Math 321 Lecture 10

Yuchong Pan

January 23, 2019

1 Stone-Weierstrass Theorem (Cont'd)

Theorem 1 (Stone-Weierstrass theorem)**.** Let (X, d) be a compact metric space. Let $\mathcal{A} \subseteq \underbrace{C(X;\mathbb{R})}_{\text{all continuous functions from } X \text{ to } \mathbb{R}}$ be a **sub-algebra** that **separates points** and **vanishes at no point**. Then $\mathcal{A}$ is dense in $C(X;\mathbb{R})$; i.e., $\overline{\mathcal{A}} = C(X;\mathbb{R})$.

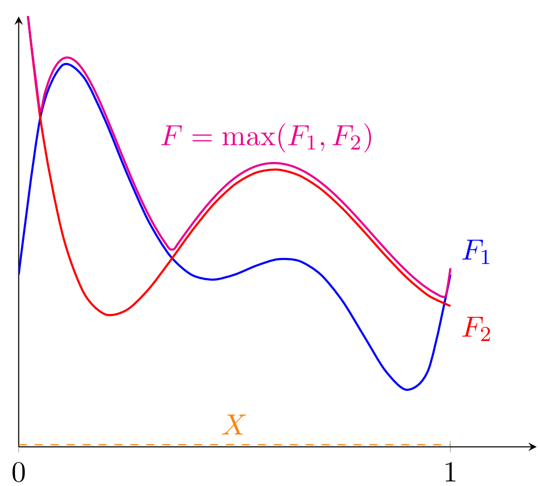
Lemma 1. Suppose $\mathcal{A}$ be any algebra of real-valued functions that separates points and vanishes at no point. Then given any $\underbrace{x_0, y_0 \in X}_{x_0 \neq y_0}$ and $a, b \in \mathbb{R}$, there exists $F \in \mathcal{A}$ such that $F(x_0) = a$ and $F(y_0) = b$.

Compactness of X is not needed. Further, continuity of functions in $\mathcal{A}$ are not assumed.

Proposition 1. Let $\mathcal{A} \subseteq C(X;\mathbb{R})$ be a sub-algebra. Then $\overline{\mathcal{A}}$ is also a sub-algebra and a **sub-lattice**; i.e., $\boxed{\text{if } F \in \mathcal{A}, \text{ then } |F| \in \mathcal{A}}$.

$\Downarrow$

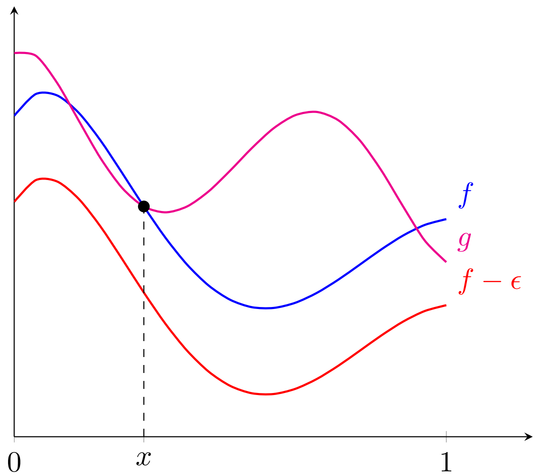
For any two functions F_1 and F_2 in $\mathcal{A}$, their maximum $F = \max(F_1, F_2) \in \mathcal{A}$.
The analogous statement holds for the minimum of two functions.



Note: $|G| = \max(G, -G)$ for any real G. Thus,

$$|F_1 - F_2| = \max(F_1 - F_2, F_2 - F_1),$$
$$\frac{1}{2}(F_1 + F_2 + |F_1 - F_2|) = \begin{cases} \frac{1}{2}(F_1 + F_2 + F_1 - F_2) = F_1, & F_1 \geq F_2, \\ \frac{1}{2}(F_1 + F_2 + F_2 - F_1) = F_2, & F_1 < F_2, \end{cases} = \max(F_1, F_2).$$

Proof of SW assuming Lemma 1 and Proposition 1. Start with $f \in C(X;\mathbb{R})$ and fix $\epsilon > 0$. We wish to find $g \in \mathcal{A}$ such that $\|f-g\|_\infty = \sup_{x\in X}|f(x)-g(x)| < \epsilon$.
Step 1: For every $x \in X$, we will find $g_x \in \mathcal{A}$ such that $g_x(x) = f(x)$ and that $g_x(y) > f(y) - \epsilon$ for all $y \in X$.



Proof of Step 1: For any $x \in X$, for every $y \neq x$, invoke Lemma 1 to find $G_y \in \mathcal{A} \subseteq C(X;\mathbb{R})$ such that

$$G_y(x) = f(x), \qquad G_y(y) = f(y). \tag{*}$$

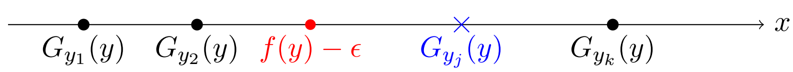
Note that $G_y - f \in C(X;\mathbb{R})$; hence its pre-images of open sets are open. This implies that

$$U_y = (G_y - f)^{-1}(-\epsilon,\infty) = \{z \in X : (G_y - f)(z) > -\epsilon\} = \{z \in X : G_y(z) > f(z) - \epsilon\}$$

is oepn and contains x and y.

Hence, $\{U_y; y \in X, y \neq x\}$ forms an open cover of X. Since X is compact, there is a finite subcover $U_{y_1}, U_{y_2}, \ldots, U_{y_M}$. Consider $g_x = \max(G_{y_1}, G_{y_2}, \ldots, G_{y_M})$. Since $X \subseteq U_{y_1} \cup U_{y_2} \cup \ldots \cup U_{y_M}$, then for every $y \in X$, there exists j such that

$$y \in U_{y_j} \Rightarrow G_{y_j}(y) - f(y) > \epsilon \Rightarrow G_{y_j}(y) > f(y) - \epsilon.$$



Verify that $g_x(x) = \max(G_{y_1}, G_{y_2}, \ldots, G_{y_M})(x) = \max(f(x), f(x), \ldots, f(x)) = f(x)$. Hence, $g_x \in \mathcal{A}$ by Proposition 1.

(Proof unfinished.)