Math 321 Lecture 15

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1 Riemann's Condition on Riemann-Stieltjes Integrability

Theorem 1. Let

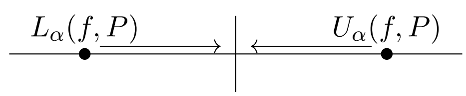
$$\begin{aligned} f &: [a,b] \xrightarrow{\text{bounded}} \mathbb{R}, \\ \alpha &: [a,b] \to \mathbb{R} \text{ be non-decreasing.} \end{aligned}$$

Then $f \in \mathcal{R}_\alpha[a,b]$ if and only if for every ϵ, there exists a partition P of $[a,b]$ such that $U_\alpha(f,P) - L_\alpha(f,P) < \epsilon$.

Proof. "$\Rightarrow$" Assume $f \in \mathcal{R}_\alpha[a,b]$. This implies that

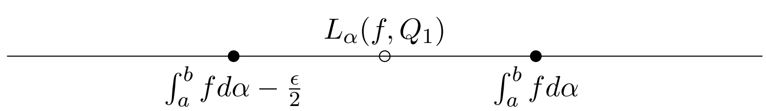
$$\sup\{L_\alpha(f,Q) : Q \text{ partition of } [a,b]\} = \inf\{U_\alpha(f,Q) : Q \text{ partition of } [a,b]\} = \underbrace{\int_a^b f d\alpha}_{\int_a^b f(x) d\alpha(x)} .$$

Fix $\epsilon > 0$.



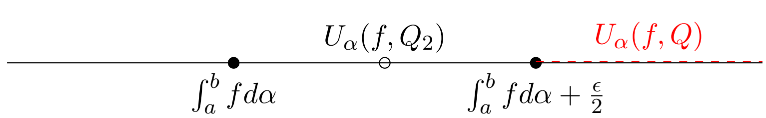
Since a supremum is a least upper bound, therefore $\int_a^b f d\alpha - \frac{\epsilon}{2}$ is *not* an upper bound for the set $\{L_\alpha(f,Q) : Q \text{ partitions}[a,b]\}$; i.e., there exists a partition Q_1 such that

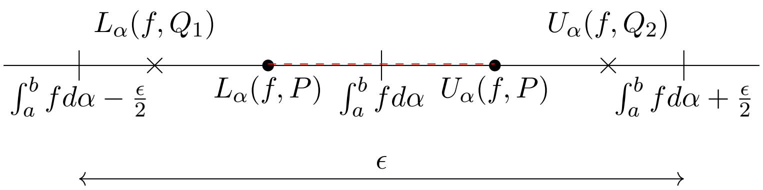
$$\int_a^b f d\alpha - \frac{\epsilon}{2} < L_\alpha(f,Q) \leq \int_a^b f d\alpha.$$



Similarly, an infimum is a greatest lower bound, so there exists a partition Q_2 such

$$\int_a^b f d\alpha + \frac{\epsilon}{2} > U_\alpha(f,Q_2) \geq \int_a^b f d\alpha.$$





Set $P = Q_1 \cup Q_2 =$ the common refinement of Q_1 and Q_2. We know that

$$L_\alpha(f, Q_1) \leq L_\alpha(f, P) \leq U_\alpha(f, P) \leq U_\alpha(f, Q_2).$$

Since $U_\alpha(f, Q_2) - L_\alpha(f, Q_1) < \epsilon$, it follows that

$$U_\alpha(f, P) - L_\alpha(f, P) < \epsilon.$$

$\Leftarrow$: Suppose that for every $\epsilon > 0$, there exists a partition P of $[a, b]$ such that

$$U_\alpha(f, P) - L_\alpha(f, P) < \epsilon. \tag{*}$$

Need to show:

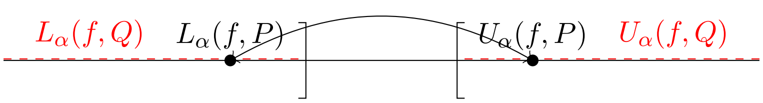
$$\sup_Q L_\alpha(f, Q) = \inf_Q U_\alpha(f, Q). \tag{**}$$

Know:

$$\text{LHS of (**)} \leq \text{RHS of (**)}.$$

Remains to prove:

$$\text{LHS of (**)} \geq \text{RHS of (**)}.$$



$$\sup_Q L_\alpha(f, Q) = \underline{\int_a^b} f\,d\alpha \overset{?}{=} \overline{\int_a^b} f\,d\alpha = \inf_Q U_\alpha(f, Q)$$

Aiming for a contradiction, suppose $\underline{\int_a^b} f\,d\alpha < \overline{\int_a^b} f\,d\alpha$; i.e.,

$$\overline{\int_a^b} f\,d\alpha - \underline{\int_a^b} f\,d\alpha = d > 0.$$

This means that

$$U_\alpha(f, P) - L_\alpha(f, P) \text{ for any partition } P.$$

This contradicts our assumption (*) if ϵ is chosen $< \delta$. $\square$

Remark.

$$\boxed{U_\alpha(f, P) - L_\alpha(f, P) < \epsilon}$$

$$\Downarrow$$

$$= \sum_{i=1}^n \underbrace{(M_i - m_i)}_{\omega(f, I_i)} \underbrace{\Delta\alpha_i}_{\omega(\alpha, I_i)}.$$

Definition 1. Given any interval $I \subseteq [a, b]$, define

$$\begin{aligned} \omega(g, I) &= \textbf{maximum oscillation} \text{ of } g \text{ on } I \\ &= \sup\{g(x) - g(y) : x, y \in I\} \\ &= \sup\{g(x) : x \in I\} - \inf\{g(y) : y \in I\}. \end{aligned}$$

Remark.

- If $g = f$ and $I = [x_{i-1}, x_i]$, then $\omega(f, I_i) = M_i - m_i$.
- If $g = \alpha$ is non-decreasing, then $\omega(\alpha, I_i) = \alpha(x_i) - \alpha(x_{i-1}) = \Delta\alpha_i$.

Corollary 1. $C[a, b] \subseteq \mathcal{R}_\alpha[a, b]$ for *any* non-decreasing α.

Proof. We will use Riemann's condition. For every $f \in C[a, b]$, any non-decreasing α and any $\epsilon > 0$, we will find a partition P of $[a, b]$ such that

$$U_\alpha(f, P) - L_\alpha(f, P) = \sum_{i=1}^{n} \omega(f, I_i)\omega(\alpha, I_i) < \epsilon.$$

f is uniformly continuous, so there exists $\delta > 0$ such that

$$|x - y| < \delta \Rightarrow |f(x) - f(y)| < \frac{\epsilon}{\alpha(b) - \alpha(a)}. \tag{***}$$

Choose

$$P = \{a = x_0 < x_1 = a + \frac{\delta}{2} < x_2 = a + \delta < \ldots < x_n = b\}.$$

Note:

$$\omega(f, I_i) < \frac{\epsilon}{\alpha(b) - \alpha(a)} \; \forall i \text{ by (***)}.$$

Hence,

$$\begin{aligned} U_\alpha(f, P) - L_\alpha(f, P) &= \sum_{i=1}^{n} \underbrace{\omega(f, I_i)}_{< \frac{\epsilon}{\alpha(b) - \alpha(a)}} \omega(\alpha, I_i) \\ &< \frac{\epsilon}{\alpha(b) - \alpha(a)} \sum_{i=1}^{n} \omega(\alpha, I_i) \\ &= \frac{\epsilon}{\alpha(b) - \alpha(a)} \underbrace{\sum_{i=1}^{n} (\alpha(x_i) - \alpha(x_{i-1}))}_{= \alpha(b) - \alpha(a)} \\ &= \epsilon. \end{aligned}$$

□

Theorem 2. If $f_n, f \in \mathcal{R}_\alpha[a, b]$ and $f_n \xrightarrow{n \to \infty} f$ uniformly on $[a, b]$, then

$$\int_a^b f_n d\alpha \xrightarrow{n \to \infty} \int_a^b f d\alpha.$$