ST202 Lent Term – Handout 2

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1 Workflow for constructing an exact confidence interval

To construct an exact (i.e. not asymptotic) confidence interval of level $1-\alpha$, we usually need to start with a pivotal quantity. Recall that a pivotal is a function of the data and the parameter of interest and its distribution does not depend on the parameter. Let $\theta \in \mathbb{R}$ be our parameter of interest and let $\alpha > 0$. The procedure to construct an $(1-\alpha)$-level confidence interval is as follows:

1. Construct a pivotal quantity $g(X_1, \ldots, X_n, \theta)$, whose distribution does not depend on θ and has CDF F_G;
2. Determine c_1 and c_2 such that $\mathbb{P}\big(c_1 \leq g(X_1, \ldots, X_n, \theta) \leq c_2\big) = 1-\alpha$. Note that c_1 and c_2 thus satisfy $F_G(c_1) + 1 - F_G(c_2) = \alpha$. The commonly used equal tail case is simply $F_G(c_1) = 1 - F_G(c_2) = \alpha/2$. We thus have $c_1 = F_G^{-1}(\alpha/2)$ and $c_2 = F_G^{-1}(1-\alpha/2)$ as the equal-tail choice. There are infinitely many choices for c_1 and c_2, though;
3. Rewrite $c_1 \leq g(X_1, \ldots, X_n, \theta) \leq c_2$ as conditions for θ by solving the two inequalities. We then have $\mathbb{P}\big(L(X_1, \ldots, X_n) \leq \theta \leq R(X_1, \ldots, X_n)\big) = 1-\alpha$;
4. We obtain an $(1-\alpha)$-level confidence interval $\big[L(X_1, \ldots, X_n), R(X_1, \ldots, X_n)\big]$ for θ.

2 Worked examples for a normal population

We now provide four worked examples. Let $X_1, \ldots, X_n$ be a random sample from $N(\mu, \sigma^2)$. In each example, we give an $(1-\alpha)$-level confidence interval for our parameter of interest.

2.1 Parameter of interest μ, with known σ

1. Observe that

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \sim N(0, 1).$$

Note that this is an exact distributional result, using the fact that the random sample is from a normal population; **NOT** an asymptotic result using the Central Limit Theorem. We thus have found a pivotal quantity.
2. We determine c_1 and c_2 such that

$$\mathbb{P}\left(c_1 \leq \frac{\sqrt{n}(\bar{X} - \mu)}{\sigma} \leq c_2\right) = 1 - \alpha.$$

We consider the equal tail case and can choose $c_1 = z_{\alpha/2} := \Phi^{-1}(\alpha/2)$ and $c_2 = z_{1-\alpha/2} := \Phi^{-1}(1-\alpha/2)$, where $\Phi(\cdot)$ is the CDF of $N(0,1)$. Since the density of $N(0,1)$ is symmetric about zero, we have $c_1 = z_{\alpha/2} = -z_{1-\alpha/2} = c_2$.

3. We can rewrite as

$$\mathbb{P}\left(\bar{X} - \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2} \leq \mu \leq \bar{X} + \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}\right) = 1 - \alpha.$$

4. An $(1-\alpha)$-level confidence interval for μ is

$$\left[\bar{X} - \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}, \bar{X} + \frac{\sigma}{\sqrt{n}} z_{1-\alpha/2}\right].$$

2.2 Parameter of interest μ, with unknown σ

1. First note that $\frac{\sqrt{n}(\bar{X}-\mu)}{\sigma}$ is no longer a pivotal, as it has dependence on an unknown quantity σ. A natural way to resolve this issue is to replace σ in the expression by a sample estimate. Recall that $S^2 = \frac{1}{n-1}\sum_{i=1}^{n}(X_i - \bar{X})^2$ is the sample variance and an estimator for the population variance σ^2. We thus consider the quantity:

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sqrt{S^2}}.$$

We further recall that for the normal population, we have (1) $\bar{X} \sim N(\mu, \sigma^2)$, (2) $(n-1)S^2 \sim \sigma^2 \chi^2_{n-1}$, and (3) $\bar{X}$ and S^2 are independent. Then

$$\frac{\sqrt{n}(\bar{X} - \mu)}{\sqrt{S^2}} = \frac{\sqrt{n}(\bar{X} - \mu)/\sigma}{\sqrt{S^2/\sigma^2}} \sim t_{n-1},$$

since the numerator of the middle quantity follows $N(0,1)$ and the denominator follows $\sqrt{\chi^2_{n-1}/(n-1)}$ and the numerator and the denominator are independent. Therefore $\frac{\sqrt{n}(\bar{X}-\mu)}{\sqrt{S^2}}$ is a pivotal quantity.

2. The density of a t-distribution is bell-shaped and symmetric about zero. We thus have

$$\mathbb{P}\left(-t_{n-1,1-\alpha/2} \leq \frac{\sqrt{n}(\bar{X} - \mu)}{S^2} \leq t_{n-1,1-\alpha/2}\right) = 1 - \alpha.$$

3. Rewrite as

$$\mathbb{P}\left(\bar{X} - \frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2} \leq \mu \leq \bar{X} + \frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}\right) = 1 - \alpha.$$

4. An $(1-\alpha)$-level confidence interval for μ is

$$\left[\bar{X} - \frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}, \bar{X} + \frac{S}{\sqrt{n}} t_{n-1,1-\alpha/2}\right].$$

Remark. This is also a valid $(1-\alpha)$-level confidence interval for μ when σ is **known**, i.e. this also works for Section 2.1. However, the confidence interval presented here is expected to be 'longer' than the one in Section 2.1, as we have used more information (known σ) to construct the pivotal there.

2.3 Parameter of interest σ^2, with known μ

We now turn our attention to the variance parameter σ^2.
1. We again start by constructing a pivotal quantity. Note that for each $i \in \{1, \ldots, n\}$, we have $(X_i - \mu)/\sigma \sim N(0, 1)$. Since the sample is i.i.d., we have

$$\sum_{i=1}^{n} \frac{(X_i - \mu)^2}{\sigma^2} \sim \chi^2_n.$$

This is a pivotal quantity for σ^2, when μ is known.
2. Recall that $\chi^2_n = \text{Gamma}(n/2, 1/2)$ has support $(0, \infty)$. We have

$$\mathbb{P}\left(\chi^2_{n,\alpha/2} \le \frac{(X_i - \mu)^2}{\sigma^2} \le \chi^2_{n,1-\alpha/2}\right) = 1 - \alpha,$$

where $\chi^2_{n,v}$ is the (lower) v-quantile of the χ^2_n distribution.
3. Rewrite as

$$\mathbb{P}\left(\frac{(X_i - \mu)^2}{\chi^2_{n,1-\alpha/2}} \le \sigma^2 \le \frac{(X_i - \mu)^2}{\chi^2_{n,\alpha/2}}\right) = 1 - \alpha.$$

4. An $(1 - \alpha)$-level confidence interval for σ^2 is

$$\left[\frac{(X_i - \mu)^2}{\chi^2_{n,1-\alpha/2}}, \frac{(X_i - \mu)^2}{\chi^2_{n,\alpha/2}}\right].$$

2.4 Parameter of interest σ^2, with unknown μ

Since μ is unknown, we cannot use $\sum_{i=1}^{n} \frac{(X_i-\mu)^2}{\sigma^2}$ as a pivotal. Again, just like in Section 2.2, we replace μ in the expression by its sample estimate $\bar{X}$:

$$\sum_{i=1}^{n} \frac{(X_i - \bar{X})^2}{\sigma^2} = \frac{n-1}{\sigma^2} S^2 \sim \chi^2_{n-1},$$

using the fact that $(n - 1)S^2 \sim \sigma^2 \chi^2_{n-1}$. We have thus found a pivotal quantity. Using very similar arguments as before, we can obtain an $(1 - \alpha)$-level confidence interval for σ^2:

$$\left[\frac{(X_i - \bar{X})^2}{\chi^2_{n-1,1-\alpha/2}}, \frac{(X_i - \bar{X})^2}{\chi^2_{n-1,\alpha/2}}\right].$$

Remark. Again, this is also a valid $(1 - \alpha)$-level confidence interval for σ^2 when μ is **known**, i.e. this also works for Section 2.3. However, the confidence interval presented here is expected to be 'longer' than the one in Section 2.3, as we have used more information (known μ) to construct the pivotal there.