Math 321 Lecture 17

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1 Riemann-Stieltjes Integrals with General Integrators

Definition 1. Suppose $f, \alpha : [a, b] \to \mathbb{R}$, where α is not necessarily non-decreasing.
Note: Lower and upper Riemann sums need not obey the same inequalities as before.

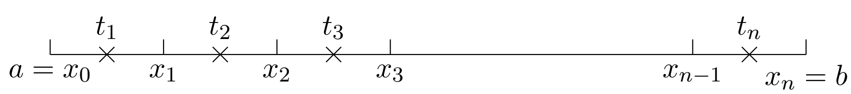
$$L_\alpha(f, P) = \sum_{i=1}^{n} m_i \Delta\alpha_i,$$

$$\not\leq$$

$$U_\alpha(f, P) = \sum_{i=1}^{n} M_i \Delta\alpha_i.$$

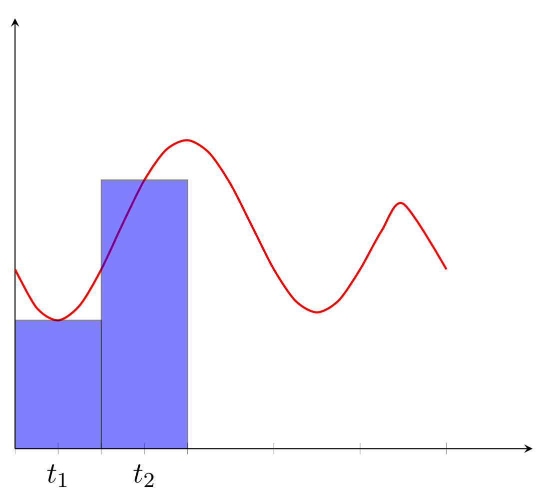
We have $m_i \leq M_i$, but $\Delta\alpha_i$ may not be ≥ 0. Also note that $P \subseteq P'$ does not imply that $L_\alpha(f, P) \leq L_\alpha(f, P') \leq U_\alpha(f, P') \leq U_\alpha(f, P)$.

Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ be any partition of $[a, b]$. Choose a selection of points $T = \{t_1 < t_2 < \ldots < t_n : t \in [x_{i-1}, x_i]\}$.



Define

$$S_\alpha(f, P, T) = \sum_{i=1}^{n} f(t_i)\Delta\alpha_i, \qquad \Delta\alpha_i = \alpha(x_i) - \alpha(x_{i-1}).$$



We say that f is **RS-integrable** with respect to α, written $f \in \mathcal{R}_\alpha[a,b]$, if there exists a real number I such that for every $\epsilon > 0$, we can find a partition $P_0 = P_0(f,\alpha,\epsilon)$ such that $|S_\alpha(f,P,T) - I| < \epsilon$ for all partitions $P \supseteq P_0$ and any selection of points T. We write

$$I = \int_a^b f d\alpha.$$

Exercise: Check that for non-decreasing integrators α, this definition is equivalent to our earlier one.

2 Functions of Bounded Variation

2.1 Examples

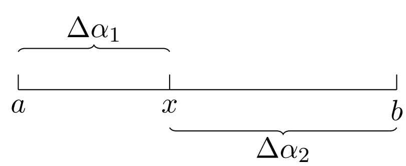
Recall: A function $\alpha : [a,b] \to \mathbb{R}$ is said to be of **bounded variation**, written $\alpha \in BV[a,b]$, if

$$V_a^b \alpha = \textbf{total variation} \text{ of } \alpha \text{ on } [a,b] = \sup_P V_a^b(\alpha, P) < \infty,$$
$$\text{where } V_a^b(\alpha,P) = \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})| = \sum_{i=1}^n |\Delta\alpha_i|.$$

Examples:

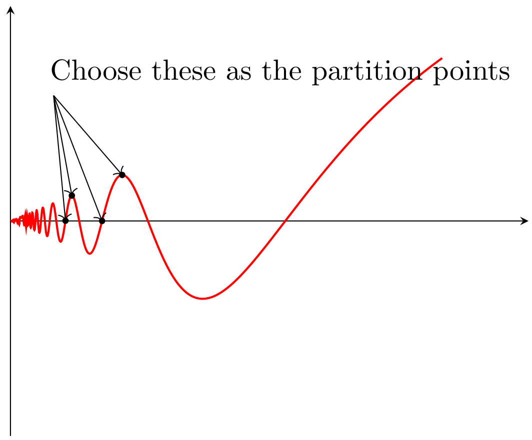
1. Constant functions are in $BV[a,b]$.
2. $BV[a,b] \subseteq \mathcal{B}[a,b] =$ space of bounded real-valued functions on $[a,b]$.
   Let $\alpha \in BV[a,b]$. Then,

$$\begin{aligned}
|\alpha(x)| &\overset{\text{triangular inequality}}{\leq} \underbrace{|\alpha(x) - \alpha(a)|}_{|\Delta\alpha_1|} + |\alpha(a)| \\
&\leq |\Delta\alpha_1| + |\Delta\alpha_2| + |\alpha(a)| \\
&= V_a^b(\alpha,P) + |\alpha(a)| \\
&\leq \underbrace{V_a^b\alpha + |\alpha(a)|}_{\text{no dependence on } x} \\
&< \infty.
\end{aligned}$$



3. Not all bounded continuous functions in $[a,b]$ are in $BV[a,b]$.

$$\alpha(x) = \begin{cases} x\sin\left(\frac{1}{x}\right), & \text{for } x \neq 0, \\ 0, & \text{at } x = 0. \end{cases}$$



$\alpha \in C[a,b]$ for all $b > 0$, but $\alpha \notin BV[a,b]$.

4. Any monotone function (increasing or decreasing, continuous or discontinuous) is in $BV[a,b]$.

5. Any step function with a finite number of jump discontinuities is in $BV[a,b]$.

2.2 Jordan's Theorem

Theorem 1 (Jordan's theorem)**.** Any $\alpha \in BV[a,b]$ can be written as

$$\alpha = \beta - \gamma,$$

where β and γ are both nondecreasing bounded functions on $[a,b]$.

Remark.

$$\int_a^b f d\alpha \xlongequal{\text{Jordan's}} \int_a^b f d(\beta - \gamma) \overset{?}{=} \underbrace{\int_a^b f d\beta - \int_a^b f d\gamma}_{\text{these are computable by our earlier discussion on non-decreasing integrators}}.$$