SICP Exercise 1.13

Yuchong Pan

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Lemma *$Fib(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$ and $\psi = \frac{1-\sqrt{5}}{2}$.*

Proof *We have the following degenerate cases:*

$$\frac{\phi^0 - \psi^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0 = Fib(0)$$

$$\frac{\phi^1 - \psi^1}{\sqrt{5}} = \frac{\sqrt{5}}{\sqrt{5}} = 1 = Fib(1)$$

Suppose we have proved that

$$Fib(i) = \frac{\phi^i - \psi^i}{\sqrt{5}}$$

$$Fib(i+1) = \frac{\phi^{i+1} - \psi^{i+1}}{\sqrt{5}}$$

According to the definition of the Fibonacci numbers, we have

$$Fib(i+2) = Fib(i) + Fib(i+1) = \frac{\left(\phi^i + \phi^{i+1}\right) - \left(\psi^i + \psi^{i+1}\right)}{\sqrt{5}}$$

Since we have $\phi^2 = \phi + 1$ and $\psi^2 = \psi + 1$, then

$$\phi^{i+2} = \phi^i \phi^2 = \phi^i(\phi + 1) = \phi^{i+1} + \phi^i$$

$$\psi^{i+2} = \psi^i \psi^2 = \psi^i(\psi + 1) = \psi^{i+1} + \psi^i$$

Hence, we have

$$Fib(i+2) = \frac{\phi^{i+2} - \psi^{i+2}}{\sqrt{5}}$$

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By induction, it can be proved that

$$Fib(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

Claim *$Fib(n)$ is the closest integer to $\frac{\phi^n}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$.*

Proof *Since $0 < \frac{1}{\sqrt{5}} < \frac{1}{2}$ and since $-\frac{1}{2} < \frac{\psi}{\sqrt{5}} < 0$, then we have*

$$-\frac{1}{2} < \frac{\psi^n}{\sqrt{5}} < \frac{1}{2}, n \in \mathbb{N}$$

By the lemma we have proved that

$$Fib(n) = \frac{\phi^n - \psi^n}{\sqrt{5}}$$

Hence, $Fib(n)$ is the closest integer to $\frac{\phi^n}{\sqrt{5}}$.