Lab 1

Written Questions

1.

	$X=0$	$X=1$
$Y=0$	$\frac{1}{4}$	$\frac{1}{4}$
$Y=1$	$\frac{1}{6}$	$\frac{1}{3}$

(a)
$$P(X=1) = \frac{1}{4} + \frac{1}{3} \boxed{= \frac{7}{12}}$$

(b)
$$P(X=1 \mid Y=1) = \frac{\frac{1}{3}}{\frac{1}{6}+\frac{1}{3}} = \frac{2}{1+2} \boxed{= \frac{2}{3}}$$

(c)
$$\mathrm{Var}[X] = E[X^2] - E[X]^2$$
$$= \sum_x \sum_y x^2 \cdot f(x,y) - \left[\sum_x \sum_y x \cdot f(x,y)\right]^2$$
$$= \left[(1)^2\left(\tfrac{1}{4}\right) + (1)^2\left(\tfrac{1}{3}\right)\right] - \left[(1)\left(\tfrac{1}{4}\right) + (1)\left(\tfrac{1}{3}\right)\right]^2$$
$$= \left[\frac{7}{12}\right] - \left[\frac{7}{12}\right]^2$$
$$= \frac{84}{144} - \frac{49}{144} = \frac{45}{144} = \frac{15}{48} \boxed{= \frac{5}{16}}$$

(d)
$$\mathrm{Var}[X \mid Y=1] = E[X^2 \mid Y=1] - [E[X \mid Y=1]]^2$$
$$= \sum_x x^2 \cdot f(x,1) - \left[\sum_x x \cdot f(x,1)\right]^2$$
$$= \left[(1)^2\left(\tfrac{1}{3}\right)\right] - \left[(1)\cdot\left(\tfrac{1}{3}\right)\right]^2$$
$$= \frac{1}{3} - \frac{1}{9} \boxed{= \frac{2}{9}}$$

(e)

$$E[X^3 + X^2 + 3Y^7 \mid Y=1]$$

$$= E[X^3 \mid Y=1] + E[X^2 \mid Y=1] + 3 \cdot E[Y^7 \mid Y=1]$$

$$= \sum_x x^3 \cdot f(x,1) + \sum_x x^2 \cdot f(x,1) + 3 \cdot (1)^7$$

$$= (1)^3\left(\frac{1}{3}\right) + (1)^2\left(\frac{1}{3}\right) + 3 = \frac{2}{3} + 3 \boxed{= \frac{11}{3}}$$

2.

$$\vec{v}_1 = [1,1,1], \quad \vec{v}_2 = [1,0,0]$$

$$p_1 = [3,3,3], \quad p_2 = [1,2,3], \quad p_3 = [0,0,1]$$

$$p_1 \Rightarrow \frac{\langle p_1, v_1 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1 + \frac{\langle p_1, v_2 \rangle}{\langle v_2, v_2 \rangle} \cdot v_2$$

$$\Rightarrow \frac{9}{3} \cdot [1,1,1] + \frac{3}{1} \cdot [1,0,0] \boxed{= [6,3,3]}$$

$$p_2 \Rightarrow \frac{\langle p_2, v_1 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1 + \frac{\langle p_2, v_2 \rangle}{\langle v_2, v_2 \rangle} \cdot v_2$$

$$\Rightarrow \frac{6}{3} \cdot [1,2,3] + \frac{1}{1} \cdot [1,0,0] \boxed{= [3,4,6]}$$

$$p_3 \Rightarrow \frac{\langle p_3, v_1 \rangle}{\langle v_1, v_1 \rangle} \cdot v_1 + \frac{\langle p_3, v_2 \rangle}{\langle v_2, v_2 \rangle} \cdot v_2$$

$$\Rightarrow \frac{1}{3} \cdot [1,1,1] + 0 \boxed{= \left[\frac{1}{3}, \frac{1}{3}, \frac{1}{3}\right]}$$

3. $P(\text{Heads}) = \frac{2}{3}$

Since $n = 100 > 30$, we approximate using Central Limit Theorem.

$$\mu = (100)\left(\frac{2}{3}\right) = \frac{200}{3}$$

$$\sigma^2 = (100)\left(\frac{2}{3}\right)\left(\frac{1}{3}\right) = \frac{200}{9}$$

$$P(\text{Heads} \le 50) = P\left(\frac{\text{Heads} - \frac{200}{3}}{\sqrt{\frac{200}{9}}} \le \frac{50 - \frac{200}{3}}{\sqrt{\frac{200}{9}}}\right)$$

$$= P\left(\frac{\text{Heads} - \frac{200}{3}}{\sqrt{\frac{200}{9}}} \le \frac{-16.667}{4.71404}\right)$$

$$= P\left(\frac{\text{Heads} - \frac{200}{3}}{\sqrt{\frac{200}{9}}} \le -3.536\right)$$

$$= \Phi(-3.536) \boxed{\approx .00020}$$