6.842 Randomness and Computation February 22, 2022

Homework 1

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1. *Collaborators and sources:* none.

Proof. We construct an approximation scheme $\mathcal{B}$ for f as follows: On input (x, ε, δ), run $\mathcal{A}(x, \varepsilon)$ independently for $k := \lceil 12 \log(1/\delta) \rceil$ times with outputs $y_1, \ldots, y_k$, and output a median of $y_1, \ldots, y_k$.

Let $t_{\mathcal{A}}(x, \varepsilon)$ be the running time of $\mathcal{A}$ on input (x, ε). Then $\mathcal{B}$ runs in $O(kt_{\mathcal{A}}(x, \varepsilon)) = O(\log(1/\delta) t_{\mathcal{A}}(x, \varepsilon))$. Since $\mathcal{A}$ runs in time polynomial in $1/\varepsilon$ and $|x|$, then $\mathcal{B}$ runs in time polynomial in $1/\varepsilon$, $|x|$ and $\log(1/\delta)$.

By the definition of medians, if more than half of $y_1, \ldots, y_k$ fall in $[f(x)/(1+\varepsilon), f(x)(1+\varepsilon)]$, then $\mathcal{B}(x, \varepsilon, \delta) \in [f(x)/(1+\varepsilon), f(x)(1+\varepsilon)]$. Let $X_1, \ldots, X_k \in \{0, 1\}$ be random variables so that $X_i = 1$ with probability $p := \mathbb{P}[\mathcal{A}(x, \varepsilon) \notin [f(x)/(1+\varepsilon), f(x)(1+\varepsilon)]] \leq 1 - 3/4 = 1/4$. Then $\sum_{i=1}^k \mathbb{E}[X_i] = kp \leq k/4$. Therefore,

$$\begin{aligned}
&\mathbb{P}\left[\mathcal{B}(x, \varepsilon, \delta) \notin \left[\frac{f(x)}{1+\varepsilon}, f(x)(1+\varepsilon)\right]\right] \\
&\leq \mathbb{P}\left[\text{at least half of } y_1, \ldots, y_k \text{ do not fall in } \left[\frac{f(x)}{1+\varepsilon}, f(x)(1+\varepsilon)\right]\right] \\
&= \mathbb{P}\left[\sum_{i=1}^k X_i \geq \frac{k}{2}\right] \\
&= \mathbb{P}\left[\sum_{i=1}^k X_i \geq (1+1) \cdot \frac{k}{4}\right] \\
&\leq e^{-\frac{k/4}{3}} && \text{(Chernoff bound)} \\
&= e^{-\frac{\lceil 12 \log \frac{1}{\delta} \rceil}{12}} \leq e^{-\frac{12 \log \frac{1}{\delta}}{12}} = \delta.
\end{aligned}$$

Therefore,

$$\mathbb{P}\left[\mathcal{B}(x, \varepsilon, \delta) \in \left[\frac{f(x)}{1+\varepsilon}, f(x)(1+\varepsilon)\right]\right] = 1 - \mathbb{P}\left[\mathcal{B}(x, \varepsilon, \delta) \notin \left[\frac{f(x)}{1+\varepsilon}, f(x)(1+\varepsilon)\right]\right] \geq 1 - \delta.$$

This completes the proof. □