# Math 321 Lecture 13

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## 1 Review (Cont'd)

### 1.1 Problem 5

Evaluate

$$\lim_{n\to\infty}\int_0^\pi \underbrace{\frac{n+\sin nx}{3n-\sin^2 nx}}_{f_n} dx.$$

**Step 0:** $f_n$ has a pointwise limit for every $x \in \mathbb{R}$.

$$\frac{n+\sin nx}{3n-\sin^2 nx} = \frac{n\left(1+\frac{\sin nx}{n}\right)}{3n\left(1-\frac{\sin^2 nx}{n}\right)} \xrightarrow{n\to\infty} \frac{1}{3} = f(x).$$

**Step 1:** Consider $C([0,\pi];\mathbb{R})$, equipped with sup norm; note $f_n, f \in C([0,\pi];\mathbb{R})$. $f_n$ is continuous because it denominator

$$3n - \sin^2 nx \geq 3n - 1, \tag{*}$$

hence never vanishes for $n \geq 1$.

*If* we can show $f_n \xrightarrow{n\to\infty} f$ uniformly on $[0,\pi]$, then by a theorem proved in class,

$$\lim_{n\to\infty}\int_0^\pi f_n(x)dx = \int_0^\pi \lim_{n\to\infty} f_n(x)dx = \int_0^\pi \frac{1}{3}dx = \frac{\pi}{3}.$$

**Step 2:**

$$|f_n(x)-f(x)| = \left|\frac{n+\sin nx}{3n-\sin^2 nx} - \frac{1}{3}\right| = \left|\frac{3\sin nx + \sin^2 nx}{3\left(3n-\sin^2 nx\right)}\right| \leq \underbrace{\frac{3|\sin nx| + \left|\sin^2 nx\right|}{3(3n-1)}}_{\text{by (*)}} \leq \underbrace{\frac{4}{3(3n-1)}}_{\text{independent of } x}.$$

Given $\epsilon > 0$, choose $N \geq 1$ so that $\frac{4}{3(3n-1)} < \epsilon\ \forall n \geq N$. Then the above implies $\|f_n - f\|_\infty < \epsilon\ \forall n \geq N$. Hence $f_n \xrightarrow{n\to\infty} f$ uniformly on $[0,\pi]$.

### 1.2 Problem 6

For every $n \in \mathbb{N}$, suppose $f_n : \mathbb{R} \xrightarrow{\text{differentiable}} \mathbb{R}$ satisfies

$$f_n(0) = 2019, \qquad \left|f_n'(t)\right| \leq 321 + |t|^{201} \qquad \text{for all } t \in \mathbb{R}. \tag{**}$$

**Show:** There exists $f : \mathbb{R} \xrightarrow{\text{continuous}} \mathbb{R}$ and a subsequence $n_1 < n_2 < \ldots < n_k < \ldots$ such that $f_{n_k} \xrightarrow{k\to\infty} f$ uniformly on every compact subset of $\mathbb{R}$.

**Hope:** Apply $\underbrace{\text{Arzela-Ascoli}}_{\text{hypothesis: } \mathcal{F} \subseteq C(K;\mathbb{C}), \text{ uniformly bounded and equicontinuous}}$; need a compact metric space for this to work.

Let $\mathcal{F} = \{f_n : \mathbb{R} \to \mathbb{R}, \text{obeying } (**)\}$. Fix a compact set $K_R = [-R, R]$.

**Question:** Is $\mathcal{F}$ uniformly bounded and equicontinuous on $K_R$?

We have

$$f_n(t) - f_n(0) = \int_0^t f_n(s) ds.$$

This implies that

$$|f_n(t)| \leq |f_n(0)| + \int_0^t |f_n'(s)| \, ds \leq 2019 + \int_0^t C_R ds \leq 2019 + C_R t.$$

Hence, $\mathcal{F} \subseteq C(K_R; \mathbb{R})$ is uniformly bounded.

We have

$$|f_n(x) - f_n(y)| = \left| \int_x^y f'(t) dt \right| \leq \int_x^y \underbrace{|f_n'(t)|}_{\leq C_R} \leq \underbrace{C_R |x - y|}_{\text{independent of } n} < \epsilon,$$

provided $|x - y| < \delta = \frac{\epsilon}{C_R}$. Hence, $\mathcal{F}$ is equicontinuous on $C(K_R; \mathbb{R})$.

AA implies that any sequence of functions in $\mathcal{F}$ must have a *uniformly convergent subsequence on* $K_R$. Note that this subsequence comes with $R$ and may not have any relation with the subsequence for $K_{R'}, R' < R$.

**Step 2:** Start with $R = 1$. Use Step 1 to get a subsequence $\underbrace{S_1}_{=\{n_1 < n_2 < n_3 < \ldots\}} \subseteq \mathbb{N}$ such that $\{f_n : n \in S_1\}$ converges uniformly on $K_1$.

Next, set $R = 2$. Look at

$$\mathcal{F}_1 = \{f_n : n \in S_1\}.$$

Note $\mathcal{F}_1$ is uniformly bounded and equicontinuous on $K_2$. By AA, there exists a subsequence $S_2 \subseteq S_1$ such that $\{f_n : n \in S_2\}$ converges uniformly on $K_2$.

Iterate. At Step $j$, get a subsequence

$$S_j = \{n_{j_1} < n_{j_2} < \ldots\} \subseteq S_{j-1} \subseteq \ldots \subseteq S_1$$

such that $\{f_n : n \in S_j\}$ converges uniformly on $K_j$.

Consider the diagonal sequence $\{f_{n_{kk}} : k \geq 1\}$.

$$\begin{array}{lllll}
S_1: & n_{11} & n_{12} & n_{13} & \ldots \\
S_2: & n_{21} & n_{22} & n_{23} & \ldots \\
\cup\!| & & & & \\
\underbrace{S_k}_{k=R}: & n_{k1} & n_{k2} & \ldots & n_{kk}
\end{array}$$

**Claim:** There exists $f \in C(\mathbb{R})$ such that $f_{n_{kk}} \xrightarrow{k \to \infty} f$ on every compact $K \subseteq \mathbb{R}$.

Start with any $K \overset{\text{compact}}{\subseteq} \mathbb{R}$. Then there exists $R \geq 1$ such that $K \subseteq K_R = [-R, R]$. Since $n_{kk} \in S_R$ for all sufficiently large $k$, we know $f_{n_{kk}}$ converges uniformly on $K_R$ and hence $K$.