3. (a)

Let $x_1 = r\cos\theta$, $x_2 = r\sin\theta$; $\mu_1 = \tau\cos\theta$, $\mu_2 = \tau\sin\theta$, then

$$f_{R,\theta}(r,\theta) = f_{X_1,X_2}(x_1,x_2)\cdot r .$$

We have $f_{X,Y}(x,y) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\frac{z}{2(1-\rho^2)}\right)$, where

$$z = \frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2^2-\mu_2)}{\sigma_2^2}$$

$$= \frac{\cos^2\theta(r-\tau)^2}{\sigma_1^2} - \frac{2\rho(r-\tau)^2\sin\theta\cos\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta(r-\tau)^2}{\sigma_2^2}$$

$$= (r-\tau)^2\left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\sin\theta\cos\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right)$$

$$\therefore\ f_{R,\theta}(r,\theta) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}(r-\tau)^2\left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\sin\theta\cos\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right)\right]\cdot r$$

$$\therefore\ f_R(r) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, r\cdot\int \exp\left[-\frac{1}{2(1-\rho^2)}(r-\tau)^2\left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\sin\theta\cos\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right)\right]d\theta$$

$$\therefore\ f_{\theta|R}(\theta|r) = \frac{f_{R,\theta}(r,\theta)}{f_R(r)} = \frac{\exp\left[-\frac{1}{2(1-\rho^2)}(r-\tau)^2\left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\cos\theta\sin\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right)\right]}{\int\exp\left[-\frac{1}{2(1-\rho^2)}(r-\tau)^2\left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\cos\theta\sin\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right)\right]d\theta}$$

Then, we have :

$$E[X_1|R] = E[R\cos\theta|R] = R\,E[\cos\theta|R]$$
$$= R\cdot\int\cos\theta\cdot f_{\theta|R}(\theta|r)\cdot d\theta$$

$$E[X_1X_2|R] = E[R^2\cos\theta\sin\theta|R] = R^2 E[\cos\theta\sin\theta/R]$$
$$= R^2\cdot\int\cos\theta\sin\theta\, f_{\theta|R}(\theta|r)\, d\theta$$

(b) (i)

Let $\gamma = (\underset{\sim}{\mu}, \Sigma)$, $\|\underset{\sim}{\mu}\| = \tau$

E-step:

$$Q(\gamma,\gamma^*) = E_{\gamma^*}\left[\log f(R,\theta;\gamma)\ \middle|\ R\right]$$
$$= \int\left(\sum_{i=1}^n \log f(R_i,\theta_i;\gamma)\right)\cdot\prod_{i=1}^n f_{\theta_i|R_i}(\theta_i|r_i;\gamma^*)\ d\underset{\sim}{\theta}$$
$$= \sum_{i=1}^n \int \log f(R_i,\theta_i;\gamma)\cdot f_{\theta_i|R_i}(\theta_i|r_i;\gamma^*)\ d\theta_i$$

We have:

$$\log f(R_i, \theta_i ; \gamma) = -\left[\log 2\pi + \log \sigma_1 + \log \sigma_2 + \frac{1}{2}\log(1-\rho^2)\right] + \log r_i$$
$$- \frac{1}{2(1-\rho^2)} (r_i - \tau_i)^2 \left(\frac{\cos^2\theta_i}{\sigma_1^2} - \frac{2\rho \sin\theta_i \cos\theta_i}{\sigma_1 \sigma_2} + \frac{\sin^2\theta_i}{\sigma_2^2}\right)$$

$$\therefore \int \log f(R_i, \theta_i ; \gamma) \cdot f_{\theta_i|R_i}(\theta_i | r_i ; \gamma^*) \, d\theta_i$$
$$= -\left[\log 2\pi + \log \sigma_1 + \log \sigma_2 + \frac{1}{2}\log(1-\rho^2)\right] + \log r_i$$
$$- \frac{1}{2(1-\rho^2)} (r_i - \tau_i)^2 \cdot \int \left(\frac{\cos^2\theta_i}{\sigma_1^2} - \frac{2\rho \sin\theta_i \cos\theta_i}{\sigma_1 \sigma_2} + \frac{\sin^2\theta_i}{\sigma_2^2}\right) f_{\theta_i|R_i}(\theta_i | r_i ; \gamma^*) \, d\theta_i ,$$

where $\int \left(\frac{\cos^2\theta_i}{\sigma_1^2} - \frac{2\rho \sin\theta_i \cos\theta_i}{\sigma_1 \sigma_2} + \frac{\sin^2\theta_i}{\sigma_2^2}\right) f_{\theta_i|R_i}(\theta_i | r_i ; \gamma^*) \, d\theta_i$ can be found using Monte Carlo method: Sample $\theta^{(k)}$ from $f_{\theta_i|R_i}(\theta_i | r_i ; \gamma^*)$ and compute

$$\frac{1}{K} \sum_{k=1}^{K} \frac{\cos^2\theta^{(k)}}{\sigma_1^2} - \frac{2\rho \sin\theta^{(k)} \cos\theta^{(k)}}{\sigma_1 \sigma_2} + \frac{\sin^2\theta^{(k)}}{\sigma_2^2}$$

M-step: obtain

$$\gamma_{(t+1)} = \underset{\gamma}{\operatorname{argmax}} \, Q(\gamma, \gamma_{(t)})$$

$$\frac{1}{2} \frac{1}{(1-\rho^2)} \cdot (-2\rho)$$

$$u = -\frac{1}{2(1-\rho^2)} \qquad u' = -\frac{\rho}{(1-\rho^2)^2}$$

$$v = \int \qquad d\theta \qquad v' = \int - \frac{2\sin\theta\cos\theta}{\sigma_1 \sigma_2}$$

(b) (ii)

$$\frac{\partial}{\partial \rho} Q(\gamma, \gamma^*) = \frac{\rho}{1-\rho^2} + (r-\tau)^2 \Bigg[-\frac{\rho}{(1-\rho^2)^2} \cdot \int \left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\cos\theta\sin\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right) f_{\theta|R}(\theta|r;\gamma^*) \, d\theta$$
$$- \frac{1}{2(1-\rho^2)} \int -\frac{2\sin\theta\cos\theta}{\sigma_1\sigma_2} f_{\theta|R}(\theta|r;\gamma^*) \, d\theta \Bigg]$$

$$\frac{\partial}{\partial \sigma_1} Q(\gamma, \gamma^*) = -\frac{1}{\sigma_1} - \frac{1}{2(1-\rho^2)} (r-\tau)^2 \int \left(\frac{-2\cos^2\theta}{\sigma_1^3} + \frac{2\rho\sin\theta\cos\theta}{\sigma_1^2\sigma_2}\right) \cdot f_{\theta|R}(\theta|r;\gamma^*) \, d\theta$$

$$\frac{\partial}{\partial \sigma_2} Q(\gamma, \gamma^*) = -\frac{1}{\sigma_2} - \frac{1}{2(1-\rho^2)} (r-\tau)^2 \int \left(\frac{2\rho\sin\theta\cos\theta}{\sigma_1\sigma_2^2} - \frac{2\sin^2\theta}{\sigma_2^3}\right) \cdot f_{\theta|R}(\theta|r;\gamma^*) \, d\theta$$

$$\frac{\partial}{\partial \tau} Q(\gamma, \gamma^*) = \frac{r-\tau}{1-\rho^2} \int \left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\sin\theta\cos\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right) f_{\theta|R}(\theta|r;\gamma^*) \, d\theta$$

$$\Rightarrow \text{Let } \underset{\sim}{q} = \nabla Q(\gamma, \gamma^*) = \begin{bmatrix} \frac{\partial}{\partial \rho} Q \\ \frac{\partial}{\partial \sigma_1} Q \\ \frac{\partial}{\partial \sigma_2} Q \\ \frac{\partial}{\partial \tau} Q \end{bmatrix}$$

then $I_i = \underset{\sim}{q}\,\underset{\sim}{q}^T$ for observation i, and $I_n = \sum_{i=1}^{n} I_i$, and the variance estimates can be found on the diagonal of I_n^{-1}.