Math 321 Lecture 5

Yuchong Pan

January 11, 2019

1 Applications of Pointwise and Uniform Convergence

1.1 Weierstrass Approximation Theorem (Take 1)

Theorem 1 (Weierstrass approximation theorem)**.** Let $a, b \in \mathbb{R}, a < b$. Every $f \in C[a, b]$ can be uniformly approximated by polynomials; i.e., given any $f : [a, b] \xrightarrow{\text{continuous}} \mathbb{R}$ or $\mathbb{C}$, there exists a sequence $\{p_n : n \geq 1\} \subseteq \mathcal{P}[a, b]$, where $\mathcal{P}[a, b]$ is the space of polynomials on $[a, b]$ with coefficients in $\mathbb{R}$ or $\mathbb{C}$, such that $p_n \xrightarrow{n\to\infty} f$ uniformly on $[a, b]$.

Corollary 1. $C[a, b]$ is **separable**; i.e., it has a countable dense subset.

Proof. $\mathcal{P}[a, b]$, i.e., the space of real/complex polynomials on $[a, b]$, is dense in $C[a, b]$ by Weierstrass approximation theorem, but not countable.

Define $\mathcal{P}^*[a, b] = \bigcup_{n=0}^{\infty} \mathcal{P}_n^*[a, b]$, where

$$\begin{aligned}\mathcal{P}_n^*[a, b] = \big\{ &p : [a, b] \to \mathbb{R} \text{ (or } \mathbb{C}); p(x) = c_0 + c_1 x + c_2 x^2 + \ldots + c_n x^n \\ &\qquad \text{for some } (c_0, c_1, \ldots, c_n) \in \mathbb{Q}^{n+1} \text{ (or } (\mathbb{Q} + i\mathbb{Q})^{n+1}) \big\}\end{aligned}$$

i.e., $\mathcal{P}_n^*[a, b]$ contains polynomials on $[a, b]$ of degree $\leq n$.

Need to show:

1. $\mathcal{P}^*$ is countable.

   It suffices to show each $\mathcal{P}_n^*$ is countable. Fix any $n \geq 1$, and define $\varphi : \mathbb{Q}^{n+1} \to \mathcal{P}_n^*$ by

   $$\varphi(c_0, c_1, \ldots, c_n) = \sum_{i=0}^{n} c_i x^i.$$

   Then φ is a bijection. Thus, $\operatorname{card}(\mathcal{P}_n^*) = \operatorname{card}\left(\mathbb{Q}^{n+1}\right)$, where $\mathbb{Q}^{n+1}$ is countable.

2. $\mathcal{P}^*$ is dense.

   Know: $\mathcal{P}^* \subsetneq \mathcal{P} \overset{\text{dense}}{\subsetneq} C[a, b]$.

   It suffices to show that $\mathcal{P}^*$ is dense in $\mathcal{P}$ (show why).

   Start with any $f \in \mathcal{P}$; i.e., $f(x) = \sum_{i=0}^{n} \alpha_i x^i, \alpha_i \in \mathbb{R}$ or $\mathbb{C}$. Since $\mathbb{Q}$ (respectively, $\mathbb{Q} + i\mathbb{Q}$) is dense in $\mathbb{R}$ (respectively, $\mathbb{C}$), we can find sequences of rationals $\left\{c_i^{(k)} : k \geq 1\right\}, 0 \leq i \leq n$ such that $c_i^{(k)} \xrightarrow{k\to\infty} \alpha_i$ for all $0 \leq i \leq n$.

Define $p_k(x) = \sum_{i=0}^n c_i^{(k)} x^i \in \mathcal{P}^*$, for all $k \geq 1$. Then

$$\begin{aligned}
\|p_k - f\|_\infty &= \sup_{x \in [a,b]} |p_k(x) - f(x)| \\
&= \sup_{x \in [a,b]} \left| \sum_{i=0}^n \left(c_i^{(k)} - \alpha_i \right) x^i \right| \\
&\leq \sup_{x \in [a,b]} \sum_{i=0}^n \left| c_i^{(k)} - \alpha_i \right| |x|^i \\
&\leq M \sum_{i=0}^n \left| c_i^{(k)} - \alpha_i \right|,
\end{aligned}$$

where $M = \max\left\{ |x|^i : x \in [a,b], 0 \leq i \leq n \right\} < \infty$. Since $\left| c_i^{(k)} - \alpha_i \right| \to 0$ as $k \to \infty$, then $\|p_k - f\|_\infty \to 0$ as $k \to \infty$.

$\square$

Proof of Theorem 1 (Bernstein). Start with $f \in C[0,1]$. Note that it suffices to consider $C[0,1]$ because $[a,b] \xrightarrow[\text{linear}]{\text{bijection}} [0,1]$ by $x \mapsto \frac{x-a}{b-a}$.

Define $p_n(f)(x) = \sum_{k=0}^\infty f\left(\frac{k}{n}\right) \underbrace{\binom{n}{k} x^k (1-x)^{n-k}}_{\text{binomial probabilities}}$. Then $p_n(f)$ is a polynomial of degree $\leq n$. If we regard $\binom{n}{k} x^k (1-x)^{n-k}$ as binomial probabilities, then $p_n(f)(x) = \mathbb{E} f\left(\frac{X}{n}\right), X \sim \text{Binomial}(n, x)$.

$f\left(\frac{i}{n}\right)$	$\mathbb{P}(X = i)$
$f\left(\frac{0}{n}\right)$	$\binom{n}{0} x^0 (1-x)^{n-0}$
$f\left(\frac{1}{n}\right)$	$\binom{n}{1} x^1 (1-x)^{n-1}$
$\vdots$	$\vdots$
$f\left(\frac{k}{n}\right)$	$\binom{n}{k} x^k (1-x)^{n-k}$

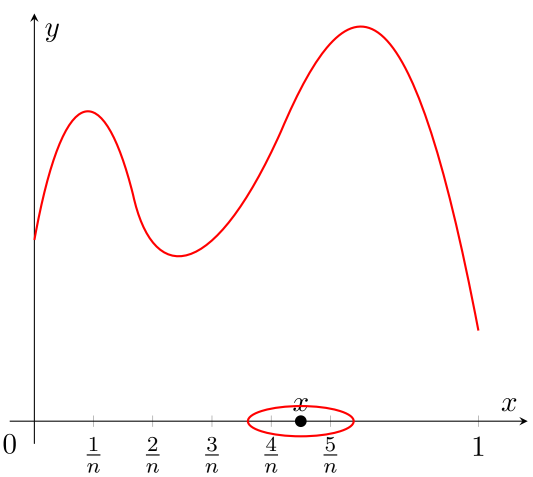
Fix $\epsilon > 0$. We want to find $N \geq 1$ such that for all $n \geq N$, $\|p_n - f\|_\infty < \epsilon$. Note that

$$\begin{aligned}
\sup_{x \in [0,1]} |p_n(f)(x) - f(x)| &= \sup_{x \in [0,1]} \left| \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} - f(x) \cdot 1 \right| \\
&= \sup_{x \in [0,1]} \left| \sum_{k=0}^n f\left(\frac{k}{n}\right) \binom{n}{k} x^k (1-x)^{n-k} - f(x) \cdot \sum_{k=0}^n \binom{n}{k} x^k (1-x)^{n-k} \right| \\
&= \sup_{x \in [0,1]} \left| \sum_{k=0}^n \left(f\left(\frac{k}{n}\right) - f(x) \right) \binom{n}{k} x^k (1-x)^{n-k} \right|.
\end{aligned}$$

By the **uniform** convergence of f, there exists $\delta > 0$ such that $|f(x) - f(y)| < \frac{\epsilon}{2}$ whenever $|x - y| < \epsilon$. Then,

$$\sup_{x\in[0,1]} |p_n(f)(x) - f(x)| \leq \sup_{x\in[0,1]} \left(\underbrace{\sum_{\substack{k=0 \\ \left|\frac{k}{n}-x\right|<\delta}}^{n} \overbrace{\left|f\left(\frac{n}{k}\right) - f(x)\right|}^{<\frac{\epsilon}{2}} \binom{n}{k} x^k (1-x)^{n-k}}_{\mathrm{I}} + \underbrace{\sum_{\substack{k=0 \\ \left|\frac{k}{n}-x\right|\geq\delta}}^{n} \left|f\left(\frac{n}{k}\right) - f(x)\right| \binom{n}{k} x^k (1-x)^{n-k}}_{\mathrm{II}} \right).$$

Note that $\mathrm{I} < \frac{\epsilon}{2}$ because $\sum_{\left|\frac{k}{n}-x\right|<\delta} \binom{n}{k} x^k (1-x)^{n-k} \leq 1$.



(Proof unfinished.)