Math 321 Lecture 4

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1 Equicontinuous Families of Functions

In Theorem 3.6, we saw that every bounded sequence of **complex numbers** admits a convergent subsequence.

Question: What about bounded sequences of **functions**?

1.1 Pointwise and Uniform Boundedness

Let $\{f_n\}$ be a sequence of functions defined on a set E.

Definition 1. We say that $\{f_n\}$ is **pointwise bounded** on E if the sequence $\{f_n(x)\}$ is bounded for every $x \in E$; i.e., there exists a finite-valued function ϕ defined on E such that

$$|f_n(x)| < \phi(x) \qquad (x \in E, n = 1, 2, 3, \ldots).$$

Remark 1. The bound $\phi(x)$ depends on x.

Definition 2. We say that $\{f_n\}$ is **uniformly bounded** on E if there exists a number M such that

$$|f_n(x)| < M \qquad (x \in E, n = 1, 2, 3, \ldots).$$

Remark 2. The bound M is independent of x.

1.2 Examples

1. $f_n(x) = \sin(nx), x \in E = [0, 2\pi], n = 1, 2, 3, \ldots$.
   (a) $|f_n(x)| = |\sin(nx)| \leq 1 < 2$. Thus, $\{f_n\}$ is uniformly bounded.
   (b) We claim that $\{f_n\}$ admits no subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E = [0, 2\pi]$.

   Proof. Suppose not; i.e., there exists a sequence $\{n_k\}_{k=1,2,3,\ldots}$ such that $\{\sin(n_k x)\}$ converges for all $x \in E = [0, 2\pi]$.
   By the Cauchy criterion (for convergence),

   $$\lim_{k\to\infty} (\sin(n_k x) - \sin(n_{k+1} x)) = 0, \qquad x \in [0, 2\pi].$$

   Hence,

   $$\lim_{k\to\infty} (\sin(n_k x) - \sin(n_{k+1} x))^2 = \left(\lim_{k\to\infty} (\sin(n_k x) - \sin(n_{k+1} x))\right)^2 = 0^2 = 0.$$

Lebesgue's Theorem states that if $f_n \to f$ pointwise as $n \to \infty$ and $|f_n(x)| < M$ for some $M \in \mathbb{R}$, then $\lim_{n\to\infty} \int_a^b f_n(x)dx = \int_a^b \lim_{n\to\infty} f_n(x)dx = \int_a^b f(x)dx$. Note that

i. $(\sin(n_k x) - \sin(n_{k+1}x))^2 \to 0$ as $k \to \infty$;

ii. $(\sin(n_k x) - \sin(n_{k+1}x))^2 \leq (1+1)^2 = 4 < 5$.

By Lebesgue's Theorem,

$$\begin{aligned}
&\lim_{k\to\infty} \int_0^{2\pi} (\sin(n_k x) - \sin(n_{k+1}x))^2 dx \\
&= \int_0^{2\pi} \lim_{k\to\infty} (\sin(n_k x) - \sin(n_{k+1}x))^2 dx \\
&= \int_0^{2\pi} 0 dx \\
&= 0.
\end{aligned} \tag{1}$$

On the other hand,

$$\begin{aligned}
&\int_0^{2\pi} (\sin(n_k x) - \sin(n_{k+1}x))^2 dx \\
&= \int_0^{2\pi} \left(\sin^2(n_k x) + \sin^2(n_{k+1}x) - 2\sin(n_k x)\sin(n_{k+1}x)\right) dx \\
&= \int_0^{2\pi} \left(\frac{1-\cos(2n_k x)}{2} + \frac{1-\cos(2n_{k+1}x)}{2} - (\cos((n_k - n_{k+1})x) - \cos((n_k + n_{k+1})x))\right) dx \\
&= \int_0^{2\pi} 1dx - \frac{1}{2}\int_0^{2\pi} \cos(2n_k x)dx - \frac{1}{2}\int_0^{2\pi} \cos(2n_{k+1}x)dx \\
&\quad - \int_0^{2\pi} \cos((n_k - n_{k+1})x)dx - \int_0^{2\pi} \cos((n_k + n_{k+1})x)dx
\end{aligned}$$

Note that $\int_0^{2\pi} \cos(mx)dx = \frac{\sin(mx)}{m}\Big|_0^{2\pi} = 0$ for $m \in \mathbb{Z}, m \neq 0$. Thus,

$$\int_0^{2\pi} (\sin(n_k x) - \sin(n_{k+1}x))^2 dx = \int_0^{2\pi} dx = 2\pi.$$

This implies that

$$\lim_{n\to\infty} \int_0^{2\pi} (\sin(n_k x) - \sin(n_{k+1}x))^2 dx = 2\pi. \tag{2}$$

(1) contradicts (2). □

2. $f_n(x) = \frac{x^2}{x^2+(1-nx)^2}, x \in [0,1], n = 1, 2, 3, \ldots$

   (a) $|f_n(x)| \leq \frac{x^2}{x^2} = 1 < 2$. Thus, $\{f_n\}$ is uniformly bounded.

   (b) We claim that no subsequences of $\{f_n\}$ converges uniformly on $[0,1]$.

   Proof. Recall that $\{f_n\}$ converges uniformly to a function f on E if for every $\epsilon > 0$, there exists an integer N such that $n \geq N$ implies $|f_n(x) - f(x)| < \epsilon$ for all $x \in E$.

The negation of this definition states that $\{f_n\}$ does not converge uniformly to f on E if there exists $\epsilon_0 > 0$ such that for every integer n, there exists $\mathcal{N}_0 \geq N$ such that $|f_{N_0}(x_0) - f(x_0)| \geq \epsilon_0$ for some $x_0 \in E$.
Note that $\lim_{n\to\infty} f_n(x) = 0$ for $0 \leq x \leq 1$. Let $f(x) = 0$ for $x \in [0, 1]$. Note that $f_n\left(\frac{1}{n}\right) = \frac{x^2}{x^2} = 1$. Take $\epsilon = \frac{1}{2}$ and $x_0 = \frac{1}{\mathcal{N}_0}$ for large $\mathcal{N}_0$. Thus,

$$\left| f_{\mathcal{N}_0}\left(\frac{1}{\mathcal{N}_0}\right) - f\left(\frac{1}{\mathcal{N}_0}\right) \right| = |1 - 0| = 1 \geq \epsilon_0 = \frac{1}{2}.$$

Hence, $\{f_n\}$ does not converge uniformly on $[0, 1]$. This proves that for any $\{n_k\} \subseteq \mathbb{N}$, $\{f_{n_k}\}$ does not converge uniformly on $[0, 1]$. $\square$

1.3 Equicontinuous Families of Functions

Definition 3. A family $\mathcal{F}$ of complex functions defined on a set E, in a metric space X, is said to be **equicontinuous** on E if for every $\epsilon > 0$, there exists $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ whenever $x \in E$, $y \in E$ and $f \in \mathcal{F}$.

Remark 3. Every member of an equicontinuous family is uniformly continuous.

1.4 Diagonal Principle

Theorem 1 (Diagonal principle)**.** If $\{f_n\}$ is a pointwise bounded sequence of complex functions on a countable set E, then $\{f_n\}$ has a subsequence $\{f_{n_k}\}$ such that $\{f_{n_k}(x)\}$ converges for every $x \in E$.

Proof sketch. Let $\{x_i\}_{i=1,2,3,\ldots}$ be the points of E.

Step 1. Since $\{f_n(x_1)\}$ is bounded, there exists a subsequence $\{f_{1,k_i}\}$ such that $\{f_{1,k_i}(x_1)\}$ converges as $k \to \infty$.

Step 2. Since $\{f_{1,k_i}(x_2)\}$ is bounded, there exists a subsequence $\{f_{2,k_{i_j}}\}$ such that $\{f_{2,k_{i_j}}(x_2)\}$ converges as $k \to \infty$.

Repeat this process.

$$\begin{array}{lllll}
\textbf{Step 1:} & f_{1,k_1} & f_{1,k_2} & f_{1,k_3} & \cdots \\
\textbf{Step 2:} & f_{2,k_{i_1}} & f_{2,k_{i_2}} & f_{2,k_{i_3}} & \cdots \\
\textbf{Step 3:} & f_{3,k_{i_{j_1}}} & f_{3,k_{i_{j_2}}} & f_{3,k_{i_{j_3}}} & \cdots \\
\vdots & & & &
\end{array}$$