PLANARITY VIA SPANNING TREE NUMBER: A LINEAR-ALGEBRAIC CRITERION[*]

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Abstract. We introduce a novel linear-algebraic planarity criterion based on the number of spanning trees. We call a matrix an *incidence submatrix* if each row has at most one 1, at most one -1, and all other entries zero. Given a connected graph G with m edges, we consider the maximum determinant $\mathsf{maxdet}(G)$ of an $m \times m$ matrix $[M|N]$, where M is the incidence matrix of G with one column removed, over all incidence submatrices N of appropriate size, and define its *excess* to be the number of spanning trees in G minus $\mathsf{maxdet}(G)$. Given a disconnected graph, we define its *excess* to be the sum of the excesses of its connected components. We show that the excess of a graph is 0 if it is planar, and at least 18 otherwise. This provides a "certificate of planarity" of a planar graph that can be verified by computing the determinant of a sparse matrix and counting spanning trees. Furthermore, we derive an upper bound on the maximum determinant of an $m \times m$ matrix $[M|N]$, where M and N are incidence submatrices. Motivated by this bound and numerical evidence, we conjecture that this maximum determinant is equal to the maximum number of spanning trees in a planar graph with m edges. We present partial progress towards this conjecture. In particular, we prove that the $\mathsf{maxdet}(\cdot)$ value of any subdivision of $K_{3,3}$ or K_5 is at most that of the best planar graph with the same number of edges.

Key words. planar graphs, planarity criteria, spanning trees, incidence matrices, counting

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1. Introduction.

There has been a series of works studying characterizations of planar graphs, including several algebraic ones; see, e.g., [10, 20, 3, 14, 19, 18, 4, 16, 5]. In this paper, we provide a new one based on linear algebra and the number of spanning trees, whose flavor is quite distinct from those of existing ones. In particular, our characterization allows one to give a "certificate of planarity" for any planar graph that can be easily verified by computing the determinant of a sparse matrix and counting spanning trees.

For clarity, we briefly define the key terms here; a more detailed treatment of the definitions can be found in Section 2. Unless specified otherwise, we allow graphs to have multiple edges between two vertices, as well as loop edges. Given a graph G, we let $\tau(G)$ denote the number of spanning trees in G, and define a *truncated incidence matrix* of G to be a matrix obtained by removing an arbitrary column from an (oriented) incidence matrix of G. We say that a matrix is an *incidence submatrix* if each row has at most one 1 and at most one -1, with all other entries zero. Given a connected graph G with n vertices and m edges (so that $m \geq n-1$), we let $\mathsf{maxdet}(G)$ denote the maximum determinant of an $m \times m$ matrix $[M|N]$, over all truncated incidence matrices M of G and over all incidence submatrices N of appropriate size. Given a connected graph G, we define $\varepsilon(G) := \tau(G) - \mathsf{maxdet}(G)$, called the *excess* of G. Given a disconnected graph G, we define its *excess*, denoted by $\varepsilon(G)$, to be the sum of the excesses of its connected components.

First, we show that the excess of a graph is always nonnegative, justifying the name "excess."

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PROPOSITION 1.1. *For any graph G, we have $\varepsilon(G) \geq 0$.*

Our contributions in this paper are threefold.

A new planarity criterion. Our main result is the following characterization of planar graphs, demonstrating a dichotomy between planar and nonplanar graphs in terms of their excesses.

THEOREM 1.2. *Let G be a graph. Then*

$$\varepsilon(G) \begin{cases} = 0, & \text{if } G \text{ is planar,} \\ \geq 18, & \text{otherwise.} \end{cases}$$

We remark that the lower bound of 18 on the excess of a nonplanar graph is tight; it is achieved by $K_{3,3}$.

Subdivisions of $K_{3,3}$ and K_5. One motivation of the $\mathsf{maxdet}(\cdot)$ function is as follows. Although the maximum number of spanning trees in a nonplanar graph with a fixed number of edges can be much greater than that in any planar graph with the same number of edges, we conjecture that the maximum $\mathsf{maxdet}(\cdot)$ value over a graph with m edges is always achieved by a planar graph, i.e., that the best planar graph "dominates" all nonplanar graphs in this linear-algebraic sense. For any $m \in \mathbb{N}$, we let τ_m denote the maximum number of spanning trees in a planar graph with m edges; this is also the maximum $\mathsf{maxdet}(\cdot)$ value of a planar graph with m edges by Theorem 1.2.

CONJECTURE 1.3. *If G is a nonplanar graph with m edges, then $\mathsf{maxdet}(G) \leq \tau_m$.*

Conjecture 1.3 is equivalent to Conjecture 1.6 which we discuss below. Our second result gives partial progress towards this conjecture.

THEOREM 1.4. *If G is a subdivision of $K_{3,3}$ or K_5 with m edges, then we have $\mathsf{maxdet}(G) \leq \tau_m$.*

Upper bounding Δ_m and τ_m. We say that a matrix is a *bi-incidence matrix* if it is the concatenation $[M|N]$ of two incidence submatrices M and N. To prove Theorem 1.2, for any connected planar graph G with m edges, we give a construction of an $m \times m$ bi-incidence matrix P such that $|\det(P)| = \tau(G)$. This construction implies τ_m is at most the maximum determinant, Δ_m, of an $m \times m$ bi-incidence matrix. By an elementary linear-algebraic argument using multilinearity of determinants and the pigeonhole principle, we show that $\Delta_m \leq \delta^m$ for all $m \in \mathbb{N}$, where $\delta \simeq 1.8393$ is the unique real root of the equation $x^3 - x^2 - x - 1 = 0$. This gives an upper bound on τ_m, which matches the current best upper bound of Stoimenow [17] who used a knot-theoretic argument. Summarizing, we have the following theorem.

THEOREM 1.5. *For all $m \in \mathbb{N}$, we have $\tau_m \leq \Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of the equation $x^3 - x^2 - x - 1 = 0$.*

We remark that an asymptotic lower bound $\exp((2G/\pi - o(1)) \cdot m) \geq 1.7916^m$ on τ_m is known (see, e.g., [21, 7]), where $G = \sum_{k=0}^{\infty} (-1)^k/(2k+1)^2 \simeq 0.9160$ is Catalan's constant. This lower bound is achieved by square grid graphs.

Theorem 1.4, together with computations of τ_m and Δ_m for small values of m (see Table 1), motivates us to propose the following conjecture, which is equivalent to Conjecture 1.3.

CONJECTURE 1.6. *For all $m \in \mathbb{N}$, we have $\tau_m = \Delta_m$.*

Table 1: Values of τ_m and Δ_m for $m = 1, \ldots, 10$.

m	1	2	3	4	5	6	7	8	9	10
τ_m	1	2	3	5	8	16	24	45	75	130
Δ_m	1	2	3	5	8	16	24	45	75	130

1.1. Outline. The paper is organized as follows. In Section 2, we introduce definitions and conventions, with results regarding operations that preserve the bi-incidence property of a matrix and the absolute value of its determinant (when the matrix is square). In Section 3, we introduce a simple yet powerful lemma called the merge-cut lemma, and prove Proposition 1.1. In Section 4, we prove Theorem 1.2. In Section 5, we prove Theorem 1.4. In Section 6, we prove Theorem 1.5. In Section 7, we give concluding remarks with future directions.

2. Preliminaries. We start with standard definitions from linear algebra and graph theory to ensure consistency; the reader familiar with these standard definitions can skip to Definition 2.8. Unless specified otherwise, we allow multiple edges between two vertices in a graph, as well as loop edges.

Definition 2.1. *Given an $m \times n$ matrix $M = (a_{i,j})$ and an $m \times k$ matrix $N = (b_{i,j})$, we define their* concatenation *to be an $m \times (n+k)$ matrix, denoted by $[M|N] = (c_{i,j})$, where*

$$c_{i,j} := \begin{cases} a_{i,j}, & \text{if } j \leq n, \\ b_{i,j-n}, & \text{otherwise}, \end{cases}$$

for all $i \in [m]$ and $j \in [n+k]$.

Definition 2.2. *Given a graph $G = (V, E)$ and $e = uv \in E$, we let G/e denote the graph obtained by* contracting *e in G, i.e., by the following procedure:*
(a) remove e from G;
(b) replace u and v with a new vertex w;
(c) for $e' \in E \setminus \{e\}$, if an endpoint of e' is u or v, replace that endpoint with w.
Note it is possible that G/e contains parallel edges between two vertices or loop edges.

Given a graph $G = (V, E)$ and $e \in E$, we let $G \setminus e$ denote the graph obtained by removing e from G.

Definition 2.3. *We define* subdivision *to be a graph operation that creates a new vertex w and replaces an edge uv with edges uw and vw. We say that a graph obtained from repeated subdivision operations on a graph G is a* subdivision *of G. We say that the vertices created in subdivision operations are* internal *vertices.*

Definition 2.4. *Given a directed graph $D = (V, A)$, define its* incidence matrix *to be an $A \times V$ matrix, denoted by $\iota_D = (a_{e,v})$, where*

$$a_{e,v} := \begin{cases} 1, & \text{if } e \text{ is not a loop arc and } e \text{ enters } v, \\ -1, & \text{if } e \text{ is not a loop arc and } e \text{ leaves } v, \\ 0, & \text{otherwise}, \end{cases}$$

for all $e \in A$ and $v \in V$. We say that a matrix is an incidence matrix *if it is the incidence matrix of some directed graph.*

Definition 2.5. *A* straight line segment *is a subset of $\mathbb{R}^2$ which has the form $\{p + \lambda(q - p) : \lambda \in [0, 1]\}$ for distinct points $p, q \in \mathbb{R}^2$.*

A polygon *is a subset of* $\mathbb{R}^2$ *which is the union of finitely many straight line segments and which is homeomorphic to the unit circle* S^1*.*

A polygonal arc *is a subset of* $\mathbb{R}^2$ *which is the union of finitely many straight line segments and which is homeomorphic to the closed unit interval* $[0, 1]$.

Definition 2.6. *A* plane graph *is a pair* (V, E) *of finite sets, where the elements of* V *are called* vertices *and the elements of* E *are called* edges*, such that*

(a) $V \subseteq \mathbb{R}^2$*;*

(b) every edge is a polygonal arc between two vertices, or a polygon containing exactly one vertex, where these vertices are called the endpoints *of the edge;*

(c) apart from its own endpoint(s), an edge contains no vertex and no point of any other edge.

A plane directed graph *is a plane graph* (V, E) *in which each* arc $e \in E$ *with two endpoints is associated with an orientation from one of the endpoints (called the* tail *of* e*) to the other (called the* head *of* e*).*

Given a plane graph or a plane directed graph G*, the regions of* $\mathbb{R}^2 \setminus G$ *are called the* faces *of* G*.*

Definition 2.7. *Given a connected plane directed graph* D*, construct its* directed plane dual graph D^* *as follows. The vertices of* D^* *are the faces of* D*. For each arc* e *in* D*, we define its* dual arc *in* D^* *as follows: as* e *is traversed from its tail to its head, one of these two (possibly identical) faces separated by* e *lies to the left of* e*, which we denote by* ℓ_e*, and the other to its right, which we denote by* r_e*; the* dual arc *of* e *in* D^* *is defined to be an arc whose tail is vertex* r_e *in* D^* *and whose head is vertex* ℓ_e *in* D^**. The arcs of* D^* *are the dual arcs of the arcs in* D*. The underlying undirected (plane) graphs of* D *and* D^* *are called the* plane dual graphs *of each other.*

Given two connected planar graphs G *and* H*, we say that* G *is a* planar dual graph *of* H *if there exist plane graphs* G' *and* H' *isomorphic to* G *and* H*, respectively, such that* G' *and* H' *are the plane dual graphs of each other.*

We define the following notions on plane graphs that simplify our exposition.

Definition 2.8. *Let* $G = (V, E)$ *be a plane graph. Let* $v \in V$*. Let* D *be an open disc on* $\mathbb{R}^2$ *around* v *with a sufficiently small radius so that it only intersects with straight edge segments of* G *that contain* v*. Enumerate these segments according to their cyclic position counterclockwise in* D *as* $s_1, \ldots, s_k$*, and let* e_i *be the edge of* G *containing* s_i *for each* $i \in [k]$*. Then* $e_1, \ldots, e_k$ *are said to form a* cyclic ordering *of the edges incident to* v *in* G*.*

Let C *be the boundary of* D*, which is a circle around* v *of the same radius. Suppose that* $e_1, \ldots, e_k$ *intersect* C *at* $p_1, \ldots, p_k$*, respectively. For each two cyclically consecutive edges* e_i *and* e_{i+1} *in* $e_1, \ldots, e_k$ *(with subscripts modulo* k*), let* S *be the open sector bounded by two radii from* v *to* p_i *and* p_{i+1}*, respectively, and the arc of* C *between* p_i *and* p_{i+1}*, and we say that the face* F *of* G *containing* S *is the face* associated with *the two consecutive edges* e_i *and* e_{i+1}*.*

These two definitions naturally extend to plane directed graphs.

Next, we define truncated incidence matrices and incidence submatrices.

Definition 2.9. *We define a* truncated incidence matrix *of a directed graph* D*, denoted by* $\tilde{\iota}_D$*, to be a matrix obtained by removing an arbitrary column from* ι_D*. We define an* incidence matrix *(resp.* truncated incidence matrix*) of an undirected graph to be an incidence matrix (resp. truncated incidence matrix) of an orientation of the graph. We say that a matrix is an* incidence submatrix *if each row has at most one* 1 *and at most one* -1*, with all other entries zero.*

The following proposition is easy to see, by appending to M a vector such that the sum of the columns of the resulting matrix is the all-zero vector.

PROPOSITION 2.10. *For each incidence submatrix M, there exists a unique loopless directed graph D such that $\tilde{\iota}_D$ is equal to M up to all-zero rows.*

Now, we are ready to define a central concept in this paper, the notion of bi-incidence matrices.

DEFINITION 2.11. *We say that a matrix P is a* bi-incidence matrix *if P is the concatenation $[M|N]$ of two incidence submatrices M and N, which we call the* left *and* right sides *of P, respectively. (Sometimes, we also refer to the left and right sides of a matrix that is not necessarily a bi-incidence matrix when they are clear from the context, e.g., when the matrix is obtained from some transformation on a bi-incidence matrix.)*

We define the $\mathsf{maxdet}(\cdot)$ function over connected graphs.

DEFINITION 2.12. *Given a connected graph G, we use $\mathsf{maxdet}(G)$ to denote the maximum determinant of a square concatenation $[M|N]$ over all truncated incidence matrices M of G and over all incidence submatrices N of appropriate size.*

Now, we define the spanning tree number and the excess of a graph.

DEFINITION 2.13. *Given a graph G, we let $\tau(G)$ denote the number of spanning trees in G. Given a connected graph G, we define the* excess *of G to be $\varepsilon(G) := \tau(G) - \mathsf{maxdet}(G)$. Given a disconnected graph G, we define its* excess*, denoted by $\varepsilon(G)$, to be the sum of the excesses of its connected components.*

Finally, we define two sequences $(\tau_m)_{m=1}^{\infty}$ and $(\Delta_m)_{m=1}^{\infty}$ for the two extremal problems considered in this paper.

DEFINITION 2.14. *For all $m \in \mathbb{N}$, we let τ_m denote the maximum number of spanning trees in a planar graph with m edges, and let Δ_m denote the maximum determinant of an $m \times m$ bi-incidence matrix.*

Since flipping the signs of all entries in a row changes the sign of the determinant, the minimum determinant of an $m \times m$ bi-incidence matrix is $-\Delta_m$. Hence, $\Delta_m \geq 0$.

2.1. Operations Preserving Bi-incidence and Determinants. We provide results on operations that preserve the bi-incidence property of a matrix and the absolute value of its determinant (when the matrix is square). We start with a list of operations that preserve the bi-incidence property of a matrix.

PROPOSITION 2.15. *Let M be a bi-incidence matrix. The following operations on M result in a bi-incidence matrix:*
- *(a) removing a column from M;*
- *(b) removing a row from M;*
- *(c) (*combination*) replacing two columns from the same side of M by their sum;*
- *(d) swapping two columns from the same side of M;*
- *(e) swapping two rows of M;*
- *(f) swapping the left and right sides of M;*
- *(g) (*realignment*) replacing a column by -1 times the sum of all columns from its side in M.*

In particular, the last four operations do not change the size of M and preserve the absolute value of the determinant of M when M is square.

The proof of Proposition 2.15 immediately follows from the definition of bi-incidence matrices.

The realignment operation allows us to work with a fixed orientation D of G with a fixed truncated vertex in $\tilde{\iota}_D$ in the definition of $\mathsf{maxdet}(\cdot)$. We formally state this fact in the next proposition.

Proposition 2.16. *Let $G = (V, E)$ be a connected graph. Let D be an arbitrary orientation of G. Let $v \in V$. Let M be the truncated incidence matrix of D in which the column corresponding to v is removed. Then $\mathsf{maxdet}(G)$ is the maximum absolute value of the determinant of a square concatenation $[M|N]$ over all incidence submatrices N of appropriate size.*

Proof. Applying the realignment operation (which preserves the absolute value of the determinant) described in Proposition 2.15 to the left side of $[M|N]$ changes the choice of the truncated vertex in a truncated incidence matrix, because the sum of the column vectors of the original incidence matrix is zero. ∎

Now, we provide an operation that eliminates a row *whose restriction to one side of a square bi-incidence matrix* has zeros only, while preserving the bi-incidence property of the matrix and not decreasing the absolute value of its determinant.

Proposition 2.17. *Let $[M|N]$ be a square bi-incidence matrix with left and right sides M and N, respectively, such that N has at least one column. Suppose that there exists a row r of $[M|N]$ whose restriction to the left side has zeros only.*

- *(a) If the restriction of row r to the right side has zeros only, then removing row r and any column on the right side of $[M|N]$ results in a square bi-incidence matrix P with $\det([M|N]) = 0 \le |\det(P)|$.*
- *(b) If row r has exactly one nonzero entry in column c, then removing row r and column c from $[M|N]$ results in a square bi-incidence matrix P with $|\det([M|N])| = |\det(P)|$.*
- *(c) If row r has exactly two nonzero entries in columns c_1 and c_2, respectively, then removing row r and replacing columns c_1 and c_2 with $c_1 + c_2$ (i.e., combining c_1 and c_2) in $[M|N]$ results in a square bi-incidence matrix P with $|\det([M|N])| = |\det(P)|$.*

Proof. The first case is trivial. The second case follows from the expansion of the determinant along row r. For the third case, since the (r, c_1)-entry and the (r, c_2)-entry are exactly one 1 and one -1, adding column c_1 to c_2 in $[M|N]$ results in a square matrix P_0 that has exactly one nonzero entry in row r, such that $|\det([M|N])| = |\det(P_0)|$. The lemma follows from the expansion of the determinant along row r. ∎

Proposition 2.17 can be repeatedly applied to eliminate all rows whose restrictions to the left side of a square bi-incidence matrix have zeros only, while preserving the bi-incidence property of the matrix and not decreasing the absolute value of its determinant.

Corollary 2.18. *Let $[M|N]$ be a square bi-incidence matrix with left and right sides M and N, respectively. Suppose that there exists a set R of rows of $[M|N]$ whose restrictions to the left side have zeros only. Let M_0 be the matrix obtained by removing rows in R from M. Then either $\det([M|N]) = 0$, or there exists a matrix N', obtained by removing rows in R from N and a sequence of $|R|$ removal and combination operations on columns of N, such that $|\det([M|N])| = |\det([M_0|N'])|$.*

Proof. Suppose that $[M|N]$ is $m \times m$ and that N has ℓ columns. If $|R| \le \ell$, then repeatedly applying Proposition 2.17 for $|R|$ times proves the lemma. Otherwise, repeatedly applying Proposition 2.17 for ℓ times results in a bi-incidence matrix P with an all-zero row such that $|\det([M|N])| \leq |\det(P)| = 0$, so $\det([M|N]) = 0$. ◻

By swapping the left and right sides in the concatenation, which does not change the absolute value of the determinant, we obtain the following analogous corollary.

COROLLARY 2.19. *Let $[M|N]$ be a square bi-incidence matrix with left and right sides M and N, respectively. Suppose that there exists a set R of rows of $[M|N]$ whose restrictions to the right side have zeros only. Let N_0 be the matrix obtained by removing rows in R from N. Then either $\det([M|N]) = 0$, or there exists a matrix M', obtained by removing rows in R from M and a sequence of $|R|$ removal and combination operations on columns of M, such that $|\det([M|N])| = |\det([M'|N_0])|$.*

3. Merge-Cut Lemma.

In this section, we introduce a simple yet powerful lemma, which we call the *merge-cut lemma*.

LEMMA 3.1 (merge-cut lemma). *For any connected graph $G = (V, E)$ and any non-loop edge $e \in E$ with $G \setminus e$ connected, we have $\mathsf{maxdet}(G) \leq \mathsf{maxdet}(G/e) + \mathsf{maxdet}(G \setminus e)$.*

The merge-cut lemma is reminiscent of the deletion-contraction relation for the number of spanning trees.

PROPOSITION 3.2. *For any graph $G = (V, E)$ and any non-loop edge $e \in E$, we have $\tau(G) = \tau(G/e) + \tau(G \setminus e)$.*

Combining Lemma 3.1 and Proposition 3.2, we obtain the following alternative form of the merge-cut lemma in terms of the excess of a graph.

COROLLARY 3.3. *For any connected graph $G = (V, E)$ and any non-loop edge $e \in E$ with $G \setminus e$ connected, we have $\varepsilon(G) \geq \varepsilon(G/e) + \varepsilon(G \setminus e)$.*

Now, we prove the merge-cut lemma.

Proof of Lemma 3.1. Let $G = (V, E)$ be a connected graph with m edges, and let $e \in E$ be such that $G \setminus e$ is connected. Let $P = [\tilde{\iota}_D\,|M]$ attain $\mathsf{maxdet}(G)$ for some fixed orientation D of G. Then $\mathsf{maxdet}(G) = \det(P)$. Let $r = (r_1, \ldots, r_m) \in \mathbb{R}^m$ be the row of P corresponding to e. Let $r^L = (r_1^L, \ldots, r_m^L), r^R = (r_1^R, \ldots, r_m^R) \in \mathbb{R}^m$ be defined by

$$r_j^L := \begin{cases} r_j, & \text{if } j \leq n-1, \\ 0, & \text{otherwise,} \end{cases} \qquad r_j^R := \begin{cases} 0, & \text{if } j \leq n-1, \\ r_j, & \text{otherwise,} \end{cases}$$

for all $j \in [m]$. Then $r = r^L + r^R$. Let P^L and P^R be the matrices obtained by replacing row r with r^L and with r^R, respectively, in P. By multilinearity of determinants, we have $\det(P) = \det(P^L) + \det(P^R)$.

First, we show that $\det(P^R) \leq \mathsf{maxdet}(G \setminus e)$. Without loss of generality, we assume that $\det(P^R) \neq 0$. Let L_0 be the matrix obtained by removing row r from $\tilde{\iota}_D$. By Corollary 2.18, there exists a matrix M', obtained by removing row r from M followed by combining two columns or removing one column, such that $|\det(P^R)| = |\det([L_0|M'])|$. It is easy to see that L_0 is a truncated incidence matrix of $G \setminus e$. Hence, $\det(P^R) \leq \mathsf{maxdet}(G \setminus e)$.

Second, we show that $\det(P^L) \leq \mathsf{maxdet}(G/e)$. Without loss of generality, we assume that $\det(P^L) \neq 0$. Let M_0 be the matrix obtained by removing row r from M. By Corollary 2.19, there exists a matrix L', obtained by removing row r from $\tilde{\iota}_D$ followed by combining two columns or removing one column, such that $|\det(P^L)| = |\det([L'|M_0])|$. We have the following two cases.

- **Case 1: one column is removed from $\tilde{\iota}_D$ to obtain L'.** Then the two endpoints of e correspond to the truncated column and the removed column. It is easy to see that L' is a truncated incidence matrix of G/e. Hence, $\det(P^L) \le \mathsf{maxdet}(G/e)$.
- **Case 2: two columns are combined in $\tilde{\iota}_D$ to obtain L'.** Then the two endpoints of e correspond to the two combined columns. It is easy to see that L' is a truncated incidence matrix of G/e. Hence, $\det(P^L) \le \mathsf{maxdet}(G/e)$.

This completes the proof. ∎

A direct application of the merge-cut lemma is Proposition 1.1.

Proof of Proposition 1.1. It suffices to prove the proposition for the case where G is connected. We proceed by induction on the number of edges in G. The base case is trivial. For the induction step, we have the following two cases. If G is a tree, then $\mathsf{maxdet}(G) = \tau(G) = 1$. Otherwise, G has a non-bridge non-loop edge e; i.e., $G \setminus e$ is connected. By the merge-cut lemma and by Proposition 3.2,

$$\mathsf{maxdet}(G) \le \mathsf{maxdet}(G/e) + \mathsf{maxdet}(G \setminus e) \le \tau(G/e) + \tau(G \setminus e) = \tau(G),$$

where the second inequality follows from the inductive hypotheses on G/e and $G \setminus e$, both of which have one edge fewer than G. This completes the proof. ∎

4. Planarity Criterion via Excess.

In this section, we prove Theorem 1.2.

4.1. Zero Excess of a Planar Graph.

Given a connected planar graph G with m edges, to prove $\mathsf{maxdet}(G) = \tau(G)$, we give a construction of an $m \times m$ bi-incidence matrix M with $|\det(M)| = \tau(G)$ whose left side is a truncated incidence matrix of G.

We remark that Maurer [15] showed, given a connected graph, the concatenation of a matrix whose columns form a cycle basis and a matrix whose columns form a cocycle basis has determinant equal to the number of spanning trees, up to the sign. Furthermore, in [15], this result was generalized to any matroid. Combined with Mac Lane's planarity criterion [14], this implies Lemma 4.1 below. We give a self-contained proof for completeness.

LEMMA 4.1. *Let G be a connected plane graph. Let D be an orientation of G. Let D^* be the directed plane dual graph of D. Suppose that, for each i, the i^{th} rows of $\tilde{\iota}_D$ and of $\tilde{\iota}_{D^*}$, respectively, correspond to the same arc in D (and its dual arc). Then $|\det([\tilde{\iota}_D \,|\, \tilde{\iota}_{D^*}])| = \tau(G)$.*

Proof. Without loss of generality, we assume that D is loopless. To justify the assumption, suppose that D has a loop arc e. Then the dual arc e^* of e is a cut arc in D^*. Hence, the row in $\tilde{\iota}_D$ corresponding to e is all-zero. By Proposition 2.17, removing from $[\tilde{\iota}_D \,|\, \tilde{\iota}_{D^*}]$ the row corresponding to e and removing or combining certain columns in $\tilde{\iota}_{D^*}$ results in a matrix M with $|\det(M)| = |\det([\tilde{\iota}_D \,|\, \tilde{\iota}_{D^*}])|$. Furthermore, observe that $M = [\tilde{\iota}_{D_1} \,|\, \tilde{\iota}_{D_1^*}]$, where D_1 is the plane directed graph obtained by removing e from D and D_1^* is the plane dual graph of D_1. This justifies our assumption.

By Euler's polyhedral formula and by the connectedness of G, $[\tilde{\iota}_D \,|\, \tilde{\iota}_{D^*}]$ is $m \times m$, where m is the number of edges in G. By Kirchhoff's matrix-tree theorem, we have $\det(\tilde{\iota}_D^{\mathsf{T}}\, \tilde{\iota}_D) = \tau(G)$ and $\det(\tilde{\iota}_{D^*}^{\mathsf{T}}\, \tilde{\iota}_{D^*}) = \tau(G^*)$, where G^* is the underlying undirected graph of D^*. Since G is planar and connected, we have $\tau(G) = \tau(G^*)$.

We show that $\tilde{\iota}_D^{\mathsf{T}}\, \tilde{\iota}_{D^*} = 0$. Let c be a column vector of $\tilde{\iota}_D$ and c' a column vector of $\tilde{\iota}_{D^*}$. Let v be the vertex in D corresponding to c, and f the face of D corresponding to c'. If there is no arc of D incident to both v and f, then $c^{\mathsf{T}} c' = 0$. Otherwise, let $e_1, \ldots, e_k$ form a cyclic ordering of the arcs incident to v in D. We re-orient $e_1, \ldots, e_k$

so that they point towards v. This is possible because $e_1, \ldots, e_k$ are non-loop and changing the orientation of an arc e in D also changes the orientation of its dual arc e^* in D^*. In other words, the signs of the components of c and c' corresponding to e and e^*, respectively, are both flipped, so $c^\mathsf{T} c'$ is preserved. Hence, the component of c corresponding to each arc incident to v is 1, so $c^\mathsf{T} c'$ is equal to the sum of the components of c' corresponding to the dual arcs of $e_1, \ldots, e_k$.

Now, every two cyclically consecutive arcs of $e_1, \ldots, e_k$ are associated with a face, and some of these faces are f. Since $e_1, \ldots, e_k$ point towards v, each pair of two consecutive arcs associated with f has one of the two dual arcs entering f and the other leaving f. Hence, the sum of the components of c' corresponding to the dual arcs of $e_1, \ldots, e_k$ is 0, so $c^\mathsf{T} c' = 0$. We illustrate this argument in Figure 1.

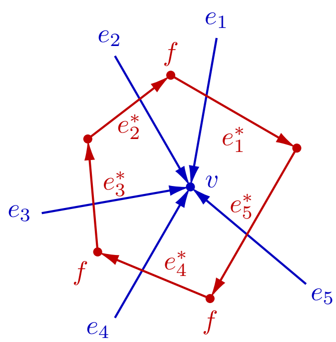


Fig. 1: An illustration of the argument that $c^\mathsf{T} c' = 0$ in the proof of Lemma 4.1, where c is a column vector of $\tilde{\iota}_D$ and c' is a column vector of $\tilde{\iota}_{D^*}$. In this instance, vector c corresponds to vertex v, and vector c' corresponds to face f. After re-orientation, arcs $e_1, \ldots, e_5$ point towards v and form a cyclic ordering of the arcs incident to v in D. Each pair of two cyclically consecutive arcs, namely $(e_1, e_2), \ldots, (e_4, e_5), (e_5, e_1)$, is associated with a face, among which the ones associated with $(e_1, e_2), (e_3, e_4), (e_4, e_5)$ are f. The components of c corresponding to $e_1, \ldots, e_5$ are $1, \ldots, 1$, respectively, and the components of c' corresponding to the dual arcs $e_1^*, \ldots, e_5^*$ of $e_1, \ldots, e_5$ are $-1, 1, -1, 0, 1$, respectively, so $c^\mathsf{T} c' = 0$. Note that e_4^* is a loop arc at f in D^*.

This proves that $\tilde{\iota}_D^\mathsf{T} \, \tilde{\iota}_{D^*} = 0$. Hence,

$$\begin{aligned}\left(\det \left[\ \tilde{\iota}_D \mid \tilde{\iota}_{D^*}\ \right]\right)^2 &= \det\left(\left[\ \tilde{\iota}_D \mid \tilde{\iota}_{D^*}\ \right]^\mathsf{T} \left[\ \tilde{\iota}_D \mid \tilde{\iota}_{D^*}\ \right]\right) = \det \begin{bmatrix} \tilde{\iota}_D^\mathsf{T} \, \tilde{\iota}_D & 0 \\ 0 & \tilde{\iota}_{D^*}^\mathsf{T} \, \tilde{\iota}_{D^*} \end{bmatrix} \\ &= \det\left(\tilde{\iota}_D^\mathsf{T} \, \tilde{\iota}_D\right) \cdot \det\left(\tilde{\iota}_{D^*}^\mathsf{T} \, \tilde{\iota}_{D^*}\right) = \tau(G) \cdot \tau\left(G^*\right) = \tau(G)^2.\end{aligned}$$

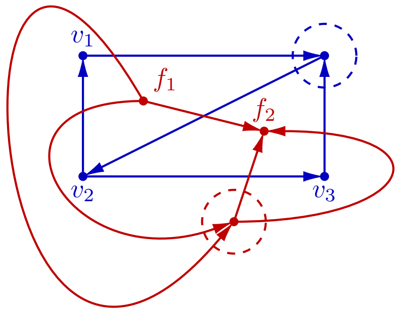
This completes the proof. ∎

Figure 2 gives an example of a connected plane graph with 5 edges and a 5×5 bi-incidence matrix illustrating Lemma 4.1.

Lemma 4.1 gives a construction that achieves the upper bound $\tau(G)$ on $\mathsf{maxdet}(G)$ from Proposition 1.1 for any connected planar graph G. It implies the following two corollaries.

Corollary 4.2. *For any planar graph G, we have $\varepsilon(G) = 0$.*

Corollary 4.3. *For all $m \in \mathbb{N}$, we have $\tau_m \le \Delta_m$.*



$$\begin{array}{c} \begin{array}{ccccc} v_1 & v_2 & v_3 & f_1 & f_2 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 & -1 & 0 \\ 1 & -1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 & 1 \\ 0 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 1 \end{bmatrix} \end{array}$$

Fig. 2: The graph on the left is a connected plane digraph (in blue) and its directed plane dual graph (in red), each with 5 edges, where the two circled vertices are the ones truncated in their truncated incidence matrices, respectively. The matrix on the right is a 5×5 matrix whose determinant has absolute value equal to the number of spanning trees in the underlying undirected graph, where each column corresponds to the vertex or face with the associated label.

4.2. Positive Excess of a Nonplanar Graph. Given a nonplanar graph, we apply the merge-cut lemma again to derive a positive lower bound on its excess.

LEMMA 4.4. *For any nonplanar graph G, we have $\varepsilon(G) \geq 18$.*

Proof. It suffices to prove the lemma for the case where G is connected. First, it can be computed that $\varepsilon(K_{3,3}) = 18$ and $\varepsilon(K_5) = 25$. (It is not practical to use the naïve brute-force algorithm to compute these two quantities within reasonable time. Our proofs are based on case work. We defer the details to Appendix A.) By Proposition 1.1, the excess of any graph is nonnegative. Let G be a nonplanar graph. The merge-cut lemma implies that, for any non-bridge non-loop edge e of G,

$$\varepsilon(G) \geq \varepsilon(G/e) + \varepsilon(G \setminus e) \geq \max\{\varepsilon(G/e), \varepsilon(G \setminus e)\}. \tag{4.1}$$

Furthermore, Corollary 2.18 implies that, for any loop edge e of G,

$$\varepsilon(G) \geq \varepsilon(G \setminus e). \tag{4.2}$$

By Wagner's theorem [19], one can obtain either $K_{3,3}$ or K_5 by a sequence of edge contractions (on non-loop edges) and edge deletions, in which every intermediate graph is connected. Hence, applying (4.1) and (4.2) inductively gives that

$$\varepsilon(G) \geq \min\{\varepsilon(K_{3,3}), \varepsilon(K_5)\} = 18.$$

This completes the proof. $\square$

Lemma 4.4 and Corollary 4.2 together prove Theorem 1.2. This illustrates a dichotomy between planar and nonplanar graphs in terms of the excess, offering a new characterization of planarity from perspectives of linear algebra and spanning trees. In addition, our characterization allows one to give a "certificate of planarity" for a planar graph that can be easily verified by computing the determinant of a sparse matrix (an $m \times m$ matrix with at most $4m$ nonzero entries, where m is the number of edges in the graph) and counting spanning trees, which can be done by computing another determinant.

Lemma 4.4 also shows that one cannot find a construction for nonplanar graphs that is similar to the one given in Lemma 4.1. However, it does not rule out the possibility that a nonplanar graph has a large number of spanning trees with a positive but small excess, resulting in a larger determinant than the maximum determinant from planar graphs. In the next section, we rule out this possibility for subdivisions of $K_{3,3}$ and K_5.

5. Subdivisions of $K_{3,3}$ and K_5.

In this section, we prove Theorem 1.4. The key insight behind the proof is to show that either the left side (in the case of K_5) or the right side (in the case of $K_{3,3}$) of the matrix attaining the $\mathsf{maxdet}(\cdot)$ value is "planar" or can be transformed to be "planar" by careful matrix operations.

LEMMA 5.1. *If G is a subdivision of $K_{3,3}$ with m edges, then* $\mathsf{maxdet}(G) \leq \tau_m$.

Proof. Let D be an orientation of G such that each internal vertex of G has one incoming edge and one outgoing edge. Then each column in $\tilde{\iota}_D$ corresponding to an internal vertex has exactly one 1 and one -1, with all other entries zero. Let $P = [\tilde{\iota}_D\,|M]$ attain $\mathsf{maxdet}(G)$, i.e., $\mathsf{maxdet}(G) = \det(P)$. Without loss of generality, we assume that $\det(P) > 0$.

Fix an edge e of $K_{3,3}$. Let $V_e \subseteq V(G)$ be the set of internal vertices in G created from subdividing e. Let S_e be the set of edges created from subdividing e. We repeatedly apply the following procedure until V_e is empty.

- Pick an internal vertex $v \in V_e$ whose corresponding column in $\tilde{\iota}_D$ has exactly two nonzero entries 1 and -1 in rows corresponding to two incident edges e_1 and e_2, respectively.
- Add row e_1 to row e_2 in P, so the (e_2, v)-entry becomes 0.
- Now, column v has exactly one nonzero entry 1, so we expand the determinant of P along this column; i.e., we remove column v and row e_1 from P.
- Remove v from V_e and replace P with the submatrix from the determinant expansion.

We apply this to every edge in $K_{3,3}$, resulting in a 9×9 matrix $P_0 = [L_0|M_0]$ (which is not necessarily a bi-incidence matrix) such that L_0 is a truncated incidence matrix of $K_{3,3}$ and $|\det(P_0)| = \det(P)$. For the left side, this process "undoes" the subdivision operations to obtain an orientation of $K_{3,3}$. By our choice of D, rows in P corresponding to edges in S_e are replaced with a single row corresponding to e, whose right side is equal to the sum of the rows in M corresponding to S_e.

By possibly exchanging columns, we assume that $\det(P_0) \geq 0$. By multilinearity of determinants, we have $\det(P) = \det(P_0) = f(r_1 + \ldots + r_k) + c = f(r_1) + \ldots + f(r_k) + c \leq k \cdot \max_{i\in[k]} f(r_i) + c$, where f is a linear form, $c \in \mathbb{R}$ is a constant, and $r_1, \ldots, r_k$ are the rows in M corresponding to S_e. Since P attains $\mathsf{maxdet}(G)$, we can assume without loss of generality that all rows in M corresponding to S_e are identical.

Then M has at most 9 distinct not-all-zero rows, in addition to some (if any) all-zero rows. Let M' be the matrix obtained by removing every all-zero row from M. By Corollary 2.19, there exists a matrix L' such that $P' := [L'|M']$ is a square bi-incidence matrix with $|\det(P')| = \det(P)$. By Proposition 2.10, there exists a directed graph D' such that $\tilde{\iota}_{D'} = M'$. Let G' be the underlying undirected graph of D'. Then G' has 5 vertices and at most m edges, among which there are at most 9 *distinct* edges. By Wagner's theorem [19], G' is planar. If G' is not connected, then

$\mathsf{maxdet}(G) = \det(P) = |\det[L' | \tilde{\imath}_{D'}]| = 0 \le \tau_m$. Otherwise,

$$\begin{aligned}\mathsf{maxdet}(G) = \det(P) &= \left|\det\left[\begin{array}{c|c} L' & \tilde{\imath}_{D'} \end{array}\right]\right| = \left|\det\left[\begin{array}{c|c} \tilde{\imath}_{D'} & L' \end{array}\right]\right| \\ &\le \mathsf{maxdet}\,(G') = \tau\,(G') \le \tau_m.\end{aligned}$$

This completes the proof. ∎

LEMMA 5.2. *If G is a subdivision of K_5 with m edges, then* $\mathsf{maxdet}(G) \le \tau_m$.

Proof. We apply the same argument as in the proof of Lemma 5.1, obtaining

- a connected graph G' with 7 vertices and m edges, among which there are at most 10 *distinct* edges;
- an orientation D' of G';
- a square bi-incidence matrix $P' = [L' | \tilde{\imath}_{D'}]$ such that $|\det(P')| = \mathsf{maxdet}(G)$;
- a 10×10 matrix $P_0 = [L_0|M_0]$ (that is not necessarily a bi-incidence matrix) with $|\det(P_0)| = |\det(P')|$ such that L_0 is a truncated incidence matrix of K_5.

(We omit the details for conciseness.) By possibly exchanging columns, we assume that $\det(P_0) = \det(P') \ge 0$. If G' is planar, then we are done.

Now, suppose that G' is nonplanar. Let H be the underlying simple graph of G', which has 7 vertices and at most 10 edges. Since the sum of the degrees of vertices in H is at most $2 \cdot 10 = 20$, there exists a vertex v in H with degree at most $\lfloor 20/7 \rfloor = 2$. Since G' is connected, so is H, implying that $\deg_H(v) \in \{1, 2\}$. Because the columns on the right side of P' are not removed in the process transforming P' to P_0, we can trace each column on the right side of P_0 to a vertex in G'. By Proposition 2.16, we can assume without loss of generality that v corresponds to a column on the right side of P', and hence to a column on the right side of P_0. We have the following two cases.

- **Case 1:** $\deg_H(v) = 1$**.** Then column v in P_0 has exactly one nonzero entry. Expanding the determinant of P_0 along column v gives a 9×9 submatrix $P_1 = [L_1|M_1]$, where L_1 is a truncated incidence matrix of $K_5 \setminus e$, i.e., the graph obtained by removing an arbitrary edge e from K_5. Since row e is eliminated in this expansion, the determinant is independent of the endpoints of e, so we modify e to coincide with another edge e' in K_5, resulting in a connected planar graph K_5' and a new 10×10 matrix P_0' in place of P_0 with the same determinant. Moreover, we re-attach the endpoints of the path in G of edges created by subdividing e to the endpoints of edge e' (see Figure 3). This gives a connected planar subdivision G'' of K_5' with m edges, while preserving $|\det(P_0)|$. Hence, $\mathsf{maxdet}(G) \le \mathsf{maxdet}(G'') = \tau(G'') \le \tau_m$.
- **Case 2:** $\deg_H(v) = 2$**.** Then column v in P_0 has exactly two nonzero entries with values $s, t \in \mathbb{Z} \setminus \{0\}$, respectively. Let $r_1, \ldots, r_{10}$ be the rows of P_0. Let e_1 and e_2 be the edges of K_5 whose rows in P_0 correspond to the two nonzero entries in column v, respectively. Without loss of generality, we assume that rows r_1 and r_2 correspond to e_1 and e_2, respectively. Then

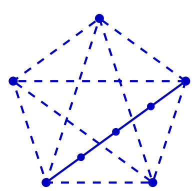
$$\begin{aligned}\det(P_0) &= \det\,[r_1, r_2, r_3, \ldots r_{10}]^{\mathsf{T}} \\ &= -st \cdot \det\left[s^{-1}r_1, -t^{-1}r_2, r_3, \ldots, r_{10}\right]^{\mathsf{T}} \\ &= -st \cdot \det\left[s^{-1}r_1, s^{-1}r_1 - t^{-1}r_2, r_3, \ldots, r_{10}\right]^{\mathsf{T}}, \qquad (5.1)\end{aligned}$$

  where we use $[u_1, \ldots, u_\ell]^{\mathsf{T}}$ to denote the matrix formed by rows $u_1, \ldots, u_\ell$. Let Q be the matrix obtained by removing column v and row $s^{-1}r_1$ from $[s^{-1}r_1, s^{-1}r_1 - t^{-1}r_2, r_3, \ldots, r_{10}]^{\mathsf{T}}$.

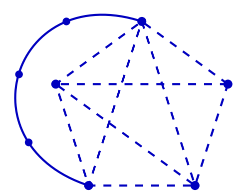
Note that the only nonzero entry of Q in column v belongs to row $s^{-1}r_1$. By the Laplace expansion along column v and by multilinearity of determinants, we have $\det(P_0) = f(s^{-1}r_1' - t^{-1}r_2') + c = s^{-1}f(r_1') - t^{-1}f(r_2') + c$, where f is a linear form, $c \in \mathbb{R}$ is a constant (depending on s and t), and r_1', r_2' are the left sides of rows r_1, r_2 in P_0, respectively. Since P' attains $\mathsf{maxdet}(G)$, we can assume without loss of generality that $r_1' = r_2'$ or $r_1' = -r_2'$. In particular, r_1' and r_2' correspond to the same edge e' in K_5.

Hence, we can modify both e_1 and e_2 to coincide with e' in K_5 (with their orientations possibly reversed), resulting in a connected planar graph K_5' and a new 10×10 matrix P_0' in place of P_0 without decreasing the determinant. Moreover, we modify the endpoints of the paths from subdividing e_1 and e_2, respectively, to coincide with the endpoints of e' (with their orientations possibly reversed), obtaining a connected planar subdivision G'' of K_5' with m edges, while preserving $|\det(P_0)|$. Figure 3 illustrates this process. Hence, $\mathsf{maxdet}(G) \leq \mathsf{maxdet}(G'') = \tau(G'') \leq \tau_m$.

This completes the proof. ∎



(a) A subdivision of K_5.



(b) A planar graph obtained from this process.

Fig. 3: Relocating one path in a subdivision of K_5 to coincide with another path results in a planar graph. The dashed edges represent paths resulting from the subdivision operations on K_5.

The proof of Lemma 5.2 follows from two useful observations. First, by Wagner's theorem [19], modifying one edge in K_5 to coincide with another edge in K_5 results in a connected planar graph with the same number of edges. Second, by certain operations on the matrix attaining $\mathsf{maxdet}(G)$, one can show that there must exist paths in the original graph created from subdividing edges in K_5 such that changing their endpoints to coincide with the endpoints of another edge in the original K_5 does not decrease the determinant of the matrix. We call this technique the "edge relocation" method. It is promising that the edge relocation method is generalizable and has further applications in extending Theorem 1.4 to a broader class of connected nonplanar graphs.

6. Upper Bounding Δ_m. We exploit the sparsity of bi-incidence matrices to obtain an upper bound on Δ_m that is exponentially stronger than the trivial upper bound, 2^m.

Lemma 6.1. *For all $m \in \mathbb{N}$, we have $\Delta_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of the equation $x^3 - x^2 - x - 1 = 0$.*

Proof. We proceed by strong induction on m. The base cases $m = 1, 2, 3$ are easy to check. Let $m \in \mathbb{N}$ with $m \geq 4$. For the induction step, we assume that

$\Delta_j \leq \delta^j$ for all $j \in [m-1]$. Let $P = [M|N] = (a_{i,j})$ be an $m \times m$ bi-incidence matrix that attains Δ_m, i.e., $\Delta_m = \det(P)$. Let c and d be the sums of the columns in M and in N, respectively. Let $M' := [-c|M]$ and $N' := [-d|N]$. Then $P' := [M'|N']$ is a bi-incidence matrix with $m+2$ columns and at most $4m$ nonzero entries. By the pigeonhole principle, there exists a column c^* of P' with at most $\lfloor 4m/(m+2) \rfloor \leq 3$ nonzero entries. By realignment (as defined in Proposition 2.15), possibly interchanging M and N, and possibly interchanging rows, we assume without loss of generality that c^* is a column in M and that the nonzero entries of c^* are the first three entries.

Let k and ℓ be the numbers of columns in M and in N, respectively. For all $i \in [3]$, let r_i be the i^{th} row of P, and let $r_i^b = ((r_i^b)_1, \dots (r_i^b)_m) \in \mathbb{R}^m$ for $b \in \{0,1\}$ be defined by

$$\left(r_i^0\right)_j := \begin{cases} a_{i,j}, & \text{if } j \leq k, \\ 0, & \text{otherwise,} \end{cases} \qquad \left(r_i^1\right)_j := \begin{cases} 0, & \text{if } j \leq k, \\ a_{i,j}, & \text{otherwise,} \end{cases}$$

for all $j \in [m]$. Then $r_i = r_i^0 + r_i^1$ for all $i \in [m]$. For all $\alpha, \beta, \gamma \in \{0,1\}$, let $P_{\alpha,\beta,\gamma}$ be the matrix formed by rows $r_1^\alpha, r_2^\beta, r_3^\gamma, r_4, \dots, r_m$. By multilinearity of determinants,

$$\det(P) = \sum_{\alpha,\beta,\gamma \in \{0,1\}} \det\left(P_{\alpha,\beta,\gamma}\right).$$

We group the summands into the following four cases.

- **Case 1: $\alpha = \beta = \gamma = 1$.** Column c^* of $P_{1,1,1}$ is all-zero, so $\det(P_{1,1,1}) = 0$.
- **Case 2: $\alpha = \beta = 1$ and $\gamma = 0$.** Repeatedly applying Corollaries 2.18 and 2.19 to the first three rows of $P_{1,1,0}$ gives an $(m-3) \times (m-3)$ bi-incidence matrix, whose determinant is at most Δ_{m-3}.
- **Case 3: $\alpha = 1$ and $\beta = 0$.** By multilinearity of determinants,

$$\sum_{\gamma \in \{0,1\}} \det\left(P_{1,0,\gamma}\right)$$

  is equal to the determinant of the matrix P' formed by rows $r_1^1, r_2^0, r_3, \dots, r_m$. Applying Corollaries 2.18 and 2.19 to the first two rows of P', respectively, gives an $(m-2) \times (m-2)$ bi-incidence matrix, whose determinant is at most Δ_{m-2}.
- **Case 4: $\alpha = 0$.** By multilinearity of determinants,

$$\sum_{\beta,\gamma \in \{0,1\}} \det\left(P_{0,\beta,\gamma}\right)$$

  is equal to the determinant of the matrix P'' formed by rows $r_1^0, r_2, \dots, r_m$. Applying Corollary 2.19 to the first row of P'' gives an $(m-1) \times (m-1)$ bi-incidence matrix, whose determinant is at most Δ_{m-1}.

Since δ is a root of the equation $x^3 - x^2 - x - 1 = 0$, we have $\delta^2 + \delta + 1 = \delta^3$. By the above case work and by the inductive hypothesis,

$$
\begin{aligned}
\Delta_m = \det(P) &= \sum_{\alpha,\beta,\gamma} \det\left(P_{\alpha,\beta,\gamma}\right) \\
&= \det\left(P_{1,1,1}\right) + \det\left(P_{1,1,0}\right) + \sum_{\gamma\in\{0,1\}} \det\left(P_{1,0,\gamma}\right) + \sum_{\beta,\gamma\in\{0,1\}} \det\left(P_{0,\beta,\gamma}\right) \\
&\le 0 + \Delta_{m-3} + \Delta_{m-2} + \Delta_{m-1} \\
&\le \delta^{m-3} + \delta^{m-2} + \delta^{m-1} \\
&= \delta^{m-3}\left(1 + \delta + \delta^2\right) \\
&= \delta^{m-3} \cdot \delta^3 = \delta^m.
\end{aligned}
$$

This completes the proof. ∎

Proposition 1.1 and Lemma 6.1 together prove Theorem 1.5. This upper bound on τ_m matches the current best upper bound of [17], who used the connection between links and spanning trees in planar graphs from knot theory. Our linear-algebraic argument is in some sense "isomorphic" to their knot-theoretic one.

7. Concluding Remarks. In this paper, we have provided a simple, linear-algebraic planarity criterion that allows one to give a "certificate of planarity" of a planar graph that can be easily verified by computing the determinant of a sparse matrix and counting spanning trees. In addition, we have proved that subdivisions of $K_{3,3}$ and K_5 "underperform" the best planar graph with the same number of edges in terms of the $\mathsf{maxdet}(\cdot)$ function. As a by-product, our linear-algebraic technique allows us to derive an upper bound on Δ_m (and hence on τ_m) that matches the current best upper bound by Stoimenow [17].

Several interesting questions remain open. The main one is Conjecture 1.6.

Problem 7.1. Does Conjecture 1.6 hold? If so, what are the asymptotic behaviors of τ_m and Δ_m?

We remark several potential approaches for proving Conjecture 1.6. First, it might be possible to generalize Theorem 1.4 to any arbitrary connected nonplanar graph, for which the edge relocation method used in proving Lemma 5.2 might help.

Problem 7.2. Can Theorem 1.4 and the edge relocation method be generalized to a broader class of connected nonplanar graphs?

Second, the application of the merge-cut lemma in the proof of Lemma 4.4 is very loose. Most nonplanar graphs contain many copies of $K_{3,3}$ and K_5 as minors.

Problem 7.3. Can the observation that most nonplanar graphs contain many copies of $K_{3,3}$ and K_5 as minors be exploited to strengthen Lemma 4.4?

In addition to Conjecture 1.6, our work raises two algorithmic questions.

Problem 7.4. Can the construction from Lemma 4.1 give rise to faster exact or approximate algorithms for counting spanning trees in a planar graph, possibly by exploiting the sparsity of the matrix?

To the best our knowledge, the fastest exact algorithm for counting spanning trees in a planar graph is the one by Lipton, Rose and Tarjan [13] using the planar separator theorem, which runs in $O(n^{1.5})$ time, where n is the number of vertices.

Problem 7.5. Can our linear-algebraic characterization of planar graphs lead to efficient algorithms for deciding and testing planarity of a graph?

Several linear-time algorithms for (exact) planarity testing are known; see, e.g., [11, 6, 2]. For property testing that allows one-sided or two-sided error, sublinear-time algorithms are known in different models; see, e.g., [12, 8, 9].

The excess can be viewed as a measure of nonplanarity of a graph. Several other measures of nonplanarity have been extensively studied, such as the crossing number, the genus and the thickness. It is not hard to show that the excess of a nonplanar graph is at least 18 times its crossing number, which follows from the merge-cut lemma. It would be interesting to further compare the excess with these measures.

Problem 7.6. How is the excess of a nonplanar graph related to other measures of nonplanarity?

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Appendix A. Computing $\varepsilon(K_{3,3})$ and $\varepsilon(K_5)$. To make the analysis on $K_{3,3}$ and K_5 easier, we introduce the concept of co-spanning trees.

Definition A.1. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $|E_1| = |E_2| = |V_1| + |V_2| - 2$. Let $f : E_1 \to E_2$ be a bijection. We say that a spanning tree T of G_1 is* annihilated *by f if $f(E_1 \setminus T)$ is not a spanning tree of G_2. We say that a spanning tree of G_1 that is not annihilated by f is a* co-spanning tree *of G_1 and G_2 under f. We let $\mathsf{cospan}(G_1, G_2)$ denote the maximum number of co-spanning trees of G_1 and G_2 over all choices of f.*

Proposition A.2. *Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be two graphs such that $|E_1| = |E_2| = |V_1| + |V_2| - 2$. Let $P = [M|N]$ be a bi-incidence matrix, where M is a truncated incidence matrix of G_1 and N is a truncated incidence matrix of G_2. Then* $\det(P) \le \mathsf{cospan}(G_1, G_2)$.

Proof. Let $M = (a_{i,j})$ and $N = (b_{i,j})$. Let $k := |V_1| - 1$ and $m := |E_1|$. For $T \subseteq [m]$, let Sym_T denote the set of all permutations on T. Then

$$\det(P) = \sum_{\sigma \in \mathsf{Sym}_{[m]}} \mathsf{sgn}(\sigma) \left(\prod_{i \in [k]} a_{\sigma(i),i} \right) \left(\prod_{j \in [m-k]} b_{\sigma(j+k),j} \right).$$

Given a subset $T \subseteq [m]$ with $|T| = k$, we use σ_T to denote the unique permutation on $[m]$ such that $T = \{\sigma_T(1), \ldots, \sigma_T(k)\}$ with $\sigma_T(1) < \ldots < \sigma_T(k)$ and $\sigma_T(k+1) < \ldots < \sigma_T(m)$. Each permutation $\sigma \in \mathsf{Sym}_{[m]}$ decomposes as $\sigma_1 \circ \sigma_2 \circ \sigma_T$, where $T = \{\sigma(1), \ldots, \sigma(k)\}$, and where σ_1 and σ_2 are permutations of T and $[m] \setminus T$, respectively.

For $i \in [k]$, we use $T(i)$ to denote the i^{th} smallest element of T. For $i \in [m-k]$, we use $\overline{T}(i)$ to denote the i^{th} smallest element of $[m] \setminus T$. Furthermore, for $T \subseteq [m]$, we use $M_{T,[k]}$ to denote the submatrix of M formed by rows T and columns $[k]$, and use $N_{[m]\setminus T,[m-k]}$ to denote the submatrix of N formed by rows $[m] \setminus T$ and columns

$[m-k]$. Hence,

$$\det(P) = \sum_{T \in \binom{[m]}{k}} \mathsf{sgn}\,(\sigma_T) \left(\sum_{\sigma_1 \in \mathsf{Sym}_T} \mathsf{sgn}\,(\sigma_1) \prod_{i \in [k]} a_{\sigma_1(T(i)),i} \right)$$

$$\left(\sum_{\sigma_2 \in \mathsf{Sym}_{[m]\setminus T}} \mathsf{sgn}\,(\sigma_2) \prod_{j \in [m-k]} b_{\sigma_2(\overline{T}(j)),j} \right)$$

$$= \sum_{T \in \binom{[m]}{k}} \mathsf{sgn}\,(\sigma_T) \det\left(M_{T,[k]}\right) \det\left(N_{[m]\setminus T,[m-k]}\right)$$

$$\leq \sum_{T \in \binom{[m]}{k}} \left|\det\left(M_{T,[k]}\right)\right| \cdot \left|\det\left(N_{[m]\setminus T,[m-k]}\right)\right|$$

$$\leq \mathsf{cospan}(G_1, G_2),$$

where the last inequality holds since $|\det(M_{T,[k]})|$ and $|\det(N_{[m]\setminus T,[m-k]})|$ are both nonzero if and only if T is a spanning tree of G_1 and $[m] \setminus T$ is a spanning tree of G_2, in which case they are both equal to 1 as M and N are totally unimodular. ∎

Now, we have the necessary machinery to prove the exact values of $\mathsf{maxdet}(K_{3,3})$ and $\mathsf{maxdet}(K_5)$. In the proofs, we repeatedly restrict the space of feasible graphs corresponding to the right incidence submatrix through various bounds. These bounds give us increasingly restrictive information about both the graph and properties of the bijection, which we then use to prove the desired results.

Proposition A.3. *We have $\varepsilon(K_{3,3}) = 18$.*

Proof. Since $\tau(K_{3,3}) = 81$, it is equivalent to show that $\mathsf{maxdet}(K_{3,3}) = 63$. First, we show that $\mathsf{maxdet}(K_{3,3}) \leq 63$. Let $\mathsf{maxdet}(K_{3,3})$ be achieved by $[M|N]$ where M is a truncated incidence matrix of $K_{3,3}$ and N is a truncated incidence matrix of a graph $G = (V, E)$ with 5 vertices. If G has fewer than 9 edges, then G is planar, so G has at most $\tau_8 = 45$ spanning trees (see Table 1); hence, $\det[M|N] \leq \mathsf{maxdet}(G) \leq \tau(G) < 63$, as desired. Otherwise, G has 9 edges, and is thus planar.

By Proposition A.2, it suffices to show that $\mathsf{cospan}(K_{3,3}, G) \leq 63$. Suppose that this is not the case. Let $f : E(K_{3,3}) \to E(G)$ be a bijection which yields the maximum number of co-spanning trees. We refer to two bijectively paired edges as the same edge. Note that $\tau(K_{3,3}) = 81$. It suffices to show that at least 18 spanning trees of $K_{3,3}$ are annihilated by f. We have the following cases.

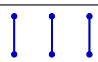
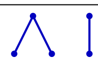
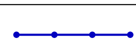
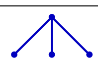
- **Case 1: G has a vertex with degree at most 2.** If the vertex has degree two, then any spanning tree of $K_{3,3}$ containing these two edges is annihilated. In particular, if these two edges are incident (resp. not incident) in $K_{3,3}$, then 21 (resp. 24) spanning trees are annihilated, as desired. If the vertex has degree one, then G has at most $\tau_8 = 45 \leq 63$ spanning trees (see Table 1), as desired.
- **Case 2: G has a triple of three parallel edge.** Applying Proposition 3.2 to any two of these edges, we have $\tau(G) \leq 2 \cdot \tau_6 + \tau_7 = 56 \leq 63$, as desired.
- **Case 3: G has two distinct pairs of two parallel edges each.** Then any spanning tree of $K_{3,3}$ missing either pair is annihilated. In particular, for each pair, if the edges are incident (resp. not incident) in $K_{3,3}$, then 12 (resp. 15) spanning trees are annihilated. There is at most one spanning tree of $K_{3,3}$ missing all four of these edges, so at least 23 spanning trees are annihilated,

as desired.

If none of the first three cases holds, then G is obtained from K_5 either by removing two edges and duplicating one edge to a pair of two parallel edges, or by removing one edge from K_5.

- **Case 4: G is obtained from K_5 by removing two edges and duplicating one edge to a pair of two parallel edges.** Since $\sum_{v\in V(G)} \deg_G(v) = 18$, there exist two vertices u_1 and u_2 of G with degree at most 3. By case 1, we assume that $\deg_G(u_1) = \deg_G(u_2) = 3$. Note that any spanning tree of $K_{3,3}$ containing either triple of edges is annihilated. Note also that $K_{3,3}$ has no triangles. We have four cases for the subgraph of $K_{3,3}$ induced by the three edges in a triple and the corresponding number of annihilated spanning trees, depicted in Table 2.

Table 2: Four cases for the subgraph of $K_{3,3}$ induced by three edges and the number of annihilated spanning trees.

the subgraph	number of annihilated spanning trees
	12
	11
	8
	9

It is impossible to obtain a graph G with 9 edges where u_1 and u_2 share two parallel edges, so there is at most one shared edge between the two triples. Hence, there is at most one spanning tree annihilated by both triples. If one of the triples annihilates at least 11 spanning trees, then at least 18 spanning trees are annihilated altogether, as desired. Hence, we may assume that the two triples are either stars or paths in $K_{3,3}$.

Now, consider the pair of parallel edges in G. Any spanning tree of $K_{3,3}$ excluding both of these edges is annihilated. If the edges are incident (resp. not incident) in $K_{3,3}$, then 12 (resp. 15) spanning trees are annihilated. In particular, there are at most 5 (resp. 4) spanning trees annihilated by both the pair and a path (resp. a star) in $K_{3,3}$. Both of these cases require the pair and the triple to be disjoint; if the pair has a shared edge with the triple, then a spanning tree annihilated by both must simultaneously contain and not contain the shared edge, which is impossible. Hence, each triple annihilates at least $\min\{8-5, 9-4\} = 3$ spanning trees (see Table 2) in addition to those also annihilated by the pair when the triple and the pair are disjoint, and at least 8 spanning trees when they are not disjoint.

If either of the two triples is not disjoint from the pair, then the total number of annihilated spanning trees is at least $12 + 8 = 20$, as desired. Otherwise, the total number of annihilated spanning trees is at least $12+3+3-1 = 17$, where the -1 term comes from the potential spanning tree annihilated by both triples but not by the pair. Note that the union of the two triples has at least 5 edges, and that a spanning tree of $K_{3,3}$ has exactly 5 edges.

This spanning tree, if it existed, would contain all of the edges in the union of the two triples, and thus not contain either edge of the pair because the pair is disjoint from both triples. Therefore, this spanning tree is already annihilated by the pair, so the number of annihilated spanning trees is at least $12 + 3 + 3 = 18$. In summary, there are always at least 18 annihilated spanning trees, as desired.

- **Case 5: G is obtained from K_5 by removing an edge.** Then G is planar, and its unique planar dual graph is the *envelope graph*, depicted in Figure 4.

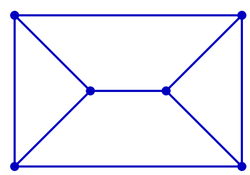


Fig. 4: The envelope graph.

If the image of a spanning tree T of $K_{3,3}$ through f and planar duality (i.e., by mapping each edge to its dual edge) in the envelope graph is not a spanning tree, then $E \setminus f(T)$ is not a spanning tree of G (by the well-known fact that, if T is a spanning tree of a connected plane graph H, then the dual of $E(H) \setminus T$ is a spanning tree in the dual of H; see, e.g., Exercise 10.2.12 in [1]), and hence T is annihilated by f. We say that a spanning tree of the envelope graph is *annihilated* by f^{-1} if the complement of its dual (by mapping every edge to its dual edge) in G is annihilated by f^{-1}.
The envelope graph has exactly 5 cycles of size at most 4, while $K_{3,3}$ has 9 cycles of size 4, every two of which share at most 2 edges. We call a 4-cycle of $K_{3,3}$ *degenerate* if its image in the envelope graph is acyclic. Each cycle of the envelope graph is contained in the image of at most one 4-cycle, so at least four of the 4-cycles are degenerate. Any spanning tree of the envelope graph containing all four edges of a degenerate 4-cycle is annihilated by f^{-1}. Two degenerated 4-cycles can have at most 2 edges in common, so they cover at least 6 edges and therefore no spanning trees is annihilated twice. Since the degree of every vertex of the envelope graph is 3, at least 3 spanning trees of the envelope graph are annihilated by f^{-1} for each degenerate 4-cycle of $K_{3,3}$. Since the envelope graph has 75 spanning trees, it follows that the number of co-spanning trees of $K_{3,3}$ and G is at most $75 - 3 - 3 - 3 - 3 = 63$, as desired.

For equality, it suffices to note that

$$\det\left[\begin{array}{ccccc|cccc} 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & -1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & -1 \\ -1 & 0 & 0 & 1 & 0 & 0 & 1 & -1 & 0 \\ -1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 \\ 0 & -1 & 1 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & -1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 & 1 & 1 & 0 & 0 & -1 \end{array}\right] = 63,$$

where the left side of the matrix is a truncated incidence matrix of $K_{3,3}$ and the right side is an incidence submatrix. This completes the proof. ∎

PROPOSITION A.4. *We have* $\varepsilon(K_5) = 25$.

Proof. Since $\tau(K_5) = 125$, it is equivalent to show that $\mathsf{maxdet}(K_5) = 100$. First, we show that $\mathsf{maxdet}(K_5) \leq 100$. Let $\mathsf{maxdet}(K_5)$ be achieved by $[M|N]$ where M is a truncated incidence matrix of K_5 and N is a truncated incidence matrix of a graph G with 7 vertices. If G has fewer than 10 edges, then G is either $K_{3,3}$ or planar, so $\det[M|N] \leq \mathsf{maxdet}(G) \leq \tau(G) \leq \max(\tau_9, \tau(K_{3,3})) = \max(75, 81) < 100$ (see Table 1). Otherwise, suppose that G has 10 edges.

By Proposition A.2, it suffices to show that $\mathsf{cospan}(K_5, G) \leq 100$. Suppose that this is not the case. Let $f : E(K_5) \to E(G)$ be a bijection which yields the maximum number of co-spanning trees. We refer to two bijectively paired edges as the same edge. Recall that $\tau(K_5) = 125$. It suffices to show that at least 25 spanning trees of K_5 are annihilated by f. We have the following cases.

- **Case 1: G has a pair of two parallel edges.** Then any spanning tree of K_5 excluding these two edges is annihilated. In particular, if these two edges are incident (resp. not incident) in K_5, then 40 (resp. 45) spanning trees are annihilated, as desired.
- **Case 2: G has a vertex with degree 1.** In this case, G has at most $\tau_9 = 75 \leq 100$ spanning trees, as desired.
- **Case 3: G has two distinct vertices with degree 2.** If these two vertices are adjacent, by Proposition 3.2 applied to the three edges incident to these vertices, G has at most $4 \cdot \tau_7 = 96 \leq 100$ spanning trees, as removing any two of the three edges disconnects G. Otherwise, consider the pair of edges at each of these two vertices. Any spanning tree of K_5 containing both edges of either pair is annihilated. In particular, for each pair, if the two edges are incident (resp. not incident) in K_5, then 15 (resp. 20) spanning trees are annihilated. There is at most one spanning tree of K_5 containing all four of these edges, so at least 29 spanning trees are annihilated, as desired.

Otherwise, since $\sum_{v \in V(G)} \deg_G(v) = 20$, it follows that G has one vertex with degree 2 and six vertices with degree 3. Hence, G can be obtained from a subdivision on a graph with 9 edges and 6 vertices, each with degree 3. We have the following two cases based on whether this 6-vertex graph is isomorphic to $K_{3,3}$.

- **Case 4: G is a subdivision of $K_{3,3}$.** See Figure 5. It can be computed that $\tau(G) = 117$. Since G is nonplanar, it follows from Lemma 4.4 that $\varepsilon(G) \geq 18$. Hence, it follows that $\det[M|N] \leq \mathsf{maxdet}(G) \leq \tau(G) - 18 = 99 < 100$, as desired.
- **Case 5: G is a subdivision of a 9-edge planar graph.** In this case,

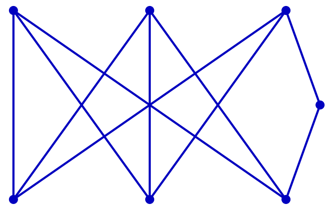


Fig. 5: The case when G is a subdivision of $K_{3,3}$.

G must be planar, so consider a planar dual graph H of G. It follows that H is a planar graph with 5 vertices and 10 edges, with exactly one pair of two parallel edges. Hence, H can be obtained from K_5 by deleting an edge and duplicating another edge into a pair of two parallel edges. In particular, there are only two isomorphism classes of H: the case when these two edges are incident (with 110 spanning trees, see Figure 6a) and the case when the edges are not incident (with 105 spanning trees, see Figure 6b).

Given a bijection $f : E(K_5) \to E(H)$, we say that a spanning tree T of H is a *bi-spanning tree* of K_5 and H under f if $f^{-1}(T)$ is a spanning tree of K_5. Let $\mathsf{bispan}(K_5, H)$ denote the maximum number of bi-spanning trees of K_5 and H over all choices of f. By Definition A.1 and by planar duality, we have $\mathsf{cospan}(K_5, G) = \mathsf{bispan}(K_5, H)$. Hence, it suffices to show that $\mathsf{bispan}(K_5, H) \leq 100$. We have the following two cases depending on the two isomorphism classes of H.

(a) The case when the deleted edge and the duplicated edge are incident.

(b) The case when the deleted edge and the duplicated edge are not incident.

Fig. 6: Two isomorphism classes of the planar dual graph H that can be obtained from K_5 by deleting an edge and duplicating another edge into a pair of two parallel edges, where the pair of parallel edges is indicated by the thick edge in each case.

- **Case 5(a): H has 110 spanning trees.** Let $f : E(K_5) \to E(H)$ be a bijection that achieves $\mathsf{bispan}(K_5, H)$. Consider a subset $F \subseteq E(H)$. Then F cannot be a spanning tree of H if F contains both of the two parallel edges in H. Note that there are exactly 40 spanning trees in the graph obtained by removing the two parallel edges from H. Hence, there are at most 40 bi-spanning trees of K_5 and H under f that contain neither of the two parallel edges.

  We count the number x of spanning trees of K_5 which contain exactly one of the two edges and for which *swapping* the contained edge to the other gives another spanning tree of K_5. Let $\mathcal{T}$ be the set of bi-spanning trees of K_5 and H under f which contain exactly one of the two edges and for which swapping gives another bi-spanning tree. Then $|\mathcal{T}| \le x$. Since there are $110 - 40 = 70$ spanning trees of H that contains exactly one of the two parallel edges, there are at most $70 - |\mathcal{T}|$ spanning trees of H which contain exactly one of the two parallel edges and which are not contained in $\mathcal{T}$. We group these spanning trees of H into pairs by identifying their edge sets *minus the two parallel edges*. In each pair, there is at most one spanning tree of H which is a bi-spanning tree. Therefore,

$$\mathsf{bispan}\,(K_5, H) \le 40 + |\mathcal{T}| + \frac{70 - |\mathcal{T}|}{2} = 75 + \frac{|\mathcal{T}|}{2} \le 75 + \frac{x}{2}.$$

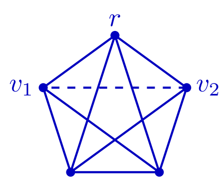
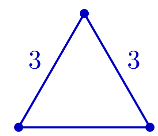
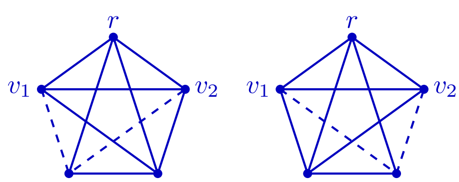
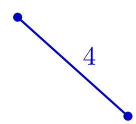
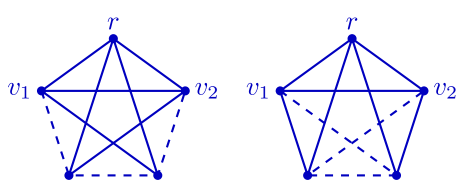
Hence, it suffices to show that $x \le 50$. We have the following two cases, depending on whether the two edges in K_5 mapping to the two parallel edges in H under f are adjacent or not.

* **Case 5(a)(i): the two edges rv_1 and rv_2 in K_5 are adjacent.** Given a spanning tree T of K_5 rooted at r, swapping one of the two edges to the other in T gives another spanning tree if and only if v_1 is in the subtree of T rooted at v_2, or vice versa. We count the number of spanning trees of K_5 rooted at r in which v_1 is a child of r and in which v_2 is a descendant of v_1 (the case where v_2 is a child of r is symmetric). We have the following three cases depending on the length of the unique v_1-v_2 path P in the spanning tree.
    * P **consists of** 1 **edge.** There is one v_1-v_2 path in K_5 which consists of 1 edge and which does not visit r. Contracting $\{rv_1\} \cup P$ in K_5 gives a graph with 15 spanning trees.
    * P **consists of** 2 **edges.** There are two v_1-v_2 paths in K_5 which consist of 2 edges and which do not visit r. Contracting $\{rv_1\} \cup P$ in K_5 gives a graph with 4 spanning trees.
    * P **consists of** 3 **edges.** There are two v_1-v_2 paths in K_5 which consist of 3 edges and which do not visit r. Contracting $\{rv_1\} \cup P$ in K_5 gives a graph with 1 spanning tree.

  Summarizing, we have $x = 2 \cdot (1 \cdot 15 + 2 \cdot 4 + 2 \cdot 1) = 50$. We illustrate the above three cases in Table 3.
* **Case 5(a)(ii): the two edges v_1v_2 and v_3v_4 in K_5 are not adjacent.** We count the number of spanning trees of K_5 in which v_1v_2 is contained and v_3v_4 is not, and in which swapping v_1v_2 to v_3v_4 in the spanning tree gives another spanning tree (the case in which v_3v_4 is contained and v_1v_2 is not is symmetric). Note that removing v_1v_2 partitions the spanning tree into two connected components, and that v_3v_4 crosses the two connected components. It follows that one of the two connected components has 2 vertices and the other has 3 vertices. There are 4 ways of assigning the five vertices to the two connected components crossed by v_1v_2 and v_3v_4.

  The subgraph of the spanning tree restricted to each of the two connected components is also a spanning tree. Note that there is one spanning tree in K_2 and 3 spanning trees in K_3. Summarizing, we have $x = 2 \cdot 4 \cdot 1 \cdot 3 = 24$.

Table 3: Three cases for counting the number of spanning trees of K_5 rooted at r in which v_1 is a child of r and in which v_2 is a descendant of v_1. In the second column, all possible v_1-v_2 paths in K_5 of the given length that do not visit r are drawn as dashed segments in each row. In the third column, the graph obtained by contracting rv_1 and any of the v_1-v_2 paths from the second column is given in each row, where the number beside an edge indicates the number of parallel edges incident to the two endpoints and where loop edges are omitted.

v_1-v_2 path length	all v_1-v_2 paths that do not visit r	contracted graph
1		
2		
3		

This completes the case where H has 110 spanning trees.

– **Case 5(b): H has 105 spanning trees.** We follow the argument for case 5(a). Let $f : E(K_5) \to E(H)$ be a bijection that achieves $\mathsf{bispan}(K_5, H)$. Note that there are exactly 45 spanning trees in the graph obtained by removing the two parallel edges from H. Let x be the number of spanning trees of K_5 which contain exactly one of the two edges and for which *swapping* the contained edge to the other gives another spanning tree of K_5. Let $\mathcal{T}$ be the set of bi-spanning trees of K_5 and H under f which contain exactly one of the two edges and for which swapping gives another bi-spanning tree. Following the same argument for case 5(a),

$$\mathsf{bispan}\,(K_5, H) \leq 45 + |\mathcal{T}| + \frac{105 - 45 - |\mathcal{T}|}{2} = 75 + \frac{|\mathcal{T}|}{2} = 75 + \frac{x}{2}.$$

Hence, it suffices to show that $x \leq 50$. This follows from cases 5(a)(i) and 5(a)(ii).

For equality, it suffices to note that

$$\det \left[\begin{array}{cccc|cccccc}
1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & -1 & 0 \\
-1 & 1 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\
-1 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 \\
0 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 \\
0 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 1 & -1
\end{array}\right] = 100,$$

where the left side of the matrix is a truncated incidence matrix of K_5 and the right side is an incidence submatrix. This completes the proof. $\square$