Third Isomorphism Theorem via Category Theory

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I. DEFINITIONS

- A *category* has a set of objects and arrows satisfying composition and identity.

- A group G can be regarded as a category $\mathbf{BG}$ because it has:

  - A one-element object set $*$,
  - Arrow set $G(*, *)$, where each group element $f : * \to *$ is an arrow/endomorphism,
  - Composition defined by multiplication,
  - Identity arrow given by the identity element e_G.

I. DEFINITIONS

- A *functor* is a morphism of categories. For categories C and B, a functor $T : C \to B$ with domain C and codomain B consists of two functions: the object function T which assigns each c of C an object Tc of B and the arrow function T which assigns to each arrow $f : c \to c'$ of C an arrow $Tf : Tc \to Tc'$ of B such that $T(1_c) = 1_{T_c}$ and $T(g \circ f) = Tg \circ Tf$ whenever $g \circ f$ is defined in C.

- E.g. A group homomorphism $f : G \to H$ is a functor between them. Suppose f maps arrows x, y, z of G to fx, fy, fz of H and $x \circ y = z$. Then $f(e_G) = e_H$ and $f(x) \circ f(y) = f(x \circ y) = f(z)$.

I. DEFINITIONS

- If $S : D \to C$ is a functor and c an object of C, a *universal arrow* from c to S is a pair $< r, u >$ consisting of an object r of D and an arrow $u : c \to Sr$ of C, such that to every pair $< d, f >$ with d an object of D and $f : c \to Sd$ an arrow of C, there is a unique arrow $f' : r \to d$ of D with $Sf' \circ u = f$.

$$\begin{tikzcd} c \arrow[r, "u"] \arrow[dr, "f"'] & Sr \arrow[d, dotted, "Sf'"] & r \arrow[d, dotted, "f'"] \\ & Sd & d \end{tikzcd}$$

- (Universal property of the quotient) Let $N \triangleleft G$, $p : G \to G/N$ sends each $g \in G$ to its coset $pg = gN$ in the quotient group G/N, and $f : G \to G'$ be a group homomorphism such that $N \subset ker(f)$. Then there exists a unique homomorphism $f' : G/N \to G'$ such that $f' \circ p = f$.

$$\begin{tikzcd} G \arrow[r, "p"] \arrow[dr, "f"'] & G/N \arrow[d, dotted, "f'"] \\ & G' \end{tikzcd}$$

II. APPLICATION: THIRD ISOMORPHISM THEOREM

Using only universality (of projections) to prove the following isomorphism of group theory:

For normal subgroups M, N of G with $M \subset N$, $(G/M)/(N/M) \cong G/N$.

$$(G/M)/(N/M) \cong G/N$$

$$\Leftrightarrow$$

$$\text{there exists } f : (G/M)/(N/M) \to G/N \text{ and } g : G/N \to (G/M)/(N/M)$$

$$\text{such that } f \circ g = g \circ f = 1_{G/N}$$

III. PROOF

$$
\begin{tikzcd}
 & G/M \arrow[d, "p_{N/M}"] \arrow[ddl, dotted, bend left=60, "\exists ! h"] \\
G \arrow[ur, "p_M"] \arrow[r, "k"] \arrow[d, "p_N"] & (G/M)/(N/M) \arrow[dl, dotted, shift left, "\exists ! f"] \\
G/N \arrow[ur, "g"] &
\end{tikzcd}
$$

III. PROOF

$$\begin{tikzcd}
 & G/M \arrow[d, "p_{N/M}"] \arrow[ddl, dotted, bend left=60, "\exists! h"] \\
G \arrow[ur, "p_M"] \arrow[r, "k"] \arrow[d, "p_N"] & (G/M)/(N/M) \arrow[dl, dotted, bend left=10, "\exists! f"] \\
G/N \arrow[ur, "g"] &
\end{tikzcd}$$

III. PROOF

$$\begin{tikzcd}
& (G/M)/(N/M) \arrow[d, "f"] \arrow[dd, dotted, bend left, "\exists ! id"] \\
G/M \arrow[ur] \arrow[r] \arrow[dr] & G/N \arrow[d, "g"] \\
& (G/M)/(N/M)
\end{tikzcd}$$

$$\begin{tikzcd}
& G/N \arrow[d, "g"] \arrow[dd, dotted, bend left, "\exists ! id"] \\
G/M \arrow[ur] \arrow[r] \arrow[dr] & (G/M)/(N/M) \arrow[d, "f"] \\
& G/N
\end{tikzcd}$$

IV. REFERENCES

Categories for the Working Mathematician by Saunders Mac Lane

Category Theory in Context by Emily Riehl