Math 321 Lecture 31

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1 Proof of the Inverse Function Theorem (Cont'd)

Proof (cont'd). (b) **Know:**

- $\mathbf{f} : \underbrace{U}_{\substack{\rotatebox{90}{$\subseteq$} \\ E \\ \rotatebox{90}{$\subseteq$} \\ \mathbb{R}^n}} \xrightarrow{\text{bijection}} \mathbf{f}(U) = V \subseteq \mathbb{R}^n$ from part (a).
- $\mathbf{a} \in U \subseteq E$.
- $\mathbf{g} = \mathbf{f}^{-1} : V \to U$.

Goal: $\mathbf{g}$ is C^1.

Let

$$\begin{aligned} \mathbf{g}(\mathbf{y} + \mathbf{k}) = \mathbf{x} + \mathbf{h} \quad &\Leftrightarrow \quad \mathbf{f}(\mathbf{x} + \mathbf{h}) = \mathbf{y} + \mathbf{k}, \qquad\qquad \mathbf{x} \in U, \quad \mathbf{y} \in V. \\ \mathbf{g}(\mathbf{y}) = \mathbf{x} \quad &\Leftrightarrow \quad \mathbf{f}(\mathbf{x}) = \mathbf{y}. \end{aligned}$$

We have shown that

$$\begin{gathered} \mathbf{A} = \mathbf{f}'(\mathbf{a}) \text{ invertible} \\ \Downarrow \\ \mathbf{f}'(x) \text{ invertible for } \underbrace{\mathbf{x} \text{ near } \mathbf{a}}_{\text{so for } \mathbf{x} \in U} \\ \Downarrow \\ \mathbf{T} = (\mathbf{f}'(\mathbf{x}))^{-1} \text{ well-defined.} \end{gathered}$$

Hence,

$$\begin{gathered} \mathbf{g} \text{ is differentiable at } \mathbf{y} \text{ with the derivative } \mathbf{T} = (\mathbf{f}(\mathbf{x}))^{-1} \\ \Leftrightarrow \quad \mathcal{A} = \boxed{\frac{\|\overbrace{\mathbf{g}(\mathbf{y} + \mathbf{k})}^{=\mathbf{x}+\mathbf{h}} - \overbrace{\mathbf{g}(\mathbf{y})}^{=\mathbf{x}} - \overbrace{\mathbf{T}}^{=(\mathbf{f}'(\mathbf{x}))^{-1}} \mathbf{k}\|}{\|\mathbf{k}\|}} \xrightarrow{k \to 0} 0. \end{gathered}$$

We have

$$\begin{aligned}
\mathcal{A} &= \frac{\|(\cancel{\mathbf{x}}+\mathbf{h}) - \cancel{\mathbf{x}} - (\mathbf{f}'(\mathbf{x}))^{-1}(\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}))\|}{\|\mathbf{k}\|} \\
&= \frac{\| \overbrace{\mathbf{h}}^{=\mathbf{I}\mathbf{h}=(\mathbf{f}'(\mathbf{x}))^{-1}(\mathbf{f}'(\mathbf{x}))\mathbf{h}} - (\mathbf{f}'(\mathbf{x}))^{-1}(\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}))\|}{\|\mathbf{k}\|} \\
&= \frac{\|(\mathbf{f}'(\mathbf{x}))^{-1}(\mathbf{f}'(\mathbf{x})\mathbf{h} - (\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x})))\|}{\|\mathbf{k}\|} \\
&\le \underbrace{\frac{\|(\mathbf{f}'(\mathbf{x}))^{-1}\|}{\|\mathbf{k}\|}\|\mathbf{h}\|}_{\text{we need to show this stays bounded}} \quad \underbrace{\frac{\|\mathbf{f}(\mathbf{x}+\mathbf{h}) - \mathbf{f}(\mathbf{x}) - \mathbf{f}'(\mathbf{x})\mathbf{h}\|}{\|\mathbf{h}\|}}_{\substack{\to 0 \text{ as } \mathbf{h}\to\mathbf{0} \Leftrightarrow \mathbf{k}\to\mathbf{0} \\ \text{because } \mathbf{f} \text{ is differentiable at } \mathbf{x}}}.
\end{aligned}$$

Observation 1. $\mathbf{f}'(\mathbf{x})$ is closed to $\mathbf{f}'(\mathbf{a})$ if $\mathbf{x}$ is close to $\mathbf{a}$ because $\underbrace{\mathbf{f} \in C^1(E)}_{x \mapsto \mathbf{f}'(x) \text{ continuous}}$.

This implies that $\left\|(\mathbf{f}'(\mathbf{x}))^{-1}\right\|$ is bounded from above for $\mathbf{x} \in U$, where $\|\mathbf{A}\| = \sup_{\mathbf{x}\neq\mathbf{0}} \frac{\|\mathbf{A}\mathbf{x}\|}{\|\mathbf{x}\|}$ for an $m \times n$ matrix $\mathbf{A}$.

Recall from (a): $\varphi_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{A}^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x}))$ is a contraction for $\mathbf{x} \in U$; more precisely,

$$\begin{aligned}
&\|\varphi_{\mathbf{y}}(\underbrace{\mathbf{x}_1}_{=\mathbf{x}+\mathbf{h}}) - \varphi_{\mathbf{y}}(\underbrace{\mathbf{x}_2}_{=\mathbf{x}})\| < \frac{1}{2}\|\underbrace{\mathbf{x}_1 - \mathbf{x}_2}_{\mathbf{h}}\| \qquad \mathbf{x}_1, x_2 \in U \\
\Rightarrow \quad &\|\cancel{\mathbf{x}} + \mathbf{h} + \mathbf{A}^{-1}(\cancel{\mathbf{y}} - \mathbf{f}(\mathbf{x}+\mathbf{h})) - (\cancel{\mathbf{x}} + \mathbf{A}^{-1}(\cancel{\mathbf{y}} - \mathbf{f}(\mathbf{x})))\| < \frac{1}{2}\|\mathbf{h}\|.
\end{aligned}$$

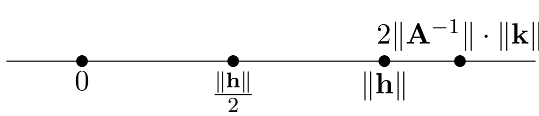
So,

$$\|\mathbf{h} - \mathbf{A}^{-1}(\underbrace{\mathbf{f}(\mathbf{x}+\mathbf{h})}_{\mathbf{y}+\mathbf{k}} - \underbrace{\mathbf{f}(\mathbf{x})}_{\mathbf{y}})\| < \frac{1}{2}\|\mathbf{h}\|.$$

This implies that

$$\begin{aligned}
\|\mathbf{h} - \mathbf{A}^{-1}\mathbf{k}\| < \frac{1}{2}\|\mathbf{h}\| \quad \Rightarrow \quad \underbrace{\|\mathbf{A}^{-1}\mathbf{k}\|}_{\substack{| \wedge \\ \|A^{-1}\|\cdot\|\mathbf{k}\|}} &\ge \|\mathbf{h}\| - \|\mathbf{h} - \mathbf{A}^{-1}\mathbf{k}\| \qquad \text{reverse triangle inequality} \\
&\ge \|\mathbf{h}\| - \frac{1}{2}\|\mathbf{h}\| \\
&= \frac{\|\mathbf{h}\|}{2}.
\end{aligned}$$

Get: $\frac{\|\mathbf{h}\|}{2} \le \|\mathbf{A}^{-1}\| \cdot \|\mathbf{k}\|$; i.e., $\frac{\|\mathbf{h}\|}{\|\mathbf{k}\|} \le 2 \cdot \|\mathbf{A}^{-1}\|$; i.e., as $\mathbf{h}, \mathbf{k} \to 0$, $\frac{\|\mathbf{h}\|}{\|\mathbf{k}\|}$ stays bounded.



□

Proof overview:

1. Show $\mathbf{f}$ is locally bijective. Find $U = B(\mathbf{a}; \epsilon)$.
   (a) $V = \mathbf{f}(U)$: $\mathbf{f}$ surjective.
   (b) CMP used to prove $\mathbf{f}$ injective.
   (c) **Exercise:** V is open.
2. Relate $\mathbf{g}'$ with $\mathbf{f}'$ where $\mathbf{g} = \mathbf{f}^{-1}$.

$$\mathbf{g}'(\mathbf{y}) = \mathbf{g}'(\mathbf{f}(\mathbf{x})) = (\mathbf{f}'(\mathbf{x}))^{-1}.$$

Critical: $\begin{matrix} \|\mathbf{h}\| \to 0 \\ \|\mathbf{k}\| \to 0 \end{matrix}$ at comparable rates; i.e., $\frac{\|\mathbf{h}\|}{\|\mathbf{k}\|}$ stays bounded.

2 Implicit Function Theorem

Question: Given any equation $f(x, y) = 0$ where $f : \mathbb{R}^2 \to \mathbb{R}$, when can we solve it to get $y = g(x)$?

Remark.

1. For arbitrarily chosen f, no guarantee of any solution; e.g., $f(x, y) = x^2 + y^2 + 1$. Need to assume $(a, b) \in \mathbb{R}^2$ such that $f(a, b) = 0$.
2. Suppose such (a, b) exists. Does this always imply that for *every* x in a neighbourhood of a, one can find $y = g(x)$ such that $f(x, g(x)) = 0$?

   Answer: No.

   Examples:

   (a) $f(x, y) = x^2 + y^2$, $(a, b) = (0, 0)$. No other solution exists for $x \neq a$.
   (b) $f(x, y) = x^2 + y^2 - 1$.