Fantastic Planar Graphs and Where to Find Them

A Linear-Algebraic Planarity Criterion

Alan Bu[1] **Yuchong Pan[2]**

[1]Harvard University **[2]MIT**

February 10, 2025
Theory Lunch
MIT CSAIL

an extremal problem

an extremal problem (Kenyon 1996)

How large is the maximum number T_m of spanning trees over m-edge planar graphs?

an extremal problem (Kenyon 1996)

How large is the maximum number T_m of spanning trees over m-edge planar graphs?

some `StackExchange` answer (user `JimT`):

> This is a very, very tough question. It is obvious that for any graph G with n edges the number of spanning trees $t(G)$ does not exceed 2^n (each edge is either included in or excluded from a subtree; this is also the upper bound of the number of connected subgraphs of G). We can somewhat improve this - if G has m vertices ($m \leqslant n + 1$) then we have
>
> $$t(G) \leqslant \binom{n}{m-1} < 2^n,$$
>
> since we need to choose $m - 1$ edges out of n (not arbitrarily, of course).

an extremal problem (Kenyon 1996)

How large is the maximum number T_m of spanning trees over m-edge planar graphs?

some `StackExchange` answer (user `JimT`):

> This is a very, very tough question. It is obvious that for any graph G with n edges the number of spanning trees $t(G)$ does not exceed 2^n (each edge is either included in or excluded from a subtree; this is also the upper bound of the number of connected subgraphs of G). We can somewhat improve this - if G has m vertices ($m \leqslant n+1$) then we have
>
> $$t(G) \leqslant \binom{n}{m-1} < 2^n,$$
>
> since we need to choose $m-1$ edges out of n (not arbitrarily, of course).

a **lower bound**: $T_m \geq 1.7916^m$ achieved by square grid graphs some **upper bounds** ($T_m \leq \tau^m$):

2 As a matter of fact, the conjecture above has a relatively elementary proof (found by yours truly) for $\tau \approx 1.96$. Of course, that is not very close to the hypothetical upper bound of $e^{2C/\pi}$, but at least it is something... – JimT Jan 8, 2021 at 4:21

2 1.9328... is the new τ - with an elementary proof. :) – JimT Jan 15, 2021 at 3:32

There was a recent improvement of 1.96^n down to 1.913^n . I could not find the proof, but the link is here: sspcdn.blob.core.windows.net/files/Documents/SEP/STS/2024/... – Igor Pak Aug 2, 2024 at 1:18

Actually, it is now at 1.8637^n. See arxiv.org/abs/2103.10523 – JimT Aug 3, 2024 at 3:34

Quick update: Here is a result by Stoimenow in knot theory, which, in their abstract, mentions giving an upper bound of 1.8393n on this number. Here: projecteuclid.org/journals/tokyo-journal-of-mathematics/... – Alien Sep 28, 2024 at 0:17

@Alien Nice! One comment though - I quickly looked thru the article; it is very non-elementary, as it uses a bunch of other results in links and knot theory, as well as related polynomial algebra. But still, it proves the even lower upper bound, so the gap between this and 1.7916... is getting smaller with every post here :) . – JimT Sep 29, 2024 at 2:32

some **upper bounds** ($T_m \leq \tau^m$):

2 As a matter of fact, the conjecture above has a relatively elementary proof (found by yours truly) for $\tau \approx 1.96$. Of course, that is not very close to the hypothetical upper bound of $e^{2C/\pi}$, but at least it is something... – JimT Jan 8, 2021 at 4:21

2 1.9328... is the new τ - with an elementary proof. :) – JimT Jan 15, 2021 at 3:32

There was a recent improvement of 1.96^n down to 1.913^n . I could not find the proof, but the link is here: sspcdn.blob.core.windows.net/files/Documents/SEP/STS/2024/... – Igor Pak Aug 2, 2024 at 1:18

Actually, it is now at 1.8637^n. See arxiv.org/abs/2103.10523 – JimT Aug 3, 2024 at 3:34

Quick update: Here is a result by Stoimenow in knot theory, which, in their abstract, mentions giving an upper bound of 1.8393n on this number. Here: projecteuclid.org/journals/tokyo-journal-of-mathematics/... – Alien Sep 28, 2024 at 0:17

@Alien Nice! One comment though - I quickly looked thru the article; it is very non-elementary, as it uses a bunch of other results in links and knot theory, as well as related polynomial algebra. But still, it proves the even lower upper bound, so the gap between this and 1.7916... is getting smaller with every post here :) . – JimT Sep 29, 2024 at 2:32

theorem (Stoimenow 2007)

$T_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

In this talk, we provide a simple, linear-algebraic proof of Stoimenow's upper bound (in contrast to his "non-elementary", knot-theoretic, polynomial-algebraic proof)

In this talk, we provide a simple, linear-algebraic proof of Stoimenow's upper bound (in contrast to his "non-elementary", knot-theoretic, polynomial-algebraic proof)

... and also some nice by-products on planarity criteria from our techniques.

another extremal problem

another extremal problem

How large is the maximum determinant Δ_m over $m \times m$ matrices that are concatenations $[M|N]$ of two submatrices of (possibly different) incidence matrices?

another extremal problem

How large is the maximum determinant Δ_m over $m \times m$ matrices that are concatenations $[M|N]$ of two submatrices of (possibly different) incidence matrices?

theorem (Bu, P. 2025)

$T_m \leq \Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

$$\Delta_m \leq \delta^m$$

definitions

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero.

definitions

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero.

A bi-incidence matrix is a concatenation $[M|N]$ of two incidence submatrices M and N.

definitions

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero.

A bi-incidence matrix is a concatenation $[M|N]$ of two incidence submatrices M and N.

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices.

definitions

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero.

A bi-incidence matrix is a concatenation $[M|N]$ of two incidence submatrices M and N.

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices.

proof of $\Delta_m \leq \delta^m$

Let $[M|N]$ achieve Δ_m.

definitions

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero.

A bi-incidence matrix is a concatenation $[M|N]$ of two incidence submatrices M and N.

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices.

proof of $\Delta_m \leq \delta^m$

Let $[M|N]$ achieve Δ_m.

Extend $[M|N]$ by two columns to $[M'|N']$ so that each row of M' (resp. each row of N') has exactly one 1 and exactly one -1.

definitions

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero.

A bi-incidence matrix is a concatenation $[M|N]$ of two incidence submatrices M and N.

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices.

proof of $\Delta_m \leq \delta^m$

Let $[M|N]$ achieve Δ_m.

Extend $[M|N]$ by two columns to $[M'|N']$ so that each row of M' (resp. each row of N') has exactly one 1 and exactly one -1.

nonzero entries in $[M'|N']$: $4m$

definitions

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero.

A bi-incidence matrix is a concatenation $[M|N]$ of two incidence submatrices M and N.

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices.

proof of $\Delta_m \leq \delta^m$

Let $[M|N]$ achieve Δ_m.

Extend $[M|N]$ by two columns to $[M'|N']$ so that each row of M' (resp. each row of N') has exactly one 1 and exactly one -1.

nonzero entries in $[M'|N']$: $4m$
columns in $[M'|N']$: $m+2$

definitions

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero.

A bi-incidence matrix is a concatenation $[M|N]$ of two incidence submatrices M and N.

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices.

proof of $\Delta_m \leq \delta^m$

Let $[M|N]$ achieve Δ_m.

Extend $[M|N]$ by two columns to $[M'|N']$ so that each row of M' (resp. each row of N') has exactly one 1 and exactly one -1.

\# nonzero entries in $[M'|N']$: $4m$
\# columns in $[M'|N']$: $m+2$

$\implies$ $[M'|N']$ has a column with at most $\lfloor 4m/(m+2) \rfloor = 3$ nonzero entries.

definitions

An incidence submatrix is a matrix where each row has at most one 1, at most one -1, and all other entries zero.

A bi-incidence matrix is a concatenation $[M|N]$ of two incidence submatrices M and N.

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices.

proof of $\Delta_m \leq \delta^m$

Let $[M|N]$ achieve Δ_m.

Extend $[M|N]$ by two columns to $[M'|N']$ so that each row of M' (resp. each row of N') has exactly one 1 and exactly one -1.

\# nonzero entries in $[M'|N']$: $4m$
\# columns in $[M'|N']$: $m+2$

$\implies$ $[M'|N']$ has a column with at most $\lfloor 4m/(m+2) \rfloor = 3$ nonzero entries.

W.l.o.g. (by column operations), this column is in $[M|N]$.

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$
\begin{array}{c}
r_1 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m
\end{array}
\left[\begin{array}{c|c}
a_{11} & \\
a_{21} & \\
a_{31} & \\
0 & \\
\vdots & \\
0 &
\end{array}\right]
$$

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$
\begin{array}{c}
r_1 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m
\end{array}
\left[\begin{array}{c|c}
a_{11} & \quad \\
a_{21} & \quad \\
a_{31} & \quad \\
0 & \quad \\
\vdots & \quad \\
0 & \quad
\end{array}\right]
$$

$$
\underset{r_i}{\left[\begin{array}{c|c} \quad & \quad \end{array}\right]} = \underset{r_i^0}{\left[\begin{array}{c|c} \quad & 0 \cdots 0 \end{array}\right]} + \underset{r_i^1}{\left[\begin{array}{c|c} 0 \cdots 0 & \quad \end{array}\right]}
$$

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\begin{array}{c} r_1 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} a_{11} & \\ a_{21} & \\ a_{31} & \\ 0 & \\ \vdots & \\ 0 & \end{array}\right]$$

$$\underbrace{\left[\begin{array}{c|c} \quad & \quad \end{array}\right]}_{r_i} = \underbrace{\left[\begin{array}{c|c} \quad & 0 \cdots \cdot 0 \end{array}\right]}_{r_i^0} + \underbrace{\left[\begin{array}{c|c} 0 \cdots \cdot 0 & \quad \end{array}\right]}_{r_i^1}$$

$$\det \left[\begin{array}{c|c} M & N \end{array}\right] = \sum_{\alpha,\beta,\gamma \in \{0,1\}} \det \begin{array}{c} r_1^\alpha \\ r_2^\beta \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & \\ & \\ & \\ 0 & \\ \vdots & \\ 0 & \end{array}\right]$$

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$
\begin{array}{c}
r_1^1 \\ r_2^1 \\ r_3^1 \\ r_4 \\ \vdots \\ r_m
\end{array}
\left[
\begin{array}{c|c}
0 \cdots 0 & \\
0 \cdots 0 & \\
0 \cdots 0 & \\
0 & \\
\vdots & \\
0 &
\end{array}
\right]
$$

case 1: $\alpha = \beta = \gamma = 1$

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\det \begin{array}{c} r_1^1 \\ r_2^1 \\ r_3^1 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{ccc|c} 0 & \cdots & 0 & \\ 0 & \cdots & 0 & \\ 0 & \cdots & 0 & \\ 0 & & & \\ \vdots & & & \\ 0 & & & \end{array}\right] = 0.$$

case 1: $\alpha = \beta = \gamma = 1$

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\sum_{\beta,\gamma\in\{0,1\}} \det \begin{array}{c} r_1^0 \\ r_2^\beta \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} \quad & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right] = \det \begin{array}{c} r_1^0 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} \quad & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right]$$

case 2: $\alpha = 0$

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\sum_{\beta,\gamma\in\{0,1\}} \det \begin{array}{c} r_1^0 \\ r_2^\beta \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right] = \det \begin{array}{c} r_1^0 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right]$$

case 2: $\alpha = 0$

lemma (Bu, P. 2025; informal)

If an $m \times m$ bi-incidence matrix has a row whose restriction to the left/right side has zeros only,

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\sum_{\beta,\gamma\in\{0,1\}} \det \begin{array}{c} r_1^0 \\ r_2^\beta \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & 0 \cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right] = \det \begin{array}{c} r_1^0 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & 0 \cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right]$$

case 2: $\alpha = 0$

lemma (Bu, P. 2025; informal)

If an $m \times m$ bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain an $(m-1) \times (m-1)$ square bi-incidence matrix, while preserving the determinant.

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\sum_{\beta,\gamma\in\{0,1\}} \det \begin{array}{c} r_1^0 \\ r_2^\beta \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right] = \det \begin{array}{c} r_1^0 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \\ & \end{array}\right] \leq \Delta_{m-1}.$$

case 2: $\alpha = 0$

lemma (Bu, P. 2025; informal)

If an $m \times m$ bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain an $(m-1) \times (m-1)$ square bi-incidence matrix, while preserving the determinant.

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\sum_{\gamma \in \{0,1\}} \det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \end{array}\right] = \det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0 \cdots\cdots 0 & \\ & 0 \cdots\cdots 0 \\ & \\ & \\ & \\ & \end{array}\right] \leq \Delta_{m-2}$$

case 3: $\alpha = 1$ and $\beta = 0$

proof of $\Delta_m \le \delta^m$ (cont'd)

$$\sum_{\gamma\in\{0,1\}} \det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0\cdots\cdots 0 & \\ & 0\cdots\cdots 0 \\ & \\ & \\ & \\ & \end{array}\right] = \det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0\cdots\cdots 0 & \\ & 0\cdots\cdots 0 \\ & \\ & \\ & \\ & \end{array}\right] \le \Delta_{m-2}$$

case 3: $\alpha = 1$ and $\beta = 0$

$$\det \begin{array}{c} r_1^1 \\ r_2^1 \\ r_3^0 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0\cdots\cdots 0 & \\ 0\cdots\cdots 0 & \\ & 0\cdots\cdots 0 \\ & \\ & \\ & \end{array}\right] \le \Delta_{m-3}$$

case 4: $\alpha = \beta = 1$ and $\gamma = 0$

proof of $\Delta_m \leq \delta^m$ (cont'd)

$$\sum_{\gamma\in\{0,1\}} \det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0\cdots\cdot 0 & \\ & 0\cdots\cdot 0 \\ & \\ & \\ & \\ & \end{array}\right] = \det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0\cdots\cdot 0 & \\ & 0\cdots\cdot 0 \\ & \\ & \\ & \\ & \end{array}\right] \leq \Delta_{m-2}$$

case 3: $\alpha = 1$ and $\beta = 0$

$$\det \begin{array}{c} r_1^1 \\ r_2^1 \\ r_3^0 \\ r_4 \\ \vdots \\ r_m \end{array} \left[\begin{array}{c|c} 0\cdots\cdot 0 & \\ 0\cdots\cdot 0 & \\ & 0\cdots\cdot 0 \\ & \\ & \\ & \end{array}\right] \leq \Delta_{m-3}$$

case 4: $\alpha = \beta = 1$ and $\gamma = 0$

$$\Delta_m \leq \Delta_{m-1} + \Delta_{m-2} + \Delta_{m-3}$$

$$T_m \leq \Delta_m$$

proof sketch of $T_m \leq \Delta_m$

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices;
T_m is the maximum number of spanning trees over m-edge planar graphs.

proof sketch of $T_m \leq \Delta_m$

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices;
T_m is the maximum number of spanning trees over m-edge planar graphs.

Goal: Given an m-edge planar graph G (achieving T_m), construct an $m \times m$ bi-incidence matrix with determinant at least the number of spanning trees in G.

proof sketch of $T_m \leq \Delta_m$

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices;
T_m is the maximum number of spanning trees over m-edge planar graphs.

Goal: Given an m-edge planar graph G (achieving T_m), construct an $m \times m$ bi-incidence matrix with determinant at least the number of spanning trees in G.

definitions

A truncated incidence matrix of a digraph is the incidence matrix of the digraph with an arbitrary column removed.

proof sketch of $T_m \leq \Delta_m$

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices;
T_m is the maximum number of spanning trees over m-edge planar graphs.

Goal: Given an m-edge planar graph G (achieving T_m), construct an $m \times m$ bi-incidence matrix with determinant at least the number of spanning trees in G.

definitions

A truncated incidence matrix of a digraph is the incidence matrix of the digraph with an arbitrary column removed.

lemma (Bu, **P.** 2025)

Let G be a connected planar graph with orientation D. Let D^* be the dual of D.

proof sketch of $T_m \leq \Delta_m$

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices;
T_m is the maximum number of spanning trees over m-edge planar graphs.

Goal: Given an m-edge planar graph G (achieving T_m), construct an $m \times m$ bi-incidence matrix with determinant at least the number of spanning trees in G.

definitions

A truncated incidence matrix of a digraph is the incidence matrix of the digraph with an arbitrary column removed.

lemma (Bu, **P.** 2025)

Let G be a connected planar graph with orientation D. Let D^* be the dual of D.

Let M and M^* be truncated incidence matrices of D and D^*, of which the i^{th} rows correspond to the same arc and its dual arc for each i.

proof sketch of $T_m \leq \Delta_m$

Recall: Δ_m is the maximum determinant over $m \times m$ bi-incidence matrices;
T_m is the maximum number of spanning trees over m-edge planar graphs.

Goal: Given an m-edge planar graph G (achieving T_m), construct an $m \times m$ bi-incidence matrix with determinant at least the number of spanning trees in G.

definitions

A truncated incidence matrix of a digraph is the incidence matrix of the digraph with an arbitrary column removed.

lemma (Bu, **P.** 2025)

Let G be a connected planar graph with orientation D. Let D^* be the dual of D.

Let M and M^* be truncated incidence matrices of D and D^*, of which the i^{th} rows correspond to the same arc and its dual arc for each i.

Then $|\det[M|M^*]|$ is equal to the number of spanning trees $\tau(G)$ in G.

proof sketch of $T_m \leq \Delta_m$ (cont'd)

lemma (Bu, P. 2025)

Let G be a connected planar graph with orientation D. Let D^* be the dual of D.

Let M and M^* be truncated incidence matrices of D and D^*, of which the i^{th} rows correspond to the same arc and its dual arc for each i.

Then $|\det[M|M^*]|$ is equal to the number of spanning trees $\tau(G)$ in G.

proof sketch of $T_m \leq \Delta_m$ (cont'd)

lemma (Bu, P. 2025)

Let G be a connected planar graph with orientation D. Let D^* be the dual of D.

Let M and M^* be truncated incidence matrices of D and D^*, of which the i^{th} rows correspond to the same arc and its dual arc for each i.

Then $|\det[M|M^*]|$ is equal to the number of spanning trees $\tau(G)$ in G.

observation

If M is a truncated incidence matrix of a graph G, then $M^\mathsf{T} M$ is a first principal minor of the Laplacian of G.

proof sketch of $T_m \leq \Delta_m$ (cont'd)

lemma (Bu, **P.** 2025)

Let G be a connected planar graph with orientation D. Let D^* be the dual of D.

Let M and M^* be truncated incidence matrices of D and D^*, of which the i^{th} rows correspond to the same arc and its dual arc for each i.

Then $|\det[M|M^*]|$ is equal to the number of spanning trees $\tau(G)$ in G.

observation

If M is a truncated incidence matrix of a graph G, then $M^{\mathsf{T}}M$ is a first principal minor of the Laplacian of G.

Kirchhoff's matrix-tree theorem

The determinant of a first principle minor of the Laplacian of a graph G is equal to $\tau(G)$.

a conjecture

corollary

For all $m \in \mathbb{N}$, we have $T_m \leq \Delta_m$.

corollary

For all $m \in \mathbb{N}$, we have $T_m \leq \Delta_m$.

T_m and Δ_m for small values of m

m	1	2	3	4	5	6	7	8	9	10
T_m	1	2	3	5	8	16	24	45	75	130
Δ_m	1	2	3	5	8	16	24	45	75	130

corollary

For all $m \in \mathbb{N}$, we have $T_m \leq \Delta_m$.

T_m and Δ_m for small values of m

m	1	2	3	4	5	6	7	8	9	10
T_m	1	2	3	5	8	16	24	45	75	130
Δ_m	1	2	3	5	8	16	24	45	75	130

conjecture

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

a planarity criterion

yet another extremal problem

Given a connected graph G with m edges, how large is the maximum determinant $\Delta(G)$ over $m \times m$ matrices $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size?

yet another extremal problem

Given a connected graph G with m edges, how large is the maximum determinant $\Delta(G)$ over $m \times m$ matrices $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size?

In other words, can we achieve better determinants than the previous construction, where N is a truncated incidence matrix of the dual?

yet another extremal problem

Given a connected graph G with m edges, how large is the maximum determinant $\Delta(G)$ over $m \times m$ matrices $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size?

In other words, can we achieve better determinants than the previous construction, where N is a truncated incidence matrix of the dual?

proposition (Bu, **P.** 2025)

For any connected graph G, we have $\Delta(G) \leq \tau(G)$.

yet another extremal problem

Given a connected graph G with m edges, how large is the maximum determinant $\Delta(G)$ over $m \times m$ matrices $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size?

In other words, can we achieve better determinants than the previous construction, where N is a truncated incidence matrix of the dual?

proposition (Bu, **P.** 2025)

For any connected graph G, we have $\Delta(G) \leq \tau(G)$.

In other words, the previous construction is the best possible.

yet another extremal problem

Given a connected graph G with m edges, how large is the maximum determinant $\Delta(G)$ over $m \times m$ matrices $[M|N]$, where M is a truncated incidence matrix of G and N is an incidence submatrix of appropriate size?

In other words, can we achieve better determinants than the previous construction, where N is a truncated incidence matrix of the dual?

proposition (Bu, **P.** 2025)

For any connected graph G, we have $\Delta(G) \leq \tau(G)$.

In other words, the previous construction is the best possible.

merge-cut lemma (Bu, **P.** 2025)

For a connected $G = (V, E)$ and a non-bridge $e \in E$, $\Delta(G) \leq \Delta(G/e) + \Delta(G \setminus e)$.

definitions

Given a connected graph G, define its excess to be $\varepsilon(G) := \tau(G) - \Delta(G)$.

definitions

Given a connected graph G, define its excess to be $\varepsilon(G) := \tau(G) - \Delta(G)$.

Given a disconnected graph G, define its excess $\varepsilon(G)$ to be the sum of the excesses of its connected components.

definitions

Given a connected graph G, define its excess to be $\varepsilon(G) := \tau(G) - \Delta(G)$.

Given a disconnected graph G, define its excess $\varepsilon(G)$ to be the sum of the excesses of its connected components.

For **planar** graphs, we have seen that the previous construction is the best possible.

corollary (Bu, **P.** 2025)

For any planar graph G, we have $\varepsilon(G) = 0$.

definitions

Given a connected graph G, define its excess to be $\varepsilon(G) := \tau(G) - \Delta(G)$.

Given a disconnected graph G, define its excess $\varepsilon(G)$ to be the sum of the excesses of its connected components.

For **planar** graphs, we have seen that the previous construction is the best possible.

corollary (Bu, **P.** 2025)

For any planar graph G, we have $\varepsilon(G) = 0$.

For **non-planar** graphs, the excesses are at least 18.

lemma (Bu, **P.** 2025)

For any non-planar graph G, we have $\varepsilon(G) \geq 18$.

definitions

Given a connected graph G, define its excess to be $\varepsilon(G) := \tau(G) - \Delta(G)$.

Given a disconnected graph G, define its excess $\varepsilon(G)$ to be the sum of the excesses of its connected components.

For **planar** graphs, we have seen that the previous construction is the best possible.

corollary (Bu, **P.** 2025)

For any planar graph G, we have $\varepsilon(G) = 0$.

For **non-planar** graphs, the excesses are at least 18.

lemma (Bu, **P.** 2025)

For any non-planar graph G, we have $\varepsilon(G) \geq 18$.

The lower bound 18 is tight and achieved by $K_{3,3}$.

towards the conjecture

conjecture

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

conjecture

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

another conjecture

If G is a connected non-planar graph with m edges, then $\Delta(G) \leq T_m$.

conjecture

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

another conjecture

If G is a connected non-planar graph with m edges, then $\Delta(G) \leq T_m$.

In other words, the maximum $\Delta(\cdot)$ value over graphs with the same number of edges is always achieved by planar graphs.

conjecture

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

another conjecture

If G is a connected non-planar graph with m edges, then $\Delta(G) \leq T_m$.

In other words, the maximum $\Delta(\cdot)$ value over graphs with the same number of edges is always achieved by planar graphs.

In other words, non-planar graphs are "dominated" by the best planar graph with the same number of edges in some linear-algebraic sense.

conjecture

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

another conjecture

If G is a connected non-planar graph with m edges, then $\Delta(G) \le T_m$.

In other words, the maximum $\Delta(\cdot)$ value over graphs with the same number of edges is always achieved by planar graphs.

In other words, non-planar graphs are "dominated" by the best planar graph with the same number of edges in some linear-algebraic sense.

These two conjectures are equivalent.

conjecture

For all $m \in \mathbb{N}$, we have $T_m = \Delta_m$.

another conjecture

If G is a connected non-planar graph with m edges, then $\Delta(G) \leq T_m$.

In other words, the maximum $\Delta(\cdot)$ value over graphs with the same number of edges is always achieved by planar graphs.

In other words, non-planar graphs are "dominated" by the best planar graph with the same number of edges in some linear-algebraic sense.

These two conjectures are equivalent.

theorem (Bu, P. 2025)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\Delta(G) \leq T_m$.

Thanks!