Math 321 Lecture 23

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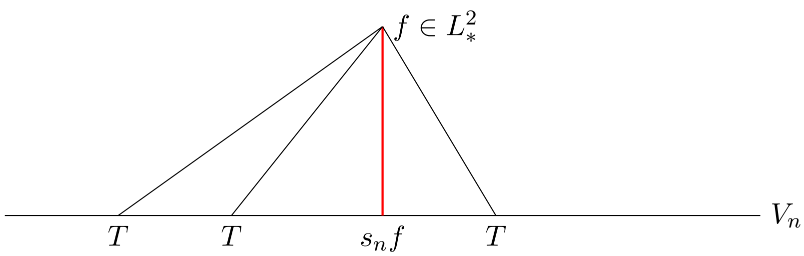
1 Fourier Series (Cont'd)

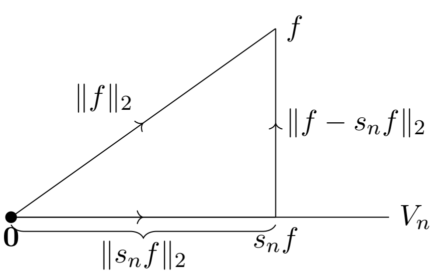
Let $f \in \mathcal{C}^{2\pi}$. Last time we saw that $s_n f = n^{\text{th}}$ partial Fourier sum of f is a "distance-minimizer" in the following sum:

$$\|f - s_n f\|_2 = \min\{\|f - T\|_2 : T \in V_n\}, \qquad V_n = \operatorname{span}\{1, \cos kx, \sin kx : 1 \le k \le n\}.$$



We showed

$$\boxed{\|f\|_2^2 = \|s_n f\|_2^2 + \|f - s_n f\|_2^2.} \tag{*}$$



We have

$$\begin{aligned} (*) \Rightarrow \|s_n f\|_2^2 &\le \|s_n f\|_2^2 + \|f - s_n f\|_2^2 \\ &= \|f\|_2^2. && \text{by } (*) \end{aligned}$$

Recall: $s_n f(x) = \alpha_0 + \sum_{k=1}^n (\alpha_k \cos kx + \beta_k \sin kx)$. Therefore,

$$\|s_n f\|_2^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |s_n f(x)|^2 dx = \underbrace{\alpha_0^2 + \frac{1}{2} \sum_{k=1}^n \left(\alpha_k^2 + \beta_k^2\right)^2}_{\text{checked last time}}.$$

Get

$$\boxed{\underbrace{\alpha_0^2 + \frac{1}{2}\sum_{k=1}^{n}\left(\alpha_k^2 + \beta_k^2\right)}_{\text{depend on } n \text{ and increases with } n} \quad \leq \quad \underbrace{\|f\|_2^2}_{\text{independent of } n}} : \text{ Bassel's inequality.}$$

Let $n \nearrow \infty$ to get: $\alpha_0^2 + \frac{1}{2}\sum_{k=1}^{n}\left(\alpha_k^2 + \beta_k^2\right)$ is convergent sum, whose value is $\leq \|f\|_2^2$.

Theorem 1 (Plancherel)**.** For every $\underbrace{f \in \mathcal{C}^{2\pi}}_{\rightsquigarrow f \text{ Riemann integrable } (\alpha(x) = x) \text{ [Exercise]}}$,

1. $\|f - s_n f\|_2 \xrightarrow{\text{as } n \to \infty} 0$.
   (In other words, $s_n f$ provides a good approximation of f in the sense of L_*^2.)
2. $\alpha_0^2 + \frac{1}{2}\sum_{k=1}^{\infty}\left(\alpha_k^2 + \beta_k^2\right) = \|f\|_2^2$ (Parseval identity).
   Say that $\{c_1, c_k \cos kx, d_k \sin kx : k \geq 1\}$ is an "*orthonormal*" basis of L_*^2.

Example from linear algebra:

$\mathbb{R}^2, \mathbf{v} \in \mathbb{R}^2, \mathbf{v} = \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \alpha_1 \mathbf{w}_1 + \alpha_2 \mathbf{w}_2$ with $\|\mathbf{v}\|^2 = \alpha_1^2 + \alpha_2^2$ *provided* $\{\mathbf{w}_1, \mathbf{w}_2\}$ is an $\boxed{\text{orthonormal}}$ basis of $\mathbb{R}^2$.

$$\mathbf{v} = \boxed{v_1}\underbrace{\begin{pmatrix} 1 \\ 0 \end{pmatrix}}_{=\mathbf{e}_1} + \boxed{v_2}\underbrace{\begin{pmatrix} 0 \\ 1 \end{pmatrix}}_{=\mathbf{e}_2} = \underbrace{\boxed{\frac{v_1+v_2}{2}}}_{=a_1}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ 1 \end{pmatrix}}_{=\mathbf{f}_1} + \underbrace{\boxed{\frac{v_1-v_2}{2}}}_{=a_2}\underbrace{\frac{1}{\sqrt{2}}\begin{pmatrix} 1 \\ -1 \end{pmatrix}}_{=\mathbf{f}_2} = \boxed{b_1}\begin{pmatrix} 1 \\ 0 \end{pmatrix} + \boxed{b_2}\begin{pmatrix} 1 \\ 1 \end{pmatrix}.$$

Then,

$$\|\mathbf{v}\| = v_1^2 + v_2^2 = a_1^2 + a_2^2 \neq b_1^2 + b_2^2.$$

Proof of Plancherel's theorem. 1. Take any $T \in V_n = \operatorname{span}\{1, \cos kx, \sin kx : 1 \leq k \leq n\}$.

$$\|f - T\|_2^2 = \frac{1}{2\pi}\underbrace{|f(x) - T(x)|^2}_{\leq \|f-T\|_\infty^2} dx \qquad \|f - T\|_\infty = \sup_{x \in [-\pi,\pi]} |f(x) - T(x)|$$

$$\Rightarrow \|f - T\|_2 \leq \|f - T\|_\infty$$

$$\Rightarrow \underbrace{\inf\{\|f - T\|_2 : T \in V_n\}}_{=\|f - s_n f\|_2} \leq \underbrace{\inf\{\|f - T\|_\infty : T \in V_n \xrightarrow{n\to\infty} 0}_{\substack{\text{this is true by Weierstrass's second theorem} \\ \text{(there exists a trignometric polynomial } P_n \\ \text{such that } \|P_n - f\|_\infty \xrightarrow{n\to\infty} 0)}} .$$

2. If $\|s_n f - f\|_2 \xrightarrow{n\to\infty} 0$ (know this by part 1), we have

$$\lim_{n\to\infty} \|s_n f\|_2 = \|f\|_2,$$

because $|\|s_n f\|_2 - \|f\|_2| \leq \|s_n f - f\|_2$ by the triangular inequality. We need to show

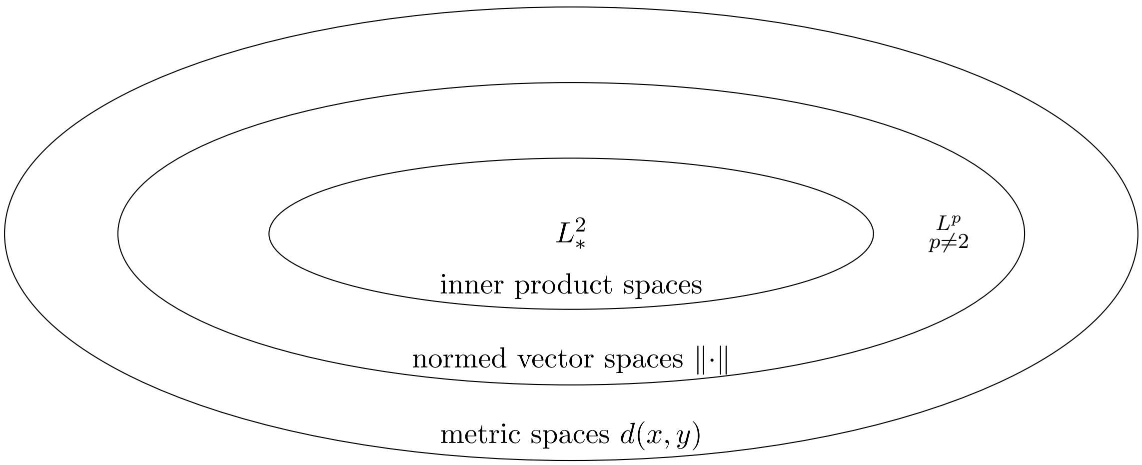
$$\|s_n f\|_2 - \|f\|_2 \leq \|s_n f - f\|_2,$$
$$\|f\|_2 - \|s_n f\|_2 \leq \|s_n f - f\|_2.$$

Therefore,

$$\lim_{n\to\infty}\|s_nf\|_2=\|f\|_2 \Rightarrow \lim_{n\to\infty}\|s_nf\|_2^2=\|f\|_2^2 \Rightarrow \lim_{n\to\infty}\left[\alpha_0^2+\frac{1}{2}\sum_{k=1}^{n}\left(\alpha_k^2+\beta_k^2\right)\right]=\|f\|_2^2.$$

$\square$

L^2-primer:



Definition 1. Let $f,g\in L^2_*$. Define

$$\langle f,g\rangle=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)g(x)dx.$$