Selected Exercise Solutions for The Art Of Computer Programming

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1 Basic Concepts

1.1 Algorithms

5. The procedure fails to be a genuine algorithm for the following three points:

   - *Finiteness.* After one finishes all the 12 chapters, the procedure suggests starting from Chapter 1.
   - *Output.* The procedure does not have outputs.
   - *Effectiveness.* It is unclear whether each step can be completed in a finite amount of time. For instance, there are open problems in the exercises which no one knows the solution.

   The procedure has the following differences in format between it and Algorithm E:

   - The procedure does not include a paragraph specifying the inputs and the purpose (outputs) of the procedure.
   - Each step of the procedure does not have a phrase that briefly summarizes the principal content of the step.

7. Let $n \in \mathbb{N}$. If $n > m$, then after E1, $r = m \neq 0$. Therefore, in E3, m and n are essentially exchanged. This reduces to the definition of T_m. Since T_m is well-defined, then so is U_m, and $U_m = T_m + 1$.

9. We say that "C_2 is a representation of C_1" or "C_2 simulates C_1" if

   - there exists $\delta : Q_1 \to 2^{Q_2}$ such that for all $q_1, q_2 \in Q_1$ with $q_1 \neq q_2$, $\delta(q_1) \cap \delta(q_2) = \emptyset$;
   - for all $q \in Q_1$ and $q' \in \delta(q)$, there exist $n \in \mathbb{N}$ and $q'_1, \ldots, q'_n \in Q_2$ such that

$$\begin{aligned} q'_1 &= f_2\left(q'\right), \\ q'_{i+1} &= f_2\left(q'_i\right), && \forall i = 1, \ldots, n-1, \\ q'_n &\in \delta\left(f_1(q)\right). \end{aligned}$$

1.2 Mathematical Preliminaries

1.2.1 Mathematical Induction

8. (a) If $n = 1$, then $1^3 = 1$.

Let $n \geq 1$. Suppose by induction that $n^3 = (2(1+\ldots+(n-1))+1)+(2(1+\ldots+(n-1)+1))+\ldots+(2(1+\ldots+n-1)+1)$. Then

$$\begin{aligned}(n+1)^3 &= n^3+3n^2+3n+1\\ &= (2(1+\ldots+(n-1))+1)+(2(1+\ldots+(n-1)+1)+1)+\ldots\\ &\quad +(2(1+\ldots+n-1)+1)+3n^2+3n+1\\ &= (2(1+\ldots+(n-1))+1+2n)+(2(1+\ldots+(n-1)+1)+1+2n)+\ldots\\ &\quad +(2(1+\ldots+n-1)+1+2n)+3n^2+3n+1-2n\cdot n\\ &= (2(1+\ldots+n)+1)+(2(1+\ldots+n+1)+1)+\ldots\\ &\quad +(2(1+\ldots+(n+1)-2)+1)+n^2+3n+1\\ &= (2(1+\ldots+n)+1)+(2(1+\ldots+n+1)+1)+\ldots\\ &\quad +(2(1+\ldots+(n+1)-2)+1)+\left(2\cdot\frac{n^2+3n}{2}+1\right).\end{aligned}$$

It remains to show that $1+\ldots+(n+1)-1=\frac{n^2+3n}{2}$. We have

$$\begin{aligned}1+\ldots+(n+1)-1 &= \frac{(1+(n+1))\cdot(n+1)}{2}-1\\ &= \frac{(n+1)(n+2)}{2}-1\\ &= \frac{n^2+3n+2}{2}-1\\ &= \frac{n^2+3n}{2}.\end{aligned}$$

This completes the proof.

(b) Adding up $1^3, 2^3, \ldots, n^3$ and applying (a),

$$\begin{aligned}1^3+2^3+\ldots+n^3 &= 1+3+5+\ldots+(2(1+\ldots+n)-1)\\ &= \frac{(1+(2(1+\ldots+n)-1))\cdot(1+\ldots+n)}{2}\\ &= \frac{2(1+\ldots+n)^2}{2}\\ &= (1+\ldots+n)^2.\end{aligned}$$

13. See Figure 1.

15. (a) Suppose for the sake of contradiction that $\mathbb{Z}$ is well-ordered by $<$. Since $\mathbb{Z} \subseteq \mathbb{Z}$, then by (iii) there exists $x \in \mathbb{Z}$ with $x \leq y$ for all $y \in \mathbb{Z}$. Note that $x-1 \in \mathbb{Z}$ and that $x-1 < x$. By (ii), $x \leq x-1$ is not true, a contradiction. This completes the proof.

(b) Define $f : \mathbb{Z} \to \mathbb{N}$ by

$$f(x) = \begin{cases} 1, & x = 0,\\ 2x, & x > 0,\\ -2x+1, & x < 0,\end{cases} \qquad x \in \mathbb{Z}.$$

Define $x \prec y$ oif $f(x) < f(y)$ for all $x, y \in \mathbb{Z}$. We show that $\mathbb{Z}$ is well-ordered by $\prec$.

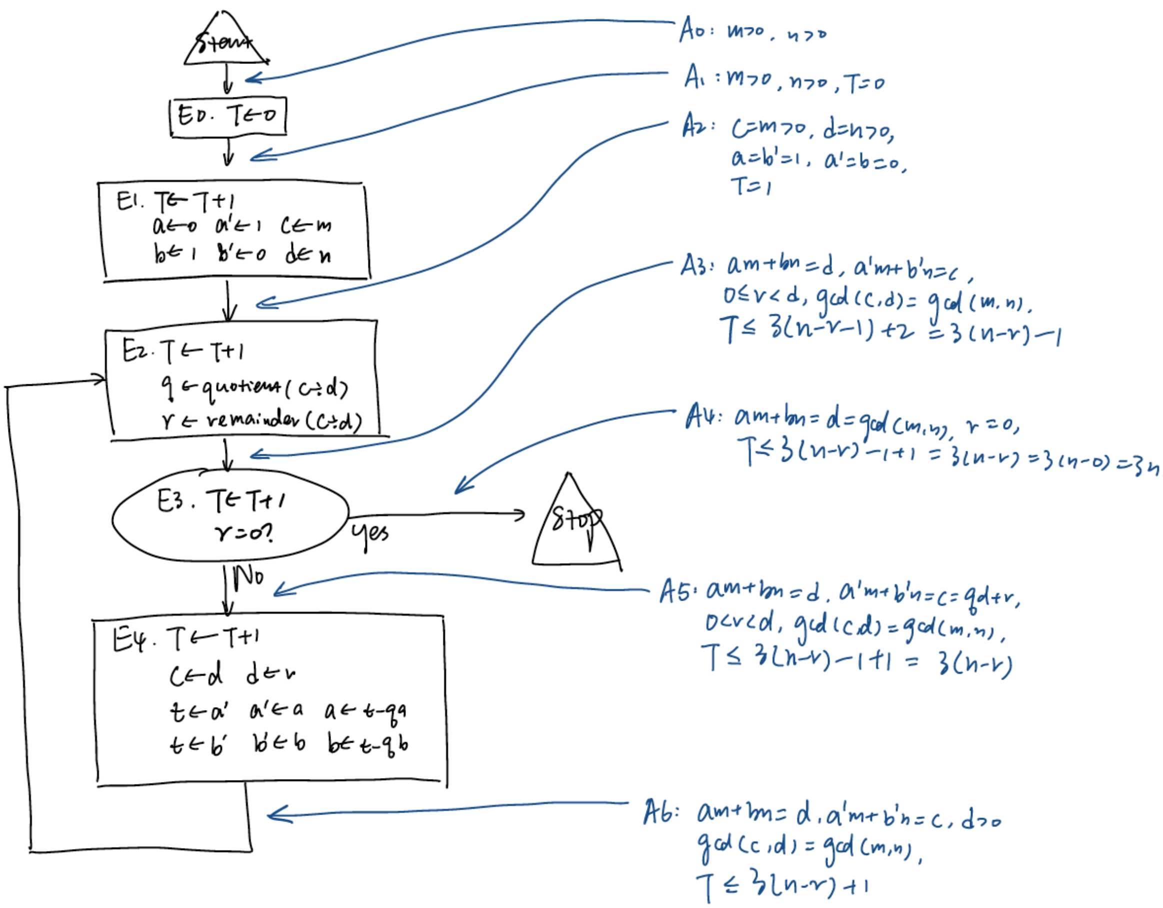

Figure 1: Flow chart for Algorithm E with assertions that prove $T \leq 3n$.

i) Let $x, y, z \in \mathbb{Z}$. Suppose that $x \prec y$ and that $y \prec z$. Then $f(x) < f(y)$ and $f(y) < f(z)$. Thus, $f(x) < f(z)$. By the definition of $\prec$, $x \prec z$.

ii) Let $x, y \in \mathbb{Z}$. Then existly one of the following tree possibilities is true: $f(x) < f(y)$, $f(x) = f(y)$, or $f(y) < f(x)$.
If $f(x) < f(y)$, then $x \prec y$. If $f(y) < f(x)$, then $y \prec x$. Suppose that $f(x) = f(y)$. If $f(x) = f(y) = 1$, then $x = y = 0$. If $f(x) = f(y)$ is even, i.e. $f(x) = f(y) = 2k$ for some $k \in \mathbb{N}$, then $x = y = k$ by the definition of f. If $f(x) = f(y)$ is odd and not equal to 1, i.e. $f(x) = f(y) = 2k+1$ for some $k \in \mathbb{N}$, then $f(x) = f(y) = -2(-k)+1$. By the definition of f, $x = y = -k$. In either case, $x = y$.
Hence, exactly one of the following three possibilities is true: $x \prec y$, $x = y$, or $y \prec x$.

iii) Let $A \subseteq Z, A \neq \emptyset$. Then there exists $a_0 \in A$. Suppose for the sake of contradiction that there does not exist $x \in A$ such that $x \preceq y$ for all $y \in A$. Thus, for all $x \in A$, there exists $y \in A$ such that $y \prec x$. Therefore, there exists $a_1 \in A$ such that $a_1 \prec a_0$. By the definition of $\prec$ $f(a_1) < f(a_0)$. Having defined $a_1, \ldots, a_k$, there exists $a_{k+1} \in A$ such that $f(a_{k+1}) < f(a_k)$.
Let $B = \{f(a_k) : k \in \mathbb{Z}, k \geq 0\} \subseteq \mathbb{N}$. Since $\mathbb{N}$ is well-ordered by $<$, then there exists $x = f(a_j) \in B$ for some $j \in \mathbb{Z}, j \geq 0$ such that $x \leq y$ for all $y \in B$. However, $f(a_{j+1}) \in B$ and $f(a_{j+1}) < f(a_j)$, a contradiction. This completes the proof.

(c) No. Suppose for the sake of contradiction that $[0, \infty)$ is well-ordered by $<$. Since

$(0, \infty) \subseteq [0, \infty)$, then by (iii), there exists $x \in (0, \infty)$ such that $x \leq y$ for all $y \in (0, \infty)$. Note that $\frac{x}{2} \in (0, \infty)$ and that $\frac{x}{2} < x$. By (ii), $x \leq \frac{x}{2}$ is not true, a contradiction. This completes the proof.

(d) Yes.

i) Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n), z = (z_1, \ldots, z_n) \in T_n$. Suppose that $x \prec y$ and that $y \prec z$. This implies that there exists $k_1, 1 \leq k_1 \leq n$ such that $x_j = y_j$ for $1 \leq j < k_1$ and that $x_{k_1} \prec y_{k_1}$, and that there exists $k_2, 1 \leq k_2 \leq n$ such that $y_j = z_j$ for $1 \leq j < k_2$ and that $y_{k_2} \prec z_{k_2}$. Let $k = \min(k_1, k_2)$. Let $j \in \mathbb{N}$ with $1 \leq j < k = \min(k_1, k_2)$. Then $1 \leq j < k_1$ and $1 \leq j < k_2$. Thus, $x_j = y_j$ and $y_j = z_j$, so $x_j = z_j$.
Without loss of generality, assume that $k = k_1$. Then $k_1 \leq k_2$. Thus, $x_k = x_{k_1} \prec y_{k_1}$. If $k_1 < k_2$, then $y_{k_1} = z_{k_1} = z_k$ and hence $x_k \prec z_k$. If $k_1 = k_2$, then $y_{k_1} = y_{k_2} \prec z_{k_2} = z_{k_1} = z_k$ and by (i), $x_k \prec z_k$. By the definition of $\prec$ on T_n, $x \prec z$.

ii) Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in T_n$. If $x_k = y_k$ for all $1 \leq k \leq n$, then $x = y$. Otherwise, there exists $k \in \mathbb{N}, 1 \leq k \leq n$ such that $x_k \neq y_k$. Let k^* be the smallest such k. Then $x_j = y_j$ for all $1 \leq j < k$. Since S is well-ordered by $\prec$, then either $x_k \prec y_k$, or $y_k \prec x_k$. By the definition of $\prec$, either $x \prec y$, or $y \prec x$, respectively.

iii) Let $A \subseteq T_n$. Let $A_1 = \{x_1 \in S : (x_1, \ldots, x_n) \in A\} \subseteq S$. Since S is well-ordered by $\prec$, then there exists $x_1^* \in A_1$ such that $x_1^* \preceq y_1$ for all $y_1 \in A_1$. Having defined $x_1^*, \ldots, x_k^*$ for $1 \leq k < n$, let $A_{k+1} = \{x_{k+1} \in S : (x_1^*, \ldots, x_k^*, x_{k+1}, \ldots, x_n) \in A\} \subseteq S$. Since S is well-ordered by $\prec$, then there exists $x_{k+1}^* \in A_{k+1}$ such that $x_{k+1}^* \preceq y_{k+1}$ for all $y_{k+1} \in A_{k+1}$.
Let $x^* = (x_1^*, \ldots, x_n^*)$. Let $y = (y_1, \ldots, y_n) \in A$. We show that $(x_1^*, \ldots, x_n^*) \preceq (y_1, \ldots, y_n)$. Suppose for the sake of contradiction that this is not true. Then by (ii), $y \prec x^*$. By the definition of $\prec$, there exists $k \in \mathbb{N}, 1 \leq k \leq n$ such that $y_k \prec x_k^*$. This however contradicts that $x_k^* \preceq y_k$. This completes the proof.

(e) No. Let $A = \{(1), (0, 1), (0, 0, 1), (0, 0, 0, 1), \ldots\} \subseteq T$. Clearly $A \neq \emptyset$. Suppose for the sake of contradiction that there exists $x \in A$ such that $x \preceq yy$ for all $y \in A$. By the definition of A, there exists $k \in \mathbb{Z}, k \geq 0$ such that $x = (\underbrace{0, \ldots, 0}_{k \text{ 0's}}, 1)$. Note that $(\underbrace{0, \ldots, 0}_{(k+1) \text{ 0's}}, 1) \in A$ and that by the definition of $\prec$, $(\underbrace{0, \ldots, 0}_{(k+1) \text{ 0's}}, 1) \prec (\underbrace{0, \ldots, 0}_{k \text{ 0's}}, 1) = x$, a contradiction. This completes the proof.

(f) ($\Longrightarrow$) Suppose that S is well-ordered by $\prec$. By the definition of well-ordering, $\prec$ satisfies (i) and (ii). Suppose for the sake of contradiction that there exists an infinite sequence $x_1, x_2, x_3, \ldots$ with $x_{j+1} \prec x_j$ for all $j \geq 1$. Let $A = \{x_1, x_2, \ldots\} \subseteq S$. Clearly $A \neq \emptyset$. Since S is well-ordered by $\prec$, then there exists $x \in A$ such that $x \preceq y$ for all $y \in A$. Suppose that $x = x_k$ for some $k \in \mathbb{N}$. Note that $x_{k+1} \in A$ and that $x_{k+1} \prec x_k$, a contradiction. This completes the proof.

($\Longleftarrow$) Suppose that $\prec$ satisfies (i) and (ii), and that there exists no infinite sequence $x_1, x_2, \ldots$ with $x_{j+1} \prec x_j$ for all $j \geq 1$. Suppose for the sake of contradiction that S is not well-ordered by $\prec$. Since $\prec$ satisfies (i) and (ii), then $\prec$ does not satisfy (iii). Thus, there exists $A \subseteq S, A \neq \emptyset$ such that for all $x \in A$, there exists $y \in A$ such that $y \prec x$. Since $A \neq \emptyset$, then there exists $x_0 \in A$. Having defined $x_0, \ldots, x_k \in A$, since $x_k \in A$, then there exists $x_{k+1} \in A$ such that $x_{k+1} \prec x_k$. This defines an infinite sequence $x_0, x_1, \ldots$ with $x_{j+1} \prec x_j$ for all $j \geq 1$, a contradiction. This completes the proof.

(g) Suppose that $P(x)$ can be proved under the assumption that $P(y)$ is true for all $y \prec x$. Suppose for the sake of contradiction that there exists $x \in S$ such that $P(x)$ is false. Let $A = \{x \in S : P(x) \text{ is false}\} \subseteq S$. Then $A \neq \emptyset$. Since S is well-ordered by $\prec$, then there exists $x^* \in A$ such that $x^* \preceq y$ for all $y \in A$. Let $z \prec x^*$. Then $P(z)$ is true. For otherwise $z \in A$ and hence $x^* \preceq z$, a contradiction by (ii). This shows that $P(z)$ is true for all $z \in x^*$. By the assumption, $P(x^*)$ can be proved to be true, a contradiction. This completes the proof.