Review:

1. Purpose: Evaluate integrals of rational functions.

- Rational functions are functions of the form $r(x) = \frac{p(x)}{q(x)}$, where $p(x)$ and $q(x)$ are polynomials. Eg, $\frac{x^2+1}{x+2}$, $\frac{5x^3-3x^2+x}{-2x^4+7}$.

For a rational function $r(x) = \frac{p(x)}{q(x)}$, if $\deg(p) < \deg(q)$, we say $r(x)$ is proper; otherwise, it is improper.

- $\frac{2}{x+1}$, $\frac{x-7}{5x^2+6}$, $\frac{2x+1}{x^{10}+x^9+x}$ are proper;
- $\frac{5x^2}{x^2+1}$, $\frac{x^3+8}{(x+3)^2}$, $\frac{(x-2)^3}{x+5}$ are improper.

Partial Fraction: Express a rational function ( a ratio of polynomials) as a sum of simpler fractions.

Previously, we can simplify $\frac{2}{x} + \frac{1}{x+1} = \frac{2(x+1)}{x(x+1)} + \frac{x}{x(x+1)} = \cdots = \frac{3x+2}{x^2+x}$.
Now, reversely, we try to decompose

$$\underbrace{\frac{3x+2}{x^2+x}}_{\text{Single fraction}} = \underbrace{\frac{2}{x} + \frac{1}{x+1}}_{\text{Partial functions}}.$$

→ easier to integrate

Partial fraction decomposition method:

1. If $r(x)$ is improper, perform the long division to get

$$r(x) = \frac{p(x)}{q(x)} = \underbrace{s(x)}_{\text{polynomial}} + \underbrace{\frac{u(x)}{v(x)}}_{\text{proper rational function}}.$$

2. With a proper $r(x) = \frac{p(x)}{q(x)}$, factor the bottom into
   1. Linear factors: if $(ax+b)$ divides $q(x)$ and $(ax+b)^n$ is the highest power that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ contains the sum $\frac{A_1}{ax+b} + \frac{A_2}{(ax+b)^2} + \cdots + \frac{A_n}{(ax+b)^n}$.

2. (Irreducible) Quadratic factors: if $ax^2+bx+c$ divides $q(x)$ and $(ax^2+bx+c)^n$ is the highest power that divides $q(x)$. Then the decomposition of $\frac{p(x)}{q(x)}$ will contain the sum

$$\frac{B_1x+C_1}{ax^2+bx+c}+\frac{B_2x+C_2}{(ax^2+bx+c)^2}+\cdots+\frac{B_nx+C_n}{(ax^2+bx+c)^n}.$$

To find the coefficients $A_i, B_i, C_i$,

1. Multiply the whole equation by the bottom $q(x)$.
2. Solve for the coefficients by
   1. either substituting zeros (roots) of the bottom.
   2. or making a system of linear equations and solving.

$$\frac{5x-4}{x^2-x-2}=\frac{5x-4}{(x-2)(x+1)}\Rightarrow\frac{5x-4}{(x-2)(x+1)}=\frac{A_1}{x-2}+\frac{A_2}{x+1}$$
$$\Rightarrow 5x-4=A_1(x+1)+A_2(x-2)$$

- substituting zeros (roots) of the bottom:
  $(x-2)(x+1)=0\Rightarrow x=2,-1$. When $x=2$,
  $5x-4=6=3A_1\Rightarrow A_1=2$. When $x=-1$
  $5x-4=-9=-3A_2\Rightarrow A_2=3$.
- making a system of linear equations and solving:
  $A_1(x+1)+A_2(x-2)=(A_1+A_2)x+(A_1-2A_2)\Rightarrow A_1+A_2=5$,
  $A_1-2A_2=-4$, so $A_1=2, A_2=3$.

Summary:

Step 1: If $f(x)$ is improper, perform the long division and get $f(x)=\frac{P(x)}{Q(x)}=s(x)+\frac{R(x)}{Q(x)}$.

Step 2: Factor the denominator as far as possible.

Step 3: Express the proper rational function $\frac{R(x)}{Q(x)}$ as a sum of partial fractions of the form

$$\frac{A}{(ax+b)^i}\text{ or }\frac{Ax+B}{(ax^2+bx+c)^j}.$$

**CASE I**: The denominator $Q(x)$ is a product of distinct linear factor:

$$Q(x) = (a_1x + b_1)(a_2x + b_2)\cdots(a_kx + b_k)$$

Then there exists $A_1, A_2, \cdots, A_k$ such that

$$\frac{R(x)}{Q(x)} = \frac{A_1}{a_1x + b_1} + \frac{A_2}{a_2x + b_2} + \cdots + \frac{A_k}{a_kx + b_k}$$

**CASE II**: $Q(x)$ is a product of linear factors, some of which are repeated.

$$Q(x) = (a_1x + b_1)^{r_1}(a_2x + b_2)^{r_2}\cdots(a_kx + b_k)^{r_k}$$

then there exists $A_{11}, \cdots, A_{1r_1}, \cdots A_{k1}, \cdots, A_{kr_k}$ such that

$$\begin{aligned}\frac{R(x)}{Q(x)} =& \frac{A_{11}}{a_1x + b_1} + \frac{A_{12}}{(a_1x + b_1)^2} + \cdots + \frac{A_{1r_1}}{(a_1x + b_1)^{r_1}} \\ &+ \cdots \\ &+ \frac{A_{k1}}{a_kx + b_k} + \cdots + \frac{A_{kr_k}}{(a_kx + b_k)^{r_k}}.\end{aligned}$$

**CASE III**: $Q(x)$ contains irreducible quadratic factors, none of which is repeated.

If $Q(x)$ has a term $ax^2 + bx + c$, where $b^2 - 4ac < 0$, then the expression for $\frac{R(x)}{Q(x)}$ will have a term $\frac{Ax+B}{ax^2+bx+c}$.

**CASE IV**: $Q(x)$ contains a repeated irreducible quadratic factor.

If $Q(x)$ has the factor $(ax^2 + bx + c)^r$, where $b^2 - 4ac < 0$, then the sum

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \cdots + \frac{A_rx + B_r}{(ax^2 + bx + c)^r}$$

occurs in the partial fraction decomposition of $\frac{R(x)}{Q(x)}$.

2.

**Rationalizing Substitutions**: Some non-rational functions can be changed into rational functions by means of appropriate substitution.

Example: Evaluate $\int \frac{\sqrt{x+4}}{x} dx$.

Solution: Let $u = \sqrt{x+4}$. Then $u^2 = x + 4$ and $x = u^2 - 4$.

$$du = \frac{1}{2}\frac{1}{\sqrt{x+4}}dx = \frac{1}{2u}dx, \quad dx = 2udu.$$

$$\begin{aligned}
\int \frac{\sqrt{x+4}}{x}dx &= \int \frac{u}{u^2-4} \cdot 2udu \\
&=2\int \frac{u^2}{u^2-4}du \\
&=2\int 1du + 8\int \frac{1}{u^2-4}du \\
&=2u + 2\int \left(\frac{1}{u-2} - \frac{1}{u+2}\right) du \\
&=2\sqrt{x+4} + 2\ln\left|\frac{u-2}{u+2}\right| + C
\end{aligned}$$

3.

**TABLE OF INTEGRATION FORMULAS** Constants of integration have been omitted.

1. $\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$
2. $\int \frac{1}{x}\,dx = \ln|x|$
3. $\int e^x\,dx = e^x$
4. $\int a^x\,dx = \frac{a^x}{\ln a}$
5. $\int \sin x\,dx = -\cos x$
6. $\int \cos x\,dx = \sin x$
7. $\int \sec^2 x\,dx = \tan x$
8. $\int \csc^2 x\,dx = -\cot x$
9. $\int \sec x \tan x\,dx = \sec x$
10. $\int \csc x \cot x\,dx = -\csc x$
11. $\int \sec x\,dx = \ln|\sec x + \tan x|$
12. $\int \csc x\,dx = \ln|\csc x - \cot x|$
13. $\int \tan x\,dx = \ln|\sec x|$
14. $\int \cot x\,dx = \ln|\sin x|$
15. $\int \sinh x\,dx = \cosh x$
16. $\int \cosh x\,dx = \sinh x$
17. $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\tan^{-1}\left(\frac{x}{a}\right)$
18. $\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\left(\frac{x}{a}\right)$

*19. $\int \frac{dx}{x^2-a^2} = \frac{1}{2a}\ln\left|\frac{x-a}{x+a}\right|$

*20. $\int \frac{dx}{\sqrt{x^2 \pm a^2}} = \ln\left|x + \sqrt{x^2 \pm a^2}\right|$

Exercises:

1. $\int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}\,dx$

Solution: Step 1: improper → long division:

$$\begin{array}{r|l} & x+1 \\ \hline x^3-x^2-x+1 & x^4-2x^2+4x+1 \\ & -(x^4-x^3-x^2+x) \\ \hline & x^3-x^2+3x+1 \\ & -(x^3-x^2-x+1) \\ \hline & 4x \end{array}$$

$$\Rightarrow x^4-2x^2+4x+1 = (x+1)(x^3-x^2-x+1)+4x$$

$$\Rightarrow \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1} = (x+1)+\frac{4x}{x^3-x^2-x+1}$$

Step 2: Factor the denominator:

$$\begin{aligned} x^3-x^2-x+1 &= x^2(x-1)-(x-1) \\ &= (x^2-1)(x-1) \\ &= (x-1)(x+1)(x-1) \\ &= (x-1)^2(x+1) \end{aligned}$$

Step 3: Decompose the rational function:

$$\frac{4x}{x^3-x^2-x+1} = \frac{A_1}{x-1}+\frac{A_2}{(x-1)^2}+\frac{A_3}{x+1}$$

① Multiply both sides by $(x-1)^2(x+1)$:

$$4x = A_1(x-1)(x+1)+A_2(x+1)+A_3(x-1)^2$$

② $4x = A_1(x^2-1) + A_2(x+1) + A_3(x^2-2x+1)$

$= (A_1+A_3)x^2 + (A_2-2A_3)x - A_1+A_2+A_3$

$$\Rightarrow \begin{cases} A_1+A_3=0 \\ A_2-2A_3=4 \\ -A_1+A_2+A_3=0 \end{cases} \Rightarrow \begin{cases} A_1=1 \\ A_2=2 \\ A_3=-1 \end{cases}$$

$$\Rightarrow \int \frac{x^4-2x^2+4x+1}{x^3-x^2-x+1}dx = \int\left[(x+1) + \frac{1}{x-1} + \frac{2}{(x-1)^2} + \frac{-1}{x+1}\right]dx$$

$$= \int(x+1)dx + \int\frac{1}{x-1}dx + \int\frac{2}{(x-1)^2}dx + \int\frac{-1}{x+1}dx$$

$$= \left(\frac{1}{2}x^2+x\right) + \ln|x-1| - \frac{2}{x-1} - \ln|x+1| + C$$

$$= \frac{1}{2}x^2+x - \frac{2}{x-1} + \ln\left|\frac{x-1}{x+1}\right| + C$$

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2. $\displaystyle\int \frac{2x^2-x+4}{x^3+4x}dx$

Solution: Step 1: proper. continue

Step 2: Factor the denominator:

$x^3+4x = x(x^2+4)$

Step 3: Decompose the rational function:

$$\frac{2x^2-x+4}{x^3+4x} = \frac{A_1}{x} + \frac{B_1x+C_1}{x^2+4}$$

① multiply both sides by $x^3+4x$:

$2x^2-x+4 = A_1(x^2+4) + (B_1x+C_1)x$

② Expand:

$2x^2-x+4 = A_1x^2 + 4A_1 + B_1x^2 + C_1x$

$= (A_1+B_1)x^2 + C_1x + 4A_1$

$$\Rightarrow \begin{cases} A_1 + B_1 = 2 \\ C_1 = -1 \\ 4A_1 = 4 \end{cases} \Rightarrow \begin{cases} A_1 = 1 \\ B_1 = 1 \\ C_1 = -1 \end{cases}$$

$$\Rightarrow \int \frac{2x^2 - x + 4}{x^3 + 4x}\,dx = \int \left(\frac{1}{x} + \frac{x-1}{x^2+4}\right) dx$$

$$= \int \frac{1}{x}\,dx + \int \frac{x-1}{x^2+4}\,dx$$

$$= \ln|x| + C + \int \frac{x}{x^2+4}\,dx - \int \frac{1}{x^2+4}\,dx$$

$$= \ln|x| + C + \int \frac{1}{x^2+4} \cdot \frac{1}{2}\,dx^2 - \int \frac{1}{\left(\frac{x}{2}\right)^2+1} \cdot \frac{1}{2}\,d\frac{x}{2}$$

$$= \ln|x| + C + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\arctan\frac{x}{2}$$

$$= \ln|x| + \frac{1}{2}\ln|x^2+4| - \frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + C. \#$$

Remark: ① $\int \frac{dx}{x^2+a^2} = \frac{1}{a}\arctan\left(\frac{x}{a}\right) + C$

② For $\int \frac{dx}{ax^2+bx+c}\,dx$, when it's irreducible $(b^2-4ac<0)$

$$ax^2+bx+c = a\left(x+\frac{b}{2a}\right)^2 + c - \frac{b^2}{4a}$$

$$\Rightarrow \int \frac{dx}{ax^2+bx+c}\,dx = \int \frac{1}{a} \cdot \frac{1}{\left(x+\frac{b}{2a}\right)^2 + \left(\frac{c-\frac{b^2}{4a}}{a}\right)}\,d\left(x+\frac{b}{2a}\right)$$

$$\underset{u=x+\frac{b}{2a}}{=} \frac{1}{a}\int \frac{1}{u^2 + \underbrace{\frac{4ac-b^2}{4a^2}}_{>0}}\,du$$

$$\underset{\text{use ①}}{=} \frac{1}{a} \cdot \frac{1}{\sqrt{\frac{4ac-b^2}{4a^2}}} \arctan\left(\frac{u}{\sqrt{\frac{4ac-b^2}{4a^2}}}\right) + C$$

$$= \frac{2|a|}{a\sqrt{4ac-b^2}} \cdot \arctan\left(\frac{2|a|}{\sqrt{4ac-b^2}} \cdot \left(x+\frac{b}{2a}\right)\right) + C$$

3. $\displaystyle\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx$

Solution: Step 1: $\deg(1-x+2x^2-x^3)=3 < 5=\deg(x(x^2+1)^2)$

$\Rightarrow$ proper, continue.

Step 2: Factor the denominator:

can't further factor $x\cdot(x^2+1)^2$

Step 3: Decompose the rational function:

$$\frac{1-x+2x^2-x^3}{x(x^2+1)^2} = \frac{A_1}{x} + \frac{A_2x+B_2}{x^2+1} + \frac{A_3x+B_3}{(x^2+1)^2}$$

① Multiply both sides by $x(x^2+1)^2$:

$$1-x+2x^2-x^3 = A_1(x^2+1)^2 + (A_2x+B_2)x(x^2+1) + (A_3x+B_3)x$$

② Expand:

$$\begin{aligned}1-x+2x^2-x^3 &= A_1(x^4+2x^2+1) + (A_2x^2+B_2x)(x^2+1) + A_3x^2+B_3x \\ &= A_1x^4 + 2A_1x^2 + A_1 + A_3x^2 + B_3x + A_2x^4 + A_2x^2 + B_2x^3 + B_2x \\ &= (A_1+A_2)x^4 + B_2x^3 + (2A_1+A_3+A_2)x^2 + (B_2+B_3)x + A_1\end{aligned}$$

$$\Rightarrow \begin{cases} A_1+A_2=0 \\ B_2=-1 \\ 2A_1+A_2+A_3=2 \\ B_2+B_3=-1 \\ A_1=1 \end{cases} \Rightarrow \begin{cases} A_1=1 \\ A_2=-1 \\ B_2=-1 \\ B_3=0 \\ A_3=1 \end{cases}$$

$$\Rightarrow \int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\,dx = \int \frac{1}{x} + \frac{-x-1}{x^2+1} + \frac{x}{(x^2+1)^2}\,dx$$

$$= \int \frac{1}{x}\,dx - \int \frac{x}{x^2+1}\,dx - \int \frac{1}{x^2+1}\,dx + \int \frac{x}{(x^2+1)^2}\,dx$$

$$= \ln|x| + C - \int \frac{1}{x^2+1} \cdot \frac{1}{2}\,dx^2 - \int \frac{1}{x^2+1}\,dx + \int \frac{1}{(x^2+1)^2} \cdot \frac{1}{2}\,dx^2$$

$$= \ln|x| + C - \frac{1}{2}\ln(x^2+1) - \arctan(x) + \frac{1}{2} \cdot \frac{-1}{x^2+1}$$

$$= \ln|x| - \frac{1}{2}\ln(x^2+1) - \tan^{-1}(x) - \frac{1}{2(x^2+1)} + C.$$

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