Math 321 Lecture 29

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1 Proof of the Inverse Function Theorem

Proof. **Given:**

$$E \overset{\text{open}}{\subseteq} \mathbb{R}^n, \mathbf{a} \in E,$$

$$\mathbf{f} : E \to \mathbb{R}^n, f \in C^1(E), \boxed{\overbrace{\underbrace{f'(\mathbf{a})}_{n\times n}}^{=\mathbf{A}}} \text{ invertible.}$$

Need to show:

1.

$$U \overset{\text{open}}{\subseteq} E, V = \mathbf{f}(U) \overset{\text{open}}{\subseteq} \mathbb{R}^n,$$
$\mathbf{f} : U \to V$ is a bijection, and hence admits an inverse $\mathbf{g} : V \to U$; $\mathbf{g}(\mathbf{v}) = \mathbf{f}^{-1}(\mathbf{v}) \in U$.

Contraction Mapping Principle:

Definition 1. Let (X, d) be any metric space. We say $\varphi : (X, d) \to (X, d)$ is a **contraction** if there exists $c < 1$ such that

$$d(\varphi(x), \varphi(y)) \leq c d(x, y), \qquad \forall x, y \in X. \tag{*}$$

Theorem 1 (Contraction mapping principle)**.** Suppose (X, d) is complete and φ is a contraction on X. Then φ admits a unique **fixed point**; i.e., there exists a unique $x_0 \in X$ such that $\varphi(x_0) = x_0$.

Step 1: Associate local bijectivity of $\mathbf{f}$ with the existence of an auxiliary function. Fix $\mathbf{y} \in \mathbb{R}^n$. Define

$$\boxed{\varphi_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} + \mathbf{A}^{-1}(\mathbf{y} - \mathbf{f}(\mathbf{x}))}, \qquad \mathbf{x} \in E.$$

Note that:

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) \Leftrightarrow \varphi_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} \Leftrightarrow \mathbf{x} \text{ is a fixed point of } \varphi_{\mathbf{y}}.$$

Define $\underbrace{U = B(\mathbf{a}; \epsilon) \subseteq E}_{\text{possible since } E \text{ is open}}$ for some $\epsilon > 0$ to be specified. Want to choose $\epsilon > 0$ so that $\varphi_{\mathbf{y}}$ is a contraction on U; i.e., need $c < 1$ such that

$$\|\varphi_{\mathbf{y}}(\mathbf{x}_1) - \varphi_{\mathbf{y}}(\mathbf{x}_2)\| \leq c\|\mathbf{x}_1 - \mathbf{x}_2\|, \qquad \forall \mathbf{x}_1, \mathbf{x}_2 \in U. \tag{**}$$

Here, $\|\mathbf{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}$.

Hope: This will hold if $\varphi_{\mathbf{y}}'$ is small.

$$\begin{aligned}
\varphi_{\mathbf{y}}(\mathbf{x}) &= \mathbf{I} - \mathbf{A}^{-1}\mathbf{f}'(\mathbf{x}), \\
\varphi_{\mathbf{y}}(\mathbf{a}) &= \mathbf{I} - \mathbf{A}^{-1}\mathbf{A} = \mathbf{0}, \\
\|\varphi_{\mathbf{y}}(\mathbf{x})\| &= \left\|\mathbf{A}^{-1}(\mathbf{A} - \mathbf{f}'(\mathbf{x}))\right\| \leq \left\|\mathbf{A}^{-1}\right\| \cdot \|\mathbf{A} - \mathbf{f}'(\mathbf{x})\|
\end{aligned}$$

Recall that for any matrix $\mathbf{B}_{m\times n}$,

$$\|\mathbf{B}\| = \sup_{\mathbb{R}^n \ni \mathbf{x} \neq \mathbf{0}} \frac{\|\mathbf{B}\mathbf{x}\|_{\mathbb{R}^m}}{\|\mathbf{x}\|_{\mathbb{R}^n}}.$$

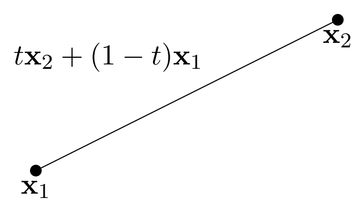
Check: $\|\mathbf{BC}\| \leq \|\mathbf{B}\| \cdot \|\mathbf{C}\|$.

$$\leq \left\|\mathbf{A}^{-1}\right\| \lambda = \frac{1}{2},$$

where $\epsilon > 0$ is chosen so that $\|\mathbf{f}'(\mathbf{x}) - \mathbf{A}\| < \lambda$ whenever $\|\mathbf{x} - \mathbf{a}\| < \epsilon$.

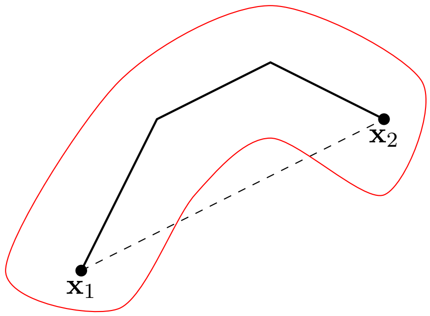
Check (**)**:**

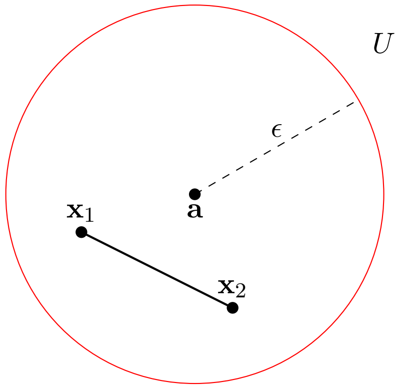
$$\|\varphi_{\mathbf{y}}(\mathbf{x}_2) - \varphi_{\mathbf{y}}(\mathbf{x}_1)\| = \left\|\int_0^1 \frac{d}{dt}\varphi_{\mathbf{y}}(t\mathbf{x}_2 + (1-t)\mathbf{x}_1)dt\right\|$$



$\boxed{\text{FTC: } \int_a^b g'(t)dt = g(b) - g(a)}$

$$\begin{aligned}
&= \left\|\int_0^1 \varphi_{\mathbf{y}}'(t\mathbf{x}_2 + (1-t)\mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1)dt\right\| \qquad \text{chain rule} \\
&\leq \int_0^1 \underbrace{\left\|\varphi_{\mathbf{y}}'(t\mathbf{x}_2 + (1-t)\mathbf{x}_1) \cdot (\mathbf{x}_2 - \mathbf{x}_1)\right\|}_{\leq \underbrace{\left\|\varphi_{\mathbf{y}}(\quad)\right\|}_{<\frac{1}{2}} \cdot \|\mathbf{x}_2 - \mathbf{x}_1\|} dt
\end{aligned}$$





$$\leq \frac{1}{2}\|\mathbf{x}_2 - \mathbf{x}_1\|.$$

Summary: $\varphi_{\mathbf{y}} : U \to \mathbb{R}^n$ is a contraction with $c = \frac{1}{2}$. By CMP, there exists $\boxed{\text{at most}}$ one point $\mathbf{x} \in U$ such that

$$\varphi_{\mathbf{y}}(\mathbf{x}) = \mathbf{x} \Leftrightarrow \mathbf{y} = \mathbf{f}(\mathbf{x}). \tag{1}$$

Set $V = \mathbf{f}(U)$. Then $f : U \to V$ is onto by definition. Further, given any $\mathbf{y} \in V$, there exists $\mathbf{x} \in U$ such that

$$\mathbf{y} = \mathbf{f}(\mathbf{x}) \Leftrightarrow \varphi_{\mathbf{y}} \text{ has a fixed point } \mathbf{x}.$$

However, (1) $\Rightarrow$ $\mathbf{x}$ is unique; i.e., $\mathbf{f} : U \xrightarrow{\text{1-1}} V$ is a bijection.

Exercise: Check that V is open.

(Proof unfinished.)