ST202/206 Extra Question - Class 3

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We study an interesting property of probability distributions: **memorylessness**. The meaning of the property is that the distribution of a 'waiting time' until a certain event is independent of how much time has already elapsed.

Definition 1 (discrete memorylessness)**.** *Suppose X is a discrete random variable with support $\{1, 2, \ldots\}$. Then the distribution of X is memoryless if for any s and t in $\{0, 1, 2, \ldots\}$, we have*

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

Definition 2 (continuous memorylessness)**.** *Suppose X is a continous random variable with support $[0, +\infty)$. Then the distribution of X is memoryless if for any $s, t \geq 0$, we have*

$$\mathbb{P}(X > s + t \mid X > s) = \mathbb{P}(X > t).$$

We recall the geometric distribution with success rate p and support $\{1, 2, \ldots\}$ from the lecture. This version of the geometric random variable counts the number of trials required to have the first success, and has PMF:

$$F_Y(y) = p(1-p)^{y-1} \qquad \text{for } y = 1, 2, \ldots$$

Question. (a) Show that geometric distributions with support $\{1, 2, \ldots\}$ are memoryless and they are indeed the **only** discrete memoryless distributions.
(b) What continuous distributions with support $[0, +\infty)$ are memoryless? Justify your answer.
(c) If you walk around Covent Garden on a busy afternoon, what is the distribution of the time until you next run into a friend likely to be?

Solution: (a) For $X \sim \text{Geom}(p)$, we know that for $s \in \mathbb{N} \cup \{0\}$

$$\mathbb{P}(X > s) = p^s.$$

See Question 1 of Problem set 3 for the derivation. Thus for $s, t \in \mathbb{N} \cup \{0\}$, we have

$$\mathbb{P}(X > s + t \mid X > s) = \frac{\mathbb{P}(X > s + t)}{\mathbb{P}(X > s)} = \frac{p^{s+t}}{p^s} = \frac{p}{t} = \mathbb{P}(X > t).$$

Hence geometric distributions are memoryless. Now we prove that geometric distributions are the only discrete memoryless distributions. If the distribution of X (which has support $\mathbb{N}$) is memoryless, then we have for any $s \in \mathbb{N}$

$$\mathbb{P}(X = s+1 \mid X > s) = 1 - \mathbb{P}(X > s+1 \mid X > s) = 1 - \mathbb{P}(X > 1) = \mathbb{P}(X = 1),$$

and thus

$$\mathbb{P}(X = s+1) = \mathbb{P}(X = 1)(1 - \mathbb{P}(X \leq s)). \tag{1}$$

Let $\tilde{p} := \mathbb{P}(X = 1)$. We now prove by induction that $\mathbb{P}(X = r) = \tilde{p}(1-\tilde{p})^{r-1}$ for any $r \in \mathbb{N}$. This is trivially true for $r = 1$, since this is how we defined $\tilde{p}$. Now, assume that this is true for all $x \leq r_0$. Then using (1), we have

$$\begin{aligned}\mathbb{P}(X = r_0+1) = \mathbb{P}(X = 1)(1 - \mathbb{P}(X \leq r_0)) &= \tilde{p}\Big(1 - \sum_{x=1}^{r_0} \tilde{p}(1-\tilde{p})^{x-1}\Big)\\ &= \tilde{p}\big(1 - \big(1 - (1-\tilde{p})^{r_0}\big)\big) = \tilde{p}(1-\tilde{p})^{r_0}.\end{aligned}$$

The induction proof is complete and we have $\mathbb{P}(X = r) = \tilde{p}(1-\tilde{p})^{r-1}$ for any $r \in \mathbb{N}$, which is the PMF for $\mathrm{Geom}(\tilde{p})$.

(b) The answer is all exponential distributions. Again, we first show that exponential distributions are indeed memoryless. Let $X \sim \mathrm{Exp}(\lambda)$, for some $\lambda > 0$. Recall that the CDF is $F_X(x) = \int_0^x \lambda e^{-\lambda x}\, dx = 1 - e^{-\lambda x}$. Thus for any $s, t \geq 0$, we have

$$\mathbb{P}(X > s+t \mid X > s) = \frac{\mathbb{P}(X > s+t)}{\mathbb{P}(X > s)} = \frac{e^{-\lambda(s+t)}}{e^{-\lambda s}} = e^{-\lambda t} = \mathbb{P}(X > t).$$

Hence, exponential distributions are memoryless. Now we prove that exponential distributions are the only continuous memoryless distributions. Let X be memoryless and we denote its survival function $S(x) = \mathbb{P}(X > x)$. By the definition of the memoryless property, we have, for any $s, t \geq 0$

$$S(s+t) = S(s)S(t).$$

Thus for any positive integer $n \in \mathbb{N}$, we have

$$S(n) = S(1)S(n-1) = S(1)^2 S(n-2) = \ldots = S(1)^n.$$

Note that since $S(n) \to 0$ as $n \to +\infty$, we must necessarily have $0 < S(1) < 1$. Now for any positive rational number $p/q \in \mathbb{Q}^+$, where p and q are positive integers, we have

$$S(1)^p = S(p) = S(p/q + p/q + \ldots + p/q) = S(p/q)^q,$$

and thus

$$S(p/q) = S(1)^{p/q}.$$

Since the CDF is a right-continuous function, the survival function is also right-continuous. Thus by right-continuity, we have $S(x) = S(1)^x = e^{-x\ln(1/S(1))}$ for all $x \in \mathbb{R}^+$. This is exactly the survival function of an $\mathrm{Exp}(\lambda)$ random variable, with $\lambda = \ln(1/S(1))$. Therefore, X must follow an exponential distribution.

(c) This will likely follow an exponential distribution due to its memoryless property.