Quantum error-correcting codes using tensor networks and investigating their ability to protect information from noise

Name: Sabrina Yue Wang

1 Introduction and background

1.1 Ads/CFT correspondence

Quantum field theory(QFT) is the combination of quantum mechanics and special relativity. It is concerned with the creation, interaction and destruction of particles because energy and matter can transform into each other depending on the situation. It is able to describe situations where non-relativistic quantum mechanics or special relativity alone breaks down, such as a precise description for the photon (relativistic quantum particle) and the creation of particle-antiparticle pairs from matter[**1**] . The unification of general relativity and QFT (quantum gravity) is one of the most desirable problems to be solved in theoretical physics.

Anti-de Sitter space (AdS_n), on the other hand, is the solution to Einsteins field equations in general relativity in a (vacuum) universe with a negative cosmological constant (decelerating expansion of the universe) and of dimension n[**2**]. AdS space can be imagined as a disk at a specific instance of time(figure 1).

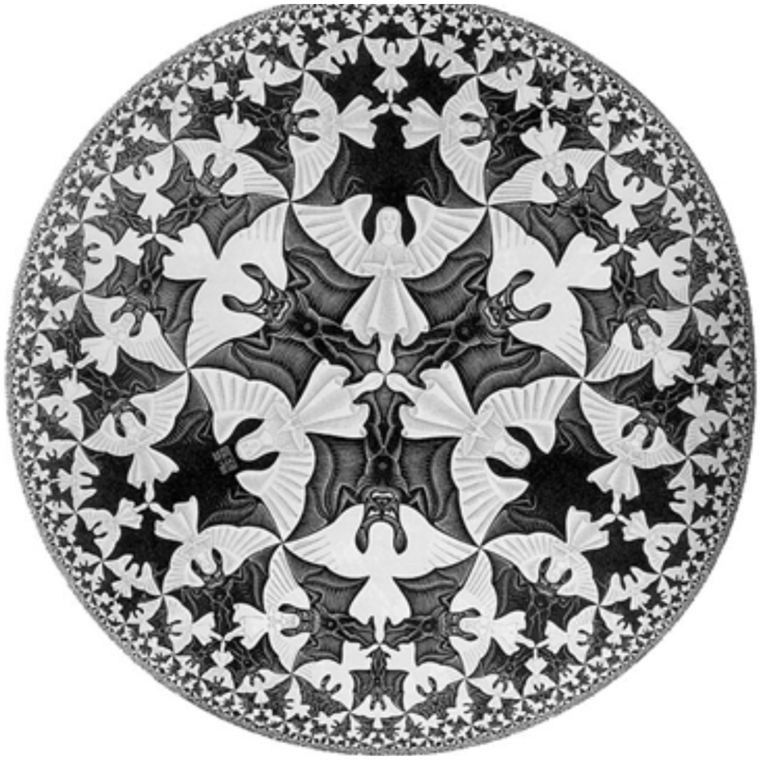

Figure 1: AdS space at a particular time[3]

Every bat shown on the disk is of the same physical size, and the outer most boundary is infinitely far away from any points in the interior of the spacetime. Stacking these disks vertically on top of each other creates a cylindrical shaped spacetime, with time along this vertical axis[**4**].

In 1999, Juan Maldacena proposed the AdS/CFT correspondence[**5**]. This was a duality/equivalence relationship between dynamics described by conformal field theory (scale invariant QFT) of d dimensions on the boundary of this cylinder, and the same dynamics described by quantum gravity in the AdS spacetime interior or 'bulk' of $d+1$ dimensions. In other words, it maps the information of this $d+1$ dimensional interior onto the exterior surface of one less dimension, also called a holographic mapping. This can be thought of as analogous as to how a 2D holographic plate can store and recreate full 3D images of an object from different angles.

Originally, the AdS/CFT correspondence was pursued mainly as a non-perturbative approach to quantum gravity. This was because perturbation theory in QFT was only successful for weakly coupled systems, and thus AdS/CFT allowed for the reformulation of specific strongly coupled QFTs in terms of classical gravity[**6**]. However, the correspondence was found to be applicable in numerous fields, such as within nuclear and condensed matter physics[**7**], making Maldacena's paper one of the most cited papers in theoretical physics. Despite the ever-increasing usage and popularity of the correspondence, it was not clear exactly how and why spacetime and gravity emerged from QFT.

In 2001, Maldacena proposed that quantum entanglement plays a key role in connecting classical spacetime and QFT descriptions[**8**]. Particles are said to be 'entangled' if their states and properties are not independent from each other, irrespective of how far apart they are. In the case of the system consisting of two spins, a basis of the joint Hilbert space is given by the tensor products of the basis vectors of each individual Hilbert space, i.e. $|\downarrow\rangle \otimes |\downarrow\rangle$, $|\downarrow\rangle \otimes |\uparrow\rangle$,$|\uparrow\rangle \otimes |\downarrow\rangle$, and $|\uparrow\rangle \otimes |\uparrow\rangle$. Thus a general two spin state is the superposition of these basis states, each term with its corresponding amplitude. A general state $|\psi\rangle$ is said to be 'entangled' if it cannot be written as a product of two single spin systems[**9**], i.e.

$$|\psi\rangle = a\,|\uparrow\rangle|\uparrow\rangle + b\,|\uparrow\rangle|\downarrow\rangle + c\,|\downarrow\rangle|\uparrow\rangle + d\,|\downarrow\rangle|\downarrow\rangle = (e\,|\uparrow\rangle + f\,|\downarrow\rangle)(g\,|\uparrow\rangle + h\,|\downarrow\rangle)$$

where a, b, c, d, e, f, g and $h \in \mathbb{C}$, is *not* an entangled state. For example, the state $\frac{|\uparrow\rangle|\uparrow\rangle+|\downarrow\rangle|\downarrow\rangle}{\sqrt{2}}$ is entangled and the spin state of the second particle can be precisely determined after measuring the first particles spin, albeit there exists an uncertainty for the first measurement.

It was argued that the product states can be naturally seen as physically belonging to separate systems and thus according to AdS/CFT, equivalent to disjoint spacetimes. However, when these states become entangled, it corresponds to the emergence of connectivity in the classical spacetime geometry[**8, 10**]. Related work in the AdS/CFT correspondence was carried out by Ryu and Takayanagi[**11**]. They showed that the entanglement entropy (quantitative measure of how much entanglement exists in a quantum state involving multiple particles) in d+1 dimensional CFTs (on the boundary region of some spacetime) can be calculated from/has a geometric interpretation in d+2 AdS space in terms of the area of minimal surfaces of d dimensions in the bulk.

Mark Van Raamsdonk consolidated these ideas in 2010 by constructing the following thought experiment[**12**]. According to the correspondence, a vacuum state (an entangled state with the lowest possible energy) on a boundary CFT has a dual on AdS spacetime. He explored what happens to the spacetime geometry when the vacuum state begins to unwind its entangled degrees of freedom. More precisely, consider the cylindrical shaped AdS spacetime as described before divided into two sections, A and B(figure 2). From Ryu and Takayanagi's work[**11**] we know that the entanglement entropy between the two regions is directly related to the minimal surface between them. Thus, if the entanglement entropy between A and B goes to zero, their spacetime will become separated. This can be visualised as the spacetime being pinched into two separate parts.

1.2 Quantum error-correction

Despite the promising prospects of the AdS/CFT correspondence, an intriguing inconsistency arises when looking into the bulk-boundary correspondence described earlier. The holographic principle requires locality (physical influences are limited by the speed of light) in the bulk, so that causality is enforced. It also requires that operators on the bulk and boundary to be commutative, but this turns out to be in direct contradiction to basic

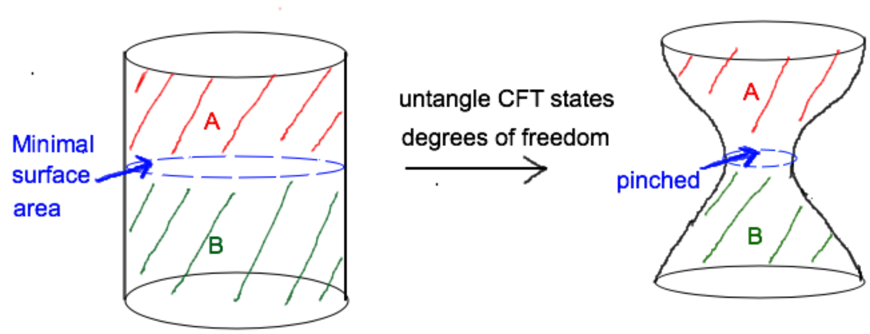

Figure 2: Cylindrical AdS spacetime with CFT living on the boundary. Two regions A and B are shown separated by a minimal surface disk. The pinching of this surface corresponds to the decrease in entanglement entropy between the (CFT description of) degrees of freedom lying in the two regions.

axioms from QFT. This paradox became the problem of bulk locality and it is related to the fact that gravity is non-local. This naturally raises the question of exactly how one can properly encode bulk states/operators as boundary CFTs as advocated by the correspondence. These issues were pointed out by Almheiri, Dong and Harlow in 2015[**13**]. They scrutinised the AdS-Rindler reconstruction, where it is sufficient to use a subregion of the entire boundary to describe an operator in the bulk as long as it is inside the causal region(regions where causality is enforced) of the subregion, and found many troubling aspects that could not be explained from a CFT point of view. One of the most important issues pointed out is that if a bulk operator $\phi(x)$ lies in multiple different causal wedges spanned by different boundary regions, then $\phi(x)$ must be mapped onto non-equivalent CFT boundary operators to ensure one does not run into the problem of contradicting QFT axioms (figure 3). They proposed that this non-unique correspondence of CFT and bulk operators could be understood and resolved from an information perspective, through quantum error-correction.

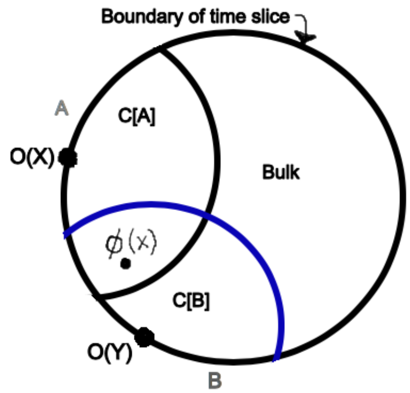

Figure 3: The spatial picture of specific time-slice, where $\phi(x)$ is an operator in the bulk, which can be holographically encoded by two different boundary regions A and B because $\phi(x)$ lies inside both causal regions. C[A] and C[B] denotes the causal regions of boundary sections A and B that houses the boundary operators O(X) and O(Y) respectively. Any operator $\phi(x)$ at some point x inside the causal regions C[A] and C[B] can be represented by unique operators O[X] and O[Y]. Image inspired from [13].

The goal of error-correcting codes is to protect information sent over noisy channels by adding surplus information so that the original information can still be recovered even if parts of the encoded message become corrupt. Classically, one can encrypt a bit 0 as 000 so that if one of the encoded bits were to be flipped by external noise (e.g. $000 \rightarrow 001$), it can still be decoded to recover the original message through a majority voting[**14**]. There are significant differences between quantum error-correction (QEC) and its classical counterpart due to additional problems arising from the laws of quantum mechanics, such as the no-cloning theorem[**14**] which prohibits the duplication of an unknown quantum state, and the collapse of the wavefunction upon any measurement.

For a quantum n-body system where each particle has two possible states, the wavefunction can take on any form within the Hilbert space of dimension 2^n. However, in many situations the states of interest are confined to a subspace of the original Hilbert space. For example, a physical state of $|\psi\rangle = \alpha|0\rangle + \beta|1\rangle, \alpha, \beta \in \mathbb{C}$ corresponds to $dim(\mathcal{H}) = 2$. In this case we can use the larger 2^n dimensional Hilbert space to mimic $|\psi\rangle$ by encoding it as $|\bar{\psi}\rangle = \alpha|0\rangle^{\otimes n} + \beta\,|1\rangle^{\otimes n}$.

More specifically, consider the 3-qutrit code. A qutrit $|\psi\rangle = \sum_{i=0}^{2} c_i\,|i\rangle$ (3 basis states in 3D Hilbert space) can be encoded as $\left|\tilde{\psi}\right\rangle = \sum c_i\,|\tilde{i}\rangle$, where $|\tilde{i}\rangle$ is a 3 qutrit state from a 3^3 dimensional Hilbert space. The explicit encoding is shown below[**14**]:

$$|0\rangle \rightarrow |\tilde{0}\rangle = |000\rangle + |111\rangle + |222\rangle \tag{1}$$

$$|1\rangle \rightarrow |\tilde{1}\rangle = |012\rangle + |120\rangle + |201\rangle \tag{2}$$

$$|2\rangle \rightarrow |\tilde{2}\rangle = |021\rangle + |102\rangle + |210\rangle \tag{3}$$

Even if one of the original qutrits ($|0\rangle, |1\rangle$ or $|2\rangle$) become 'erased' or lost, it can still be recovered from the two remaining encoded qutrits as they all include entangled states of the original basis. This addition of redundant information should be reminiscent of the principle concept of holography from before. Removing a piece of the holographic film still allows for the reconstruction of the 3D images.

In a realistic scenario, quantum information will actively/continuously experience noise from the environment and/or through different computations. The QECC must ensure that the errors from these computations do not grow uncontrollably[**14**]. For example, if a system was disturbed by unknown perturbation or even being measured, one could lose the information completely. This is another reason why it is important to encode the basis states. Figure 4 shows the circuit version of the 3-qutrit code implementation. It can be viewed as spreading the information of the single input $|\psi\rangle$ onto multiple output lines such that the input information can be recovered by measuring all the output states. This is

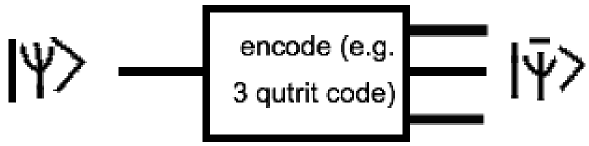

Figure 4: The LHS represents the initial physical system in some state, $|\psi\rangle$ while $|\bar{\psi}\rangle$ represents the encoded state.

very good because a perturbation on all of $|\psi\rangle$ would be more destructive than a perturbation on a portion of the output used to reconstruct $|\bar{\psi}\rangle$. The fundamental basis vectors (in this case $|0\rangle$, $|1\rangle$ and $|2\rangle$) are the same, they are just being mapped onto a subset of basis vectors in a larger Hilbert space.

It turns out that quantum error-correcting codes (QECCs) address the issues found in the AdS-Rindler reconstruction[**13**]. Going back to the 3-qutrit code, the single physical qutrit ($|i\rangle$ for $i = 0, 1, 2$) before it is encoded can be viewed as an example of a bulk operator $\phi(x)$, while a logical qutrit ($|\tilde{i}\rangle$ for $i = 0, 1, 2$) obtained from encoding can be viewed as a boundary operator corresponding to $\phi(x)$. The eradication of parts of the boundary can be seen as the 'error' that can arise in this QECC. Erasing different sections of the boundary allows for the reconstruction of $\phi(x)$ in different ways[**13**].

2 QECCs and AdS/CFT correspondence

In a follow-up paper by Pastawski, Yoshida, Harlow and Preskill in 2015[**15**], the connection between QECC and the AdS/CFT correspondence was solidified when they explicitly constructed QECCs (called holographic codes) with a tensor network structure as toy models for the AdS/CFT correspondence. In these models, the mappings of bulk to boundary operators are exactly shown, so that the AdS/CFT correspondence is explicitly realised as a QECC.

Tensor networks are useful for simulating the states for many body wavefunctions through a geometric representation of nodes and edges. Systems quickly become exponentially more difficult to deal with with increasing system size. For example, the wavefunction for a N-body system $|\psi\rangle = \sum a_{\phi_1\phi_2\ldots\phi_N} |\phi_1\rangle \otimes |\phi_2\rangle \otimes \cdots \otimes |\phi_N\rangle$ resides in a d^N dimensional Hilbert space, where d is the dimensionality of a single state. Tensor networks allow for the representation and extraction of information from the wavefunction that reduces the exponentially growing number of operations needed to of order $\mathcal{O}(poly(N))$[**16**]. This allows for the simulation of a quantum system on a classical computer. A tensor of rank n is geometrically represented as a circle with n attached lines/legs (figure 5), which interact to perform algebraic operations.

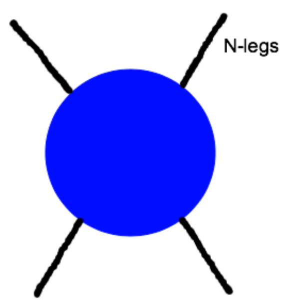

Figure 5: Tensor of rank four (four un-contracted/free legs).

The geometric structure of these toy models are constructed using n-gons as tiles on a bounded 2D hyperbolic space(figure 6). This space will be inhabited by tensors (also n-gons) connected in a network (tensor contraction is represented by joining two tensors with a line). Holographic codes are realised by placing tensors of $n + 1$ legs on each of these n-gons so that the one un-contracted leg (red circles) becomes the logical user input. The outputs (white circles) are the un-contracted legs on the boundary.

The construction of these holographic codes/toy models rely on the property of isom-

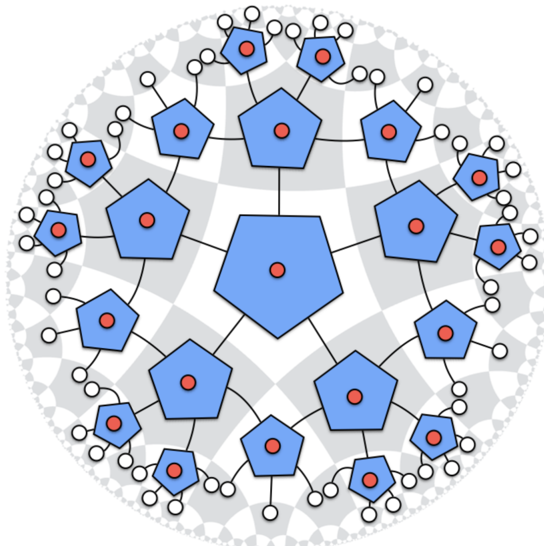

Figure 6: 2D hyperbolic space tiled using pentagons[15]. Each pentagon is inhabited by a rank 6 tensor. Red dots represent un-contracted legs (input for QECC) and white circles represent boundary un-contracted legs (output for QECC).

etry, which is a linear mapping between two Hilbert spaces, $T : \mathcal{H}_A \rightarrow \mathcal{H}_B$ such that the inner product is preserved and $dim(\mathcal{H}_B) \geqslant dim(\mathcal{H}_A)$. T can also be represented as a tensor of two indices, $T : |a\rangle \rightarrow \sum_b T_{ba} |b\rangle$ that maps the basis $\{|a\rangle\}$ for $\mathcal{H}_A$ onto the basis $\{|b\rangle\}$ for $\mathcal{H}_B$[**15**] . This isometry embeds a lower dimensional object in a higher dimensional space, like a point embedded on a line.

To reconnect this geometric picture back to QEC, the red dots(figure 6) correspond to the physical systems expressed in their basis states (i.e. $|\psi\rangle$). The tensor network corresponds to the embedding of (or isometry between) the bulk representation of $|\psi\rangle$ and the boundary representation of $|\bar{\psi}\rangle$ (white dots). A physical state/operator in the bulk encoded in the boundary of larger dimension allows for the recovery of the physical state/operator, despite having pieces of the boundary erased/become faulty, and the QECC is simply the map between bulk and boundary[**15**].

References

[**1**] A. Zee, *Quantum field theory in a nutshell.* Princeton university press, 2010.

[**2**] I. Bengtsson, "Anti-de sitter space." Lecture Notes, 1998.

[**3**] M. C. Escher, "Circle limit iv (heaven and hell)." Woodcut art piece, 1960.

[**4**] L. Susskind, "Entanglement and complexity: Gravity and quantum mechanics (lecture)." Stanford Institude for Theoretical Physics, 2015. Lecture.

[**5**] J. Maldacena, "The large n limit of superconformal field theories and supergravity," in *AIP Conference Proceedings CONF-981170*, vol. 484, pp. 51–63, AIP, 1999.

[**6**] A. Zaffaroni, "Introduction to the ads-cft correspondence," *Classical and Quantum Gravity*, vol. 17, no. 17, p. 3571, 2000.

[**7**] S. A. Hartnoll, "Lectures on holographic methods for condensed matter physics," *Classical and Quantum Gravity*, vol. 26, no. 22, p. 224002, 2009.

[**8**] J. Maldacena, "Eternal black holes in anti-de sitter," *Journal of High Energy Physics*, vol. 2003, no. 04, p. 021, 2003.

[**9**] R. Horodecki, P. Horodecki, M. Horodecki, and K. Horodecki, "Quantum entanglement," *Reviews of modern physics*, vol. 81, no. 2, p. 865, 2009.

[**10**] M. van Raamsdonk, "Gravity and entanglement (lecture)." Stanford Institude for Theoretical Physics, 2015.

[**11**] S. Ryu and T. Takayanagi, "Holographic derivation of entanglement entropy from the anti–de sitter space/conformal field theory correspondence," *Physical review letters*, vol. 96, no. 18, p. 181602, 2006.

[**12**] M. Van Raamsdonk, "Building up space–time with quantum entanglement," *International Journal of Modern Physics D*, vol. 19, no. 14, pp. 2429–2435, 2010.

[**13**] A. Almheiri, X. Dong, and D. Harlow, "Bulk locality and quantum error correction in ads/cft," *Journal of High Energy Physics*, vol. 2015, no. 4, p. 163, 2015.

[**14**] M. A. Nielsen and I. Chuang, *Quantum computation and quantum information*. AAPT, 2002.

[**15**] F. Pastawski, B. Yoshida, D. Harlow, and J. Preskill, "Holographic quantum error-correcting codes: Toy models for the bulk/boundary correspondence," *Journal of High Energy Physics*, vol. 2015, no. 6, pp. 1–55, 2015.

[**16**] D. R. Ramanujam, "Frontiers in quantum science 2012: introduction to tensor networks states." Lecture, 2015.