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ECO 518 – Econometric Theory II, Spring 2022

Problem Set 1, due February 1

1. Let $\varepsilon_t \sim$i.i.d.$\mathcal{N}(0,1)$, $t = 0, 1, 2 \ldots$, $y_0 = 0$ and let a and $b > 0$ real numbers. Are the following stochastic processes $\{y_t\}_{t=1}^{\infty}$ martingale difference sequences with respect to the information set generated past values of (y_t, ε_t) (i.e. relative to $\mathfrak{F}_{t-1} = \sigma(y_{t-1}, \varepsilon_{t-1}, y_{t-2}, \varepsilon_{t-2}, \ldots, y_0, \varepsilon_0)$ for $t \geq 0$)? Are they White Noise? Are they covariance stationary?

(Just to make sure you don't get confused by the notation in the lecture notes: For $(y_t, \mathfrak{F}_t)$ to be a mds, it must be the case that $E[y_t|\mathfrak{F}_{t-1}] = 0$ and $E[|y_t|]$ exists.)

(a) $y_t = a + b\varepsilon_0$

(b) $y_t = t\varepsilon_t$

(c) $y_t = \varepsilon_t + \varepsilon_{t-1}^2 - 1$

(d) $y_t = (\mathbf{1}[|y_{t-1}| > 2] + 1)\varepsilon_t$

(e) $y_t = \varepsilon_t \varepsilon_{t-1}$

2. Suppose the conditional probability density function of y_t given $\{y_s\}_{s=1}^{t-1}$ is $f_t(\cdot|\theta, y_{t-1}, \cdots y_1)$, where θ is a scalar parameter. Let $s_t(\theta) = \partial \ln f_t(y_t|\theta, y_{t-1}, \cdots y_1)/\partial\theta$. Suppose the data is generated with $\theta = \theta_0$. Show that $\{s_t(\theta_0)\}_{t=1}^{\infty}$ is a mds relative to $\mathfrak{F}_{t-1} = \sigma(\{y_s\}_{s=1}^{t-1})$. (You may differentiate under the integral sign, but note where you need this to work).

3. Show that a covariance stationary (mean zero) AR(p) process $y_t = \phi_1 y_{t-1} + \cdots + \phi_p y_{t-p} + \varepsilon_t$ with $\varepsilon_t \sim iid(0, \sigma^2)$ has autocovariances $\gamma(k)$ that satisfy

$$\gamma(k) = \begin{cases} \phi_1\gamma(k-1) + \ldots + \phi_p\gamma(k-p) & \text{for } k > 0 \\ \phi_1\gamma(1) + \ldots + \phi_p\gamma(p) + \sigma^2 & \text{for } k = 0 \end{cases}.$$

(For $k > 0$, the first equation shows that autocovariances satisfy the same difference equation as the underlying AR process. They are useful, because the system of equations for $k =$

$0, \cdots, p$ can be solved to obtain an expression for $(\gamma(0), \gamma(1), \cdots, \gamma(p))$ in terms of the ϕ's and σ^2, similar to the equation $\Sigma_0 = \Phi\Sigma_0\Phi' + D_e$ derived in the lecture notes.)

4. Let y_t be the stationary AR(2) $y_t = 1.2y_{t-1} - 0.5y_{t-2} + \varepsilon_t$, where $\varepsilon_t \sim iid(0, \sigma^2)$.

(a) Compute the eigenvalues of the companion matrix Φ.

(b) Compute the variance of y_t.

(c) Compute the roots of the lag polynomial, and their modulus, and compare your answer to what you obtained in (a).

5. Program a function in matlab (or any other programming language) that takes $(\phi_1, \ldots, \phi_p)$ and σ^2 as inputs, and that yields $\gamma(0), \ldots, \gamma(p-1)$ of the corresponding stationary AR(p) as output. Check that it you get the right answer with the parameters from question 4.

6. Suppose daily stock prices at close, y_t, follow a random walk, with t denoting trading days. Now consider the weekly series x_s of the average stock price over the week (5 trading days), with s denoting weeks, computed as an average of y_t. Does this series also follow a random walk? If not, what is the autocovariance function of the first differences $x_s - x_{s-1}$?

7. Suppose u_t follows the stationary ARMA(1,1) process $u_t = \phi u_{t-1} + \varepsilon_t - \theta\varepsilon_{t-1}$, where $\varepsilon_t \sim iid(0, 1)$, and let $\gamma(k) = E[u_t u_{t-k}]$.

(a) Derive the moving average representation for u_t, that is find the values of c_i such that $y_t = c_0\varepsilon_t + c_1\varepsilon_{t-1} + c_2\varepsilon_{t-2} + \cdots$.

(b) Show that $\gamma(k) = \phi\gamma(k-1)$ for $k \geq 2$.

8. Is the MA(2) process $\varepsilon_t + 1.2\varepsilon_{t-1} - 0.4\varepsilon_{t-2}$ invertible? Compute the autocovariances of the process for $\varepsilon_t \sim iid(0, 4)$.

9. Suppose $\{y_t\}_{t=-\infty}^{\infty}$ is generated by the non-invertible MA(1) process $y_t = \varepsilon_t - 2\varepsilon_{t-1}$ where $\varepsilon_t \sim iid(0, \sigma^2)$. Suppose you mistakenly think the process is an invertible MA(1) $y_t = \tilde{\varepsilon}_t - \theta_1\tilde{\varepsilon}_{t-1}$ with $\theta_1 = 1/2$, and you solve for $\tilde{\varepsilon}_t$ by imposing this assumption on y_t. Find an expression of $\tilde{\varepsilon}_t$ as a function of ε_t. Compute the autocovariances of $\tilde{\varepsilon}_t$.