ST202/206 Extra Question - Class 2

Yudong Chen
y.chen276@lse.ac.uk

(October 13, 2022)

We consider an interesting extension to Question 2 of Problem set 2. In (a), we find that

$$\mathbb{P}(\text{first ball is black}) = 1/2;$$

in the (c) part, we have also calculated that

$$\mathbb{P}(\text{first two balls both black}) = 1/3.$$

If we continue calculating, it is not too difficult to verify that

$$\mathbb{P}(\text{first three balls all black}) = 1/4.$$

Have you spotted a pattern here?

Question. There are n urns, of which the kth contains $k-1$ red balls and $n-k$ black balls, for $k = 1, \ldots, n$. Let m be an integer not greater than $n-1$. You pick an urn at random and remove m balls without replacement from this urn. Find the probability that all m balls are black.

Solution: For the kth urn, it includes $k-1$ red balls and $n-k$ black balls. We label these $k-1$ red balls from 1 to $k-1$ and label black balls from $k+1$ to n. Note that we have intentionally left out the label k, the reason for which shall be clear in a moment. Now, pulling out m balls which are all black from the kth urn **is equivalent to** the randomly picked m numbers (without replacement) from $\{1, \ldots, k-1, k+1, \ldots, n\}$ being all bigger than k.

Recall that we need to first randomly choose one of the n urns. This means that we shall first randomly choose a number k from $\{1, \ldots, n\}$. Then, we perform the procedure above, i.e. randomly choose m numbers without replacement from $\{1, \ldots, k-1, k+1, \ldots, n\}$. We want to calculate the probability of all these m numbers being bigger than k.

Equivalently, this is asking that, if we randomly choose $m+1$ numbers (without replacement) from $\{1, \ldots, n\}$, what is the probability that the first chosen number is the smallest among these $m+1$ numbers. By symmetry, the answer is $1/(m+1)$.