Math 321 Lecture 11

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1 Proof of Stone-Weierstrass Theorem (Cont'd)

Proof (cont'd). Let (X, d) be a compact metric space. Let $\mathcal{A} \subseteq C(X; \mathbb{R})$ be a subalgebra that separates points and vanishes at no point. We need to show that $\overline{\mathcal{A}} = C(X; \mathbb{R})$.

Step 1: Given $f \in C(X; \mathbb{R}), \epsilon > 0, x \in X$, there exists $g_x \in \overline{\mathcal{A}}$ such that $g_x(x) = f(x)$ and $g_x(y) > f(y) - \epsilon$ for all $y \in X$.

Step 2: Goal is to find $g \in \overline{\mathcal{A}}$ such that $f(y) - \epsilon < g(y) < f(y) + \epsilon$ for all $y \in X$.
Proof: Given $\epsilon > 0$, $(-\infty, \epsilon)$ is an open set in $\mathbb{R}$. For any $x \in X$, consider $g_x - f$, which is in $C(X; \mathbb{R})$. Hence $V_x = (\underbrace{g_x}_{\text{from step 1}} - f)^{-1}(-\infty, \epsilon)$ is open in X. Then,

$$V_x = \{z \in X; (g_x - f)(z) < \epsilon\} = \{z \in X; g_x(z) < f(z) + \epsilon\}.$$

Note: $x \in V_x$ because $g_x(x) \underbrace{=}_{\text{by step 1}} f(x) < f(x) + \epsilon$.

Therefore, $\{V_x; x \in X\}$ forms an open cover of X. Since X is compact, we can find $x_1, x_2, \ldots, x_J \in X$ such that $X = \bigcup_{j=1}^J V_{x_j}$. This means that for every $x \in X$, there exists $1 \leq j \leq J$ such that $x \in V_{x_j}$; i.e., $g_{x_j}(x) - f(x) < \epsilon$. This implies that

$$g_{x_j}(x) < f(x) + \epsilon. \tag{*}$$

Set $g = \min(g_{x_1}, g_{x_2}, \ldots, g_{x_J}) \in \overline{\mathcal{A}}$ by Proposition. Then (*) implies that $g(x) \leq g_{x_j}(x) < f(x) + \epsilon$ for all $x \in X$.

On the other hand, **for every** $1 \leq j \leq J$, $g_{x_j}(x) > f(x) - \epsilon$ for all $x \in X$, by step 1. This implies that $g(x) > f(x) - \epsilon$ because for every x, $g(x) = g_{x_j}(x)$ for some x_j depending on x.

To do:

1. $g \in \overline{\mathcal{A}}$; not $\mathcal{A}$, but this suffices.

2. Lemma and Proposition remain to be proved.

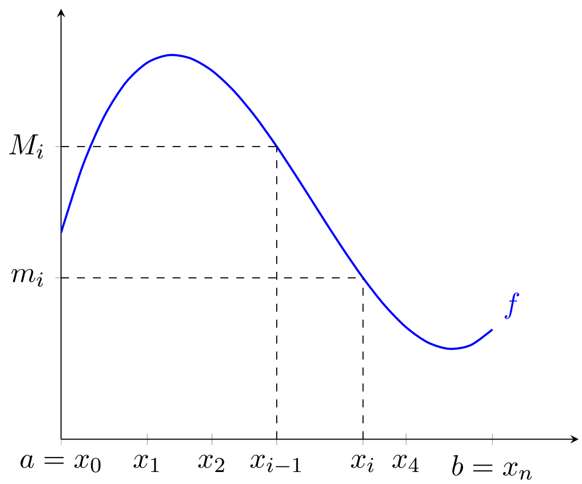
Steps 1 and 2 show that for every $\epsilon > 0$, there exists $g \in \overline{\mathcal{A}}$ such that $\|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)| < \epsilon$. Thus,

$$\overline{\mathcal{A}} \text{ is dense in } C(X; \mathbb{R}) \Leftrightarrow \underbrace{\overline{\overline{\mathcal{A}}}}_{=\overline{\mathcal{A}}} = C(X; \mathbb{R}).$$

$\square$

2 Riemann-Stieltjes Integral

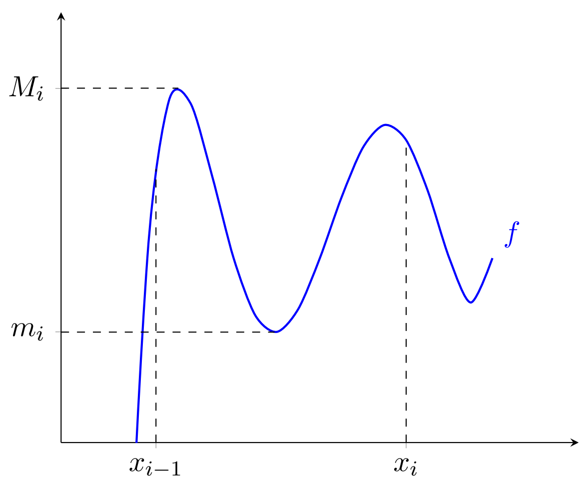
Recall classical Riemann integration: Let $f : [a, b] \xrightarrow{\text{bounded}} \mathbb{R}$.



Let $P = \{a = x_0 < x_1 < \ldots < x_n = b\}$ denote a (finite) partition of $[a, b]$. Let

$$\begin{aligned}\Delta x_i &= \text{length of the } i^{\text{th}} \text{ subinterval} = x_i - x_{i-1},\\ m_i &= \inf\{f(x); x_{i-1} \leq x \leq x_i\},\\ M_i &= \sup\{f(x); x_{i-1} \leq x \leq x_i\}.\end{aligned}$$

Both m_i and M_i are finite because f is bounded.



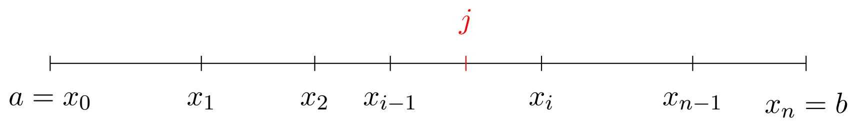
Definition 1.

$$L(f, P) = \textbf{lower Riemann sum} \text{ of } f \text{ associated to the partition } P = \sum_{i=1}^{n} m_i \Delta x_i,$$

$$U(f, P) = \textbf{upper Riemann sum} \text{ of } f \text{ associated to the partition } P = \sum_{i=1}^{n} M_i \Delta x_i.$$

Note that $L(f, P) \leq U(f, P)$.

Definition 2. Let P, Q be two partitions of $[a, b]$. Say Q is a **refinement** of P if $P \subseteq Q$.



Exercise: Show:

1. $\underbrace{L(f, P) \le L(f, Q) \le U(f, Q) \le U(f, P)}_{\text{if } Q \text{ is a refinement of } P}$.

2. If P, Q are arbitrary partitions,

$$\max(L(f, P), L(f, Q)) \le L(f, P \cup Q) \le U(f, P \cup Q) \le \min(U(f, P), U(f, Q)).$$

Definition 3.

$$\textbf{Lower Riemann integral } \underline{\int_a^b} f\, dx = \sup_{\substack{P \\ \text{partition of } [a,b]}} L(f, P),$$

$$\textbf{Upper Riemann integral } \overline{\int_a^b} f\, dx = \inf_{\substack{P \\ \text{partition of } [a,b]}} U(f, P).$$

Say f is **Riemann integrable** if $\underline{\int_a^b} f\, dx = \overline{\int_a^b} f\, dx = \int_a^b f\, dx$.