ST202 Lent Term – Handout 4
Extra exercise on hypothesis testing

Yudong Chen
y.chen276@lse.ac.uk

March, 2023

1. Let $Y_1, \ldots, Y_n$ be i.i.d exponential random variables with rate parameter $\theta > 0$.
(a) Suppose we want to test $H_0 : \theta = \theta_0$ vs. $H_1 : \theta = \theta_1$, where $\theta_0 < \theta_1$. Find the most powerful test of size α.
(b) Show that this is the uniformly most powerful test (UMPT) for $H_0 : \theta = \theta_0$ vs. $H_1 : \theta > \theta_0$.
(c) Is this also the UMPT for $H_0 : \theta \leq \theta_0$ vs. $H_1 : \theta > \theta_0$?
(Hint: you may find the following result helpful: if $X_1, \ldots, X_n$ are i.i.d. Exp(θ) random variable, then $2\theta \sum_{i=1}^n X_i \sim \chi^2_{2n}$.)

2. This question is based on PS6Q5. Suppose that $X_1, \ldots, X_n$ and $Y_1, \ldots, Y_n$ are two independent random samples from two exponential distributions with rate θ_1 and θ_2, respectively. We want to test $H_0 : \theta_1 = \theta_2$ vs. $H_1 : \theta_1 \neq \theta_2$. Recall that the likelihood ratio statistic is

$$r(\mathbf{X}, \mathbf{Y}) = \left[\frac{(\bar{X} + \bar{Y})^2}{4\bar{X}\bar{Y}} \right]^n.$$

We have established in PS6Q5 that $2 \log r(\mathbf{X}, \mathbf{Y})$ is approximately χ^2_1 under H_0, when n is large (i.e. χ^2_1 is the asymptotic distribution) and we can use this to construct a test with asymptotic size α. In this question, though, we would like to derive an exact (non-asymptotic) test of size α based on the above likelihood ratio statistic.
(a) Show that the likelihood ratio $r(\mathbf{X}, \mathbf{Y})$ is a monotone function of $\left|T(\mathbf{X}, \mathbf{Y}) - 1/2\right|$, where

$$T(\mathbf{X}, \mathbf{Y}) = \frac{\bar{X}}{\bar{X} + \bar{Y}}.$$

Is it monotonically increasing or decreasing?
(b) What is the distribution of T under H_0? (Hint: if $U \sim \text{Gamma}(\alpha, \lambda)$ and $V \sim \text{Gamma}(\beta, \lambda)$ are independent, then $U/(U + V) \sim \text{Beta}(\alpha, \beta)$.)
(c) Construct a test of exact size α using $|T - 1/2|$.