Math 321 Lecture 32

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1 Implicit Function Theorem

Examples:

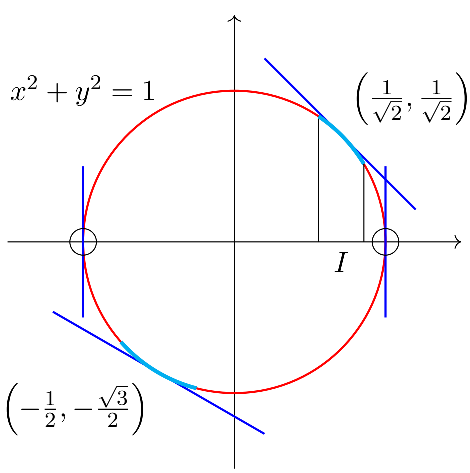
- $x^2 + y^2 + 1 = 0$: no solution $(x, y) \in \mathbb{R}^2$.
- $x^2 + y^2 = 0$: exactly one solution $(0, 0)$; no solution of the form $y = g(x), x \in$ interval.
- $x^2 + y^2 - 1 = 0$. Then,

$$2x + 2y\frac{dy}{dx} = 0,$$
$$\boxed{\frac{dy}{dx} = -\frac{x}{y}.} \tag{*}$$

Case 1 (Good): $(a, b) = \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Then we can solve (*): $y = \sqrt{1 - x^2}$.

$(a, b) = \left(-\frac{1}{2}, -\frac{\sqrt{3}}{2}\right)$: $y = -\sqrt{1 - x^2}$.

$\frac{dy}{dx}$ well-defined.

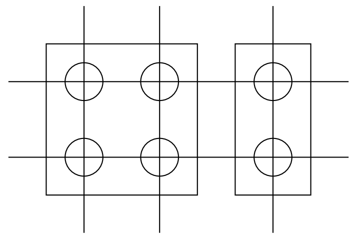


Case 2 (Bad): $(a, b) = (1, 0)$ or $(-1, 0)$. Solutions to $x^2 + y^2 = 1$ are *not* unique in any neighbourhood of these points; $y = \pm\sqrt{1 - x^2}$ are both possible.

$\frac{dy}{dx}$ is not meaningful.

Theorem 1 (Implicit function theorem)**.** Let $E \overset{\text{open}}{\subseteq} \mathbb{R}^{n+m}, n, m \geq 1$ and $\mathbf{f} : E \to \mathbb{R}^n, \mathbf{f} \in C^1(E)$. Suppose $(\underbrace{\mathbf{a}}_{\in\mathbb{R}^n}, \underbrace{\mathbf{b}}_{\in\mathbb{R}^m}) \in E$ such that $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$.

Set $\mathbf{A} = \mathbf{f}'(\mathbf{a}, \mathbf{b})_{n\times(n+m)}$. Write $\mathbf{A} = \left[\begin{array}{c|c} \underbrace{\mathbf{A}_x}_{n\times n} & \underbrace{\mathbf{A}_y}_{n\times m} \end{array}\right]_{n\times(n+m)}$. Assume $\mathbf{A}_x$ is invertible.



$\mathbf{A}_x$ $\mathbf{A}_y$

Conclusion: There exist open sets $\boxed{U \subseteq E \subseteq \mathbb{R}^{n+m}}$ and $W \subseteq \mathbb{R}^m$, $(\mathbf{a}, \mathbf{b}) \in U$ such that

1. For every $\mathbf{y} \in W$, there exists a unique $(\mathbf{x}, \mathbf{y}) \in U$ which satisfies the equation $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$. Define this $\mathbf{x}$ as $\mathbf{g}(\mathbf{y})$.
2. $\mathbf{g} : \underbrace{W}_{\subseteq\mathbb{R}^m} \to \mathbb{R}^n$ with $\mathbf{y} \mapsto \mathbf{x}$. Then $\mathbf{g} \in C^1$, with

$$\mathbf{g}'(\mathbf{b}) = \underbrace{-(\overbrace{\mathbf{A}_x}^{n\times n})^{-1} \overbrace{\mathbf{A}_y}^{n\times m}}_{n\times m \text{ matrix}}.$$

Proof. **Strategy:** Define an auxiliary function $\mathbf{F} : E \subseteq \mathbb{R}^{n+m} \to \mathbb{R}^{n+m}$ on which the inverse function theorem can be applied:

$$\mathbf{F}(\mathbf{x}, \mathbf{y}) = (\underbrace{\mathbf{f}(\mathbf{x}, \mathbf{y})}_{\in\mathbb{R}^n}, \underbrace{\mathbf{y}}_{\in\mathbb{R}^m}), \quad \mathbf{x} \in \mathbb{R}^n, \quad \mathbf{y} \in \mathbb{R}^m, \quad (\mathbf{x}, \mathbf{y}) \in E \subseteq \mathbb{R}^{n+m}.$$

Need to verify the hypothesis of the inverse function theorem, namely:

- $\underbrace{\mathbf{F}}_{=(\mathbf{f}(\mathbf{x},\mathbf{y}),\mathbf{y})} \in C^1(E)$ because each entry is C^1;
- $\mathbf{F}'(\mathbf{a}, \mathbf{b})$ is invertible:

$$\mathbf{F}'(\mathbf{a}, \mathbf{b}) = \left[\begin{array}{c|c} \multicolumn{2}{c}{\mathbf{A} = \mathbf{f}'(\mathbf{a}, \mathbf{b})_{n\times(n+m)}} \\ \hline \mathbf{0}_{m\times n} & \mathbf{I}_{m\times m} \end{array}\right] = \underbrace{\left[\begin{array}{c|c} (\mathbf{A}_x)_{n\times n} & (\mathbf{A}_y)_{n\times m} \\ \hline \mathbf{0}_{m\times n} & \mathbf{I}_{m\times m} \end{array}\right]}_{(n+m)\times(n+m)} = \mathbf{X}.$$

Note that

$$\mathbf{X}\begin{pmatrix}\mathbf{u} \\ \mathbf{v}\end{pmatrix} = \mathbf{0} \quad \Leftrightarrow \quad \begin{array}{ccc} \mathbf{u} & = & \mathbf{0} \\ \mathbf{v} & = & \mathbf{0} \end{array},$$

because

$$\mathbf{A}_x\mathbf{u} + \mathbf{A}_y\mathbf{v} = \mathbf{0} \quad \Rightarrow \quad \mathbf{A}_x\mathbf{u} = \mathbf{0} \quad \Rightarrow \quad \mathbf{A}_x^{-1}\mathbf{A}_x\mathbf{u} = \mathbf{0} \quad \text{i.e.} \quad \mathbf{u} = \mathbf{0},$$
$$\mathbf{0u} + \mathbf{Iv} = \mathbf{0} \quad \text{i.e.} \quad \mathbf{v} = \mathbf{0}.$$

(Proof unfinished.)