# Randomization in Recent Progress on Traveling Salesman Problem

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MIT 6.842

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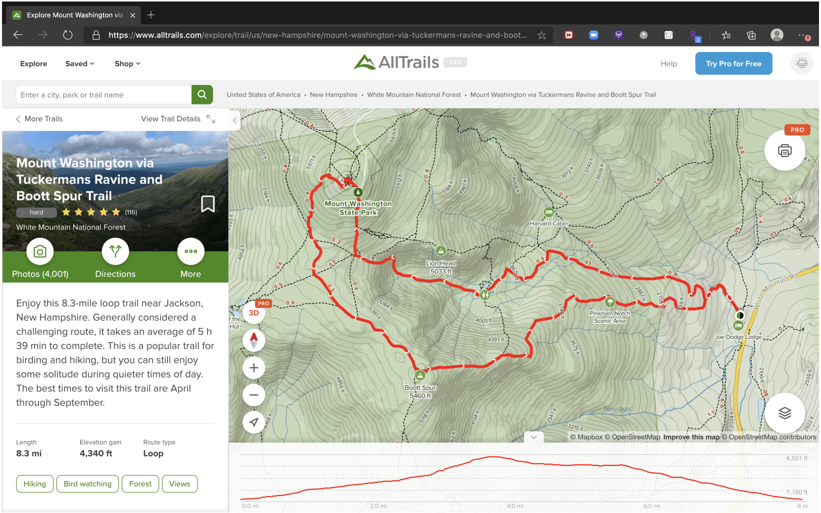
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hard
(116)
White Mountain National Forest
Photos (4,001)
Directions
More
Enjoy this 8.3-mile loop trail near Jackson, New Hampshire. Generally considered a challenging route, it takes an average of 5 h 39 min to complete. This is a popular trail for birding and hiking, but you can still enjoy some solitude during quieter times of day. The best times to visit this trail are April through September.
Length
8.3 mi
Elevation gain
4,340 ft
Route type
Loop
Hiking
Bird watching
Forest
Views
Mount Washington State Park
Harvard Cabin
Lion Head 5033 ft
Pinkham Notch Scenic Area
Joe Dodge Lodge
Boott Spur 5460 ft
White Mountain Rd
PRO
3D
© Mapbox © OpenStreetMap Improve this map © OpenStreetMap contributors
6,551 ft
1,780 ft
0.0 mi
2.0 mi
4.0 mi
6.0 mi

# Metric Traveling Salesman Problem

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**Input:** a set $V$ of vertices and pairwise symmetric costs $c : V \times V \to \mathbb{R}_+$ which satisfies the triangle inequality

**Output:** a shortest tour that visits each vertex exactly once

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*It belongs to the most seductive problems in combinatorial optimization, thanks to a blend of complexity, applicability, and appeal to imagination.*

— *Lex Schrijver*

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$O \leftarrow \{v \in V : \deg_T(v) \text{ is odd}\}$<br>
$M \leftarrow$ a minimum-weight perfect matching in $G[O]$<br>
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shortcut repeated vertices to form a Hamiltonian tour

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**the "$4/3$-conjecture"**

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**Theorem (Karlin, Klein and Oveis Gharan, 2021)**

*For some absolute constant $\varepsilon > 10^{-36}$, there exists a* **randomized** *algorithm that outputs a tour with expected cost at most $3/2 - \varepsilon$ times the cost of the optimum solution.*

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- We assume that the metric is a **graph metric**, i.e., there exists an unweighted graph where the metric is the shortest path metric of that graph. This special case is due to Oveis Gharan, Saberi and Singh (2011).

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- To focus on the randomization part of the proof, we assume that all proper cuts with respect to $x$ have size at least $2 + \varepsilon$. In the general case, we exploit the structure of near-min-cuts, called the **deformable polygon representation** developed by Benczúr and Goemans.

# Subtour Elimination LP

$$
\begin{aligned}
\text{minimize} \quad & \sum_{u,v \in V} x_{\{u,v\}} c(u,v) \\
\text{subject to} \quad & \sum_{\{u,v\} \in \delta(S)} x_{\{u,v\}} \geq 2 && \forall S \subsetneq V, S \neq \emptyset \\
& \sum_{\{u,v\} \in \delta(\{v\})} x_{\{u,v\}} = 2 && \forall v \in V \\
& x_{\{u,v\}} \in [0,1] && \forall u, v \in V
\end{aligned}
$$

# $O$-Joins

**Definition ($O$-joins)**

A set $F \subset E$ is an $O$-**join** if every vertex $v \in O$ has odd degree in $F$, and every other vertex has even degree in $F$.

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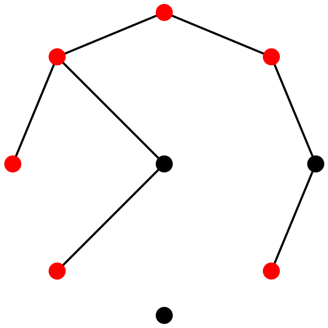

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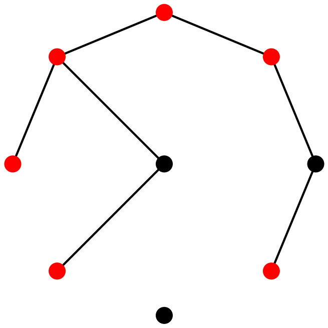

**observation:** an $O$-join is a union of paths connecting red vertices

# $O$-Joins

**Theorem (Edmonds and Johnson, 1973)**

*For any graph $G = (V, E)$ with edge costs $c : E \to \mathbb{R}_+$ and for any $O \subset V$ with $|O|$ even, the minimum cost of an $O$-join equals the optimum value of the following LP:*

$$\begin{array}{lrl} \textit{minimize} & c(y) & \\ \textit{subject to} & y(\delta(S)) \geq 1 & \forall S \subsetneq V, |S \cap O| \textit{ odd} \\ & y_e \geq 0 & \forall e \in E \end{array}$$

# A Randomized Rounding Algorithm

## Definition ($\lambda$-uniform distributions)

For $\lambda : \mathbb{E} \to \mathbb{R}_{\geq 0}$, a $\lambda$-**uniform** distribution $\mu_\lambda$ on spanning trees on a graph $G$ satisfies that for any spanning tree $T$,

$$\mathbb{P}_\mu[T] \propto \prod_{e \in E(T)} \lambda_e.$$

## Algorithm (Oveis Gharan, Saberi and Singh, 2011)

$x \leftarrow$ an optimal solution to the subtour elimination LP
$\mu \leftarrow$ a $z$-uniform distribution with marginal $z = (1 - 1/n)x$
$T \sim \mu$
$O \leftarrow$ set of odd degree vertices in $T$
find a minimum cost $O$-join $F$
**return** $T \cup F$

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- Hence, $c(F) \leq c(y) = c(x)/2$.
- However, we only need to satisfy $y(\delta(S)) \geq 1$ for all $S \subset V$ with $|S \cap O|$ odd.
- The idea is to use the randomness of $T$ to assign a slightly value $y(e) = (1/2 - \varepsilon')x(e)$ to some of the edges (with constant probability) while preserving the feasibility of $y$.

# Even Edges, Good Edges

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**Definition (good edges)**

An edge $e$ is **good** if

$$\mathbb{P}[e \text{ is even}] \geq \gamma,$$

for some constant $\gamma$ to be determined later.

# Even Edges, Good Edges

## Lemma

*Suppose that every edge e is good. Then*

$$\mathbb{E}[c(F)] \leq \left(\frac{1}{2} - \Omega(\varepsilon \cdot \gamma)\right) c(x).$$

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## Proof.

- For any $e \in E$, let

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- If $v$ has odd degree, then $y(\delta(\{v\})) = x(\delta(\{v\})) = 1$.
- Since every edge is good,
$$\mathbb{E}[c(y)] \leq \sum_{e \in E} \frac{x(e)}{2}\left(1 - \mathbb{P}[e \text{ even}]\frac{\varepsilon}{4}\right) \leq \left(\frac{1}{2} - \Omega(\varepsilon \cdot \gamma)\right) c(x).$$

$\square$

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**Theorem (Hoeffding, 1956)**

*For any Bernoulli's $B_1, \ldots, B_n$ with success probabilities $p_1, \ldots, p_n$ and any $g : \mathbb{R} \to \mathbb{R}$, $\mathbb{E}[g(B_1 + \ldots + B_m)]$ is minimized when $p_1, \ldots, p_m \in \{0, p, 1\}$ for some fixed $p$.*