MATH 54 SUMMARY

– CHECK IF LINEAR TRANSFORMATION

↳ $A(\underline{u}+\underline{v}) = A\underline{u} + A\underline{v}$

↳ $A(c\underline{u}) = cA\underline{u}$

– IMPORTANT TERMS:

INJ LI ① injective: one-to-one = # pivots = # columns → LI (only trivial)

SU onto span ② surjective: onto = # pivots = # rows → span

③ bijective: one-to-one and onto ⇒ isomorphism

– HOW TO FIND INVERSE:

① 2x2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

② $[A \,\vdots\, I] \xrightarrow{\text{row reduce}} [I \,\vdots\, A^{-1}]$

③ cofactor espansion → transpose → $\times \frac{1}{\det A}$

– PROPERTIES OF MATRICES (mxn)

$A(BC) = (AB)C$

$A(B+C) = AB + AC$

$AB \neq BA$

$AB = AC \rightarrow B \neq C$

$AB = 0 \rightarrow A \neq 0, B \neq 0$

$r(AB) = (rA)B = A(rB)$

$I_m A = A = A I_n$

– PROPERTIES OF TRANSPOSE / INVERSE

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$ $\qquad (A^{-1})^{-1} = A$

$r(A^T) = (rA)^T$ $\qquad (AB)^{-1} = B^{-1}A^{-1}$ $\qquad (\text{Col}\,A)^{\perp} = \text{Nul}\,A^T$

$(AB)^T = B^T A^T$ $\qquad (A^T)^{-1} = (A^{-1})^T$ $\qquad (\text{Row}\,A)^{\perp} = \text{Nul}\,A$

$(AB)^{-1} = B^{-1}A^{-1}$ $\qquad \text{Col}(A)^{\perp} = \text{Nul}(A^T)$

- INVERTIBLE MATRIX THRM:

↳ $A_{n \times n}$: all true or all false

1. A = invertible
2. A = row equivalent to I_n
3. A has n pivot points
4. $Ax = 0$ only has trivial soln
5. Columns of A = LI set
6. LI transf : $x \mapsto Ax$ is one to one ~~has one soln for each b in~~ $\mathbb{R}^n$
7. Eqn : $Ax = b$ has one soln for each b in $\mathbb{R}^n$
8. Columns of A span $\mathbb{R}^n$
9. Lin transf $x \mapsto Ax$ maps $\mathbb{R}^n$ to $\mathbb{R}^n$
10. $C_{n \times n}$ exists such that $CA = I$
11. $D_{n \times n}$ exists such that $AD = I$
12. A^T is an invertible matrix
13. Columns of A = basis for $\mathbb{R}^n$
14. $\text{Col}(A) = \mathbb{R}^n$
15. $\text{Dim}(A) = n$
16. rank $A = n$
17. Nul $A = \{0\}$
18. $\dim(\text{Nul} A) = 0$
19. $\lambda = 0$ = not an e-value

- PROPERTIES OF DETERMINANTS

$\det(AB) = \det A \det B$

$\det(A) = \det A^T$

- HOW TO CALCULATE DETERMINANTS

1. cofactor expansion
2. Row reduce ⇒ upper triangular ⇒ multiply diag
   - adding rows → nothing
   - interchanging rows ⇒ $\times -1$
   - scaling by k ⇒ $\times$ k

- RANK-NULLITY THRM
  ↳ rank A + Nul A = n
  dim Col A + Nul A = n
  # pivot + # free = #columns

- TEST VECTOR SPACE
  ① Addition
  ② Scalar Mult
  ③ zero

  $\left.\begin{array}{l} ① \text{ Addition} \\ ② \text{ Scalar Mult} \\ ③ \text{ zero} \end{array}\right]$ — NOT.

  - W in V (V=V.S.)
  - span {something}
  - show = Nul(A) for some matrix A

- POLYNOMIAL
  ① use std {1, t, t², t³, etc}
  ② put w/ constants a[ ] + b[ ] - - -
  ③ group by term + solve for a

- STD VECTORS : $\begin{bmatrix}1\\0\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\1\\0\\0\\0\\0\end{bmatrix}, \begin{bmatrix}0\\0\\1\\0\\0\\0\end{bmatrix} \ldots$

- $X = P_B [X]_B$

  vector · std coords/ matrix of bases · coordinate vector

- BASIS
  ↳ must be LI and span

$\star$ $\left[\begin{array}{l} \text{Col } A = \text{Rank } A = \#\text{ pivots} = \mathbb{R}^M \\ \text{Nul } A = \text{kernel} = \#\text{ free} = \mathbb{R}^N \\ \text{Row } A = \quad = \#\text{ pivots} = \mathbb{R}^N \end{array}\right]$ $\left[\underset{\text{domain}}{\mathbb{R}^N} \longrightarrow \underset{\text{codomain}}{\mathbb{R}^M}\right]$

- $\text{Dim}(P_N) = N+1$
- $\text{Dim}(\mathbb{R}^N) = N$

- CHANGE OF BASES

$$[x]_C = \underset{C \leftarrow B}{P} [x]_B$$

$$\left[C \,\vdots\, B\right] \xrightarrow[\text{reduce}]{\text{row}} \left[I \,\vdots\, \underset{C \leftarrow B}{P}\right]$$

$$T: \left[T(b)\right]_C$$

- E-VALUES + E-VALUES
  - ↳ $\text{Det}(A - \lambda I) = 0$

- DIAGONALIZE
  - ↳ $A = PDP^{-1} \longrightarrow$ P = e-vectors D = λ diagonal
  - can use to find A^N
  - complex: $A = PCP^{-1} \rightarrow P = [\text{real} \;\; \text{imag}]$ $C = \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$ $\lambda = a \pm ib$

- ORTHOGONAL PROJECTIONS
  - ↳ $y = \underset{\text{in } W}{\hat{y}} + \underset{\text{in } W^{\perp}}{z}$

- hug against orm basis / col W

- LEAST SQUARES
  - ↳ for $AX=b \Rightarrow$ inconsistent

① $(A^TA)\tilde{X} = (A^Tb)$
  ↳ then use new $\hat{X}$ to solve for $\hat{b}$

② If already orthogonal:
$\hat{b} = \frac{b \cdot u_1}{u_1 \cdot u_1} u_1 + \cdots$

③ if already QR factorized
  ↳ $\hat{X} = R^{-1}Q^Tb$

- QR FACTORIZATION
  - ↳ $A_{m \times n}$ = LI
    - ↳ $Q_{m \times n}$ = orthonormal basis

$A = QR$
$Q^TA = R$
$Q^{-1}A = R$

R = upper triangular w/ ⊕ entries on diagonal

- PROPERTIES OF SIMILARITY
  - ↳ $A = PDP^{-1}$
  - same characteristic polynomial, λ, multiplicity
  - A sim B
    A^2 sim B^2
    A^{-1} sim B^{-1}
    A^T sim B^T

- SPECTRAL THRM
  ↳ A = symmetric

① n real λ (w/ mult)
② each multiplicity ⇒ has e-vector
③ e-space = mutually orthogonal
④ A = orthogonally diagonalizable

- WRONKIAN

$$\begin{bmatrix} f & g & h \\ f' & g' & h' \\ f'' & g'' & h'' \end{bmatrix}$$

⇒ det ≠ 0 LI
⇒ det = 0
- check near pt
- prove LD otherwise

- REST OF DIFF EQS

HEAT EQN

$$\frac{\partial u}{\partial t}(x,t) = \beta \frac{\partial^2 u}{\partial x^2}(x,t)$$ gen eqn

$u(0,t) = u(L,t) = 0$ boundary

$u(x,0) = f(x)$ initial

SOLN : $\sum c_n e^{-\beta\left(\frac{n\pi}{L}\right)^2 t} \sin\left(\frac{n\pi}{L}x\right)$

WAVE EQN →

$$\frac{\partial^2 u}{\partial t^2} = \alpha \frac{\partial^2 u}{\partial x^2}$$ gen eqn

$u(0,t) = u(L,t) = 0$ boundary

$u(x,0) = f(x)$

$\partial u/\partial t(x,0) = g(x)$ initial

$\sum \left(A_m \cos\left(\frac{n\pi}{L}t\right) + B_m \sin\left(\frac{n\pi}{L}t\right)\right)$

cos/sin $\left(\frac{n\pi}{L}x\right)$

SOLN

MATH 54

LA CH. 1

① For each b in $\mathbb{R}^m$ the eqn $AX=b$ has a soln
② Each b in $\mathbb{R}^n$ is a lin combo of the columns in A
③ The columns of A span $\mathbb{R}^n$
④ A has a pivot in every row

(All of the above ①-④ are equivalent)

- The soln set of $AX=b$ is shown as $w = p + v_h$ where v_h is the soln of $AX=0$
  ↳ kinda like general + particular soln (diff eq)

- set w/ just one vector = automatically LI (as long as it's not the zero matrix)
- LD if:
  - one vector = multiple or lin combo of other
  - # col > # row cuz free variables
  - if contains zero vector

- Transformation = linear if: (can combine)
  ① $T(u+v) = T(u) + T(v)$
  ② $T(cu) = cT(u)$
- Must also have $T(0) = 0$
  ↳ aka no constants
- All matrix transformations = linear transformations (but not necessarily other way around)

- the range of T is all linear combinations of A in AX not the whole codomain of $\mathbb{R}^m$

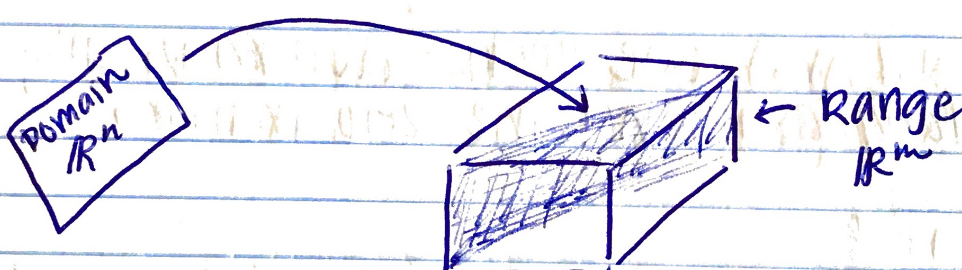
- Standard Matrix

$$A = [T(e_1) \cdots T(e_n)]$$

w/ $e_1 = \begin{bmatrix}1\\0\\0\end{bmatrix}$, $e_2 = \begin{bmatrix}0\\1\\0\end{bmatrix}$, $e_3 = \begin{bmatrix}0\\0\\1\end{bmatrix}$, etc.

$= 1$, t, t^2 etc.

- <u>onto</u>: if for each b in $\mathbb{R}^m$ (codomain) there is at least one soln of $T(x) = b$



- <u>one to one</u>: if each b in $\mathbb{R}^m$ is in the image of at most one x in $\mathbb{R}^n$
  - ↳ free variable? LD? can't be one-to-one
  - ↳ pivot in each row? LI? is one-to-one and maps $\mathbb{R}^n$ to $\mathbb{R}^m$

- If $T: \mathbb{R}^n \to \mathbb{R}^m$ is a linear transf. and A is a standard matrix for T then:
  1. T maps $\mathbb{R}^n$ onto $\mathbb{R}^m$ $\iff$ columns of A spans $\mathbb{R}^m$ (pivot in each row: onto)
  2. T is one to one $\iff$ columns of A are LI (pivot in each col: one-to-one)

$\partial u/\partial t(x,0) = g(x)$

- ONTO: span = pivot in each row
- ONE TO ONE: LI = pivot in each column;
  ↳ $T(x) = 0$ is only trivial soln
- Image will always be the span of the columns of A

☆☆

ONTO	SURJECTIVE	span	Pivot in each row
1 to 1	INJECTIVE	LI	Pivot in each col
BOTH	BIJECTIVE	both	Pivots everywhere $(n \times n)$

- How to generally write solns:

$A = [A] \qquad B = [b_1 \; b_2 \; b_3]$

$AB = \quad b_1[A] \quad b_2[A] \quad b_3[A]$

$\quad\quad\quad\quad \downarrow \quad\quad \downarrow \quad\quad \downarrow$

$C = [\; c_1 \quad c_2 \quad c_3 \;]$

- PROPERTIES OF MATRICES: $(m \times n)$

$A(BC) = (AB)C$

$A(B+C) = AB + AC$

$AB \neq BA$

$AB = AC \longrightarrow B \neq C$

$AB = 0 \longrightarrow A \neq 0, B \neq 0$

$r(AB) = (rA)B = A(rB)$

$I_m A = A = A I_n$

- PROPERTIES OF TRANSPOSE:

$(A^T)^T = A$

$(A+B)^T = A^T + B^T$

$r(A^T) = (rA)^T$

$\boxed{\begin{aligned}(AB)^T &= B^T A^T \\ (AB)^{-1} &= B^{-1} A^{-1}\end{aligned}}$

INVERSE MATRICES:

↳ 2x2: $\begin{bmatrix} a & b \\ c & d \end{bmatrix} = A$

if determinant $(ad-bc) \neq 0$ then matrix is invertible

$\frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = A^{-1}$

must be square

↳ not 2x2;

$$[A \,\vdots\, I] \xrightarrow[\text{reduce}]{\text{row}} [I \,\vdots\, A^{-1}]$$

- If A is invertible then for each b in $\mathbb{R}^n$, $Ax=b$ has the unique soln $x = A^{-1}b$
  ↳ aka can just put inverse on other side

☆ INVERTIBLE MATRIX THRM:

↳ $A_{n \times n}$, either all true or all false

1. A = invertible
2. A = row equivalent to I_n (identity)
3. A has n pivot points
4. $Ax=0$ only has the trivial soln
5. columns of A = LI set
6. Lin transf $x \to Ax$ is one-to-one
7. Eqn $Ax=b$ has one soln for each b in $\mathbb{R}^n$
8. columns of A span $\mathbb{R}^n$
9. Lin transf $x \to Ax$ maps $\mathbb{R}^n$ onto $\mathbb{R}^n$

cont'd ⤴

⑩ $C_{n\times n}$ exists such that $CA = I$
⑪ $D_{n\times n}$ exists such that $AD = I$
⑫ A^T is an invertible matrix

contd

- lin transf $T: \mathbb{R}^n \to \mathbb{R}^n$ is invertible if A is invertible and $T^{-1} = S$

$$S(x) = T^{-1}(x) = A^{-1}x$$

- OTHER PROPERTIES of INERSE

$(A^{-1})^{-1} = A$
$(AB)^{-1} = B^{-1}A^{-1}$
$(A^T)^{-1} = (A^{-1})^T$

☆ - Subspace H in $\mathbb{R}^n$ has 3 properties:

need to prove this.

① $\underline{0}$ in H (aka no constants)
② $\underline{u}, \underline{v}$ in H, then $\underline{u} + \underline{v}$ in H (closed under addition)
③ $\underline{u}$ in H, $c = \#$, the $c\underline{u}$ in H (closed under multiplicate)

☆ - BASIS for a subspace H in $\mathbb{R}^n$ is:
① Linearly Independent
② Spans $\mathbb{R}^n$

☆ — COLUMN SPACE

↳ all linear combinations of A (Range/image)

- Basis (Col A) = { pivots columns of A }

↳ **How to find:** row reduce A to echelon form, mark pivot columns, go back to original matrix and extract.

☆ — NULL SPACE

↳ set of all soln for homo eqn $AX = 0$

- Basis (Nul A) = { free vars stuff }

↳ **How to find:** row reduce, set equal to zero vector, solve for x, y, z..., put into list of eqns w/ free variables and then write as: $\vec{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix} = z\begin{pmatrix} \# \\ \# \\ 1 \end{pmatrix}$

— Column space = subspace of $\mathbb{R}^m$
— Null space = subspace of $\mathbb{R}^n$

— COORDINATES

$$[X]_B = \begin{bmatrix} c_1 \\ \vdots \\ c_p \end{bmatrix} \qquad [X]_B [A] = X$$

- Dimension of nonzero subspace H (dim H) is the # of vectors in any basis for H
  ↳ Dim H = # columns

- Rank A = Dim (Col A)

- Rank-Nullity Thrm:

  Rank A + Dim (Nul A) = n
  Dim (Col A) + Dim (Col A) = # columns

- Basis Theorem:
  ↳ H = p-dimensional subspace of $\mathbb{R}^n$
  then any LI set w/ p dimensions = basis for H
  ↳ set of p elements that span H = basis for H

- aka Goldilocks

Too Few (can't span)	Just Right	Too Many (Lin Dep)
(is LI)	basis!	(does span)

- IMT CONT'D
  13. Columns of A = a basis for $\mathbb{R}^n$
  14. Col (A) = $\mathbb{R}^n$
  15. dim (A) = n
  16. rank A = n
  17. Nul (A) = $\{0\}$
  18. dim (Nul A) = 0

- Basis allows us to express $\underline{x}$ ~~in~~ wrt the basis

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- DETERMINANTS
  ↳ 2 ways to calculate:

  ① Cofactor expansion (bomberman)

  ② upper Δ-ler $\Rightarrow$ multiply diagonal

- Determinants w/ row operations on matrices
  ① Adding multiple of other row → nothing
  ② interchanging rows → $\times -1$
  ③ Scaling row by k → $\times k$

- Invertible @ $Det \neq 0$

☆ - Properties: $\det(A^T) = \det(A)$ | $\det(A+B) \neq \det A + \det B$
  $\det(AB) = \det(A)\det(B)$ | $\det(AB) = \det(BA)$

- CRAMER'S RULE: solve systems w/ determinants
  ↳ only for invertible matrices (A)

$$Ax = b$$

① Replace 1^{st} col of A w/ b = A_1
② Replace 2^{nd} col of A w/ b = A_2
③ $x_1 = \dfrac{\det A_1}{\det A}$ $\quad x_2 = \dfrac{\det A_2}{\det A}$

- INVERSE FORMULA:

$$A^{-1} = \frac{1}{\det A} (\text{adj } A)$$

↳ adj A = transpose of the matrix of cofactors : aka replace each spot w/ cofactor
- eh. just use $[A \vdots I]$ then switch col + row

- physical meaning of determinants
① 2x2 : $|\det A|$ = area of a parallelogram
② 3x3 : $|\det A|$ = volume of a parallelepiped

- Linear transformations
↳ if $T: \mathbb{R}^2 \to \mathbb{R}^2$ and S = parallelogram in $\mathbb{R}^2$

① area $T(S) = |\det A| \cdot \{\text{area of } S\}$
② volume $T(S) = |\det A| \cdot \{\text{volume of } S\}$

- If $v_1 \dots v_p$ in a vector space V, then span $\{v_1 \dots v_p\}$ is a subspace of V

- Spanning Set Thrm:
↳ Let $S = \{v_1 \dots v_p\}$ set in V and $H = \text{span}\{v_1 \dots v_p\}$

① can remove vectors that are lin combos of the other vectors w/o changing the span
② if $H \neq \{0\}$ some subset S is a basis for H

- aka goldilocks

- CHANGE OF COORDS

$x = P_\beta [x]_\beta$

↳ P_β: change of coords matrix (invertible)
↳ $\beta = \{b_1, \dots b_n\}$ = basis for vector space

- Need to contain polynomials
  ↳ not just t^3 but $t^3, t^2, t, 1$

- NEW COORD SYSTEM

$B = \{b_1, \dots b_n\}$ $C = \{c_1 \dots c_n\}$

★ $$\boxed{[x]_C = \underset{C \leftarrow B}{P} [x]_B}$$

$$\underset{C \leftarrow B}{P} = \left(\underset{B \leftarrow C}{P}\right)^{-1}$$

$$P_B [x]_B = P_C [x]_C = x$$

↳ HOW to find Δ in coords matrix:

$$[c_1 \ c_2 \ \vdots \ b_1 \ b_2] \sim [I \ \vdots \ \underset{C \leftarrow B}{P}]$$

- columns of $\underset{C \leftarrow B}{P}$ are C-coordinate vector wrt basis B

- Eigenvalue λ if $A\underline{x} = A\underline{\lambda}$

- Row Reduction changes λ
  - ↳ can only use values on diagonal if already in upper triangular form (aka don't force it)

- How to find λ: (eigenvalues)
  - ↳ $\det(A-\lambda I)=0$ or $\det(\lambda I - A)=0$

- How to find e-vectors
  - ↳ put in λ into $A-\lambda I$ then find Null space

- set of e-vectors = L.I.

- A matrix is invertible if $\lambda=0$ is not an e-value
  - ↳ aka $\mathrm{Det}(A) \neq 0$

- Similar matrices ($A=PDP^{-1}$) share the same characteristic polynomials and therefore same λs and their multiplicities

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- DIAGONALIZABLE
  - ↳ follows $A=PDP^{-1}$
    - ↳ A = original matrix
      D = diagonal matrix of e-values
      P = matrix of e-vectors

- only diagonalizable if there are n L.I. e-vectors
  - ↳ aka each multiplicity has to match up

- if $\#\lambda = n$; diagonalizable
- A matrix being diagonalizable has nothing to do w/ it being invertible
- Diagonizable @ Dim(espaces) $= n$

- Matrix relative to bases B and C

$$M = \left[[T(b_1)]_C \cdots [T(b_n)]_C \right] = \text{matrix of } T \text{ relative to } B = B \text{ matrix}$$

↳ $[T(x)]_C = r_1 [T_1(b_1)]_C + \cdots r_n [T(b_n)]_C$

since $[x]_B = \begin{bmatrix} r_1 \\ r_2 \\ \vdots \\ r_n \end{bmatrix}$

$$\begin{tikzcd} x \arrow[r] \arrow[d] & T(x) \arrow[d] \\ {[x]_B} \arrow[r, "M"] & {[T(x)]_C} \end{tikzcd} \qquad \begin{tikzcd} V \arrow[r, "T"] \arrow[d, "{[-]_B}"'] & W \arrow[d, "{[-]_C}"] \\ \mathbb{R}^n \arrow[r, "{[T]}"] & \mathbb{R}^m \end{tikzcd}$$

5.5

- aka do transformation on bases B then put in terms of new bases C

↳ @ $W = V$; B same as C (aka represent same subspace)

- B matrix for $T: V \to W$

$[T(x)]_B = [T]_B [x]_B$

- can think of $A = PDP^{-1}$ as transformation
  - ↳ If B is the basis for $\mathbb{R}^n$ formed by columns of P then D is the matrix for the transformation $x \mapsto Ax$

$$\begin{tikzcd}
x \arrow[r, "\text{multiply by } A"] \arrow[d, "\text{multiply by } P^{-1}"'] & Ax \\
{[x]_B} \arrow[r, "\text{multiply by } C"] & {[Ax]_B} \arrow[u, "\text{multiply by } P"']
\end{tikzcd}$$

- for complex e-values, use $A = PCP^{-1}$
  - ↳ A: original matrix

$$C = \begin{vmatrix} a & -b \\ b & a \end{vmatrix} \quad \text{w/ } \lambda = a \pm ib$$

$$P = \begin{bmatrix} \text{real} & \text{imag} \end{bmatrix}$$

- can rewrite complex #s w/ sines/cosines

$$C = \begin{vmatrix} r & 0 \\ 0 & r \end{vmatrix} \begin{vmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{vmatrix}$$

$r = \sqrt{a^2+b^2}$

$b = r\sin\theta$

$a = r\cos\theta$

Ch. 6

- DOT PRODUCT $u \cdot v = uv^T = u_1v_1 + \cdots u_nv_n$
  - ↳ length: $\|u\| = \sqrt{u \cdot u}$
  - ↳ distance: $\text{dist}(u,v) = \|(u-v)\|$

- $u \cdot v = 0$ @ $\perp$ / orthogonal
- orthogonal if $\|u+v\|^2 = \|u\|^2 + \|v\|^2$

- $(\text{ROW } A)^\perp = \text{Nul}(A)$
- $\text{Col}(A)^\perp = \text{Nul}(A^T)$

$\left[\begin{array}{l} \perp = \text{perp} \\ T = \text{transpose} \end{array}\right]$

- If $\{u_1 \dots u_p\}$ is an orthogonal set of non zero vectors the LI and a basis for the subspace spanned by it

- orthogonal ~~subspace~~ basis for a subspace is a basis for W that is also an ortho set

~~$y = c_1 u_1 + \dots + c_p u_p$ then~~
~~then $y \cdot u$~~

- orthogonal projection = hugging formula

$$\hat{y} = \frac{y \cdot u_1}{u_1 \cdot u_1} u_1$$

- $U_{m \times n}$ = orthonormal columns @ $U^T U = I$
  - ↳ set of orthonormal columns = orthogonal matrix
    - ↳ must preserve angle and length
  - ↳ $\boxed{U^{-1} = U^T}$

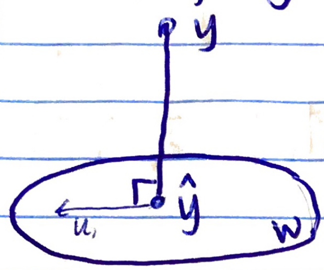
- ORTHOGONAL PROPERTIES

$\|Ux\| = \|x\|$

$(Ux)\cdot(Uy) = x\cdot y$

$(Ux)\cdot(Uy) \Leftrightarrow x\cdot y = 0$

- can project things orthogonally



$\hat{y} = \text{proj}_W(y) = \frac{y\cdot u_1}{u_1\cdot u_1} u_1 + \cdots$

$W \Rightarrow$ orth basis $\{u_1 \ldots u_p\}$

- If orthogonal basis $\Rightarrow \hat{y} = (y\cdot u_1)u_1$
  - ↳ cuz $u_1\cdot u_1 = 1$ (must be normalized)
  - ↳ ☆ $\text{proj}_W y = UU^T y$

- If A is similar to B then:

A^2 " B^2

A^{-1} " B^{-1}

A^T " B^T

- GRAM-SCHMIDT PROCESS
  - ↳ algorithm for finding orthonormal basis

- given basis $\{u_1, u_2, u_3 \cdots\}$

$V_1 = U_1$

$V_2 = U_2 - \frac{U_2\cdot V_1}{V_1\cdot V_1} V_1$ = scale + cross out others

$V_3 = U_3 - \frac{U_3\cdot V_1}{V_1\cdot V_1} V_1 - \frac{U_3\cdot V_2}{V_2\cdot V_2} -$

NORMALIZE @ END!

- QR FACTORIZATION
  - ↳ $A_{m \times n}$ w/ LI columns can be factored $A = QR$
  - ↳ $Q_{m \times n}$ has orthonormal basis

$A = QR$

$Q^T A = R$ (cuz $Q^T = Q^{-1}$)

↳ R = upper triangular matrix w/ ⊕ entries on the diagonal.

- least squares
  - ↳ $AX = b$ → inconsistent

$(A^T A)\tilde{\hat{x}} = (A^T b)$

↳ solve for $\hat{x}$

↳ solve $A\tilde{x} = \hat{b}$

[alt method] * If already orthogonal columns

$u = [u_1 \ u_2]$ $b = [\ ]$

$$\hat{b} = \frac{b \cdot u_1}{u_1 \cdot u_1} u_1 + \frac{b \cdot u_2}{u_2 \cdot u_2} u_2 \cdots$$

* If already have QR factorization

$$\hat{x} = R^{-1} Q^T b$$

- INNER PRODUCT SPACES
  ↳ $\langle u, v \rangle = u \cdot v =$ some real #

$$\left[u \cdot v = \int_{-1}^{1} uv\,dt \right]$$

- satisfies axioms:
  ① $\langle u, v \rangle = \langle v, u \rangle$
  ② $\langle u+v, w \rangle = \langle u, w \rangle + \langle v, w \rangle$
  ③ $\langle cu, v \rangle = c\langle u, v \rangle$
  ④ $\langle u, u \rangle \geq 0$
  $\langle u, u \rangle = 0$ only if $\underline{u} = \underline{0}$

- A v.s w/ an inner product is an inner product space

$$\| u \| = \sqrt{\langle u, u \rangle}$$

$$\langle u, v \rangle \leq \| u \| \, \| v \|$$

- symmetric matrices $A^T = A$
  ↳ if symmetric then any 2 e-vectors from different e-values are orthogonal

$A_{n \times n}$ orth diagonalizable $\Leftrightarrow$ $A_{n \times n}$ symmetric
↳ $A = PDP^T = A = PDP^{-1}$
w/ $P =$ orthogonal matrix of e-vectors

- don't have n $\lambda \Rightarrow$ Gram Schmidt!

- Spectral Thrm: $A_{n \times n}$ has a basis of orthogonal e-vectors $\Leftrightarrow$ A is a symmetric matrix

DIFF EQs

- HOMOGENEOUS LINEAR EQN
  ↳ $ay'' + by' + cy = 0$

  ↳ auxilliary eqn: $ar^2 + br + c = 0$

- GENERAL SOLN: 3 cases

  1. 2 distinct roots: $y = Ae^{r_1 t} + Be^{r_2 t}$
  2. 1 repeated root: $y = Ae^{r_1 t} + Bte^{r_2 t}$
  3. 2 complex roots: $y = Ae^{(\alpha + i\beta)} + Be^{(\alpha - i\beta)}$
     $y = Ae^{\alpha t}\cos \beta t + Be^{\alpha t}\sin \beta t$

- NONHOMOGENEOUS ⇒ Method of undetermined Coefficients
  $ay'' + by' + cy = f(t)$

  1. $f(t) = e^{rt} \Rightarrow y_p = Ae^{rt}$
  2. $f(t) =$ polynomial $\Rightarrow y_p =$ Countdown: $At^2 + Bt + C$
  3. $f(t) =$ sine or cosine $\Rightarrow y_p = A\cos(\#t) + B\sin(\#t)$
     ↳ if one need to include other as well
  4. $f(t) =$ combo $\Rightarrow y_p = (\text{poly})e^{\square}\cos + (\text{poly})e^{\square}\sin$

- After you pick a a y_p, derive it a bunch of times then put it into your actual eqn

- IF y_p coincides w/ homo ⇒ add factor of t til it works

- IF adding stuff ⇒ calculate separately then add together

- Superposition principle: $\boxed{y_{gen} = y_p + y_{homo}}$

- WRONSKIAN

↳ $$\tilde{W}(t) = \tilde{W}[f_1, f_2, f_3] = \begin{bmatrix} f_1 & f_2 & f_3 \\ f_1' & f_2' & f_3' \\ f_1'' & f_2'' & f_3'' \end{bmatrix}$$

$w(t)$ = determinant

↳ If wronskian $\neq 0$ then LI

↳ If wronskian $= 0$ then

① Pick another point that $\neq 0$, LI

② Prove LD another way (ex: lin combo / multiples)

- HIGHER ORDER DIFF EQs

↳ same thing :)

- can reduce to a lower order diff eq if you rename + replace

↳ (see other notes) $\Rightarrow y' = Ay + f$

- HOW TO SOLVE $X' = AX$
  ① Find λ/e-values + e-vectors
  ② Put in form: $x(t) = Ae^{\lambda t}$ [e-vector] + . . .

- If n-distinct e-values; e-vectors = LI form fundamental soln set

- If A = symmetric $A = A^T$ then there will be n - LI e-vectors

- If A = not symmetric, can have repeated e-value but not to have 2 LI corresponding e-vectors

- If repeated root; take form:

check $y(t) = \cancel{Ae^{rt}[\,]+B(te^{rt}+ue^{rt})[\,]} = Ae^{rt}[\;] + Be^{rt}[t[\;]+u]$

w/ $u \Rightarrow [A - \lambda I]u = $ [e-vector]

- If complex root: $r = \alpha \pm bi$; evector $= |a| \pm |B| i$

$X(t) = e^{(\alpha + \beta i)t}$ [vector]

☆ $\boxed{\begin{aligned} &= A\left(e^{\alpha t}\cos\beta t\,[\text{real}] - e^{\alpha t}\sin\beta t\,[\text{imaginary}]\right) \\ &+ B\left(e^{\alpha t}\sin\beta t\,[\text{real}] + e^{\alpha t}\cos\beta t\,[\text{imaginary}]\right)\end{aligned}}$

– HEAT EQN (β, L, f(x) ⟹ vari's)

$\boxed{\text{QUESTION}}$
$$\begin{cases} \bullet\ \frac{\partial u}{\partial t}(x,t) = \beta \frac{\partial^2 u}{\partial x^2}(x,t) & \text{general eqn} \\ \bullet\ u(0,t) = u(L,t) = 0 & \text{boundary conditions} \\ \bullet\ u(x,0) = f(x) & \text{initial conditions} \end{cases}$$

$\boxed{\text{STEP \#1}}$ SEPARATION OF VARIABLES

↳ $u(x,t) = X(x)\,T(t)$

- put into general eqn

$$\Downarrow$$

$$X''(x) = \lambda X(x), \quad T'(t) = \beta \lambda T(t)$$

$\boxed{\text{STEP \#2}}$ SET UP BOUNDARY EQNS

- $u(0,t) = X(0)T(t) = 0 \longrightarrow X(0) = 0$
- $u(L,t) = X(L)T(t) = 0 \longrightarrow X(L) = 0$

$\boxed{\text{STEP \#3}}$ 3 CASES TO FIND E-VALUES/FUNCTIONS

① @ $\lambda > 0$; $\lambda = \omega^2$

$r^2 = \omega^2$

$r = \pm\omega \longrightarrow X(x) = Ae^{\omega x} + Be^{-\omega x}$

$\Downarrow$

$\omega = 0$ contradict

② @ $\lambda = 0$

$r = 0$ (mult 2) $\longrightarrow X(x) = Ax + B$

$\Downarrow$

(sometimes not 0) $X(x) = 0$ nope.

@ $\lambda < 0$; $\lambda = -W^2$

$r^2 = -\omega^2$

$r = \pm \omega i \longrightarrow X(x) = A\cos \omega x + B\sin \omega x$

$\Downarrow$

$A = 0 \qquad B = ☺$

① if $L = 1$; $\lambda = -W^2 = -(\pi m)^2$

$X(x) = \sin(\pi m x)$

② if $L = \pi$; $\lambda = -w^2 = -m^2$

$X(x) = \sin(mx)$

$\boxed{\text{STEP 4}}$ USE λ TO FIND $T(t)$

$T'(t) = \beta\lambda T(t)$

$r = \beta\lambda \Rightarrow$ ① $\beta(-m\pi)^2$ ② $\beta(-m^2)$

$\Downarrow$

$T(t) = A_m e^{-\sqrt{\beta} m\pi t}$ OR $A_m e^{-\sqrt{\beta} mt}$

$\boxed{\text{STEP 5}}$ LIN COMBOS $\Rightarrow$ $u(x,t) = X(x)T(t)$

$u(x,t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=1}^{\infty} A_m e^{\square t} \sin \square$

(depending on λ)

$\boxed{\text{STEP 6}}$ PUT IN INITIAL CONDITIONS to SOLVE FOR A_m

① If of form $\sin(\# x)$ be happy

↳ just plug in @ $m = \#$, blah

↳ $m =$ anything else, $A_m = 0$

② if of form x be sad

↳ use fourier series

STEP2

$$A_m = \frac{f \cdot \sin}{\sin \cdot \sin} = \frac{2\int_0^L f \cdot \sin}{\int_{-L}^{L} \sin^2} = \frac{2}{L}\int_0^L f \sin$$

$$\Downarrow$$

$$\frac{\square(-1)^M}{\square}$$

STEP 7 SUB BACK INTO OVERALL EQN

$$\boxed{\text{soln: } \sum C_m e^{-\beta\left(\frac{m\pi}{L}\right)^2 t} \sin\left(\frac{m\pi}{L}x\right)}$$

— WAVE EQN

QUESTION

- $\frac{\partial^2 u}{\partial t^2}(x,t) = \alpha \frac{\partial^2 u}{\partial x^2}$ general eqn
- $u(0,t) = u(L,t) = 0$ boundary condition
- $u(x,0) = f(x)$ position
- $\frac{\partial u}{\partial t}(x,0) = g(x)$ velocity

— initial condition

STEP #1 SEPARATION OF VARIABLES: $u(x,t) = X(x)T(t)$

↳ Put into gen eqn

$$\Downarrow$$

$$X''(x) = \lambda X(x) \qquad T''(t) = \alpha\lambda T(t)$$

$\boxed{\text{STEP 2}}$ BOUNDARY CONDITION

$$u(0,t) = 0 = X(0)T(t) \rightarrow X(0) = 0$$
$$u(L,t) = 0 = X(L)T(t) \rightarrow X(L) = 0$$

$\boxed{\text{STEP 3}}$ 3 CASES TO FIND E-VALUES/E-FUNCTIONS

① @ $\lambda > 0$; $\lambda = \omega^2$

$$r^2 = \omega^2$$
$$r = \pm\omega \rightarrow X(x) = Ae^{\omega x} + Be^{-\omega x}$$
$$\Downarrow$$
$\omega = 0$; contradiction!

② @ $\lambda = 0$

$$r = 0 \text{ (mult 2)} \longrightarrow X(x) = AX + B$$
$$\Downarrow$$
① $X(x) = B$:(

② $X(x) = 0$:)

③ @ $\lambda < 0$; $\lambda = -\omega^2$

$$r^2 = -\omega^2$$
$$r = \pm\omega i \rightarrow X(x) = A\cos\omega x + B\sin\omega x$$
$$\Downarrow$$
$A = 0$ $B = \square$

① if $L = 1$; $\lambda = -\omega^2 = -(\pi m)^2$

$$X(x) = \sin(\pi m x)$$

② if $L = \pi$; $\lambda = -\omega^2 = -m^2$

$$X(x) = \sin(mx)$$

$\boxed{\text{STEP \#4}}$ use λ to find $T''(t) = \alpha\lambda\, T(t)$

① $T(t) = A_m \cos(mt) + B_m \sin(mt)$
② $T(t) = A_m \cos(\pi m t) + B_m \sin(\pi m t)$

$\boxed{\text{STEP \#5}}$ LIN COMBOS

$u(x,t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=1}^{\infty} (A_m \cos(mt) + B_m \sin(mt)) \sin(mx)$

$\boxed{\text{STEP \#6}}$ USE THE 1st INITIAL CONDITION ⇒ (A_m)

$u(x,0) = f(x)$

↳ if of form __ $\sin(\#x)$ = celebrate (plug + chug)

↳ if something else ⇒ fourier it up

$\boxed{\text{STEP \#7}}$ USE 2nd INITIAL CONDITION ⇒ B_m

↳ same as above
- happy if sine
- sad if fourier

$\boxed{\text{STEP \#8}}$ Plug in coefficients A_m / B_m

$u(x,t) = \sum_{m=1}^{\infty} (\tilde{A}_m \cos(mt) + \tilde{B}_m \sin(mt)) \sin(mx)$