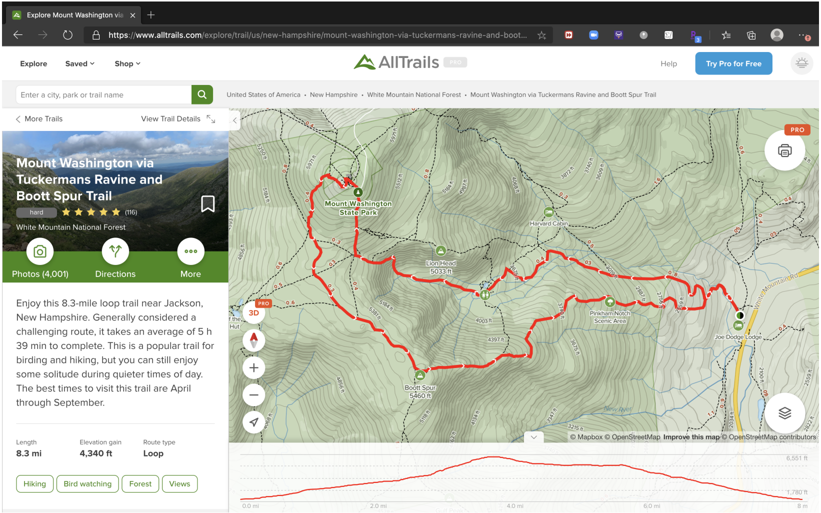
Randomization in Recent Progress on Traveling Salesman Problem

Yuchong Pan

MIT 6.842

May 9, 2022


Explore Mount Washington via
https://www.alltrails.com/explore/trail/us/new-hampshire/mount-washington-via-tuckermans-ravine-and-boot...
Explore
Saved
Shop
AllTrails
PRO
Help
Try Pro for Free
Enter a city, park or trail name
United States of America • New Hampshire • White Mountain National Forest • Mount Washington via Tuckermans Ravine and Boott Spur Trail
More Trails
View Trail Details
Mount Washington via Tuckermans Ravine and Boott Spur Trail
hard
(116)
White Mountain National Forest
Photos (4,001)
Directions
More
Enjoy this 8.3-mile loop trail near Jackson, New Hampshire. Generally considered a challenging route, it takes an average of 5 h 39 min to complete. This is a popular trail for birding and hiking, but you can still enjoy some solitude during quieter times of day. The best times to visit this trail are April through September.
Length
8.3 mi
Elevation gain
4,340 ft
Route type
Loop
Hiking
Bird watching
Forest
Views
Mount Washington State Park
Harvard Cabin
Lion Head 5033 ft
Pinkham Notch Scenic Area
Joe Dodge Lodge
Boott Spur 5460 ft
White Mountain Rd
PRO
3D
© Mapbox © OpenStreetMap Improve this map © OpenStreetMap contributors
6,551 ft
1,780 ft
0.0 mi
2.0 mi
4.0 mi
6.0 mi
8 m

Metric Traveling Salesman Problem

Metric Traveling Salesman Problem (TSP)

Input:	a set V of vertices and pairwise symmetric costs $c : V \times V \to \mathbb{R}_+$ which satisfies the triangle inequality
Output:	a shortest tour that visits each vertex exactly once

Metric Traveling Salesman Problem

Metric Traveling Salesman Problem (TSP)

Input: a set V of vertices and pairwise symmetric costs $c : V \times V \to \mathbb{R}_+$ which satisfies the triangle inequality

Output: a shortest tour that visits each vertex exactly once

> *It belongs to the most seductive problems in combinatorial optimization, thanks to a blend of complexity, applicability, and appeal to imagination.*
>
> — *Lex Schrijver*

The Christofides-Serdyukov Algorithm

The Christofides-Serdyukov Algorithm (1976, 1978)

$T \leftarrow$ a minimum spanning tree of G
$O \leftarrow \{v \in V : \deg_T(v) \text{ is odd}\}$
$M \leftarrow$ a minimum-weight perfect matching in $G[O]$
find a Eulerian tour in $T \cup M$
shortcut repeated vertices to form a Hamiltonian tour

The Christofides-Serdyukov Algorithm

The Christofides-Serdyukov Algorithm (1976, 1978)

$T \leftarrow$ a minimum spanning tree of G
$O \leftarrow \{v \in V : \deg_T(v) \text{ is odd}\}$
$M \leftarrow$ a minimum-weight perfect matching in $G[O]$
find a Eulerian tour in $T \cup M$
shortcut repeated vertices to form a Hamiltonian tour

approximation ratio: $3/2$

The Christofides-Serdyukov Algorithm

The Christofides-Serdyukov Algorithm (1976, 1978)

$T \leftarrow$ a minimum spanning tree of G
$O \leftarrow \{v \in V : \deg_T(v) \text{ is odd}\}$
$M \leftarrow$ a minimum-weight perfect matching in $G[O]$
find a Eulerian tour in $T \cup M$
shortcut repeated vertices to form a Hamiltonian tour

approximation ratio: $3/2$

the "$4/3$**-conjecture"**

… 45 Years Later

... 45 Years Later

Theorem (Karlin, Klein and Oveis Gharan, 2021)

For some absolute constant $\varepsilon > 10^{-36}$, there exists a **randomized** *algorithm that outputs a tour with expected cost at most $3/2 - \varepsilon$ times the cost of the optimum solution.*

… 45 Years Later

Theorem (Karlin, Klein and Oveis Gharan, 2021)

For some absolute constant $\varepsilon > 10^{-36}$, there exists a **randomized** *algorithm that outputs a tour with expected cost at most $3/2 - \varepsilon$ times the cost of the optimum solution.*

- We assume that the metric is a **graph metric**, i.e., there exists an unweighted graph where the metric is the shortest path metric of that graph. This special case is due to Oveis Gharan, Saberi and Singh (2011).

… 45 Years Later

Theorem (Karlin, Klein and Oveis Gharan, 2021)

For some absolute constant $\varepsilon > 10^{-36}$, there exists a **randomized** *algorithm that outputs a tour with expected cost at most $3/2 - \varepsilon$ times the cost of the optimum solution.*

- We assume that the metric is a **graph metric**, i.e., there exists an unweighted graph where the metric is the shortest path metric of that graph. This special case is due to Oveis Gharan, Saberi and Singh (2011).
- To focus on the randomization part of the proof, we assume that all proper cuts with respect to x have size at least $2 + \varepsilon$. In the general case, we exploit the structure of near-min-cuts, called the **deformable polygon representation** developed by Benczúr and Goemans.

Subtour Elimination LP

$$
\begin{aligned}
\text{minimize} \quad & \sum_{u,v \in V} x_{\{u,v\}} c(u,v) \\
\text{subject to} \quad & \sum_{\{u,v\} \in \delta(S)} x_{\{u,v\}} \geq 2 && \forall S \subsetneq V, S \neq \emptyset \\
& \sum_{\{u,v\} \in \delta(\{v\})} x_{\{u,v\}} = 2 && \forall v \in V \\
& x_{\{u,v\}} \in [0,1] && \forall u, v \in V
\end{aligned}
$$

O-Joins

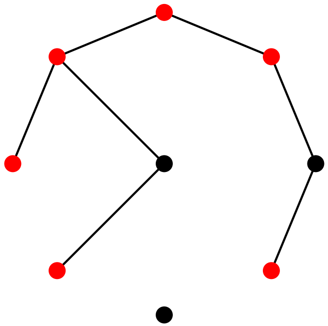
Definition (O-joins)

A set $F \subset E$ is an O-**join** if every vertex $v \in O$ has odd degree in F, and every other vertex has even degree in F.

O-Joins

Definition (O-joins)

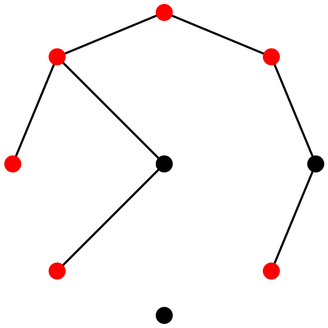
A set $F \subset E$ is an O-**join** if every vertex $v \in O$ has odd degree in F, and every other vertex has even degree in F.



O-Joins

Definition (O-joins)

A set $F \subset E$ is an O-**join** if every vertex $v \in O$ has odd degree in F, and every other vertex has even degree in F.



observation: an O-join is a union of paths connecting red vertices

O-Joins

Theorem (Edmonds and Johnson, 1973)

For any graph $G = (V, E)$ with edge costs $c : E \to \mathbb{R}_+$ and for any $O \subset V$ with $|O|$ even, the minimum cost of an O-join equals the optimum value of the following LP:

$$
\begin{array}{lrl}
\textit{minimize} & c(y) & \\
\textit{subject to} & y(\delta(S)) \geq 1 & \forall S \subsetneq V, |S \cap O| \textit{ odd} \\
& y_e \geq 0 & \forall e \in E
\end{array}
$$

A Randomized Rounding Algorithm

Definition (λ-uniform distributions)

For $\lambda : \mathbb{E} \to \mathbb{R}_{\geq 0}$, a λ-**uniform** distribution μ_λ on spanning trees on a graph G satisfies that for any spanning tree T,

$$\mathbb{P}_\mu[T] \propto \prod_{e \in E(T)} \lambda_e.$$

A Randomized Rounding Algorithm

Definition (λ-uniform distributions)

For $\lambda : \mathbb{E} \to \mathbb{R}_{\geq 0}$, a λ-**uniform** distribution μ_λ on spanning trees on a graph G satisfies that for any spanning tree T,

$$\mathbb{P}_\mu[T] \propto \prod_{e \in E(T)} \lambda_e.$$

Algorithm (Oveis Gharan, Saberi and Singh, 2011)

$x \leftarrow$ an optimal solution to the subtour elimination LP
$\mu \leftarrow$ a z-uniform distribution with marginal $z = (1 - 1/n)x$
$T \sim \mu$
$O \leftarrow$ set of odd degree vertices in T
find a minimum cost O-join F
return $T \cup F$

A (Very) High Level Idea

A (Very) High Level Idea

- $\mathbb{E}[c(T)] = \sum_{e \in E} c(e)z(e) = c(z) = (1 - 1/n)c(x)$.

A (Very) High Level Idea

- $\mathbb{E}[c(T)] = \sum_{e \in E} c(e)z(e) = c(z) = (1 - 1/n)c(x)$.
- First, let's see $c(F) \leq c(x)/2$.

A (Very) High Level Idea

- $\mathbb{E}[c(T)] = \sum_{e \in E} c(e)z(e) = c(z) = (1 - 1/n)c(x)$.
- First, let's see $c(F) \leq c(x)/2$.
- Let $y(e) = x(e)/2$ for all $e \in E$.

A (Very) High Level Idea

- $\mathbb{E}[c(T)] = \sum_{e \in E} c(e)z(e) = c(z) = (1 - 1/n)c(x)$.
- First, let's see $c(F) \leq c(x)/2$.
- Let $y(e) = x(e)/2$ for all $e \in E$.
- Since $x(\delta(S)) \geq 2$ for all proper $S \subset V$, then $y(\delta(S)) \geq 1$.

A (Very) High Level Idea

- $\mathbb{E}[c(T)] = \sum_{e \in E} c(e)z(e) = c(z) = (1 - 1/n)c(x)$.
- First, let's see $c(F) \leq c(x)/2$.
- Let $y(e) = x(e)/2$ for all $e \in E$.
- Since $x(\delta(S)) \geq 2$ for all proper $S \subset V$, then $y(\delta(S)) \geq 1$.
- Hence, $c(F) \leq c(y) = c(x)/2$.

A (Very) High Level Idea

- $\mathbb{E}[c(T)] = \sum_{e \in E} c(e)z(e) = c(z) = (1 - 1/n)c(x)$.
- First, let's see $c(F) \leq c(x)/2$.
- Let $y(e) = x(e)/2$ for all $e \in E$.
- Since $x(\delta(S)) \geq 2$ for all proper $S \subset V$, then $y(\delta(S)) \geq 1$.
- Hence, $c(F) \leq c(y) = c(x)/2$.
- However, we only need to satisfy $y(\delta(S)) \geq 1$ for all $S \subset V$ with $|S \cap O|$ odd.

A (Very) High Level Idea

- $\mathbb{E}[c(T)] = \sum_{e \in E} c(e)z(e) = c(z) = (1 - 1/n)c(x)$.
- First, let's see $c(F) \leq c(x)/2$.
- Let $y(e) = x(e)/2$ for all $e \in E$.
- Since $x(\delta(S)) \geq 2$ for all proper $S \subset V$, then $y(\delta(S)) \geq 1$.
- Hence, $c(F) \leq c(y) = c(x)/2$.
- However, we only need to satisfy $y(\delta(S)) \geq 1$ for all $S \subset V$ with $|S \cap O|$ odd.
- The idea is to use the randomness of T to assign a slightly smaller value $y(e) = (1/2 - \varepsilon')x(e)$ to some of the edges (with constant probability) while preserving the feasibility of y.

Even Edges, Good Edges

Definition (even edges)

An edge is **even** if both of its endpoints have degree 2 in T.

Even Edges, Good Edges

Definition (even edges)

An edge is **even** if both of its endpoints have degree 2 in T.

Definition (good edges)

An edge e is **good** if

$$\mathbb{P}[e \text{ is even}] \geq \gamma,$$

for some constant γ to be determined later.

Even Edges, Good Edges

Lemma

Suppose that every edge e is good. Then

$$\mathbb{E}[c(F)] \leq \left(\frac{1}{2} - \Omega(\varepsilon \cdot \gamma)\right) c(x).$$

Even Edges, Good Edges

Lemma

Suppose that every edge e is good. Then

$$\mathbb{E}[c(F)] \leq \left(\frac{1}{2} - \Omega(\varepsilon \cdot \gamma)\right) c(x).$$

Proof.

- For any $e \in E$, let

$$y(e) = \begin{cases} x(e)/(2+\varepsilon), & \text{if } e \text{ is even,} \\ x(e)/2, & \text{otherwise.} \end{cases}$$

Even Edges, Good Edges

Lemma

Suppose that every edge e is good. Then

$$\mathbb{E}[c(F)] \leq \left(\frac{1}{2} - \Omega(\varepsilon \cdot \gamma)\right) c(x).$$

Proof.

- For any $e \in E$, let
$$y(e) = \begin{cases} x(e)/(2+\varepsilon), & \text{if } e \text{ is even,} \\ x(e)/2, & \text{otherwise.} \end{cases}$$
- For any proper S, we have $y(\delta(S)) \geq x(\delta(S))/(2+\varepsilon) \geq 1$.

Even Edges, Good Edges

Lemma

Suppose that every edge e is good. Then

$$\mathbb{E}[c(F)] \leq \left(\frac{1}{2} - \Omega(\varepsilon \cdot \gamma)\right) c(x).$$

Proof.

- For any $e \in E$, let
$$y(e) = \begin{cases} x(e)/(2+\varepsilon), & \text{if } e \text{ is even,} \\ x(e)/2, & \text{otherwise.} \end{cases}$$
- For any proper S, we have $y(\delta(S)) \geq x(\delta(S))/(2+\varepsilon) \geq 1$.
- If v has odd degree, then $y(\delta(\{v\})) = x(\delta(\{v\})) = 1$.

Even Edges, Good Edges

Lemma

Suppose that every edge e is good. Then

$$\mathbb{E}[c(F)] \leq \left(\frac{1}{2} - \Omega(\varepsilon \cdot \gamma)\right) c(x).$$

Proof.

- For any $e \in E$, let

$$y(e) = \begin{cases} x(e)/(2+\varepsilon), & \text{if } e \text{ is even,} \\ x(e)/2, & \text{otherwise.} \end{cases}$$

- For any proper S, we have $y(\delta(S)) \geq x(\delta(S))/(2+\varepsilon) \geq 1$.
- If v has odd degree, then $y(\delta(\{v\})) = x(\delta(\{v\})) = 1$.
- Since every edge is good,

$$\mathbb{E}[c(y)] \leq \sum_{e \in E} \frac{x(e)}{2}\left(1 - \mathbb{P}[e \text{ even}]\frac{\varepsilon}{4}\right) \leq \left(\frac{1}{2} - \Omega(\varepsilon \cdot \gamma)\right) c(x).$$

$\square$

Existence of Good Edges (Very, Very High Level)

Existence of Good Edges (Very, Very High Level)

- Trivally, we cannot expect a tree to have an even number of edges in every cut.

Existence of Good Edges (Very, Very High Level)

- Trivally, we cannot expect a tree to have an even number of edges in every cut.
- However, using randomness, we can expect to have an even number of edges in at least $1/3$ of the cuts.

Existence of Good Edges (Very, Very High Level)

- Trivally, we cannot expect a tree to have an even number of edges in every cut.
- However, using randomness, we can expect to have an even number of edges in at least $1/3$ of the cuts.
- The crux of the proof is that the degree distribution of a vertex equals the sum of independent Bernoulli random variables $B_1, \ldots, B_m$ (using real stable polynomials).

Existence of Good Edges (Very, Very High Level)

- Trivally, we cannot expect a tree to have an even number of edges in every cut.
- However, using randomness, we can expect to have an even number of edges in at least $1/3$ of the cuts.
- The crux of the proof is that the degree distribution of a vertex equals the sum of independent Bernoulli random variables $B_1, \ldots, B_m$ (using real stable polynomials).
- It remains to answer what is the minimum possible value of $\mathbb{P}[B_1 + \ldots + B_m = 2]$.

Existence of Good Edges (Very, Very High Level)

- Trivally, we cannot expect a tree to have an even number of edges in every cut.
- However, using randomness, we can expect to have an even number of edges in at least $1/3$ of the cuts.
- The crux of the proof is that the degree distribution of a vertex equals the sum of independent Bernoulli random variables $B_1, \ldots, B_m$ (using real stable polynomials).
- It remains to answer what is the minimum possible value of $\mathbb{P}[B_1 + \ldots + B_m = 2]$.

Theorem (Hoeffding, 1956)

For any Bernoulli's $B_1, \ldots, B_n$ with success probabilities $p_1, \ldots, p_n$ and any $g : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[g(B_1 + \ldots + B_m)]$ is minimized when $p_1, \ldots, p_m \in \{0, p, 1\}$ for some fixed p.