Math 321 Lecture 14

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February 4, 2019

1 Riemann-Stieltjes Integration

1.1 Definitions

Definition 1. Let

$$
\begin{aligned}
&[a,b] \overset{\text{compact}}{\subseteq} \mathbb{R},\\
&\alpha : [a,b] \to \mathbb{R} \text{ be } \underbrace{\text{non-constant}}_{\exists x,y\in[a,b], \alpha(x)\neq\alpha(y)}, \underbrace{\text{non-decreasing}}_{\text{if } x<y \text{ then } \alpha(x)\leq\alpha(y)}, \text{ called the } \textbf{integrator},\\
&f : [a,b] \to \mathbb{R} \text{ be bounded, called the } \textbf{integrand}.
\end{aligned}
$$

Given any partition $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ of $[a,b]$, define

$$
\left.\begin{array}{rcl} L_\alpha(f,P) & = & \sum_{i=1}^n m_i \Delta\alpha_i, \\ U_\alpha(f,P) & = & \sum_{i=1}^n M_i \Delta\alpha_i, \end{array}\right\} \text{ where } \begin{array}{rcl} m_i & = & \inf\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ M_i & = & \sup\{f(x) : x_{i-1} \leq x \leq x_i\}, \\ \Delta\alpha_i & = & \alpha(x_i) - \alpha(x_{i-1}). \end{array}
$$

Exercises:

1. $L_\alpha(f,P) \leq U_\alpha(f,P)$.
2. If Q refines P, i.e. $P \subsetneq Q$, then
$$L_\alpha(f,P) \leq L_\alpha(f,Q) \leq U_\alpha(f,Q) \leq U_\alpha(f,P). \tag{*}$$

Definition 2.

Lower RS integral of f with respect to α on $[a,b] = \underline{\int_a^b} f d\alpha = \underline{\int_a^b} f(x) d\alpha(x) \overset{\text{def}}{=} \sup_{\text{partition } P \text{ of } [a,b]} L_\alpha(f,P)$,

Upper RS integral of f with respect to α on $[a,b] = \overline{\int_a^b} f d\alpha = \overline{\int_a^b} f(x) d\alpha(x) \overset{\text{def}}{=} \inf_{\text{partition } P \text{ of } [a,b]} U_\alpha(f,P)$.

Remark.

$$(*) \Rightarrow \underline{\int_a^b} f d\alpha \leq \overline{\int_a^b} f d\alpha \text{ for any } f.$$

Definition 3. We say that f is **Riemann-Stieltjes integrable** with respect to α, written $f \in \mathcal{R}_\alpha[a,b]$, if

$$\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha.$$

In this case,

$$\underline{\int_a^b} f d\alpha = \overline{\int_a^b} f d\alpha \overset{\text{notation}}{=\!=\!=} \int_a^b f d\alpha = \text{the } \textbf{RS integral} \text{ of } f \text{ with respect to } \alpha \text{ on } [a,b].$$

1.2 Examples

1. $\alpha(x) = x$; the standard Riemann integration.

   Applications:

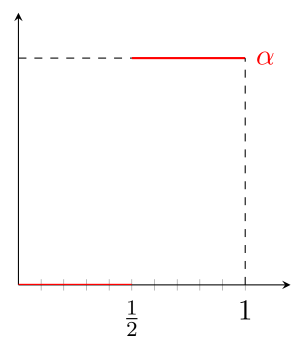
   (a) Represents signed area under a curve.

   (b)
   $$\frac{d}{dx}\int_a^x f(t)dt = f(x).$$
   Indefinite Riemann integrals are "antiderivatives", i.e., provide inverses to differentiation.

   (c) Errors in approximation of a function by its linear expansion involve a Riemann integral.

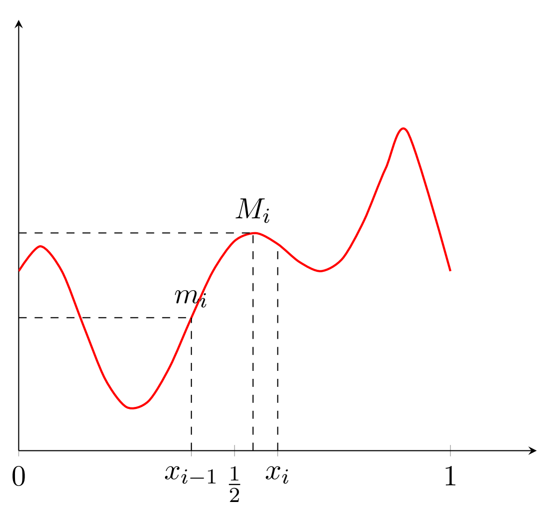
2.
$$\begin{aligned}
&[a, b] = [0, 1],\\
&\alpha(x) = \begin{cases} 0, & \text{if } 0 \leq x < \frac{1}{2},\\ 1, & \text{if } \frac{1}{2} \leq x \leq 1. \end{cases}
\end{aligned}$$

   Is **every continuous** f on $[0, 1]$ in $\mathcal{R}_\alpha[0, 1]$ for this α?



$$\Delta\alpha_i = \begin{cases} 0, & \text{if } \frac{1}{2} \notin [x_{i-1}, x_i],\\ 1, & \text{if } \frac{1}{2} \in [x_{i-1}, x_i]. \end{cases}$$

   Note: There exist at least one and at most two choices of i for which $\Delta\alpha_i$ can be 1.



Recall f is continuous on $[0,1]$, hence uniformly continuous. Therefore, given any $\epsilon > 0$, there exists $\delta > 0$ such that $|x-y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{2}$.

$$U_\alpha(f,P) - L_\alpha(f,P) = \underbrace{\sum_{i=1}^n \overbrace{(M_i - m_i)}^{<\frac{\epsilon}{2}} \Delta\alpha_i}_{\text{at most 2 nonzero terms where } \Delta\alpha_i = 1} < \frac{\epsilon}{2} + \frac{\epsilon}{2} = \epsilon \Rightarrow f \in \mathcal{R}_\alpha[0,1],$$

provided $\Delta x_i = x_i - x_{i-1} < \delta$ for all i. Choose any $x, y \in [x_{i-1}, x_i]$. Since $\Delta x_i < \delta$, then $|f(x) - f(y)| < \frac{\epsilon}{2}$. This implies that $M_i - m_i < \frac{\epsilon}{2}$.

Claim 1.

$$\int_0^1 f \underbrace{d\alpha}_{\text{Dirac delta at } x = \frac{1}{2}} = f\left(\frac{1}{2}\right).$$

Proof. Fix $\epsilon > 0$. It suffices to show that

$$\left| U_\alpha(f,P) - f\left(\frac{1}{2}\right) \right| < \epsilon \text{ for sufficiently fine partitions } P.$$

We have

$$\begin{aligned}
U_\alpha(f,P) - f\left(\frac{1}{2}\right) &= \sum_{i=1}^n M_i \Delta\alpha_i - f\left(\frac{1}{2}\right) \\
&= \underbrace{M_i^* \overbrace{\Delta\alpha_{i^*}}^{=1}}_{\text{where } i^* \text{ is the unique index where } \Delta\alpha_{i^*} = 1} - f\left(\frac{1}{2}\right) \\
&= \sup\{f(x) : x \in [x_{i^*-1}, x_{i^*}]\} - f\left(\frac{1}{2}\right) \\
&= \sup\left\{ \underbrace{f(x) - f\left(\frac{1}{2}\right)}_{<\epsilon} : x \in [x_{i^*-1}, x_{i^*}] \right\},
\end{aligned}$$

if $x_{i^*} - x_{i^*-1} < \delta$ by the continuity of f. □

Exercise: Try this for

$$\alpha(x) = \begin{cases} 0, & \text{if } x \in \left[0, \frac{1}{2}\right), \\ \frac{1}{2}, & \text{if } x = \frac{1}{2}, \\ 1, & \text{if } x \in \left(\frac{1}{2}, 1\right]. \end{cases}$$

Check that:

$$f \in \mathcal{R}_\alpha[0,1] \Leftrightarrow f \text{ is continuous at } x = \frac{1}{2}.$$