On the Sensitivity of Boolean Functions

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UBC CPSC 531F

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Paul Erdős famously spoke of a book, maintained by God, in which was written the simplest, most beautiful proof of each theorem. The highest compliment Erdős could give a proof was that it "came straight from the book." In this case, I find it hard to imagine that even God knows how to prove the Sensitivity Conjecture in any simpler way than this.
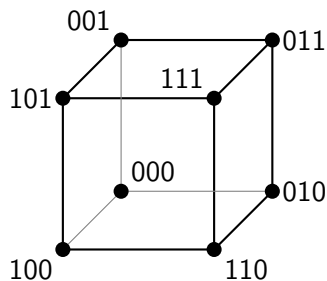
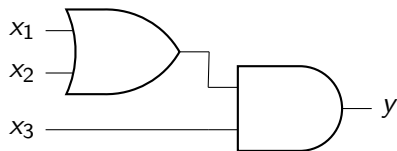*— Scott Aaronson*[1]

[1] https://www.scottaaronson.com/blog/?p=4229
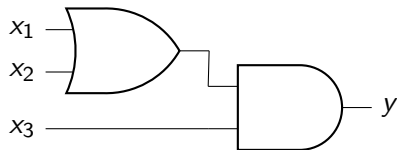
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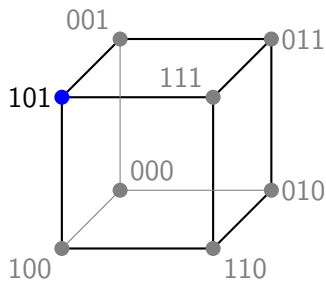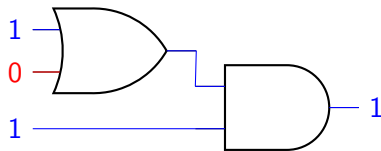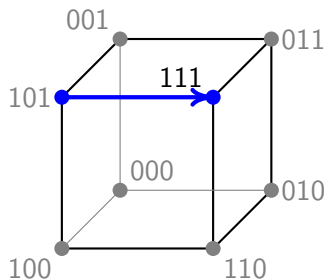# Sensitivity of Boolean Functions
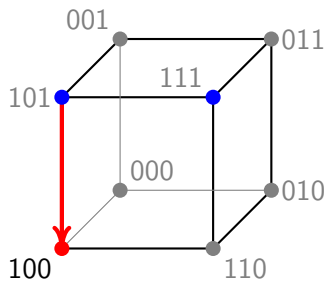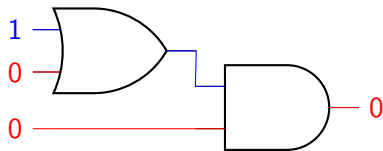
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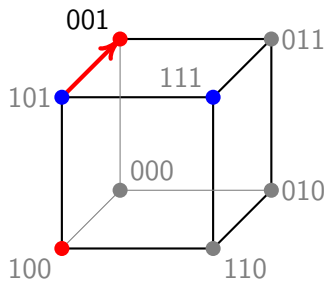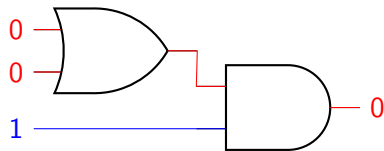
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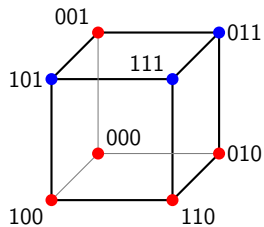
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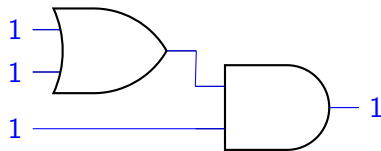
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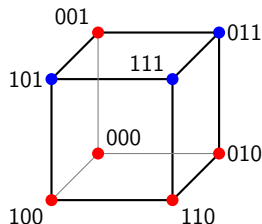
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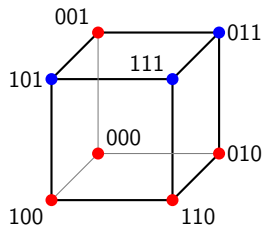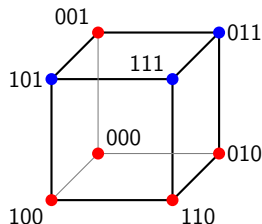# Block Sensitivity of Boolean Functions

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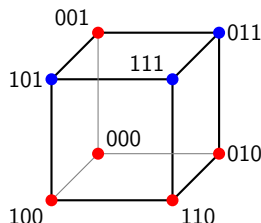
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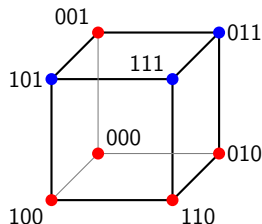
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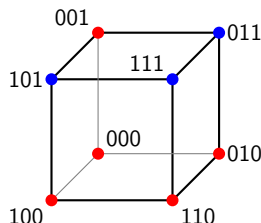
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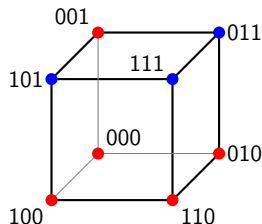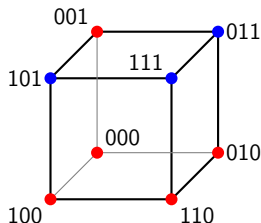

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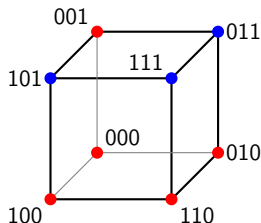
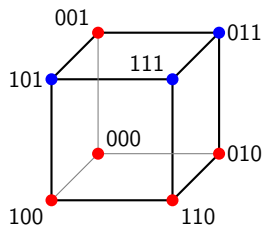$$bs(f, 101) = 2$$
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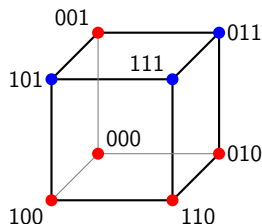
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- For $x = 11110000$, then $\{1\}, \{2\}, \{3\}, \{4\}, \{5,6\}, \{7,8\}$ are 6 disjoint, **sensitive** blocks for f, so $bs(f) \geq 6 > s(f)$.

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Theorem (Rubinstein 1995)

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Define $f : \{0,1\}^{n^2} \to \{0,1\}$ as

$$f\left(x_{11}, \ldots, x_{nn}\right) = \bigvee_{i=1}^{n} g\left(x_{i1}, \ldots, x_{in}\right),$$

where $g(x_1, \ldots, x_n)$ if and only if $x_j = x_{j+1} = 1$ for some $j \in [n-1]$, and all other $x_k = 0$.

Rubinstein's Function

Claim

$bs(f) \geq bs(f, 0) = \Omega(n^2).$

Proof.

$$
\begin{array}{ccccccc}
0 & 0 & 0 & 0 & \cdots & \rightarrow & 0 \\
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\vdots & \vdots & \vdots & \vdots & & & \vdots \\
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  Hence, $s(f, x) \leq 2n$.
- **Case 2:** $f(x) = 1$.
  - If two rows output 1, $s(f, x) = 0$.
  - If only one row outputs 1, $s(f, x) \leq n$.

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Theorem (Huang 2019)

For every Boolean function f,

$$bs(f) \leq s(f)^4.$$

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Complexity measures α, β of Boolean functions are **polynomially related** if there exist polynomials p_1, p_2 such that for every Boolean function f,

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Theorem (Hatami, Kulkarni, and Pankratov 2010)

The following complexity measures are polynomially related:
- *block sensitivity*
- *decision tree complexity*
- *certificate complexity*
- *degree as polynomial*
- *approximate degree*
- *randomized query complexity*
- *quantum query complexity*

Why?

So if, as is conjectured, sensitivity and block-sensitivity are polynomially related, then sensitivity—arguably the most basic of all Boolean function complexity measures—ceases to be an outlier and joins a large and happy flock.

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- Low-sensitivity Boolean functions have low degrees as real polynomials.
- Any randomized algorithm to guess the parity of an *n*-bit string, which succeeds with probability $\geq \frac{2}{3}$ on the majority of strings, must make at least $\sim \sqrt{n}$ queries to the string, while any such quantum algorithm must make at least $\sim n^{1/4}$ queries (Aaronson et al. 2014).

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- Given a Boolean function f, we use $\deg(f)$ to denote the **degree** of f, i.e., degree of the unique multilinear real polynomial that represents f.

Huang (2019)'s Roadmap

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Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \to \mathbb{R}$.

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For $n \in \mathbb{N}$, if H is a $(2^{n-1} + 1)$-vertex induced subgraph of $\mathbb{B}^n$, then

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Idea 1: Largest Eigenvalue

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Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1, 0, 1\})$ be symmetric with $A_{ij} = 0$ if i, j are not adjacent in G. Then

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○ Hence $|\lambda_1| \leq \Delta(G)$. $\qquad\square$

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Theorem (Cauchy's Interlace Theorem)

Let $A \in \mathcal{M}_m(\mathbb{R})$ be symmetric. Let B be a $k \times k$ principal submatrix of A for some $m < n$. Then for all $i \in [m]$,

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○ If A be a signed adjacency matrix of $\mathbb{B}^n$, and if H an induced subgraph of $\mathbb{B}^n$ with $|V(H)| = 2^{n-1} + 1$, then

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○ **Magic!** Find a signed adjacency matrix A of $\mathbb{B}^n$ with

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Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Then $A_n \in \mathcal{M}_{2^n}(\mathbb{R})$ whose eigenvalues are $\sqrt{n}$ of multiplicity 2^{n-1}, and $-\sqrt{n}$ of multiplicity 2^{n-1}.

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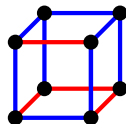

Uses Hadamard's inequality. See Huang's talk at Simons Institute.[3]

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[3] https://www.youtube.com/watch?v=EJoe4qH6kLs.

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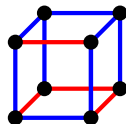# Idea 3: Magic Signed Adjacency Matrix

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○ Prove by induction that $A_n^2 = nI$.

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- Hence, the eigenvalues of A_n are either $\sqrt{n}$ or $-\sqrt{n}$.
- Since $\sum_{\lambda \text{ eigenvalue of } A} \lambda = \operatorname{tr}(A) = 0$, then exactly half of the eigenvalues of A are $\sqrt{n}$, and the rest are $-\sqrt{n}$.

$\square$

Open Questions

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- ○ Let G be a "nice" graph with high symmetry. Denote by $\alpha(G)$ the independence number of G, i.e., the size of the largest independent vertex set. Let $f(G)$ be the minimum $\Delta(H)$ over $(\alpha(G) + 1)$-vertex induced subgraphs H of G vertices. What can we say about $f(G)$? For which graphs, Huang's method would provide a tight bound?

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- The best separation between block sensitivity and sensitivity is $bs(f) = \frac{2}{3}s(f)^2 - \frac{1}{3}s(f)$ (Ambainis and Sun 2011). Close the gap between this and the quartic upper bound.

Huang's Timeline

Nov 2012: I was introduced to this problem by Michael Saks when I was a postdoc at the IAS, and got immediately attracted by the induced subgraph reformulation. And of course, in the next few weeks, I exhausted all the combinatorial techinques that I am aware of, yet I could not even improve the constant factor from the Chung–Furedi–Graham–Seymour paper, or give an alternative proof without using the isoperimetric inequality.

Around mid-year 2013: I started to believe that the maximum eigenvalue is a better parameter to look at, actually it is polynomially related to the max degree, i.e $\sqrt{\Delta(G)} \le \lambda(G) \le \Delta(G)$. And in some sense it reflects some kind of "average degree" (unfortunately the average degree itself could be very small, something like $\sqrt{n}/2^n$).

2013–2018: I revisited this conjecture every time when I learn a new tool, without any success though. But at least thinking about it helps me quickly fall asleep many nights.

Late 2018: After working on a project (with Pohoata and Klurman) that uses Cvetkovic's inertia bound to re-prove Kleitman's isodiametric theorem (it is another cute proof using algebra solving extremal combinatoiral problems), and several semesters of teaching a graduate combinatorics course, I started to have a better understanding of eigenvalue interlacing, and believe that it might help this problem. For example, applying interlacing to the original adjacency matrix, one can already show that with $(1/2+c)$ proportion of vertices, the induced subgraph has maximum degree $c'*\sqrt{n}$. I don't think this statement could follow easily from combinatorial arguments. Yet at that time, I was hoping for developing something more general using the eigenspace decomposition of the adjacency matrix, like in this unanswered MO question:

https://mathoverflow.net/questions/331825/generalization-of-cauchys-eigenvalue-interlacing-theorem

June 2019: In a Madrid hotel when I was painfully writing a proposal and trying to make the approaches sound more convincing, I finally realized that the maximum eigenvalue of any pseudo-adjacency matrix of a graph provides lower bound on the maximum degree. The rest is just a bit of trial-and-error and linear algebra.

Thank you!