Math 321 Lecture 18

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1 Functions of Bounded Variation

1.1 Jordan's Theorem

Definition 1. $\alpha : [a,b] \to \mathbb{R}$ is in $BV[a,b]$ if $V_a^b \alpha = \sup_{P \text{ partition}} V_a^b(\alpha, P) = \sup_P \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})| < \infty$.

Theorem 1 (Jordan's theorem)**.** $\alpha \in BV[a,b]$ if and only if α can be written as $\alpha = \beta - \gamma$, with β, γ nondecreasing.

Proof. "$\Leftarrow$": Assume $\alpha = \beta - \gamma$ with $\beta, \gamma \xrightarrow{\text{nondecreasing}} \mathbb{R}$. Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$. Then,

$$\begin{aligned}
V_a^b(\alpha, P) &= \sum_{i=1}^n |\alpha(x_i) - \alpha(x_{i-1})| \\
&= \sum_{i=1}^n |[\beta(x_i) - \beta(x_{i-1})] - [\gamma(x_i) - \gamma(x_{i-1})]| \\
&\underbrace{\leq \sum_{i=1}^n |\beta(x_i) - \beta(x_{i-1})| + \sum_{i=1}^n |\gamma(x_i) - \gamma(x_{i-1})|}_{\text{triangular inequality}} \\
&= V_a^b(\beta, P) + V_a^b(\gamma, P) \leq V_a^b \beta + V_a^b \gamma \\
&= \underbrace{[\beta(b) - \beta(a)] + [\gamma(b) - \gamma(a)]}_{\text{independent of } P} < \infty.
\end{aligned}$$

Hence, $V_a^b \alpha = \sup_P V_a^b(\alpha, P) \leq [\beta(b) - \beta(a)] + [\gamma(b) - \gamma(a)] < \infty$.

Exercise: If $f, g \in BV[a,b]$, then $V_a^b(f \pm g) leq V_a^b f + V_a^b g$.

"$\Rightarrow$": Assume that $\alpha \in BV[a,b]$; need to find β, γ nondecreasing such that $\underbrace{\alpha = \beta - \gamma}_{\Rightarrow\ \gamma = \beta - \alpha}$. Introduce the variation function $v(x) = V_a^x \alpha =$ total variation of α on $[a,x]$.

Exercise: $v(x)$ is well-defined because $V_a^b \alpha \geq V_a^c \alpha$ for any interval $[c,d] \subseteq [a,b]$.

Question: Is v nondecreasing? Yes.

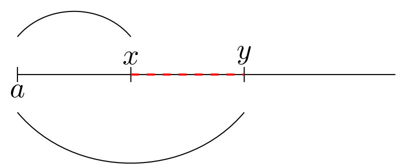
Let $x < y$. Need to verify if $\underbrace{v(x)}_{= V_a^x \alpha} \leq \underbrace{v_y(y)}_{= V_a^y \alpha}$.

True by Exercise above since $[a,x] \subseteq [a,y]$.

Question: Is $v(\cdot) - \alpha(\cdot)$ nondecreasing? Yes.

Let $x < y$; need to verify if

$$\begin{aligned} v(x) - \alpha(x) &\leq v(y) - \alpha(y), \\ \alpha(y) - \alpha(x) &\leq \underbrace{v(y) - v(x)}_{V_a^y \alpha - V_a^x \alpha = V_x^y \alpha = \sup_{P \text{ partition of } [x,\,y]} V_x^y(\alpha, P)}. \end{aligned}$$



Note: $|\alpha(y) - \alpha(x)| = V_x^y \alpha(x, P_0)$, where $P_0 = \{x, y\}$. Thus,

$$\alpha(y) - \alpha(x) \leq |\alpha(y) - \alpha(x)| = V_x^y(\alpha, P_0) \leq V_x^y \alpha = v(y) - v(x).$$

Exercise: Check that for any $c \in (a, b)$ and any $f \in BV[a, b]$,

$$V_a^c f + V_c^b f = V_a^b f.$$

$\square$

Remark. Jordan's theorem suggests that $BV[a, b]$ could be a good source of Riemann-Stieltjes integrators, with respect to which continuous functions on $[a, b]$ can be integrated.

$$\int f d \underbrace{\alpha}_{\in BV} = \int f d(\beta - \gamma) \overset{?}{=\!=} \int f d\beta - \int f d\gamma.$$

1.2 Interchanging Integrands and Integrators

Recall:

$$\int_a^b (\underbrace{udv}_{=u(x)v'(x)dx} + \underbrace{vdu}_{=v(x)u'(x)dx}) = u(b)v(b) - u(a)v(a). \qquad \longrightarrow \text{integration by parts}$$

Theorem 2 (Integration by parts for Riemann-Stieltjes integrals)**.** Let $f, \alpha \to \mathbb{R}$ be arbitrary functions. Then,

$$\underbrace{f \in \mathcal{R}_\alpha[a, b]}_{\substack{\exists I \text{ s.t. } \forall \epsilon > 0, \exists \text{a partition } P_0 \text{ s.t. } |S_\alpha(f, P, T) - I| < \epsilon \\ \text{for } P \supseteq P_0 \text{ and any selection of points } T \text{ subordinate to } P}} \qquad \Leftrightarrow \alpha \in \mathcal{R}_f[a, b].$$

In either case,

$$\int_a^b f d\alpha + \int_a^b \alpha df = \alpha(b) f(b) - \alpha(a) f(a).$$

Corollary 1. $C[a, b] \subseteq \mathcal{R}_\alpha[a, b]$ for all $\alpha \in BV[a, b]$.

Proof. Use Jordan's theorem to write $\alpha = \beta - \gamma$ with β, γ nondecreasing.

Note by a previous theorem, $C[a,b] \subseteq \mathcal{R}_\beta[a,b] \cap \mathcal{R}_\gamma[a,b]$. This implies that if $f \in C[a,b]$, then $\int_a^b f d\beta$ and $\int_a^b f d\gamma$ are well-defined.

By IBP, $\int_a^b \beta df$ and $\int_a^b \gamma df$ are well-defined too; i.e., $\beta, \gamma \in \mathcal{R}_f[a,b]$. But it is easy to see that $\mathcal{R}_f[a,b]$ is a vector space, with

$$\int_a^b (\beta \pm \gamma) df = \int_a^b \beta df \pm \int_a^b \gamma df.$$

This implies $\beta \pm \gamma \in \mathcal{R}_f[a,b]$; in particular, $\alpha = \beta - \gamma \in \mathcal{R}_f[a,b]$.

Use IBP again; get $f \in \mathcal{R}_\alpha[a,b]$. □