

On the Sensitivity of Boolean Functions

Yuchong Pan

UBC CPSC 531F

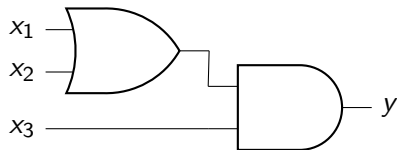
April 13, 2021

Paul Erdős famously spoke of a book, maintained by God, in which was written the simplest, most beautiful proof of each theorem. The highest compliment Erdős could give a proof was that it “came straight from the book.” In this case, I find it hard to imagine that even God knows how to prove the Sensitivity Conjecture in any simpler way than this.

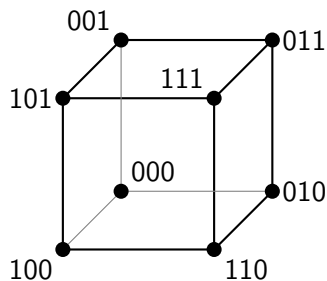
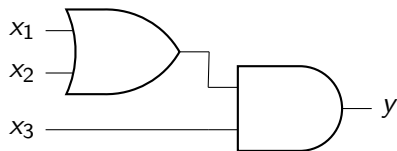
— Scott Aaronson¹

¹<https://www.scottaaronson.com/blog/?p=4229>

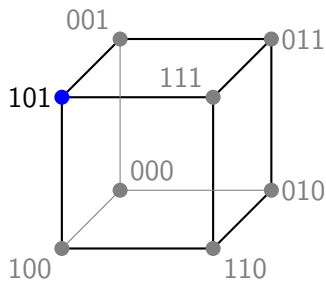
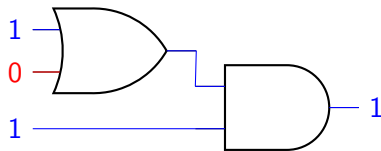
Sensitivity of Boolean Functions



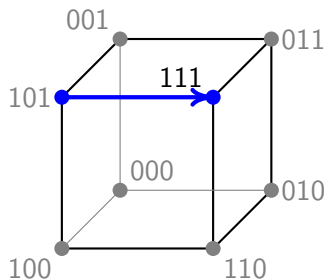
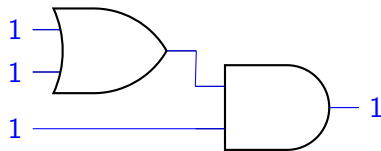
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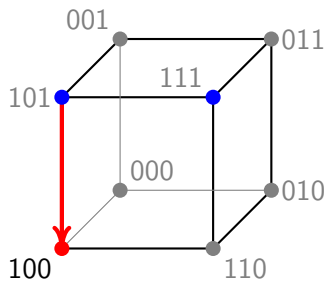
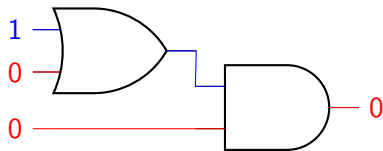
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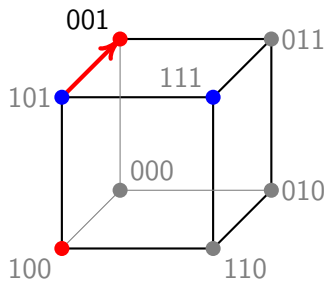
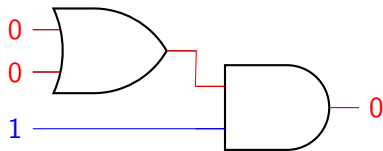
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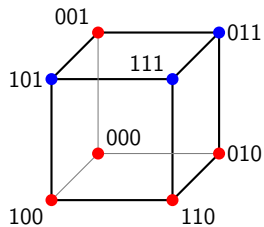
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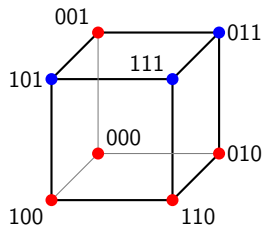
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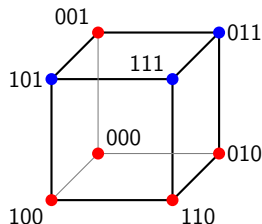
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Definition

For $x \in \{0, 1\}^n$ and $S \subseteq [n]$, we denote by x^S the binary vector obtained from x by flipping indices from S .

Sensitivity of Boolean Functions



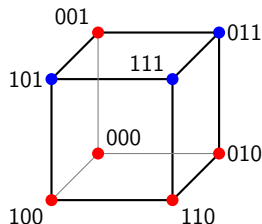
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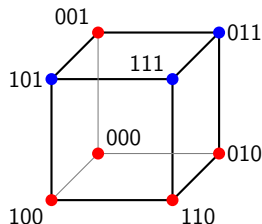
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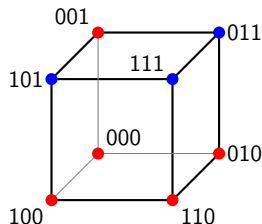
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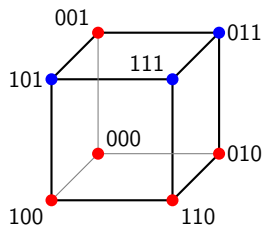
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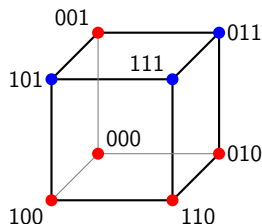
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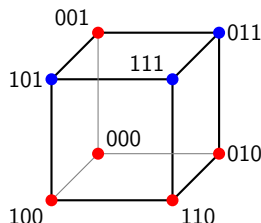
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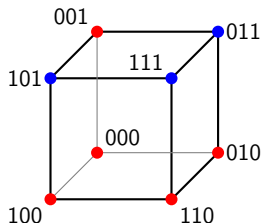
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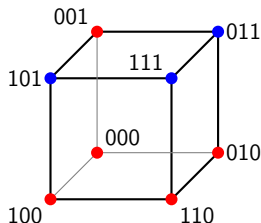
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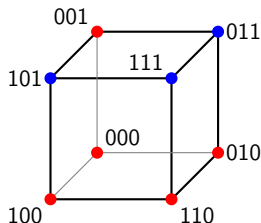
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- For $x = 11100000$, $f(x^{\{i\}}) \neq f(x)$ for $i \in \{4, \dots, 8\}$, so $s(f, x) = 5$. Indeed, $s(f) = 5$.
- For $x = 11110000$, then $\{1\}, \{2\}, \{3\}, \{4\}, \{5, 6\}, \{7, 8\}$ are 6 disjoint, **sensitive** blocks for f , so $bs(f) \geq 6 > s(f)$.

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Theorem (Rubinstein 1995)

There exists an infinite family of Boolean functions f such that

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Define $f : \{0, 1\}^{n^2} \rightarrow \{0, 1\}$ as

$$f(x_{11}, \dots, x_{nn}) = \bigvee_{i=1}^n g(x_{i1}, \dots, x_{in}),$$

where $g(x_1, \dots, x_n)$ if and only if $x_j = x_{j+1} = 1$ for some $j \in [n-1]$, and all other $x_k = 0$.

Rubinstein's Function

Claim

$$bs(f) \geq bs(f, 0) = \Omega(n^2).$$

Proof.

$$\begin{array}{ccccccc} 0 & 0 & 0 & 0 & \dots & \rightarrow & 0 \\ 0 & 0 & 0 & 0 & \dots & \rightarrow & 0 \\ 0 & 0 & 0 & 0 & \dots & \rightarrow & 0 \\ 0 & 0 & 0 & 0 & \dots & \rightarrow & 0 \\ \vdots & \vdots & \vdots & \vdots & & & \vdots \\ & & & & & & \downarrow \\ & & & & & & 0 \end{array}$$



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Hence, $s(f, x) \leq 2n$.

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Hence, $s(f, x) \leq 2n$.

- **Case 2:** $f(x) = 1$.

- If two rows output 1, $s(f, x) = 0$.
- If only one row outputs 1, $s(f, x) \leq n$.



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Question 3 (Nisan and Szegedy 1992)

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Theorem (Huang 2019)

For every Boolean function f ,

$$bs(f) \leq s(f)^4.$$

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Complexity measures α, β of Boolean functions are **polynomially related** if there exist polynomials p_1, p_2 such that for every Boolean function f ,

$$\alpha(f) \leq p_1(\beta(f)), \quad \beta(f) \leq p_2(\alpha(f)).$$

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Theorem (Hatami, Kulkarni, and Pankratov 2010)


The following complexity measures are polynomially related:

- *block sensitivity*
- *decision tree complexity*
- *certificate complexity*
- *degree as polynomial*
- *approximate degree*
- *randomized query complexity*
- *quantum query complexity*

Why?

So if, as is conjectured, sensitivity and block-sensitivity are polynomially related, then sensitivity—arguably the most basic of all Boolean function complexity measures—ceases to be an outlier and joins a large and happy flock.

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- Low-sensitivity Boolean functions have low degrees as real polynomials.
- Any randomized algorithm to guess the parity of an n -bit string, which succeeds with probability $\geq \frac{2}{3}$ on the majority of strings, must make at least $\sim \sqrt{n}$ queries to the string, while any such quantum algorithm must make at least $\sim n^{1/4}$ queries (Aaronson et al. 2014).

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- Given a Boolean function f , we use $\deg(f)$ to denote the **degree** of f , i.e., degree of the unique multilinear real polynomial that represents f .

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Theorem (Gotsman and Linial 1992)

T.F.A.E. for any monotone function $h : \mathbb{N} \rightarrow \mathbb{R}$.

- *For any induced subgraph H of \mathbb{B}^n with $|V(H)| \neq 2^{n-1}$, we have $\Gamma(H) \geq h(n)$.*
- *For any Boolean function f , we have $s(f) \geq h(\deg(f))$.*

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For $n \in \mathbb{N}$, if H is a $(2^{n-1} + 1)$ -vertex induced subgraph of \mathbb{B}^n , then

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Idea 1: Largest Eigenvalue

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Lemma

Let G be an m -vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1, 0, 1\})$ be symmetric with $A_{ij} = 0$ if i, j are not adjacent in G . Then

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- Hence $|\lambda_1| \leq \Delta(G)$.



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Theorem (Cauchy's Interlace Theorem)

Let $A \in \mathcal{M}_m(\mathbb{R})$ be symmetric. Let B be a $k \times k$ principal submatrix of A for some $m < n$. Then for all $i \in [m]$,

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- If A be a signed adjacency matrix of \mathbb{B}^n , and if H an induced subgraph of \mathbb{B}^n with $|V(H)| = 2^{n-1} + 1$, then

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- **Magic!** Find a signed adjacency matrix A of \mathbb{B}^n with

$$\lambda_{2^{n-1}}(A) = \sqrt{n}.$$

Idea 3: Magic Signed Adjacency Matrix

³<https://www.youtube.com/watch?v=EJoe4qH6kLs>.

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Lemma

Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Then $A_n \in \mathcal{M}_{2^n}(\mathbb{R})$ whose eigenvalues are \sqrt{n} of multiplicity 2^{n-1} , and $-\sqrt{n}$ of multiplicity 2^{n-1} .

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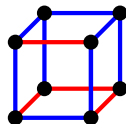
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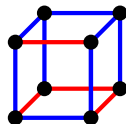
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Uses Hadamard's inequality. See Huang's talk at Simons Institute.³

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- Prove by induction that $A_n^2 = nI$.

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Proof.

- Prove by induction that $A_n^2 = nl$. For $n = 1$, $A_1^2 = I$.
- Suppose $A_{n-1}^2 = (n-1)I$. Then

$$A_n^2 = \begin{bmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{bmatrix} = \begin{bmatrix} nl & 0 \\ 0 & nl \end{bmatrix} = nl.$$

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- Hence, the eigenvalues of A_n are either \sqrt{n} or $-\sqrt{n}$.
- Since $\sum_{\lambda \text{ eigenvalue of } A} \lambda = \text{tr}(A) = 0$, then exactly half of the eigenvalues of A are \sqrt{n} , and the rest are $-\sqrt{n}$.



Open Questions

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- Let G be a “nice” graph with high symmetry. Denote by $\alpha(G)$ the independence number of G , i.e., the size of the largest independent vertex set. Let $f(G)$ be the minimum $\Delta(H)$ over $(\alpha(G) + 1)$ -vertex induced subgraphs H of G vertices. What can we say about $f(G)$? For which graphs, Huang’s method would provide a tight bound?

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- The best separation between block sensitivity and sensitivity is $bs(f) = \frac{2}{3}s(f)^2 - \frac{1}{3}s(f)$ (Ambainis and Sun 2011). Close the gap between this and the quartic upper bound.

Huang's Timeline

Nov 2012: I was introduced to this problem by Michael Saks when I was a postdoc at the IAS, and got immediately attracted by the induced subgraph reformulation. And of course, in the next few weeks, I exhausted all the combinatorial techniques that I am aware of, yet I could not even improve the constant factor from the Chung–Furedi–Graham–Seymour paper, or give an alternative proof without using the isoperimetric inequality.

Around mid-year 2013: I started to believe that the maximum eigenvalue is a better parameter to look at, actually it is polynomially related to the max degree, i.e. $\sqrt{\Delta(G)} \leq \lambda(G) \leq \Delta(G)$. And in some sense it reflects some kind of “average degree” (unfortunately the average degree itself could be very small, something like $\sqrt{n}/2^n$).

2013–2018: I revisited this conjecture every time when I learn a new tool, without any success though. But at least thinking about it helps me quickly fall asleep many nights.

Late 2018: After working on a project (with Pohoata and Klurman) that uses Cvetkovic's inertia bound to re-prove Kleitman's isodiametric theorem (it is another cute proof using algebra solving extremal combinatorial problems), and several semesters of teaching a graduate combinatorics course, I started to have a better understanding of eigenvalue interlacing, and believe that it might help this problem. For example, applying interlacing to the original adjacency matrix, one can already show that with $(1/2+c)$ proportion of vertices, the induced subgraph has maximum degree $c\sqrt{n}$. I don't think this statement could follow easily from combinatorial arguments. Yet at that time, I was hoping for developing something more general using the eigenspace decomposition of the adjacency matrix, like in this unanswered MO question:

<https://mathoverflow.net/questions/331825/generalization-of-cauchys-eigenvalue-interlacing-theorem>

June 2019: In a Madrid hotel when I was painfully writing a proposal and trying to make the approaches sound more convincing, I finally realized that the maximum eigenvalue of any pseudo-adjacency matrix of a graph provides lower bound on the maximum degree. The rest is just a bit of trial-and-error and linear algebra.

Thank you!