KKM-Type Theorems and Their Applications[*]

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1 Two Problems

First, we consider two seemingly unrelated problems.

1.1 A Colorful Gallai's Theorem

Definition 1. Let $\mathcal{F}$ be a finite family of compact intervals in $\mathbb{R}$. Define the *matching number* of $\mathcal{F}$, denoted by $\nu(\mathcal{F})$, to be the maximum number of pairwise disjoint intervals in $\mathcal{F}$. Define the *covering number* of $\mathcal{F}$, denoted by $\tau(\mathcal{F})$, to be the minimum number of points needed to intersect all intervals in $\mathcal{F}$. A *matching* is a set of pairwise disjoint intervals.

Figure 1 gives two examples.

(a) $\nu(\mathcal{F}) = \tau(\mathcal{F}) = 2$.

(b) $\nu(\mathcal{F}) = \tau(\mathcal{F}) = 4$.

Figure 1: Two examples of the matching number and the covering number of a finite family of compact intervals in $\mathbb{R}$, where red vertical lines denote a minimum set of points that intersect all intervals in $\mathcal{F}$, and blue intervals denote a maximum set of pairwise disjoint intervals.

Theorem 2 (Gallai, 1960)**.** *If $\mathcal{F}$ is a finite family of compact intervals in $\mathbb{R}$, then $\tau(\mathcal{F}) = \nu(\mathcal{F})$.*

It is trivial to see that $\tau(\mathcal{F}) \geq \nu(\mathcal{F})$, as any set of k pairwise disjoint intervals requires k points to intersect all the intervals in the set. We reformulate Gallai's theorem as follows:

Theorem 3 (Gallai, 1960)**.** *Let $\mathcal{F}$ be a finite family of compact intervals in $\mathbb{R}$. If $\tau(\mathcal{F}) > k$, then there exists a matching in $\mathcal{F}$ of size $k + 1$.*

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We consider the following colorful version of Gallai's theorem:

Problem 4. *Let $\mathcal{F}_1, \ldots, \mathcal{F}_{k+1}$ be $k+1$ families of intervals in $\mathbb{R}$ such that $\tau(\mathcal{F}_i) > k$ for all $i \in [k+1]$. Can one find a* rainbow matching*, where a* rainbow matching *is a matching $\mathcal{M}$ such that $|\mathcal{M} \cap \mathcal{F}_i| = 1$ for all $i \in [k+1]$?*

Figure 2 gives an example of the colorful version of Gallai's theorem.

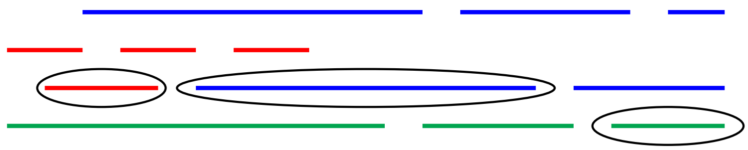

Figure 2: An example of the colorful Gallai's theorem, where each family is denoted by a distinct color, the covering number of each family is greater than 2, and a rainbow matching is circled.

1.2 Fair Division of a Cake

Consider a cake identified by $[0, 1]$ and n players. Given any partition of the cake, each player gives a list of pieces they prefer in that partition. We say that a player is *hungry* if they prefer a piece of positive length in any partition. Figure 3 gives a partition of the cake with an empty piece.

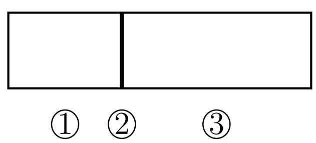

Figure 3: A partition of the cake, where the second piece is an empty piece.

We say that a preference list is *closed* if the following holds: if the player refers piece i in a converging sequence of partitions, then they prefer piece i also in the limiting partition. See Figure 4 for an illustration.

Moreover, we say that a division (i.e., a partition) is *envy-free* if each player has a distinct piece in their preference list.

Theorem 5 (Su, 1980)**.** *If every player is hungry, and if the preference list of each player is closed, then an envy-free division exists.*

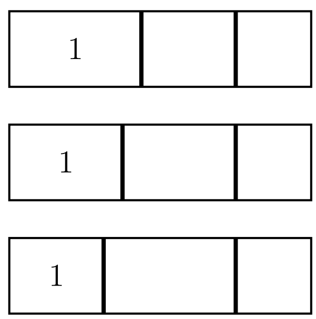

Figure 4: Illustrating the concept of a closed preference list, where the three partitions indicate a converging sequence of partitions. If the first player prefers the first piece in every partition in this sequence, then they must prefer the first piece in the limiting partition.