# Math 321 Lecture 33

Yuchong Pan

March 29, 2019

## 1 Proof of the Implicit Function Theorem

*Proof (cont'd).* **Last time: Given:** $\mathbf{f} : E \to \mathbb{R}^n$, $E \overset{\text{open}}{\subseteq} \mathbb{R}^{n+m}$, $\mathbf{f} \in C^1(E)$, $\underbrace{(\overbrace{\mathbf{a}}^{=\mathbf{x}}, \overbrace{\mathbf{b}}^{=\mathbf{y}})}_{\mathbb{R}^{n+m}} \in E$, $\mathbf{f}(\mathbf{a}, \mathbf{b}) = \mathbf{0}$, $\mathbf{A} = \mathbf{f}'(\mathbf{a}, \mathbf{b}) = \left[ \boxed{\mathbf{A}_x}_{n\times n} \middle| \boxed{\mathbf{A}_y}_{n\times m} \right]$, $\mathbf{A}_x$ invertible.

(a) **Goal:** Given $\mathbf{y}$ near $\mathbf{b}$, want to find a unique $\underbrace{\mathbf{x}}_{\text{near } \mathbf{a}}$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}$; i.e., $\mathbf{x} = \mathbf{g}(\mathbf{y})$.

We defined $\mathbf{F}(\mathbf{x}, \mathbf{y}) \overset{\text{def}}{=} (\mathbf{f}(\mathbf{x}, \mathbf{y}), \mathbf{y})$, $\mathbf{F} : \underbrace{E}_{\substack{\cap \\ \mathbb{R}^{n+m}}} \to \mathbb{R}^{n+m}$ and checked the hypotheses of the $\boxed{\text{inverse function theorem}}$ for $\mathbf{F}$. $\mathbf{F}'(\mathbf{a}, \mathbf{b})$ is invertible.

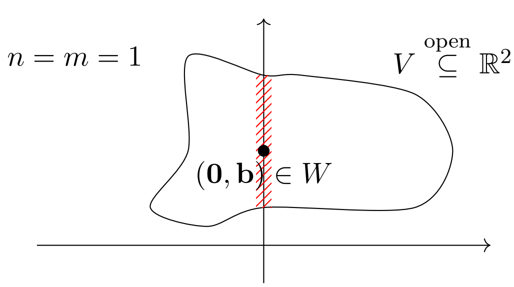
By the inverse function theorem, we know that there exist open sets $U, V \subseteq \mathbb{R}^{n+m}$ such that $\mathbf{F} : \underbrace{U}_{\substack{\cup \\ (\mathbf{a},\mathbf{b})}} \to \underbrace{V}_{\substack{\cap \\ \mathbb{R}^{n+m}}} = \mathbf{F}(U)$ is a bijection, and admits a $C^1$-inverse $\mathbf{G} = \mathbf{F}^{-1}$.

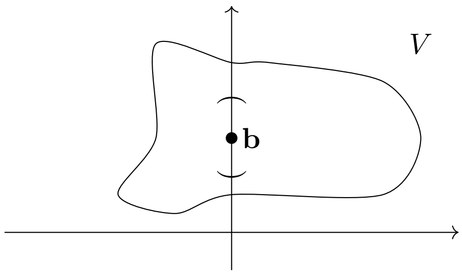
Let $\boxed{W = \{\mathbf{y} \in \mathbb{R}^m : (\mathbf{0}, \mathbf{y}) \in \underbrace{V}_{=F(U)}\}}$. $W$ is nonempty because $\mathbf{b} \in W$.

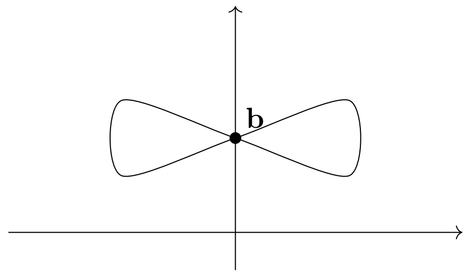
**Claim:** $\boxed{W \text{ is open in } \mathbb{R}^m. \text{ (Assume for now.)}}$

**Fact:** For any open set $O \subseteq \mathbb{R}^{k+k'}$, show $\{\mathbf{y} : (\underbrace{\mathbf{0}}_{\in \mathbb{R}^k}, \underbrace{\mathbf{y}}_{\in \mathbb{R}^{k'}}) \in O\} \subseteq \mathbb{R}^{k'}$ is open.

**Hint:** Study the set $\{\mathbf{y} : (\mathbf{0}, \mathbf{y}) \in \underbrace{B}_{\text{open ball in } \mathbb{R}^{k'}}\}$.






Choose $\mathbf{y} \in W$.

$$\begin{aligned}
&\Leftrightarrow \quad (\mathbf{0}, \mathbf{y}) \in V = F(U) \\
&\Leftrightarrow \quad \exists (\mathbf{x}, \mathbf{y}) \in U \text{ s.t. } \underbrace{\mathbf{F}(\mathbf{x}, \mathbf{y})}_{=(\mathbf{f}(\mathbf{x},\mathbf{y}),\mathbf{y})} = (\mathbf{0}, \mathbf{y}) \\
&\Leftrightarrow \quad \exists \mathbf{x} \in \mathbb{R}^n \text{ s.t. } \mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{0}.
\end{aligned}$$

Observe that $\mathbf{x}$ is unique: if there exist $\mathbf{x} \neq \mathbf{x}'$ such that $\mathbf{f}(\mathbf{x}, \mathbf{y}) = \mathbf{f}(\mathbf{x}', \mathbf{y}) = \mathbf{0}$ and that $(\mathbf{x}, \mathbf{y}), (\mathbf{x}', \mathbf{y}) \in U$, then $\mathbf{F}(\mathbf{x}, \mathbf{y}) = \mathbf{F}(\mathbf{x}', \mathbf{y}) = (\mathbf{0}, \mathbf{y})$, contradicting the bijectivity of $\mathbf{F}$ on $U$.

(b) Note that so far for every $\mathbf{y} \in W$, we have a unique $\mathbf{x} = \mathbf{g}(\mathbf{y})$ such that $(\mathbf{x}, \mathbf{y}) \in U$ and $\mathbf{f}(\mathbf{g}(\mathbf{y}), \mathbf{y}) = \mathbf{0}$. In other words, $\mathbf{x}$ is *implicitly* a function of $\mathbf{y}$, hence the name of the theorem.

e.g., $y^2 + x^3 = 0 \Rightarrow x = \left(-y^2\right)^{\frac{1}{3}}$.

However, $y^2 + xy + x^3 \sin x = 0$ cannot be solved explicitly as a function of $y$ but the implicit function theorem ensures that near certain $(a, b)$ such solutions exist.

**Goal:** $\mathbf{g} \in C^1(W)$ and $\mathbf{g}'(\mathbf{b}) = -\mathbf{A}_x^{-1} \mathbf{A}_y$, where $\mathbf{g} : \mathbb{R}^m \to \mathbb{R}^n$.

Note that

$$\mathbf{F} : (\underbrace{\mathbf{x}}_{=\mathbf{g}(\mathbf{y})}, \mathbf{y}) \mapsto (\underbrace{\mathbf{f}(\mathbf{x}, \mathbf{y})}_{=\mathbf{0}}, \mathbf{y}).$$

Define

$$\underbrace{\Phi(\mathbf{y})}_{\in C^1(W)} = (\underbrace{\mathbf{g}(\mathbf{y})}_{\in C^1(W)}, \mathbf{y}) = \mathbf{F}^{-1}(\mathbf{0}, \mathbf{y}).$$

Then,

$$\Phi'(\mathbf{y}) = \begin{pmatrix} \mathbf{g}'(\mathbf{y})_{n \times m} \\ \mathbf{I}_{m \times m} \end{pmatrix}.$$

By the inverse function theorem, we know $\mathbf{F}^{-1}$ is $C^1$ on $V$. Thus,

$$\begin{aligned}
&\mathbf{f}(\underbrace{\mathbf{g}(\mathbf{y}), \mathbf{y}}_{=\Phi(\mathbf{y})}) = \mathbf{0} \\
\boxed{\overset{\text{chain}}{\underset{\text{rule}}{\Longrightarrow}}} \quad &\underbrace{\mathbf{f}'(\Phi(\mathbf{y}))}_{n \times (n+m)} \Phi'(\mathbf{y}) = \mathbf{0} \\
\Rightarrow \quad &\underbrace{\mathbf{f}'(\mathbf{g}(\mathbf{y}), \mathbf{y})}_{n \times (n+m)} \begin{pmatrix} \mathbf{g}'(\mathbf{y}) \\ \mathbf{I} \end{pmatrix}_{(n+m) \times m} = \mathbf{0}.
\end{aligned}$$

Why is $\mathbf{g}$ or $\Phi$ differentiable?

Set $\mathbf{y} = \mathbf{b}$; get

$$\underbrace{\begin{bmatrix} \mathbf{A}_x & \mathbf{A}_y \end{bmatrix}}_{=\mathbf{f}'(\underbrace{\mathbf{g}(\mathbf{b})}_{=\mathbf{a}}, \mathbf{b})} \begin{pmatrix} \mathbf{g}'(\mathbf{b}) \\ \mathbf{I} \end{pmatrix} = \mathbf{0};$$

i.e.,

$$\mathbf{A}_x \mathbf{g}'(\mathbf{b}) + \mathbf{A}_y = \mathbf{0} \quad \Rightarrow \quad \mathbf{g}'(\mathbf{b}) = -\mathbf{A}_x^{-1} \mathbf{A}_y \qquad \text{since } \mathbf{A}_x \text{ is known to be invertible.}$$

$\square$