Math 321 Lecture 9

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1 Weierstrass Theorem (Take 2)

Theorem 1 (Classical Weierstrass)**.** $\underbrace{\mathcal{P}_{\mathbb{R} \text{ or } \mathbb{C}}}_{\substack{\text{space of polynomials on } [a,b] \\ \text{with coefficients in } \mathbb{R} \text{ or } \mathbb{C}}} \overset{\text{dense}}{\subseteq} \underbrace{C([a,b]; \mathbb{R} \text{ or } \mathbb{C})}_{\substack{\text{class of } \mathbb{R}\text{-valued or } \mathbb{C}\text{-valued} \\ \text{functions on } [a,b]}}$.

Question: Given a compact metric space X, can we determine whether a subset $S \subseteq C(X;\mathbb{R})$ is dense in $C(X;\mathbb{R})$?

Remark. Polynomials are not always well-defined on a general metric space X.

Exercise: What is an example of a compact metric space X (not $[a,b]$) on which polynomials can be defined?

1. $X = \mathbb{Z} \pmod p = \{0, 1, \ldots, p-1\}$ is finite, hence compact.
2. $X = [a,b] \cap \mathbb{Q}^c$ is not compact.
3. $K \subseteq \mathbb{R}^n$ that is compact (i.e., closed and bounded); e.g., $K = [0,1]^n$ or $B(0;1)$ or a sphere $\mathbb{S}^{n-1}$.

   Example: $\boxed{n = 2,\ P(x,y) = xy}$.

   A general polynomial in n variables of degree $\leq R$ is of the form

   $$P(\underbrace{x_1, x_2, \ldots, x_n}_{\mathbf{x}}) = \sum_{\substack{\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{Z}^n_{\geq 0} \\ |\alpha| = \alpha_1 + \ldots + \alpha_n \leq R}} \underbrace{c_\alpha}_{\in \mathbb{R}} x^\alpha,$$

   where

   $$\mathbf{x}^\alpha \overset{\text{def}}{=} x_1^{\alpha_1} x_2^{\alpha_2} \ldots x_n^{\alpha_n}.$$

Definition 1. Let A be a **vector space** over $\mathbb{R}$. Say that A is an **algebra** if there exists an operation $\times : A \times A \to A$, denoted by $(f,g) \mapsto fg$, obeying

1. $f(gh) = (fg)h$;
2. $f(g+h) = fg + fh$;
3. $(g+h)f = gf + hf$;
4. $\alpha(fg) = (\alpha f)g = f(\alpha g)$ for all $\alpha \in \mathbb{R}$.

A is called **commutative** if $fg = gf$.

Say that a commutative algebra A has an **identity element** e if there exists $e \in A$ such that $fe = ef = f$ for all $f \in A$.

If A is a normed vector space and an albegra, we call A a **normed algebra** if $\|fg\| \leq \|f\| \cdot \|g\|$ (called a **Banach algebra** if A is complete).

Examples:

1. $\mathbb{R}$ is a Banach algebra.
2. X is compact; $\mathcal{B}(X; \mathbb{R})$, the space of bounded real-valued functions on X, is also a Banach algebra.
3. $A = C(X; \mathbb{R})$, where X is compact, is a commutative Banach algebra with e equal to the constant function 1.
4. For compact $K \subseteq \mathbb{R}$, $\mathcal{P} = \{\text{polynomials on } K\}$ is a commutative algebra, but not Banach. Note that $\overline{\mathcal{P}} = C(X; \mathbb{R})$.

Question revised: Let $S \overset{\text{sub-algebra}}{\subseteq} \underbrace{C(X; \mathbb{R})}_{\text{Banach algebra}}$. When is S dense in $C(X; \mathbb{R})$?

Theorem 2 (Stone-Weierstrass theorem)**.** An algebra $S \subseteq C(X; \mathbb{R})$ is dense in $C(X; \mathbb{R})$ if S **separates points** and **vanishes at no point**.

Definition 2. Say a set $\mathcal{A} \subseteq C(X; \mathbb{R})$ **separates points** if for any two points $x, y \in X, x \neq y$, there exists $f \in \mathcal{A}$ such that $f(x) \neq f(y)$.

Definition 3. Say $\mathcal{A} \subseteq C(X; \mathbb{R})$ **vanishes at no point** if for all $x \in X$, there exists $f \in \mathcal{A}$ such that $f(x) \neq 0$.