# Math 321 Lecture 2

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## 1 Pointwise and Uniform Convergence

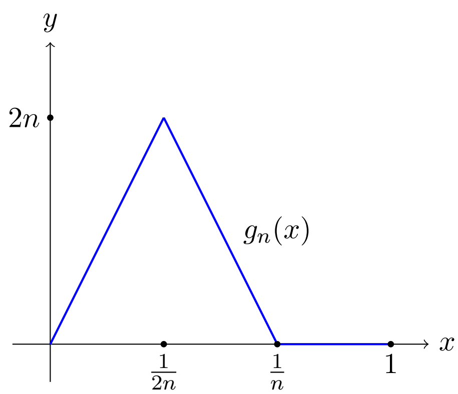
### 1.1 Examples of Pointwise and Uniform Convergence (Cont'd)

1. Last time we checked

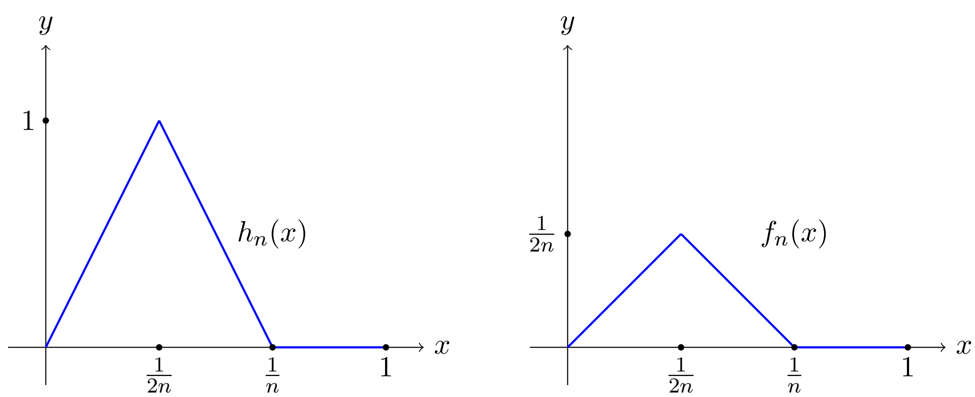
   (a) $g_n \longrightarrow 0$ pointwise on $[0, 1]$;

   (b) $g_n \not\longrightarrow 0$ uniformly on $[0, 1]$ (neither on $(0, 1)$);

   (c) $\int_0^1 g_n(x)dx = 1 \not\longrightarrow \int_0^1 0dx = 0$.



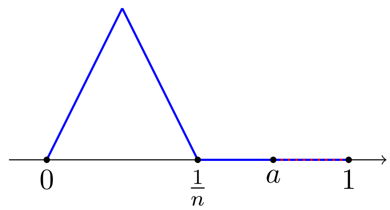

   Similar sequences of functions:




(a) $h_n \not\longrightarrow 0$ uniformly on $[0, 1]$ (b) $f_n \longrightarrow 0$ uniformly on $[0, 1]$

**Claim 1.** Fix any $0 < a < 1$. Then $g_n \to 0$ uniformly on $[a, 1]$.

*Proof.* Choose $N \geq 1$ integer so that $\frac{1}{n} < a$ for all $n \geq N$.




Then $g_n(x) = 0$ for all $x \in [a, 1]$. Thus,

$$\sup_{x \in [a,1]} |g_n(x)| = 0 \xrightarrow{n \to \infty} 0.$$

□

2. $h_n(x) = \frac{x^{n+1}}{n+1}, x \in [0, 1]$.

   (a) $|h_n(x) = \frac{x^{n+1}}{n+1} \leq \frac{1}{n} \xrightarrow{\text{uniformly}} 0$ as $n \to \infty$. Thus,

   $$\sup_{x \in [0,1]} |h_n(x)| = \frac{1}{n+1} \xrightarrow{n \to \infty} 0 = h(x).$$

   (b)

   $$\int_0^1 h_n(x)dx = \int_0^1 \frac{x^{n+1}}{n+1}dx = \frac{1}{n+1}\left[\frac{x^{n+2}}{n+2}\right]_{x=0}^{x=1} = \frac{1}{(n+1)(n+2)} \xrightarrow{n \to \infty} 0 = \int_0^1 0dx = \int_0^1 h(x)dx.$$

   (c)

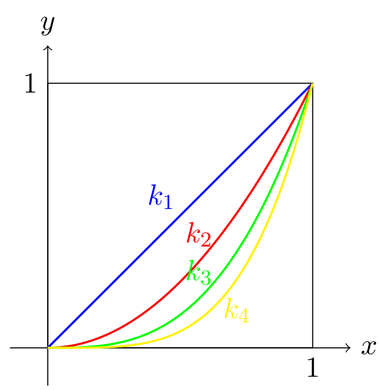
   $$h_n'(x) = k_n(x) = x^n \xrightarrow[n \to \infty]{\text{pointwise}} k(x) = \begin{cases} 0, & x \neq 1, \\ 1, & x = 1, \end{cases}, x \in [0, 1].$$

   **Claim 2.** $k_n \not\to k$ uniformly.

   *Proof.* Note that

   $$\sup_{x \in [0,1]} |k_n(x) - k(x)| \geq \left|k_n\left(1 - \frac{1}{n}\right) - k\left(1 - \frac{1}{n}\right)\right| = \left(1 - \frac{1}{n}\right)^n \to \frac{1}{e} \neq 0.$$

   Therefore, $\sup_{x \in [0,1]} |k_n(x) - k(x)| \geq \frac{1}{e}$ and hence $\not\to 0$, proving the claim. □



## 1.2 Theorems

**Theorem 1.** Let $f_n, f : (X, d) \to (Y, \rho)$. Assume

1. $f_n$ is continuous for every $n \geq 1$;
2. $f_n \to f$ uniformly on $X$.

Then, $f$ is continuous.

**Theorem 2.** Let $f_n \in C[a, b]$; i.e., $f_n : [a, b] \xrightarrow{\text{continuous}} \mathbb{R}$. Assume $f_n \xrightarrow{n\to\infty} f$ uniformly on $[a, b]$. (Hence, $f \in C[a, b]$ by Theorem 1.) Then,

$$\int_a^b f_n(x)dx \xrightarrow{n\to\infty} \int_a^b f(x)dx.$$

**Remark 1.** Example 1 shows that "uniform convergence" is necessary.

**Remark 2.** Both theorems have a point in common; they involve **interchanging limits**.

1. Theorem 1:

$$\begin{aligned} & f \text{ is } \textbf{continuous} \text{ at a point } x \in X \\ \Leftrightarrow\ & \text{for every sequence } \{x_k\} \subseteq X \text{ with } x_k \to x, \text{ we have } f(x_k) \xrightarrow{k\to\infty} f(x) \\ \Leftrightarrow\ & \lim_{k\to\infty} f(x_k) = f(x) \\ \Leftrightarrow\ & \lim_{k\to\infty} \underbrace{\lim_{n\to\infty} f_n(x_k)}_{(2)} = \lim_{n\to\infty} f_n(x) = \lim_{n\to\infty} f_n\left(\lim_{k\to\infty} x_k\right) \xlongequal{f \text{ is continuous}} \lim_{n\to\infty}\lim_{k\to\infty} f_n(x_k). \end{aligned}$$

2. Theorem 2:

$$\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b f(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx.$$

*Proof of Theorem 2.* Since $f_n \to f$ uniformly on $[a, b]$, we know that given any $\epsilon > 0$, there exists $N \geq 1$ such that

$$\sup_{x\in[a,b]} |f_n(x) - f(x)| < \epsilon \quad \forall n \geq N. \tag{*}$$

Thus,

$$\begin{aligned} \left|\int_a^b f_n(x)dx - \int_a^b f(x)dx\right| &= \left|\int_a^b [f_n(x) - f(x)]dx\right| \\ &\leq \int_a^b |f_n(x) - f(x)|dx \\ &\leq \int_a^b \left(\sup_{x\in[a,b]} |f_n(x) - f(x)|\right) dx \qquad \text{(by (*))} \\ &< \int_a^b \epsilon dx \\ &= \epsilon(b - a). \end{aligned}$$

□