Math 321 Lecture 27

Yuchong Pan

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1 HW 9, 3 (b)

HW 9, 3 (b): Find $f \in \mathcal{C}^{2\pi}$ such that $\sup_N |s_N f(0)| = \infty$.

Assume 3 (a): Given any $n \geq 1$, there exists $f_n \in \mathcal{C}^{2\pi}$ such that $\|f_n\|_\infty = 1$ and $\sup_j |s_j f_n(0)| > n$.

Step 0: Without loss of generality, we can choose f_n to be a trignometric polynomial, say of degree d_n.

Proof. Fix $n \geq 1$. Given any $f_n \in \mathcal{C}^{2\pi}$, we know $\sigma_N(f_n) = N^{\text{th}}$ Cesàro sum of $f_n \xrightarrow[\text{uniformly}]{N \to \infty} f_n$. Fix $N = N_n$ such that

$$\| \underbrace{\sigma_{N_n} f_n}_{\text{a trignometric polynomial}} - f_n\|_\infty < 1.$$

Note: $g_n = \sigma_{N_n} f_n$ is a trignometric polynomial.

We have

$$|g_n(x)| \leq \underbrace{|g_n(x) - f_n(x)|}_{\leq 1} + \underbrace{|f_n(x)|}_{\leq 1} \leq 2,$$

or

$$\begin{aligned}
g_n(x) &= \sigma_{N_n} * f_n(x) = \frac{1}{2\pi} \int_{-\pi}^{\pi} f_n(x-y) K_n(y) dy, \\
|g_n(x)| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \underbrace{|f_n(x-y)|}_{\leq 1} \cdot |K_n(y)| dy \leq 1 \qquad \text{because } \|K_n\|_1 = 1.
\end{aligned}$$

Suppose $\underbrace{j}_{=e^{cn}}$ is an index such that $\|s_{\underbrace{j}_{=e^{cn}}} f_n(0)\| > n$. Then,

$$|s_n g_n(0)| \geq |s_j f_n(0)| - |s_j(f_n - g_n)(0)| > n - 1,$$

where

$$\begin{aligned}
s_j(f_n - g_n)(\underbrace{0}_{=x}) &= \sum_{k=-j}^{j} \left[\widehat{f}_n(k) - \widehat{g}_n(k)\right] \underbrace{e^{ik \overbrace{x}^{=0}}}_{=1}, \\
|s_j(f_n - g_n)(0)| &\leq \sum_{k=-j}^{j} \left|\widehat{f}_n(k) - \widehat{g}_n(k)\right| \leq 2e^{cn} \cdot \frac{1}{2} e^{cn} = 1, \\
\left|\widehat{f}_n(k) - \widehat{g}_n(k)\right| &\leq \frac{1}{2\pi} \int_{-\pi}^{\pi} \left|(f_n(x) - g_n(x)) e^{-ikx}\right| dx < 1.
\end{aligned}$$

$\square$

Step 1: We now have a sequence of trignometric polynomials $\{g_n : n \geq 1\}$, $\deg(g_n) = d_n$, $\|g_n\|_\infty \leq 1$, and

$$|s_{k_n} g_n(0)| > n - 1, \qquad k_n = e^{cn} \ll d_n.$$

Goal: Define $f(x) = \sum_{n=1}^\infty \frac{1}{n^2} \boxed{g_{\lambda_n}(\lambda_n x)}$ for a fast-growing sequence of integers $\lambda_n \nearrow \infty$ to be specified.

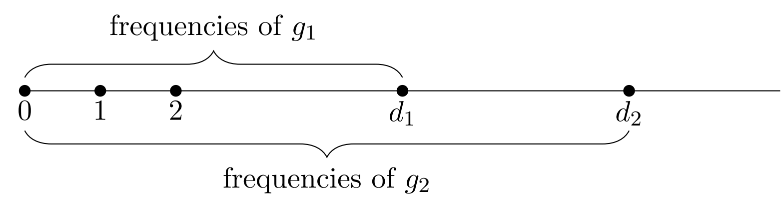
Choose $\lambda_n \nearrow \infty$ large enough so that $\lambda_2 > d_1\lambda_1, \lambda_3 > d_2\lambda_2, \ldots, \lambda_{n+1} > d_n\lambda_n$, so non-zero frequencies of individual summands are disjoint.

Show: $\sup_N |s_N f(0)| = \infty$.

Note that

$$g_n(x) = \sum_{k=-d_n}^{d_n} \widehat{g}_n(k) e^{ikx} \qquad \text{because } g_n \text{ is a trignometric polynomial,}$$

$$g_n(\lambda_n x) = \sum_{k=-d_n}^{d_n} \widehat{g}_n(k) e^{i\boxed{k\lambda_n}x} \qquad \text{has frequencies } \{k\lambda_n : k = 0, 1, \ldots, d_n\}.$$



$$\boxed{\begin{array}{rl} & h_1 + h_2 + \ldots + h_n, \\ \text{Frequency } k\text{:} & \widehat{h}_1(k) + \widehat{h}_2(k) + \ldots + \widehat{h}_n(k). \end{array}}$$

Choose $\lambda_N \gg N^3$. Then,

$$s_{\lambda_N^2} f(x) = \underbrace{s_{\lambda_N^2} \sum_{n=1}^\infty \frac{1}{n^2} g_{\lambda_n}(\lambda_n x)}_{=a+b},$$

$$|s_{\lambda_N^2} f(0)| \geq \underbrace{\frac{1}{N^2} s_{\lambda_N^2} g_{\lambda_N}(\lambda_n \cdot 0)}_{\text{main}} - \underbrace{\left| \sum_{n<N} \frac{1}{n^2} s_{\lambda_n^2} g_{\lambda_n}(0) + \sum_{n>N} \frac{1}{n^2} s_{\lambda_n^2} g_{\lambda_n}(0) \right|}_{\text{error}} \quad \text{because } |a+b| \geq |a| - |b|$$

$$\geq \frac{1}{N^2} s_{\lambda_N} g_{\lambda_N}(0) \geq \frac{\lambda_N}{N^2} \nearrow \infty.$$

Note that

$$s_{\lambda_N^2} g_N(\lambda_N x) = s_{\lambda_N} g_{\lambda_N}(x).$$