Math 321 Lecture 30

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1 Proof of the Inverse Function Theorem (Cont'd)

Proof (Cont'd). Let

$$\mathbf{f}: E \to \mathbb{R}^n, \qquad E \overset{\text{open}}{\subseteq} \mathbb{R}^n, \qquad \mathbf{a} \in E, \qquad \mathbf{A} = \mathbf{f}'(\mathbf{a}) \text{ invertible}, \qquad \underbrace{\mathbf{f} \in C^1(E)}_{\substack{\Leftrightarrow x \mapsto \mathbf{f}'(x) \text{ is continuous on } E: \\ \text{given any } \lambda > 0, \text{ there exists } \delta > 0 \\ \text{such that } \|\mathbf{f}'(\mathbf{x}) - \mathbf{A}\| < \lambda \text{ for } \|\mathbf{x} - \mathbf{a}\| < \delta}}.$$

1. Last time, we found $U = B(\mathbf{a}, \epsilon)$ and $V = \mathbf{f}(U)$ such that $\mathbf{f}: U \to V$ is a bijection. We showed $\mathbf{f}$ is 1-1 on U using the CMP. Defined $\varphi_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{A}^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x})), \mathbf{x} \in E$; showed that for $\mathbf{x}_1, \mathbf{x}_2 \in U$,

$$\boxed{\|\varphi_{\mathbf{y}}(\mathbf{x}_1) - \varphi_{\mathbf{y}}(\mathbf{x}_2)\| \leq \frac{1}{2}\|\mathbf{x}_1 - \mathbf{x}_2\|.} \tag{*}$$

Note that

$$\|\varphi_{\mathbf{y}}(\mathbf{x}_1) - \varphi_{\mathbf{y}}(\mathbf{x}_2)\| \leq \|\varphi_{\mathbf{y}}'\| \cdot \|\mathbf{x}_1 - \mathbf{x}_2\|,$$

where

$$\|\varphi_{\mathbf{y}}'(\mathbf{x})\| = \|\mathbf{I} - \mathbf{A}^{-1}\mathbf{f}'(\mathbf{x})\| = \left\|\mathbf{A}^{-1}(\mathbf{A} - \mathbf{f}'(\mathbf{x}))\right\|.$$

Pick $\epsilon > 0$ so that

$$\underbrace{\|\mathbf{x} - \mathbf{a}\| < \epsilon}_{\mathbf{x} \in U} \Rightarrow \|\mathbf{A} - \mathbf{f}'(\mathbf{x})\| < \frac{1}{2\|\mathbf{A}^{-1}\|}.$$

Thus,

$$\|\varphi_{\mathbf{y}}'(\mathbf{x})\| \leq \left\|\mathbf{A}^{-1}\right\| \cdot \|\mathbf{A} - \mathbf{f}'(\mathbf{x})\| < \left\|\mathbf{A}^{-1}\right\| \cdot \lambda = \frac{1}{2}.$$

Remark: The ϵ chosen to define $U = B(\mathbf{a}, \epsilon)$ is independent of $\mathbf{y}$, but dependent on $\mathbf{a}$ and $\mathbf{f}$.

Remark: Let V, W be vector spaces.

$\mathbf{B}: V \xrightarrow{\text{linear}} W$ where $\dim(V) < \infty, \dim(W) < \infty \Rightarrow \mathbf{B}$ is continuous and bounded; i.e., $\|\mathbf{B}\| < \infty$.

(*) is a contraction condition, but $\varphi_{\mathbf{y}}$ need not map U to itself.

Note: U is not complete, but $\overline{U}$ is. (*) continues to hold for $\mathbf{x}_1, \mathbf{x}_2 \in \overline{U}$.

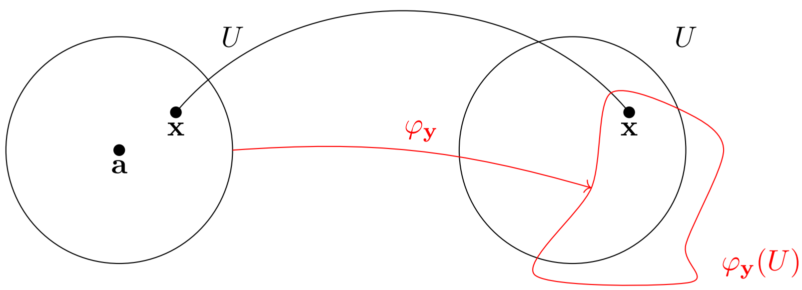
Can $\varphi_{\mathbf{y}}$ have a fixed point in U?

If there exists such $\mathbf{x}$, then

$$\varphi_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} \in U \Rightarrow \varphi_{\mathbf{y}}(\mathbf{x}) \in U.$$

We do not know if $\varphi_{\mathbf{y}}(U) \subseteq U$, so cannot guarantee the *existence* of a fixed point of $\varphi_{\mathbf{y}}$. However, $\boxed{\text{if a fixed point exists}}$, by CMP it would have to be unique.

Take $\mathbf{y} \in V = \mathbf{f}(U)$. Then there exists $\mathbf{x} \in U$ such that $\mathbf{f}(\mathbf{x}) = \mathbf{y}$. Thus, $\mathbf{x}$ is a fixed point of $\varphi_{\mathbf{y}}$. By CMP, there exists only one such $\mathbf{x}$; hence $\mathbf{f} : U \to V$ is injective.



Exercise: V is open.

2. Let $\mathbf{g} : V \to U$ where $\mathbf{g} = \mathbf{f}^{-1}$ defined by part (a). Want to show $\mathbf{g}$ is differentiable on V; in fact, $\mathbf{g} \in C^1(V)$.

Proof. Need to find $\mathbf{T} = \mathbf{T}(\mathbf{y})$ continuous on V such that

$$\frac{\|\mathbf{g}(\mathbf{y}+\mathbf{k}) - \mathbf{g}(\mathbf{y}) - \mathbf{T}\mathbf{k}\|}{\|\mathbf{k}\|} \xrightarrow{\|\mathbf{k}\| \to 0} 0.$$

In $\mathbb{R}$,

$$g \circ f(x) = x \Rightarrow g'(f(x)) \cdot f'(x) = 1 \Rightarrow g'\left(\underbrace{f(x)}_{y}\right) = \frac{1}{\underbrace{f'(x)}_{f'(f^{-1}(y))}}.$$

Choose $\mathbf{T} = (\mathbf{f}'(\mathbf{g}(\mathbf{y})))^{-1}$. We will show that $\mathbf{T}$ is well-defined.

Recall: A matrix $\mathbf{B}_{n \times n}$ is invertible if and only if $\underbrace{\det(\mathbf{B})}_{\substack{\text{a polynomial in the entries of } \mathbf{B} \\ \text{hence continuous in the entries of } \mathbf{B}}} \neq 0.$

Hence, $\mathbf{x} \in E \subseteq \mathbb{R}^n \mapsto \det(\mathbf{f}'(\mathbf{x})) \in \mathbb{R}$ is a continuous map, which assumes a nonzero value at $\mathbf{x} = \mathbf{a}$. Hence it remains nonzero on $U = B(\mathbf{a}, \epsilon)$ for $\epsilon > 0$ small.

Set

$$\mathbf{f}(\mathbf{x}) = \mathbf{y} \in V \Leftrightarrow \mathbf{g}(\mathbf{y}) = \mathbf{x},$$
$$\mathbf{f}(\mathbf{x}+\mathbf{h}) = \mathbf{y}+\mathbf{k} \Leftrightarrow \mathbf{g}(\mathbf{y}+\mathbf{k}) = \mathbf{x}+\mathbf{k}.$$

Then,

$$\mathbf{g}(\mathbf{y}+\mathbf{k}) - \mathbf{g}(\mathbf{y}) - \mathbf{T}\mathbf{k} = \mathbf{h} - \mathbf{T}\mathbf{k} = \mathbf{T}\left(\mathbf{T}^{-1}\mathbf{h} - \mathbf{k}\right) = \mathbf{T}(\mathbf{f}'(\mathbf{x})\mathbf{h} - \mathbf{k}) = \mathbf{T}\left(\underbrace{(\mathbf{f}(\mathbf{x}) - \mathbf{f}'(\mathbf{a}))\mathbf{h}}_{\mathrm{I}} + \underbrace{\mathbf{f}'(\mathbf{a})\mathbf{h} - \mathbf{k}}_{\mathrm{II}}\right).$$

Want to show that

$$\frac{\|\mathrm{II}\|}{\|\mathbf{k}\|} = \frac{\mathbf{T}(\mathbf{A}\mathbf{h} - \mathbf{k})}{\|\mathbf{k}\|} \xrightarrow[??]{\mathbf{k} \to \mathbf{0}} 0. \tag{1}$$

Note that

$$\underbrace{\varphi_{\mathbf{y}}(\mathbf{x}+\mathbf{h})}_{=\mathbf{x}+\mathbf{h}+\mathbf{A}^{-1}(\mathbf{y}-\mathbf{f}(\mathbf{x}+\mathbf{h}))} - \varphi_{\mathbf{y}}(\mathbf{x}) = \mathbf{h} + \mathbf{A}^{-1}(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}+\mathbf{h})).$$

Since $\varphi_{\mathbf{y}}$ is a contraction,

$$\left\|\mathbf{h} + \mathbf{A}^{-1}(\mathbf{f}(\mathbf{x}) - \mathbf{f}(\mathbf{x}+\mathbf{h}))\right\| \leq \frac{1}{2}\|\mathbf{h}\| \Rightarrow \left\|\mathbf{h} + \mathbf{A}^{-1}\mathbf{k}\right\| \leq \frac{1}{2}\|\mathbf{h}\|.$$

Use this to prove (1).

(Proof unfinished.)