On the Sensitivity of Boolean Functions

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UBC CPSC 531F

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Paul Erdős famously spoke of a book, maintained by God, in which was written the simplest, most beautiful proof of each theorem. The highest compliment Erdős could give a proof was that it "came straight from the book." In this case, I find it hard to imagine that even God knows how to prove the Sensitivity Conjecture in any simpler way than this.

— *Scott Aaronson*[1]

[1] https://www.scottaaronson.com/blog/?p=4229.

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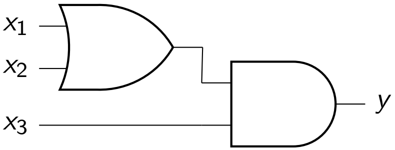

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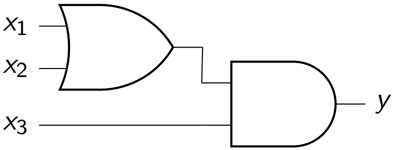

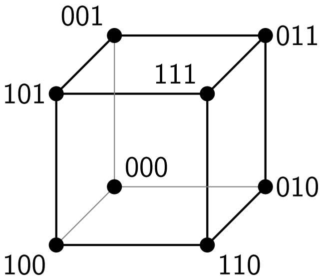

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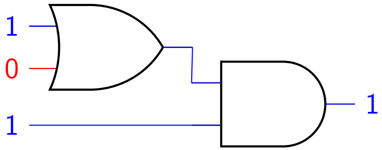

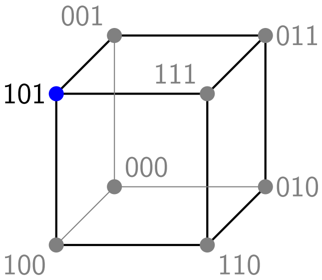

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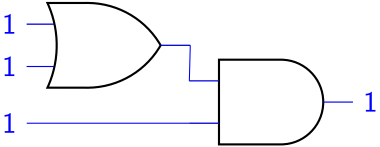

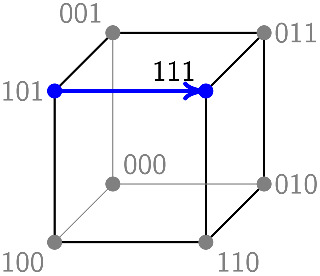

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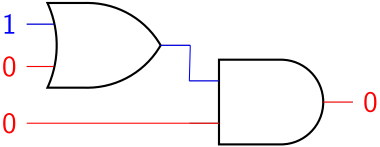

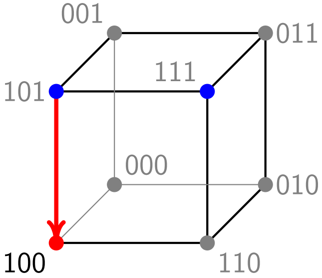

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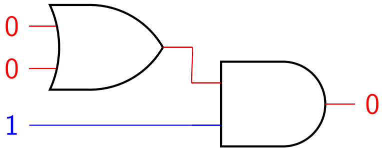

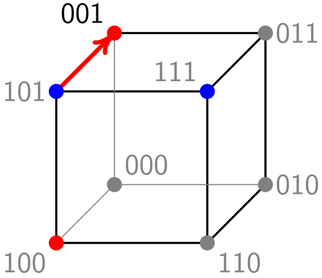

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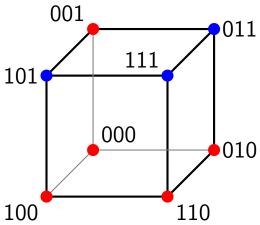

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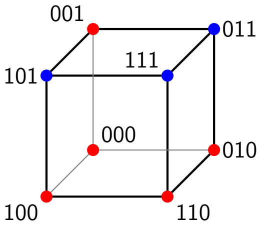

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For $x \in \{0,1\}^n$ and $S \subseteq [n]$, we denote by x^S the binary vector obtained from x by flipping indices from S.

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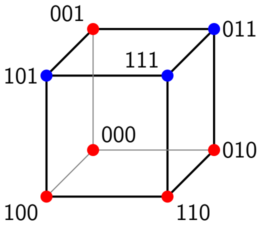

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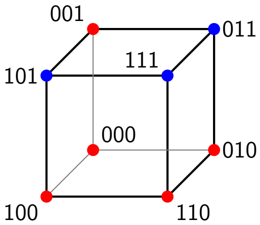

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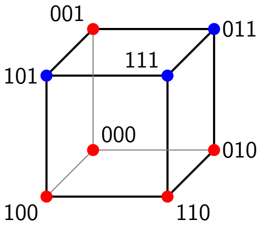

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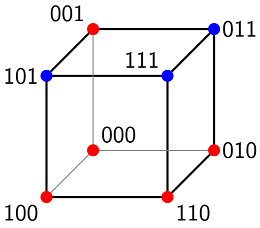

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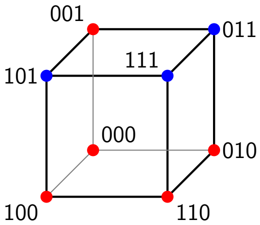

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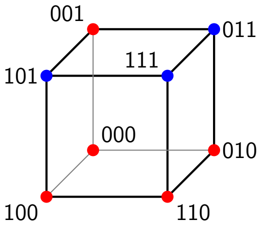

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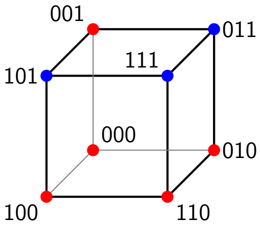

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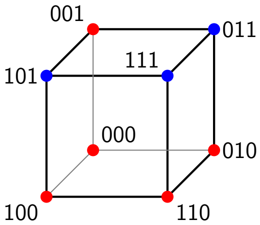

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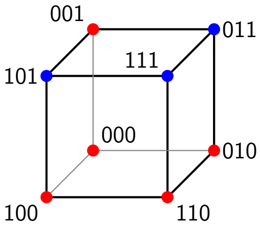

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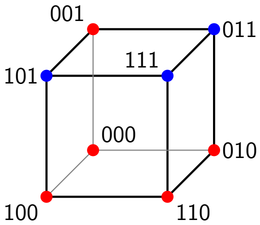

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- For $x = 11110000$, then $\{1\}, \{2\}, \{3\}, \{4\}, \{5,6\}, \{7,8\}$ are 6 disjoint, **sensitive** blocks for f, so $bs(f) \geq 6 > s(f)$.

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Theorem (Rubinstein 1995)

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Define $f : \{0,1\}^{n^2} \to \{0,1\}$ as

$$f\left(x_{11}, \ldots, x_{nn}\right) = \bigvee_{i=1}^{n} g\left(x_{i1}, \ldots, x_{in}\right),$$

where $g(x_1, \ldots, x_n)$ if and only if $x_j = x_{j+1} = 1$ for some $j \in [n-1]$, and all other $x_k = 0$.

Rubinstein's Function

Claim

$bs(f) \geq bs(f, 0) = \Omega(n^2)$.

Proof.

$$
\begin{array}{cccccccc}
0 & 0 & 0 & 0 & \cdots & \to & 0 \\
0 & 0 & 0 & 0 & \cdots & \to & 0 \\
0 & 0 & 0 & 0 & \cdots & \to & 0 \\
0 & 0 & 0 & 0 & \cdots & \to & 0 \\
\vdots & \vdots & \vdots & \vdots & & & \vdots \\
 & & & & & & \downarrow \\
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$$

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  Hence, $s(f, x) \leq 2n$.
- **Case 2:** $f(x) = 1$.
  - If two rows output 1, $s(f, x) = 0$.
  - If only one row outputs 1, $s(f, x) \leq n$.

$\square$

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Is $bs(f)$ always polynomial in $s(f)$?

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Theorem (Huang 2019)

For every Boolean function f,

$$bs(f) \leq s(f)^4.$$

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Complexity measures α, β of Boolean functions are **polynomially related** if there exist polynomials p_1, p_2 such that for every Boolean function f,

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Theorem (Hatami, Kulkarni, and Pankratov 2010)

The following complexity measures are polynomially related:

- *block sensitivity*
- *decision tree complexity*
- *certificate complexity*
- *degree as polynomial*
- *approximate degree*
- *randomized query complexity*
- *quantum query complexity*

Why?

So if, as is conjectured, sensitivity and block-sensitivity are polynomially related, then sensitivity—arguably the most basic of all Boolean function complexity measures——ceases to be an outlier and joins a large and happy flock.

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- Low-sensitivity Boolean functions have low degrees as real polynomials.
- Any randomized algorithm to guess the parity of an n-bit string, which succeeds with probability $\geq \frac{2}{3}$ on the majority of strings, must make at least $\sim \sqrt{n}$ queries to the string, while any such quantum algorithm must make at least $\sim n^{1/4}$ queries (Aaronson et al. 2014).

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Notations

- Given an undirected graph G with $|V(G)| = m$, a matrix $A \in \mathcal{M}_m(\{-1, 0, 1\})$ is a **signed adjacency matrix** of G when $A_{ij} = 0$ if and only if i, j are not adjacent in G.
- For an undirected graph G, we use $\Delta(G)$ to denote the maximum degree of G.
- For an undirected graph G and an induced subgraph H of G, we use $G \setminus H$ to denote the subgraph of G induced by the vertex set $V(G) \setminus V(H)$.
- We use $\mathbb{B}^n$ to denote the n-dimensional Boolean hypercube.
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- Given a Boolean function f, we use $\deg(f)$ to denote the **degree** of f, i.e., degree of the unique multilinear real polynomial that represents f.

Huang (2019)'s Roadmap

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Theorem (Gotsman and Linial 1992)

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For $n \in \mathbb{N}$, if H is a $(2^{n-1}+1)$-vertex induced subgraph of $\mathbb{B}^n$, then

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Idea 1: Largest Eigenvalue

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Lemma

Let G be an m-vertex undirected graph. Let $A \in \mathcal{M}_m(\{-1,0,1\})$ be symmetric with $A_{ij} = 0$ if i,j are not adjacent in G. Then

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- Hence $|\lambda_1| \leq \Delta(G)$.

$\square$

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Theorem (Cauchy's Interlace Theorem)

Let $A \in \mathcal{M}_m(\mathbb{R})$ be symmetric. Let B be a $k \times k$ principal submatrix of A for some $m < n$. Then for all $i \in [m]$,

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- If A be a signed adjacency matrix of $\mathbb{B}^n$, and if H an induced subgraph of $\mathbb{B}^n$ with $|V(H)| = 2^{n-1} + 1$, then

$$\lambda_1\left(A_H\right) \geq \lambda_{1+2^n-(2^{n-1}+1)}(A) = \lambda_{2^{n-1}}(A),$$

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  where A_H is the corresponding principal submatrix of A.
- **Magic!** Find a signed adjacency matrix A of $\mathbb{B}^n$ with

$$\lambda_{2^{n-1}}(A) = \sqrt{n}.$$

Idea 3: Magic Signed Adjacency Matrix

[3] https://www.youtube.com/watch?v=EJoe4qH6kLs.

Idea 3: Magic Signed Adjacency Matrix

Lemma

Let

$$A_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \qquad A_n = \begin{bmatrix} A_{n-1} & I \\ I & -A_{n-1} \end{bmatrix}.$$

Then $A_n \in \mathcal{M}_{2^n}(\mathbb{R})$ whose eigenvalues are $\sqrt{n}$ of multiplicity 2^{n-1}, and $-\sqrt{n}$ of multiplicity 2^{n-1}.

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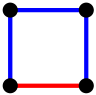

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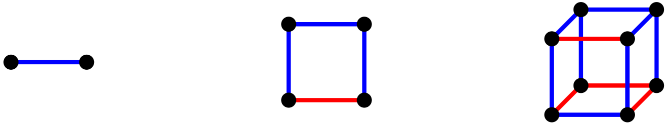

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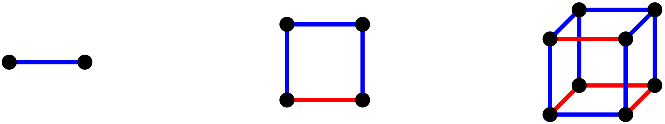

Uses Hadamard's inequality. See Huang's talk at Simons Institute.[3]

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- Prove by induction that $A_n^2 = nI$.

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- Prove by induction that $A_n^2 = nI$. For $n = 1$, $A_1^2 = I$.
- Suppose $A_{n-1}^2 = (n-1)I$. Then

$$A_n^2 = \begin{bmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{bmatrix} = \begin{bmatrix} nI & 0 \\ 0 & nI \end{bmatrix} = nI.$$

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- Hence, the eigenvalues of A_n are either $\sqrt{n}$ or $-\sqrt{n}$.

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- Prove by induction that $A_n^2 = nI$. For $n = 1$, $A_1^2 = I$.
- Suppose $A_{n-1}^2 = (n-1)I$. Then

$$A_n^2 = \begin{bmatrix} A_{n-1}^2 + I & 0 \\ 0 & A_{n-1}^2 + I \end{bmatrix} = \begin{bmatrix} nI & 0 \\ 0 & nI \end{bmatrix} = nI.$$

- Hence, the eigenvalues of A_n are either $\sqrt{n}$ or $-\sqrt{n}$.
- Since $\sum_{\lambda \text{ eigenvalue of } A} \lambda = \mathrm{tr}(A) = 0$, then exactly half of the eigenvalues of A are $\sqrt{n}$, and the rest are $-\sqrt{n}$.

□

Open Questions

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- Let $g(n,k)$ be the minimum t such that every t-vertex induced subgraphs H of $\mathbb{B}^n$ has $\Delta(H) \geq k$. Huang (2019) shows $g(n,\sqrt{n}) = 2^{n-1}+1$.

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- The best separation between block sensitivity and sensitivity is $bs(f) = \frac{2}{3}s(f)^2 - \frac{1}{3}s(f)$ (Ambainis and Sun 2011). Close the gap between this and the quartic upper bound.

Huang's Timeline

Nov 2012: I was introduced to this problem by Michael Saks when I was a postdoc at the IAS, and got immediately attracted by the induced subgraph reformulation. And of course, in the next few weeks, I exhausted all the combinatorial techinques that I am aware of, yet I could not even improve the constant factor from the Chung-Furedi-Graham-Seymour paper, or give an alternative proof without using the isoperimetric inequality.

Around mid-year 2013: I started to believe that the maximum eigenvalue is a better parameter to look at, actually it is polynomially related to the max degree, i.e \sqrt{\Delta(G)} \le \lambda(G) \le \Delta(G). And in some sense it reflects some kind of "average degree" (unfortunately the average degree itself could be very small, something like \sqrt{n}/2^n).

2013-2018: I revisited this conjecture every time when I learn a new tool, without any success though. But at least thinking about it helps me quickly fall asleep many nights.

Late 2018: After working on a project (with Pohoata and Klurman) that uses Cvetkovic's inertia bound to re-prove Kleitman's isodiametric theorem (it is another cute proof using algebra solving extremal combinatoiral problems), and several semesters of teaching a graduate combinatorics course, I started to have a better understanding of eigenvalue interlacing, and believe that it might help this problem. For example, applying interlacing to the original adjacency matrix, one can already show that with (1/2+c) proportion of vertices, the induced subgraph has maximum degree c'*\sqrt{n}. I don't think this statement could follow easily from combinatorial arguments. Yet at that time, I was hoping for developing something more general using the eigenspace decomposition of the adjacency matrix, like in this unanswered MO question:

https://mathoverflow.net/questions/331825/generalization-of-cauchys-eigenvalue-interlacing-theorem

June 2019: In a Madrid hotel when I was painfully writing a proposal and trying to make the approaches sound more convincing, I finally realized that the maximum eigenvalue of any pseudo-adjacency matrix of a graph provides lower bound on the maximum degree. The rest is just a bit of trial-and-error and linear algebra.

Thank you!