Math 321 Lecture 21

Yuchong Pan

February 27, 2019

1 Fourier Series

Consider the interval $[-\pi, \pi]$.

Definition 1. A function $T : [-\pi, \pi] \to \mathbb{C}$ is called a **trignometric polynomial** if T is of the form

$$T = \alpha_0 + \sum_{k=1}^{n} \left(\underbrace{\alpha_k}_{\textbf{amplitude}} \cos kx + \underbrace{\beta_k}_{\textbf{amplitude}} \sin kx \right),$$

where $\underbrace{\alpha_k}_{0 \le k \le n}, \underbrace{\beta_k}_{1 \le k \le n} \in \mathbb{C}$.

Say T has a **frequency** $k \in \{\pm 1, \ldots, \pm n\}$ if at least one of α_k or β_k does not equal 0.

Alternatively, T can also be expressed as

$$T(x) = \sum_{k=-n}^{n} a_k e^{ikx}, a_k \in \mathbb{C}, \tag{*}$$

because $e^{i\theta} = \cos\theta + i \sin\theta$ so $\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2}$.

Recall Weierstrass's second theorem: Any $f \in \mathcal{C}^{2\pi}$ = the space of complex-valued, 2π-periodic, continuous functions on $[-\pi, \pi]$ can be uniformly approximated by a sequence of trignometric polynomials.

Question: Given a trignometric polynomial T with unspecified coefficients α_k, β_k (or a_k), is it possible to obtain its frequencies and corresponding amplitudes?

$$T(x) = \alpha_0 + \sum_{k=1}^{n} (\alpha_k \cos kx + \beta_k \sin kx), \qquad n, \alpha_k, \beta_k \text{ unknown}.$$

Fact 1. Let $\{1, \cos mx, \sin mx : m = 1, 2, 3, \ldots\} = \mathcal{F}^*$.

1. For any two $f, g \in \mathcal{F}^*, f \neq g$,

$$\int_{-\pi}^{\pi} f(x) g(x) dx = 0.$$

   Note that

$$\cos mx \sin nx = \frac{1}{2}(\sin(n+m)x + \sin(n-m)x),$$
$$\cos mx \cos nx = \frac{1}{2}(\cos(n+m)x + \cos(n-m)x).$$

2. If $f = g$,

$$\int_{-\pi}^{\pi} 1^2 dx = 2\pi,$$

$$\int_{-\pi}^{\pi} \cos^2 mx dx = \int_{-\pi}^{\pi} (1 + \cos 2mx) dx = \pi = \int_{-\pi}^{\pi} \sin^2 mx dx, \qquad m = 1, 2, 3, \ldots.$$

Integrate both sides on $[-\pi, \pi]$,

$$\int_{-\pi}^{\pi} T(x) dx = \alpha_0 \int_{-\pi}^{\pi} d(x) + \sum_{k=1}^{n} \left(\alpha_k \underbrace{\int_{-\pi}^{\pi} \cos kx dx}_{=0} + \beta_k \underbrace{\int_{-\pi}^{\pi} \sin kx dx}_{=0} \right).$$

Thus,

$$\alpha_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} T(x) dx = \frac{1}{2\pi} \langle T, 1 \rangle.$$

To solve for α_m, multiply both sides of (*) by $\cos mx$:

$$\int_{-\pi}^{\pi} T(x) \cos mx dx = \alpha_m \int_{-\pi}^{\pi} \cos^2 mx + \underbrace{\cdots}_{\text{all remaining integrals equal to 0 by Fact 1}},$$

i.e.,

$$\alpha_m = \frac{1}{\pi} \int_{-\pi}^{\pi} T(x) \cos mx dx = \frac{1}{\pi} \langle T(\cdot), \cos(m\cdot) \rangle,$$
$$\beta_m = \frac{1}{\pi} \int_{-\pi}^{\pi} T(x) \sin mx dx = \frac{1}{\pi} \langle T(\cdot), \sin(m\cdot) \rangle.$$

Here,

$$\langle f, g \rangle = \int_{-\pi}^{\pi} f(x) g(x) dx.$$

We have just shown that for a trignometric polynomial T,

$$T(x) = \underbrace{\frac{\langle T, 1 \rangle}{2\pi}}_{=\alpha_0} + \sum_{k=1}^{n} \left(\underbrace{\frac{\langle T(\cdot), \cos(k\cdot) \rangle}{\pi}}_{=\alpha_k} \cos kx + \underbrace{\frac{\langle T(\cdot), \sin(k\cdot) \rangle}{\pi}}_{=\beta_k} \sin kx \right).$$

1. Is the representation unique?
2. Can we do this for larger classes of functions, say $\mathcal{C}^{2\pi}$?

Definition 2. Let $f \in \mathcal{C}^{2\pi}$, or more generally $f \in \mathcal{R}[-\pi, \pi]$. Define

$$\begin{cases} \alpha_0 &= \frac{\langle f, 1 \rangle}{2\pi} = \frac{1}{2\pi} \int_{-\pi}^{\pi} f(x) dx, \\ \alpha_m &= \frac{\langle f(\cdot), \cos(m\cdot) \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos mx dx, \qquad m = 1, 2, \ldots, \\ \beta_m &= \frac{\langle f(\cdot), \sin(m\cdot) \rangle}{\pi} = \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin mx dx, \qquad m = 1, 2, \ldots. \end{cases} \tag{**}$$

Note: α_m, β_m could be nonzero for infinitely many m (unlike trignometric polynomials T).

Question: Is it possible that the **Fourier series** of f, where α_k, β_k are defined in (**), defined by $\boxed{\alpha_0 + \sum_{k=1}^{\infty}(\alpha_k \cos kx + \beta_k \sin kx)}$, converges for every $f \in \mathcal{C}^{2\pi}$ (or $\mathcal{R}[-\pi, \pi]$)?

If so, can it be true that

$$f(x) = \alpha_0 + \sum_{k=0}^{\infty}(\alpha_k \cos kx + \beta_k \sin kx), \qquad \forall x \in [-\pi, \pi]. \tag{1}$$

Remark. False, even for $f \in \mathcal{C}^{2\pi}$: there exists $f \in \mathcal{C}^{2\pi}$ such that the partial sums of the right-hand side of (1) goes to ∞.

Why does this not contradict WS 2^{nd} theorem?