Roundtrip Spanners and Roundtrip Routing in Directed Graphs

by Roditty, Thorup, and Zwick (2008)

Yuchong Pan

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Spanners in **Undirected** Graphs

Definition (Spanner)

Let $G = (V, E)$ be a weighted **undirected** graph. A subgraph H of G is a t-**spanner** if $d_H(u, v) \leq t \cdot d_G(u, v)$ for all $u, v \in V$.

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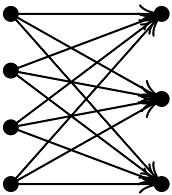

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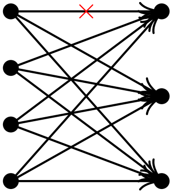

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Let G be a weighted **directed** graph. A subgraph H of G is a t-**roundtrip-spanner** if $d_H(u \rightleftarrows v) \leq t \cdot d_G(u \rightleftarrows v)$ for $u, v \in V$.

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Theorem (Roditty, Thorup, and Zwick, 2008)

For any $k \in \mathbb{N}, \varepsilon > 0$, any weighted **directed** *graph on n vertices with edge weights from $[1, W]$ has a $(2k + \varepsilon)$-roundtrip-spanner with $O((k^2/\varepsilon)n^{1+1/k} \log(nW))$ edges.*

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Procedure $\textsc{PartialSpanner}(R)$

Input: $R \in [1, 2nW]$.
Output: A subgraph H of G with $O(kn^{1+1/k})$ edges such that $d_H(u \rightleftarrows v) \leq 2kR$ for all $u, v \in V$ with $d_G(u \rightleftarrows v) \leq R$.

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The Approach of Cohen (1998)

foreach $i \leftarrow 1, \ldots, \log_{1+\varepsilon}(2nW)$ **do**
    $R_i \leftarrow (1+\varepsilon)^i$
    $H_i \leftarrow \textsc{PartialSpanner}(R_i)$
$H \leftarrow \bigcup\{H_i : i \in [\log_{1+\varepsilon}(2nW)]\}$
return H

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Claim: H is a $2k(1+\varepsilon)$-roundtrip-spanner of G with $O(kn^{1+1/k} \log_{1+\varepsilon}(nW))$ edges.

Constructing Partial Spanners

Definition (Ball)

For every $V' \subset V$, $v \in V$ and $r \geq 0$, let

$$\mathrm{ball}_{V'}(v, r) = \left\{ u \in V' : d_{G[V']}(v \rightleftarrows u) \leq r \right\},$$

called a **ball** centered at v of radius r.

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Definition ((k, R)-cover)

A set $\mathcal{C}$ of balls is a (k, R)-**cover** of a directed graph $G = (V, E)$ if each ball in $\mathcal{C}$ is of radius at most kR, and for all $u, v \in V$ with $d_G(u \rightleftarrows v) \leq R$, there is a ball $B \in \mathcal{C}$ such that $u, v \in B$.

Constructing Partial Spanners

Procedure $\textsc{Cover}(G, k, R)$

$\mathcal{C} \leftarrow \emptyset$
$V_{k-1} \leftarrow V(G)$
foreach $i \leftarrow k-1, \ldots, 0$ **do**
    $S_i \leftarrow \textsc{Sample}(V_i, n^{-i/k})$
    $\mathcal{C} \leftarrow \mathcal{C} \cup \{\mathrm{ball}_{V_i}(v, (i+1)R) : v \in S_i\}$
    $V_{i-1} \leftarrow V_i \setminus \bigcup_{v \in S_i} \mathrm{ball}_{V_i}(v, iR)$
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Theorem

The collection $\mathcal{C} = \textsc{Cover}(G, k, R)$ is a (k, R)-cover of a directed graph $G = (V, E)$. For each $v \in V$, the **expected** *number of balls in $\mathcal{C}$ containing v is at most $kn^{1/k}$.*

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- Let i be the largest index so that a vertex u' on c is contained in the **core** $\text{ball}_{V_i}(w, iR)$ of $\text{ball}_{V_i}(w, (i+1)R)$ added to $\mathcal{C}$.

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- By the triangle inequality,

$$\begin{aligned} d_{G[V_i]}(w \rightleftarrows v) &\leq d_{G[V_i]}(w \rightleftarrows u) + d_{G[V_i]}(u \rightleftarrows v) \\ &\leq iR + R = (i+1)R. \end{aligned}$$

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- Therefore, $v \in \mathrm{ball}_{V_i}(w, (i+1)R)$.
- Similarly, $u \in \mathrm{ball}_{V_i}(w, (i+1)R)$, completing the proof.

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- $\mathbb{P}[u \in S_i \wedge v \in \text{ball}_{V_i}(u, (i+1)R)] = \left(1 - n^{-(i+1)/k}\right)^{|B|} n^{-i/k}$.

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- The expected number of balls containing v in iteration i is $|B| \left(1 - n^{-(i+1)/k}\right)^{|B|} n^{-i/k}$.

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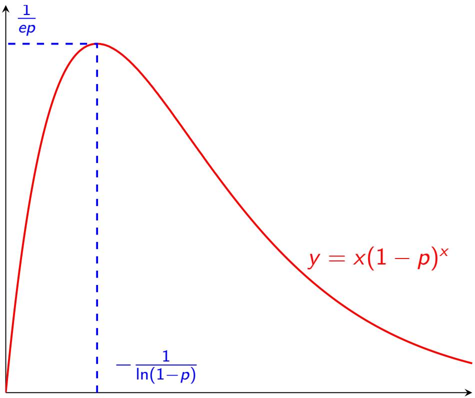

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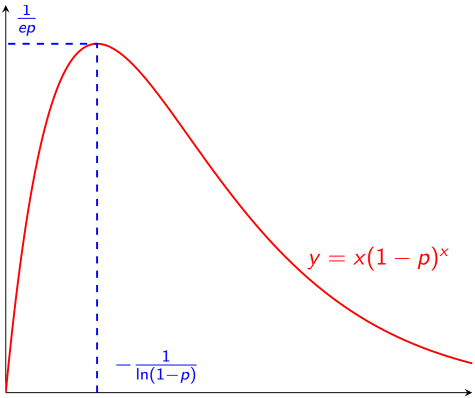

$$\mathbb{E}[\#\text{ balls containing } v \text{ in iteration } i]$$

$$= |B| \left(1 - n^{-(i+1)/k}\right)^{|B|} n^{-i/k} \leq \frac{n^{-i/k}}{en^{-(i+1)/k}} = \frac{n^{1/k}}{e}.$$

Double-Trees

Definition (Double-tree)

Let $G = (V, E)$ be a weighted directed graph. Let $V' \subset V$, $v \in V$ and $r \geq 0$. Let $B = \text{ball}_{V'}(v, r)$.
Let $\text{OutTree}(B, v)$ be a tree containing directed shortest paths in $G[V']$ from v to all the vertices in B.
Let $\text{InTree}(B, v)$ be a tree containing directed shortest paths in $G[V']$ from all the vertices in B to v.
Let $\text{InOutTrees}(B, v) = \text{InTree}(B, v) \cup \text{OutTree}(B, v)$, refered to as a **double-tree**.

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Let $V' \subset V$ and $v \in V$. If $u \in B := \text{ball}_{V'}(v, r)$ and w is on a shortest path in $G[V']$ from u to v, or from v to u, then $w \in B$.

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Corollary

$V(\mathrm{InOutTrees}(B, v)) \subset B$, *so* $e(\mathrm{InOutTrees}(B, v)) \leq 2(|B| - 1)$.

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- Therefore, H is a $(2k + \varepsilon)$-roundtrip-spanner of G with $O((k^2/\varepsilon)n^{1+1/k}\log(nW))$ edges **in expectation**.

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- Therefore, H is a $(2k+\varepsilon)$-roundtrip-spanner of G with $O((k^2/\varepsilon)n^{1+1/k}\log(nW))$ edges **in expectation**.

Application: Compact roundtrip routing schemes.

State-of-the-Art

Theorem (Cen, Duan, and Gu, 2019)

For any $k \in \mathbb{N}$, any weighted directed graph on n vertices has a $(2k-1)$-roundtrip-spanner with $O(kn^{1+1/k} \log n)$ edges.

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Proposition

If the Erdős girth conjecture is true, there is an **undirected** *graph on n vertices such that any $(2k-1)$-spanner has $\Omega(n^{1+1/k})$ edges.*

Thank you and happy holidays!