Math 321 Lecture 7

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January 16, 2019

1 The Space $C(X)$ (Cont'd)

Let (X, d) be any metric space. Then $C(X)$ is the space of real-valued or complex-valued continuous functions on X. $C(X)$ is always a vector space on $\mathbb{R}$ or $\mathbb{C}$. If X is **compact**, then $C(X)$ is also a metric space, with respect to the uniform norm: $\|f - g\|_\infty = \sup_{x \in X} |f(x) - g(x)|$.

Questions:

1. Suppose X is compact, so that $C(X)$ is a metric space. What are the compact subsets of $C(X)$?

2. **Know:**

$$\text{uniform convergence} \Rightarrow \text{pointwise convergence}$$
$$\text{pointwise convergence} \not\Rightarrow \text{uniform convergence}$$

$$\text{pointwise convergence} + \boxed{\text{equicontinuity}} \Rightarrow \text{uniform convergence.}$$

1.1 Arzela-Ascoli Theorem

Theorem 1 (Arzela-Ascoli)**.** Let (X, d) be a **compact** metric space. Let $\mathcal{F} \subseteq C(X)$. Then $\mathcal{F}$ is compact if and only if $\mathcal{F}$ is closed, uniform bounded ($\exists M > 0$ s.t. $\|F\| < M \ \forall F \in \mathcal{F}$) and equicontinuous.

Recall that $\mathcal{G} \subseteq C(X)$ where X is compact is **equicontinuous** if for all $\epsilon > 0$, there exists $\delta = \delta(\epsilon)$ such that for all $x, y \in X$ with $d(x, y) < \delta$, one has $|f(x) - f(y)| < \epsilon$ for all $f \in \mathcal{G}$.

Examples:

1. $\mathcal{G} = \{$ all constant functions $\}$ is equicontinuous but not uniformly bounded; AA $\Rightarrow \mathcal{G}$ is not compact.

2. $\mathcal{G} = \{f_n(x) = x^n; x \in [0, 1]\}$ is uniformly bounded by 1, but not equicontinuous (Exercise). Check that δ_n has to go to zero as $n \to \infty$ (near 1).

3. $\mathcal{G} = \{$ all constant functions $f(x) = k, k \in [0, 1]\}$ is equicontinuous and uniformly bounded.

4. Any finite collection of continuous functions on a compact metric space is both equicontinuous and uniformly bounded.

5. $\mathcal{G} = \left\{x^n; x \in \left[0, \frac{1}{2}\right]\right\} \subseteq C\left[0, \frac{1}{2}\right]$ and $\mathcal{G} = \left\{\frac{x}{n}; x \in [0,1]\right\}$ are equicontinuous and uniformly bounded.

6. The space of polynomials on $[0,1]$ is neither equicontinuous nor uniformly bounded.

Exercise: Show that if $f_n, f \in C(X)$ where X is compact, $f_n \to f$ uniformly on X, then $\mathcal{F} = \{f_n : n \geq 1\} \cup \{f\}$ is equicontinuous and uniformly bounded.

Proof of Theorem 1. **Step 1:** Assume $\mathcal{F}$ is compact. Show that $\mathcal{F}$ is closed, uniformly bounded and equicontinuous.

- $\mathcal{F}$ is closed because compactness implies closedness in any metric space.
- $\mathcal{F}$ is uniformly bounded: $\mathcal{F}$ compact $\Rightarrow \mathcal{F}$ $\underbrace{\text{totally bounded}}_{\text{i.e., } \forall \epsilon > 0, \exists f_1, f_2, \ldots, f_M \in C(X) \text{ s.t. } \mathcal{F} \subseteq \bigcup_{j=1}^M B(f_j; \epsilon)}$.

  Choose $\epsilon = 1$. Then
  $$\mathcal{F} \subseteq \bigcup_{j=1}^M \underbrace{\{f \in C(X) : \|f - f_j\|_\infty < 1\}}_{=B(f_j;1)}.$$
  Thus, for any $f \in \mathcal{F}$, there exists f_j such that $\|f - f_j\|_\infty < 1$; i.e.,
  $$|f(x)| \leq |f(x) - f_j(x)| + |f_j(x)| \leq \|f - f_j\|_\infty + \|f_j\| < 1 + N,$$
  where $N = \max\{\|f_j\| : 1 \leq j \leq M\} < \infty$.

- $\mathcal{F}$ is equicontinuous: Fix any $\epsilon > 0$.

  Since $\mathcal{F}$ is totally bounded, identify $f_1, \ldots, f_M \in C(X)$ such that
  $$\mathcal{F} \subseteq \bigcup_{j=1}^M B\left(f_j; \frac{\epsilon}{3}\right). \tag{*}$$
  For every $1 \leq j \leq M$, there exists $\delta_j = \delta_j(\epsilon) > 0$ such that
  $$d(x,y) < \delta_j \xRightarrow{f \text{ is continuous}} |f_j(x) - f_j(y)| < \frac{\epsilon}{3} \quad \forall x, y \in X. \tag{**}$$
  Set $\delta = \min(\delta_1, \delta_2, \ldots, \delta_M) > 0$. We need to show that this δ works for all $f \in \mathcal{F}$:
  $$|f(x) - f(y)| \leq |f(x) - f_j(x)| + |f_j(x) - f_j(y)| + |f_j(y) - f(y)|,$$
  where $1 \leq j \leq M$ is chosen so that $\|f - f_j\| < \frac{\epsilon}{3}$ (by (*)). Thus,
  $$\begin{aligned} |f(x) - f(y)| &< 2\|f - f_j\| + \frac{\epsilon}{3} \qquad \text{by (**)} \\ &< 2 \cdot \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon. \end{aligned}$$
  whenever $d(x,y) < \delta$.

(Proof unfinished.)