Perturbation-Stable Maximum Cut

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UBC Beyond Worst-Case Analysis Seminars

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Maximum Cut

Problem (Maximum Cut)

Input: An undirected graph $G = (V, E)$ with edge weights $w_e > 0$ for each $e \in E$.

Goal: A cut (A, B) that maximizes the weight of the *crossing* edges.

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- Maximum Cut is a type of 2-clustering problem (e.g. weights measure dissimilarities).

MAXIMUM CUT Is *NP*-Hard

Problem (MAXIMUM CUT, Decision Version)

Input: An undirected graph $G = (V, E)$ with edge weights $w_e > 0$ for each $e \in E$, and a positive integer W.

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Problem (PARTITION, Decision Version)

Input: $(c_1, \ldots, c_n) \in \mathbb{Z}^n$.

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Proof Sketch (Partition $\leq_P$ Maximum Cut)

- $G = K_n$.
- $w_{ij} = c_i c_j$ for all $i, j \in V, i \neq j$.
- $W = \lceil \frac{1}{4} \sum c_i^2 \rceil$.

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- **Question:** Can't we negate the edge weights, yielding a Minimum Cut instance?
- No! Polynomial-time algorithms solving Minimum Cut require nonnegative edge weights.

Beyond Worst-Case: Exact Recovery

- **Theme:** To recover the optimal solution in polynomial time in γ-*perturbation-stable* instances, where γ is as small as possible.

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Definition (γ-Perturbation-Stability)

For $\gamma \geq 1$, an instance of MAXIMUM CUT is γ-*perturbation-stable* if a cut (A, B) is the *unique* optimal solution to all γ-*perturbations*, where each original edge weight w_e is replaced with an edge weight $w'_e \in [\frac{1}{\gamma} w_e, w_e]$.

LP Relaxation, Take 1

- **Question:** Can we use an LP relaxation similar to the one for MINIMUM CUT, i.e.

$$
\begin{aligned}
\max \quad & \sum_{e \in E} w_e x_e \\
\text{s.t.} \quad & x_e \geq |d_u - d_v|, && \forall e = uv \in E. \\
& x_e \in [0,1], && \forall e \in E. \\
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- No! $x_e = 1$ for each $e \in E$ is a feasiable solution and maximizes the objective value.

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- No! $x_e = 1$ for each $e \in E$ is a feasiable solution and maximizes the objective value.
- **Question:** What about $x_e \leq d_u - d_v$ and $x_e \leq d_v - d_u$?
- This forces $x_e = 0$, instead of $x_e \leq |d_u - d_v|$.

LP Relaxation, Take 2

- Let $x_{ij} \in \{0, 1\}$ denote whether or not i, j are on different sides of the cut, for all distinct $i, j \in V$. We denote by x_{ij} and x_{ji} the same variable.

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$$x_{jk} \le x_{ij} + x_{ik}, \qquad \forall i, j, k \in V \text{ distinct.}$$

LP Relaxation, Take 2

- Hence we obtain the LP relaxation $(\text{LP-}\textsc{MaxCut})$:

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\text{s.t.} \quad & x_e \geq |d_u - d_v|\,, & \forall e = uv \in E. \\
& x_{ij} + x_{ik} + x_{jk} \leq 2, & \forall i,j,k \in V \text{ distinct.} \\
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& x_{ij} \in [0,1], & \forall i,j \in V \text{ distinct.}
\end{aligned}
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Main Theorem

Theorem

There is a constant $c > 0$ such that in every $(c \log n)$-perturbation-stable instance of Maximum Cut *with n vertices,* (LP-MaxCut) *solves to integers.*