Math 321 Lecture 12

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1 Proofs of Lemma and Proposition in Stone-Weierstrass

Lemma 1. Let $\mathcal{A}$ be a subalgebra of real-valued functions on a set X. Suppose that $\mathcal{A}$ separates points and vanishes at no point.
Show: Given any two points $x_0, y_0 \in X, x_0 \neq y_0$ and $a, b \in \mathbb{R}$, there exists $f \in \mathcal{A}$ such that $f(x_0) = a, f(y_0) = b$.

Proof. **Take 1:** Since $\mathcal{A}$ vanishes at no point, there exists $g \in \mathcal{A}$ such that $g(x_0) \neq 0$. Define $f_1(x) = \underbrace{\alpha}_{\in\mathbb{R}} g(x)$. Then $f_1 \in \mathcal{A}$ because $\mathcal{A}$ is a vector space.

Choose $\alpha \in \mathbb{R}$ so that $f_1(x_0) = a$; i.e., $\alpha g(x_0) = a \Rightarrow \alpha = \frac{a}{g(x_0)}$. The function $f_1 = \frac{a}{g_0(x)} g(x)$ lies in $\mathcal{A}$ and takes the value a at x.

Since $\mathcal{A}$ separates points, there exists $h \in \mathcal{A}$ such that $h(x_0) \neq h(y_0)$. Define

$$f_2(x) = \frac{h(x) - h(x_0)}{\underbrace{h(y_0) - h(x_0)}_{\text{need not be in } A}}.$$

Then $f_2(x_0) = 0, f_2(y_0) = b$.

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Take 2: Since $\mathcal{A}$ separates points, there exists $g \in \mathcal{A}$ such that $g(x_0) \neq g(y_0)$. Since $\mathcal{A}$ vanishes at no point, there exist $h, k \in \mathcal{A}$ such that $h(x_0) \neq 0, k(y_0) \neq 0$.

Define:

$$u(x) = [g(x) - g(x_0)]k(x) = \underbrace{g(x)k(x)}_{\in\mathcal{A}} - \underbrace{g(x_0)k(x)}_{\in\mathcal{A}} \Rightarrow u \in A.$$

Then $u(x_0) = 0, u(y_0) = \underbrace{[g(y_0) - g(x_0)]}_{\neq 0} \underbrace{k(y_0)}_{\neq 0} \neq 0$.

Set

$$v(x) \stackrel{\text{def}}{=} [g(x) - g(y_0)]h(x) \in \mathcal{A}.$$

Similarly, $v(x_0) \neq 0, v(y_0) = 0$.

Define

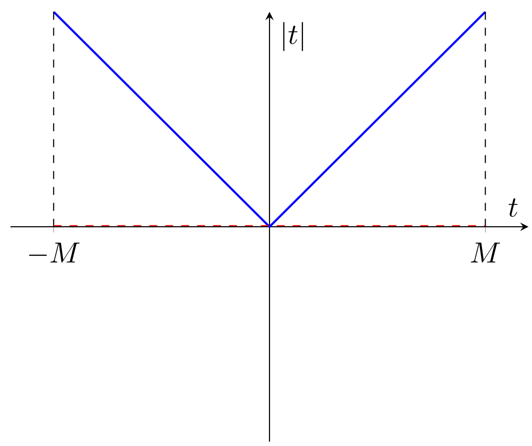
$$f(x) = \frac{b}{\underbrace{u(y_0)}_{\in\mathbb{R}}} u(x) + \frac{a}{\underbrace{v(x_0)}_{\in\mathbb{R}}} v(x).$$

Note that $f \in \mathcal{A}$. Then $f(x_0) = \frac{a}{v(x_0)} \cdot v(x_0) = a$ and similarly $f(y_0) = b$. □

Proposition 1. Let X be a metric space. Let $\mathcal{A} \subseteq \mathcal{B}(X;\mathbb{R})$ be a subalgebra.
Show: $\overline{\mathcal{A}}$ is a sub-lattice; i.e., if $f \in \overline{\mathcal{A}}$, then $|f| \in \overline{\mathcal{A}}$.

Proof. Given $\epsilon > 0$, we need to find $g \in \overline{\mathcal{A}}$ such that

$$\||f| - g\|_\infty = \sup_{x \in X} ||f(x)| - g(x)| < \epsilon.$$



Note that

$$f \in \mathcal{B}(X;\mathbb{R}) \Rightarrow \underbrace{\|f\|_\infty}_{=\sup_{x\in X}|f(x)|} = M < \infty.$$

By the classical Weierstrass, polynomials are dense in $C([-M,M];\mathbb{R})$. Thus, there exists a polynomial P such that

$$\sup_{t\in[-M,M]} ||t| - P(t)| < \epsilon. \tag{*}$$

Note that as x ranges over X, $f(x)$ takes values within $[-M,M]$. Replacing $f(x) = t$, we have

$$\sup_{x\in X} \left| |f(x)| - \underbrace{|P(f(x))|}_{g(x)} \right| \le \sup_{t\in[-M,M]} ||t| - P(t)| \underbrace{<}_{(*)} \epsilon.$$

If $P(t) = a_0 + a_1 t + a_2 t^2 + \ldots + a_n t^n$, then

$$P(f(x)) = a_0 + a_1 \underbrace{f(x)}_{\in\overline{\mathcal{A}}} + \ldots + a_n \underbrace{(f(x))^n}_{\in\overline{\mathcal{A}}}.$$

Note that $P(0)$ can be **chosen** to be 0. Hence, $P \circ f \in \overline{\mathcal{A}}$ because $\overline{\mathcal{A}}$ is a subalgebra. $\square$