Math 321 Lecture 1

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1 Pointwise and Uniform Convergence

1.1 Definitions

Let (X, d) and (Y, ρ) be two metric spaces. Given $f_n, f : X \to Y, n \geq 1$.

Definition 1. Say $f_n \xrightarrow{n\to\infty} f$ **pointwise on** X if $f_n(x) \xrightarrow{n\to\infty} f(x)$ in Y for every $x \in X$;

i.e., for every $x \in X$, $\rho(f_n(x), f(x)) \to 0$ as $n \to \infty$;

i.e., for every $x \in X$ and every $\epsilon > 0$, there exists $N = N(\epsilon, x) \geq 1$ such that $\rho(f_n(x), f(x)) < \epsilon$ whenever $n \geq N$.

Definition 2. Say $f_n \to f$ **uniformly on** X if $\sup[\rho(f_n(x), f(x)) : x \in X] \xrightarrow{n\to\infty} 0$;

i.e., for every $\epsilon > 0$, there exists $N = N(\epsilon) \geq 1$ such that $\sup_{x\in X} \rho(f_n(x), f(x)) < \epsilon$ for all $n \geq N$;

i.e., for every $\epsilon > 0$, there exists $N = N(\epsilon) \geq 1$ such that $\rho(f_n(x), f(x)) < \epsilon$ for all $n \geq N$ and for all $x \in X$.

Exercise 1. Show that if $f_n \to f$ uniformly, then $f_n \to f$ pointwise.

1.2 Motivation of Math 321

Examples of metric spaces:

- $\mathbb{R}$ with the usual Euclidean metric;
- $\mathbb{R}$ with the discrete metric;
- $C[0, 1]$ with the supremum norm metric:

$$C_{\mathbb{R}}[0,1] = \left\{ f : [0,1] \xrightarrow{\text{continuous}} \mathbb{R} \right\},$$

$$d(f, g) = \sup_{x\in[0,1]} |f(x) - g(x)| = \|f - g\|_\infty;$$

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$$\ell^1(\mathbb{R}) = \left\{ \text{infinite real sequences } \vec{a} = (a_1, a_2, a_3, \ldots) \text{ that are absolutely summable; i.e., } \sum_{n=1}^\infty |a_n| < \infty \right\},$$

$$\|\vec{a}\|_1 = \sum_{n=1}^\infty |a_n|, \qquad d(\vec{a}, \vec{b}) = \sum_{n=1}^\infty |a_n - b_n| \quad (\text{finite on } \ell^1(\mathbb{R})).$$

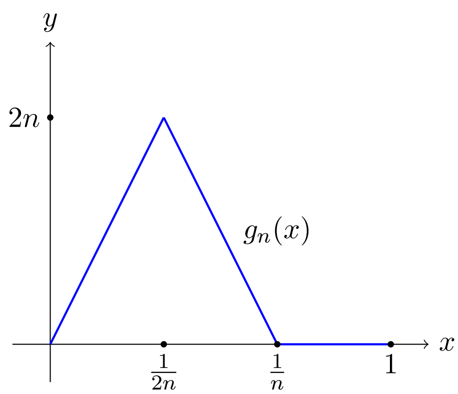
This term, we will explore special properties of metric spaces like $C[0,1]$, or more generally $C(X)$ (for a metric space (X,d), $C(X) = \left\{f : X \xrightarrow{\text{continuous}} \mathbb{R} \text{ or } \mathbb{C}\right\}$). The intended applications involve properties of $C[0,1]$ not necessarily enjoyed by general metric spaces.

Remark 1. Note that the notion of uniform convergence, though meaningful in any metric space, takes on a special meaning in $C[0,1]$, or $C[a,b], a < b, a, b \in \mathbb{R}$.

If $f_n, f \in (C[0,1], \|\cdot\|_\infty), f_n, f : [0,1] \xrightarrow{\text{continuous}} \mathbb{R}$, then $f_n \xrightarrow{n\to\infty} f$ uniformly if and only if $\|f_n - f\|_\infty \xrightarrow{n\to\infty} 0$; i.e., $f_n \xrightarrow{n\to\infty} f$ on $C[0,1]$.

1.3 Example

$g_n(x)$ is a continuous, piecewise linear function on $[0,1]$.

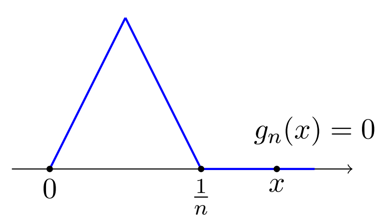


1. Does $\{g_n\}$ have a pointwise limit?

   We claim $g_n \to g \equiv 0$ pointwise. We need to check: For every $x \in [0,1]$, $g_n(x) \xrightarrow{n\to\infty} 0$.

   This is easy for $x = 0$ and $x = 1$ because $g_n(0) = g_n(1) = 0$ for all n.

   Fix any $x \in (0,1)$. Then there exists $N \geq 1$ such that $\frac{1}{n} < x$ for all $n \geq N$ (by the archimedean property of real numbers).



2. Does $g_n \to 0$ uniformly? No! If possible, this would mean

   (a) $\sup_{x\in[0,1]} |g_n(x) - g(x)| \xrightarrow{n\to\infty} 0$;

   (b) $\sup_{x\in[0,1]} |g_n(x)| \geq g_n(\frac{1}{2n}) = 2n \xrightarrow{n\to\infty} \infty$.

   (a) contradicts (b).

3. $$\int_0^1 g_n(x)dx = 1 \not\to \int_0^1 g(x)dx = 0.$$