Math 321 Lecture 3

Yuchong Pan

January 7, 2019

1 Theorem and Consequences

1.1 Theorem and Proof

Theorem 1. Let (X, d) and (Y, ρ) be metric spaces. Assume that $f_n : X \to Y$ is continuous for $n \geq 1$ and that $f_n \to f$ uniformly on X. Then f is continuous on X.

Proof. Fix $x_0 \in X$ and $\epsilon > 0$. We need to find $\delta = \delta(\epsilon, x_0) > 0$ such that

$$\rho(f_(x), f(x_0)) < \epsilon$$

whenever $d(x, x_0) < \delta$.

Given:

$$\begin{aligned} & f_n \xrightarrow{n\to\infty} f \textbf{ uniformly on } X \\ \Leftrightarrow\quad & \sup_{x\in X} \rho(f_n(x), f(x)) \xrightarrow{n\to\infty} 0 \\ \Leftrightarrow\quad & \text{given any } \epsilon > 0, \text{ there exists } N \geq 1 \text{ such that} \\ & \sup_{x\in X} \rho(f_n(x), f(x)) < \frac{\epsilon}{3} \qquad \text{whenever } n \geq N. \end{aligned} \tag{*}$$

Thus, by the triangular inequality applied twice,

$$\rho(f(x), f(x_0)) \leq \underbrace{\rho(f(x), f_N(x))}_{(1)} + \underbrace{\rho(f_N(x), f_N(x_0))}_{(2)} + \underbrace{\rho(f_N(x_0), f(x_0))}_{(3)}.$$

Both (1) and (3) are buonded above by $\sup_{y\in X} \rho(f_N(y), f(y))$, which is $< \frac{\epsilon}{3}$ by (*). Therefore,

$$\rho(f(x), f(x_0)) < \frac{2\epsilon}{3} + \underbrace{\rho(f_N(x), f_N(x_0))}_{(2)} < \frac{2\epsilon}{3} + \frac{\epsilon}{3},$$

if we choose $\delta = \delta_N(\epsilon, x_0) > 0$ such that $\rho(f_N(x), f_N(x_0)) < \frac{\epsilon}{3}$ whenever $d(x, x_0) < \delta_N$, by the ϵ-δ definition of the continuity of f_N at $x_0 \in X$. $\square$

Example: $k_n(x) = x^n$ is continuous on $[0, 1]$. Last time, we showed that $k_n \xrightarrow{\text{pointwise}} k$, where

$$k(x) = \begin{cases} 0, & x \neq 1, \\ 1, & x = 1, \end{cases}$$

which is discontinuous.

1.2 Consequences

1. Suppose $X = [a, b]$ and $Y = \mathbb{R}$ or $\mathbb{C}$. Then Theorem 1 implies that $C[a, b]$ is closed under **uniform limits**.

   That is, any sequence in $C[a, b]$, if it converges uniformly, admits the limit in $C[a, b]$; i.e., if $\{f_n\} \subseteq C[a, b]$ and $f_n \xrightarrow{n\to\infty} f$ uniformly on $C[a, b]$, then $f \in C[a, b]$.

   Note that $C[a, b]$ is closed in $\mathcal{B}[a, b] = \left\{ f : [a, b] \xrightarrow{\text{bounded}} \mathbb{R} \text{ or } \mathbb{C} \right\}$, a metric space with metric $d(f, g) = \sup_{x\in a[a,b]} |f(x) - g(x)|$.

2. $C[a, b]$ is **complete**.

   That is, every Cauchy sequence $\{f_n\} \subseteq C[a, b]$ (i.e., $\sup_{x\in[a,b]} |f_n(x) - f_m(x)| \xrightarrow{n,m\to\infty} 0$) converges.

Step 1: Find a candidate for $\lim_{n\to\infty} f_n$.
Know: $\|f_n - f_m\|_\infty = \sup_{x\in[a,b]} |f_n(x) - f_m(x)| \xrightarrow{n,m\to\infty} 0$.
Fix any $x \in [a, b]$. Then $\{f_n(x)\}$ is a sequence of real (or complex) numbers, of $|f_n(x) - f_m(x)| \leq \|f_n - f_m\|_\infty \xrightarrow{n,m\to\infty} 0$.
Since $\mathbb{R}$ and $\mathbb{C}$ are complete, $\lim_{n\to\infty} f_n(x)$ exists for all $x \in [a, b]$. Set $f(x) = \lim_{n\to\infty} f_n(x)$.
Note that f is pointwise convergent so far.

Step 2: We need to show:

i. $f_n \xrightarrow{n\to\infty} f$ uniformly on $[a, b]$, and
ii. $f \in C[a, b]$.

i.

$$\begin{aligned}
\sup_{x\in[a,b]} |f_n(x) - f(x)| &= \sup_{x\in[a,b]} \left| f_n(x) - \lim_{m\to\infty} f_m(x) \right| \\
&= \sup_{x\in[a,b]} \left| \lim_{m\to\infty} (f_n(x) - f_m(x)) \right| \\
&= \sup_{x\in[a,b]} \lim_{m\to\infty} |f_n(x) - f_m(x)| \\
&\leq \sup_{x\in[a,b]} \underbrace{\lim_{m\to\infty} \|f_n - f_m\|_\infty}_{\text{does not depend on } x} \\
&= \lim_{m\to\infty} \|f_n - f_m\|_\infty \\
&< \epsilon,
\end{aligned}$$

if $n \geq N_\epsilon$ (using the fact that $\{f_n\} \subseteq C[a, b]$ is Cauchy).

ii. $f \in C[a, b]$ by Theorem 1.

3. **Weierstrass M-test**:
   Let $\{g_n\} \subseteq \mathcal{B}(X) = \left\{ f : X \xrightarrow{\text{bounded}} \mathbb{R} \text{ or } \mathbb{C} \right\}$ be such that $\sum_{n=1}^\infty \|g_n\|_\infty < \infty$.

   Then $g = \sum_{n=1}^\infty g_n$ converges uniformly on X; i.e., $\left\{ S_n(x) = \sum_{n=1}^N g_n(x) : N \geq 1 \right\}$ converges uniformly on X. In addition, $g \in \mathcal{B}(X)$.

   Moreover, if $\{g_n\} \subseteq C(X)$, then $g \in C(X)$.