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ECO 518 – Econometric Theory II, Spring 2022

Problem Set 2, due February 8

You don't need to answer the questions in parantheses.

1. Suppose that y_t is generated by

$$\begin{aligned} y_t &= x_t + u_t \\ x_t &= \phi x_{t-1} + \varepsilon_t \end{aligned}$$

where ε_t and u_t are mutually independent i.i.d. processes.

(a) Show that y_t has an ARMA(1,1) representation: $(1-\phi L)y_t = (1-\theta L)e_t$. (Hint: What are the autocovariances of $\varepsilon_t + u_t - \phi u_{t-1}$? What are the autcovariances of $e_t - \theta e_{t-1}$?)

(b) How are the sets of parameters $(\sigma_u^2, \sigma_\varepsilon^2, \phi)$ and $(\theta, \sigma_e^2, \phi)$ related?

(c) Can any ARMA(1,1) model be written as an AR(1) plus independent white noise?

(d) Suppose that you have data on y_t, $t \leq T$, but you don't observe x_t or u_t. How would you forecast y_{T+4}?

(e) How would your answer in (d) change if you did observe (x_t, u_t) for $t \leq T$?

2. Show that the process

$$y_t = \alpha \cos(\lambda t) + \beta \sin(\lambda t)$$

where λ is fixed, but (α, β) is random with $E[\alpha] = E[\beta] = E[\alpha\beta] = 0$ and $E[\alpha^2] = E[\beta^2] = \sigma^2$, is perfectly linearly predictable. *Hint*: $\cos(\lambda t)\cos(\lambda(t-k)) + \sin(\lambda t)\sin(\lambda(t-k)) = \cos(k\lambda)$ for $k \geq 0$, and $\cos(2\lambda) = \cos(\lambda)^2 - \sin(\lambda)^2$.

[Think about what would happen in our Wold-representation proof for this process].

3. Suppose y_t follows the MA(1) process $y_t = \mu + \varepsilon_t - \theta\varepsilon_{t-1}$, where ε_t is $iid(0, \sigma^2)$. Show that $T^{-1}\sum_{t=1}^T y_t \xrightarrow{p} \mu$, without invoking a LLN for dependent data.

(4.) Suppose that $X_i \sim iid\mathcal{N}(0,1)$ and let $s_n^2 = n^{-1}\sum_{i=1}^n X_i^2$.

(a) Show that $s_n^2 \xrightarrow{p} 1$.

(b) Show that $\sqrt{n}(s_n^2 - 1) \Rightarrow \mathcal{N}(0,\omega^2)$, and derive an expression for ω^2.

(c) Suppose $n = 15$, find $P(s_n^2 \leq .73)$ using an exact calculation.

(e) Redo part (c) using the normal approximation that you developed in (b).

(f) Let $A_n^2 = (n-1)^{-1}\sum_{i=1}^n (X_i - \bar{X}_n)^2$, where $\bar{X}_n = n^{-1}\sum_{i=1}^n X_i$. Show that $A_n^2 \xrightarrow{p} 1$.

(5.) Suppose that $X_n \sim \mathcal{N}(1, n^{-1})$ for all n. Find the limiting distribution of $\sqrt{n}(\exp(X_n) - \exp(1))$.

6. Consider the linear instrumental variable model $y_t = x_t'\beta + \varepsilon_t$, where x_t is $k \times 1$, where in general $E[x_t\varepsilon_t] \neq 0$, but we also observe the $m \geq k$ vector z_t with $E[z_t\varepsilon_t] = 0$. Assuming (y_t, z_t, x_t) to be stationary and ergodic, make assumptions similar to those in the derivations of OLS estimator and prove the asymptotic normality of the 2SLS estimator under your assumptions.

7. Suppose y_t is a stationary stochastic process that is well-described by an AR(1) model: $y_t = c + \phi y_{t-1} + \varepsilon_t$ where $\varepsilon_t \sim iid(0,\sigma^2)$. Suppose you have data $y_1, \ldots, y_T$.

(a) Let $\mu = E[y_t]$, and let $\hat{\mu} = T^{-1}\sum_{t=1}^T y_t$. Show that $T^{1/2}(\hat{\mu} - \mu) \Rightarrow \mathcal{N}(0, V)$, and derive an expression for V.

(b) How would you estimate V? Is this estimator consistent? Explain.

(c) Explain how you would construct a 95% confidence interval for μ using $\hat{\mu}$ and the estimator (c).

(8.) Consider the stationary AR(p) process

$$y_t = \phi_1 y_{t-1} + \phi_2 y_{t-2} + \cdots + \phi_p y_{t-p} + \varepsilon_t, \quad \varepsilon_t \sim iid(0,\sigma^2). \tag{1}$$

You will show that selecting $\hat{p}$ using the BIC information criterion

$$\text{BIC}(\tilde{p}) = \ln(T^{-1}\text{SSR}(\tilde{p})) + \tilde{p}\ln(T)/T$$

is consistent for the true number of lags (=p). Let $SSR(\tilde{p})$ be the sum of squared residuals of the OLS regression (1) with p chosen as $\tilde{p}$ with sample size T (so we have sufficient presample observations $y_0, y_{-1}, \ldots$ to run the regression).

(a) Show that for $\tilde{p} > p$, $(\text{SSR}(p) - \text{SSR}(\tilde{p}))/\sigma^2 \Rightarrow \chi^2_{\tilde{p}-p}$.

(b) Show that for $\tilde{p} < p$, $T^{-1}\text{SSR}(\tilde{p}) \xrightarrow{p} \omega^2(\tilde{p}) > \sigma^2$.

(c) Show that for any $\tilde{p} > p$,

$$P(\text{BIC}(\tilde{p}) - \text{BIC}(p) > 0) \to 1.$$

(d) Show that for any $\tilde{p} < p$,

$$P(\text{BIC}(\tilde{p}) - \text{BIC}(p) > 0) \to 1.$$

(e) Conclude from (c) and (d) that for $p_{\max} \geq p$,

$$\hat{p} = \arg\min_{0 \leq \tilde{p} \leq p_{\max}} \text{BIC}(\tilde{p}) \xrightarrow{p} p.$$