Math 321 Lecture 24

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1 Fourier Series (Cont'd)

1.1 Inner Product Spaces

Definition 1. Let V be a vector space over $\mathbb{C}$ (or $\mathbb{R}$). Say $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{C}$ or $\mathbb{R}$ is an **inner product** if it obeys the following properties:

1. Conjugate symmetry: $\langle \mathbf{v}, \mathbf{w} \rangle = \overline{\langle \mathbf{w}, \mathbf{v} \rangle}$ for all $\mathbf{v}, \mathbf{w} \in V$.
2. Linearity in the first coordinate:
$$\begin{aligned} \langle \alpha \mathbf{v}, \mathbf{w} \rangle &= \alpha \langle \mathbf{v}, \mathbf{w} \rangle, && \forall \alpha \in \mathbb{C}, \mathbf{v}, \mathbf{w} \in V, \\ \langle \mathbf{v}_1 + \mathbf{v}_2, \mathbf{w} \rangle &= \langle \mathbf{v}_1, \mathbf{w} \rangle + \langle \mathbf{v}_2, \mathbf{w} \rangle, && \forall \mathbf{v}_1, \mathbf{v}_2, \mathbf{w} \in V. \end{aligned}$$
3. Positive definiteness: For any $\mathbf{v} \in V$, $\langle \mathbf{v}, \mathbf{v} \rangle$ is a non-negative real number. $\langle \mathbf{v}, \mathbf{v} \rangle = 0$ if and only if $\mathbf{v} = \mathbf{0}$.

Examples:

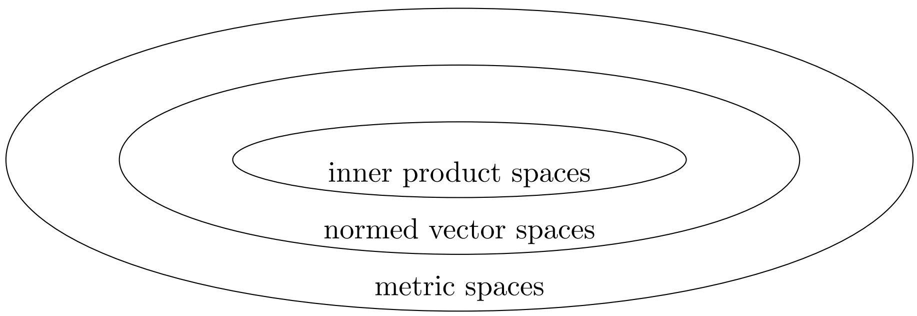
1. $\mathbb{R}^n$ or $\mathbb{C}^n$. Let $\mathbf{v} = (v_1, \ldots, v_n)$ and $\mathbf{w} = (w_1, \ldots, w_n)$. Then,
$$\langle \mathbf{v}, \mathbf{w} \rangle = \sum_{j=1}^{n} v_j \overline{w_j} = \text{Euclidean dot product of } \mathbf{v} \text{ and } \mathbf{w}.$$
2. $\ell^2(\mathbb{N}) = \left\{ \mathbf{x} = (x_1, x_2, x_3, \ldots) : x_j \in \mathbb{C}, \sum_{j=1}^{\infty} |x_j|^2 < \infty \right\}$. Define
$$\langle \mathbf{x}, \mathbf{y} \rangle = \sum_{j=1}^{\infty} x_j \overline{y_j}.$$
This is well-defined as an absolutely convergent infinite series because
$$\sum_{j=1}^{n} \|x_j \overline{y_j}\| \underbrace{\leq}_{\text{Cauchy-Schwarz}} \left(\sum_{j=1}^{n} |x_j|^2 \right)^{\frac{1}{2}} \left(\sum_{j=1}^{n} |y_j|^2 \right)^{\frac{1}{2}},$$
and let $n \to \infty$.

3. $L_*^2[-\pi, \pi] = \mathcal{C}^{2\pi}$, equipped with the norm

$$\|f\|_2 = \left[\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(x)|^2 dx\right]^{\frac{1}{2}}.$$

(Aside: $L^2[0,1] = \left\{f : \int_0^1 |f(x)|^2 dx < \infty\right\}$.) Define

$$\langle f, g\rangle = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)\overline{g(x)}dx.$$



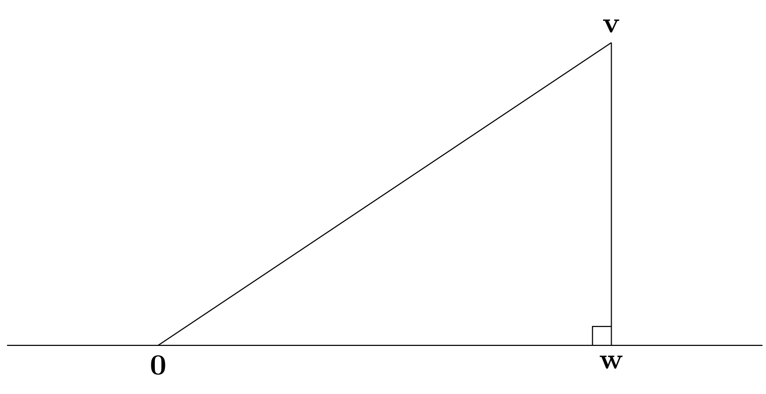
Facts:

1. Every inner product generates a norm on V; $\|\mathbf{v}\| = \sqrt{\langle \mathbf{v}, \mathbf{v}\rangle}$. Check that this obeys the properties of a norm.
2. Every inner product obeys the Cauchy-Swartz inequality

$$|\langle \mathbf{v}, \mathbf{w}\rangle| \leq \|\mathbf{v}\| \cdot \|\mathbf{w}\|. \qquad \text{(Exercise)}$$

3. Note that 2 offers a way to define the notion of "angle" between two vectors $\mathbf{v}$ and $\mathbf{w}$; we say that the angle between $\mathbf{v}$ and $\mathbf{w}$ is θ if

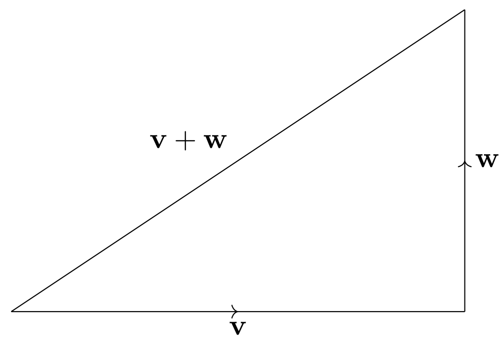
$$\cos\theta = \frac{|\langle \mathbf{v}, \mathbf{w}\rangle|}{\|\mathbf{v}\| \cdot \|\mathbf{w}\|}.$$



4. Many identities from Eucliean geometry carry over to an inner product space V.

(a) **Pythagorean theorem:** Suppose $\mathbf{v}, \mathbf{w} \in V$ with $\mathbf{v} \perp \mathbf{w}$ (i.e., $\langle \mathbf{v}, \mathbf{w} \rangle = 0$). Then,

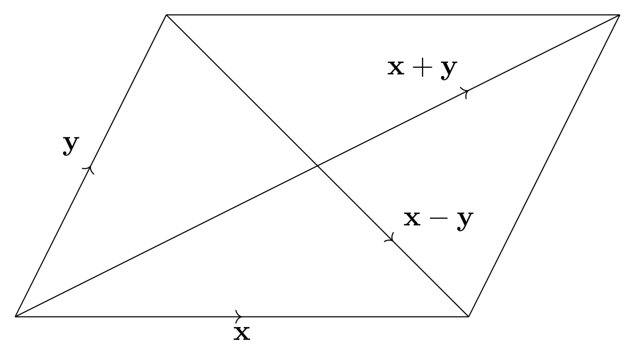
$$\|\mathbf{v} + \mathbf{w}\|^2 = \|\mathbf{v}\|^2 + \|\mathbf{w}\|^2.$$



(b) **Parallelogram law:**

$$\|\mathbf{x} + \mathbf{y}\|^2 + \|\mathbf{x} - \mathbf{y}\|^2 = 2\left[\|\mathbf{x}\| + \|\mathbf{y}\|^2\right],$$

for all $\mathbf{x}, \mathbf{y}$ in an inner product space.



Exercise: Use the parallelogram law to show that if ℓ^p or L^p_* is an inner product space, then $p = 2$.

1.2 Uniform Convergence (or Lackthereof) of Fourier Series

Message:

1. If f is merely known to be continuous and 2π-periodic, then it is *not* in general true that $\boxed{S_n f \xrightarrow{n\to\infty} f \text{ uniformly}}$ (i.e., $\sup_{x\in[-\pi,\pi]} |s_n f(x) - f(x)| = \|s_n f - f\|_\infty \xrightarrow{n\to\infty} 0$) or even pointwise.

2. Plancherel: $\|s_n f - f\|_2 \xrightarrow{n\to\infty} 0$.

$$\underbrace{\textbf{uniform convergence}}_{\substack{f_n \to f \text{ uniformly} \Leftrightarrow \\ \sup_x |f_n(x) - f(x)| \xrightarrow{n\to\infty} 0}} \Rightarrow \underbrace{\textbf{pointwise convergence}}_{\text{For every } x,\ f_n(x) \xrightarrow{n\to\infty} f(x)},$$

$$L^2 \textbf{ convergence: } \int_{-\pi}^{\pi} |f_n(x) - f(x)|^2 \to 0.$$