6.842 Randomness and Computation April 27, 2022

Homework 5

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1. *Collaborators and sources:* none.

Proof. Let $f : \{\pm 1\}^n \to \{\pm 1\}$. Let $\varepsilon \in (0, 1/2)$. Then

$$\begin{aligned}
NS_\varepsilon(f) &= \mathop{\mathbb{P}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[f(x) \neq f\left(N_\varepsilon(x)\right)\right] \\
&= \mathop{\mathbb{P}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[f(x) f\left(N_\varepsilon(x)\right) = -1\right] \\
&= \mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[\frac{1}{2} - \frac{1}{2} f(x) f\left(N_\varepsilon(x)\right)\right] \\
&= \frac{1}{2} - \frac{1}{2} \mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[f(x) f\left(N_\varepsilon(x)\right)\right] \\
&= \frac{1}{2} - \frac{1}{2} \mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[\left(\sum_{S\subset[n]} \hat{f}(S)\chi_S(x)\right)\left(\sum_{T\subset[n]} \hat{f}(T)\chi_T\left(N_\varepsilon(x)\right)\right)\right] \\
&= \frac{1}{2} - \frac{1}{2} \sum_{S,T\subset[n]} \hat{f}(S)\hat{f}(T) \mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[\chi_S(x)\chi_T\left(N_\varepsilon(x)\right)\right].
\end{aligned}$$

For all $x \in \{\pm 1\}^n$ and $i \in [n]$, we denote by x_i and $N_\varepsilon(x)_i$ the i^{th} coordinates of x and $N_\varepsilon(x)$, respectively. For all $S \subset [n]$,

$$\begin{aligned}
\mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[\chi_S(x)\chi_S\left(N_\varepsilon(x)\right)\right] &= \mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[\left(\prod_{i\in S} x_i\right)\left(\prod_{i\in S} N_\varepsilon(x)_i\right)\right] \\
&= \mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[\prod_{i\in S} x_i N_\varepsilon(x)_i\right] \\
&= \prod_{i\in S} \mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[x_i N_\varepsilon(x)_i\right] && (1) \\
&= \left(\varepsilon \cdot (-1) + (1-\varepsilon)\cdot 1\right)^{|S|} && (2) \\
&= (1-2\varepsilon)^{|S|}.
\end{aligned}$$

Note that (1) is due to the independence of each bit in $N_\varepsilon(x)$, and (2) is due to the fact that each bit of x uniformly chosen from $\{\pm 1\}^n$ is uniform in $\{\pm 1\}$. For all $S, T \subset [n]$ with $S \neq T$,

$$\mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}} \left[\chi_S(x)\chi_T\left(N_\varepsilon(x)\right)\right]$$

$$= \mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}}\left[\left(\prod_{i\in S} x_i\right)\left(\prod_{i\in T} N_\varepsilon(x)_i\right)\right]$$

$$= \mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}}\left[\left(\prod_{i\in S\cap T} x_i N_\varepsilon(x)_i\right)\left(\prod_{i\in S\setminus T} x_i\right)\left(\prod_{i\in T\setminus S} N_\varepsilon(x)_i\right)\right]$$

$$= \left(\prod_{i\in S\cap T}\mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}}[x_i N_\varepsilon(x)_i]\right)\left(\prod_{i\in S\setminus T}\mathop{\mathbb{E}}_{x\in\{\pm1\}^n}[x_i]\right)\left(\prod_{i\in T\setminus S}\mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}}[N_\varepsilon(x)_i]\right). \tag{3}$$

Note that (3) is again due to the independence of each bit in $N_\varepsilon(x)$. For $S,T\subset[n]$ with $S\neq T$, either $S\setminus T\neq\emptyset$ or $T\setminus S\neq\emptyset$. Note that each bit of x uniformly chosen from $\{\pm1\}^n$ is uniform in $\{\pm1\}$. Therefore, if $S\setminus T\neq\emptyset$,

$$\prod_{i\in S\setminus T}\mathop{\mathbb{E}}_{x\in\{\pm1\}^n}[x_i] = \left(\mathop{\mathbb{E}}_{b\in\{\pm1\}}[b]\right)^{|S\setminus T|} = 0^{|S\setminus T|} = 0.$$

Moreover, if $T\setminus S\neq\emptyset$,

$$\prod_{i\in T\setminus S}\mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}}[N_\varepsilon(x)_i] = \left(\frac{1}{2}(\varepsilon(-1)+(1-\varepsilon)\cdot1)+\frac{1}{2}(\varepsilon\cdot1+(1-\varepsilon)(-1))\right)^{|T\setminus S|} = 0^{|T\setminus S|} = 0.$$

Therefore, for all $S,T\subset[n]$ with $S\neq T$,

$$\mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}}[\chi_S(x)\chi_T(N_\varepsilon(x))] = 0.$$

It follows that

$$NS_\varepsilon(f) = \frac{1}{2}-\frac{1}{2}\sum_{S,T\subset[n]}\hat{f}(S)\hat{f}(T)\mathop{\mathbb{E}}_{\substack{x\in\{\pm1\}^n\\ N_\varepsilon}}[\chi_S(x)\chi_T(N_\varepsilon(x))] = \frac{1}{2}-\frac{1}{2}\sum_{S\subset[n]}\hat{f}(S)^2(1-2\varepsilon)^{|S|}.$$

This completes the proof. $\square$

2. (a) *Collaborators and sources:* none.

Proof. Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be monotone. Let $i \in [n]$. WLOG, assume $i = 1$. Then

$$\begin{aligned}
\hat{f}(\{1\}) &= \langle f, \chi_{\{1\}} \rangle \\
&= \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} f(x) \chi_{\{1\}}(x) \\
&= \frac{1}{2^n} \sum_{x = (x_1, \ldots, x_n) \in \{\pm 1\}^n} f(x) x_1 \\
&= \frac{1}{2^n} \left(\sum_{\substack{x = (x_1, \ldots, x_n) \in \{\pm 1\}^n \\ x_1 = 1}} f(x) \cdot 1 + \sum_{\substack{x = (x_1, \ldots, x_n) \in \{\pm 1\}^n \\ x_1 = -1}} f(x) \cdot (-1) \right) \\
&= \frac{1}{2^n} \left(\sum_{x' \in \{\pm 1\}^{n-1}} f\left(1, x'\right) - \sum_{x' \in \{\pm 1\}^{n-1}} f\left(-1, x'\right) \right) \\
&= \frac{1}{2^n} \sum_{x' \in \{\pm 1\}^{n-1}} \left(f\left(1, x'\right) - f\left(-1, x'\right) \right) \\
&= \frac{1}{2^n} \sum_{\substack{x' \in \{\pm 1\}^{n-1} \\ f(1, x') \neq f(-1, x')}} \left(f\left(1, x'\right) - f\left(-1, x'\right) \right).
\end{aligned}$$

Since f is monotone, then $f(1, x') \geq f(-1, x')$ for all $x' \in \{\pm 1\}^{n-1}$. Hence, for all $x' \in \{\pm 1\}^{n-1}$, if $f(1, x') \neq f(-1, x')$, then $f(1, x') = 1$ and $f(-1, x') = -1$, so $f(1, x') - f(-1, x') = 1 - (-1) = 2$. Therefore,

$$\begin{aligned}
\hat{f}(\{1\}) &= \frac{1}{2^n} \sum_{\substack{x' \in \{\pm 1\}^{n-1} \\ f(1, x') \neq f(-1, x')}} 2 \\
&= \frac{1}{2^n} \cdot 2 \left| \left\{ x' \in \{\pm 1\}^{n-1} : f\left(1, x'\right) \neq f\left(-1, x'\right) \right\} \right| \\
&= \frac{1}{2^{n-1}} \left| \left\{ x' \in \{\pm 1\}^{n-1} : f\left(1, x'\right) \neq f\left(-1, x'\right) \right\} \right|.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
\mathit{Inf}_1(f) &= \mathop{\mathbb{P}}_{x \in \{\pm 1\}^n} \left[f(x) \neq f\left(x^{\oplus 1}\right) \right] \\
&= \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} \mathbb{1} \left[f(x) \neq f\left(x^{\oplus 1}\right) \right] \\
&= \frac{1}{2^n} \cdot 2 \sum_{x' \in \{\pm 1\}^{n-1}} \mathbb{1} \left[f\left(1, x'\right) \neq f\left(-1, x'\right) \right] \\
&= \frac{1}{2^{n-1}} \left| \left\{ x' \in \{\pm 1\}^{n-1} : f\left(1, x'\right) \neq f\left(-1, x'\right) \right\} \right| \\
&= \hat{f}(\{1\}).
\end{aligned}$$

This completes the proof. $\square$

(b) *Collaborators and sources:* none.

Proof. Let $n \in \mathbb{N}$ be odd. Let $f : \{\pm 1\}^n \to \{\pm 1\}$ be the majority function, i.e., $f(x) = \text{sign}(\sum_{i=1}^n x_i)$ for all $x = (x_1, \ldots, x_n) \in \{\pm 1\}^n$. First, we show that f is monotone. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \{\pm 1\}^n$ be such that $x_i \leq y_i$ for all $i \in [n]$. Then $\sum_{i=1}^n x_i \leq \sum_{i=1}^n y_i$, so $f(x) = \text{sign}(\sum_{i=1}^n x_i) \leq \text{sign}(\sum_{i=1}^n y_i) = f(y)$. This proves that f is monotone.

Second, let $g : \{\pm 1\}^n \to \{\pm 1\}$ be monotone. Then

$$\begin{aligned}
\mathit{Inf}(g) &= \sum_{i=1}^n \mathit{Inf}_i(g) \\
&= \sum_{i=1}^n \hat{g}(\{i\}) && \text{(part (a))} \\
&= \sum_{i=1}^n \langle g, \chi_{\{i\}} \rangle \\
&= \sum_{i=1}^n \frac{1}{2^n} \sum_{x \in \{\pm 1\}^n} g(x)\chi_{\{i\}}(x) \\
&= \frac{1}{2^n} \sum_{x=(x_1,\ldots,x_n) \in \{\pm 1\}^n} g(x) \sum_{i=1}^n x_i \\
&\leq \left| \frac{1}{2^n} \sum_{x=(x_1,\ldots,x_n) \in \{\pm 1\}^n} g(x) \sum_{i=1}^n x_i \right| \\
&\leq \frac{1}{2^n} \sum_{x=(x_1,\ldots,x_n) \in \{\pm 1\}^n} |g(x)| \left| \sum_{i=1}^n x_i \right| && \text{(triangle inequality)} \\
&= \frac{1}{2^n} \sum_{x=(x_1,\ldots,x_n) \in \{\pm 1\}^n} \left| \sum_{i=1}^n x_i \right|. && \text{(since } g(x) \in \{\pm 1\} \text{ for all } x \in \{\pm 1\}^n\text{)}
\end{aligned}$$

Third, since f is monotone,

$$\mathit{Inf}(f) = \frac{1}{2^n} \sum_{x=(x_1,\ldots,x_n) \in \{\pm 1\}^n} f(x) \sum_{i=1}^n x_i.$$

Since n is odd, then $\sum_{i=1}^n x_i \neq 0$. If $\sum_{i=1}^n x_i < 0$, then $f(x) = \text{sign}(\sum_{i=1}^n x_i) < 0$, so

$$\mathit{Inf}(f) = \frac{1}{2^n} \sum_{x=(x_1,\ldots,x_n) \in \{\pm 1\}^n} |f(x)| \left| \sum_{i=1}^n x_i \right| = \frac{1}{2^n} \sum_{x=(x_1,\ldots,x_n) \in \{\pm 1\}^n} \left| \sum_{i=1}^n x_i \right|, \tag{4}$$

since $f(x) \in \{\pm 1\}$ for all $x \in \{\pm 1\}^n$. Otherwise, $\sum_{i=1}^n x_i > 0$, so $f(x) = \text{sign}(\sum_{i=1}^n x_i) > 0$, implying that (4) holds. Hence, (4) holds in both cases. It follows that $\mathit{Inf}(g) \leq \mathit{Inf}(f)$ for any monotone $g : \{\pm 1\}^n \to \{\pm 1\}$, completing the proof. $\square$