Review:

1. Integrals of sine and cosine:

Strategy for Evaluating $\int \sin^m x \cos^n x\, dx$

(a) If the power of cosine is odd ($n = 2k + 1$), save one cosine factor and use $\cos^2 x = 1 - \sin^2 x$ to express the remaining factors in terms of sine:

$$\int \sin^m x \cos^{2k+1} x\, dx = \int \sin^m x\, (\cos^2 x)^k \cos x\, dx$$
$$= \int \sin^m x\, (1 - \sin^2 x)^k \cos x\, dx$$

Then substitute $u = \sin x$. See Example 1.

(b) If the power of sine is odd ($m = 2k + 1$), save one sine factor and use $\sin^2 x = 1 - \cos^2 x$ to express the remaining factors in terms of cosine:

$$\int \sin^{2k+1} x \cos^n x\, dx = \int (\sin^2 x)^k \cos^n x \sin x\, dx$$
$$= \int (1 - \cos^2 x)^k \cos^n x \sin x\, dx$$

Then substitute $u = \cos x$. See Example 2.

[Note that if the powers of both sine and cosine are odd, either (a) or (b) can be used.]

(c) If the powers of both sine and cosine are even, use the half-angle identities

$$\sin^2 x = \tfrac{1}{2}(1 - \cos 2x) \qquad \cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

See Examples 3 and 4.

It is sometimes helpful to use the identity

$$\sin x \cos x = \tfrac{1}{2}\sin 2x$$

$$\Rightarrow \int \sin^{2k} x \cos^{2l} x\, dx = \int \left[\tfrac{1}{2}(1-\cos 2x)\right]^k \cdot \left[\tfrac{1}{2}(1+\cos 2x)\right]^l dx$$

Then we expand each one and we continue using the same identities if necessary until there are no more powers in the sine and cosine.

2. Integrals of tangent and secant:

$$\frac{d \sec x}{dx} = \sec x \cdot \tan x$$

$$\frac{d \tan x}{dx} = \sec^2 x$$

$$\tan^2 x + 1 = \sec^2 x$$

Strategy for Evaluating $\int \tan^m x \sec^n x \, dx$

(a) If the power of secant is even ($n = 2k$, $k \geqslant 2$), save a factor of $\sec^2 x$ and use $\sec^2 x = 1 + \tan^2 x$ to express the remaining factors in terms of $\tan x$:

$$\int \tan^m x \sec^{2k} x \, dx = \int \tan^m x \, (\sec^2 x)^{k-1} \sec^2 x \, dx$$

$$= \int \tan^m x \, (1 + \tan^2 x)^{k-1} \sec^2 x \, dx$$

Then substitute $u = \tan x$. See Example 5.

(b) If the power of tangent is odd ($m = 2k + 1$), save a factor of $\sec x \tan x$ and use $\tan^2 x = \sec^2 x - 1$ to express the remaining factors in terms of $\sec x$:

$$\int \tan^{2k+1} x \sec^n x \, dx = \int (\tan^2 x)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

$$= \int (\sec^2 x - 1)^k \sec^{n-1} x \, \sec x \, \tan x \, dx$$

Then substitute $u = \sec x$. See Example 6.

For other cases, the guidelines are not as clear-cut. We may need to use identities, integration by parts, and occasionally a little ingenuity. We will sometimes need to be able to integrate $\tan x$ by using the formula established in (5.5.5):

$$\boxed{\int \tan x \, dx = \ln |\sec x| + C}$$

We will also need the indefinite integral of secant:

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$$\boxed{\int \sec x \, dx = \ln |\sec x + \tan x| + C}$$

3. Integrals of sine and cosine evaluated at multiples of x:

Using Product Identities

The following product identities are useful in evaluating certain trigonometric integrals.

2 To evaluate the integrals (a) $\int \sin mx \cos nx\, dx$, (b) $\int \sin mx \sin nx\, dx$, or (c) $\int \cos mx \cos nx\, dx$, use the corresponding identity:

$$\text{(a)}\quad \sin A \cos B = \tfrac{1}{2}[\sin(A - B) + \sin(A + B)]$$

$$\text{(b)}\quad \sin A \sin B = \tfrac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\text{(c)}\quad \cos A \cos B = \tfrac{1}{2}[\cos(A - B) + \cos(A + B)]$$

4. Trigonometric substitution:

Table of Trigonometric Substitutions

Expression	Substitution	Identity
$\sqrt{a^2 - x^2}$	$x = a \sin\theta,\quad -\frac{\pi}{2} \leqslant \theta \leqslant \frac{\pi}{2}$	$1 - \sin^2\theta = \cos^2\theta$
$\sqrt{a^2 + x^2}$	$x = a \tan\theta,\quad -\frac{\pi}{2} < \theta < \frac{\pi}{2}$	$1 + \tan^2\theta = \sec^2\theta$
$\sqrt{x^2 - a^2}$	$x = a \sec\theta,\quad 0 \leqslant \theta < \frac{\pi}{2}$ or $\pi \leqslant \theta < \frac{3\pi}{2}$	$\sec^2\theta - 1 = \tan^2\theta$

Exercises:

1. Use suitable trigonometric identities and substitutions to evaluate the following integrals.

(i) $\displaystyle\int \frac{\sin^2(\sqrt{x})}{\sqrt{x}}dx$ (ii) $\displaystyle\int_0^{\pi} \sin^2 t \cos^4 t dt$ (iii) $\displaystyle\int \tan^2(2x)\sec^5(2x)dx$ (iv) $\displaystyle\int_0^{\pi/3} \tan^5 x \sec^6 x dx$

(i) $$\int \frac{\sin^2(\sqrt{x})}{\sqrt{x}}dx = \int \sin^2(\sqrt{x})\, d(2\sqrt{x}) \overset{u=\sqrt{x}}{=\!=} \int 2\sin^2 u\, du$$

$$= \int (1-\cos 2u)\, du = u - \frac{1}{2}\sin 2u + C = \sqrt{x} - \frac{1}{2}\sin(2\sqrt{x}) + C$$

(ii) $$\int_0^{\pi} \sin^2 t \cos^4 t\, dt \overset{①}{=} \int_0^{\pi} \sin^2 t \cdot (\cos^2 t)^2\, dt = \int_0^{\pi} \frac{1-\cos 2t}{2}\cdot\left(\frac{1+\cos 2t}{2}\right)^2 dt$$

$$= \frac{1}{8}\int_0^{\pi} (1-\cos 2t)(1+2\cos 2t+\cos^2 2t)\, dt$$

$$= \frac{1}{8}\int_0^{\pi} (1-\cos 2t)\left[1+2\cos 2t+\frac{1}{2}(1+\cos 4t)\right] dt$$

$$= \frac{1}{8}\int_0^{\pi}\left(\frac{3}{2}+\frac{1}{2}\cos 2t+\frac{1}{2}\cos 4t-2\cos^2 2t-\frac{1}{2}\cos 2t\cos 4t\right) dt$$

$$= \frac{1}{8}\int_0^{\pi}\left[\frac{3}{2}+\frac{1}{2}\cos 2t+\frac{1}{2}\cos 4t-(\cos 4t+1)-\frac{1}{4}(\cos 6t+\cos 2t)\right] dt$$

$$= \frac{1}{8}\int_0^{\pi}\left[\frac{1}{2}+\frac{1}{4}\cos 2t-\frac{1}{2}\cos 4t-\frac{1}{4}\cos 6t\right] dt$$

$$= \frac{1}{8}\left[\frac{1}{2}t+\frac{1}{8}\sin 2t-\frac{1}{8}\sin 4t-\frac{1}{24}\sin 6t\right]\Bigg|_0^{\pi}$$

$$= \frac{\pi}{16}$$

$$\int_0^{\pi} \sin^2 t \cos^4 t\, dt \overset{②}{=\!=} \int_0^{\pi} (\sin t\cos t)^2\cdot\cos^2 t\, dt = \int_0^{\pi}\frac{1}{4}\sin^2 2t\cdot\cos^2 t\, dt$$

$$= \int_0^{\pi}\frac{1}{4}(\sin 2t\cos t)^2 dt = \int_0^{\pi}\frac{1}{4}\left(\frac{\sin 3t+\sin t}{2}\right)^2 dt$$

$$= \frac{1}{16}\int_0^{\pi}(\sin^2 3t+2\sin 3t\sin t+\sin^2 t)\, dt$$

$$= \frac{1}{16}\int_0^{\pi}\left[\frac{1-\cos 6t}{2}+(\cos 2t-\cos 4t)+\frac{1-\cos 2t}{2}\right] dt$$

$$= \frac{1}{16}\int_0^{\pi}\left(1+\frac{1}{2}\cos 2t-\cos 4t-\frac{1}{2}\cos 6t\right) dt$$

$$= \frac{1}{16}\left[t+\frac{1}{4}\sin 2t-\frac{1}{4}\sin 4t-\frac{1}{12}\sin 6t\right]\Bigg|_0^{\pi} = \frac{\pi}{16}.$$

(iii) $\int \tan^2(2x)\sec^5(2x)\,dx = \int (\sec^2(2x)-1)\sec^5(2x)\,dx$

$$= \int (\sec^7(2x) - \sec^5(2x))\,dx$$

$$\overset{u=2x}{=\!=} \int (\sec^7 u - \sec^5 u)\,d\frac{u}{2}$$

$$= \frac{1}{2}\left(\int \sec^7 u\,du - \int \sec^5 u\,du\right) \cdots (1)$$

First let's prove the reduction formula:

$$\int \sec^n x dx = \frac{\tan x \sec^{n-2} x}{n-1} + \frac{n-2}{n-1}\int \sec^{n-2} x dx$$

Solution For any integer $n \geq 2$,

$$\underline{\int \sec^n x dx} = \int \sec^{n-2} x \sec^2 x dx = \int \sec^{n-2} x d\tan x = \sec^{n-2} x \tan x - \int \tan x\, d\sec^{n-2} x$$

$$= \sec^{n-2} x \tan x - \int (n-2)\tan^2 x \sec^{n-2} x dx = \sec^{n-2} x \tan x - (n-2)\int (\sec^2 x - 1)\sec^{n-2} x dx$$

$$= \sec^{n-2} x \tan x - (n-2)\underline{\int \sec^n x dx} + (n-2)\int \sec^{n-2} x dx$$

i.e.,

$$(n-1)\int \sec^n x dx = \sec^{n-2} x \tan x + (n-2)\int \sec^{n-2} x dx$$

$$\int \sec^n x dx = \frac{1}{n-1}\sec^{n-2} x \tan x + \frac{n-2}{n-1}\int \sec^{n-2} x dx$$

$$\Rightarrow \int \sec^7 x\,dx - \int \sec^5 x\,dx = \frac{\tan x \sec^5 x}{6} - \frac{1}{6}\int \sec^5 x\,dx$$

$$= \frac{1}{6}\tan x \sec^5 x - \frac{1}{6}\left[\frac{1}{4}\tan x \sec^3 x + \frac{3}{4}\int \sec^3 x\,dx\right)$$

$$= \frac{1}{6}\tan x\cdot\sec^5 x - \frac{1}{24}\tan x\cdot\sec^3 x - \frac{3}{24}\left(\frac{1}{2}\tan x\cdot\sec x + \frac{1}{2}\int \sec x\,dx\right)$$

$$= \frac{1}{6}\tan x\cdot\sec^5 x - \frac{1}{24}\tan x\cdot\sec^3 x - \frac{3}{48}\tan x\cdot\sec x - \frac{3}{48}\cdot\ln|\sec x + \tan x|$$

$$\Rightarrow (1) = \frac{1}{12}\tan 2x\cdot\sec^5 2x - \frac{1}{48}\tan 2x\cdot\sec^3 2x - \frac{1}{32}\tan x\cdot\sec x \quad +C$$

$$- \frac{1}{32}\ln|\sec 2x + \tan 2x| + C$$

(iv). $\int \tan^5 x \sec^6 x \, dx = \int \tan^4 x \sec^5 x \cdot \tan x \sec x \, dx$

$= \int \tan^4 x \sec^5 x \, d\sec x = \int (\sec^2 x - 1)^2 \sec^5 x \, d\sec x$

$= \int (\sec^4 x - 2\sec^2 x + 1) \sec^5 x \, d\sec x$

$= \frac{1}{10}\sec^{10} x - \frac{1}{4}\sec^8 x + \frac{1}{6}\sec^6 x + C$

$\Rightarrow \int_0^{\frac{\pi}{3}} \tan^5 x \sec^6 x \, dx = \left(\frac{1}{10}\sec^{10} x - \frac{1}{4}\sec^8 x + \frac{1}{6}\sec^6 x\right)\Big|_0^{\frac{\pi}{3}}$

$= \frac{981}{20}$

2. Use suitable trigonometric identities to help show that:

(i) $\displaystyle\int_{-\pi}^{\pi} \sin mx \sin nx dx = \begin{cases} 0 & \text{if } m \neq n \\ \pi & \text{if } m = n \end{cases}$ for any integers m, n.

(ii) A *finite Fourier series* is given by the sum

$$f(x) = \sum_{i=1}^{N} a_n \sin nx = a_1 \sin x + a_2 \sin(2x) + \cdots + a_N \sin(Nx) \; .$$

Show that the m-th coefficient a_m is given by the formula $a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin mx dx$.

(i) $\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \int_{-\pi}^{\pi} \frac{1}{2}(\cos(m-n)x - \cos(m+n)x) \, dx$

$= \begin{cases} \frac{1}{2}\left(x - \frac{1}{m+n}\sin(m+n)x\right)\Big|_{-\pi}^{\pi}, & m = n \\ \frac{1}{2}\left(\frac{1}{m-n}\sin(m-n)x - \frac{1}{m+n}\sin(m+n)x\right)\Big|_{-\pi}^{\pi}, & m \neq n \end{cases}$

$= \begin{cases} \pi, & m = n \\ 0, & m \neq n \end{cases}$

(ii) $\frac{1}{\pi}\int_{-\pi}^{\pi} \sum_{i=1}^{N} a_n \sin nx \sin mx \, dx$

$= \frac{1}{\pi}\sum_{i=1}^{N} a_n \int_{-\pi}^{\pi} \sin nx \sin mx \, dx$

$\overset{(i)}{=} \frac{1}{\pi} a_n \cdot \pi = a_n$

3. Evaluate the following integrals by suitable trigonometric substitutions.

(i) $\displaystyle\int_0^2 x^2\sqrt{x^2+4}\,dx$ (ii) $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx$, where $a>0$ is a constant. (iii) $\displaystyle\int_{\sqrt{2}/3}^{2/3} \frac{dx}{x^5\sqrt{9x^2-1}}$,

(iv) $\displaystyle\int \frac{x^2}{(3+4x-4x^2)^{3/2}}\,dx$ (v) $\displaystyle\int \frac{x^2}{(x^2+a^2)^{3/2}}\,dx$

(i) $\displaystyle\int_0^2 x^2\sqrt{x^2+4}\,dx \overset{x=2\tan u}{=\!=} \int_0^{\frac{\pi}{4}} 4\tan^2 u\sqrt{4\tan^2 u+4}\cdot 2\sec^2 u\,du$

$$= 16\int_0^{\frac{\pi}{4}} \tan^2 u\cdot \sec^3 u\,du$$

$$= \frac{16}{3}\int_0^{\frac{\pi}{4}} \tan u\; d\sec^3 u$$

$$= \frac{16}{3}\left[\tan u\cdot\sec^3 u\Big|_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}}\sec^3 u\; d\tan u\right]$$

$$= \frac{32}{3}\sqrt{2} - \frac{16}{3}\cdot\int_0^{\frac{\pi}{4}}\sec^5 u\,du$$

$$\overset{\text{reduction}}{\underset{\text{formula}}{=\!=}} \frac{32}{3}\sqrt{2} - \frac{16}{3}\left[\frac{1}{4}\tan u\cdot\sec^3 u\Big|_0^{\frac{\pi}{4}} + \frac{3}{4}\int_0^{\frac{\pi}{4}}\sec^3 u\,du\right]$$

$$= \frac{32}{3}\sqrt{2} - \frac{8}{3}\sqrt{2} - 4\cdot\left[\frac{1}{2}\tan u\cdot\sec u\Big|_0^{\frac{\pi}{4}} + \frac{1}{2}\int_0^{\frac{\pi}{4}}\sec u\,du\right]$$

$$= 8\sqrt{2} - 4\cdot\left[\frac{\sqrt{2}}{2} + \frac{1}{2}\ln|\sec u+\tan u|\Big|_0^{\frac{\pi}{4}}\right]$$

$$= 6\sqrt{2} - 2\ln(\sqrt{2}+1)$$

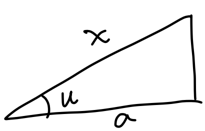
(ii) $\displaystyle\int \frac{\sqrt{x^2-a^2}}{x^4}\,dx \overset{x=a\sec u}{=\!=} \int \frac{\sqrt{a^2\sec^2 u-a^2}}{a^4\sec^4 u}\,a\sec u\tan u\,du$

$$= \int \frac{1}{a^2}\cdot\frac{\tan^2 u}{\sec^3 u}\,du$$

$x = a\sec u$

$\Leftrightarrow \frac{x}{a} = \frac{1}{\cos u}$

$\Leftrightarrow \cos u = \frac{a}{x}$



$\Rightarrow \sin u = \sqrt{1-\frac{a^2}{x^2}}$

$$= \frac{1}{a^2}\int\frac{\sin^2 u}{\cos^2 u}\cdot\cos^3 u\,du$$

$$= \frac{1}{a^2}\int\sin^2 u\cos u\,du$$

$$= \frac{1}{a^2}\int\sin^2 u\; d\sin u$$

$$= \frac{1}{3a^2}\sin^3 u + C = \frac{1}{3a^2}\left(\sqrt{1-\frac{a^2}{x^2}}\right)^3 + C$$

$$= \frac{1}{3a^2}\cdot\frac{(x^2-a^2)^{\frac{3}{2}}}{x^3} + C$$

(iii) $$\int_{\frac{\sqrt{2}}{3}}^{\frac{2}{3}} \frac{dx}{x^5\sqrt{9x^2-1}} \overset{x=\frac{1}{3}\sec u}{=\!=\!=} \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{3^5}{\sec^5 u\cdot \tan u}\cdot \frac{1}{3}\sec u\cdot \tan u\, du$$

$$= 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^4 u\, du$$

$$= 81\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(\frac{1+\cos 2u}{2}\right)^2 du$$

$$= \frac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} (1+2\cos 2u+\cos^2 2u)\, du$$

$$= \frac{81}{4}\int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \left(1+2\cos 2u+\frac{1+\cos 4u}{2}\right) du$$

$$= \frac{81}{4}\left[\frac{3}{2}u+\sin 2u+\frac{1}{8}\sin 4u\right]\Big|_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{81}{4}\left[\frac{1}{8}\pi+\frac{7\sqrt{3}}{16}-1\right]$$

$$= \frac{567}{64}\sqrt{3}+\frac{81}{32}\pi-\frac{81}{4}$$

(iv) $$\int \frac{x^2}{(3+4x-4x^2)^{\frac{3}{2}}}dx = \int \frac{x^2}{(-(4x^2-4x+1)+4)^{\frac{3}{2}}}dx$$

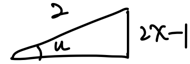
$$= \int \frac{x^2}{(4-(2x-1)^2)^{\frac{3}{2}}}dx \overset{2x-1=2\sin u}{=\!=\!=} \int \frac{\left(\frac{2\sin u+1}{2}\right)^2}{(4-4\sin^2 u)^{\frac{3}{2}}}\cdot \cos u\, du$$

$$= \frac{1}{32}\int \frac{(2\sin u+1)^2}{\cos^2 u}du = \frac{1}{32}\int (4\tan^2 u+4\tan u\cdot\sec u+\sec^2 u)\, du$$

$$= \frac{1}{32}\cdot\int [4(\sec^2 u-1)+4\tan u\cdot\sec u+\sec^2 u]\, du$$

$$= \frac{1}{32}\int (5\sec^2 u+4\tan u\cdot\sec u-4)\, du$$

$$= \frac{1}{32}\cdot[5\tan u+4\sec u-4u]+C$$



$$= \frac{5}{32}\cdot\frac{2x-1}{\sqrt{4-(2x-1)^2}}+\frac{1}{8}\cdot\frac{2}{\sqrt{4-(2x-1)^2}}-\frac{1}{8}\arcsin\frac{2x-1}{2}+C$$

$$= \frac{1}{32}\cdot\frac{10x+3}{\sqrt{3+4x-4x^2}}-\frac{1}{8}\arcsin\frac{2x-1}{2}+C$$

(v) $$\int \frac{x^2}{(x^2+a^2)^{\frac{3}{2}}}\,dx \xlongequal{x=a\tan u} \int \frac{a^2\tan^2 u}{(a^2\sec^2 u)^{\frac{3}{2}}}\cdot a\sec^2 u\,du$$

$$= \int \frac{\tan^2 u}{\sec u}\,du = \int \frac{\sin^2 u}{\cos^2 u}\cdot \cos u\,du = \int \frac{\sin^2 u}{\cos u}\,du$$

$$= \int \frac{1-\cos^2 u}{\cos u}\,du = \int (\sec u - \cos u)\,du$$

$$= \ln|\sec u + \tan u| - \sin u + C$$

$$= \ln\left|\frac{\sqrt{a^2+x^2}}{a} + \frac{x}{a}\right| - \frac{x}{\sqrt{a^2+x^2}} + C$$