Math 321 Lecture 22

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1 Fourier Series

Let $f \in \mathcal{R}[-\pi, \pi]$. Recall: $\alpha_0 + \sum_{k=1}^n (\alpha_k \cos kx + \beta_k \sin kx)$ is called the **Fourier series** of f if

$$\begin{cases} \alpha_0 &= \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, \\ \alpha_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx, \quad k \geq 1, \\ \beta_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx, \quad k \geq 1. \end{cases} \tag{*}$$

As of now, we do not know if this series converges for *any* $x \in [-\pi, \pi]$.

Definition 1. The n^{th} **partial Fourier series/sum** of f is given by

$$(S_n f)(x) = \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx),$$

where $\alpha_0, \alpha_k, \beta_k$ are as in (*) for $1 \leq k \leq n$. Note that $S_n f$ is a well-defined trignometric polynomial for every $n \geq 1$.

Question: What do $S_n f$ represent, especially since $\{S_n f : n \geq 1\}$ need not converge to f pointwise?

Consider the following space of functions in $[-\pi, \pi]$. Fix any $p \in [1, +\infty)$,

$$L_*^p[-\pi, \pi] := \left\{ f : [-\pi, \pi] \xrightarrow[2\pi\text{-periodic}]{\text{continuous}} \mathbb{C} \right\},$$

equipped with the norm

$$\|f\|_p = \left[\frac{1}{2\pi} \int_{-\pi}^{\pi} |f(x)|^p dx \right]^{\frac{1}{p}}, \qquad 1 \leq p < \infty.$$

For $p = \infty$,

$$\|f\|_\infty \overset{\text{def}}{=\!=} \sup\{|f(x)| : x \in [-\pi, \pi]\}.$$

$\boxed{\textbf{Special case: } p = 2.}$

Theorem 1. Consider the following minimization problem: Take any $f \in L_*^2[-\pi, \pi]$.

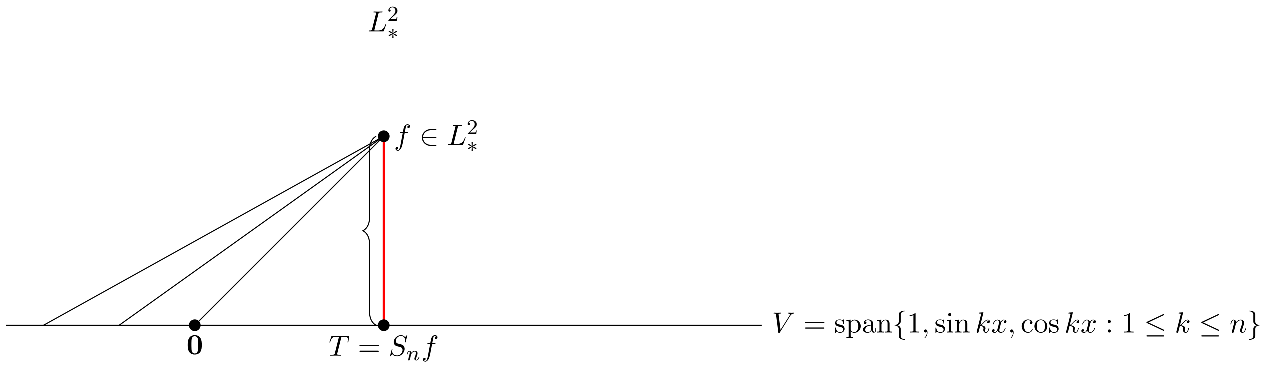
What is $\min\{\|f - T\| : T \in \operatorname{span}\{1, \cos kx, \sin kx : 1 \leq k \leq n\}$?

When is the minimum attained?

Answer: $\min\{\|f - T\| : T \in \operatorname{span}\{1, \cos kx, \sin kx : 1 \leq k \leq n\}$ is attained if and only if $T = S_n f = n^{\text{th}}$ Fourier series.

Note that

$$\underbrace{\dim\left(L^2_*[-\pi,\pi]\right)}_{\{1,\sin kx,\cos kx:k\geq 1\}}=\infty.$$



Proof. We have

$$\|f-T\|_2^2=\int_{-\pi}^{\pi}(f(x)-T(x))^2dx=\int_{-\pi}^{\pi}(f(x))^2dx-2\int_{-\pi}^{\pi}f(x)T(x)dx+\int_{-\pi}^{\pi}(T(x))^2dx.$$

Suppose $T(x)=a_0+\sum_{k=1}^n(a_k\cos kx+b_k\cos kx)$, $a_0,a_k,b_k\in\mathbb{C}$. Then,

$$\begin{aligned}
\|f-T\|_2^2&=\int_{-\pi}^{\pi}(f(x))^2dx-2\int_{-\pi}^{\pi}f(x)\left[a_0+\sum_{k=1}^n(a_k\cos kx+b_k\cos kx)\right]dx+\int_{-\pi}^{\pi}(T(x))^2dx\\
&=\int_{-\pi}^{\pi}(f(x))^2dx-2\left[a_0\underbrace{\int_{-\pi}^{\pi}f(x)dx}_{=2\pi\alpha_0}+\sum_{k=1}^n\left(a_k\underbrace{\int_{-\pi}^{\pi}f(x)\cos kxdx}_{=\pi\alpha_k}+b_k\underbrace{\int_{-\pi}^{\pi}f(x)\sin kxdx}_{=\pi\beta_k}\right)\right]\\
&\quad+\int_{-\pi}^{\pi}(T(x))^2dx\\
&=\int_{-\pi}^{\pi}(f(x))^2dx-2\pi\left[2\alpha_0a_0+\sum_{k=1}^n(a_k\alpha_k+b_k\beta_k)\right]+a_0^2\cdot 2\pi+\pi\sum_{k=1}^n\left(a_k^2+b_k^2\right)\\
&=\int_{-\pi}^{\pi}(f(x))^2dx+2\pi\underbrace{\left[a_0^2-2\alpha_0a_0\right]}_{=(a_0-\alpha_0)^2-\alpha_0^2}+\pi\sum_{k=1}^n\underbrace{\left(a_k^2-2\alpha_ka_k\right)}_{=(a_k-\alpha_k)^2-\alpha_k^2}+\pi\sum_{k=1}^n\underbrace{\left(b_k^2-2\beta_kb_k\right)}_{=(b_k-\beta_k)^2-\beta_k^2}\\
&=\underbrace{\int_{-\pi}^{\pi}(f(x))^2dx-\pi\left[2\alpha_0^2+\sum_{k=1}^n\left(\alpha_k^2+\beta_k^2\right)\right]}_{\text{independent of }a_k,b_k\text{ (dependent on }f)}+\underbrace{\pi\left[(2(a_0-\alpha_0)^2+\sum_{k=1}^n\left((a_k-\alpha_k)^2+(b_k-\beta_k)^2\right)\right]}_{\geq 0}\\
&\geq\underbrace{\int_{-\pi}^{\pi}(f(x))^2dx-\pi\left[2\alpha_0^2+\sum_{k=1}^n\left(\alpha_k^2+\beta_k^2\right)\right]}_{\text{minimum value, attained when }a_0=\alpha_0,a_k=\alpha_k,b_k=\beta_k}.
\end{aligned}$$

□

Exercise: $\|f-S_nf\|_2^2=\|f\|_2^2-\|S_nf\|_2^2$.