Handout 3 Solutions

1. (correction: density $f(y;\lambda,\theta)=\lambda e^{-\lambda(y-\theta)}$ for $y \geq \theta$)

(a) $$f(\underline{y};\lambda,\theta)=\prod_{i=1}^{n}\left(\lambda e^{-\lambda(y_i-\theta)}\mathbb{1}\{y_i\geq\theta\}\right)$$
$$=\lambda^n e^{n\lambda\theta-\lambda\sum_{i=1}^{n}y_i}\prod_{i=1}^{n}\mathbb{1}\{y_i\geq\theta\}$$
$$=\lambda^n e^{n\lambda\theta-\lambda\sum_{i=1}^{n}y_i}\mathbb{1}\{y_1\geq\theta,y_2\geq\theta,\dots,y_n\geq\theta\}$$
$$=\underbrace{\lambda^n e^{n\lambda\theta-\lambda\sum_{i=1}^{n}y_i}\mathbb{1}\{\min_{1\leq i\leq n}y_i\geq\theta\}}_{b(\lambda,\theta,h(\underline{y}))}\qquad c(\underline{y})=1$$

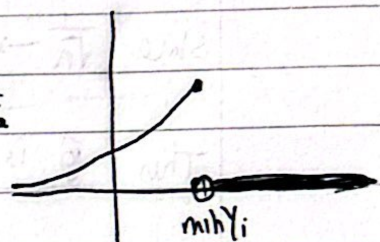
by factorisation criterion, a sufficient statistic is $\boxed{\begin{pmatrix}\sum_{i=1}^{n}Y_i\\ \min_{1\leq i\leq n}Y_i\end{pmatrix}}$.

(b) $$L(\lambda,\theta;\underline{Y})=\lambda^n e^{n\lambda\theta-\lambda\sum_{i=1}^{n}Y_i}\mathbb{1}\{\min_{1\leq i\leq n}Y_i\geq\theta\}$$

fix λ, we want to maximise over θ

$$=\lambda^n e^{-\lambda\sum_{i=1}^{n}Y_i}\underbrace{e^{n\lambda\theta}\mathbb{1}\{\theta\leq\min_{1\leq i\leq n}Y_i\}}$$

plot



The maximum is taken at $\boxed{\hat{\theta}=\min_{1\leq i\leq n}Y_i}$

$$L(\lambda,\hat{\theta};\underline{Y})=\lambda^n e^{-\lambda(\sum_{i=1}^{n}Y_i-n\cdot\min_{1\leq i\leq n}Y_i)}$$

$$\ell(\lambda,\hat{\theta};\underline{Y})=n\log\lambda-\left(\sum_{i=1}^{n}Y_i-n\cdot\min_{1\leq i\leq n}Y_i\right)\lambda$$

$$\frac{\partial\ell}{\partial\lambda}=\frac{n}{\lambda}-\left(\sum_{i=1}^{n}Y_i-n\cdot\min_{1\leq i\leq n}Y_i\right)=0 \rightsquigarrow \hat{\lambda}=\frac{n}{\sum_{i=1}^{n}Y_i-n\cdot\min_{1\leq i\leq n}Y_i}=\frac{1}{\bar{Y}-\min_{1\leq i\leq n}Y_i}$$

$$\frac{\partial^2\ell}{\partial\lambda^2}=-\frac{n}{\lambda^2}<0$$

Thus $\boxed{\hat{\lambda}=\frac{1}{\bar{Y}-\min_{1\leq i\leq n}Y_i}}$ is indeed the maximiser.

$$L(\hat{\lambda},\hat{\theta})\geq L(\lambda,\hat{\theta})\geq L(\lambda,\theta)\quad\text{for any }(\lambda,\theta).$$

Thus $\begin{pmatrix}\hat{\lambda}\\ \hat{\theta}\end{pmatrix}=\begin{pmatrix}\frac{1}{\bar{Y}-\min_{1\leq i\leq n}Y_i}\\ \min_{1\leq i\leq n}Y_i\end{pmatrix}$ is an MLE.

(c) For any $\varepsilon > 0$ small

$$P(\sqrt{n}|\hat{\theta}-\theta| > \varepsilon) = P(\sqrt{n}|\min_{1\le i\le n} Y_i - \theta| > \varepsilon)$$

$$= P(\min_{1\le i\le n} Y_i - \theta > \frac{\varepsilon}{\sqrt{n}}) = P(\min_{1\le i\le n} Y_i > \theta + \frac{\varepsilon}{\sqrt{n}})$$

since $Y_i \ge \theta$

$$= P(Y_1 > \theta + \frac{\varepsilon}{\sqrt{n}}, Y_2 > \theta + \frac{\varepsilon}{\sqrt{n}}, \dots, Y_n > \theta + \frac{\varepsilon}{\sqrt{n}})$$

$$= \left[P(Y_1 > \theta + \frac{\varepsilon}{\sqrt{n}})\right]^n$$

$$= \left[\int_{\theta+\frac{\varepsilon}{\sqrt{n}}}^{\infty} \lambda e^{-\lambda(y-\theta)} dy\right]^n = \left[e^{-\frac{\lambda\varepsilon}{\sqrt{n}}}\right]^n = e^{-\underbrace{\sqrt{n}\lambda\varepsilon}_{\to\infty}} \to 0$$

as $n \to \infty$

Thus $\sqrt{n}(\hat{\theta}-\theta) \xrightarrow{P} 0$

since $\frac{1}{\sqrt{n}} \to 0$, by Slutsky's Lemma: $\underbrace{\frac{1}{\sqrt{n}} \cdot \sqrt{n}(\hat{\theta}-\theta)}_{=\hat{\theta}-\theta} \xrightarrow{P} 0$

Thus $\hat{\theta}$ is consistent

(d) $\hat{\lambda} = \dfrac{1}{\bar{Y} - \min Y_i}$. By weak Law of Large numbers: $\bar{Y} \xrightarrow{P}$ population mean $= \theta + \frac{1}{\lambda}$

$\min_{1\le i\le n} Y_i \xrightarrow{P} \theta$ (by (c))

Thus $\bar{Y} - \min Y_i \xrightarrow{P} \theta + \frac{1}{\lambda} - \theta = \frac{1}{\lambda}$

by Continuous mapping theorem $\hat{\lambda} \xrightarrow{P} \frac{1}{1/\lambda} = \lambda$, which proves consistency of $\hat{\lambda}$.

* (e) $$\sqrt{n}(\hat{\lambda}-\lambda) = \sqrt{n}\left(\frac{1-\lambda(\bar{Y}-\min Y_i)}{\bar{Y}-\min Y_i}\right) = \frac{-\lambda[\sqrt{n}(\bar{Y}-\theta-\frac{1}{\lambda}) + \sqrt{n}(\theta - \min Y_i)]}{\bar{Y}-\min Y_i}$$

by the Central Limit Theorem, we have: $\sqrt{n}[\bar{Y} - (\theta+\frac{1}{\lambda})] \xrightarrow{d} N(0, var(Y_1)) = N(0, \frac{1}{\lambda^2})$

by (c), we have $\sqrt{n}(\min Y_i - \theta) \xrightarrow{P} 0$

In addition, $\bar{Y} \xrightarrow{P} \theta + \frac{1}{\lambda}$ $\min Y_i \xrightarrow{P} \theta$ by (d)

Thus by Slutsky's Lemma, we have

$$\sqrt{n}(\hat{\lambda}-\lambda) \xrightarrow{d} \frac{-\lambda[N(0,\frac{1}{\lambda^2})+0]}{\theta+\frac{1}{\lambda}-\theta} = \frac{-\lambda N(0,\frac{1}{\lambda^2})}{\frac{1}{\lambda}} = N(0,\lambda^2)$$

$\sqrt{n}(\hat{\lambda}-\lambda) \xrightarrow{d} N(0,\lambda^2)$ .. $\hat{\lambda}$ satisfies the asymptotic normality

2. (a) $L(\lambda; \underline{Y}) = \lambda^n e^{-\lambda \sum_{i=1}^n Y_i}$

$\ell(\lambda; \underline{Y}) = n \log \lambda - \lambda \sum_{i=1}^n Y_i$

$\frac{\partial \ell}{\partial \lambda} = s(\lambda; \underline{Y}) = \frac{n}{\lambda} - \sum_{i=1}^n Y_i = 0 \rightsquigarrow \hat{\lambda} = \frac{n}{\sum_{i=1}^n Y_i} = \frac{1}{\bar{Y}}$

$\frac{\partial^2 \ell}{\partial \lambda^2} = \frac{\partial s}{\partial \lambda} = -\frac{n}{\lambda^2} < 0$

$\boxed{\hat{\lambda} = \frac{1}{\bar{Y}} \text{ is indeed MLE}}$

By Jensen's inequality $\mathbb{E}\left[\frac{1}{\bar{Y}}\right] > \frac{1}{\mathbb{E}\bar{Y}} = \frac{1}{1/\lambda} = \lambda$. The inequality is strict since $x \mapsto \frac{1}{x}$ is strictly convex, and $\bar{Y}$ is not a constant.

Thus $\hat{\lambda}$ is biased. $\left[\mathbb{E}\left[\frac{1}{\bar{Y}}\right] > \lambda\right]$

(b) By the weak law of large number, we have $\bar{Y} \xrightarrow{p} \mathbb{E}Y_1 = \frac{1}{\lambda}$

Then by continuous mapping theorem, we have $\hat{\lambda} = \frac{1}{\bar{Y}} \xrightarrow{p} \frac{1}{1/\lambda} = \lambda$, i.e. consistent

(c) $I_Y(\lambda) = -\mathbb{E}\left[\frac{\partial s(\lambda; \underline{Y})}{\partial \lambda}\right] = \boxed{\frac{n}{\lambda^2}}$

(d) Asymptotic normality of MLE: $\sqrt{n}(\hat{\lambda} - \lambda) \xrightarrow{d} N(0, I_{Y_1}^{-1}(\lambda)) = N(0, \lambda^2)$

use observed F.I. instead: $\sqrt{n}(\hat{\lambda} - \lambda) \approx N(0, \hat{\lambda}^2)$

$\sqrt{n}\left(\frac{1}{\bar{Y}} - \lambda\right) \approx N\left(0, \frac{1}{\bar{Y}^2}\right) \rightsquigarrow \frac{1}{\bar{Y}} - \lambda \approx N\left(0, \frac{1}{n\bar{Y}^2}\right)$

An approximate $(1-\alpha)$-level CI for λ:

$$\left[\frac{1}{\bar{Y}} - \frac{Z_{1-\alpha/2}}{\sqrt{n}\,\bar{Y}}, \frac{1}{\bar{Y}} + \frac{Z_{1-\alpha/2}}{\sqrt{n}\,\bar{Y}}\right].$$

3. $L(\mu, \sigma^2; \underline{Y}) = (2\pi\sigma^2)^{-n/2} \exp\left(-\frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i-\mu)^2\right)$

$\ell(\mu, \sigma^2; \underline{Y}) = -\frac{n}{2}\log\sigma^2 - \frac{1}{2\sigma^2}\sum_{i=1}^{n}(Y_i-\mu)^2 + \text{constant}$

(a) under H_0: $\mu = \mu_0$

$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(Y_i-\mu_0)^2 = 0 \rightsquigarrow \hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\mu_0)^2$

$\left.\frac{\partial^2 \ell}{\partial(\sigma^2)^2} = \frac{n}{2\sigma^4} - \frac{1}{\sigma^6}\sum_{i=1}^{n}(Y_i-\mu_0)^2\right|_{\sigma^2=\hat{\sigma}_0^2} < 0$

$$\boxed{\begin{aligned}&\text{MLE under } H_0: \quad \hat{\mu}_0 = \mu_0 \ (\text{given})\\ &\hat{\sigma}_0^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\mu_0)^2\end{aligned}}$$

under H_1: $\mu \neq \mu_0$

$\frac{\partial \ell}{\partial \mu} = \frac{1}{\sigma^2}\sum_{i=1}^{n}(Y_i-\mu) = 0 \rightsquigarrow \hat{\mu}_1 = \bar{Y}$

$\frac{\partial^2 \ell}{\partial \mu^2} < 0$

sample variance

$\frac{\partial \ell}{\partial \sigma^2} = -\frac{n}{2\sigma^2} + \frac{1}{2\sigma^4}\sum_{i=1}^{n}(Y_i-\hat{\mu}_1)^2 = 0 \rightsquigarrow \hat{\sigma}_1^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2 = \frac{n-1}{n}S^2$

$\left.\frac{\partial^2 \ell}{\partial \sigma^2}\right|_{\sigma^2=\hat{\sigma}_1^2} < 0$

$$\boxed{\begin{aligned}&\text{MLE under } H_1: \quad \hat{\mu}_1 = \bar{Y}\\ &\hat{\sigma}_1^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar{Y})^2 = \frac{n-1}{n}S^2\end{aligned}}$$

(b) $r(\underline{Y}) = \frac{L(\hat{\mu}_1, \hat{\sigma}_1^2; \underline{Y})}{L(\hat{\mu}_0, \hat{\sigma}_0^2; \underline{Y})} = \frac{(\hat{\sigma}_1^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\hat{\sigma}_1^2}\sum_{i=1}^{n}(Y_i-\hat{\mu}_1)^2\right)}{(\hat{\sigma}_0^2)^{-\frac{n}{2}} \exp\left(-\frac{1}{2\hat{\sigma}_0^2}\sum_{i=1}^{n}(Y_i-\hat{\mu}_0)^2\right)}$

$= \frac{(\hat{\sigma}_1^2)^{-\frac{n}{2}} \cancel{\exp\left(-\frac{n}{2}\right)}}{(\hat{\sigma}_0^2)^{-\frac{n}{2}} \cancel{\exp\left(-\frac{n}{2}\right)}} = \left(\frac{\hat{\sigma}_0^2}{\hat{\sigma}_1^2}\right)^{\frac{n}{2}} = \left(\frac{\frac{1}{n}\sum_{i=1}^{n}(Y_i-\mu_0)^2}{\frac{n-1}{n}S^2}\right)^{n/2}$

Wilks' Theorem:

$2\log r(\underline{Y}) \xrightarrow{d} \chi^2_{|\Theta_1|-|\Theta_0|}$ as $n\to\infty$ under H_0

$|\Theta_1| = 2$ (both μ, σ^2 unconstrained) $\quad |\Theta_0| = 1$ (μ fixed, σ^2 unconstrained)

$\boxed{|\Theta_1| - |\Theta_0| = 1}$: $2\log r(\underline{Y}) \xrightarrow{d} \chi_1^2$ as $n\to\infty$ under H_0

We reject H_0 when $2\log r(\underline{Y}) > \chi^2_{1,1-\alpha}$.

*(C) $r(\underline{Y}) = \left(\frac{\hat\sigma_0^2}{\hat\sigma_1^2}\right)^{\frac{n}{2}}$ $\hat\sigma_0^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\mu_0)^2 = \frac{1}{n}\sum_{i=1}^{n}(Y_i-\bar Y+\bar Y-\mu_0)^2$

$$= \frac{1}{n}\left(\sum_{i=1}^{n}(Y_i-\bar Y)^2 + \cancel{2(\bar Y-\mu_0)\underbrace{\sum_{i=1}^{n}(Y_i-\bar Y)}_{0}} + n(\bar Y-\mu_0)^2\right)$$

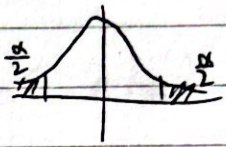
$$= \hat\sigma_1^2 + (\bar Y-\mu_0)^2$$

$$r(\underline{Y}) = \left(1+\frac{(\bar Y-\mu_0)^2}{\hat\sigma_1^2}\right)^{n/2} = \left[1+\frac{(\sqrt{n}(\bar Y-\mu_0))^2}{(n-1)S^2}\right]^n = \left[1+\frac{1}{n-1}\cdot\frac{(\sqrt{n}(\bar Y-\mu_0))^2}{S^2}\right]^n$$

we reject H_0 when $r(\underline{Y})$ is big, or equivalently, when $\left|\frac{\sqrt{n}(\bar Y-\mu_0)}{\sqrt{S^2}}\right|$ is big

under the null $\sqrt{n}(\bar Y-\mu_0) \sim N(0,1)$
$(n-1)S^2 \sim \chi^2_{n-1}$ and $\bar Y \perp\!\!\!\perp S^2$

Thus $\frac{\sqrt{n}(\bar Y-\mu_0)}{\sqrt{S^2}} \sim t_{n-1}$



Hence a size α test (LRT) is: we reject H_0 when $\frac{\sqrt{n}|\bar Y-\mu_0|}{\sqrt{S^2}} > t_{1-\frac{\alpha}{2}}$.

Handout 4 solutions

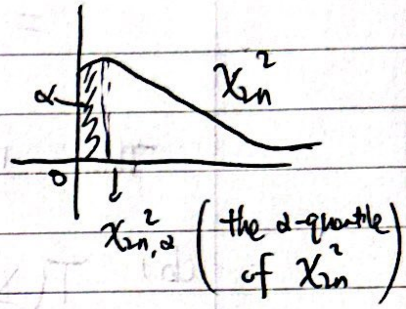
1. (a) $H_0: \theta=\theta_0$ vs $H_1: \theta=\theta_1$ $\quad (\theta_0<\theta_1)$

$$r(\underline{Y}) = \frac{L(\theta_1;\underline{Y})}{L(\theta_0;\underline{Y})} = \frac{\theta_1^n e^{-\theta_1 \sum_{i=1}^n Y_i}}{\theta_0^n e^{-\theta_0 \sum_{i=1}^n Y_i}} = \left(\frac{\theta_1}{\theta_0}\right)^n e^{(\theta_0-\theta_1)\sum_{i=1}^n Y_i}$$

This is a <u>decreasing</u> function of $\sum_{i=1}^n Y_i$

We reject H_0 when $\sum_{i=1}^n Y_i < k_\alpha'$, and this is the MPT by Neyman-Pearson lemma.

To determine k_α', we use the size α condition



$$\alpha = P_{H_0}(\text{reject } H_0) = P_{\theta_0}\left(\sum_{i=1}^n Y_i < k_\alpha'\right)$$

$$= P_{\theta_0}\Big(\underbrace{2\theta_0 \sum_{i=1}^n Y_i}_{\sim \chi^2_{2n} \text{ by the hint}} < 2\theta_0 k_\alpha'\Big)$$

$$\Rightarrow 2\theta_0 k_\alpha' = \chi^2_{2n,\alpha} \Rightarrow k_\alpha' = \frac{\chi^2_{2n,\alpha}}{2\theta_0}$$

<u>MPT of size α is that we reject H_0 when $\sum_{i=1}^n Y_i < \frac{\chi^2_{2n,\alpha}}{2\theta_0}$</u>.

(b) The likelihood ratio $r(\underline{Y})$ derived in (a) is always a decreasing function of $\sum_{i=1}^n Y_i$ when $\theta_1>\theta_0$. Thus the LRT remains the same as long as $\theta_1>\theta_0$. Therefore the test derived in (a) is indeed uniformly most powerful for $H_0: \theta=\theta_0$ vs $H_1: \theta>\theta_0$.

(c) $H_0: \theta\le\theta_0$ vs $H_1: \theta>\theta_0$

The only thing to check here is that for any $\theta\le\theta_0$, we have $P_\theta(\text{reject } H_0)\le\alpha$, i.e. (uniform) size control for the above test.

$$P_\theta(\text{reject } H_0) = P_\theta\left(\sum_{i=1}^n Y_i < \frac{\chi^2_{2n,\alpha}}{2\theta_0}\right) = P_\theta\Big(\underbrace{2\theta\sum_{i=1}^n Y_i}_{\sim \chi^2_{2n} \text{ (by the hint)}} < \frac{\theta}{\theta_0}\chi^2_{2n,\alpha}\Big)$$

$$\underset{\theta\le\theta_0}{\le} P(\chi^2_{2n} < \chi^2_{2n,\alpha}) = \alpha.$$

therefore, the test derived in (a) is [redacted] still UMPT for $H_0: \theta\le\theta_0$ vs $H_1: \theta>\theta_0$.

2. LR statistic is $r(\underline{X}, \underline{Y}) = \left(\dfrac{(\bar{X}+\bar{Y})^2}{4\bar{X}\bar{Y}} \right)^n$

$T(\underline{X}, \underline{Y}) = \dfrac{\bar{X}}{\bar{X}+\bar{Y}}$

(a) $r(\underline{X}, \underline{Y}) = \left(\frac{1}{4} \cdot \dfrac{\bar{X}+\bar{Y}}{\bar{X}} \cdot \dfrac{\bar{X}+\bar{Y}}{\bar{Y}} \right)^n = \left(\frac{1}{4} \dfrac{1}{T(\underline{X},\underline{Y})} \cdot \dfrac{1}{1-T(\underline{X},\underline{Y})} \right)^n$

$$= \left[\frac{1}{4\,T(\underline{X},\underline{Y})\,(1-T(\underline{X},\underline{Y}))} \right]^n = \left[\frac{1}{1-4(T(\underline{X},\underline{Y})-\frac{1}{2})^2} \right]^n$$

$$= \left(\frac{1}{1-4\,|T(\underline{X},\underline{Y})-\frac{1}{2}|^2} \right)^n$$

Thus, $r(\underline{X}, \underline{Y})$ is an increasing function of $|T(\underline{X}, \underline{Y}) - \frac{1}{2}|$.

(b) $T(\underline{X}, \underline{Y}) = \dfrac{\bar{X}}{\bar{X}+\bar{Y}}$ $\quad \bar{X} \sim \text{Gamma}(n, \theta_1)$, $\bar{Y} \sim \text{Gamma}(n, \theta_2)$ and independent

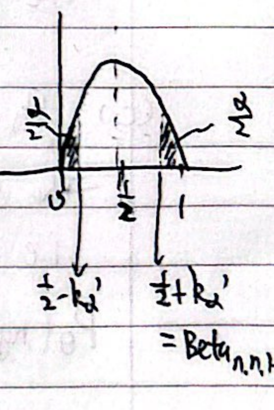
Under $H_0: \theta_1 = \theta_2$, by hint, $\boxed{\dfrac{\bar{X}}{\bar{X}+\bar{Y}} \sim \text{Beta}(n, n)}$

(c) We reject H_0 when $r(\underline{X}, \underline{Y})$ is big, equivalently when $|T - \frac{1}{2}|$ is big.

Hence we reject H_0 when $|T - \frac{1}{2}| > k_\alpha'$

We determine k_α' by the size requirement

Under H_0: $T \sim \text{Beta}(n, n) \longrightarrow$ density $\propto x^{n-1}(1-x)^{n-1}$ (symmetric about $\frac{1}{2}$)



$\alpha = P_{H_0}(|T - \frac{1}{2}| > k_\alpha')$

by the figure on the right, we need $\frac{1}{2} + k_\alpha' = \text{Beta}_{n,n,1-\frac{\alpha}{2}}$

$k_\alpha' = \text{Beta}_{n,n,1-\frac{\alpha}{2}} - \frac{1}{2}$

A test of exact size α:

reject H_0 when $\left| \dfrac{\bar{X}}{\bar{X}+\bar{Y}} - \frac{1}{2} \right| > \text{Beta}_{n,n,1-\frac{\alpha}{2}} - \frac{1}{2}$.