Fantastic Planar Graphs and Where to Find Them

A Linear-Algebraic Planarity Criterion

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February 10, 2025 Theory Lunch MIT CSAIL an extremal problem

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some StackExchange answer (user JimT):

This is a very, very tough question. It is obvious that for any graph G with n edges the number of spanning trees t(G) does not exceed 2^n (each edge is either included in or excluded from a subtree; this is also the upper bound of the number of connected subgraphs of G). We can somewhat improve this - if G has m vertices ($m \le n + 1$) then we have

$$t(G) \leqslant \binom{n}{m-1} < 2^n,$$

since we need to choose m-1 edges out of n (not arbitrarily, of course).



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a **lower bound**: $T_m \ge 1.7916^m$ achieved by square grid graphs

some **upper bounds** ($T_m \leq \tau^m$):

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@Alien Nice! One comment though - I quickly looked thru the article; it is very non-elementary, as it uses a bunch of other results in links and knot theory, as well as related polynomial algebra. But still, it proves the even lower upper bound, so the gap between this and 1.7916... is getting smaller with every post here:). - JimT Sep 29, 2024 at 2:32 \mathscr{E}

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theorem (Stoimenow 2007)

 $T_m \leq \delta^m$, where $\delta \simeq 1.8393$ is the unique real root of $x^3 - x^2 - x - 1 = 0$.

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... and also some nice by-products on planarity criteria from our techniques.

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 $\Delta_m \leq \delta^m$

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proof of $\Delta_m \leq \delta^m$

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W.l.o.g. (by column operations), this column is in [M|N].

r_1	a ₁₁	_
r ₂	a ₂₁	
r ₃	a ₃₁	
r ₄	0	
:	:	
	:	
r _m	0	

$$\begin{bmatrix} & & & \\ & & & \\ & & & \\ r_i & & & \\ & & & \\ r_i^0 & & & \\$$

$$\begin{array}{c|cccc}
r_1 & a_{11} & & & \\
r_2 & a_{21} & & & \\
r_3 & a_{31} & & & \\
\vdots & \vdots & & \vdots & \\
r_m & 0 & & & \\
\end{array}$$

$$\begin{bmatrix} & & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & & & & \\ & &$$

$$\det \left[\begin{array}{c|c} M \mid N \end{array}\right] = \sum_{\substack{\alpha,\beta,\gamma \in \{0,1\}\\ r_1 \\ \vdots \\ r_m \end{array}} \det \begin{bmatrix} r_1^{\alpha} \\ r_2^{\beta} \\ r_4 \\ \vdots \\ r_m \end{bmatrix} 0$$

case 1:
$$\alpha = \beta = \gamma = 1$$

$$\det \begin{array}{c}
r_1^1 \\ r_2^1 \\ r_3^2 \\ r_4^2 \\ \vdots \\ r_m \\ 0
\end{array}
= 0.$$

$$\sum_{\beta,\gamma\in\{0,1\}} \det \begin{matrix} r_1^0 \\ r_2^\beta \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{matrix} \begin{bmatrix} 0 \cdots 0 \\ 0 \cdots 0 \\ \end{bmatrix} = \det \begin{matrix} r_1^0 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{matrix} \begin{bmatrix} 0 \cdots 0 \\ 0 \cdots 0 \\ \end{bmatrix}$$

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lemma (Bu, P. 2025; informal)

If an $m \times m$ bi-incidence matrix has a row whose restriction to the left/right side has zeros only,

$$\sum_{\beta,\gamma\in\{0,1\}} \det \begin{matrix} r_1^0 \\ r_2^\beta \\ r_3^\gamma \\ r_4 \\ \vdots \\ r_m \end{matrix} \begin{bmatrix} 0 \cdots 0 \\ 0 \cdots 0 \\ \end{bmatrix} = \det \begin{matrix} r_1^0 \\ r_2 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{matrix} \begin{bmatrix} 0 \cdots 0 \\ 0 \cdots 0 \\ \end{bmatrix}$$

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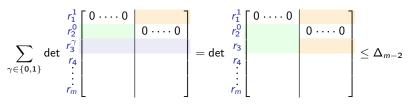
If an $m \times m$ bi-incidence matrix has a row whose restriction to the left/right side has zeros only, then one can remove this row and perform column operations to obtain an $(m-1)\times(m-1)$ square bi-incidence matrix, while preserving the determinant.

$$\sum_{\substack{\beta,\gamma\in\{0,1\}\\\beta,\gamma\in\{0,1\}}}\det \begin{matrix} r_1^0\\r_2^\beta\\r_3^\gamma\\r_4\\\vdots\\r_m^r \end{matrix} \begin{bmatrix} 0\cdots0\\0\cdots0\\\\\vdots\\r_m^r \end{bmatrix} = \det \begin{matrix} r_1^0\\r_2\\r_3\\\vdots\\\vdots\\r_m^r \end{bmatrix}$$

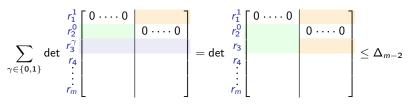
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$$\det \begin{bmatrix} r_1^1 & 0 & \cdots & 0 \\ r_2^1 & 0 & \cdots & 0 \\ r_3^n & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ r_m & \vdots & \vdots & \vdots \\ \end{bmatrix} \leq \Delta_{m-3}$$

case 4: $\alpha = \beta = 1$ and $\gamma = 0$

$$\sum_{\gamma \in \{0,1\}} \det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3^{\gamma} \\ \vdots \\ r_m \end{array} \begin{bmatrix} 0 \cdots 0 \\ 0 \cdots 0 \\ \end{bmatrix} = \det \begin{array}{c} r_1^1 \\ r_2^0 \\ r_3 \\ r_4 \\ \vdots \\ r_m \end{bmatrix} \begin{bmatrix} 0 \cdots 0 \\ 0 \cdots 0 \\ \end{bmatrix} \leq \Delta_{m-2}$$

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case 4:
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$$\Delta_m \leq \Delta_{m-1} + \Delta_{m-2} + \Delta_{m-3}$$

 $T_m \leq \Delta_m$

proof sketch of $T_m \leq \Delta_m$

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A truncated incidence matrix of a digraph is the incidence matrix of the digraph with an arbitrary column removed.

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Then $|\det[M|M^*]|$ is equal to the number of spanning trees $\tau(G)$ in G.

proof sketch of $T_m \leq \Delta_m$ (cont'd)

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If M is a truncated incidence matrix of a graph G, then $M^{\mathsf{T}}M$ is a first principal minor of the Laplacian of G.

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Kirchhoff's matrix-tree theorem

The determinant of a first principle minor of the Laplacian of a graph G is equal to $\tau(G)$.

corollary

For all $m \in \mathbb{N}$, we have $T_m \leq \Delta_m$.

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 T_m and Δ_m for small values of m

m	1	2	3	4	5	6	7	8	9	10
T_m	1	2	3	5	8	16	24	45	75	130
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conjecture

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a planarity criterion

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proposition (Bu, P. 2025)

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merge-cut lemma (Bu, P. 2025)

For a connected G = (V, E) and a non-bridge $e \in E$, $\Delta(G) < \Delta(G/e) + \Delta(G \setminus e)$.

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For planar graphs, we have seen that the previous construction is the best possible.

corollary (Bu, P. 2025)

For any planar graph G, we have $\varepsilon(G) = 0$.

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For any planar graph G, we have $\varepsilon(G) = 0$.

For non-planar graphs, the excesses are at least 18.

lemma (Bu, P. 2025)

For any non-planar graph G, we have $\varepsilon(G) \geq 18$.

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lemma (Bu, P. 2025)

For any non-planar graph G, we have $\varepsilon(G) > 18$.

The lower bound 18 is tight and achieved by $K_{3,3}$.

towards the conjecture

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another conjecture

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theorem (Bu, P. 2025)

If G is a subdivision of K_5 or $K_{3,3}$ with m edges, then $\Delta(G) \leq T_m$.

Thanks!