1. $\cancel{g(y|\theta) = C^n(\theta) e^{\sum_{i=1}}}$

1. $\cancel{g(y|\theta) = \frac{1}{a(\theta)} e^{\theta t^T(y)} b(y)}$ $\longrightarrow$ form of exponential family

~~where~~ $\cancel{a(\theta) = \int_y e^{\theta t^T(y)} b(y)\, dy}$.

$$z_{i,k} = \begin{cases} 1 & \text{if } y_i \text{ is from distribution } k. \\ 0 & \text{o.w.} \end{cases}$$

$$f_i(y_i | z_i, \theta) = \prod_{k=1}^{K} g_k(y_i | \theta_k)^{z_{i,k}}.$$

$$p_i(y_i, z_i | \theta, \pi) = f_i(y_i | z_i, \theta) \cdot q(z_i | \pi) = \prod_{k=1}^{K} \{\pi_k g_k(y_i | \theta_k)\}^{z_{i,k}}$$

E-step:

$$Q(\theta, \pi | \theta_p, \pi_p) = \sum_{i=1}^{n} E_{z|y}(\log p_i(y_i, z_i | \theta, \pi) \mid \theta_p, \pi_p)$$

$$= \sum_{i=1}^{n} E_{z|y}[\log f_i(y_i | z_i, \theta) \mid \theta_p, \pi_p] + \sum_{i=1}^{n} E_{z|y}[\log q(z_i | \pi) \mid \theta_p, \pi_p]$$

$$E_{z|y}[\log f_i(y_i | z_i, \theta) \mid \theta_p, \pi_p] = E_{z|y}\left[\sum_{k=1}^{K} \log g_k(y_i | \theta_k) \cdot z_{ik} \mid \theta_p, \pi_p\right]$$

$$= \sum_{k=1}^{K} \log g_k(y_i | \theta_k) E_{z|y}(z_{ik} \mid \theta_p, \pi_p) \qquad ①$$

$$E_{z|y}[\log q(z_i | \pi) \mid \theta_p, \pi_p] = E_{z|y}\left[\sum_{k=1}^{K} z_{ik} \log(\pi_k) \mid \theta_p, \pi_p\right]$$

$$= \sum_{k=1}^{K} \log \pi_k E_{z|y}(z_{ik} \mid \theta_p, \pi_p) \qquad ②.$$

$\cancel{E_{z|y}(z_{ik} | \theta_p, \theta)}$

$$E_{z|y}(z_i | \theta_p, \pi_p) = \sum_{z_i \in \Omega_z} \frac{\prod_{k=1}^{K} \{\pi_k g_k(y_i | \theta_k)\}^{z_{i,k}}}{\sum_{k=1}^{K} \pi_k g_k(y_i | \theta_k)} z_i$$

$$\Rightarrow E_{z|y}(z_{ik} | \theta_p, \pi_p) = \frac{\pi_{kp} g_k(y_i | \theta_{kp})}{\sum_{j=1}^{K} \pi_{jp} g_j(y_i | \theta_{jp})}$$

$$① + ② \Rightarrow Q(\theta, \pi | \theta_p, \pi_p) = \sum_{i=1}^{n} \sum_{k=1}^{K} [\log g_k(y_i | \theta_k) + \log(\pi_k)] \cdot \frac{\pi_{kp} g_k(y_i | \theta_{kp})}{\sum_{j=1}^{K} \pi_{jp} g_j(y_i | \theta_{jp})}.$$

M-step: Since $\sum_{k=1}^{K} \pi_k = 1$, use a Lagrange multiplier.

$$\frac{\partial}{\partial \pi_k}\left(Q(\theta, \pi | \theta_p, \pi_p) - \lambda\left(\sum_{k=1}^{K} \pi_k - 1\right)\right) = \sum_{i=1}^{n} \frac{1}{\pi_k} \frac{\pi_{kp} g_k(y_i | \theta_{kp})}{\sum_{j=1}^{K} \pi_{jp} g_j(y_i | \theta_{jp})} - \lambda$$

$$\Rightarrow \quad \hat{\pi}_k = \frac{1}{n}\sum_{i=1}^{n} \frac{\pi_{kp}\, g_k(y_i \mid \theta_{kp})}{\sum_{j=1}^{K} \pi_{jp}\, g_j(y_i \mid \theta_{jp})}$$

Use

$$\frac{d}{d\theta} Q(\theta, \pi \mid \theta_p, \pi_p) \text{ to get } \theta_{p+1}$$

$$\frac{d}{d\theta_k} Q(\theta, \pi \mid \theta_p, \pi_p) = \sum_{i=1}^{n} \left(\frac{\pi_{kp}\, g_k(y_i \mid \theta_{kp})}{\sum_{j=1}^{K} \pi_{jp}\, g_j(y_i \mid \theta_{jp})} \cdot \frac{\frac{d}{d\theta_k} g_k(y_i \mid \theta_k)}{g_k(y_i \mid \theta_k)} \right)$$

Now suppose we have a Beta distribution.

$$g(y; \theta) = \frac{y^{\alpha-1}(1-y)^{\beta-1}}{B(\alpha, \beta)}$$

$$\hat{\pi}_k = \frac{1}{n}\sum_{i=1}^{n} \frac{\pi_{k0}\, g_k(y_i \mid \theta_{k0})}{\sum_{j=1}^{K} \pi_{k0}\, g_j(y_i \mid \theta_{j0})}$$

For α and β, there is no closed form since there is a gamma function in the likelihood, we can use gradient descent to solve.

2. $y_j \sim N(X_j\beta, \underbrace{\tau^2 Z_j Z_j^T + \sigma_j^2 I}_{\Sigma})$ $\quad b_j | y_j \sim N(\tau^2 Z_j^T \Sigma^{-1}(y_j - X_j\beta), \tau^2 I - \tau^4 Z_j \Sigma_j^{-1} Z_j)$

$$\begin{pmatrix} b_j \\ y_j \end{pmatrix} \sim N\left(\begin{bmatrix} 0 \\ X_j\beta \end{bmatrix}, \begin{bmatrix} \tau^2 I & \tau^2 Z_j \\ \tau^2 Z_j & \tau^2 Z_j Z_j^T + \sigma_j^2 I \end{bmatrix}\right)$$

E-step:

$$Q = \sum_{j=1}^m E[\log(y_j, b_j) | y_j, \theta^p]$$

$$= \sum_{j=1}^m E\left(-\frac{n_j}{2}\log(2\pi\sigma_j^2) - \frac{1}{2}\log \tau^2 - \frac{1}{2\sigma_j^2}\left[(y_j - X_j\beta)^T(y_j - X_j\beta) + (Z_j b_j)^T(Z_j b_j) + (Z_j b_j)^T(y_j - X_j\beta)\right]\right.$$

$$\left. - \frac{1}{2\tau^2} b_j^T b_j \;\middle|\; y_j, \theta^p\right),$$

where $\theta = \{\beta, \tau, \sigma_j\}, j \in \{1 \dots m\}$

M-step: $\frac{\partial Q}{\partial \beta}, \frac{\partial Q}{\partial \tau}, \frac{\partial Q}{\partial \sigma_j}$

$$Q = \sum_{j=1}^m \left[-\frac{n_j}{2}\log(2\pi) - \log\tau - n_j\log\sigma_j - \frac{(y_j - X_j\beta)^T(y_j - X_j\beta)}{2\sigma_j^2} - \frac{(y_j - X_j\beta)^T}{\sigma_j^2}(\tau^p)^2 Z_j Z_j^T (\Sigma^p)^{-1}(y_j - X_j\beta^p)\right.$$

$$- \frac{1}{2\sigma_j^2} \underbrace{\operatorname{trace}(Z_j^T Z_j (\tau^p)^2 - (\tau^p)^4 Z_j^T (\Sigma)^{-1} Z_j) + (\tau^p)^4 (y_j - X_j\beta^p)^T \Sigma^{p^{-1}} Z_j Z_j^T Z_j Z_j^T (\Sigma^p)^{-1}(y_j - X_j\beta^p)}$$

$$\left.- \frac{1}{2\tau^2} \underbrace{\operatorname{trace}((\tau^p)^2 I - (\tau^p)^4 Z_j^T \Sigma^{-1} Z_j) + (\tau^p)^4 (y_j - X_j\beta^p)^T \Sigma^{-1} Z_j Z_j^T \Sigma^{p^{-1}}(y_j - X_j\beta^p)}_{A}\right]$$

$$\Rightarrow \frac{\partial Q}{\partial \beta} = -\sum_{j=1}^m \frac{(\tau^p)^2 Z_j Z_j^T (\Sigma^p)^{-1}(y_j - X_j\beta^p)}{(\sigma_j^p)^2} X_j - \frac{1}{2}\sum_{j=1}^m \sigma_j^{p^{-2}}(2\beta(X_j^T X_j) - 2y_j^T X_j)$$

$$\hat\beta^{p+1} = \sum_{j=1}^m \frac{X_j^T X_j}{(\sigma_j^p)^2} \cdot \sum_{j=1}^m \frac{X_j^T(y_j - (\tau^p)^2 Z_j Z_j^T (\Sigma^p)^{-1}(y_j - X_j\beta^p))}{(\sigma_j^p)^2}$$

$$\Rightarrow \frac{\partial Q}{\partial \tau} = \sum_{j=1}^m \frac{A_j^p}{\tau^3} - \frac{m}{\tau}$$

$$\hat\tau^{p+1} = \sqrt{\frac{\sum_{j=1}^m A_j^p}{m}}$$

$$\Rightarrow \frac{\partial Q}{\partial \sigma_j} = \frac{1}{(\sigma_j)^3}\left[\sigma_j^p - (y_j - X_j\beta^p)^T(y_j - X_j\beta^p) - 2(y_j - X_j\beta^p)^T(\tau^p)^2 Z_j Z_j^T (\Sigma^p)^{-1}(y_j - X_j\beta^p)\right]$$

$$- \frac{n_j}{\sigma_j}$$

$$\hat\sigma_j^{p+1} = \sqrt{\frac{\sigma_j^p - (y_j - X_j\beta^p)^T(y_j - X_j\beta^p) - 2(y_j - X_j\beta^p)^T(\tau^p)^2 Z_j Z_j^T (\Sigma^p)^{-1}(y_j - X_j\beta^p)}{n_j}}$$

hw3

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10/28/2021

q2

```
Vj <- function(tau, sigma, z, nj)
    (sigma^2) * diag(nj) + (tau^2) * z %*% t(z)

Aj <- function(tau, sigma, beta, x, y, z, q, nj, Vj)
  sum(diag(t(z) %*% z %*% ((tau^2)*diag(q) - (tau^4)*t(z) %*%
                             solve(Vj(tau, sigma, z, nj))%*%z))) +
  (tau^4)*t(y-x %*% beta) %*% solve(Vj(tau, sigma, z, nj)) %*% z %*%
  t(z) %*% z %*% t(z) %*% solve(Vj(tau, sigma, z, nj)) %*% (y-x %*% beta)

Cj <- function(tau, sigma, beta, x, y, z, q, nj, Vj)
  (tau^4)*t(y-x %*% beta) %*% solve(Vj(tau, sigma, z, nj)) %*%
  z %*% t(z) %*% solve(Vj(tau, sigma, z, nj)) %*% (y-x %*% beta) +
  sum(diag((tau^2)*diag(q)-(tau^4)*t(z) %*% solve(Vj(tau, sigma, z, nj)) %*% z))

Dj <- function(tau, sigma, beta, x, y, z, nj, Vj)
  (tau^2)*z %*% t(z) %*% solve(Vj(tau, sigma, z, nj)) %*% (y-x %*% beta)


em <- function(beta0, tau0, sigma0, X, Y, Z, n, p, q, tol = 1e-5){
  beta_new=beta0
  tau_new=tau0
  sigma_new=sigma0
  N = cumsum(n)
  m <- length(n)

  z = list()
  y = list()
  x = list()
  for (j in 1:m) {
    zs=matrix(nrow = n[j],ncol = q)
    xs=matrix(nrow = n[j],ncol = p)
    ys = matrix(nrow = n[j],ncol = 1)
    if(j==1) {
      for (i in 1:n[1])
        for (k in 1:q)
          zs[i,k]=Z[i,k]
      for (i in 1:n[1])
        for (k in 1:p)
```

```r
        xs[i,k]=X[i,k]
    for (i in 1:n[1])
        ys[i,1]=Y[i,1]
  }
  if(j>1) {
    for (i in 1:n[j])
      for (k in 1:q)
        zs[i,k]=Z[(N[j-1]+i),(q*(j-1))+k]
    for (i in 1:n[j])
      for (k in 1:p)
        xs[i,k]=X[(N[j-1]+i),k]
    for (i in 1:n[j])
        ys[i,1]=Y[(N[j-1]+i),1]
  }
  x[[j]] = xs
  z[[j]] = zs
  y[[j]] = ys
}

######E-Step
A=matrix(nrow = m,ncol = 1)
C=matrix(nrow = m,ncol = 1)
for (j in 1:m) {
  A[j] = Aj(tau_new, sigma_new[j], beta_new, x[[j]], y[[j]], z[[j]], q, n[j], Vj)
  C[j] = Cj(tau_new, sigma_new[j], beta_new, x[[j]], y[[j]], z[[j]], q, n[j], Vj)
}
siginv = diag(rep((1/sigma_new)^2,n))
tmp = c()
for (j in 1:m)
  tmp=c(tmp, Dj(tau_new,sigma_new[j],beta_new,x[[j]],y[[j]],z[[j]], n[j], Vj))

######M-Step
beta_hat=solve(t(X)%*%siginv%*%X)%*%t(X)%*%siginv%*%(Y-tmp)
tau_hat=sqrt(sum(C)/(m*q))
sigma_hat=sigma_new
for (j in 1:m)
  sigma_hat[j] =
  sqrt((A[j]-2*t(y[[j]]-(x[[j]]%*%beta_new)) %*%
          Dj(tau_new,sigma_new[j],beta_new,x[[j]],y[[j]],z[[j]], n[j], Vj)+
          t(y[[j]]-(x[[j]]%*%beta_new)) %*% (y[[j]]-(x[[j]]%*%beta_new)))/n[j])

while((abs(tau_new-tau_hat) >= tol) ||
      (abs(sigma_new-sigma_hat) >= tol) || (abs(beta_new-beta_hat) >= tol)) {
  beta_new <- beta_hat
  sigma_new <- sigma_hat
  tau_new <- tau_hat
  ######E-Step
  for (j in 1:m) {
    A[j]<-Aj(tau_new, sigma_new[j],
             beta_new, x[[j]], y[[j]], z[[j]], q, n[j], Vj)
    C[j]<-Cj(tau_new, sigma_new[j],
             beta_new, x[[j]], y[[j]], z[[j]], q, n[j], Vj)
  }
```

```r
    tmp = c()
    for (j in 1:m)
        tmp=c(tmp, Dj(tau_new,sigma_new[j],
                      beta_new,x[[j]],y[[j]],z[[j]], n[j], Vj))

    siginv=diag(rep((1/sigma_new)^2,n))
    ######M-Step
    beta_hat <- solve(t(X)%*%siginv%*%X)%*%t(X)%*%siginv%*%(Y-tmp)
    for (j in 1:m)
      sigma_hat[j] = sqrt((A[j]-2*t(y[[j]]-(x[[j]]%*%beta_new))
      %*% Dj(tau_new,sigma_new[j],beta_new,x[[j]],y[[j]],z[[j]], n[j], Vj) +
        t(y[[j]]-(x[[j]]%*%beta_new)) %*% (y[[j]]-(x[[j]]%*%beta_new)))/n[j])
    tau_hat=sqrt(sum(C)/(m*q))
  }

  par <- list(bate = beta_hat, tau = tau_hat, sigma = sigma_hat)
  return(par)
}
```

```r
n <- c(10, 10, 10)
p <- 2
q <- 2
m <- length(n)

X <- matrix(runif(sum(n) * p), nrow = sum(n), ncol = p, byrow = T)
Z <- bdiag(matrix(rnorm(n[1] * q), nrow = n[1], ncol = q, byrow = T),
          matrix(rnorm(n[2] * q), nrow = n[2] , ncol = q, byrow = T),
          matrix(rnorm(n[3] * q), nrow = n[3], ncol = q, byrow = T))
Z <- as.matrix(Z)
beta <- matrix(rnorm(p), nrow = p, ncol = 1, byrow = T)
sigma <- rnorm(m)
tau <- rnorm(1)
b <- matrix(rnorm(m * q, 0, tau^2), nrow = m * q, ncol = 1, byrow = T)
e <- matrix(c(rmvnorm(1, rep(0, n[1]), (sigma[1])^2*diag(n[1])),
            rmvnorm(1, rep(0, n[2]), (sigma[2])^2*diag(n[2])),
            rmvnorm(1, rep(0, n[3]), (sigma[3])^2*diag(n[3]))),
            nrow = sum(n), ncol = 1, byrow = T)
Y <- X %*% beta + Z %*% b + e
beta0 <- matrix(c(1, 1),nrow = p,ncol = 1,byrow = T)
tau0 <- 1
sigma0 <- c(1, 1, 1)
em(beta0, tau0, sigma0, X, Y, Z, n, q = q, p = p)
```

```
## $bate
##             [,1]
## [1,]  1.2873534
## [2,] -0.8636941
##
## $tau
## [1] 0.01362274
##
## $sigma
## [1] 0.007755035 0.664061092 0.652090124
```

3. (a)

Let $x_1 = r\cos\theta$, $x_2 = r\sin\theta$; $\mu_1 = \tau\cos\theta$, $\mu_2 = \tau\sin\theta$, then

$$f_{R,\theta}(r,\theta) = f_{X_1,X_2}(x_1,x_2)\cdot r .$$

We have $f_{X,Y}(x,y) = \dfrac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left(-\dfrac{z}{2(1-\rho^2)}\right)$, where

$$\begin{aligned} z &= \frac{(x_1-\mu_1)^2}{\sigma_1^2} - \frac{2\rho(x_1-\mu_1)(x_2-\mu_2)}{\sigma_1\sigma_2} + \frac{(x_2^2-\mu_2)}{\sigma_2^2} \\ &= \frac{\cos^2\theta(r-\tau)^2}{\sigma_1^2} - \frac{2\rho(r-\tau)^2\sin\theta\cos\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta(r-\tau)^2}{\sigma_2^2} \\ &= (r-\tau)^2\left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\sin\theta\cos\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right) \end{aligned}$$

$$\therefore\ f_{R,\theta}(r,\theta) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\exp\left[-\frac{1}{2(1-\rho^2)}(r-\tau)^2\left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\sin\theta\cos\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right)\right]\cdot r$$

$$\therefore\ f_R(r) = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}}\, r\cdot\int \exp\left[-\frac{1}{2(1-\rho^2)}(r-\tau)^2\left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\sin\theta\cos\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right)\right]d\theta$$

$$\therefore\ f_{\theta|R}(\theta|r) = \frac{f_{R,\theta}(r,\theta)}{f_R(r)} = \frac{\exp\left[-\frac{1}{2(1-\rho^2)}(r-\tau)^2\left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\cos\theta\sin\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right)\right]}{\int\exp\left[-\frac{1}{2(1-\rho^2)}(r-\tau)^2\left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho\cos\theta\sin\theta}{\sigma_1\sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right)\right]d\theta}$$

Then, we have :

$$\begin{aligned} E[X_1\mid R] &= E[R\cos\theta\mid R] = R\,E[\cos\theta\mid R] \\ &= R\cdot\int\cos\theta\cdot f_{\theta|R}(\theta|r)\cdot d\theta \end{aligned}$$

$$\begin{aligned} E[X_1X_2\mid R] &= E[R^2\cos\theta\sin\theta\mid R] = R^2 E[\cos\theta\sin\theta / R] \\ &= R^2\cdot\int\cos\theta\sin\theta\, f_{\theta|R}(\theta|r)\, d\theta \end{aligned}$$

(b) (i)

Let $\gamma = (\underset{\sim}{\mu}, \Sigma)$, $\|\underset{\sim}{\mu}\| = \tau$

E-step:

$$\begin{aligned} Q(\gamma,\gamma^*) &= E_{\gamma^*}\left[\log f(R,\theta;\gamma)\mid R\right] \\ &= \int\left(\sum_{i=1}^n \log f(R_i,\theta_i;\gamma)\right)\cdot\prod_{i=1}^n f_{\theta_i|R_i}(\theta_i|r_i;\gamma^*)\, d\underset{\sim}{\theta} \\ &= \sum_{i=1}^n\int\log f(R_i,\theta_i;\gamma)\cdot f_{\theta_i|R_i}(\theta_i|r_i;\gamma^*)\, d\theta_i \end{aligned}$$

We have:

$$\log f(R_i, \theta_i ; \gamma) = -\left[\log 2\pi + \log \sigma_1 + \log \sigma_2 + \frac{1}{2}\log(1-\rho^2)\right] + \log r_i$$
$$- \frac{1}{2(1-\rho^2)} (r_i - \tau_i)^2 \left(\frac{\cos^2\theta_i}{\sigma_1^2} - \frac{2\rho \sin\theta_i \cos\theta_i}{\sigma_1 \sigma_2} + \frac{\sin^2\theta_i}{\sigma_2^2}\right)$$

$$\therefore \int \log f(R_i, \theta_i ; \gamma) \cdot f_{\theta_i|R_i}(\theta_i | r_i ; \gamma^*) \, d\theta_i$$
$$= -\left[\log 2\pi + \log \sigma_1 + \log \sigma_2 + \frac{1}{2}\log(1-\rho^2)\right] + \log r_i$$
$$- \frac{1}{2(1-\rho^2)} (r_i - \tau_i)^2 \cdot \int \left(\frac{\cos^2\theta_i}{\sigma_1^2} - \frac{2\rho \sin\theta_i \cos\theta_i}{\sigma_1 \sigma_2} + \frac{\sin^2\theta_i}{\sigma_2^2}\right) f_{\theta_i|R_i}(\theta_i | r_i ; \gamma^*) \, d\theta_i,$$

where $\int \left(\frac{\cos^2\theta_i}{\sigma_1^2} - \frac{2\rho \sin\theta_i \cos\theta_i}{\sigma_1 \sigma_2} + \frac{\sin^2\theta_i}{\sigma_2^2}\right) f_{\theta_i|R_i}(\theta_i | r_i ; \gamma^*) \, d\theta_i$ can be found using Monte Carlo method: Sample $\theta^{(k)}$ from $f_{\theta_i|R_i}(\theta_i | r_i ; \gamma^*)$ and compute

$$\frac{1}{K}\sum_{k=1}^{K} \frac{\cos^2\theta^{(k)}}{\sigma_1^2} - \frac{2\rho \sin\theta^{(k)} \cos\theta^{(k)}}{\sigma_1 \sigma_2} + \frac{\sin^2\theta^{(k)}}{\sigma_2^2}$$

M-step: obtain $\gamma_{(t+1)} = \underset{\gamma}{\operatorname{argmax}}\, Q(\gamma, \gamma_{(t)})$ using numerical methods like the R function optim.

(b) (ii)

$$\frac{\partial}{\partial \rho} Q(\gamma, \gamma^*) = \frac{\rho}{1-\rho^2} + (r-\tau)^2 \left[-\frac{\rho}{(1-\rho^2)^2} \cdot \int \left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho \cos\theta \sin\theta}{\sigma_1 \sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right) f_{\theta|R}(\theta | r ; \gamma^*) \, d\theta \right.$$
$$\left. - \frac{1}{2(1-\rho^2)} \int -\frac{2\sin\theta\cos\theta}{\sigma_1 \sigma_2} f_{\theta|R}(\theta | r ; \gamma^*) \, d\theta \right]$$

$$\frac{\partial}{\partial \sigma_1} Q(\gamma, \gamma^*) = -\frac{1}{\sigma_1} - \frac{1}{2(1-\rho^2)} (r-\tau)^2 \int \left(\frac{-2\cos^2\theta}{\sigma_1^3} + \frac{2\rho \sin\theta\cos\theta}{\sigma_1^2 \sigma_2}\right) \cdot f_{\theta|R}(\theta | r ; \gamma^*) \, d\theta$$

$$\frac{\partial}{\partial \sigma_2} Q(\gamma, \gamma^*) = -\frac{1}{\sigma_2} - \frac{1}{2(1-\rho^2)} (r-\tau)^2 \int \left(\frac{2\rho \sin\theta\cos\theta}{\sigma_1 \sigma_2^2} - \frac{2\sin^2\theta}{\sigma_2^3}\right) \cdot f_{\theta|R}(\theta | r ; \gamma^*) \, d\theta$$

$$\frac{\partial}{\partial \tau} Q(\gamma, \gamma^*) = \frac{r-\tau}{1-\rho^2} \cdot \int \left(\frac{\cos^2\theta}{\sigma_1^2} - \frac{2\rho \sin\theta\cos\theta}{\sigma_1 \sigma_2} + \frac{\sin^2\theta}{\sigma_2^2}\right) f_{\theta|R}(\theta | r ; \gamma^*) \, d\theta$$

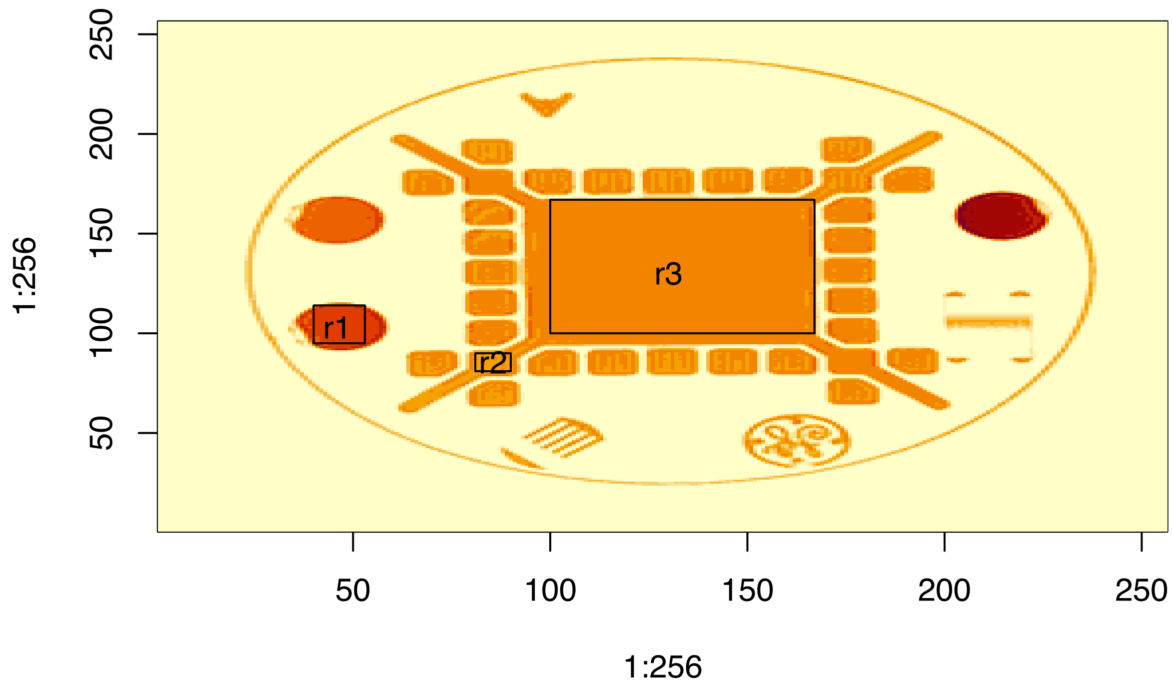
$$\Rightarrow \text{Let } \underset{\sim}{q} = \nabla Q(\gamma, \gamma^*) = \begin{bmatrix} \frac{\partial}{\partial \rho} Q \\ \frac{\partial}{\partial \sigma_1} Q \\ \frac{\partial}{\partial \sigma_2} Q \\ \frac{\partial}{\partial \tau} Q \end{bmatrix}$$

then $I_i = \underset{\sim}{q}\, \underset{\sim}{q}^T$ for observation i, and $I_n = \sum_{i=1}^{n} I_i$, and the variance estimates can be found on the diagonal of I_n^{-1}.

q3

3 The three selected regions are:

```
d = oro.nifti::readNIfTI(fname = "new_phantom.nii.gz")
data = d@.Data
data = drop(data)
r1 = data[40:53,95:114 , ]
r2 = data[81:90, 81:90, ]
r3 = data[100:167, 100:167, ]
```



The functions for the EM algorithm and all other functions used for q3:

```
logf <- function(x, r, rho, sigma1, sigma2, tau) {
  -1/(2*(1-rho^2)) * (r - tau)^2 *
    (cos(x)^2/sigma1^2 -
       (2*rho*cos(x)*sin(x))/(sigma1*sigma2) +
       sin(x)^2/sigma2^2)
}

Qfunc.each <- function(par, par.cur, r, sample.size = 10000, FUNC = logf){
  theta.cond <- arms(sample.size, FUNC, -pi, pi, metropolis = TRUE,
                      arguments = list(
                        r=r, sigma1=par.cur[1], sigma2=par.cur[2],
                        rho=par.cur[3], tau=par.cur[4]
                      ))
```

```r
  sigma1 <- par[1]
  sigma2 <- par[2]
  rho <- par[3]
  tau <- par[4]

  intt <-
    cos(theta.cond)^2/sigma1^2 -
    (2*rho*sin(theta.cond)*cos(theta.cond))/(sigma1*sigma2) +
    sin(theta.cond)^2/sigma2^2
  intt <- mean(intt)

  result <-
    -log(sigma1) - log(sigma2) - 0.5*log(1-rho^2) + log(r) -
    1/(2*(1-rho^2)) * (r-tau)^2 * intt

  result
}

Qfunc <- function(par, par.cur, r.vec, sample.size=1000) {
  result <- sapply(r.vec, function(r) {
    Qfunc.each(par, par.cur, r, sample.size)
  })

  return(-sum(result))
}

em_q3 <- function(r, iter.max=40, tol=1e-4, sample.size=1000){
  par.cur <- c(sd(r), sd(r), 0.5, mean(r))

  iter <- 0
  while(TRUE){
    iter <- iter + 1

    par.old <- par.cur
    # obtain the estimates
    opt <- optim(par.cur, Qfunc, par.cur=par.cur,
                 r.vec=r, sample.size=sample.size,
                 method = "L-BFGS-B", lower = c(0.0001,0.0001,-0.99,0.0001),
                 upper=c(1000, 1000, 0.99, 5000))

    # update the current value
    par.cur <- opt$par

    l2.diff <- sum((par.cur-par.old)^2)

    cat(paste("iter:", iter, "l2.diff:", l2.diff, "Q value:", opt$value, "\n"))

    if(l2.diff < tol){
      cat("less than the tolerance. exit\n")
      break
    } else if (iter > iter.max) {
      cat("reach max iteration. exit\n")
      break
```

```
    }
  }

  return(par.cur)

}

variance.est.each <- function(par, r, sample.size = 1000, FUNC = logf){
  sigma1 <- par[1]
  sigma2 <- par[2]
  rho <- par[3]
  tau <- par[4]

  theta.cond <- arms(sample.size, FUNC, -pi, pi, metropolis = TRUE,
                     arguments = list(
                       r=r, sigma1=sigma1, sigma2=sigma2, rho=rho, tau=tau
                     ))

  part.sigma1 <-
    mean(-2*cos(theta.cond)^2/sigma1^3 +
           (2*rho*sin(theta.cond)*cos(theta.cond))/(sigma1^2*sigma2))
  part.sigma2 <-
    mean(-2*sin(theta.cond)^2/sigma2^3 +
           (2*rho*sin(theta.cond)*cos(theta.cond))/(sigma1*sigma2^2))
  part.rho1 <-
    mean(cos(theta.cond)^2/sigma1^2 -
           (2*rho*cos(theta.cond)*sin(theta.cond))/(sigma1*sigma2) +
    sin(theta.cond)^2/sigma2^2)
  part.rho2 <- mean(-2*sin(theta.cond)*cos(theta.cond)/(sigma1*sigma2))
  part.tau <- part.rho1

  dev.sigma1 <- -1/sigma1 - 1/(2*(1-rho^2))*(r-tau)^2*part.sigma1
  dev.sigma2 <- -1/sigma2 - 1/(2*(1-rho^2))*(r-tau)^2*part.sigma2
  dev.rho <-
    rho/(1-rho^2) +
    (r-tau)^2 * (-rho/(1-rho^2)^2 * part.rho1 - 1/(2*(1-rho^2)) * part.rho2)
  dev.tau <- (r-tau)/(1-rho^2) * part.tau

  q <- c(dev.sigma1, dev.sigma2, dev.rho, dev.tau)

  I.matrix <- q %*% t(q)

  return(I.matrix)
}

variance.est <- function(par, r.vec, sample.size = 1000, return_inv = TRUE){
  result <- lapply(r.vec, function(r) {
    variance.est.each(par, r, sample.size)
  })

  result <- Reduce("+", result)

  if(return_inv) {
```

```
    return(solve(result))
  } else {
    return(result)
  }

}
```

Let $\tau = ||\mu||$, the estimates and variance estimates of σ_1, σ_2, ρ, and τ is showing in the following table

image	region	parameter	estimates	variance	lower	upper
1	1	sigma1	172.8668	972.8182	111.7354	233.9981
1	1	sigma2	170.5234	1294.8561	99.9959	241.0510
1	1	rho	0.9692	0.0001	0.9551	0.9833
1	1	tau	1751.2732	18.7354	1742.7896	1759.7568
1	2	sigma1	36.4765	750.3110	-17.2104	90.1635
1	2	sigma2	36.5039	723.2372	-16.2056	89.2133
1	2	rho	0.9454	0.0012	0.8787	1.0120
1	2	tau	1228.0466	3.7083	1224.2723	1231.8209
1	3	sigma1	38.6535	77.8482	21.3604	55.9466
1	3	sigma2	37.8958	82.3696	20.1076	55.6840
1	3	rho	0.7796	0.0001	0.7586	0.8006
1	3	tau	1308.3897	0.2555	1307.3990	1309.3804
2	1	sigma1	105.9257	218.6615	76.9433	134.9081
2	1	sigma2	114.4804	134.0315	91.7895	137.1712
2	1	rho	0.9900	0.0000	0.9844	0.9956
2	1	tau	1250.0448	2.7651	1246.7857	1253.3040
2	2	sigma1	30.6110	10130.3104	-166.6582	227.8803
2	2	sigma2	30.6110	9540.5085	-160.8295	222.0516
2	2	rho	0.5000	0.0520	0.0532	0.9468
2	2	tau	1015.1200	9.5989	1009.0476	1021.1924
2	3	sigma1	32.1237	4.2511	28.0826	36.1648
2	3	sigma2	32.2347	4.1522	28.2408	36.2285
2	3	rho	0.9828	0.0000	0.9804	0.9852
2	3	tau	1079.3206	0.0259	1079.0054	1079.6358
3	1	sigma1	149.7907	2311.5452	55.5585	244.0229
3	1	sigma2	149.9697	2353.2047	54.8921	245.0472
3	1	rho	0.9771	0.0000	0.9671	0.9871
3	1	tau	1546.0981	11.2722	1539.5177	1552.6785
3	2	sigma1	32.2603	2346.8384	-62.6885	127.2091
3	2	sigma2	32.1726	2109.7667	-57.8528	122.1981
3	2	rho	0.8205	0.0068	0.6590	0.9820
3	2	tau	1149.2288	6.1511	1144.3678	1154.0898
3	3	sigma1	35.3751	5.6038	30.7354	40.0148
3	3	sigma2	36.4938	5.2187	32.0164	40.9713
3	3	rho	0.9799	0.0000	0.9773	0.9825
3	3	tau	1235.9117	0.0351	1235.5445	1236.2790
4	1	sigma1	91.7492	1829.0065	7.9276	175.5707
4	1	sigma2	92.4030	1692.5092	11.7699	173.0362
4	1	rho	0.9732	0.0000	0.9615	0.9849
4	1	tau	954.9988	4.7016	950.7490	959.2486
4	2	sigma1	23.3054	1559.7548	-54.1009	100.7117
4	2	sigma2	24.1364	1293.0455	-46.3418	94.6147

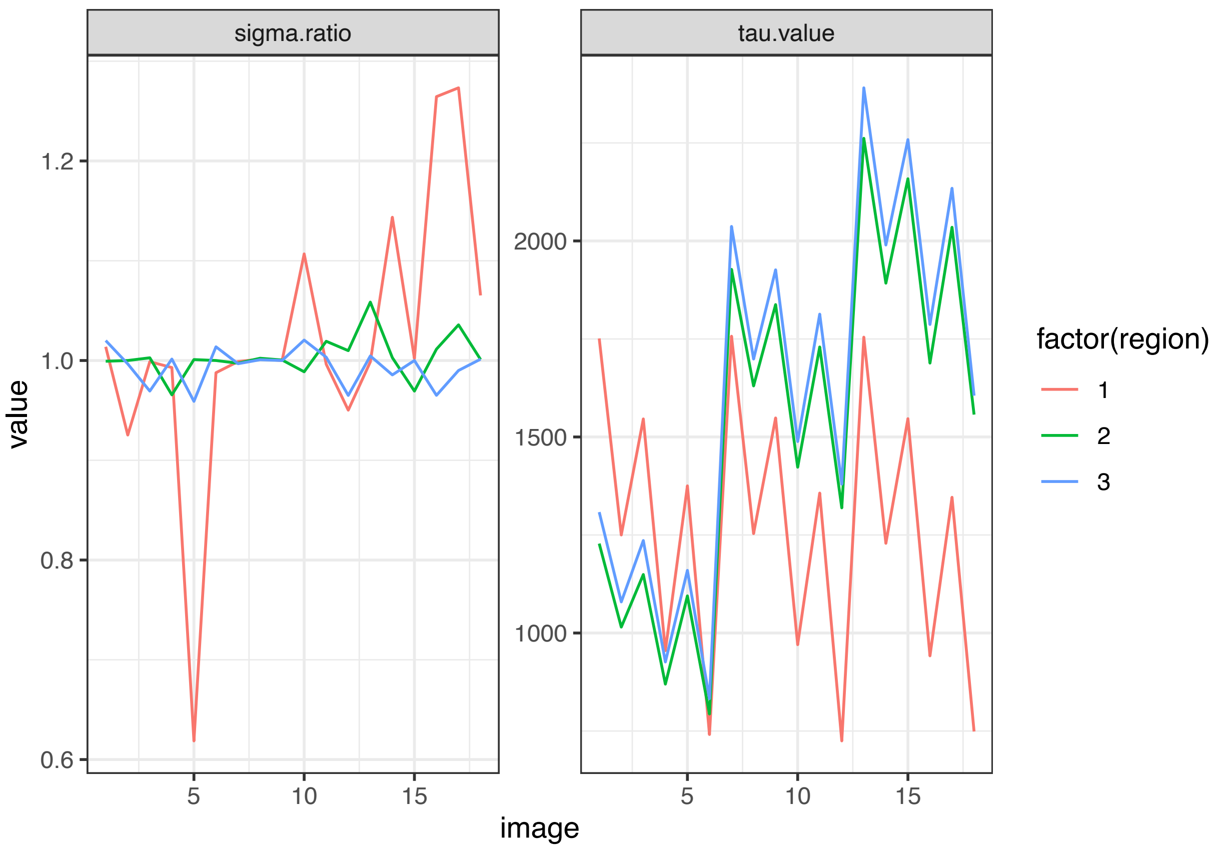
image	region	parameter	estimates	variance	lower	upper
4	2	rho	0.6750	0.0394	0.2860	1.0640
4	2	tau	869.6367	4.1366	865.6504	873.6230
4	3	sigma1	28.4894	4.6577	24.2594	32.7193
4	3	sigma2	28.4488	4.5940	24.2479	32.6497
4	3	rho	0.9689	0.0000	0.9652	0.9727
4	3	tau	926.1581	0.0311	925.8124	926.5038
5	1	sigma1	88.7581	7510.0904	-81.0939	258.6101
5	1	sigma2	143.4657	2807.3466	39.6182	247.3132
5	1	rho	0.6634	0.2646	-0.3447	1.6715
5	1	tau	1375.3600	89.4089	1356.8273	1393.8927
5	2	sigma1	30.8775	5528.4234	-114.8524	176.6074
5	2	sigma2	30.8487	5740.7612	-117.6534	179.3509
5	2	rho	0.6314	0.0313	0.2849	0.9780
5	2	tau	1094.7087	7.2510	1089.4310	1099.9865
5	3	sigma1	31.2342	3.4768	27.5797	34.8888
5	3	sigma2	32.5680	2.9101	29.2245	35.9115
5	3	rho	0.9848	0.0000	0.9827	0.9870
5	3	tau	1159.8642	0.0231	1159.5662	1160.1622
6	1	sigma1	71.6738	655.7060	21.4854	121.8621
6	1	sigma2	72.5632	580.1050	25.3567	119.7696
6	1	rho	0.9722	0.0000	0.9597	0.9848
6	1	tau	741.1539	2.9326	737.7975	744.5103
6	2	sigma1	22.5099	2831.8319	-81.7895	126.8093
6	2	sigma2	22.5099	2734.1864	-79.9756	124.9953
6	2	rho	0.5000	0.0808	-0.0570	1.0570
6	2	tau	793.5500	4.6440	789.3263	797.7737
6	3	sigma1	27.3865	8.7217	21.5983	33.1748
6	3	sigma2	27.0155	9.3124	21.0344	32.9965
6	3	rho	0.9097	0.0000	0.9004	0.9191
6	3	tau	832.4242	0.0632	831.9316	832.9167
7	1	sigma1	168.4028	14912.8256	-70.9443	407.7498
7	1	sigma2	168.5830	16875.8787	-86.0304	423.1964
7	1	rho	0.9717	0.0000	0.9589	0.9845
7	1	tau	1756.9999	17.3031	1748.8470	1765.1527
7	2	sigma1	49.5991	9433.3081	-140.7628	239.9611
7	2	sigma2	49.7350	10183.4233	-148.0508	247.5207
7	2	rho	0.5006	0.0753	-0.0374	1.0385
7	2	tau	1927.4637	21.1872	1918.4421	1936.4854
7	3	sigma1	57.1891	73.9748	40.3318	74.0465
7	3	sigma2	57.3743	73.9176	40.5235	74.2252
7	3	rho	0.8892	0.0000	0.8797	0.8986
7	3	tau	2036.9860	0.3929	2035.7574	2038.2146
8	1	sigma1	105.5890	1263.1388	35.9306	175.2475
8	1	sigma2	105.4428	1317.0921	34.3123	176.5734
8	1	rho	0.9803	0.0000	0.9714	0.9892
8	1	tau	1253.5531	4.5946	1249.3519	1257.7543
8	2	sigma1	38.8348	4639.6879	-94.6687	172.3382
8	2	sigma2	38.7468	4572.3840	-93.7848	171.2784
8	2	rho	0.7688	0.0147	0.5309	1.0067
8	2	tau	1630.2189	8.8564	1624.3861	1636.0517
8	3	sigma1	49.6102	28.7190	39.1067	60.1137
8	3	sigma2	49.5727	28.3345	39.1398	60.0056

image	region	parameter	estimates	variance	lower	upper
8	3	rho	0.9497	0.0000	0.9445	0.9548
8	3	tau	1698.7909	0.1362	1698.0677	1699.5141
9	1	sigma1	144.5341	5491.9618	-0.7145	289.7826
9	1	sigma2	144.5341	5548.8671	-1.4650	290.5332
9	1	rho	0.5000	0.0030	0.3929	0.6071
9	1	tau	1548.5000	320.6022	1513.4061	1583.5939
9	2	sigma1	44.9617	4558.0793	-87.3624	177.2859
9	2	sigma2	44.9392	4944.9922	-92.8867	182.7651
9	2	rho	0.6069	0.0433	0.1989	1.0148
9	2	tau	1837.6550	15.8000	1829.8643	1845.4457
9	3	sigma1	57.5551	182.5874	31.0711	84.0391
9	3	sigma2	57.5551	183.0233	31.0395	84.0707
9	3	rho	0.5000	0.0002	0.4706	0.5294
9	3	tau	1926.3352	1.1747	1924.2110	1928.4595
10	1	sigma1	81.4150	26.2058	71.3816	91.4483
10	1	sigma2	73.5549	37.6981	61.5210	85.5888
10	1	rho	0.8633	0.0004	0.8227	0.9039
10	1	tau	970.2726	15.8921	962.4592	978.0860
10	2	sigma1	35.5073	4926.4989	-102.0607	173.0752
10	2	sigma2	35.9139	4712.0104	-98.6260	170.4538
10	2	rho	0.7714	0.0153	0.5287	1.0141
10	2	tau	1422.9334	7.2336	1417.6621	1428.2048
10	3	sigma1	47.0160	23.6150	37.4915	56.5405
10	3	sigma2	46.0728	26.0671	36.0660	56.0796
10	3	rho	0.8943	0.0000	0.8838	0.9048
10	3	tau	1488.3486	0.2159	1487.4379	1489.2594
11	1	sigma1	127.3234	81576.7880	-432.4746	687.1215
11	1	sigma2	127.8040	81527.1660	-431.8238	687.4318
11	1	rho	0.5012	0.0031	0.3926	0.6097
11	1	tau	1356.9157	241.8609	1326.4346	1387.3969
11	2	sigma1	44.2364	2904.6021	-61.3946	149.8674
11	2	sigma2	43.4000	3576.4868	-73.8131	160.6132
11	2	rho	0.7184	0.0334	0.3603	1.0765
11	2	tau	1729.5467	12.5723	1722.5971	1736.4962
11	3	sigma1	55.9941	150.7748	31.9276	80.0606
11	3	sigma2	55.8112	148.7484	31.9070	79.7154
11	3	rho	0.8269	0.0001	0.8123	0.8416
11	3	tau	1813.2962	0.4675	1811.9561	1814.6363
12	1	sigma1	57.5720	1675.6581	-22.6588	137.8027
12	1	sigma2	60.5875	1594.3334	-17.6721	138.8471
12	1	rho	0.7886	0.0019	0.7023	0.8750
12	1	tau	724.8224	11.2361	718.2526	731.3923
12	2	sigma1	33.8474	9251.8717	-154.6750	222.3698
12	2	sigma2	33.5162	9303.9012	-155.5355	222.5680
12	2	rho	0.4976	0.1071	-0.1438	1.1391
12	2	tau	1319.1868	10.6269	1312.7976	1325.5761
12	3	sigma1	44.7066	5.7664	40.0001	49.4132
12	3	sigma2	46.3287	5.1340	41.8877	50.7696
12	3	rho	0.9852	0.0000	0.9829	0.9874
12	3	tau	1379.6723	0.0507	1379.2309	1380.1137
13	1	sigma1	163.1989	24501.6613	-143.5944	469.9921
13	1	sigma2	163.4228	24326.4390	-142.2715	469.1172

image	region	parameter	estimates	variance	lower	upper
13	1	rho	0.9158	0.0002	0.8868	0.9449
13	1	tau	1754.5953	51.2710	1740.5612	1768.6294
13	2	sigma1	50.3444	5891.8100	-100.0987	200.7876
13	2	sigma2	47.5610	6652.9062	-112.3041	207.4262
13	2	rho	0.7778	0.0201	0.5003	1.0553
13	2	tau	2261.7455	13.1163	2254.6472	2268.8438
13	3	sigma1	68.6787	114.3574	47.7193	89.6382
13	3	sigma2	68.3673	115.6790	47.2871	89.4475
13	3	rho	0.8820	0.0000	0.8719	0.8922
13	3	tau	2390.4667	0.5780	2388.9767	2391.9568
14	1	sigma1	91.5993	149.5217	67.6331	115.5656
14	1	sigma2	80.0982	248.0687	49.2284	110.9680
14	1	rho	-0.6612	0.0034	-0.7762	-0.5461
14	1	tau	1228.8387	50.0725	1214.9697	1242.7078
14	2	sigma1	46.4667	16254.1746	-203.4127	296.3462
14	2	sigma2	46.3357	15449.1302	-197.2771	289.9485
14	2	rho	0.5658	0.0676	0.0563	1.0753
14	2	tau	1892.2336	18.1145	1883.8918	1900.5754
14	3	sigma1	58.8446	104.8702	38.7734	78.9158
14	3	sigma2	59.7065	101.2922	39.9806	79.4323
14	3	rho	0.8663	0.0000	0.8548	0.8777
14	3	tau	1989.6063	0.5116	1988.2043	1991.0082
15	1	sigma1	145.7206	2264.7023	52.4481	238.9931
15	1	sigma2	145.4268	1755.1151	63.3159	227.5377
15	1	rho	0.9836	0.0000	0.9744	0.9929
15	1	tau	1546.6990	7.6311	1541.2847	1552.1133
15	2	sigma1	47.2490	5714.0061	-100.9068	195.4047
15	2	sigma2	48.7438	5511.1092	-96.7577	194.2454
15	2	rho	0.5980	0.0422	0.1955	1.0004
15	2	tau	2158.5487	16.0645	2150.6931	2166.4044
15	3	sigma1	66.5225	304.6244	32.3143	100.7307
15	3	sigma2	66.5225	303.1279	32.3984	100.6466
15	3	rho	0.5000	0.0002	0.4724	0.5276
15	3	tau	2258.4276	1.8826	2255.7384	2261.1168
16	1	sigma1	87.0000	10.8454	80.5453	93.4546
16	1	sigma2	68.8029	47.7472	55.2597	82.3461
16	1	rho	0.7098	0.0015	0.6342	0.7854
16	1	tau	941.8164	34.3546	930.3285	953.3043
16	2	sigma1	36.1672	2123.9573	-54.1605	126.4949
16	2	sigma2	35.7607	1949.4222	-50.7761	122.2976
16	2	rho	0.5599	0.0349	0.1935	0.9262
16	2	tau	1688.4327	8.7227	1682.6441	1694.2213
16	3	sigma1	44.0039	4.9935	39.6242	48.3837
16	3	sigma2	45.5963	4.8714	41.2704	49.9222
16	3	rho	0.9868	0.0000	0.9848	0.9887
16	3	tau	1786.8755	0.0437	1786.4659	1787.2851
17	1	sigma1	132.3303	4119.0916	6.5395	258.1211
17	1	sigma2	103.9252	7414.7497	-64.8452	272.6957
17	1	rho	0.5971	0.1297	-0.1088	1.3030
17	1	tau	1346.1967	112.1349	1325.4419	1366.9515
17	2	sigma1	50.2326	4591.0289	-82.5689	183.0342
17	2	sigma2	48.4986	4774.8815	-86.9359	183.9332

image	region	parameter	estimates	variance	lower	upper
17	2	rho	0.7434	0.0213	0.4571	1.0296
17	2	tau	2034.8892	15.5318	2027.1649	2042.6135
17	3	sigma1	62.5900	46.5575	49.2166	75.9634
17	3	sigma2	63.2224	46.4301	49.8673	76.5776
17	3	rho	0.9419	0.0000	0.9361	0.9476
17	3	tau	2134.3202	0.2586	2133.3235	2135.3169
18	1	sigma1	76.1752	148.4617	52.2940	100.0563
18	1	sigma2	71.5100	258.4399	40.0015	103.0185
18	1	rho	0.9285	0.0003	0.8953	0.9616
18	1	tau	748.9538	7.6502	743.5327	754.3748
18	2	sigma1	39.4612	13877.1328	-191.4250	270.3474
18	2	sigma2	39.4213	14067.8569	-193.0461	271.8887
18	2	rho	0.4997	0.0691	-0.0156	1.0150
18	2	tau	1557.1009	13.8762	1549.7998	1564.4019
18	3	sigma1	49.7309	55.8783	35.0799	64.3820
18	3	sigma2	49.6656	56.0487	34.9922	64.3390
18	3	rho	0.8472	0.0001	0.8326	0.8617
18	3	tau	1605.4034	0.3002	1604.3295	1606.4774

We also plotted σ_1/σ_2 and τ for all three regions over the 18 images. It seems that there is some pattern for the normed mean intensity



The variance estimates of ρ have been shown in the previous table, and the variance estimates of $\hat{\sigma}_1 - \hat{\sigma}_2$ are:

image	region	sig1__sig2	var.sig1__sig2	lower	upper
1	1	2.3434	4477.1884	-128.8014	133.4881
1	2	-0.0273	2899.1689	-105.5595	105.5048
1	3	0.7576	320.2963	-34.3195	35.8348
2	1	-8.5546	687.0855	-59.9298	42.8206
2	2	0.0000	39320.5081	-388.6491	388.6491
2	3	-0.1110	16.7472	-8.1318	7.9098
3	1	-0.1790	9307.8535	-189.2708	188.9129
3	2	0.0877	8889.5934	-184.7068	184.8822
3	3	-1.1187	21.5738	-10.2223	7.9848
4	1	-0.6539	7032.4247	-165.0156	163.7078
4	2	-0.8310	5686.8341	-148.6340	146.9720
4	3	0.0406	18.4627	-8.3810	8.4622
5	1	-54.7076	19486.0065	-328.3034	218.8883
5	2	0.0287	22516.3184	-294.0725	294.1299
5	3	-1.3338	12.7096	-8.3212	5.6535
6	1	-0.8894	2464.2276	-98.1840	96.4051
6	2	0.0000	11117.8849	-206.6613	206.6613
6	3	0.3710	36.0126	-11.3908	12.1329
7	1	-0.1803	63490.3915	-494.0383	493.6777
7	2	-0.1359	39168.9072	-388.0350	387.7633
7	3	-0.1852	295.6608	-33.8864	33.5160
8	1	0.1462	5153.8744	-140.5605	140.8530
8	2	0.0880	18377.8779	-265.6146	265.7906
8	3	0.0375	114.0077	-20.8899	20.9649
9	1	0.0000	22045.0239	-291.0070	291.0070
9	2	0.0225	18947.2013	-269.7642	269.8093
9	3	0.0000	730.9137	-52.9884	52.9884
10	1	7.8601	120.5287	-13.6575	29.3777
10	2	-0.4066	19236.0213	-272.2418	271.4286
10	3	0.9432	99.2021	-18.5781	20.4645
11	1	-0.4806	326180.2194	-1119.8589	1118.8978
11	2	0.8364	12864.2323	-221.4639	223.1367
11	3	0.1829	598.8847	-47.7815	48.1474
12	1	-3.0155	6536.0091	-161.4700	155.4389
12	2	0.3312	37076.1581	-377.0633	377.7256
12	3	-1.6220	21.6729	-10.7465	7.5024
13	1	-0.2240	97632.2470	-612.6375	612.1895
13	2	2.7834	25019.9071	-307.2374	312.8043
13	3	0.3114	459.8825	-41.7198	42.3426
14	1	11.5011	778.4870	-43.1846	66.1868
14	2	0.1310	63339.2582	-493.1388	493.4009
14	3	-0.8619	412.1439	-40.6517	38.9280
15	1	0.2938	7985.1767	-174.8482	175.4358
15	2	-1.4949	22405.9021	-294.8741	291.8843
15	3	0.0000	1215.1089	-68.3212	68.3212
16	1	18.1970	98.2705	-1.2324	37.6265
16	2	0.4065	8125.4689	-176.2674	177.0803
16	3	-1.5924	19.6868	-10.2887	7.1040
17	1	28.4051	22560.8201	-265.9866	322.7968
17	2	1.7340	18652.2422	-265.9446	269.4126
17	3	-0.6325	185.8087	-27.3491	26.0841
18	1	4.6652	791.0822	-50.4611	59.7915

image	region	sig1_sig2	var.sig1_sig2	lower	upper
18	2	0.0399	55854.5255	-463.1694	463.2492
18	3	0.0653	223.7318	-29.2512	29.3818

It seems that the corresponding confidence intervals for ρ do not include 0, but those for $\hat{\sigma}_1 - \hat{\sigma}_2$ include 0.