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ECO 518 – Econometric Theory II, Spring 2022

Problem Set 5, due March 1

1. Consider the series JCXFE (quarterly price deflator of personal consumption expenditures less food and energy). Let p_t denote the value of this series. Compute $\pi_t = 400 \times \ln(p_t/p_{t-1})$ which is the "core" rate of inflation (in percentage points at an annual rate). Let $y_t = (1-L)\pi_t$.

(a) Using the data on y_t from 1959:Q3 – 2021:Q4, compute the sample variance of y_t and the first autocovariance (i.e. $\hat{\gamma}(0)$ and $\hat{\gamma}(1)$).

(b) Suppose that y_t follows the MA(1) process $y_t = (1-\theta L)\varepsilon_t$, where $\varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$. Compute estimates of θ and σ_ε^2 using the autocovariances from (a).

(c) Using the MA(1) model with parameters estimated in (b), compute $y_{t+1|t}$ (the one-quarter ahead forecasts of y_{t+1} based on $\{y_s\}_{s=1}^t$) and the implied value of $\pi_{t+1|t}$ for t =2022:Q1, 2022:Q2,. . ., 2022:Q4.

(d) Given the MA(1) model for $y_t = (1-\theta L)\varepsilon_t$, show that π_t can be represented as

$$\pi_t = e_t + \tau_t, \text{ where } \tau_t = \tau_{t-1} + a_t$$

where $\{e_t\}$ and $\{a_t\}$ are uncorrelated white noise processes with variances σ_e^2 and σ_a^2. Derive the function relating (σ_e^2, σ_a^2) and (θ, σ_e^2). Estimate (σ_e^2, σ_a^2) using the estimates of (θ, σ_e^2) from part (b).

(e) Compute $\tau_{t|t}$ and $P_{t|T}$ using the Kalman filter. Initialize the filter using $\tau_{0|0} = 0$ and $P_{0|0} = 10000$.

(f) Compare the values of $\tau_{t|t}$ computed in (e) and $\pi_{t+1|t}$ computed in (c).

(g) Use the Kalman smoother to compute $\tau_{t|T}$ for t =1959:Q3 – 2021:Q4.

2. Consider the linear panel model

$$y_{it} = \alpha_i + x_{it}'\beta + \varepsilon_{it},\ i = 1, \ldots, n,\ t = 1, \ldots, T$$

and suppose (Y_i, X_i) is i.i.d. across $i = 1, \ldots n$ with $Y_i = (y_{1i}, \ldots, y_{iT})'$ and $X_i = (x_{i1}, \ldots, x_{iT})'$.

(a) Verify the claim in the lecture notes that $\hat{\beta}_{FE}$ corresponds to the MLE with $\varepsilon_{it} \sim iid\mathcal{N}(0, \sigma_\varepsilon^2)$ when we treat the α_i as parameters.

(b) Show that the first difference estimator corresponds to the MLE in the model where $\varepsilon_{it} - \varepsilon_{i,t-1} \sim iid\mathcal{N}(0, \sigma_\varepsilon^2)$ (again treating the α_i as parameters.)

(c) Let G be a rank $T-1$ transformation matrix with $G'e = 0$, as in the lecture notes, and define $\tilde{Y}_i = G'Y_i$ and $\tilde{X}_i = G'X_i$. Consider the (infeasible) GLS estimator of $\tilde{Y}_i$ on $\tilde{X}_i$ under the assumption $E[\varepsilon_{it}|X_i] = 0$. Does this estimator depend on the choice of G?

3. Suppose

$$y_{it} = \alpha_i + x_{it}\beta + u_{it},\ i = 1, \ldots, n,\ t = 1, \ldots, T$$

with independence across i, $E[u_{it}|x_{i1}, \ldots, x_{iT}] = 0$ and suppose x_{it} and u_{it} are stationary and weakly dependent when viewed as a time series. Consider the first difference estimator $\hat{\beta}$ of β, so that by assumption, $g_{it} = \Delta x_{it}\Delta u_{it}$ is a mean-zero weakly dependent stationary time series.

We will consider asymptotics where n is fixed, and $T \to \infty$. Assume that all necessary regularity conditions hold.

(a) Argue that for each i,

$$\frac{1}{T}\sum_{t=2}^{T}(\Delta x_{it})^2 \xrightarrow{p} H$$

with H independent of i, so that also $\frac{1}{nT}\sum_{i=1}^{n}\sum_{t=2}^{T}(\Delta x_{it})^2 \xrightarrow{p} H$.

(b) Argue that for each i,

$$\eta_{iT} = T^{-1/2}\sum_{t=2}^{T} g_{it} \Rightarrow \mathcal{N}(0, V)$$

with V independent of i, so that also $n^{-1/2}\sum_{i=1}^{n}\eta_{iT} \Rightarrow \mathcal{N}(0, V)$.

(c) Argue that

$$(nT)^{1/2}(\hat{\beta} - \beta) = H^{-1}n^{-1/2}\sum_{i=1}^{n}\eta_{iT} + o_p(1) \tag{1}$$

so that $(nT)^{1/2}(\hat{\beta}-\beta) \Rightarrow \mathcal{N}(0, H^{-1}VH^{-1})$.

(d) Argue that with $\hat{g}_i = T^{-1/2}\sum_{t=1}^{T}\hat{g}_{it}$ where $\hat{g}_{it} = \Delta x_{it}\widehat{\Delta u_{it}}$ and $\widehat{\Delta u_{it}}$ is the residual of the first difference OLS estimator,

$$\begin{aligned}\hat{V} &= (nT)^{-1}\sum_{i=1}^{n}\left(\sum_{t=2}^{T}\hat{g}_{it}\right)^2 \\ &= n^{-1}\sum_{i=1}^{n}\hat{g}_i^2 \\ &= n^{-1}\sum_{i=1}^{n}(\eta_{iT}-\bar{\eta}_T)^2 + o_p(1)\end{aligned}$$

where $\bar{\eta}_T = n^{-1}\sum_{i=1}^{n}\eta_{iT}$. [Hint: Use the approximation in (1) and exploit that n is fixed.]

(e) Conclude that the usual t-statistic

$$\frac{(nT)^{1/2}(\hat{\beta}-\beta_0)}{\hat{\sigma}_\beta}$$

using the robust variance estimator $\hat{\sigma}_\beta^2 = \hat{H}^{-1}\hat{V}\hat{H}^{-1}$ with $\hat{H} = (nT)^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T}(\Delta x_{it})^2$ and $\hat{V} = (nT)^{-1}\sum_{i=1}^{n}\left(\sum_{t=2}^{T}\hat{g}_{it}\right)^2$ is asymptotically student-t with $n-1$ degrees of freedom under the null hypothesis. [Note that the scaling by T cancels, so this t-statistic is numerically identical to

$$\frac{n^{1/2}(\hat{\beta}-\beta_0)}{\tilde{\sigma}_\beta}$$

with $\tilde{\sigma}_\beta^2 = \tilde{H}^{-1}\tilde{V}\tilde{H}^{-1}$, $\tilde{H} = n^{-1}\sum_{i=1}^{n}\sum_{t=2}^{T}(\Delta x_{it})^2$ and $\tilde{V} = n^{-1}\sum_{i=1}^{n}\left(\sum_{t=2}^{T}\hat{g}_{it}\right)^2$, so it is the exact same statistic that we would have computed under $n\to\infty$, T fixed asymptotics.]