Math 321 Lecture 20

Yuchong Pan

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1 Riesz Representation Theorem

Theorem 1 (Riesz representation theorem)**.** Given any continuous linear functional $L : C[a, b] \to \mathbb{R}$, there exists $\alpha \in BV[a, b]$ with $\underbrace{\|L\| = V_a^b \alpha}_{\text{(Recall: } \|L\|_{op} = \|L\| = \sup_{0 \neq f \in C[a,b]} \frac{|L(f)|}{\|f\|_\infty}\text{)}}$ such that $L(f) = \int_a^b f d\alpha$ for all $f \in C[a, b]$.

We can choose α to be right continuous on $[a, b]$ with $\alpha(a) = 0$. In this case, α is unique.

Proof. **Step 1:** Find approximations for α. Fix $n \geq 1, 0 \leq k \leq n$. Define

$$\begin{aligned} P_{n,k}(x) &= \binom{n}{k} x^k (1-x)^{n-k} \\ &= \text{binomial probability for } k \text{ sucesses in } n \text{ tosses of a coin with success probability } x, \\ B_n(f)(x) &= \sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}(x) \\ &= \text{expected value of } f \text{ with respect to the above binomial distribution.} \end{aligned}$$

Know from proof of classical Weierstrass:

$$B_n(f) \xrightarrow{n \to \infty} L(f) \text{ uniformly}, \qquad \forall f \in C[0, 1].$$

Since L is *continuous*,

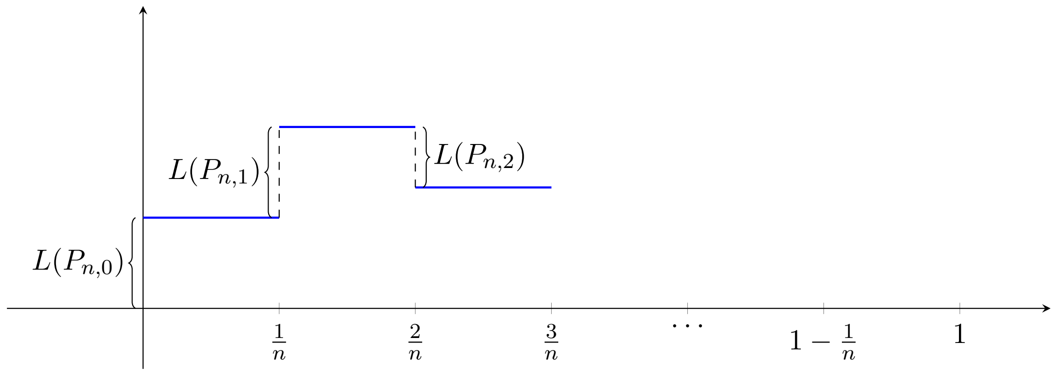
$$L(B_n(f)) \xrightarrow{n \to \infty} L(f).$$

Note that

$$L(B_n(f)) = L\left(\sum_{k=0}^{n} f\left(\frac{k}{n}\right) P_{n,k}\right) \xlongequal{\text{since } L \text{ is linear}} \sum_{k=0}^{n} \underbrace{f\left(\frac{k}{n}\right)}_{\text{scalars}} \underbrace{L(P_{n,k})}_{\text{functions in } C[0,1]} = \int_0^1 f d\alpha_n,$$

where α_n is a step function (WLOG can be chosen to be right continuous with $\alpha(0) = 0$) with possible discontinuities at the points $x_k = \frac{k}{n}, 0 \leq k \leq n$ with a jump of $L(P_{n,k})$ at x_k.
$\to$ See HW 6 ($\sum f(x_i) \Delta\alpha_i = \int f d\alpha$).

Question: Is $\alpha_n \in BV[a, b]$?
Yes, because each a_n is a well-defined step function in $\mathcal{B}[a, b]$.



Step 2: Use Helly's theorems to find $\alpha \in BV[a,b]$ from the sequence $\{\alpha_n : n \geq 0\}$. Recall Helly's second theorem:

Theorem 2 (Helly's second theorem)**.** *Suppose* $\{a_n\} \subseteq BV[0,1]$*. Assume:*

- $\alpha_n \xrightarrow{n\to\infty} \alpha$ *pointwise on* $[0,1]$*;*
- $\boxed{V_a^b \alpha_n \leq K \text{ for all } \forall n \geq 1.}$ *— (*)*

Then $\alpha \in BV[a,b]$ *and* $\int_0^1 f d\alpha_n \xrightarrow{n\to\infty} \int_0^1 f d\alpha$*.*

Check (*):

$$\begin{aligned}
V_0^1 \alpha_n &= \text{sum of the magnitudes of the jumps of } \alpha_n \\
&= \sum_{k=0}^n |L(P_{n,k})| && \text{If } L(P_{n,k}) \geq 0 \text{, then} \\
& && \sum_{k=0}^n L(P_{n,k}) = L\left(\sum_{k=0}^n P_{n,k}\right) = L(1). \\
&= \sum_{k=0}^n \epsilon_{n,k} L(P_{n,k}) && \text{where } \epsilon_{n,k} = \begin{cases} 1, & \text{if } L(P_{n,k}) \geq 0, \\ -1, & \text{if } L(P_{n,k}) < 0. \end{cases} \\
&= L\left(\sum_{k=0}^n \epsilon_{n,k} P_{n,k}\right) && \text{since } L \text{ is linear} \\
&\leq \underbrace{\|L\|_{op}}_{\text{independent of } n} \left\|\sum_{k=0}^n \epsilon_{n,k} P_{n,k}\right\|_\infty && \text{since } L \text{ is continuous, linear and bounded} \\
&\leq \|L\|_{op} \underbrace{\left\|\sum_{k=0}^n |\cancel{\epsilon_{n,k}} \underbrace{P_{n,k}}_{\geq 0}|\right\|_\infty}_{\text{triangular inequality}} = \|L\|_{op} = K < \infty.
\end{aligned}$$

Helly's first theorem says (*) $\Rightarrow \exists n_k \nearrow \infty$ such that $\alpha_{n_k} \to \alpha$ pointwise. Recall:

$$L(B_n(f)) \to L(f).$$

Therefore,

$$L(B_{n_k}(f)) \xrightarrow{k\to\infty} L(f).$$

Note that $L(B_{n_k}(f)) = \int_0^1 f d\alpha_{n_k}$ and that $\alpha_{n_k} \to \alpha$. This implies that

$$\int_0^1 f d\alpha_{n_k} \xrightarrow{\text{by Helly's two theorems}} \int_0^1 f d\alpha.$$

This proves that $L(f) = \int_0^1 f d\alpha$. $\square$