Math 321 Lecture 26

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1 Dirichlet Kernel

Recall:

$$D_N(x) = \frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{2\sin\left(\frac{x}{2}\right)} = \sum_{k=-N}^{N} e^{ikx}.$$

Claim 1. There exists a constant $c_1 > 0$ such that for all $N \geq 1$,

$$\|D_N\|_1 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |D_N(x)| dx \geq c_1 \log N.$$

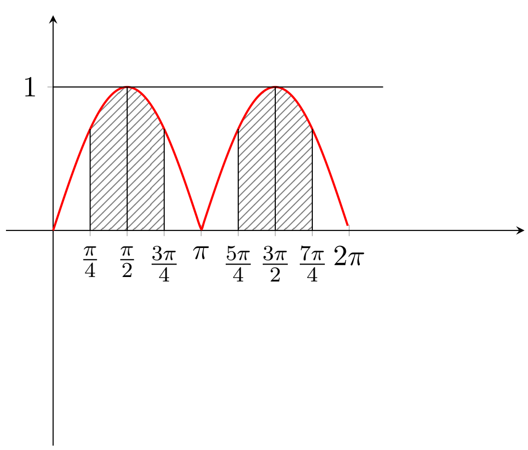
Proof.

$$\|D_N\|_1 = \frac{1}{4\pi}\int_{-\pi}^{\pi} \frac{\left|\sin\left(\left(N+\frac{1}{2}\right)x\right)\right|}{\left|\sin\left(\frac{x}{2}\right)\right|} dx.$$

Facts:

1. There exists $c > 0$ such that for $x \in [-\pi, \pi]$, $c^{-1} \leq \frac{\left|\sin\left(\frac{x}{2}\right)\right|}{|x|} \leq c$ ($c = 100$ will do).

2.

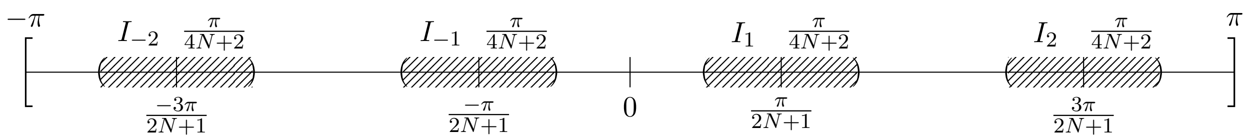


Since $|\sin t| \geq \frac{1}{\sqrt{2}}$ for $t \in (2k+1)\frac{\pi}{2} + \left[-\frac{\pi}{4}, \frac{\pi}{4}\right]$, $k \in \mathbb{Z}$, we conclude that

$$\left|\sin\left(\left(N+\frac{1}{2}\right)x\right)\right| \geq \frac{1}{\sqrt{2}} \text{ whenever } \left|\left(N+\frac{1}{2}\right)x - (2k+1)\frac{\pi}{2}\right| < \frac{\pi}{4} \text{ for some } k \in \mathbb{Z}.$$

Note that

$$\begin{aligned}
&\left|\left(N+\frac{1}{2}\right)x-(2k+1)\frac{\pi}{2}\right|<\frac{\pi}{4}\\
\Leftrightarrow\ &\left|x-\frac{2k+1}{N+\frac{1}{2}}\cdot\frac{\pi}{2}\right|<\frac{\pi}{4\left(N+\frac{1}{2}\right)}\\
\Leftrightarrow\ &\underbrace{\boxed{\left|x-\frac{2k+1}{2N+1}\cdot\pi\right|<\frac{\pi}{4N+2}}}_{\text{call this interval } I_k}.
\end{aligned}$$



Need to ensure that these intervals fall within $[-\pi,\pi]$, so suffices to impose the condition

$$\begin{aligned}
-\pi<\frac{2k+1}{2N+1}\pi<\pi &\Rightarrow -1<\frac{2k+1}{2N+1}<1\\
&\Rightarrow -2N-1<2k+1<2N+1\\
&\Rightarrow \boxed{-N-1<k<N}.
\end{aligned}$$

Combine Facts 1 and 2,

$$\begin{aligned}
\|D_N\|_1 &\geq \sum_{k=N}^{N-1}\int_{I_k}\left|\frac{\sin\left(\left(N+\frac{1}{2}\right)x\right)}{\sin\left(\frac{x}{2}\right)}\right|dx\\
&\overset{(2)}{\geq} \frac{1}{\sqrt{2}}\sum_{k=-N}^{N}\int_{I_k}\frac{1}{\left|\sin\left(\frac{x}{2}\right)\right|}dx\\
&\geq \frac{c^{-1}}{\sqrt{2}}\sum_{k=-N}^{N}\int_{I_k}\frac{dx}{|x|}\\
&\geq \frac{c^{-1}}{\sqrt{2}}\sum_{k=-N}^{N}\frac{1}{\frac{2k+1}{2N+1}\pi+\frac{\pi}{4N+2}}\cdot\frac{2\pi}{4N+2}\\
&\geq c_0\sum_{k=-N}^{N}\underbrace{\frac{1}{2k+\frac{3}{2}}}_{\text{comparable to the harmonic series}}\\
&\geq c_1'\sum_{k=-N}^{N-1}\frac{1}{k}\geq c_1\log N.
\end{aligned}$$

□

Remark.

1. Note that
$$c_1 \log N \leq \|D_N\|_1 \leq \underbrace{\|D_N\|_2}_{\text{Plancherel: } \sqrt{\text{sum of squares of the Fourier coefficients of } D_N} = \sqrt{2N+1}}.$$

2. An example of a function $f \in \mathcal{C}^{2\pi}$ whose Fourier series does not converge uniformly: In HW 9, Q3 (b), you found $f \in \mathcal{C}^{2\pi}$ such that
$$\sup_N |s_N f(0)| = \infty. \tag{*}$$
If $s_N f \to f$ uniformly on $[-\pi, \pi]$, then $s_N f(0) \xrightarrow{N\to\infty} f(0)$; not possible by (*).

3. Question: What is $\|D_N\|_\infty$? $\|D_N\|_\infty = 2N + 1$.
$$|D_N(x)| = \frac{\left|\sin\left(\left(N+\frac{1}{2}\right)x\right)\right|}{\left|\sin\left(\frac{x}{2}\right)\right|} = \left|\sum_{k=-N}^{N} e^{ikx}\right| \underbrace{\leq}_{\text{equality when } x=0} \sum_{k=-N}^{N} \underbrace{\left|e^{ikx}\right|}_{=1} = 2N+1.$$

2 Convergence of Fourier Series

Theorem 1. Suppose $f \in \mathcal{C}^{2\pi}$ is twice continuously differentiable. Then $s_N f \xrightarrow{N\to\infty} f$ uniformly.

Proof.
$$\begin{aligned}
\widehat{f}(k) &= \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)e^{-ikx}dx \\
&\overset{\text{IBP}}{=} \frac{1}{2\pi}\left[\cancelto{0}{f(x)\frac{e^{-ikx}}{-ik}\Big|_{-\pi}^{\pi}} - \int_{-\pi}^{\pi} f'(x)\frac{e^{-ikx}}{-ik}dx\right] \\
&= \frac{1}{2\pi ik}\int_{-\pi}^{\pi} f'(x)e^{-ikx}dx \\
&\overset{\text{IBP}}{=} \frac{1}{2\pi ik}\left[f'(x)\frac{e^{-ikx}}{-ik}\Big|_{-\pi}^{\pi} - \int_{-\pi}^{\pi} f''(x)\frac{e^{-ikx}}{-ik}dx\right].
\end{aligned}$$

$$\begin{aligned}
&\Rightarrow \left|\widehat{f}(k)\right| \leq \frac{c}{k^2} \\
&\xRightarrow{\text{by } M\text{-test}} \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{ikx} \text{ converges uniformly to some } g \in \mathcal{C}^{2\pi}.
\end{aligned}$$

$$\begin{aligned}
\text{Plancherel} &\Rightarrow \sum_{k\in\mathbb{Z}} \widehat{f}(k)e^{ikx} \text{ converges in } L^2 \text{ to } f \in \mathcal{C}^{2\pi} \\
&\Rightarrow \|f-g\|_2 = 0 \\
&\Rightarrow f \equiv g \text{ since } f, g \in \mathcal{C}^{2\pi}.
\end{aligned}$$

□