# Math 321 Lecture 19

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# 1 Linear Functionals on Normed Vector Spaces

## 1.1 Definition and Examples

Let $V$ be a vector space on $\mathbb{R}$, equipped with a norm $\|\cdot\|_V$. $\|\cdot\|_V$ generates a metric on $V$ via $d(\mathbf{v}, \mathbf{w}) = \|\mathbf{v} - \mathbf{w}\|_V$. This allows to define *continuous functions* on $X$.

**Recall:** A function $f : V \to \mathbb{R}$ is continuous (by definition) if $f^{-1}$(open set in $\mathbb{R}$) is open in $V$.

**Definition 1.** Say $T : V \to \mathbb{R}$ is a **continuous linear functional** on $V$ if

1. $T$ is linear: $T(\alpha\mathbf{v} + \beta\mathbf{w}) = \alpha T(\mathbf{v}) + \beta T(\mathbf{w})$ for all $\mathbf{v}, \mathbf{w} \in V, \alpha, \beta \in \mathbb{R}$;
2. $T$ is continuous.

**Examples:**

1. $V = \mathbb{R}^n$; what is a continuous linear functional on $V$?

   $T : \mathbb{R}^n \xrightarrow{\text{linear}} \mathbb{R} \Leftrightarrow \exists \mathbf{a} \in \mathbb{R}^n$ s.t. $T(\mathbf{x}) = \sum_{j=1}^n a_j x_j \ \forall \mathbf{x} = (x_1, \ldots, x_n)$. These are continuous.

   **Exercise:** Check that if $V$ is finite-dimensional, then any linear functional on $V$ is continuous.

2. Is this fact necessarily true if $V$ is infinite-dimensional?

   (a)
   $$\begin{aligned} V &= \{\text{all infinite sequences with all but finitely many entries } 0\} \\ &= \{\mathbf{x} = (x_1, x_2, x_3, \ldots) : \exists N \geq 1 \text{ s.t. } x_j = 0 \ \forall j \geq N\}. \end{aligned}$$

   $V$ is infinite-dimensional because $\left\{ \mathbf{e}_j = (0, 0, \ldots, \underbrace{j}_{j^{\text{th}} \text{ entry}}, 0, \ldots) : j \geq 1 \right\}$ is an infinite linearly independent set in $V$.

   Define $T(\mathbf{x}) = \sum_{j=1}^\infty x_j$. Recall that $\|\mathbf{x}\|_1 = \sum_{j=1}^\infty |x_j|$. Note that $T$ is linear and $|T(\mathbf{x})| \leq \sum_{j=1}^\infty |x_j| = \|\mathbf{x}\|_1$.

   **Claim 1.** $T$ is continuous and linear with respect to $\|\cdot\|_1$.

   Define $\|\mathbf{x}\|^* = \sum_{j=1}^\infty 2^{-j}|x_j|$. Note that $T(\mathbf{e}_j) = 1$ for every $j \geq 1$ and $\|\mathbf{e}_j\|^* = 2^{-j}$. This implies that
   $$\frac{|T(\mathbf{e}_j)|}{\|\mathbf{e}_j\|^*} = \frac{1}{2^{-j}} = 2^j \xrightarrow{j \to \infty} \infty.$$

(b) $\mathcal{P} = \{\text{all polynomials on } [0,1]\}$, equipped with the sup norm.

$\mathcal{P}$ is infinite-dimensional because $\left\{ \underbrace{x_n}_{=P_n} : n \geq 0 \right\}$ form a linearly independent set.

Let $T : \mathcal{P} \to \mathbb{R}$ be defined by $f \mapsto f'(1)$.

**Claim 2.** $T$ is linear but discontinuous.

Note that $T(P_n) = nx^{n-1}\big|_{x=1} = n$ and that $\|P_n\|_\infty = 1$. This implies that

$$\frac{|T(P_n)|}{\|P_n\|} \xrightarrow{n\to\infty} \infty.$$

**Theorem 1.** Let $V$ be a normed vector space. A linear map $T : V \to \mathbb{R}$ is continuous if and only if $T$ obeys

$$\sup \left\{ \frac{|T(\mathbf{x})|}{\|\mathbf{x}\|} : \mathbf{x} \in V \right\} = K < \infty \Leftrightarrow |T(\mathbf{x})| \leq K\|\mathbf{x}\|. \tag{*}$$

*Proof.* "$\Leftarrow$": **Exercise:** If $T$ obeys (*), show that $T$ is Lipschitz and hence continuous.

"$\Rightarrow$": Assume $T$ is linear and continuous on $V$, hence at $\mathbf{0}$; i.e., given any $\epsilon > 0$ (choose $\epsilon = 1$), there exists $\delta > 0$ such that

$$\|\mathbf{x}\| = \|\mathbf{x} - \mathbf{0}\| < \delta \Rightarrow \left| T(\mathbf{x}) - \underbrace{T(\mathbf{0})}_{=0} \right| < \epsilon = 1.$$

**Shown:**

$$\|\mathbf{x}\| < \delta \Rightarrow |T(\mathbf{x})| < 1. \tag{**}$$

Choose any $\mathbf{y} \in V, \mathbf{y} \neq \mathbf{0}$. Set $\mathbf{v} = \frac{\delta}{2}\frac{\mathbf{y}}{\|\mathbf{y}\|}$. Then, $\|\mathbf{v}\| = 1 \cdot \frac{\delta}{2} = \frac{\delta}{2} < \delta$. Note that

$$\begin{aligned}
(**) &\Rightarrow |T(\mathbf{v})| < 1 \\
&\Rightarrow \left| T\left( \frac{\delta}{2} \frac{\mathbf{y}}{\|\mathbf{y}\|} \right) \right| < 1 \\
&\Rightarrow \frac{\delta}{2} \frac{1}{\|\mathbf{y}\|} |T(\mathbf{y})| < 1 \\
&\Rightarrow |T(\mathbf{y})| < \underbrace{\frac{2}{\delta}}_{=K} \|\mathbf{y}\|.
\end{aligned}$$

This is (*) with $K = \frac{2}{\delta}$. □

## 1.2 Riesz Representation Theorem

**Definition 2.** Let $V$ be a normed vector space. Then $V^* = \left\{ T : V \xrightarrow[\text{linear}]{\text{continuous}} \mathbb{R} \right\}$ is called the **dual** of $V$.

**Question:** What is $C[a,b]^*$? i.e., what does a continuous linear functional on $C[a,b]$ look like?

**Theorem 2** (Riesz representation theorem)**.** $C[a,b]^* \cong BV[a,b]$.

In other words, $L : C[a,b] \to \mathbb{R}$ is a continuous linear functional if and only if there exists $\alpha \in BV[a,b]$ such that

$$L(f) = \int_a^b f d\alpha \ \forall f \in C[a,b].$$

**Remark.**

1. Is $\alpha$ unique?

   **Answer:** No. If $\alpha$ works, $\alpha +$ constant will absolutely work.

   In fact, there exists a choice of $\alpha \in BV[a,b]$ such that $\alpha$ is right continuous on $[a,b]$ and $\alpha(a) = 0$. Such a choice is unique.

2. $\alpha$ is, in the language of measure, the *distribution function* of a random variable; i.e., if $X$ is a random variable, $\alpha(x) = \mathbb{P}(X \leq x)$.