Math 321 Lecture 16

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1 Theorem

Theorem 1. $\mathcal{R}_\alpha[a,b]$ is closed under uniform convergence; i.e., if $\{f_n : n \geq 1\} \subseteq \mathcal{R}_\alpha[a,b]$ and $f_n \xrightarrow{n\to\infty} f$ uniformly on $[a,b]$, then $f \in \mathcal{R}_\alpha[a,b]$ and $\int_a^b f_n d\alpha \xrightarrow{n\to\infty} \int_a^b f d\alpha$.

Proof. **Step 1:** To show that $f \in \mathcal{R}_\alpha[a,b]$. Will invoke Riemann's condition for this. Fix $\epsilon > 0$. Need to find a partition $P = P_\epsilon$ such that $U_\alpha(f,P) - L_\alpha(f,P) < \epsilon$.

For any partition Q of $[a,b]$,

$$U_\alpha(f,Q) - L_\alpha(f,Q) = \sum_{i=1}^n \underbrace{\omega(f,I_i)}_{=(M_i - m_i)} \omega(\alpha, I_i),$$

where

$$Q = \{x_0 = a < x_1 < \ldots < x_n = b\}, \qquad I_i = [x_{i-1}, x_i].$$

Recall:

$$\begin{aligned} & f_n \xrightarrow{n\to\infty} f \text{ uniformly if and only if} \\ \Leftrightarrow\ & \exists N \text{ s.t. } \sup_{x\in[a,b]} |f_n(x) - f(x)| < \frac{\epsilon}{10}\ \forall n \geq N. \end{aligned} \tag{1}$$

Choose $n = N$.

$$\begin{aligned} |f(s) - f(t)| &\overset{\text{triangular inequality}}{\leq} \underbrace{|f_n(s) - f(s)|}_{<\frac{\epsilon}{10} \text{ by (1)}} + \underbrace{|f_n(t) - f(t)|}_{<\frac{\epsilon}{10} \text{ by (1)}} + |f_n(s) - f_n(t)| \\ &< \frac{\epsilon}{5} + |f_n(s) - f_n(t)|. \end{aligned}$$

If $s, t \in I_i = [x_{i-1}, x_i]$, then the above implies

$$\omega(f, I_i) \leq \omega(f_n, I_i) + \frac{\epsilon}{5}. \tag{*}$$

Use (*).

$$U_\alpha(f,Q) - L_\alpha(f,Q) \leq \sum_{i=1}^n \left(\frac{\epsilon}{5} + \omega(f_N, I_i)\right) \omega(\alpha, I_i). \tag{**}$$

Note that

$$\omega(f, I_i) = \sup\{|f(s) - f(t)| : s, t \in I_i\},$$

where

$$|f(s) - f(t)| < \frac{\epsilon}{5} + |f_n(s) - f_n(t)| \leq \underbrace{\frac{\epsilon}{5} + \overbrace{\omega(f_n, I_i)}^{\text{independent of } s,t}}_{\text{an upper bound for } \{|f(s)-f(t)| : s,t \in I_i\}}.$$

This implies that

$$\underbrace{\omega(f, I_i)}_{\text{least upper bound}} \leq \frac{\epsilon}{5} + \omega(f_n, I_i).$$

Choose the partition $Q = P_N$ such that

$$U_\alpha(f_N, P_N) - L_\alpha(f_N, P_N) < \frac{\epsilon}{5}.$$

Such a partition P_N exists by Riemann's condition, since $f_N \in \mathcal{R}_\alpha[a, b]$.

$$\begin{aligned}
U_\alpha(f, P_N) - L_\alpha(f, P_N) &\leq \frac{\epsilon}{5} \sum_{i=1}^{n} \omega(\alpha, I_i) + \underbrace{[U_\alpha(f_N, P_N) - L_\alpha(f_N, P_N)]}_{< \frac{\epsilon}{5}} \\
&= \frac{\epsilon}{5} \underbrace{(\alpha(b) - \alpha(a))}_{C_\epsilon} + \frac{\epsilon}{5}.
\end{aligned}$$

Step 2: To show that

$$\begin{aligned}
&\int_a^b f_n d\alpha \xrightarrow{n\to\infty} \int_a^b f d\alpha \\
\Leftrightarrow\ &\int_a^b (f_n - f) d\alpha \xrightarrow{n\to\infty} 0.
\end{aligned}$$

Use $\|f_n - f\|_\infty < \epsilon$ for all $n \geq N$ to show

$$\left|\int_a^b (f_n - f) d\alpha\right| \underbrace{\leq}_{\text{Justify!}} \int_a^b |f_n - f| d\alpha < \epsilon \int_a^b d\alpha = \epsilon(\alpha(b) - \alpha(a)).$$

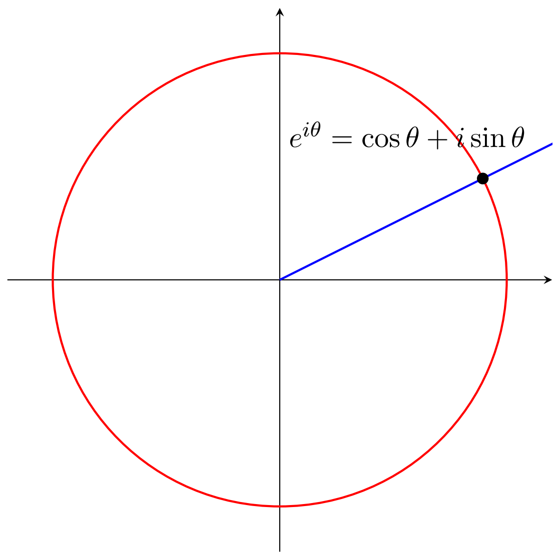
□

2 Bounded Variation

Remark. So far:

1. We have defined $\int_a^b f d\alpha$ only on compact intervals $[a, b]$.
2. Depends on α.
3. Computability issues?
4. Immediate applications to physical problems are unclear.
5. Can $\int_a^x f d\alpha$ be interpreted as some kind of antiderivative?

Example: Let $\Gamma = \left\{e^{i\theta} : \theta \in [0, 2\pi]\right\}$.



Note: A typical path integral

$$\int_P f \underbrace{dz}_{\text{integral of } f \text{ on } \Gamma}.$$

does not correspond to any $\mathcal{R}_\alpha[a, b]$ developed so far.

Need to expand our class of integrators.

Definition 1. Let $\alpha : [a, b] \to \mathbb{R}$. Let P be a partition of $[a, b]$. Define

$$V_a^b(\alpha, P) = \sum_{i=1}^{n} |\alpha(x_i) - \alpha(x_{i-1})|.$$

The **total variation** of α on $[a, b]$ is defined as

$$V_a^b(\alpha) = \sup_P V_a^b(\alpha, P).$$

Call $\alpha = BV[a, b]$ if $V_a^b(\alpha) < \infty$.

Show: $V_a^b(\alpha, P) \leq V_a^b(\alpha, Q)$ if $P \subseteq Q$.