Tolerance Allocation Based on a Rate-Sigma-Cost Model

Nomenclature

C_i	Unit cost of the *i*-th process	F_i	Constant of *i*-th process
A_i	Constant of the *i*-th process	r	Manufacturing rate
B_i	Constant of *i*-th process	G_i	Constant
E_i	Constant of *i*-th process	σ	Standard deviation
V_i	Total cost of component type *i*	C_T	Total cost of products
U	Average cost of a product	Q	Number of satisfactory components
N_i	Number of manufactured component type *i*	M	Number of satisfactory products
m	Number of different components in a product	X_i	Dimension of component *i*
Y	Dimension of assembly	n	Order of polynomial

1. Problem Description

To model the relationship between tolerance and manufacturing cost, we build two models: a cost-rate model and a σ-rate model. For the *i*-th component, the two models are given by Eq. (1) and Eq. (2).

r is the rate, which can be computed by the volume affected divided by operation time.

cost-rate model for the *i*th component:

$$C_i = A_i + \frac{B_i}{r_i} \tag{1}$$

σ-rate model for the *i*th component:

$$\sigma_i = E_i + F_i r^2 \tag{2}$$

To compare different processes, we use rate as a metric to evaluate the precision and cost relationship, as shown in Fig. 1.

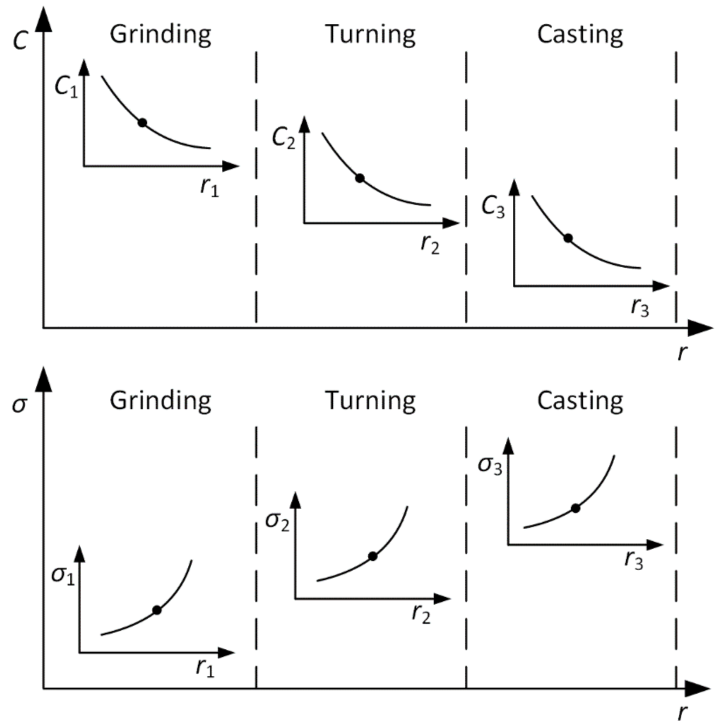

Fig. 1

Using Eq.(1), and Eq. (2), we can get the relationship between cost and σ, as shown in Eq. (3).

$$C_i = A_i + B_i \sqrt{\frac{Fi}{\sigma_i - E_i}} \tag{3}$$

Equation (3) is a reciprocal powered model, which is used in some literature.

We need to find constants to build the cost-rate model and the σ-rate model.

2. Minimize unit cost of products by optimizing manufacturing rate of components

Total cost of component type *i*:

$$V_i(r) = C_i(r) N_i \tag{4}$$

Total cost of products

$$C_T = \sum_{i=1}^{m} V_i = \sum_{i=1}^{m} C_i N_i \tag{5}$$

Average cost of a product

$$U = \frac{C_T}{M} = \frac{\sum_{i=1}^{m} C_i N_i}{M} \tag{6}$$

Number of satisfactory products

$$M = Q\beta(r_1, r_2, ..., r_m) \tag{7}$$

in which, β is the satisfaction rate of products.

$$U(r_1, r_2, ..., r_m) = \frac{\sum_{i=1}^{m} C_i(r_i) N_i}{Q\beta(r_1, r_2, ..., r_m)} \tag{8}$$

The product characteristic value, Y, is computed from the component characteristic values, X_i, by the design function f:

$$Y = f(X_1, X_2, ..., X_m) \tag{9}$$

Variable X_i, follows a normal distribution with a standard deviation σ_i. We use first order Taylor series to approximate variable Y. And the distribution of Y is approximated using a normal distribution with a standard deviation σ_y, as shown in Fig. Fig. 2.

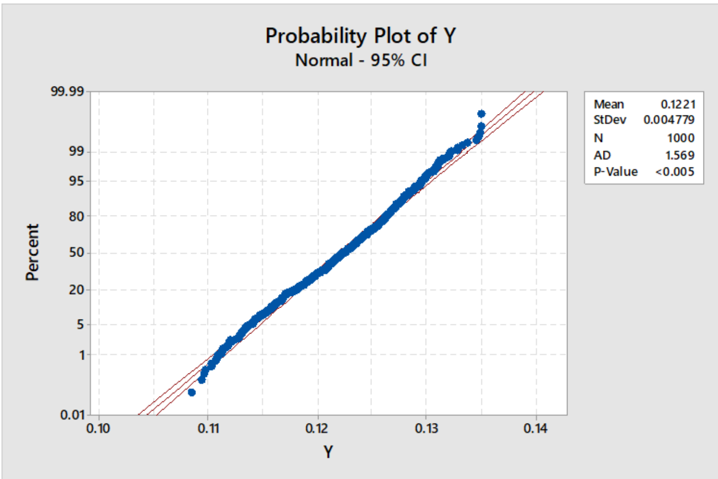

Fig. 2

First order Taylor series in several variables:

$$T(x_1, x_2, ..., x_m) = f(a_1, a_2, ..., a_m) + \sum_{j=1}^{m} \frac{\partial f(a_1, a_2, ..., a_m)}{\partial x_j}(x_j - a_j) \tag{10}$$

Variance of *Y*, σ^2_Y

$$\sigma_y{}^2 = \sum_{i=1}^{m} (\frac{\partial f}{\partial X_i}\Big|_{\mu_1, \mu_2, ..., \mu_m})^2 \sigma_i{}^2 = \sum_{i=1}^{m} D_i^2 \sigma_i{}^2 \tag{11}$$

in which, D_i is an intermediate variable, which can be computed by:

$$D_i = \frac{\partial f}{\partial X_i}\Big|_{\mu_1, \mu_2, ..., \mu_m} \tag{12}$$

Equations to estimate the satisfactory rate of products:

$$\beta_i(r_1, r_2, ..., r_m) = \int_{LS}^{US} \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{(t-u)^2}{2\sigma^2{}_y}} dt \tag{13}$$

We assume that the assembly process is under statistical process control, i.e., the mean of the assembly equals the nominal value of *Y*, thus *β* can be simplified:

$$\beta_i(r_1, r_2, ..., r_m) = 2\int_0^{US'} \frac{1}{\sqrt{2\pi}\sigma_y} e^{-\frac{t^2}{2\sigma^2}} dt = \frac{2}{\sqrt{\pi}} \int_0^{\frac{US'}{\sqrt{2}\sigma}} e^{-t^2} dt = erf(\frac{US'}{\sqrt{2}\sigma_y}) \tag{14}$$

in which, *US'* is the shifted upper specification (constant), and can be computed using the following equation:

$$US' = US - \mu_Y \tag{15}$$

The specification limits of products, LS and US are fixed (given by design).

The error function, erf(x) can be approximated using power series expansion (we can compute the order needed for a given accuracy):

$$erf(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2} dt = \frac{2}{\sqrt{\pi}}(x - \frac{x^3}{3} + \frac{x^5}{5 \cdot 2!} - \frac{x^7}{7 \cdot 3!} + ...) = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} \frac{(-1)^i x^{2i+1}}{(2i+1)i!} \tag{16}$$

3. Two Scenarios

If components are inspected, and unsatisfactory components are scraped, the satisfactory components would follow a truncated normal distribution. Though, dimensions of products roughly follow a normal distribution, its standard deviation is smaller than the value computed by Eq. (11).

Based on whether components are inspected, we analyze two scenarios.

3.1. Scenario One: do not scrap component

If components are not inspected, all components (both satisfactory components and unsatisfactory components) will be assembled. For each type of components, N_i is the given constant Q, thus, the unit cost, U, can be computed by:

$$U(r_1, r_2, ..., r_m) = \frac{1}{Q\beta(r_1, r_2, ..., r_m)} \sum_{i=1}^{m} C_i(r_i)Q = \frac{1}{erf\left(\frac{US'}{\sqrt{2}\sigma_y}\right)} \sum_{i=1}^{m} C_i(r_i) \tag{17}$$

Average cost of a product, U, is a function of manufacturing rates, r_1, r_2, …, r_m of components. We optimize these manufacturing rates to minimize U. This is an unconstrained multivariate optimization problem. The problem can be solved using gradient based optimization algorithms or using TensorFlow.

3.2. Scenario Two: scrap components

If unsatisfactory components are scrapped, the dimensions of products still roughly follow a normal distribution, as shown in Fig. 3. The standard deviation of Y, σ_y, estimated by Eq. (12) is larger than real value. We can compensate this error using the following equation

$$\sigma'_y(r_1, r_2, ..., r_m) = \lambda \sigma_y(r_1, r_2, ..., r_m) \tag{18}$$

in which, λ is the adjustment constant, the value of which can be estimated by simulation.

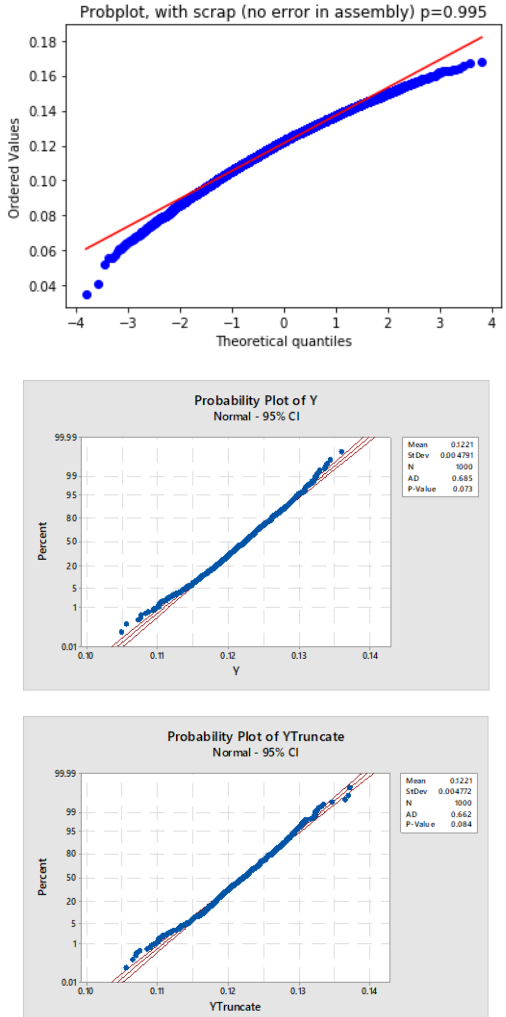

Fig. 3 probability plot

The number of manufactured component type *i* (including unsatisfactory components), N_i, can be estimated,

$$N_i = \frac{Q}{\alpha_i(r)} \tag{19}$$

in which, α_i is the satisfaction rate of component type *i*, it can be estimated by:

$$\alpha_i(r) = \int_{LS_i}^{US_i} \frac{1}{\sqrt{2\pi}\sigma_i} e^{\frac{-(t-u)^2}{2\sigma^2{}_i}} dt = \int_{LS_i'}^{US_i'} \frac{1}{\sqrt{2\pi}\sigma_i(r)} e^{\frac{-t^2}{2\sigma^2{}_i(r)}} dt \tag{20}$$

Lower specification limit:

$$LS_i = \mu - k_i \sigma_i(r) \tag{21}$$

Upper specification limit:

$$US_i = \mu + k_i \sigma_i(r) \tag{22}$$

Shifted lower specification limit:

$$LS_i' = LS_i - \mu = -k_i \sigma_i(r) \tag{23}$$

Shifted upper specification limit:

$$US_i' = US_i - \mu = k_i \sigma_i(r) \tag{24}$$

Even function integration,

$$\alpha_i(r) = \int_{LS_i'}^{US_i'} \frac{1}{\sqrt{2\pi}\sigma_i(r)} e^{\frac{-t^2}{2\sigma^2{}_i(r)}} dt = \frac{2}{\sqrt{2\pi}\sigma_i(r)} \int_0^{US_i'} e^{\frac{-t^2}{2\sigma^2{}_i(r)}} dt = \frac{2}{\sqrt{\pi}} \int_0^{\frac{k_i}{\sqrt{2}}} e^{-t^2} dt = erf\left(\frac{k_i}{\sqrt{2}}\right) \tag{25}$$

in which, erf(x) is the error function. This equation has been verified by an example.

The unit cost, U, is a function of r_1, r_2, …, r_m, and k_1, k_2, …, k_m.

$$U = \frac{1}{\beta(r_1, r_2, ..., r_m)} \sum_{i=1}^{m} \frac{C_i(r_i)}{\alpha_i} = \frac{1}{erf\left(\frac{US'}{\sqrt{2}\sigma_y'}\right)} \sum_{i=1}^{m} \left(\frac{C_i(r_i)}{erf\left(\frac{k_i}{\sqrt{2}}\right)}\right) \tag{26}$$

4. Case study

$$Y = f(X_1, X_2, X_3) = \arccos\left(\frac{X_1 + X_2}{X_3 - X_2}\right) \tag{27}$$

$$D_1 = \frac{\partial f}{\partial X_1}\Big|_{\mu_1, \mu_2, ..., \mu_m} = \left(\frac{-1}{\sqrt{1 - \left(\frac{X_1 + X_2}{X_3 - X_2}\right)^2}} \frac{1}{X_3 - X_2}\right)\Big|_{\mu_1, \mu_2, ..., \mu_m} \tag{28}$$

$$D_2 = \frac{\partial f}{\partial X_2}\Big|_{\mu_1, \mu_2, ..., \mu_m} = \left(\frac{-1}{\sqrt{1 - \left(\frac{X_1 + X_2}{X_3 - X_2}\right)^2}} \frac{X_1 + X_3}{(X_3 - X_2)^2}\right)\Big|_{\mu_1, \mu_2, ..., \mu_m} \tag{29}$$

$$D_3 = \frac{\partial f}{\partial X_3}\Big|_{\mu_1, \mu_2, ..., \mu_m} = \frac{1}{\sqrt{1 - \left(\frac{X_1 + X_2}{X_3 - X_2}\right)^2}} \frac{X_1 + X_2}{(X_3 - X_2)^2}\Big|_{\mu_1, \mu_2, ..., \mu_m} \tag{30}$$

$$\frac{\partial \sigma_y}{\partial r_i} = (\sum_{j=1}^{m} D_j{}^2 \sigma_j{}^2)^{-\frac{1}{2}} \bullet D_i{}^2 \sigma_i \frac{d\sigma_i}{dr_i} \tag{31}$$

$$\frac{\partial \sigma_i}{\partial r_i} = 2F_i r_i \tag{32}$$

$$\frac{dc_i}{dr_i} = -B_i r_i^{-2} \tag{33}$$

$$\frac{derf}{dx} = \frac{2}{\sqrt{\pi}}(1 - x^2 + \frac{x^4}{2} - \frac{x^6}{6}) = \frac{2}{\sqrt{\pi}} \sum_{i=0}^{\infty} (-1)^i \frac{x^{2i}}{i!} \approx \frac{2}{\sqrt{\pi}} \sum_{i=0}^{n} (-1)^i \frac{x^{2i}}{i!} \tag{34}$$

Derivatives of Scenario One

$$\frac{\partial U}{\partial r_i} = erf^{-2}(\frac{US'}{\sqrt{2}\sigma_y}) \bullet \frac{derf}{dx}\Bigg|_{x=\frac{US'}{\sqrt{2}\sigma_y}} \bullet \frac{US'}{\sqrt{2}\sigma_y{}^2} \bullet \frac{\partial \sigma_y}{\partial r_i}(\sum_{j=1}^{m} C_j) + erf^{-1}(\frac{US'}{\sqrt{2}\sigma_y})\frac{dC_i}{dr_i} \tag{35}$$

Derivatives of Scenario Two

$$\frac{\partial U}{\partial r_i} = erf^{-2}(\frac{US'}{\sqrt{2}\sigma'_y}) \bullet \frac{derf}{dx}\Bigg|_{x=\frac{US'}{\sqrt{2}\lambda\sigma_y}} \bullet \frac{US'}{\sqrt{2}\sigma'_y{}^2} \bullet \lambda \bullet \frac{\partial \sigma_y}{\partial r_i}(\sum_{j=1}^{m} \frac{C_j}{erf(\frac{k_j}{\sqrt{2}})}) + erf^{-1}(\frac{US'}{\sqrt{2}\sigma'_y}) erf^{-1}(\frac{k_i}{\sqrt{2}})\frac{dC_i}{dr_i} \tag{36}$$

$$\frac{\partial U}{\partial k_i} = -erf^{-1}(\frac{US'}{\sqrt{2}\sigma'_y}) erf^{-2}(\frac{k_i}{\sqrt{2}}) \bullet \frac{derf}{dx}\Bigg|_{x=\frac{k_i}{\sqrt{2}}} \bullet \frac{C_i}{\sqrt{2}} \tag{37}$$