

Homework 5

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1. *Collaborators and sources:* none.

Proof. Let $f : \{\pm 1\}^n \rightarrow \{\pm 1\}$. Let $\varepsilon \in (0, 1/2)$. Then

$$\begin{aligned}
NS_\varepsilon(f) &= \mathbb{P}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} [f(x) \neq f(N_\varepsilon(x))] \\
&= \mathbb{P}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} [f(x)f(N_\varepsilon(x)) = -1] \\
&= \mathbb{E}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} \left[\frac{1}{2} - \frac{1}{2} f(x)f(N_\varepsilon(x)) \right] \\
&= \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} [f(x)f(N_\varepsilon(x))] \\
&= \frac{1}{2} - \frac{1}{2} \mathbb{E}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} \left[\left(\sum_{S \subset [n]} \hat{f}(S) \chi_S(x) \right) \left(\sum_{T \subset [n]} \hat{f}(T) \chi_T(N_\varepsilon(x)) \right) \right] \\
&= \frac{1}{2} - \frac{1}{2} \sum_{S, T \subset [n]} \hat{f}(S) \hat{f}(T) \mathbb{E}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} [\chi_S(x) \chi_T(N_\varepsilon(x))].
\end{aligned}$$

For all $x \in \{\pm 1\}^n$ and $i \in [n]$, we denote by x_i and $N_\varepsilon(x)_i$ the i^{th} coordinates of x and $N_\varepsilon(x)$, respectively. For all $S \subset [n]$,

$$\begin{aligned}
\mathbb{E}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} [\chi_S(x) \chi_S(N_\varepsilon(x))] &= \mathbb{E}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} \left[\left(\prod_{i \in S} x_i \right) \left(\prod_{i \in S} N_\varepsilon(x)_i \right) \right] \\
&= \mathbb{E}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} \left[\prod_{i \in S} x_i N_\varepsilon(x)_i \right] \\
&= \prod_{i \in S} \mathbb{E}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} [x_i N_\varepsilon(x)_i] \tag{1}
\end{aligned}$$

$$\begin{aligned}
&= (\varepsilon \cdot (-1) + (1 - \varepsilon) \cdot 1)^{|S|} \tag{2} \\
&= (1 - 2\varepsilon)^{|S|}.
\end{aligned}$$

Note that (1) is due to the independence of each bit in $N_\varepsilon(x)$, and (2) is due to the fact that each bit of x uniformly chosen from $\{\pm 1\}^n$ is uniform in $\{\pm 1\}$. For all $S, T \subset [n]$ with $S \neq T$,

$$\mathbb{E}_{\substack{x \in \{\pm 1\}^n \\ N_\varepsilon}} [\chi_S(x) \chi_T(N_\varepsilon(x))]$$

$$\begin{aligned}
&= \mathbb{E}_{x \in \{\pm 1\}^n, N_\varepsilon} \left[\left(\prod_{i \in S} x_i \right) \left(\prod_{i \in T} N_\varepsilon(x)_i \right) \right] \\
&= \mathbb{E}_{x \in \{\pm 1\}^n, N_\varepsilon} \left[\left(\prod_{i \in S \cap T} x_i N_\varepsilon(x)_i \right) \left(\prod_{i \in S \setminus T} x_i \right) \left(\prod_{i \in T \setminus S} N_\varepsilon(x)_i \right) \right] \\
&= \left(\prod_{i \in S \cap T} \mathbb{E}_{x \in \{\pm 1\}^n, N_\varepsilon} [x_i N_\varepsilon(x)_i] \right) \left(\prod_{i \in S \setminus T} \mathbb{E}_{x \in \{\pm 1\}^n} [x_i] \right) \left(\prod_{i \in T \setminus S} \mathbb{E}_{x \in \{\pm 1\}^n, N_\varepsilon} [N_\varepsilon(x)_i] \right). \quad (3)
\end{aligned}$$

Note that (3) is again due to the independence of each bit in $N_\varepsilon(x)$. For $S, T \subset [n]$ with $S \neq T$, either $S \setminus T \neq \emptyset$ or $T \setminus S \neq \emptyset$. Note that each bit of x uniformly chosen from $\{\pm 1\}^n$ is uniform in $\{\pm 1\}$. Therefore, if $S \setminus T \neq \emptyset$,

$$\prod_{i \in S \setminus T} \mathbb{E}_{x \in \{\pm 1\}^n} [x_i] = \left(\mathbb{E}_{b \in \{\pm 1\}} [b] \right)^{|S \setminus T|} = 0^{|S \setminus T|} = 0.$$

Moreover, if $T \setminus S \neq \emptyset$,

$$\prod_{i \in T \setminus S} \mathbb{E}_{x \in \{\pm 1\}^n, N_\varepsilon} [N_\varepsilon(x)_i] = \left(\frac{1}{2}(\varepsilon(-1) + (1 - \varepsilon) \cdot 1) + \frac{1}{2}(\varepsilon \cdot 1 + (1 - \varepsilon)(-1)) \right)^{|T \setminus S|} = 0^{|T \setminus S|} = 0.$$

Therefore, for all $S, T \subset [n]$ with $S \neq T$,

$$\mathbb{E}_{x \in \{\pm 1\}^n, N_\varepsilon} [\chi_S(x) \chi_T(N_\varepsilon(x))] = 0.$$

It follows that

$$NS_\varepsilon(f) = \frac{1}{2} - \frac{1}{2} \sum_{S, T \subset [n]} \hat{f}(S) \hat{f}(T) \mathbb{E}_{x \in \{\pm 1\}^n, N_\varepsilon} [\chi_S(x) \chi_T(N_\varepsilon(x))] = \frac{1}{2} - \frac{1}{2} \sum_{S \subset [n]} \hat{f}(S)^2 (1 - 2\varepsilon)^{|S|}.$$

This completes the proof. \square