ST202/206 Extra Question - Class 1

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We introduce the famous ‘stars and bars’ problem.

Question. Suppose we have n identical (indistinguishable) balls, and k different boxes.
(a) How many ways can we put these n balls into these k boxes, if each box must have at least 1 ball? Note that in this case we require $n \geq k$.
(b) How many ways can we put these n balls into these k boxes, if we allow empty boxes?

Before we present the solution, it is worth mentioning that applications of this problem include resource allocation. For example, if we have n fire trucks, and k fire emergencies to respond to. The corresponding question is how many ways can we assign these n fire engines. Of course, in real life, there will be extra constraints due to the severity of each fire emergency, e.g. some fire sites may require a minimum number of fire trucks.

Solution: Say we put x_i many balls into the ith box. The question is equivalent to counting the number of solutions to the following equations:

$$\begin{aligned} (a) &: x_1 + x_2 + \cdots + x_k = n \qquad \text{subject to } x_i \in \mathbb{N} \text{ for all } i; \\ (b) &: x_1 + x_2 + \cdots + x_k = n \qquad \text{subject to } x_i \in \mathbb{N} \cup \{0\} \text{ for all } i. \end{aligned}$$

Let us solve (a) first. Consider that we have n stars and we need to separate them into k groups, i.e. we need to insert $k-1$ bars (dividers) to separate these stars. Since we have n stars, there are $n-1$ gaps in between them. Note that, since every group should have at least one star (each x_i at least 1), we can not insert a bar before the first star or after the last star, nor can we insert multiple bars into one gap. Thus we simply have $\binom{n-1}{k-1}$ many ways to insert the bars and this is the answer to part (a). We illustrate the idea with two figures.

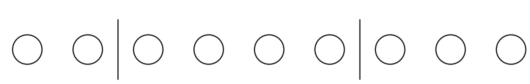

Figure 1: $n = 9, k = 3 : x_1 = 2, x_2 = 4, x_3 = 3$

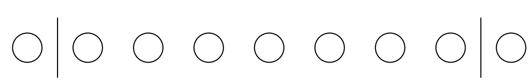

Figure 2: $n = 9, k = 3 : x_1 = 1, x_2 = 7, x_3 = 1$

For (b), there are two ways to tackle the question. The first approach is to add 1 to each x_i. We rewrite (b) as

$$(x_1 + 1) + (x_2 + 1) + \cdots + (x_k + 1) = n + k \qquad \text{subject to } x_i \in \mathbb{N} \cup \{0\} \text{ for all } i.$$

This is the same as

$$y_1 + y_2 + \cdots + y_k = n + k \qquad \text{subject to } y_i \in \mathbb{N} \text{ for all } i.$$

Now, this is in the exact same form as (a), with $n+k$ taking the place of n. Hence the number of solutions is $\binom{n+k-1}{k-1}$.

We now discuss an alternative way to solve part (b), without using this +1 trick. We go back to look at these stars and bars. Now instead of having n stars already in place before inserting $k-1$ bars, we consider a total of $n+k-1$ places to put either stars or bars. We then choose $k-1$ places to put in the bars. This achieves the goal of separating the n stars into k groups, but now also allowing 0 ball in a group. The total number of ways is thus again $\binom{n+k-1}{k-1}$. We illustrate the idea with three figures.

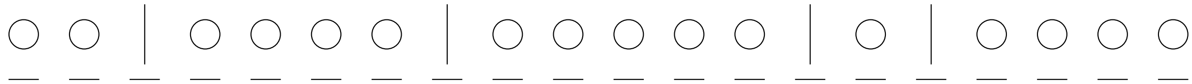

Figure 3: $n = 16, k = 5 : x_1 = 2, x_2 = 4, x_3 = 5, x_4 = 1, x_5 = 4$

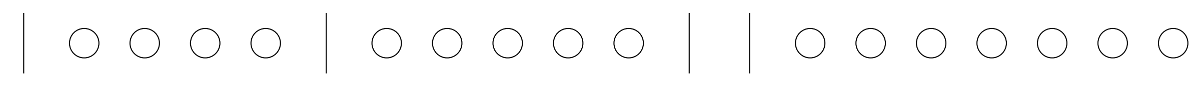

Figure 4: $n = 16, k = 5 : x_1 = 0, x_2 = 4, x_3 = 5, x_4 = 0, x_5 = 7$

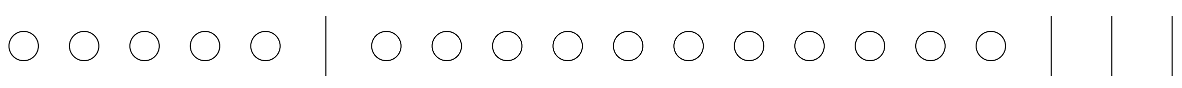

Figure 5: $n = 16, k = 5 : x_1 = 5, x_2 = 11, x_3 = 0, x_4 = 0, x_5 = 0$