Math 321 Lecture 25

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1 Uniform Convergence and Non-Convergence of Fourier Series

$$\begin{tikzcd}
 & \text{Uniform convergence on } [-\pi,\pi] \arrow[ddl, Rightarrow] \arrow[ddr, Rightarrow] & \\
 & \times \quad \checkmark \ (\text{Weierstrass } 2^{\text{nd}}) & \\
L^2\text{-convergence on } [-\pi,\pi] \arrow[rr, Rightarrow, "/" marking, shift left] & & \text{Pointwise convergence on } [-\pi,\pi] \arrow[ll, Rightarrow, "/" marking, shift left] \\
\checkmark \quad \checkmark & & \times \quad \checkmark \\
\boxed{\text{Plancherel}} & &
\end{tikzcd}$$

1. How does the Fourier series of $f \in \mathcal{C}^{2\pi}$ interact with these different notions of convergence?
2. Does there exist a sequence of trignometric polynomials P_n such that $P_n \to f \in \mathcal{C}^{2\pi}$?

1.1 Dirichlet and Fejér Kernels

Recall:

$$\begin{aligned}
s_N f(x) &= N^{\text{th}} \text{ partial Fourier series of } f = \sum_{k=-N}^{N} \widehat{f}(k) e^{ikx} \\
&\overset{\text{HW 8}}{\underset{\text{Problem 5}}{=\!=\!=}} f * D_N(x), \qquad \text{where} \\
&\boxed{D_N(x) = \sum_{k=-N}^{N} e^{ikx} = \frac{\sin\left[\left(N+\frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)}} : \text{Dirichlet kernel,} \\
f * g(x) &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x-t) g(t) dt.
\end{aligned}$$

Theorem 1. 1. D_N is a trignometric polynomial by definition.

2. $\frac{1}{2\pi}\int_{-\pi}^{\pi} D_N(x)dx = 1$ for all $N \geq 0$ [because $\int_{-\pi}^{\pi} D_N(x)dx = \sum_{k=-N}^{N} \underbrace{\int_{-\pi}^{\pi} e^{ikx}dx}_{\begin{cases} 0, & \forall k \in \mathbb{Z}\setminus\{0\}, \\ 2\pi, & k = 0. \end{cases}} = 2\pi$].

3. There exist absolute constants $c_1, c_2 > 0$ such that for all $N \geq 1$,

$$\boxed{c_1 \log N \overset{(*)}{\leq} \frac{1}{2\pi} \int_{-\pi}^{\pi} |D_N(x)| dx \leq c_2 \log N} \Rightarrow \sup_N \|D_N\|_1 = \infty.$$

Recall:

$$\begin{aligned}\sigma_N f(x) &= \frac{1}{N}[s_1 f(x) + s_2 f(x) + \ldots + s_N f(x)] \\ &= N^{\text{th}} \text{ partial Cesàro sum of } f = f * K_N(x).\end{aligned}$$

Moral: $\sigma_N f$ has a better chance of converging than $s_N f$.

$\{a_N\}: \quad a_1 = 1, \quad a_2 = -1, \quad a_3 = 1, \quad a_4 = -1, \quad \ldots.$

$\{s_N\}: \quad s_1 = a_1 = 1, \quad s_2 = a_1 + a_2 = 0, \quad s_3 = a_1 + a_2 + a_3 = 1, \quad s_4 = 0, \quad \ldots \quad \lim_N s_N$ does not exist.

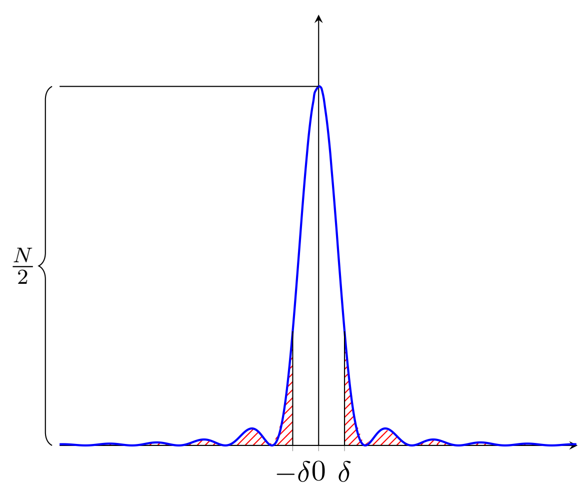
$\{\sigma_N\}: \quad \sigma_1 = s_1 = 1, \quad \sigma_2 = \frac{1}{2}(s_1 + s_2) = \frac{1}{2}, \quad \sigma_3 = \frac{1}{3}(s_1 + s_2 + s_3) = \frac{2}{3}, \quad \ldots \quad \lim_N \sigma_N = 1.$

In HW 8, we saw:

$$K_N(t) = \frac{\sin^2\left(\frac{Nt}{2}\right)}{2N \sin^2\left(\frac{t}{2}\right)} = \frac{1}{N}(D_1(t) + \ldots + D_N(t)).$$

Theorem 2 (Fejér)**.**

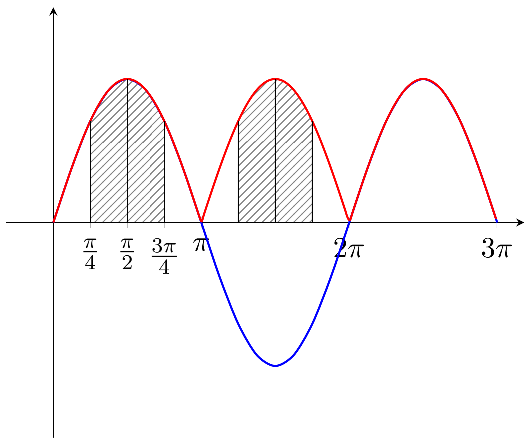
1. K_N is a trignometric polynomial.
2. $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x) dx = 1$ (because $\frac{1}{2\pi}\int_{-\pi}^{\pi} K_N(x) dx = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{2\pi}\int_{-\pi}^{\pi} D_k(x) dx = \frac{1}{N}\underbrace{(1 + \ldots + 1)}_{N \text{ times}} = 1$).
3. $\frac{1}{2\pi}\int_{-\pi}^{\pi} |K_N(x)| dx = 1$ for any $N \Rightarrow \sup_N \|K_N\|_1 = 1 < \infty$.
4. For every $\delta > 0$, $\int_{|x|>\delta} K_N(x) dx \xrightarrow{N\to\infty} 0$.



Corollary 1. Use Fejér's theorem $1-4$ to show that for every $f \in \mathcal{C}^{2\pi}$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(x-y) K_N(y) dy = f * K_N = \sigma_N f \xrightarrow{N\to\infty} f \text{ uniformly}.$$

Proof of (*).



Note that

$$\sin\left(\frac{x}{2}\right) \leq Cx, \qquad x \in [-\pi, \pi],$$

and that

$$\frac{k\pi}{2} - \frac{\pi}{4} \leq \left(N + \frac{1}{2}\right)\frac{x}{2} \leq \frac{k\pi}{2} + \frac{\pi}{4} \text{ for some odd integer } k.$$

Therefore, we have

$$\int_{-\pi}^{\pi} \left| \frac{\sin\left[\left(N+\frac{1}{2}\right)x\right]}{\sin\left(\frac{x}{2}\right)} \right| dx \geq \sum_{\substack{k \text{ odd integer} \\ k \leq N}} \int_{\frac{k\pi}{2}-\frac{\pi}{4}}^{\frac{k\pi}{2}+\frac{\pi}{4}} \frac{c_0}{|x|} dx \geq c_0 \sum_{\substack{k \text{ odd integer} \\ k \leq N}} \frac{1}{k} \geq c_1 \log N.$$

□