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ECO 518 – Econometric Theory II, Spring 2022

Problem Set 3, due February 15

You don't need to answer the questions in parentheses.

1. Suppose $\{x_t\}$ and $\{y_t\}$ are two uncorrelated stationary processes with spectral density functions f_x and f_y. What is the spectral density of $\{ax_t + by_t\}$ for a and b real constants?

2. Find the spectral density of a stationary AR(1) process with coefficient $|\rho| < 1$ and variance σ^2 by directly evaluating the expression $f(\lambda) = (2\pi)^{-1} \sum_{j=-\infty}^{\infty} \gamma(j) e^{-\mathbf{i}\lambda j}$. Compare with what we claimed in the lecture notes.

(3). Consider the stationary AR(1) process $y_t = 0.4y_{t-1} + \varepsilon_t$, $t = 1, \ldots, T = 101$. What is the largest correlation between the T random variables $\sqrt{T}(\bar{y} - \mu)$, $\{Z_l^{\sin}\}$ and $\{Z_l^{\cos}\}$ in this example?

5. Consider the series GDPC96 (real GDP for the United States). Let $x_t = 400 \ln(\mathrm{GDP}_t/\mathrm{GDP}_{t-1})$ denote the growth rate of real GDP (in percentage points at an annual rate). Using data on x_t from 1948:Q1 through 2021:Q4:

(a) Estimate an AR(4) model for x_t (including a constant) by OLS.

((b)) Optional: Estimate the AR coefficients by maximizing the Whittle likelihood, and compare your estimates to (a).

(c) Use the estimated AR model to generate forecasts x_t in the periods 2022:Q1-2022:Q4.

(d) Use the estimated AR model for x_t to construct an estimate of the spectrum for x_t. Plot the spectrum and discuss what its shape tells you about the stochastic process for x_t.

(e) Suppose we are interested in the spectral density at frequency $\lambda = 0.5$, $f(0.5)$. Using again the AR(4) model, show that $\sqrt{T}(\hat{f}_{AR(4)}(0.5) - f(0.5)) \Rightarrow \mathcal{N}(0, V)$, and derive an expression for V. (Hint: Recall the delta method).

(f) Now consider $y_t = \frac{1}{4}(x_t + x_{t-1} + x_{t-2} + x_{t-3}) = 100 \ln(\text{GDP}_t/\text{GDP}_{t-4})$. What is the gain of this filter? Without re-estimating anything, what is the spectrum of y_t based on your answer in (d)?

6. Suppose that $y_t = x_t\beta + u_t$, where x_t and u_t are stationary AR(1) processes $x_t = \rho x_{t-1} + \varepsilon_t$, $u_t = \phi u_{t-1} + \eta_t$, $\varepsilon_t \sim iid(0, \sigma_\varepsilon^2)$, $\eta_t \sim iid(0, \sigma_\eta^2)$, and $\{\varepsilon_t\}$ and $\{\eta_t\}$ are independent. Let $\hat\beta$ be the OLS estimator of β. Show that $T^{1/2}(\hat\beta - \beta) \Rightarrow \mathcal{N}(0, V)$, and find an expression for V as a function of $(\rho, \sigma_\varepsilon^2, \phi, \sigma_\eta^2)$. (Don't worry about the technical conditions for CLTs and LLNs).

7. Consider the spread between 10 year and 3 months treasury yields from 1983:1 - 2022:1. You are asked to construct a 95% confidence interval for the (population) mean, based on alternative long-run variance estimators:

(a) A parametric long-run variance estimator based on an AR specification with 12 lags (using 1982:1-1982:12 as "pre sample data" for the initial conditions) in the AR estimator.

(b) A nonparametric long-run variance estimator based on sample autocovariances

$$\hat\omega^2 = \sum_{k=-T+1}^{T-1} w(k/b_T)\hat\gamma(k)$$

"Newey-West" weighting function $w(x) = (1 - |x|)\mathbf{1}[|x| < 1]$ and $b_T = 0.75T^{1/3}$ (a rule of thumb choice advocated by Stock and Watson's undergraduate textbook for series with moderate persistence, for instance). What is the implicit assumption about the flatness of the spectrum for this estimator?

(c) A prewhitened long-run variance estimator based on an AR(1) prewhitening model with a non-parametric estimator for the residuals as in part (b).

(d) An estimator that averages the first 10 periodogram coordinates (not counting frequency zero), and is treated as "consistent".

(e) An estimator that averages the first 3 periodogram coordinates (not counting frequency zero), and a corresponding confidence interval that uses the student-t approximation discussed in the lecture notes.