Math 321 Lecture 6

Yuchong Pan

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1 Bernstein's Proof of Weierstrass Approximation Theorem (Cont'd)

1.1 Proof

Proof (Bernstein). Let $f \in C[0,1]$. Set $p_n(f)(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}$, a polynomial of degree $\leq n$. We want to show that $p_n \xrightarrow{n\to\infty} f$ uniformly on $[0,1]$.

Fix $\epsilon > 0$. Last time, we showed that

$$\sup_{x\in[0,1]} |p_n(f)(x) - f(x)| \leq \mathrm{I} + \mathrm{II},$$

where

$$\mathrm{I} = \sup_{x\in[0,1]} \sum_{k\in F} \left| f\left(\frac{n}{k}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k},$$
$$\mathrm{II} = \sup_{x\in[0,1]} \sum_{k\in F^c} \left| f\left(\frac{n}{k}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k},$$

and

$$F = F_x = \left\{0 \leq k \leq n : \left|\frac{k}{n} - x\right| < \delta\right\},$$
$$F^c = \left\{0 \leq k \leq n : \left|\frac{k}{n} - x\right| \geq \delta\right\}.$$

Checked: For $k \in F$, $\left|f\left(\frac{k}{n}\right) - f(x)\right| < \frac{\epsilon}{2}$, by the continuity of f. Thus,

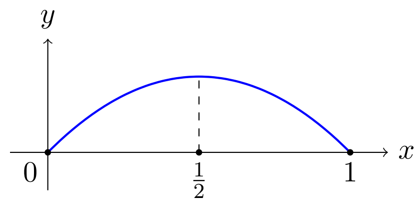
$$\mathrm{I} < \sum_{k\in F} \frac{\epsilon}{2}\binom{n}{k}x^k(1-x)^{n-k} \leq \frac{\epsilon}{2} \qquad \left(\text{since } \sum_{k=0}^{n}\binom{n}{k}x^k(1-x)^{n-k} = (x+1-x)^n\right).$$

Lemma 1. $\sum_{k=0}^{n}\left(\frac{k}{n} - x\right)^2\binom{n}{k}x^k(1-x)^{n-k} = \frac{x(1-x)}{n}$ (HW 3, (1)).

Assume Lemma 1 for now.

This implies that

$$\delta^2 \sum_{k\in F^c}\binom{n}{k}x^k(1-x)^k \leq \sum_{x\in F^c}\underbrace{\left(\frac{k}{n} - x\right)^2}_{\geq\delta^2}\binom{n}{k}x^k(1-x)^{n-k}$$
$$\leq \sum_{k=0}^{n}\underbrace{\left(\frac{k}{n} - x\right)^n\binom{n}{k}x^k(1-x)^{n-k}}_{\text{non-negative}} = \frac{x(1-x)}{n}.$$



Summary:

$$\sum_{k \in F^c} \binom{n}{k} x^k (1-x)^{n-k} \leq \frac{x(1-x)}{n\delta^2} \leq \frac{1}{4n\delta^2}. \tag{*}$$

Back to II. Recall that

$$f : \underbrace{[0,1]}_{\text{compact}} \xrightarrow{\text{continuous}} \mathbb{R} \text{ or } \mathbb{C}$$

$$\Rightarrow \ f \text{ is bounded}$$

$$\Rightarrow \ \sup_{x \in [0,1]} |f(x)| = M < \infty.$$

This implies that

$$\mathrm{II} = \sum_{k \in F^c} \underbrace{\left| f\left(\frac{k}{n}\right) - f(x) \right|}_{\leq |f(\frac{k}{n})| + |f(x)| \leq 2M} \binom{n}{k} x^k (1-x)^{n-k} \leq 2M \sum_{k \in F^c} \binom{n}{k} x^k (1-x)^{n-k} \leq 2M \cdot \frac{1}{4n\delta^2} = \frac{M}{2n\delta^2}.$$

Choose $n \geq N$ large enough so that $\frac{M}{2n\delta^2} \leq \frac{\epsilon}{2}$ for all $n \geq N$. Then $\mathrm{II} \leq \frac{\epsilon}{2}$ for all $n \geq N$. Hence, for all $n \geq N$,

$$\|p_n - f\|_\infty \leq \mathrm{I} + \mathrm{II} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon.$$

$\square$

1.2 Discussion

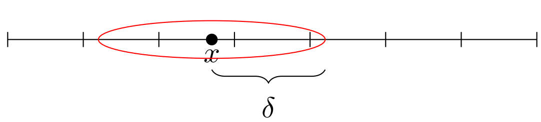
$\boxed{\epsilon > 0, f \in C[0,1]}$

$\boxed{\text{define } p_n(f)}$

$\boxed{\text{write } |p_n - f| \text{ as a sum of two terms, where I is a sum over } k \text{ such that } \left|\frac{k}{n} - x\right| < \delta}$

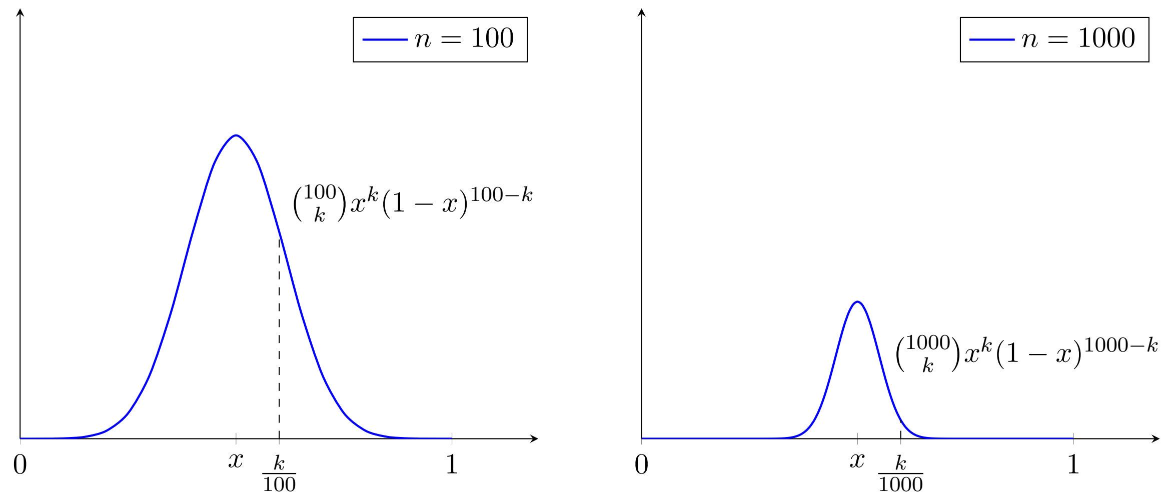
Eliminate I
Continuity of f crucial here

$$I = \sum_{k \in F} \underbrace{\left| f\left(\frac{k}{n}\right) - f(x) \right|}_{< \frac{\epsilon}{2}} \binom{n}{k} x^k (1-x)^{n-k} < \frac{\epsilon}{2}$$



Eliminate II
Continuity not really important,
only needed for boundedness
Behavior of binomial distribution
as $n \to \infty$ crucial here

Question: How do binomial probabilities $\binom{n}{k}x^k(1-x)^k$ behave as $n \to \infty$?
Answer: They sum to 1, but their weight $\xrightarrow{n\to\infty}$ 0 away from x.



All the mass of binomial probabilities eventually **concentrates** near x:

$$p_n(x) = \sum_{k=0}^{n} f\left(\frac{k}{n}\right)\binom{n}{k}x^k(1-x)^{n-k}.$$

$\frac{B(n,x)}{n}$ converges weakly to the **Dirac delta** at x as $n \to \infty$.

Remark. Suppose $f \in C(\mathbb{R})$ or $C(X)$, for an arbitrary metric space X. Polynomials $p(x) = a_0 + a_1x + \ldots + a_nx^n \to \pm\infty$ as $n \to \infty$, so uniform convergence is not possible in general.