Perturbation-Stable Maximum Cut

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UBC Beyond Worst-Case Analysis Reading Group
(Based on Tim Roughgarden's Notes for Stanford CS264)

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Maximum Cut

Problem (Maximum Cut)

Input: An undirected graph $G = (V, E)$ with edge weights $w_e > 0$ for each $e \in E$.

Goal: A cut (A, B) that maximizes the weight of the **crossing** edges.

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- ▶ MAXIMUM CUT is a type of 2-clustering problem (e.g. weights measure dissimilarities).

MAXIMUM CUT Is *NP*-Hard

Problem (MAXIMUM CUT, Decision Version)

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Output: Yes iff. there is a set $S \subseteq V$ such that the weight of the **crossing** edges is at least W.

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Problem (PARTITION, Decision Version)

Input: $(c_1, \ldots, c_n) \in \mathbb{Z}^n$.

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Proof Sketch (Partition $\leq_P$ Maximum Cut)

- $G = K_n$.
- $w_{ij} = c_i c_j$ for all $i, j \in V, i \neq j$.
- $W = \lceil \frac{1}{4} \sum c_i^2 \rceil$.

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- ▶ Minimum Cut is **not** *NP*-hard and can be solved by the Maximum-Flow Minimum-Cut Theorem.
- ▶ **Question:** Can't we negate the edge weights, yielding a Minimum Cut instance?
- ▶ No! Polynomial-time algorithms solving Minimum Cut require nonnegative edge weights.

Exact Recovery

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Definition (γ-Perturbation-Stability)

For $\gamma \geq 1$, an instance of MAXIMUM CUT is γ-**perturbation-stable** if a cut (A, B) is the **unique** optimal solution to all γ-**perturbations**, where each original edge weight w_e is replaced with an edge weight $w'_e \in [\frac{1}{\gamma} w_e, w_e]$.

LP Relaxation, Take 1

▶ **Question:** Can we use an LP relaxation similar to the one for MINIMUM CUT, i.e.

$$\max \quad \sum_{e \in E} w_e x_e$$

$$\text{s.t.} \quad x_e \geq |d_u - d_v|, \quad \forall e = uv \in E,$$

$$x_e \in [0,1], \quad \forall e \in E,$$

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▶ **Question:** What about $x_e \leq d_u - d_v$ and $x_e \leq d_v - d_u$?

▶ This forces $x_e = 0$, instead of $x_e \leq |d_u - d_v|$.

LP Relaxation, Take 2

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$$x_{jk} \leq x_{ij} + x_{ik}, \qquad \forall i, j, k \in V \text{ distinct.}$$

LP Relaxation, Take 2

▶ Hence we obtain the LP relaxation (LP-MAXCUT):

$$\max \quad \sum_{(i,j) \in E} w_{ij} x_{ij}$$

$$\begin{aligned}
\text{s.t.} \quad & x_e \geq |d_u - d_v|, && \forall e = uv \in E, \\
& x_{ij} + x_{ik} + x_{jk} \leq 2, && \forall i, j, k \in V \text{ distinct}, \\
& x_{jk} \leq x_{ij} + x_{ik}, && \forall i, j, k \in V \text{ distinct}, \\
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- ▶ MINIMUM s-t CUT: $A = \{v \in V : \hat{d}_v \leq r\}$ and $B = V \setminus A$, where $r \sim \mathrm{Uniform}(0, 1)$.

- ▶ MINIMUM MULTIWAY CUT: For each iteration, a group and a threshold are chosen uniformly randomly.

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LP algorithms (e.g. the ellipsoid method) always return an extreme point of the feasible region.

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    Exercise 2. *Show how to find (in polytime) a bfs with objective value within the range. You may use the LP oracle.*

- ▶ Since all of the extreme points of the feasible region are integral and correspond to a cut, then LP algorithms always solve (LP-MaxCut) to an integral optimal solution.

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- ▶ Formally, we define $\Delta(C)$ to be the total cost of C that exceeds that of C^* and $\Delta(\hat{x})$ to be total cost of C^* that exceeds the objective function value of $\hat{x}$.
- ▶ We show that $\mathbb{E}[\Delta(C)] \leq 0$ by the probability properties of the cut generated by the randomized rounding algorithm.
- ▶ Since $\Delta(C) \geq 0$ and since the equality holds iff. C is an optimal cut, it follows that the randomized rounding algorithm outputs an optimal cut w.p.1.

Randomized Rounding Algorithm

Lemma

Fix an instance of the MAXIMUM CUT *problem, with F^* the edges in the optimal cut, and $\hat{x}$ the optimal solution to* (LP-MAXCUT). *Then there exists a randomized algorithm that generates a random cut (A, B) and a scaling parameter $\sigma > 0$ such that:*

1. *For every edge $e = ij \notin F^*$,*

$$\mathbb{P}[e \text{ cut by } (A, B)] \geq \sigma \cdot \frac{\hat{x}_{ij}}{\alpha},$$

   where $\alpha = \Theta(\log n)$;

2. *For every edge $e = ij \in F^*$,*

$$\mathbb{P}[e \text{ not cut by } (A, B)] \leq \sigma \cdot (1 - \hat{x}_{ij});$$

3. *The rounding algorithm is deterministic iff. $\hat{x}$ is integral.*

Randomized Rounding Algorithm, Roadmap

▶ **Exercise:** Show that this lemma implies the main theorem (outlined above, Homework #4).

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Proposition

Fix an instance of MAXIMUM CUT, a cut C, and a feasible solution $\hat{x}$ to (LP-MAXCUT). For distinct $i, j \in V$, define

$$\hat{y}_{ij} = \begin{cases} \hat{x}_{ij}, & \text{if } i, j \text{ are on the same side of } C, \\ 1 - \hat{x}_{ij}, & \text{if } i, j \text{ are on different sides of } C. \end{cases}$$

Then $\hat{y}$ satisfies the triangle inequality:

$$\hat{y}_{jk} \leq \hat{y}_{ij} + \hat{y}_{ik}$$

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- That is, $\hat{x}, \hat{y}$ are both **pseudometrics** (i.e. metrics except that distinct points may have zero distances).

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Theorem (Bourgain's Theorem)

*For every n-point **pseudometric** space (X, d), there exists a randomized algorithm that generates a random partition (A, B) of X and a scaling parameter $\sigma > 0$ such that, for all distinct $i, j \in X$,*

$$\mathbb{P}[i, j \text{ on different sides of } (A, B)] \in \sigma \cdot \left[\frac{d(i,j)}{\alpha}, d(i,j) \right],$$

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▶ That is, every *n*-point metric space admits a randomized partitioning algorithm so that the sepration probabilities between pairs of points are **proportional** to the distances, up to a $\Theta(\log n)$ factor.

▶ The $\Theta(\log n)$ approximation factor is the best possible for **arbitrary** pseudometric spaces.

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Proof (Proposition & Bourgain's Theorem $\implies$ Lemma).

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▶ Fix an instance of MAXIMUM CUT. Let C^* denote an optimal cut, cutting the edges F^*.

▶ Let $\hat{x}$ be an optimal solution to (LP-MAXCUT). Define $\hat{y}$ as in Proposition (with C^* being the cut).

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▶ By Proposition, $\hat{y}$ is a pseudometric.

▶ By Bourgain's Theorem, there is a randomized algorithm that outputs a partition (A, B) and $\sigma > 0$ such that

$$\mathbb{P}[i, j \text{ on different sides of } (A, B)] = \sigma \cdot \left[\frac{\hat{y}_{ij}}{\alpha}, \hat{y}_{ij} \right],$$

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    1. If i, j are on the same side of C^*, then

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▶ Lemma follows.

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▶ **Exercise:** Prove Proposition and Bourgain's Theorem (Homework #4). For Bourgain's Theorem see e.g. CPSC 531F Tools for Modern Algorithm Analysis.

Dimensionality Reduction

Definition (α-Embeddings)

Let $(X, d_X), (Y, d_Y)$ be metric spaces. We say that $\phi : X \to Y$ is an α-**embedding** if there exists $r > 0$ such that

$$r \cdot d_X(u, v) \leq d_Y(\phi(u), \phi(v)) \leq r \cdot \alpha \cdot d_X(u, v).$$

for all $u, v \in X$.

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- **Dimensionality reduction** is the process of mapping a high dimensional dataset to a lower dimensional space, while preserving much of the important structure.

- For instance, let $X \subseteq \mathbb{R}^d$ and $Y = \mathbb{R}^t$ with $t < d$ and d_X, d_Y being the Euclidean distance.

Dimensionality Reduction

Theorem (Johnson-Lindenstrauss, 1984)

Let $x_1, \ldots, x_n \in \mathbb{R}^d$. Let $\epsilon \in (0, 1)$. Then for some $t = O(\frac{\log(n)}{\epsilon^2})$, there exist $y_1, \ldots, y_n \in \mathbb{R}^t$ such that

$$
\begin{aligned}
(1 - \epsilon) \|x_j\| &\leq & \|y_j\| &\leq & (1 + \epsilon) \|x_j\|, && \forall j \in [n], \\
(1 - \epsilon) \|x_j - x_{j'}\| &\leq & \|y_j - y_{j'}\| &\leq & (1 + \epsilon) \|x_j - x_{j'}\|, && \forall j, j' \in [n].
\end{aligned}
$$

Notation: $\|v\| = \sqrt{\sum_{i=1}^n v_i^2}$.

Dimensionality Reduction

Theorem (Johnson-Lindenstrauss, 1984)

Let $x_1, \ldots, x_n \in \mathbb{R}^d$. Let $\epsilon \in (0, 1)$. Then for some $t = O(\frac{\log(n)}{\epsilon^2})$, there exist $y_1, \ldots, y_n \in \mathbb{R}^t$ such that

$$
\begin{array}{rclccl}
(1 - \epsilon) \|x_j\| & \leq & \|y_j\| & \leq & (1 + \epsilon) \|x_j\|, & \forall j \in [n], \\
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Notation: $\|v\| = \sqrt{\sum_{i=1}^{n} v_i^2}$.

- **Remark:** There is a **random linear map** such that for any $x_1, \ldots, x_n$ the above condition holds with probability at least $\frac{1}{2n}$. This linear map is **oblivious**: it does not depend on $x_1, \ldots, x_n$ at all! In fact, the linear map is just a matrix whose entries are independent Gaussians.

Bourgain's Theorem & SPARSEST CUT

Theorem (Bourgain's Metric Embedding Theorem)

For any metric space (V, d), there exists an $O(\log n)$-embedding into $\mathbb{R}^{O(\log^2 n)}$ with the ℓ_1-norm that is computable with high probability by a randomized polynomial-time algorithm.

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▶ This result is the best possible; i.e., there exists a metric that cannot be embedded into ℓ_1 with distortion less than $\Omega(\log n)$.

Definition (Cut Metrics)

A metric (X, d) is a **cut metric** if there exists $S \subseteq X$ such that $d(x, y) = 0$ whenever $x, y \in S$ or $x, y \notin S$, and $d(x, y) = 1$ otherwise.

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Lemma

A metric (X, d) is an ℓ_1 metric if and only if it is a nonnegative linear combination of cut metrics.

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A metric (X, d) is an ℓ_1 metric if and only if it is a nonnegative linear combination of cut metrics.

- ▶ **Exercise:** Prove Lemma.
- ▶ Lemma and Bourgain's Metric Embedding Theorem imply Bourgain's Theorem in the proof of the main theorem.

Bourgain's Theorem & Sparsest Cut

Problem (Sparsest Cut)

Input: An undirected graph $G = (V, E)$ with edge weights $w_e > 0$ for each $e \in E$, and k pairs of vertices (s_i, t_i) each with demand d_i.

Goal: A set of vertices S that minimizes

$$\rho(S) \equiv \frac{\sum_{e \in \delta(S)} c_e}{\sum_{i:|S \cap \{s_i, t_i\}| = 1} d_i}.$$

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Corollary

There is a randomized $O(\log n)$-approximation algorithm for Sparsest Cut.

Tree Metric Embedding

Definition (Tree Metrics)

A metric (X, d) is a **tree metric** if there exists a tree $T = (V, E)$ with edge costs c_e for each $e \in E$ such that $d(u, v)$ is the cost of the unique path from u to v in T.

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Theorem (Fakcharoenphol-Rao-Talwar)

For any metric (V, d) such that $d(u, v) \geq 1$ for all $u, v \in V$ with $u \neq v$, there exists a randomized, polynomial-time algorithm that produces a tree metric $(V', T), V \subseteq V'$ such that for all $u, v \in V$, we have $d(u, v) \leq T_{uv}$ and $\mathbb{E}[T_{uv}] \leq O(\log n) d(u, v)$.

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▶ The above result is obtained via the method of **hierarchical tree decomposition**.

Semidefinite Programming

Definition (Positive Semidefinite Matrices)

A matrix $X \in \mathbb{R}^{n \times n}$ is **positive semidefinite** (or **psd**), denoted $X \succeq 0$, if $y^T X y \geq 0$ for all $y \in \mathbb{R}^n$.

Fact

If $X \in \mathbb{R}^{n \times n}$ is a symmetric matrix, then the following statements are equivalent:

1. *X is psd;*
2. *X has nonnegative eigenvalues;*
3. *$X = V^T V$ for some $V \in \mathbb{R}^{m \times n}$ where $m \leq n$;*
4. *$X = \sum_{i=1}^{n} \lambda_i w_i w_i^T$ for some $\lambda_i \geq 0$ and $w_i \in \mathbb{R}^n$ such that $w_i^T w_i = 1$ and $w_i^T w_j = 0$ for all $i \neq j$.*

Semidefinite Programming

Definition (Semidefinite Programming, SDP)

A **semidefinite program**, or **SDP**, is a mathematical program with real-valued variables, a linear objective function, linear constraints, and a square symmetric matrix of variables constrained to be psd.

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▶ Below is an example of SDP with variables x_{ij} for $i, j \in [n]$:

$$\text{max or min} \quad \sum_{i,j \in [n]} c_{ij} x_{ij} \tag{1}$$

$$\text{s.t.} \quad \sum_{i,j \in [n]} a_{ijk} x_{ij} = b_k, \qquad \forall k \in [n],$$

$$x_{ij} = x_{ji}, \qquad \forall i, j \in [n],$$

$$X = (x_{ij}) \succeq 0.$$

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▶ SDP can be solved to within an additive error of ϵ in polynomial time in the size of the input and $\log(\frac{1}{\epsilon})$.

Equivalence of SDP and Vector Programming

> ## Definition (Vector Programming)
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> A **vector program** is a mathematical program with variables $v_i \in \mathbb{R}^n$, where n is the number of vectors, and an objective function and constraints linear in the inner products of the vectors.

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- Below is an example of vector programming with variables $v_i \in \mathbb{R}^n$ for $i \in [n]$:

$$
\text{max or min} \quad \sum_{i,j \in [n]} c_{ij} \left(v_i \cdot v_j \right) \tag{2}
$$

$$
\text{s.t.} \quad \sum_{i,j \in [n]} a_{ijk} \left(v_i \cdot v_j \right) = b_k, \qquad \forall k \in [n],
$$

$$
v_i \in \mathbb{R}^n, \qquad \forall i \in [n].
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The SDP (1) *and the vector program* (2) *are equivalent.*

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- ($\impliedby$) Given a solution v_i's to (2), construct a matrix V with i^{th} column v_i, and let $X = V^T V$.

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- ($\Longleftarrow$) Given a solution v_i's to (2), construct a matrix V with i^{th} column v_i, and let $X = V^T V$.
- Then X is symmetric and psd by Fact, with $x_{ij} = v_i \cdot v_j$.

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$\square$

Quadratic Programming Formulation

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- Let $z_i \in \{-1, +1\}$ for each $i \in V$ indicate which side of the cut i belongs to.
- Hence, $z_i - z_j$ is 0 if i, j are on the same side of the cut, and ± 2 otherwise.

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- Equivalently, $(z_i - z_j)^2$ is 0 if i, j are on the same side of the cut, and 4 otherwise.

Quadratic Programming Formulation

- Hence we obtain the **exact** quadratic programming formulation of MAXIMUM CUT:

$$\max \quad \frac{1}{4} \sum_{ij \in E} w_{ij}(z_i - z_j)^2$$

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- This quadratic program is **equivalent** to MAXIMUM CUT. Hence optimizing this program is *NP*-hard.

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- **Recall:** Vector programming and SDP are equivalent and hence can be solved to within small error in polynomial time.

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- Hence relaxing each $z_i \in \{-1, 1\}$ to a **unit vector** $v_i \in \mathbb{R}^n$ and therefore replacing quadratic terms with inner products yield a vector program that is computationally tractable:

$$\max \quad \frac{1}{4} \sum_{ij \in E} w_{ij} \|v_i - v_j\|^2$$

$$\|v_i\|^2 = 1, \qquad \forall i \in V,$$

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- The quadratic terms $\|v_i - v_j\|^2$ and $\|v_i\|^2$ expands to sums of inner products $v_i \cdot v_i - 2 \cdot v_i \cdot v_j + v_j \cdot v_j$ and $v_i \cdot v_i$, respectively.

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- ▶ This vector program is a relaxation of the quadratic program by setting $v_i = (z_i, 0, \ldots, 0) \in \mathbb{R}^n$.

Goemans-Williamson Approximation Algorithm

- ▶ The above vector programming relaxation already leads to the best-known approximation algorithm for MAXIMUM CUT:

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Theorem (Goemans-Williamson Approximation Algorithm)

There is a randomized .878-approximation algorithm for MAXIMUM CUT.

Theorem

Given the unique game conjecture there is no α-approximation for MAXIMUM CUT with constant $\alpha > .878$ unless $P = NP$.

Vector Programming Relaxation

▶ For our purposes we want the vector programming relaxation to generalize the LP relaxation. Hence we want the analogs of the following two sets of constraints:

$$x_{ij} + x_{ik} + x_{jk} \leq 2, \qquad \forall i, j, k \in V \text{ distinct},$$
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▶ This implies the following two sets of constraints in the form of inner products:

$$\|v_i - v_j\|^2 + \|v_i - v_k\|^2 + \|v_j - v_k\|^2 \leq 8, \quad \forall i, j, k \in V \text{ distinct,}$$
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▶ The extended vector program, which we call (VP-MaxCut), is still a relaxation for Maximum Cut by setting v_i to $\pm e_1$ according to i's side.

Improved Exact Recovery, Roadmap

▶ Recall the roadmap for the proof of the exact recovery for
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- ▶ We first proved a proposition saying that if $\hat{x}$ is a pseudometric, then $\hat{y}$ defined below is also a pseudometric:

$$\hat{y}_{ij} = \begin{cases} \hat{x}_{ij}, & \text{if } i, j \text{ are on the same side of } C, \\ 1 - \hat{x}_{ij}, & \text{if } i, j \text{ are on different sides of } C. \end{cases}$$

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- ▶ This proposition and Bourgain's Theorem imply the lemma for the existence of randomized rounding algorithm that outputs a (random) cut such that the probability of an edge being cut is approximately the same as the value of the corresponding decision variable.

- ▶ The lemma then implies the exact recovery theorem by a common pattern used in perturbation-stable MINIMUM CUT and MINIMUM MULTIWAY CUT.

- Fix an instance of MAXIMUM CUT. Let $\hat{v}_i$'s be an optimal solution to the vector program. Let $\hat{x}_{ij} = \frac{1}{4}\|v_i - v_j\|^2$ for any distinct $i, j \in V$.

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Definition (ℓ_2^2 Metrics)

A metric (X, d) is an ℓ_2^2 **metric** if it represents squared Euclidean distances between points in $\mathbb{R}^k$ for some k.

That is, there exists an embedding from X to $\mathbb{R}^k$ for some k such that $d(\cdot, \cdot)$'s are squared Euclidean distances between points in $\mathbb{R}^k$.

▶ Not every metric is an ℓ_2^2 metric, e.g. the discrete metric.

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▶ **Question:** Can the approximation factor α be improved for the more restricted ℓ_2^2 metrics?

Improved Exact Recovery, Roadmap

- ▶ Not every metric is an ℓ_2^2 metric, e.g. the discrete metric.
- ▶ The $\Theta(\log n)$ approximation in Bourgain's Theorem is the best possible for **arbitrary** pseudometric spaces.
- ▶ **Question:** Can the approximation factor α be improved for the more restricted ℓ_2^2 metrics?

Theorem (Arora, Lee, and Naor, 2005)

For every n-point ℓ_2^2 metric space (X, d), there exists a randomized algorithm that generates a random partition (A, B) of X and a scaling parameter $\sigma > 0$ such that, for all distinct $i, j \in X$,

$$\mathbb{P}[i, j \text{ on different sides of } (A, B)] \in \sigma \cdot \left[\frac{d(i,j)}{\alpha}, d(i,j) \right],$$

where $\alpha = O(\sqrt{\log n} \log \log n)$.

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Proposition

Fix an instance of MAXIMUM CUT, a cut C, and a feasible solution $\hat{v}_i$'s to (VP-MaxCut). Let $\hat{x}$ be the induced ℓ_2^2 metric $\hat{x}_{ij} = \frac{1}{4}\|v_i - v_j\|^2$. For distinct $i, j \in V$, define

$$\hat{y}_{ij} = \begin{cases} \hat{x}_{ij}, & \text{if } i, j \text{ are on the same side of } C, \\ 1 - \hat{x}_{ij}, & \text{if } i, j \text{ are on different sides of } C. \end{cases}$$

Then $\hat{y}$ is also an ℓ_2^2 metric.

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> **Proof.**
>
> - $\hat{y}$ is a metric by the triangle inequality constraints of (VP-MaxCut).

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- To prove that $\hat{y}$ is an ℓ_2^2 metric, we need to exhibit an embedding of V into $\mathbb{R}^k$ for some k such that $\hat{y}_{ij}$'s are squared Euclidean distances between points in $\mathbb{R}^k$.

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- Define
$$\hat{u}_i = \begin{cases} \hat{v}_i, & \text{if } i \in A, \\ -\hat{v}_i, & \text{if } i \in B. \end{cases}$$

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- Then $\hat{y}_{ij} = \frac{1}{4}\|\hat{u}_i - \hat{u}_j\|^2$ for any distinct $i, j \in V$.

$\square$

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Lemma

Fix an instance of MAXIMUM CUT, *with F^* the edges in the optimal cut, and v_i's the optimal solution to* (VP-MAXCUT). *Then there exists a randomized algorithm that generates a random cut (A, B) and a scaling parameter $\sigma > 0$ such that:*

1. *For every edge $e = ij \notin F^*$,*

$$\mathbb{P}[e \text{ cut by } (A, B)] \geq \sigma \cdot \frac{\frac{1}{4}\|v_i - v_j\|^2}{\alpha},$$

   where $\alpha = \Theta(\sqrt{\log n} \log \log n)$;

2. *For every edge $e = ij \in F^*$,*

$$\mathbb{P}[e \text{ not cut by } (A, B)] \leq \sigma \cdot \left(1 - \frac{1}{4}\|v_i - v_j\|\right);$$

3. *The rounding algorithm is deterministic iff. $\hat{v}_i$'s are integral.*

▶ By $\hat{v}_i$'s being **integral**, we mean that there exist antipodal unit vectors $w, -w$ such that $\hat{v}_i \in \{w, -w\}$ for each $i \in V$.

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▶ As the lemma implies the main theorem in the exact recovery with $\gamma = \Theta(\log n)$, the above lemma implies the following theorem by an analagous argument:

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- As the lemma implies the main theorem in the exact recovery with $\gamma = \Theta(\log n)$, the above lemma implies the following theorem by an analogous argument:

Theorem

There is a constant $c > 0$ such that in every $(c\sqrt{\log n}\log\log n)$-perturbation-stable instance of MAXIMUM CUT with n vertices, every optimal solution to (VP-MAXCUT) is integral.

Can We Do Better?

Theorem

There exist $O(\sqrt{\log n})$-perturbation-stable instances of MAXIMUM CUT *for which the optimal solution to* (VP-MAXCUT) *is not integral.*

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Assuming the Unique Game Conjecture, for every constant $\gamma \geq 1$, there is no polynomial-time algorithm for certifiable exact recovery in γ-perturbation-stable instances of MAXIMUM CUT.

Can We Do Better?

Theorem

There exist $O(\sqrt{\log n})$-perturbation-stable instances of MAXIMUM CUT for which the optimal solution to (VP-MAXCUT) is not integral.

Theorem

Assuming the Unique Game Conjecture, for every constant $\gamma \geq 1$, there is no polynomial-time algorithm for certifiable exact recovery in γ-perturbation-stable instances of MAXIMUM CUT.

- ▶ Both results are based upon a reduction from SPARSEST CUT to MAXIMUM CUT. See *Bilu–Linial Stable Instances of Max Cut and Minimum Multiway* by Makarychev, Makarychev, and Vijayaraghavan (2013) and Homework #4.