Math 321 Lecture 8

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1 Proof of Arzela-Ascoli Theorem (Cont'd)

Proof (cont'd). Let (X, d) be a compact metric space.

Step 2: Assume $\mathcal{F} \subseteq C(X)$ is closed, uniformly bounded and equicontinuous. We need to show that $\mathcal{F}$ is compact.

(A)

X compact $\Rightarrow$ X separable i.e., X admits a countable **dense set** Y.

X totally bounded $\Rightarrow X \subseteq \bigcup_{i=1}^{K_n} B\left(x_i^{(n)}; \frac{1}{n}\right)$. $Y = \bigcup_{n=1}^{\infty} \left\{x_i^{(n)}; 1 \leq i \leq K_n\right\}$ is countable and dense.

(B)

Suppose $\{f_n; n \geq 1\}$ is a collection of functions on any metric space X, with the property that $\lvert f_n(x)\rvert \leq M$ for all $n \geq 1$ and for all $x \in X$. Let $Y \subseteq X$ be any countable subset. Then there exists a subsequence $n_k \nearrow \infty$ such that $\{f_{n_k}(y); k \geq 1\}$ converges to some limit, for every $y \in Y$.

(HW 1, problem 1)

Outline of proof: Recall (Math 320) that a set A in a metric space $\mathcal{A}$ is compact if and only if every sequence $\{a_n; n \geq 1\} \subseteq A$ admits a convergent subsequence $\{a_{n_k}; k \geq 1\}, a_{n_k} \xrightarrow{k \to \infty} a \in A$. Accordingly, pick any sequence $\{f_n; n \geq 1\} \subseteq \mathcal{F}$. Our job is to find a subsequence $\{n_k; k \geq 1\} \subseteq \{f_n; n \geq 1\}$ such that $f_{n_k} \xrightarrow{k \to \infty} f$ uniformly on X.

Claim: (A) and (B) $\Rightarrow$

(C)

The subsequence $\{f_{n_k}; k \geq 1\}$ is uniformly Cauchy on X; i.e., $\lVert f_{n_k} - f_{n_{k'}}\rVert_\infty \xrightarrow{k,k' \to \infty} 0$.

$\Downarrow C(X)$ is compact

There exists $f \in C(X)$ such that $f_{n_k} \xrightarrow{k \to \infty} \underbrace{f}_{\text{a limit point of } \mathcal{F}}$ uniformly on X. $f \in \mathcal{F}$ because $\mathcal{F}$ is closed, hence contains all its limit points.

Proof of claim: (A) and (B) $\Rightarrow$ (C).

Start with any $\epsilon > 0$. We need to find $K_0 = K_0(\epsilon) \geq 1$ such that

$$\|f_{n_k} - f_{n_{k'}}\|_\infty = \sup_{x \in X} |f_{n_k}(x) - f_{n_{k'}}(x)| < \epsilon \ \forall k, k' \geq K_0.$$

By equicontinuity of $\mathcal{F}$, there exists $\delta > 0$ such that

$$d(x, x') < \delta \Rightarrow |g(x) - g(x')| < \frac{\epsilon}{3} \ \forall g \in \mathcal{F}. \tag{*}$$

Recall X is compact, hence totally bounded. So we can cover X by finitely many balls of radius δ; i.e., $y_1, y_2, \ldots, y_R \in Y$ such that

$$X \subseteq \bigcup_{i=1}^{R} B(y_i; \delta).$$

Given any $x \in X$, there exsts $y_i \in Y, 1 \leq i \leq R$ such that $d(x, y_i) < \delta$. Thus,

$$|f_{n_k}(x) - f_{n_{k'}}(x)| \leq \underbrace{|f_{n_k}(x) - f_{n_k}(y_i)|}_{\mathrm{I}} + \underbrace{|f_{n_k}(y_i) - f_{n_{k'}}(y_i)|}_{\mathrm{II}} + \underbrace{|f_{n_{k'}}(y_i) - f_{n_{k'}}(x)|}_{\mathrm{III}}.$$

By (*), I and III are each bounded above by $\frac{\epsilon}{3}$, provided $d(x, y) < \delta$.

We want to choose $y \in Y$ and use the pointwise convergence of $\{f_{n_k}(y); k \geq 1\}$. Since (B) implies that $\{f_{n_k}(y); k \geq 1\}$ is convergent for every $y \in Y$, we have $\{f_{n_k}(y); k \geq 1\}$ is Cauchy for all $y \in Y$; i.e., given any $\epsilon > 0$, there exists $K = K(y, \epsilon) \geq 1$ such that $|f_{n_k}(y) - f_{n_{k'}}(y)| < \frac{\epsilon}{3}$ for all $k, k' \geq K(y, \epsilon)$. Thus, II $< \frac{\epsilon}{3}$ provided $k, k' \geq K_0 \stackrel{\text{def}}{=} \max(K(y_1, \epsilon), \ldots, K(y_R, \epsilon))$. $\square$