulli's B_1, \dots, B_n with success probabilities p_1, \dots, p_n and any $g : \mathbb{R} \rightarrow \mathbb{R}$, $\mathbb{E}[g(B_1 + \dots + B_m)]$ is minimized when $p_1, \dots, p_m \in \{0, p, 1\}$ for some fixed p .