

# Third Isomorphism Theorem via Category Theory

Yucong Chen

(Dated: June 5, 2023)

## I. DEFINITIONS

---

- A *category* has a set of **objects** and **arrows** satisfying **composition** and **identity**.
- A group  $G$  can be regarded as a category **BG** because it has:
  - A one-element object set  $*$ ,
  - Arrow set  $G(*, *)$ , where each group element  $f : * \rightarrow *$  is an arrow/endomorphism,
  - Composition defined by multiplication,
  - Identity arrow given by the identity element  $e_G$ .

## I. DEFINITIONS

---

- A *functor* is a morphism of categories. For categories  $C$  and  $B$ , a functor  $T : C \rightarrow B$  with domain  $C$  and codomain  $B$  consists of two functions: **the object function**  $T$  which assigns each  $c$  of  $C$  an object  $Tc$  of  $B$  and **the arrow function**  $T$  which assigns to each arrow  $f : c \rightarrow c'$  of  $C$  an arrow  $Tf : Tc \rightarrow Tc'$  of  $B$  such that  $T(1_c) = 1_{Tc}$  and  $T(g \circ f) = Tg \circ Tf$  whenever  $g \circ f$  is defined in  $C$ .
- E.g. A group homomorphism  $f : G \rightarrow H$  is a functor between them. Suppose  $f$  maps arrows  $x, y, z$  of  $G$  to  $fx, fy, fz$  of  $H$  and  $x \circ y = z$ . Then  $f(e_G) = e_H$  and  $f(x) \circ f(y) = f(x \circ y) = f(z)$ .

## I. DEFINITIONS

---

- If  $S : D \rightarrow C$  is a functor and  $c$  an object of  $C$ , a *universal arrow* from  $c$  to  $S$  is a pair  $\langle r, u \rangle$  consisting of an object  $r$  of  $D$  and an arrow  $u : c \rightarrow Sr$  of  $C$ , such that to every pair  $\langle d, f \rangle$  with  $d$  an object of  $D$  and  $f : c \rightarrow Sd$  an arrow of  $C$ , there is a unique arrow  $f' : r \rightarrow d$  of  $D$  with  $Sf' \circ u = f$ .

$$\begin{array}{ccc}
 c & \xrightarrow{u} & Sr \\
 & \searrow f & \downarrow Sf' \\
 & & Sd
 \end{array}
 \qquad
 \begin{array}{ccc}
 r & & \\
 & \downarrow f' & \\
 & d &
 \end{array}$$

- (Universal property of the quotient) Let  $N \triangleleft G$ ,  $p : G \rightarrow G/N$  sends each  $g \in G$  to its coset  $pg = gN$  in the quotient group  $G/N$ , and  $f : G \rightarrow G'$  be a group homomorphism such that  $N \subset \ker(f)$ . Then there exists a unique homomorphism  $f' : G/N \rightarrow G'$  such that  $f' \circ p = f$ .

$$\begin{array}{ccc}
 G & \xrightarrow{p} & G/N \\
 & \searrow f & \downarrow f' \\
 & & G'
 \end{array}$$

## II. APPLICATION: THIRD ISOMORPHISM THEOREM

---

Using only universality (of projections) to prove the following isomorphism of group theory:

For normal subgroups  $M, N$  of  $G$  with  $M \subset N$ ,  $(G/M)/(N/M) \cong G/N$ .

### III. PROOF

---

$$(G/M)/(N/M) \cong G/N$$

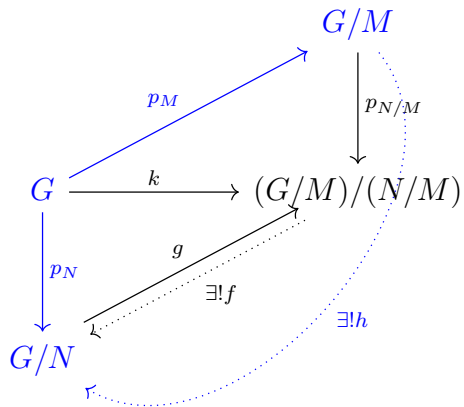
$$\Leftrightarrow$$

there exists  $f : (G/M)/(N/M) \rightarrow G/N$  and  $g : G/N \rightarrow (G/M)/(N/M)$

such that  $f \circ g = g \circ f = 1_{G/N}$

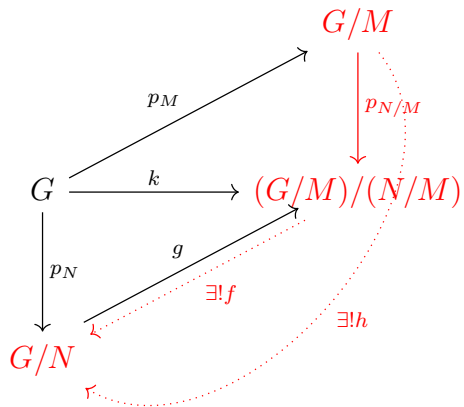
### III. PROOF

---



### III. PROOF

---





### III. PROOF

---

$$\begin{array}{ccccc}
 & & (G/M)/(N/M) & & \\
 & \nearrow & \downarrow f & & \\
 G/M & \longrightarrow & G/N & \exists ! id & \\
 & \searrow & \downarrow g & & \\
 & & (G/M)/(N/M) & &
 \end{array}$$

$$\begin{array}{ccccc}
 & & G/N & & \\
 & \nearrow & \downarrow g & & \\
 G/M & \longrightarrow & (G/M)/(N/M) & \exists ! id & \\
 & \searrow & \downarrow f & & \\
 & & G/N & \leftarrow &
 \end{array}$$

## IV. REFERENCES

---

*Categories for the Working Mathematician* by Saunders Mac Lane

*Category Theory in Context* by Emily Riehl