Math 321 Lecture 28

Yuchong Pan

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1 Inverse and Implicit Function Theorems

1.1 Differentiability

Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m, m, n \geq 1$.
Question: Where is $\mathbf{f}$ *differentiable*?

Definition 1. Say that $\mathbf{f}$ is **differentiable** at $\mathbf{x}_0 \in \mathbb{R}^n$ if there exists a linear transformation $\underbrace{\mathbf{A} : \mathbb{R}^n \to \mathbb{R}^m}_{\text{can be represented as an } m \times n \text{ matrix}}$ such that

$$\frac{\left\| \underbrace{\mathbf{f}(\overbrace{\mathbf{x}_0}^{\in\mathbb{R}^n} + \overbrace{\mathbf{h}}^{\in\mathbb{R}^n})}_{\in\mathbb{R}^m} - \underbrace{\mathbf{f}(\mathbf{x}_0)}_{\in\mathbb{R}^m} - \underbrace{\mathbf{A}}_{m\times n}\underbrace{\mathbf{h}}_{n\times 1} \right\|}{\|\mathbf{h}\|} \xrightarrow{\mathbf{h}\to\mathbf{0}} 0. \qquad (*)$$

Call $A = \mathbf{f}'(\mathbf{x}_0)$, the "**partial derivative** of $\mathbf{f}$ at $\mathbf{x}_0$".

Examples:

1. If $m = n = 1$, our standard definition of differentiability says that

$$\begin{aligned} & f'(x) \lim_{h\to 0} \frac{f(x_0+h) - f(x_0)}{h} \text{ exists} \\ \Leftrightarrow\ & \lim_{h\to 0} \left(\frac{f(x_0+h) - f(x_0)}{h} - f'(x_0) \right) = 0 \\ \Leftrightarrow\ & \lim_{h\to 0} \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{h} = 0 \\ \Leftrightarrow\ & \lim_{h\to 0} \left| \frac{f(x_0+h) - f(x_0) - f'(x_0)h}{h} \right| = 0 \qquad \text{which agrees with Definition (*).} \end{aligned}$$

2. **Observation:** If $\mathbf{A}$ exists, it is unique.

   Suppose there exist $\mathbf{A}, \mathbf{B} : \mathbb{R}^n \to \mathbb{R}^m$ obeying (*).

$$\begin{aligned} \frac{\|(\mathbf{A} - B)\mathbf{h}\|}{\|\mathbf{h}\|} &= \frac{1}{\|\mathbf{h}\|} \left\| \mathbf{A}\mathbf{h} - \underbrace{(\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0))} + \underbrace{(\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0))} - \mathbf{B}\mathbf{h} \right\| \\ &\leq \frac{\|\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - \mathbf{A}\mathbf{h}\|}{\|\mathbf{h}\|} + \frac{\|\mathbf{f}(\mathbf{x}_0+\mathbf{h}) - \mathbf{f}(\mathbf{x}_0) - \mathbf{A}\mathbf{h}\|}{\|\mathbf{h}\|} \xrightarrow[\mathbf{h}\to\mathbf{0}]{\text{by (*)}} 0. \end{aligned}$$

Hence:

$$\lim_{\mathbf{h}\to\mathbf{0}} \frac{\|(\mathbf{A}-\mathbf{B})\mathbf{h}\|}{\|\mathbf{h}\|} = 0. \tag{**}$$

Conclusion: $\mathbf{A} = \mathbf{B}$. (If *not*, then there exists $\mathbf{v} \in \mathbb{R}^n$ such that $(\mathbf{A}-\mathbf{B})\mathbf{v} \neq \mathbf{0}$. Choose $\mathbf{h} = t\mathbf{v}, t \to 0$. Then,

$$\frac{\|(\mathbf{A}-\mathbf{B})\mathbf{h}\|}{\|\mathbf{h}\|} = \frac{\|(\mathbf{A}-\mathbf{B})\mathbf{v}\|}{\|\mathbf{v}\|} \neq \mathbf{0}, \text{ a contradiction.}$$

3. **Exercises:**

   (a) Show that if $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^m$ is *differentiable* at $\mathbf{x}_0$, then for every $1 \leq j \leq n$,

   $$\lim_{t\to 0} \frac{\mathbf{f}(\mathbf{x}_0 + t\mathbf{e}_j) - \mathbf{f}(\mathbf{x}_0)}{t} = \underbrace{\frac{\partial \mathbf{f}}{\partial \mathbf{x}_j}(\mathbf{x}_0)}_{m\text{-dimensional vectors}} \qquad \text{exists,}$$

   called the j^{th} **partial derivative** of $\mathbf{f}$ at $\mathbf{x}_0$, where

   $$\mathbf{e}_j = \begin{pmatrix} 0 \\ \vdots \\ 0 \\ 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \to \text{the } j^{\text{th}} \text{ entry.}$$

   (b) Show:

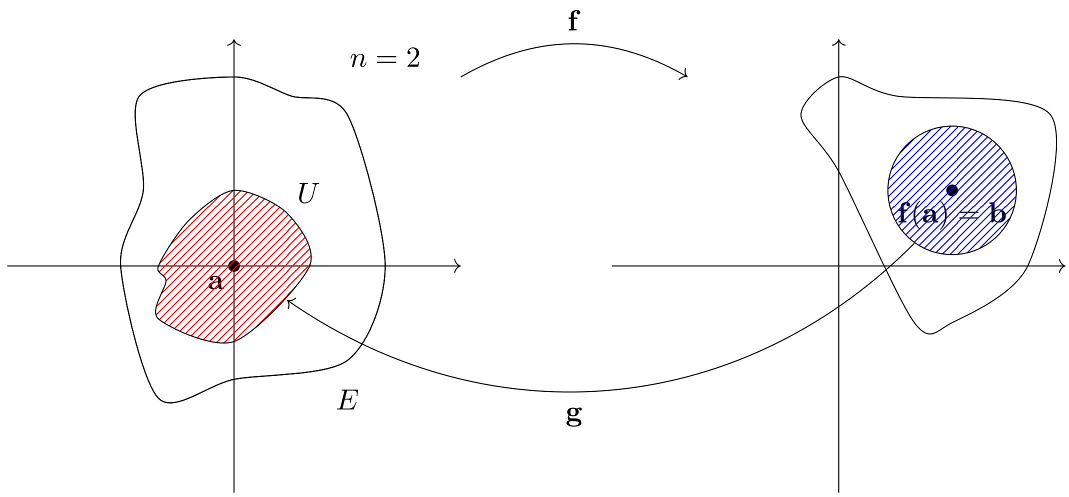
   $$\underbrace{\mathbf{A}}_{m\times n} = \left(\frac{\partial \mathbf{f}}{\partial x_1}(\mathbf{x}_0) \quad \frac{\partial \mathbf{f}}{\partial x_2}(\mathbf{x}_0) \quad \ldots \quad \frac{\partial \mathbf{f}}{\partial x_n}(\mathbf{x}_0) \right).$$

   (c) However, the converse need not be true. Show that there exists $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}, n \geq 2$ such that all partial derivatives of $\mathbf{f}$ exists at $\mathbf{0}$, but $\mathbf{f}$ is not differentiable at $\mathbf{0}$.

1.2 Inverse Function Theorem

Theorem 1. Let $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$. Let $E \overset{\text{open}}{\subseteq} \mathbb{R}^n$ and $\mathbf{a} \in E$. Assume that $\mathbf{f} \in C^1(E)$ (i.e., $\underbrace{x}_{\in E} \mapsto \mathbf{f}'(x)$ is continuous) and $\underbrace{\overbrace{\mathbf{f}'(\mathbf{a})}^{n\times n} \text{ is invertible}}_{\text{sufficient but not necessary, as the example } h \text{ shows}}$. Set $\mathbf{b} = \mathbf{f}(\mathbf{a})$.

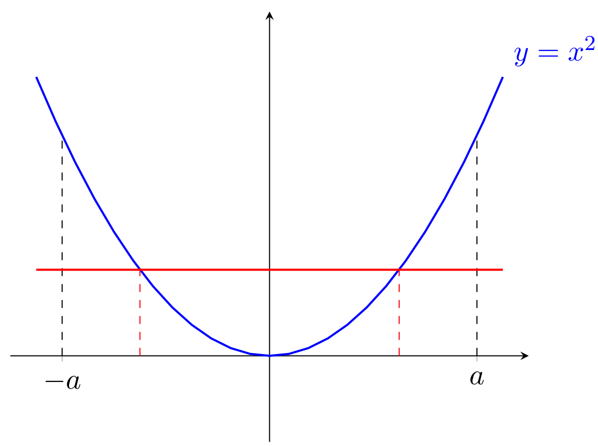
1. We can invert $\mathbf{f}$ *locally*: There exists $U \overset{\text{open}}{\subseteq} E \subseteq \mathbb{R}^n$, $\mathbf{b} \in V \overset{\text{open}}{\subseteq} \mathbb{R}^n$ and $\underbrace{g}_{=f^{-1}} : V \xrightarrow[\text{onto}]{\text{1-1}} U$ such that $f \circ g = \text{id}$ and $g \circ f = \text{id}$.



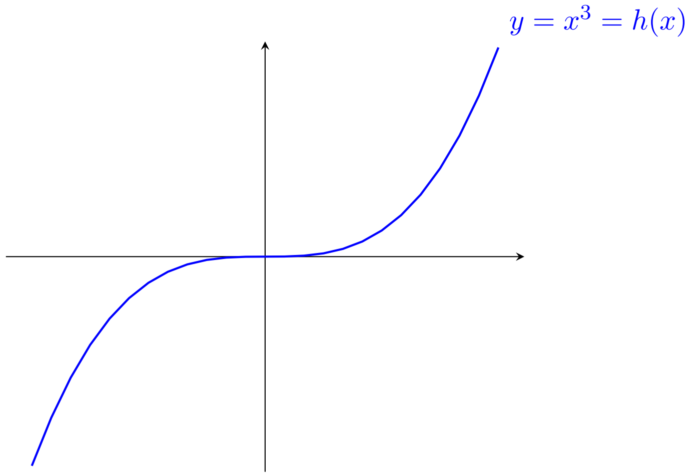
$$\begin{aligned} \mathbf{g} \circ \mathbf{f}(\mathbf{u}) &= \mathbf{u} \qquad \forall \mathbf{u} \in U, \\ \mathbf{f} \circ \mathbf{g}(\mathbf{v}) &= \mathbf{v} \qquad \forall \mathbf{v} \in V. \end{aligned}$$

2. $g \in C^1(V)$.

Example: Suppose $n = 1$. Then $f'(a)$ invertible means that $f'(a) \neq 0$. Let $f(x) = x^2, x \in (-a, a)$. Then $f'(0) = 0$ and f is not invertible in any neighborhood of the origin.



Let $h(x) = x^3$. Then $h'(0) = 0$. However, h is invertible near 0.


$y = x^3 = h(x)$